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ASYMPTOTIC ANALYSIS AND TOPOLOGICAL DERIVATIVE FOR 3D QUASI-LINEAR MAGNETOSTATICS

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Abstract. In this paper we study the asymptotic behaviour of the quasilinear curl-curl equation of 3D magnetostatics with respect to a singular perturbation of the differential operator and prove the existence of the topological derivative using a Lagrangian approach. We follow the strategy proposed in Gangl and Sturm (*ESAIM: COCV* **26** (2020) 106) where a systematic and concise way for the derivation of topological derivatives for quasi-linear elliptic problems in H^1 is introduced. In order to prove the asymptotics for the state equation we make use of an appropriate Helmholtz decomposition. The evaluation of the topological derivative at any spatial point requires the solution of a nonlinear transmission problem. We discuss an efficient way for the numerical evaluation of the topological derivative in the whole design domain using precomputation in an offline stage. This allows us to use the topological derivative for the design optimization of an electrical machine.

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1. Introduction

The main result of this paper is the computation of the topological derivative for the tracking-type cost function

$$J(\Omega) = \int_{\Omega_g} |\operatorname{curl} u - B_d|^2 \, \mathrm{d}x \tag{1.1}$$

subject to the constraint that $u \in V(\mathsf{D})^3 := \{u \in H_0(\mathsf{D}, \operatorname{curl}) : \operatorname{div}(u) = 0 \text{ in } \mathsf{D}\}$ solves

$$\int_{\mathsf{D}} \mathcal{A}_\Omega(x, \operatorname{curl} u) \cdot \operatorname{curl} v \, \mathrm{d}x = \langle F, v \rangle \quad \text{for all} \quad v \in V(\mathsf{D})^3, \tag{1.2}$$

where $\mathcal{A}_\Omega : \mathsf{D} \times \mathbf{R}^3 \to \mathbf{R}^3$ is a piecewise nonlinear function defined by

$$\mathcal{A}_\Omega(x, y) := \begin{cases} a_1(y) & \text{for} \quad x \in \Omega, \\ a_2(y) & \text{for} \quad x \in \mathsf{D} \setminus \Omega, \end{cases} \tag{1.3}$$

with two continuously differentiable functions $a_1, a_2 : \mathbf{R}^3 \to \mathbf{R}^3$ satisfying the following assumption:

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Assumption A. *There are constants $c_1, c_2, c_3 > 0$ such that the functions $a_i : \mathbf{R}^3 \to \mathbf{R}^3$, $i = 1, 2$ are differentiable and satisfy:*

(i) $(a_i(x) - a_i(y)) \cdot (x - y) \geq c_1|x - y|^2$, *for all* $x, y \in \mathbf{R}^3$,
(ii) $|a_i(x) - a_i(y)| \leq c_2|x - y|$ *for all* $x, y \in \mathbf{R}^3$,
(iii) $|\partial a_i(x) - \partial a_i(y)| \leq c_3|x - y|$ *for all* $x, y \in \mathbf{R}^3$.

The right hand side F is a linear and continuous functional on $V(\mathsf{D})^3$ defined by

$$\langle F, v\rangle := \int_{\Omega_1} J \cdot v \, \mathrm{d}x + \int_{\Omega_2} M \cdot \operatorname{curl} v \, \mathrm{d}x \quad \text{for} \quad v \in V(\mathsf{D})^3,$$

where $\Omega_1, \Omega_2 \subset \mathsf{D}$ are open sets (see Fig. 1) and $J, M \in L_2(\mathsf{D})^3$. Properties (i) and (ii) of Assumption A imply that the operator $A_\Omega : V(\mathsf{D})^3 \to \mathcal{L}(V(\mathsf{D})^3, \mathbf{R})$ defined by $\langle A_\Omega\varphi, \psi\rangle := \int_{\mathsf{D}} \mathcal{A}_\Omega(x, \operatorname{curl}\varphi) \cdot \operatorname{curl}\psi \, \mathrm{d}x$ is Lipschitz continuous and strongly monotone for all measurable $\Omega \subset \mathsf{D}$. Hence the state equation (1.2) admits a unique solution by the theorem of Zarantonello; see page 504, Theorem 25.B of [36].

Among other applications the set of equations (1.2) models a 3D electrical machine and captures nonlinear physical effects. A realistic physical model for which the above assumption are satisfied in practice will be presented in the last section.

The topological derivative has already been computed for many linear PDEs and also the literature on its numerical implementation is rich. We refer to the monograph [23] for many examples and also references therein. For nonlinear PDEs the literature is far less complete and only few articles dealing with nonlinear constraints exist. Here we would like to mention [2, 20], and more recently [33], where semilinear problems were studied.

Concerning quasi-linear problems, in which the topological perturbation enters in the main part of the nonlinearity, even less work has been done. Here we mention [3] where the authors consider a regularised version of the p-Poisson equation and also [4] where the topological derivative for the quasi-linear equation of 2D magnetostatics was derived. More recently, in [13] the topological derivative for a class of quasi-linear equations under fairly general assumptions in an H^1 setting was presented.

Shape optimisation for the linear Maxwell's equation has been studied in the context of time-harmonic electromagnetic waves [17], magnetic impedance tomography [18], in electromagentic scattering [9] and [19], where the last work takes a geometric viewpoint using differential forms. All these articles deal with linear problems and as far as the present authors knowledge no work has been done in the nonlinear case. In the context of optimal control in a quasi-linear $H(\text{curl})$ setting we mention [34], where also numerical analysis is presented.

The topological sensitivity of 2D nonlinear magnetostatics, which is a simplification of Maxwell's equation in 3D, was treated in [4]. The topological sensitivity of three dimensional linear Maxwell's equations has been studied in [22] and is based on asymptotics derived in [1]. In the nonlinear context it seems no work has been done so far.

To our knowledge the asymptotics for (1.2) with respect to a singular perturbation of the operator is unknown. Accordingly also the topological derivative for the functional (1.1) and its numerical implementation are new. These are the main contribution of this paper.

The structure of the paper is as follows. In Section 2 we recall a regular Helmholtz decomposition and prove a Helmholtz-type decomposition in $\mathbf{R}^3$ which will be essential for the asymptotic analysis of the next section. In Section 3 we present the asymptotic analysis of the state equation (1.2). In Section 4 we compute the topological derivative for the cost function (1.1) using a Lagrangian method. In Section 5 we discuss the efficient numerical realisation of the obtained topological derivative. Finally, in the last section, we apply our results to a 3D electric machine and verify the pertinence of our approach in several numerical experiments.

Notation and definitions

Standard L^p spaces and Sobolev spaces on an open set $\mathsf{D} \subset \mathbf{R}^3$ are denoted $L_p(\mathsf{D})$ and $W^k_p(\mathsf{D})$, respectively, where $p \geq 1$ and $k \geq 1$. In case $p = 2$ and $k \geq 1$ we set as usual $H^k(\mathsf{D}) := W^k_2(\mathsf{D})$. Vector valued spaces are denoted $L_p(\mathsf{D})^3 := L_p(\mathsf{D},\mathbf{R}^3)$ and $W_p^k(\mathsf{D})^3 := W_p^k(\mathsf{D},\mathbf{R}^3)$. Given a normed vector space V we denote by $\mathcal{L}(V,\mathbf{R})$ the space of linear and continuous functions on V. We recall the definition of the space $H(\mathsf{D},\mathrm{curl}) = \{u \in L_2(\mathsf{D})^3 : \mathrm{curl} \in L_2(\mathsf{D})^3\}$ and also

$$H_0(\mathsf{D},\mathrm{curl}) = \left\{u \in H(\mathsf{D},\mathrm{curl}) : \int_{\mathsf{D}} \mathrm{curl}\, u \cdot v = \int_{\mathsf{D}} u \cdot \mathrm{curl}\, v \quad \text{for all} \quad v \in H^1(\mathsf{D})^3\right\} \tag{1.4}$$

equipped with the norm $\|u\|^2_{H(\mathsf{D},\mathrm{curl})} := \|u\|^2_{L_2(\mathsf{D})^3} + \|\,\mathrm{curl}\, u\|^2_{L_2(\mathsf{D})^3}$. It can be shown that $H_0(\mathsf{D},\mathrm{curl}) = \{u \in L^2(\mathsf{D})^3 | \,\mathrm{curl}\, u \in L^2(\mathsf{D})^3$ and $u \times n = 0$ on $\partial\mathsf{D}\}$. Moreover, we define the subspace

$$V(\mathsf{D})^3 := \{u \in H_0(\mathsf{D},\mathrm{curl}) : \mathrm{div}(u) = 0 \quad \text{on} \quad \mathsf{D}\}. \tag{1.5}$$

Recall that the Friedrich's inequality $\|u\|_{L_2(\mathsf{D})^3} \le C\|\,\mathrm{curl}\, u\|_{L_2(\mathsf{D})^3}$ holds for all $u \in V(\mathsf{D})^3$ provided D is a simply connected bounded Lipschitz domain; see [30], Corollary 3.2 or [5], Theorem 5.1.

We let $\mathrm{BL}(\mathbf{R}^3) := \{u \in H^1_{\mathrm{loc}}(\mathbf{R}^3) : \nabla u \in L_2(\mathbf{R}^3)^3\}$ and define the *Beppo–Levi space* or *homogeneous Sobolev space* as the quotient space $\dot{\mathrm{BL}}(\mathbf{R}^3) := \mathrm{BL}(\mathbf{R}^3)/\mathbf{R}$, where $/\mathbf{R}$ means that we quotient out the constant functions. We denote by $[u]$ the equivalence classes of $\dot{\mathrm{BL}}(\mathbf{R}^3)$. Equipped with the norm

$$\|[u]\|_{\dot{\mathrm{BL}}(\mathbf{R}^3)} := \|\nabla u\|_{L_2(\mathbf{R}^3)^3}, \quad u \in [u], \tag{1.6}$$

the Beppo–Levi space is a Hilbert space (see [11, 24]) and $C_c^\infty(\mathbf{R}^3)/\mathbf{R}$ is dense in $\dot{\mathrm{BL}}(\mathbf{R}^3)$. The vector valued Beppo–Levi space $\dot{\mathrm{BL}}(\mathbf{R}^3,\mathbf{R}^3)$ will be denoted by $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$ and equipped with the standard norm. Whenever no confusion is possible we will not distinguish between an equivalence class $[u]$ and a representative u and use the same notation. This will be clear from the context.

In the whole paper we equip $\mathbf{R}^d$ with the Euclidean norm $|\cdot|$ and use the same notation for the corresponding matrix (operator) norm. We denote by $B_\delta(x)$ the Euclidean ball centred at x with radius $\delta > 0$.

Remark 1.1. As remarked in Remark 2.2 of [13], it follows from Assumption A that the non-linearity a_i satisfies:

$$|a_i(x)| \le |a_i(0)| + c_2|x|, \tag{1.7}$$

$$|\partial a_i(x)| \le |\partial a_i(0)| + c_3|x|, \tag{1.8}$$

$$|\partial a_i(x)v| \le c_2|v|, \tag{1.9}$$

for $i = 1, 2$ and for all $x, v \in \mathbf{R}^3$.

2. Helmholtz-type decompositions in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$

In this section we develop the function space setting for the exterior equation that will appear in the asymptotic expansion of the state equation (see Sect. 3). In particular we will study a subspace of the Beppo–Levi space $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$ and derive a Helmholtz-type decomposition, which will be essential later on. We recall the following regular Helmholtz decomposition of functions in $H_0(\mathsf{D},\mathrm{curl})$; see, *e.g.*, [16], Lemma 3.4, [28, 30], Theorem 29. Throughout this section we assume that $\mathsf{D} \subset \mathbf{R}^3$ is a simply connected bounded Lipschitz domain.

Lemma 2.1 (Regular decomposition of $H_0(\mathsf{D},\mathrm{curl})$)**.** *For every $u \in H_0(\mathsf{D},\mathrm{curl})$ there exist $\phi \in H_0^1(\mathsf{D})$, $u^* \in H_0^1(\mathsf{D})^3$ such that*

$$u = \nabla\phi + u^*.$$

Moreover, the following estimates hold:

$$\|\phi\|_{H^1(\mathsf{D})} \le C\|u\|_{H(\mathsf{D},\mathrm{curl})} \quad \text{and} \quad \|u^*\|_{H^1(\mathsf{D})^3} \le C\|\,\mathrm{curl}\, u\|_{L^2(\mathsf{D})^3}.$$

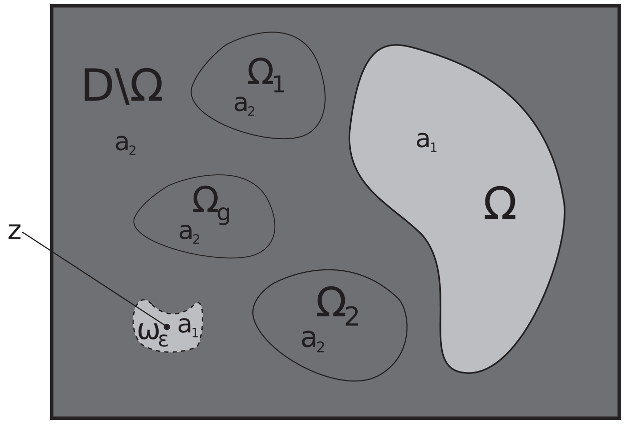

FIGURE 1. Setting for topological derivative: Inclusion ω_ε of radius $\varepsilon > 0$ containing material a_1 around point $z \in \mathsf{D} \setminus \bar{\Omega}$ (where material a_2 is present).

The following Helmholtz decomposition is standard.

Lemma 2.2. *For every* $u \in H^1_0(\mathsf{D})^3$ *we find* $\phi \in H^1_0(\mathsf{D})$ *and* $\psi \in V(\mathsf{D})^3$*, such that*

$$u = \nabla\phi + \psi. \tag{2.1}$$

Proof. This follows directly by solving for given $u \in H^1_0(\mathsf{D})^3$: find $\phi \in H^1_0(\mathsf{D})$, such that

$$\int_{\mathsf{D}} \nabla\phi \cdot \nabla v \,\mathrm{d}x = \int_{\mathsf{D}} u \cdot \nabla v \,\mathrm{d}x \quad \text{for all} \quad v \in H^1_0(\mathsf{D}). \tag{2.2}$$

Then $\psi := u - \nabla\phi$ satisfies (2.1) and $\operatorname{div}(\psi) = 0$. To see the boundary condition note that since $u \in H^1_0(\mathsf{D})^3$, we have by partial integration

$$\int_{\mathsf{D}} \operatorname{curl}\psi \cdot v \,\mathrm{d}x = \int_{\mathsf{D}} \operatorname{curl} u \cdot v \,\mathrm{d}x = \int_{\mathsf{D}} u \cdot \operatorname{curl} v \,\mathrm{d}x = \int_{\mathsf{D}} u \cdot \operatorname{curl} v \,\mathrm{d}x - \int_{\mathsf{D}} \nabla\phi \cdot \operatorname{curl} v \,\mathrm{d}x \tag{2.3}$$

for all $v \in H^1(\mathsf{D})^3$. Here we used that the last integral vanishes, which can be seen by partial integration due to $\phi \in H^1_0(\mathsf{D})$. Noting that $\psi + \nabla\phi = u$, it follows $\psi \in V(\mathsf{D})^3$; see (1.4). This finishes the proof. □

We will now introduce a subspace of the space $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$. The reason why we cannot work with $H(\mathbf{R}^3, \operatorname{curl})$ directly is that we do not have control over the function u itself, but only over its curl. In order to get around this difficulty we introduce the following function space. We also refer to [1] for a different approach using weighted spaces.

Definition 2.3. We define the space

$$\mathrm{BLC}(\mathbf{R}^3) := \overline{\{\varphi \in C^\infty_c(\mathbf{R}^3)^3 : \operatorname{div}(\varphi) = 0\}}^{|\cdot|_{H(\mathbf{R}^3,\operatorname{curl})}}, \tag{2.4}$$

where $|\varphi|^2_{H(\mathbf{R}^3,\operatorname{curl})} := \int_{\mathbf{R}^3} |\operatorname{curl}\varphi|^2 \,\mathrm{d}x$. We set $\dot{\mathrm{BLC}}(\mathbf{R}^3) := \mathrm{BLC}(\mathbf{R}^3)/\mathbf{R}$, where $/\mathbf{R}$ means that we quotient out constants.

We have the following result.

Lemma 2.4. (i) *We have* $\mathrm{BLC}(\mathbf{R}^3) \subset \mathrm{BL}(\mathbf{R}^3)^3$ *and hence* $\dot{\mathrm{BLC}}(\mathbf{R}^3) \subset \dot{\mathrm{BL}}(\mathbf{R}^3)^3$.

(ii) *The space* $\dot{\mathrm{BLC}}(\mathbf{R}^3)$ *becomes a Hilbert space when equipped with* $|\cdot|_{H(\mathbf{R}^3,\mathrm{curl})}$.

(iii) *We have* $\dot{\mathrm{BLC}}(\mathbf{R}^3) = \{u \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3 : \mathrm{div}(u) = 0\}$.

Proof. We start by observing that (see [30], Rem. 1.1)

$$\int_{\mathbf{R}^3} |\,\mathrm{div}(\varphi)|^2 + |\,\mathrm{curl}(\varphi)|^2 \,\mathrm{d}x = \int_{\mathbf{R}^3} |\nabla\varphi|^2 \,\mathrm{d}x \tag{2.5}$$

holds for all test functions $\varphi \in C_c^\infty(\mathbf{R}^3)^3$. Therefore we have

$$|\varphi|^2_{H(\mathbf{R}^3,\mathrm{curl})} = \int_{\mathbf{R}^3} |\,\mathrm{curl}(\varphi)|^2 \,\mathrm{d}x = \int_{\mathbf{R}^3} |\nabla\varphi|^2 \,\mathrm{d}x \tag{2.6}$$

for all test functions $\varphi \in C_c^\infty(\mathbf{R}^3)^3$ satisfying $\mathrm{div}(\varphi) = 0$. Let (φ_n) be a sequence in $C_c^\infty(\mathbf{R}^3)^3$ with $\mathrm{div}(\varphi_n) = 0$ that is Cauchy with respect to $|\cdot|_{H(\mathbf{R}^3,\mathrm{curl})}$. Then in view of (2.6) it also converges in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$ and hence its limit belongs to $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$, which shows the inclusion (i). Also (ii) follows at once since a closed subspace of a Hilbert space is a Hilbert space itself.

To see (iii) we can use standard mollifier techniques; see page 21 of [37]. Let $u \in L_{2,loc}(\mathbf{R}^3)^3$ with $\nabla u \in L_2(\mathbf{R}^3)^{3\times 3}$ and $\mathrm{div}(u) = 0$. Let $\xi \in C_c^\infty(\overline{B_1(0)})$ with $\int_{\mathbf{R}^3} \xi \,\mathrm{d}x = 1$. Set $\xi_\varepsilon(x) := \varepsilon^{-3}\xi(x/\varepsilon)$ and define the convolution of u with ξ_ε by $u_\varepsilon(x) := (\xi_\varepsilon * u)(x) := \int_{\mathbf{R}^3} \xi_\varepsilon(x-y)u(y)\,\mathrm{d}y$. Then u_ε is smooth, has compact support and satisfies $\partial_{x_i} u_\varepsilon(x) = \xi_\varepsilon * (\partial_{x_i} u)(x)$ and thus $\mathrm{div}(u_\varepsilon) = \xi_\varepsilon * (\mathrm{div}(u)) = 0$. Since $\partial_{x_i} u \in L_2(\mathbf{R}^3)^3$ we conclude from [37], Theorem 1.6.1, (iii) that $\partial_{x_i} u_\varepsilon \to \partial_{x_i} u$ strongly in $L_2(\mathbf{R}^3)^3$ as $\varepsilon \searrow 0$. But this means that $u \in \dot{\mathrm{BLC}}(\mathbf{R}^3)$ and finishes the proof. ∎

We now prove a Helmholtz-type decomposition in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$. It can be seen as an analogue of Lemma 2.2 in case $\mathsf{D} = \mathbf{R}^3$. We also refer to [31, 32] for Helmholtz decompositions in exterior domains.

Let us introduce

$$\mathrm{BL}^2(\mathbf{R}^3) := \{\varphi \in L_{2,loc}(\mathbf{R}^3) : \partial^2_{x_i x_j}\varphi \in L_2(\mathbf{R}^3),\ i,j \in \{1,2,3\}\}, \tag{2.7}$$

and the associated second order Beppo–Levi space $\dot{\mathrm{BL}}^2(\mathbf{R}^3) := \mathrm{BL}^2(\mathbf{R}^3)/P$, where $P := \{x \mapsto b + x\cdot a : b \in \mathbf{R}, a \in \mathbf{R}^3\}$ denotes the space of linear functions in $\mathbf{R}^3$. The function

$$\|\phi\|_{\mathrm{BL}^2} := \|\partial^2\phi\|_{L_2(\mathbf{R}^3)^{3\times 3}}, \quad \phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3) \tag{2.8}$$

is a norm on $\dot{\mathrm{BL}}^2(\mathbf{R}^3)$ and makes it a Hilbert space; see [11], Section III and Theorem 2.1.

Remark 2.5. We note that it makes sense to say that an equivalence class $\varphi \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ has zero divergence $\mathrm{div}(\varphi) = 0$, since the divergence of a constant function is zero and hence the divergence free property is independent of the representative.

Lemma 2.6. *For every* $u \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ *there is* $\phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3)$ *and* $u^* \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ *with* $\mathrm{div}(u^*) = 0$, *such that*

$$u = \nabla\phi + u^* \quad (\text{in} \quad \dot{\mathrm{BL}}(\mathbf{R}^3)^3). \tag{2.9}$$

In fact, we have the direct sum $\dot{\mathrm{BL}}(\mathbf{R}^3)^3 = \nabla(\dot{\mathrm{BL}}^2(\mathbf{R}^3)) \oplus \dot{\mathrm{BLC}}(\mathbf{R}^3)^3$.

Proof. We will use arguments from [27], Theorem 3.3. Given $u \in C_c^\infty(\mathbf{R}^3)^3$ we define $\phi \in C^\infty(\mathbf{R}^3)$ as

$$\phi(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{\mathrm{div}\, u(y)}{|x-y|}\,\mathrm{d}y, \quad x \in \mathbf{R}^3. \tag{2.10}$$

Since u is smooth and has compact support we have $\Delta\phi = \operatorname{div} u$ pointwise in $\mathbf{R}^3$ (see [12], p. 21, Thm. 1). The Caldéron-Zygmund theorem (see [7,8] and also [15], Para. 9.4) implies that

$$\|\partial^2\phi\|_{L_2(\mathbf{R}^3)^{3\times 3}} \le C\|\operatorname{div} u\|_{L_2(\mathbf{R}^3)}. \tag{2.11}$$

However, this means that $\phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3)$ and hence $u^* := \nabla\phi - u$ satisfies $\operatorname{div}(u^*) = 0$ and $\nabla u^* \in L_2(\mathbf{R}^3)^3$. Therefore $u^* \in \dot{\mathrm{BLC}}(\mathbf{R}^3)$ and u^* satisfies (2.9).

Let now $u \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ and $(u_n) \subset C_c^\infty(\mathbf{R}^3)^3$ with $\nabla u_n \to \nabla u$ strongly in $L_2(\mathbf{R}^3)^{3\times 3}$ as $n \to \infty$. The first part of the proof shows that we can split $u_n = \nabla\phi_n + u_n^*$ with $\phi_n \in \dot{\mathrm{BL}}(\mathbf{R}^3)$ and $u_n^* \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ satisfying $\operatorname{div}(u_n^*) = 0$. In view of (2.11) it follows that (ϕ_n) is a Cauchy sequence in $\dot{\mathrm{BL}}^2(\mathbf{R}^3)$ and thus converging to some $\phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3)$. From this it follows that also (u_n^*) is a Cauchy sequence in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$ and converges to some $u^* \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ satisfying $\operatorname{div}(u^*) = 0$. Now we can pass to the limit in $\partial_{x_i} u_n = \partial_{x_i}\nabla\phi_n + \partial_{x_i} u_n^*$, $i = 1, 2, 3$ with respect to the $L_2(\mathbf{R}^3)^3$ norm to obtain

$$\partial_{x_i}(u - \nabla\phi - u^*) = 0, \quad a.e. \quad \text{on} \quad \mathbf{R}^3, \quad i = 1, 2, 3. \tag{2.12}$$

It follows that $u - \nabla\phi - u^*$ is constant on $\mathbf{R}^3$. Therefore $u = \nabla\phi + u^*$ in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$.

To show that the sum is direct, we let $\tilde{\phi}, \phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3)$ and $\tilde{u}^*, u^* \in \dot{\mathrm{BLC}}(\mathbf{R}^3)^3$, such that $u = \nabla\phi + u^* = \nabla\tilde{\phi} + \tilde{u}^*$. Set $\hat{\phi} := \tilde{\phi} - \phi$ and $\hat{u}^* := \tilde{u}^* - u^*$. We have $\nabla\hat{\phi} = -\hat{u}^*$ and thus since $\hat{u}^*$ is divergence free, $\Delta\hat{\phi} = 0$, that is, $\hat{\phi}$ is harmonic. By Weyl's lemma $\hat{\phi}$ is smooth. Since $\hat{\phi}$ is harmonic $v := \partial^2_{x_i x_j}\hat{\phi}$ is harmonic, too and hence enjoys the mean value property (see [12], Thm. 2, p. 25):

$$v(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} v \,\mathrm{d}x, \quad r > 0,\ x_0 \in \mathbf{R}^3. \tag{2.13}$$

Fix $x_0 \in \mathbf{R}^3$. Then we obtain from Hölder's inequality

$$|v(x_0)| \le \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |v| \,\mathrm{d}x \le \frac{1}{|B_r(x_0)|^{1/2}}\|v\|_{L_2(B_r(x_0))} \le Cr^{-\frac{3}{2}}\|\partial^2\hat{\phi}\|_{L_2(\mathbf{R}^3)^{3\times 3}}. \tag{2.14}$$

Passing to the limit $r \to \infty$ we see that $v(x_0) = 0$ and since x_0 was arbitrary we have $v = \partial^2_{x_i x_j}\hat{\phi} = 0$ on $\mathbf{R}^3$. Hence $\hat{\phi}(x) = a \cdot x + b$ for some $a \in \mathbf{R}^3, b \in \mathbf{R}$ and thus the corresponding equivalence class $\hat{\phi} = 0$ in $\dot{\mathrm{BL}}^2(\mathbf{R}^3)$ or equivalently $\phi = \tilde{\phi}$ as elements in $\dot{\mathrm{BL}}^2(\mathbf{R}^3)$. In view of $a = \nabla\hat{\phi} = -\hat{u}^*$ it follows that $\hat{u}^* = 0$ in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$ or equivalently $u^* = \tilde{u}^*$. This shows that we have a direct sum. □

The following example illustrates the usefulness of the function space $\dot{\mathrm{BLC}}(\mathbf{R}^3)$ and the Helmholtz-type decomposition.

Example 2.7. Let $\zeta \in \mathbf{R}^3$ be a vector and let $\omega \subset \mathbf{R}^3$ be an open and bounded set. Consider the problem: find $K \in \dot{\mathrm{BLC}}(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3} \beta_\omega \operatorname{curl} K \cdot \operatorname{curl} v \,\mathrm{d}x = \int_\omega \zeta \cdot \operatorname{curl} v \,\mathrm{d}x \quad \text{for all} \quad v \in \dot{\mathrm{BLC}}(\mathbf{R}^3), \tag{2.15}$$

where $\beta_\omega := \beta_1\chi_\omega + \beta_2\chi_{\mathbf{R}^3\setminus\omega}$ with $\beta_1, \beta_2 > 0$. This system appears in the derivation of the topological derivative for Maxwell's equation in the linear case; see [22] on page 553. Thanks to the theorem of Lax–Milgram there exists a unique solution K of (2.15) in $\dot{\mathrm{BLC}}(\mathbf{R}^3)$. Moreover, for given $v \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ we find according to Lemma 2.6 the decomposition $v = \nabla\phi + v^*$ with $\phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3)$ and $v^* \in \dot{\mathrm{BLC}}(\mathbf{R}^3)$. Therefore plugging $v^* = v - \nabla\phi$ as test function in (2.15) and using $\operatorname{curl}(\nabla\phi) = 0$, we obtain

$$\int_{\mathbf{R}^3} \beta_\omega \operatorname{curl} K \cdot \operatorname{curl} v \,\mathrm{d}x = \int_\omega \zeta \cdot \operatorname{curl} v \,\mathrm{d}x \quad \text{for all} \quad v \in \mathrm{BL}(\mathbf{R}^3)^3. \tag{2.16}$$

Remark 2.8. We remark that there are alternatives to the choice of the space $\mathrm{BLC}(\mathbf{R}^3)$ defined in (2.4). We mention the use of weighted Sobolev spaces as it was done in [1], where the L_2-part of the norm is weighted by the function $x \mapsto \frac{1}{\sqrt{\|x\|^2+1}}$. Another alternative is to use the Beppo–Levi type factor space

$$\dot{H}(\mathrm{curl}, \mathbf{R}^3) := \{[v + \nabla \dot{\mathrm{BL}}(\mathbf{R}^3)] : v \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbf{R}^3) : \mathrm{curl}\, v \in L_2(\mathbf{R}^3)\} \tag{2.17}$$

together with the norm

$$\|[v]\|_{\dot{H}(\mathrm{curl},\mathbf{R}^3)} := \|\,\mathrm{curl}\, v\|_{L_2(\mathbf{R}^3)^3}. \tag{2.18}$$

3. Asymptotics of the state equation

3.1. Main result for direct state

In this section we study the behaviour of $u_\varepsilon - u_0$, where, for $\varepsilon > 0$, $u_\varepsilon \in V(\mathsf{D})^3$ is the solution to

$$\int_{\mathsf{D}} \mathcal{A}_\varepsilon(x, \mathrm{curl}\, u_\varepsilon) \cdot \mathrm{curl}\, v \; \mathrm{d}x = \langle F, v\rangle \quad \text{for all} \quad v \in V(\mathsf{D})^3, \tag{3.1}$$

with $\mathcal{A}_\varepsilon := \mathcal{A}_{\Omega_\varepsilon}$ and $\Omega_\varepsilon(z) := \Omega \cup \omega_\varepsilon(z)$ and u_0 is the solution to (1.2). Here, $\Omega \Subset \mathsf{D}$ is open and the scaled inclusion $\omega_\varepsilon(z) := z + \varepsilon\omega$ is defined by an open and bounded set $\omega \subset \mathbf{R}^d$ satisfying $0 \in \omega$ and the center of the inclusion $z \in \Omega \cup \mathsf{D} \setminus \overline{\Omega}$. For simplicity and without loss of generality, we will assume $z := 0 \in \mathsf{D} \setminus \overline{\Omega}$ throughout this paper. Moreover, for simplicity we assume that $\Omega = \emptyset$ (the general case can be readily retrieved by minor modifications).

Using Lemma 2.1 we find the regular decomposition

$$u_\varepsilon = \nabla \phi_\varepsilon + u_\varepsilon^*, \quad \phi_\varepsilon \in H_0^1(\mathsf{D}),\; u_\varepsilon^* \in H_0^1(\mathsf{D})^3. \tag{3.2}$$

Definition 3.1. The variation of u_ε is defined by

$$K_\varepsilon := \left(\frac{u_\varepsilon - u_0}{\varepsilon}\right) \circ T_\varepsilon \in V(\varepsilon^{-1}\mathsf{D})^3, \quad \varepsilon > 0, \tag{3.3}$$

and the variation of u_ε^* is defined by

$$K_\varepsilon^* := \left(\frac{u_\varepsilon^* - u_0^*}{\varepsilon}\right) \circ T_\varepsilon \in H_0^1(\varepsilon^{-1}\mathsf{D})^3, \quad \varepsilon > 0, \tag{3.4}$$

where $T_\varepsilon(x) := \varepsilon x$ for $x \in \mathbf{R}^3$. By extending u_ε^* by zero outside of D we can view K_ε^* as a function in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$.

Now we can state our first main theorem.

Main Theorem 1. *Assume that* $\mathrm{curl}\, u_0 \in C^\alpha(\overline{B_\delta(z)})^3$ *for some* $\delta > 0$ *and* $0 < \alpha < 1$. *Then we have*

(i) *There exists a unique* $K \in \dot{\mathrm{BLC}}(\mathbf{R}^3)$, *such that*

$$\begin{aligned}\int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \mathrm{curl}\, K + U_0) - \mathcal{A}_\omega(x, U_0)) \cdot \mathrm{curl}\, \varphi \; \mathrm{d}x \\ = -\int_\omega \big(a_1(U_0) - a_2(U_0))\big) \cdot \mathrm{curl}\, \varphi \; \mathrm{d}x\end{aligned} \tag{3.5}$$

for all $\varphi \in \mathrm{BLC}(\mathbf{R}^3)$. *Here* $U_0 := \mathrm{curl}(u_0)(z)$ *and* $\mathcal{A}_\omega(x, y) := a_1(y)\chi_\omega(x) + a_2(y)\chi_{\mathbf{R}^3\setminus\omega}(x)$.

(ii) *The family* (K_ε^*) *defined in* (3.4), *satisfies*

$$\mathrm{curl}(K_\varepsilon^*) \to \mathrm{curl}(K) \quad \text{strongly in} \quad L_2(\mathbf{R}^3)^3 \quad \text{as} \quad \varepsilon \searrow 0. \tag{3.6}$$

Proof. Proof of (i): Thanks to Assumption A the operator $B_\omega : \dot{\mathrm{BLC}}(\mathbf{R}^3) \to \dot{\mathrm{BLC}}(\mathbf{R}^3)^*$ defined by $\langle B_\omega \varphi, \psi\rangle := \int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl}\varphi + U_0) - \mathcal{A}_\omega(x, U_0)) \cdot \operatorname{curl}\psi \, \mathrm{d}x$ is strongly monotone and Lipschitz continuous and hence the unique solvability follows by the theorem of Zarantonello; see [36], Theorem 25.B, p. 504.

The proof of (ii) is given in the subsequent sections. □

Before turning our attention to the proof of (ii) let us make two remarks.

Remark 3.2. Notice that the regular decomposition (3.2) is not necessarily unique. However, if we find another $\tilde{\phi}_\varepsilon \in H^1_0(\mathsf{D})$ and $\tilde{u}^*_\varepsilon \in H^1_0(\mathsf{D})^3$ with $u_\varepsilon = \nabla\tilde{\phi}_\varepsilon + \tilde{u}^*_\varepsilon$, then $\operatorname{curl}(u_\varepsilon) = \operatorname{curl}(u^*_\varepsilon) = \operatorname{curl}(\tilde{u}^*_\varepsilon)$, so $\operatorname{curl}(u_\varepsilon)$ and accordingly $\operatorname{curl}(K^*_\varepsilon)$ does not depend on the choice of decomposition in (3.2).

Remark 3.3. Let us make an important remark. Equation (3.5) is actually only allowed to be tested with functions $v \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ with $\operatorname{div}(v) = 0$. However, we can in fact test this equation with all functions in $\dot{\mathrm{BL}}(\mathbf{R}^3)^3$. To see this let $v \in \dot{\mathrm{BL}}(\mathbf{R}^3)^3$ be arbitrary. Thanks to Lemma 2.6 we find $\phi \in \dot{\mathrm{BL}}^2(\mathbf{R}^3)$ and $v^* \in \mathrm{BL}(\mathbf{R}^3)^3$, such that $v = \nabla\phi + v^*$. Since $v^* \in \dot{\mathrm{BLC}}(\mathbf{R}^3)$ we can use $v^* = v - \nabla\phi$ as test function in (3.5) and using $\operatorname{curl}\nabla\phi = 0$ we obtain

$$\begin{aligned}\int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} K + U_0) - \mathcal{A}_\omega(x, U_0)) \cdot \operatorname{curl} v \, \mathrm{d}x \\ = -\int_\omega \big(a_1(U_0) - a_2(U_0))\big) \cdot \operatorname{curl} v \, \mathrm{d}x\end{aligned} \tag{3.7}$$

for all $v \in \mathrm{BL}(\mathbf{R}^3)^3$. This will be used later on.

3.2. Analysis of the perturbed state equation

We assume in the whole section that $\operatorname{curl} u_0 \in C(\overline{B_\delta(z)})^3$ for some $\delta > 0$. Moreover we assume that Assumption A(i), (ii) are satisfied. Let u_ε denote the solution to (3.1).

Lemma 3.4. *There is a constant $C > 0$, such that for all small $\varepsilon > 0$,*

$$\|u_\varepsilon - u_0\|_{L_2(\mathsf{D})^3} + \|\operatorname{curl}(u_\varepsilon - u_0)\|_{L_2(\mathsf{D})^3} \le C\varepsilon^{3/2}. \tag{3.8}$$

Proof. Subtracting (3.1) for $\varepsilon > 0$ and $\varepsilon = 0$ yields

$$\begin{aligned}\int_{\mathsf{D}} (\mathcal{A}_\varepsilon(x, \operatorname{curl} u_\varepsilon) - \mathcal{A}_\varepsilon(x, \operatorname{curl} u_0)) \cdot \operatorname{curl}\varphi \, \mathrm{d}x \\ = -\int_{\omega_\varepsilon} (a_1(\operatorname{curl} u_0) - a_2(\operatorname{curl} u_0)) \cdot \operatorname{curl}\varphi \, \mathrm{d}x,\end{aligned} \tag{3.9}$$

for all $\varphi \in V(\mathsf{D})^3$. Hence choosing $\varphi = u_\varepsilon - u_0$ as a test function, using Hölder's inequality and the monotonicity of $\mathcal{A}_\varepsilon$, yield

$$c\|\operatorname{curl}(u_\varepsilon - u_0)\|^2_{L_2(\mathsf{D})^3} \le C\sqrt{|\omega_\varepsilon|}(1 + \|\operatorname{curl} u_0\|_{C(\overline{B}_\delta(z))^3})\|\operatorname{curl}(u_\varepsilon - u_0)\|_{L_2(\mathsf{D})^3}, \tag{3.10}$$

where we used (1.7). Now the result follows from $|\omega_\varepsilon| = |\omega|\varepsilon^3$ and the Friedrich's inequality. □

A direct consequence of Lemmas 3.4 and 2.1 is the following. Recall the splitting $u_\varepsilon = \nabla\phi_\varepsilon + u^*_\varepsilon$ introduced in (3.2).

Corollary 3.5. *Under the assumptions of Lemma 3.4, there are constants C_1, C_2, such that for all small $\varepsilon > 0$ we have*

$$\|u^*_\varepsilon - u^*_0\|_{L_2(\mathsf{D})^3} + \|\nabla(u^*_\varepsilon - u^*_0)\|_{L_2(\mathsf{D})^{3\times 3}} \le C_1\varepsilon^{3/2} \tag{3.11}$$

and

$$\|\phi_\varepsilon - \phi\|_{L_2(\mathsf{D})} + \|\nabla(\phi_\varepsilon - \phi)\|_{L_2(\mathsf{D})^3} \le C_2\varepsilon^{3/2}. \tag{3.12}$$

The proof of Main Theorem 1 is split into several lemmas. The outline of the proof is as follows:

– introduce an auxiliary function H_ε and decompose it into $H_\varepsilon = \nabla\tilde{\phi}_\varepsilon + H_\varepsilon^*$
– split $K_\varepsilon^* - K = K_\varepsilon^* - H_\varepsilon^* + H_\varepsilon^* - K$
– show $\operatorname{curl}(H_\varepsilon^* - K) \to 0$ strongly in $L_2(\mathbf{R}^3)^3$
– show $\operatorname{curl}(H_\varepsilon^* - K_\varepsilon^*) \to 0$ strongly in $L_2(\mathbf{R}^3)^3$

The proof is following the main arguments of [13], Theorem 4.3. The main difference is that we cannot directly work with K_ε and H_ε but have to work with the functions K_ε^* and H_ε^* coming from the regular Helmholtz decomposition as in Lemma 2.1.

Let us first investigate the variation $H_\varepsilon^* - K$. We start by changing variables in (3.9) to obtain an equation for $K_\varepsilon^* \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$:

$$\begin{aligned}\int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} K_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon))) \cdot \operatorname{curl}\varphi \, \mathrm{d}x \\ = -\int_\omega (a_1(\operatorname{curl} u_0(x_\varepsilon)) - a_2(\operatorname{curl} u_0(x_\varepsilon))) \cdot \operatorname{curl}\varphi \, \mathrm{d}x \end{aligned} \tag{3.13}$$

for all $\varphi \in V(\varepsilon^{-1}\mathsf{D})^3$. Here $x_\varepsilon := \varepsilon x$ and $\operatorname{curl} u_0(x_\varepsilon)$ denotes the curl of u_0 evaluated at x_ε.

We now introduce an approximation H_ε of K_ε.

Definition 3.6. We define $H_\varepsilon \in V(\varepsilon^{-1}\mathsf{D})^3$ as the solution to

$$\begin{aligned}\int_{\varepsilon^{-1}\mathsf{D}} (\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon + U_0) - \mathcal{A}_\omega(x, U_0)) \cdot \operatorname{curl}\varphi \, \mathrm{d}x \\ = -\int_\omega (a_1(U_0) - a_2(U_0)) \cdot \operatorname{curl}\varphi \, \mathrm{d}x \quad \text{for all} \quad \varphi \in V(\varepsilon^{-1}\mathsf{D})^3. \end{aligned} \tag{3.14}$$

Remark 3.7. We can replace $V(\varepsilon^{-1}\mathsf{D})^3$ as test space in (3.13) and also in (3.14) by $H_0^1(\varepsilon^{-1}\mathsf{D})^3$. Indeed in view of Lemma 2.2 we can decompose every $v \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$ as $v = \nabla\phi + \psi$ with $\phi \in H^1(\varepsilon^{-1}\mathsf{D})$ and $\psi \in V(\varepsilon^{-1}\mathsf{D})^3$. Hence we may test (3.14) with $\varphi = \psi$ and using $\operatorname{curl}(\nabla\phi) = 0$ implies that we can test (3.14) with all functions in $v \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$. Compare the $\mathbf{R}^3$ analogue discussed in Remark 3.3.

Again we invoke Lemma 2.1 to decompose $H_\varepsilon = \nabla\phi_\varepsilon + H_\varepsilon^*$, $H_\varepsilon^* \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$ and $\phi_\varepsilon \in H_0^1(\varepsilon^{-1}\mathsf{D})$. We now introduce the projection of K into the space $H_0^1(\varepsilon^{-1}\mathsf{D})^3$:

Definition 3.8. We define $\hat{K}_\varepsilon^* \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$ as the minimiser of

$$\min_{\substack{\varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3 \\ \operatorname{div}\varphi = 0}} \|\operatorname{curl}(\varphi - K)\|_{L_2(\varepsilon^{-1}\mathsf{D})^3}. \tag{3.15}$$

The minimisation problem (3.15) admits indeed a unique solution. To see this, we let $\varphi_n \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$ be a minimising sequence, such that $\operatorname{div}(\varphi_n) = 0$ and

$$\lim_{n\to\infty} \|\operatorname{curl}(\varphi_n - K)\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} = \inf_{\substack{\varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3 \\ \operatorname{div}\varphi = 0}} \|\operatorname{curl}(\varphi - K)\|_{L_2(\varepsilon^{-1}\mathsf{D})^3}. \tag{3.16}$$

Since the infimum on the right hand side is finite we conclude that there is $C > 0$, such that $\|\operatorname{curl}\varphi_n\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} \le C$ for all n. On the other hand in view of (2.5) and $\operatorname{div}(\varphi_n) = 0$ we have

$$\|\operatorname{curl}\varphi_n\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} = \|\nabla\varphi_n\|_{L_2(\varepsilon^{-1}\mathsf{D})^{3\times 3}}. \tag{3.17}$$

Therefore (φ_n) is bounded in $H_0^1(\varepsilon^{-1}\mathsf{D})^3$ and we find a weakly converging subsequence (denoted the same) converging to some element $\varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$ satisfying $\operatorname{div}(\varphi) = 0$. Since also $\operatorname{curl}(\varphi_n) \rightharpoonup \operatorname{curl}(\varphi)$ weakly in $L_2(\varepsilon^{-1}\mathsf{D})^3$, we conclude

$$\|\operatorname{curl}(\varphi - K)\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} \le \varliminf_{n\to\infty} \|\operatorname{curl}(\varphi_n - K)\|_{L_2(\varepsilon^{-1}\mathsf{D})^3}, \tag{3.18}$$

which together with (3.16) shows that (3.15) admits a solution. The uniqueness follows from that fact that $\varphi \mapsto \|\operatorname{curl}(\varphi)\|^2_{L_2(\varepsilon^{-1}\mathsf{D})^3}$ is strictly convex on $\{\varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3 : \operatorname{div}(\varphi) = 0\}$.

As for K_ε^*, we can also view H_ε^* and $\hat{K}_\varepsilon^*$ as elements of $\operatorname{BL}(\mathbf{R}^3)^3$ by extending them by 0 outside of $\varepsilon^{-1}\mathsf{D}$.

Lemma 3.9. *It holds that*

$$\operatorname{curl} \hat{K}_\varepsilon^* \to \operatorname{curl} K \quad \text{strongly in } L_2(\mathbf{R}^3)^3 \text{ as } \varepsilon \searrow 0. \tag{3.19}$$

Proof. We readily check that the minimiser to (3.15) satisfies

$$\int_{\varepsilon^{-1}\mathsf{D}} \operatorname{curl} \hat{K}_\varepsilon^* \cdot \operatorname{curl} \varphi \, dx = \int_{\varepsilon^{-1}\mathsf{D}} \operatorname{curl} K \cdot \operatorname{curl} \varphi \, dx \quad \text{for all} \quad \varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3, \ \operatorname{div}(\varphi) = 0. \tag{3.20}$$

Choosing $\varphi = \hat{K}_\varepsilon^*$ and using Hölder's inequality and the fact that (see (2.5))

$$\|\operatorname{curl} v\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} = \|\nabla v\|_{L_2(\varepsilon^{-1}\mathsf{D})^{3\times 3}} \quad \text{for all} \quad v \in H_0^1(\varepsilon^{-1}\mathsf{D})^3 \text{ with } \operatorname{div}(v) = 0, \tag{3.21}$$

we obtain

$$\begin{aligned} \|\nabla \hat{K}_\varepsilon^*\|^2_{L_2(\varepsilon^{-1}\mathsf{D})^{3\times 3}} &= \|\operatorname{curl} \hat{K}_\varepsilon^*\|^2_{L_2(\varepsilon^{-1}\mathsf{D})^3} \\ &\le \|\operatorname{curl} K\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} \|\operatorname{curl} \hat{K}_\varepsilon^*\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} \\ &= \|\operatorname{curl} K\|_{L_2(\varepsilon^{-1}\mathsf{D})^3} \|\nabla \hat{K}_\varepsilon^*\|_{L_2(\varepsilon^{-1}\mathsf{D})^{3\times 3}}. \end{aligned} \tag{3.22}$$

This implies $\|\nabla \hat{K}_\varepsilon^*\|_{L_2(\mathbf{R}^3)^{3\times 3}} \le C$ for all $\varepsilon > 0$. Now fix $\tilde{\varepsilon} > 0$ and let $\varepsilon \in (0, \tilde{\varepsilon})$. Then we obtain from (3.20) (by extending K and $\hat{K}_\varepsilon^*$ by zero outside of $\varepsilon^{-1}\mathsf{D}$),

$$\int_{\mathbf{R}^3} \operatorname{curl} \hat{K}_\varepsilon^* \cdot \operatorname{curl} \varphi \, dx = \int_{\mathbf{R}^3} \operatorname{curl} K \cdot \operatorname{curl} \varphi \, dx \quad \text{for all} \quad \varphi \in H_0^1(\tilde{\varepsilon}^{-1}\mathsf{D})^3, \ \operatorname{div}(\varphi) = 0. \tag{3.23}$$

Let (ε_n) be a null-sequence. In view of the boundedness of $(\hat{K}_{\varepsilon_n}^*)$ in $\dot{\operatorname{BL}}(\mathbf{R}^3)^3$, we can extract a subsequence (denoted the same) and find $\tilde{K} \in \dot{\operatorname{BL}}(\mathbf{R}^3)^3$, such that $\nabla \hat{K}_{\varepsilon_n}^* \rightharpoonup \nabla \tilde{K}$ and thus also $\operatorname{curl} \hat{K}_{\varepsilon_n}^* \rightharpoonup \operatorname{curl} \tilde{K}$ weakly in $L_2(\mathbf{R}^3)^3$. Therefore, selecting $\varepsilon = \varepsilon_n$ in (3.23) we can pass to the limit $n \to \infty$ to obtain

$$\int_{\mathbf{R}^3} \operatorname{curl} \tilde{K} \cdot \operatorname{curl} \varphi \, dx = \int_{\mathbf{R}^3} \operatorname{curl} K \cdot \operatorname{curl} \varphi \, dx \quad \text{for all} \quad \varphi \in H_0^1(\tilde{\varepsilon}^{-1}\mathsf{D})^3, \ \operatorname{div}(\varphi) = 0. \tag{3.24}$$

Since $\operatorname{div}(\hat{K}_\varepsilon^*) = 0$ for all $\varepsilon > 0$ and in view of the weak convergence $\nabla \hat{K}_{\varepsilon_n}^* \rightharpoonup \nabla \tilde{K}$, it is also readily checked that $\operatorname{div}(\tilde{K}) = 0$. Since $\tilde{\varepsilon}$ was arbitrary and since $C_c^\infty(\mathbf{R}^3)/\mathbf{R}$ is dense in $\dot{\operatorname{BL}}(\mathbf{R}^3)$ it follows that (3.24) holds for test functions in $\operatorname{BL}(\mathbf{R}^3)^3$ from which we conclude that $\tilde{K} = K$. Therefore $\hat{K}_\varepsilon^* \rightharpoonup K$ weakly in $\dot{\operatorname{BL}}(\mathbf{R}^3)^3$. The strong convergence follows by testing (3.20) with $\varphi = \hat{K}_\varepsilon^*$ and passing to the limit $\varepsilon \searrow 0$. This shows that $\|\operatorname{curl} \hat{K}_\varepsilon^*\|_{L_2(\mathbf{R}^3)^3} \to \|\operatorname{curl} K\|_{L_2(\mathbf{R}^3)^3}$ as $\varepsilon \searrow 0$. Since in a Hilbert space norm convergence together with weak convergence implies strong convergence we finish the proof. ☐

Lemma 3.10. *We have*

$$\operatorname{curl} H_\varepsilon^* \to \operatorname{curl} K \quad \text{strongly in} \quad L_2(\mathbf{R}^3)^3 \quad \text{as} \quad \varepsilon \searrow 0. \tag{3.25}$$

Proof. Subtracting (3.14) from (3.5) and introducing a zero term leads to

$$\begin{aligned}\int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} \hat{K}_\varepsilon^* + U_0) - \mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + U_0)) \cdot \operatorname{curl} \varphi \, \mathrm{d}x \\ = \int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} \hat{K}_\varepsilon^* + U_0) - \mathcal{A}_\omega(x, \operatorname{curl} K + U_0)) \cdot \operatorname{curl} \varphi \, \mathrm{d}x \end{aligned} \tag{3.26}$$

for all $\varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$. Here we used the observation of Remark 3.3 and $H_0^1(\varepsilon^{-1}\mathsf{D})^3 \subset \dot{\mathrm{BL}}(\mathbf{R}^3)^3$. Now we test this equation with $\varphi = \hat{K}_\varepsilon^* - H_\varepsilon^* \in H_0^1(\varepsilon^{-1}\mathsf{D})^3 \subset \dot{\mathrm{BL}}(\mathbf{R}^3)^3$, use the monotonicity of $\mathcal{A}_\omega$ and Hölder's inequality:

$$\begin{aligned} &C\| \operatorname{curl}(\hat{K}_\varepsilon^* - H_\varepsilon^*)\|_{L_2(\mathbf{R}^3)^3}^2 \\ &\le \int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} \hat{K}_\varepsilon^* + U_0) - \mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + U_0)) \cdot \operatorname{curl}(\hat{K}_\varepsilon^* - H_\varepsilon^*) \, \mathrm{d}x \\ &\overset{(3.26)}{=} \int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} \hat{K}_\varepsilon^* + U_0) - \mathcal{A}_\omega(x, \operatorname{curl} K + U_0)) \cdot \operatorname{curl}(\hat{K}_\varepsilon^* - H_\varepsilon^*) \, \mathrm{d}x \\ &\le \| \operatorname{curl}(\hat{K}_\varepsilon^* - K)\|_{L_2(\mathbf{R}^3)^3} \| \operatorname{curl}(\hat{K}_\varepsilon^* - H_\varepsilon^*)\|_{L_2(\mathbf{R}^3)^3}. \end{aligned} \tag{3.27}$$

It follows from Lemma 3.9, we have $\operatorname{curl} \hat{K}_\varepsilon^* \to \operatorname{curl} K$ strongly in $L_2(\mathbf{R}^3)^3$. Therefore (3.27) implies $\operatorname{curl}(\hat{K}_\varepsilon^* - H_\varepsilon^*) \to 0$ strongly in $L_2(\mathbf{R}^3)^3$ and therefore also $\| \operatorname{curl}(H_\varepsilon^* - K)\|_{L_2(\mathbf{R}^3)^3} \le \| \operatorname{curl}(H_\varepsilon^* - \hat{K}_\varepsilon^*)\|_{L_2(\mathbf{R}^3)^3} + \| \operatorname{curl}(\hat{K}_\varepsilon^* - K)\|_{L_2(\mathbf{R}^3)^3} \to 0$ as $\varepsilon \searrow 0$. ☐

We now prove that $\operatorname{curl}(H_\varepsilon^* - K_\varepsilon^*) \to 0$ strongly in $L_2(\mathbf{R}^3)^3$.

Lemma 3.11. *Assume there are $\delta > 0$ and $\alpha > 0$, such that $\operatorname{curl} u_0 \in C^\alpha(\overline{B_\delta(z)})^3$. Then we have*

$$\operatorname{curl}(H_\varepsilon^* - K_\varepsilon^*) \to 0 \quad \text{strongly in} \quad L_2(\mathbf{R}^3)^3 \quad \text{as} \quad \varepsilon \searrow 0. \tag{3.28}$$

Proof. Subtracting (3.13) and (3.14) we obtain

$$\begin{aligned} &\int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} K_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - \mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + U_0)) \cdot \operatorname{curl} \varphi \, \mathrm{d}x \\ &\quad + \int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, U_0) - \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon))) \cdot \operatorname{curl} \varphi \, \mathrm{d}x \\ &= - \int_\omega (a_1(\operatorname{curl} u_0(x_\varepsilon)) - a_2(\operatorname{curl} u_0(x_\varepsilon))) \cdot \operatorname{curl} \varphi - (a_1(U_0) - a_2(U_0)) \cdot \operatorname{curl} \varphi \, \mathrm{d}x \end{aligned} \tag{3.29}$$

for all $\varphi \in H_0^1(\varepsilon^{-1}\mathsf{D})^3$ where we recall the notation $x_\varepsilon = \varepsilon x$. We want to use the monotonicity of $\mathcal{A}_\omega$ and therefore we rewrite the previous equation as follows

$$\begin{aligned} &\int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, \operatorname{curl} K_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - \mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)))) \cdot \operatorname{curl} \varphi \, \mathrm{d}x \\ &= \underbrace{- \int_{\mathbf{R}^3} ((\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - (\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + U_0) \cdot \operatorname{curl} \varphi \, \mathrm{d}x}_{=:I_1(\varepsilon,\varphi)} \\ &\quad \underbrace{- \int_{\mathbf{R}^3} (\mathcal{A}_\omega(x, U_0) - \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon))) \cdot \operatorname{curl} \varphi \, \mathrm{d}x}_{=:I_2(\varepsilon,\varphi)} \\ &\quad \underbrace{- \int_\omega (a_1(\operatorname{curl} u_0(x_\varepsilon)) - a_2(\operatorname{curl} u_0(x_\varepsilon))) \cdot \operatorname{curl} \varphi \, \mathrm{d}x - (a_1(U_0) - a_2(U_0)) \cdot \operatorname{curl} \varphi \, \mathrm{d}x}_{=:I_3(\varepsilon,\varphi)}. \end{aligned} \tag{3.30}$$

Now the a_i are Lipschitz continuous and $\operatorname{curl} u_0 \in C^\alpha(\overline{B_\delta(z)})^3$ with $\alpha, \delta > 0$, we immediately obtain that $|I_3(\varepsilon, \varphi)| \le C\varepsilon^\alpha \|\operatorname{curl}\varphi\|_{L_2(\mathbf{R}^3)^3}$ for a suitable constant $C > 0$. We now show that also $|I_1(\varepsilon,\varphi) + I_2(\varepsilon,\varphi)| \le C(\varepsilon)\|\operatorname{curl}\varphi\|_{L_2(\mathbf{R}^3)^3}$ and $C(\varepsilon) \to 0$ as $\varepsilon \searrow 0$. We write for arbitrary $r \in (0,1)$,

$$\begin{aligned} I_1(\varepsilon,\varphi) + I_2(\varepsilon,\varphi) = &-\int_{B_{\varepsilon^{-r}}} ((\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - (\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + U_0)\cdot \operatorname{curl}\varphi \,\mathrm{d}x \\ &- \int_{B_{\varepsilon^{-r}}} (\mathcal{A}_\omega(x, U_0) - \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon)))\cdot \operatorname{curl}\varphi\,\mathrm{d}x \\ &- \int_{\mathbf{R}^3\setminus B_{\varepsilon^{-r}}} ((\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - (\mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon))\cdot \operatorname{curl}\varphi\,\mathrm{d}x \\ &+ \int_{\mathbf{R}^3\setminus B_{\varepsilon^{-r}}} ((\mathcal{A}_\omega(x, \operatorname{curl} H_\varepsilon^* + U_0) - \mathcal{A}_\omega(x, U_0))\cdot \operatorname{curl}\varphi\,\mathrm{d}x. \end{aligned} \tag{3.31}$$

Now we can estimate the right hand side of (3.31) using the Lipschitz continuity of a_i (see Assumption A(ii)) as follows

$$\begin{aligned} |I_1(\varepsilon,\varphi) + I_2(\varepsilon,\varphi)| &\le 2C\int_{B_{\varepsilon^{-r}}} |U_0 - \operatorname{curl} u_0(x_\varepsilon)||\operatorname{curl}\varphi|\,\mathrm{d}x + 2C\int_{\mathbf{R}^3\setminus B_{\varepsilon^{-r}}} |\operatorname{curl} H_\varepsilon^*|\,|\operatorname{curl}\varphi|\,\mathrm{d}x \\ &\le C\int_{B_{\varepsilon^{-r}}} |x_\varepsilon|^\alpha|\operatorname{curl}\varphi|\mathrm{d}x + 2C\int_{\mathbf{R}^3\setminus B_{\varepsilon^{-r}}} |\operatorname{curl} H_\varepsilon^*|\,|\operatorname{curl}\varphi|\,\mathrm{d}x \\ &\le \varepsilon^{-r\alpha}\varepsilon^\alpha\varepsilon^{-3r/2}C\|\operatorname{curl}\varphi\|_{L_2(\mathbf{R}^3)^3} + 2C\|\operatorname{curl} H_\varepsilon^*\|_{L_2(\mathbf{R}^3\setminus B_{\varepsilon^{-r}})^3}\|\operatorname{curl}\varphi\|_{L_2(\mathbf{R}^3\setminus B_{\varepsilon^r})^3} \end{aligned} \tag{3.32}$$

For r sufficiently close to 0, we have $\varepsilon^{-r\alpha}\varepsilon^\alpha\varepsilon^{-3r/2} = \varepsilon^{\alpha - r(\frac{3}{2}+\alpha)} \to 0$. Moreover, by the triangle inequality we have

$$\|\operatorname{curl} H_\varepsilon^*\|_{L_2(\mathbf{R}^3\setminus B_{\varepsilon^{-r}})^3} \le \|\operatorname{curl}(H_\varepsilon^* - K)\|_{L_2(\mathbf{R}^3\setminus B_{\varepsilon^{-r}})^3} + \|\operatorname{curl} K\|_{L_2(\mathbf{R}^3\setminus B_{\varepsilon^{-r}})^3}. \tag{3.33}$$

The first term on the right hand side goes to zero in view of Lemma 3.10. The second term goes to zero since $\operatorname{curl} K \in L_2(\mathbf{R}^3)^3$ thus $\|\operatorname{curl} K\|_{L_2(\mathbf{R}^3\setminus B_{\varepsilon^{-r}})^3} \to 0$ as $\varepsilon \searrow 0$. Using $K_\varepsilon^* - H_\varepsilon^*$ as test function in (3.30), using the monotonicity of $\mathcal{A}_\omega$ and employing $|I_1(\varepsilon,\varphi) + I_2(\varepsilon,\varphi) + I_3(\varepsilon,\varphi)| \le C(\varepsilon)\|\operatorname{curl}\varphi\|_{L_2(\mathbf{R}^3)^3}$ with $C(\varepsilon) \to 0$ as $\varepsilon \searrow 0$, shows the result. □

Combining Lemmas 3.10 and 3.11 proves the Main Theorem 1(ii). ■

4. The topological derivative

In this section we show that the hypotheses of Theorem A.4 are satisfied for the Lagrangian G given by (4.1).

Let $\ell(\varepsilon) := |\omega_\varepsilon|$, and introduce the Lagrangian $G : [0,\tau] \times H_0(\mathsf{D}, \operatorname{curl}) \times H_0(\mathsf{D}, \operatorname{curl}) \to \mathbf{R}$ associated with the perturbation ω_ε by

$$G(\varepsilon, u, p) := \int_{\Omega_g} |\operatorname{curl}(u) - B_d|^2\,\mathrm{d}x + \int_{\mathsf{D}} \mathcal{A}_{\Omega_\varepsilon}(x, \operatorname{curl} u)\cdot \operatorname{curl} p\,\mathrm{d}x - \langle F, p\rangle. \tag{4.1}$$

Here, the operator $\mathcal{A}_{\Omega_\varepsilon}$ is defined according to (1.3) with $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon$. It is clear from Assumption A that the Lagrangian G is ℓ-differentiable in the sense of Definition A.3 with $X = Y = V(\mathsf{D})^3$ and $\ell(\varepsilon) := |\omega_\varepsilon|$.

Main Theorem 2. *Let Assumption A be satisfied. Let $\Omega \subset D$ open and u_0 the solution to (1.2) and p_0 the solution to (4.6). Let $z \in \mathsf{D} \setminus \overline{\Omega}$, such that $z \notin (\Omega_1 \cup \Omega_2 \cup \Omega_g)$. Further assume that* $\operatorname{curl} u_0 \in C^\alpha(\overline{B_\delta(z)})^3$ *for some $\delta > 0$ and $0 < \alpha < 1$ and also* $\operatorname{curl} p_0 \in C(\overline{B_\delta(z)})^3 \cap L_\infty(\mathsf{D})^3$.

(a) *Then the assumptions of Theorem A.4 are satisfied for the Lagrangian G given by (4.1) and hence the topological derivative at $z \in \mathsf{D} \setminus \overline{\Omega}$ is given by*

$$\mathrm{d}J(\Omega)(z) = \partial_\ell G(0, u_0, p_0) + R_1(u_0, p_0) + R_2(u_0, p_0). \tag{4.2}$$

(b) *We have*

$$\partial_\ell G(0, u_0, p_0) = ((a_1(U_0) - a_2(U_0)) \cdot P_0 \tag{4.3}$$

and

$$R_1(u_0, p_0) = \frac{1}{|\omega|}\left(\int_{\mathbf{R}^3} \left[\mathcal{A}_\omega(x, \operatorname{curl} K + U_0) - \mathcal{A}_\omega(x, U_0) - \partial_u \mathcal{A}_\omega(x, U_0)(\operatorname{curl} K)\right] \cdot P_0 \,\mathrm{d}x \right) \tag{4.4}$$

and

$$R_2(u_0, p_0) = \frac{1}{|\omega|} \int_\omega \left[\partial_u a_1(U_0) - \partial_u a_2(U_0)\right] (\operatorname{curl} K) \cdot P_0 \,\mathrm{d}x \tag{4.5}$$

where $U_0 := \operatorname{curl} u_0(z)$, $P_0 := \operatorname{curl} p_0(z)$ and $\mathcal{A}_\omega(x, y) := a_1(y)\chi_\omega(x) + a_2(y)\chi_{\mathbf{R}^3\setminus\omega}(x)$, and K is the unique solution to (3.5) and $p_0 \in V(\mathsf{D})^3$ solves

$$\int_{\mathsf{D}} \partial_u \mathcal{A}_\Omega(x, \operatorname{curl} u_0)(\operatorname{curl} \varphi) \cdot \operatorname{curl} p_0 \,\mathrm{d}x = -\int_{\mathsf{D}} 2(\operatorname{curl} u_0 - B_d) \cdot \operatorname{curl} \varphi \,\mathrm{d}x \tag{4.6}$$

for all $\varphi \in V(\mathsf{D})^3$.

Remark 4.1.
- We restrict ourselves to the case where $z \in D \setminus \overline{\Omega}$ without loss of generality. However, the exact same proof can be conducted in the case where $z \in \Omega$ and $z \notin (\Omega_1 \cup \Omega_2 \cup \Omega_g)$. In that case, the formula for the topological derivative is obtained by just switching the roles of a_1 and a_2 in the theorem above (in particular also in the definition of $\mathcal{A}_\omega$).
- Also the case where $z \in \Omega_1 \cup \Omega_2 \cup \Omega_g$ can be dealt with in a similar manner. Indeed the derivation of [13] shows that for instance if $z \in \Omega_g$ an additional term $\int_{\mathbf{R}^3} |\nabla K|^2 \,\mathrm{d}x$ in $\mathrm{d}J(\Omega)(z)$ appears. The case $z \in \Omega_1$ and/or $z \in \Omega_2$ have to be treated separately since in this case the right hand side F becomes domain dependent.
- The assumption $z = 0$ is without loss of generality, too. In the general case, this situation can be obtained by a simple change of the coordinate system.
- Recall that we made the assumption $\Omega = \emptyset$. The general case can be treated similarly by small modifications.

4.1. Computation of $R_1(u_0, p_0)$ and $R_2(u_0, p_0)$

It remains to check that the limits of $R_1(u_0, p_0)$ and $R_2(u_0, p_0)$ exist. For this we use Assumption A(i)–(iii). Using the change of variables $T_\varepsilon(x) = \varepsilon x$ and the definition $\ell(\varepsilon) = |\omega_\varepsilon| = \varepsilon^3|\omega|$, we have

$$\begin{aligned}
R_1^\varepsilon(u_0, p_0) &= \frac{1}{\ell(\varepsilon)} \int_0^1 \int_{\mathsf{D}} \left(\partial_u \mathcal{A}_\varepsilon(x, \operatorname{curl}(s u_\varepsilon + (1-s)u_0)) - \partial_u \mathcal{A}_\varepsilon(x, \operatorname{curl} u_0)\right) (\operatorname{curl}(u_\varepsilon - u_0)) \cdot \operatorname{curl} p_0 \,\mathrm{d}x \,\mathrm{d}s \\
&\quad + \frac{1}{\ell(\varepsilon)} \int_{\Omega_g} |\operatorname{curl}(u_\varepsilon - u_0)|^2 \,\mathrm{d}x \\
&= \underbrace{\frac{1}{|\omega|} \int_0^1 \int_{\mathbf{R}^3} \left(\partial_u \mathcal{A}_\omega(x, s \operatorname{curl} K_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - \partial_u \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon))\right) (\operatorname{curl} K_\varepsilon^*) \cdot \operatorname{curl} p_0(x_\varepsilon) \,\mathrm{d}x \,\mathrm{d}s}_{=:I_\varepsilon} \\
&\quad + \underbrace{\frac{1}{|\omega|} \int_{\varepsilon^{-1}\Omega_g} |\operatorname{curl} K_\varepsilon^*|^2 \,\mathrm{d}x}_{=:II_\varepsilon} \\
&\to \frac{1}{|\omega|} \int_0^1 \int_{\mathbf{R}^3} \left(\partial_u \mathcal{A}_\omega(x, s \operatorname{curl} K + U_0) - \partial_u \mathcal{A}_\omega(x, U_0)\right) (\operatorname{curl} K) \cdot P_0 \,\mathrm{d}x \,\mathrm{d}s.
\end{aligned} \tag{4.7}$$

Since $\operatorname{curl} K_\varepsilon^* \to \operatorname{curl} K$ strongly in $L_2(\mathbf{R}^3)^3$ as $\varepsilon \searrow 0$ and since $\varepsilon^{-1}\Omega_g$ goes to "infinity" because $z \notin \Omega_g$ it readily follows that $II_\varepsilon \to 0$ as $\varepsilon \searrow 0$. To see the convergence of the first term, we may write $I_\varepsilon$as follows

$$\begin{aligned}
&\int_0^1 \int_{\mathbf{R}^3} (\partial_u \mathcal{A}_\omega(x, s \operatorname{curl} K_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - \partial_u \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon)))(\operatorname{curl} K_\varepsilon^*) \cdot \operatorname{curl} p_0(x_\varepsilon)\, \mathrm{d}x\, \mathrm{d}s \\
&\quad = \int_0^1 \int_{\mathbf{R}^3} (\partial_u \mathcal{A}_\omega(x, s \operatorname{curl} K_\varepsilon^* + \operatorname{curl} u_0(x_\varepsilon)) - \partial_u \mathcal{A}_\omega(x, s\operatorname{curl} K + \operatorname{curl} u_0(x_\varepsilon)))(\operatorname{curl} K_\varepsilon^*) \cdot \operatorname{curl} p_0(x_\varepsilon)\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad + \int_0^1 \int_{\mathbf{R}^3} (\partial_u \mathcal{A}_\omega(x, s \operatorname{curl} K + \operatorname{curl} u_0(x_\varepsilon)) - \partial_u \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon)))(\operatorname{curl}(K_\varepsilon^* - K)) \cdot \operatorname{curl} p_0(x_\varepsilon)\, \mathrm{d}x\, \mathrm{d}s \\
&\qquad + \int_0^1 \int_{\mathbf{R}^3} (\partial_u \mathcal{A}_\omega(x, s \operatorname{curl} K + \operatorname{curl} u_0(x_\varepsilon)) - \partial_u \mathcal{A}_\omega(x, \operatorname{curl} u_0(x_\varepsilon)))(\operatorname{curl} K) \cdot \operatorname{curl} p_0(x_\varepsilon)\, \mathrm{d}x\, \mathrm{d}s.
\end{aligned}$$

Using Assumption A(iii) and $\operatorname{curl} p_0 \in L^\infty(\mathsf{D})^3$, we see that the absolute value of the first and second term on the right hand side can be bounded by $C\|\operatorname{curl}(K_\varepsilon^* - K)\|_{L_2(\mathbf{R}^3)^3}\|\operatorname{curl} K\|_{L_2(\mathbf{R}^3)^3}$ and hence using $\operatorname{curl} K_\varepsilon^* \to \operatorname{curl} K$ in $L_2(\mathbf{R}^3)^3$ as $\varepsilon \searrow 0$ they disappear in the limit. The last term converges to the desired limit by using Lebesgue's dominated convergence theorem. Using the fundamental theorem, we obtain the expression in (4.4). Similarly, using (1.8), the continuity of $\operatorname{curl} u_0$ and $\operatorname{curl} p_0$ at z, the continuity of $\partial_u a_1, \partial_u a_2$, and again $\operatorname{curl} K_\varepsilon^* \to \operatorname{curl} K$ strongly in $L_2(\mathbf{R}^3)^3$, we obtain by Lebesgue's dominated convergence theorem

$$\begin{aligned}
R_2^\varepsilon(u,p) &= \frac{1}{\ell(\varepsilon)} \int_{\omega_\varepsilon} (\partial_u a_1(\operatorname{curl} u_0) - \partial_u a_1(\operatorname{curl} u_0))(\operatorname{curl}(u_\varepsilon - u_0)) \cdot \operatorname{curl} p_0\, \mathrm{d}x \\
&= \frac{1}{|\omega|} \int_\omega (\partial_u a_1(\operatorname{curl} u_0(x_\varepsilon)) - \partial_u a_2(\operatorname{curl} u_0(x_\varepsilon)))(\operatorname{curl} K_\varepsilon^*) \cdot \operatorname{curl} p_0(x_\varepsilon)\, \mathrm{d}x \\
&\to \frac{1}{|\omega|} \int_\omega (\partial_u a_1(U_0) - \partial_u a_2(U_0))(\operatorname{curl} K) \cdot P_0\, \mathrm{d}x. \qquad (4.8)
\end{aligned}$$

Therefore all Hypotheses of Theorem A.4 are satisfied. This finishes the proof of our Main Theorem 2.

5. Numerical realization

Formula (4.2) together with (4.3)–(4.6) states the topological derivative for problem (1.1) and (1.2) at a single spatial point z. Note that the evaluation of the topological derivative involves the solution of problem (3.5), which in turn depends on the point z *via* the vector $U_0 = \operatorname{curl}(u_0)(z)$. When using the topological derivative (4.2) in a numerical optimization algorithm, it has to be evaluated at every point in the design area in every iteration of the algorithm. Therefore, a direct evaluation of (4.2) is unfeasible and an efficient technique for numerical approximation is indispensable. In this section, we show a way to approximately evaluate formula (4.2) by first precomputing certain values in an offline phase and looking them up and interpolating them during the online phase of the optimization algorithm. We proceed in an analogous way to [4], Section 7.

For this, we need the following additional assumption:

Assumption B. (i) *For all orthogonal matrices $R \in \mathbf{R}^{3\times 3}$ and all $y \in \mathbf{R}^3$, it holds that*

$$a_i(Ry) = R\, a_i(y) \quad \text{for} \quad i = 1, 2. \qquad (5.1)$$

(ii) *The inclusion is the unit ball: $\omega = B_1(0)$.*

We will show a concrete application that satisfies this assumption in Section 6. We note that the topological derivative (4.2) depends on the spatial point z only *via* U_0, P_0 and $K = K_{U_0}$. Let us make this dependence more clear by introducing the notation

$$
\begin{aligned}
\mathrm{d}J(\Omega)(U_0,P_0) := {} & ((a_1(U_0) - a_2(U_0)) \cdot P_0 && (5.2)\\
& + \frac{1}{|\omega|}\left(\int_{\mathbf{R}^3} \left[\mathcal{A}_\omega(x, \operatorname{curl} K_{U_0} + U_0) - \mathcal{A}_\omega(x,U_0) - \partial_u \mathcal{A}_\omega(x,U_0)(\operatorname{curl} K_{U_0})\right] \cdot P_0 \,\mathrm{d}x \right) && (5.3)\\
& + \frac{1}{|\omega|} \int_\omega \left[\partial_u a_1(U_0) - \partial_u a_2(U_0)\right] (\operatorname{curl} K_{U_0}) \cdot P_0 \,\mathrm{d}x. && (5.4)
\end{aligned}
$$

Remark 5.1. Recall that e_i, $i = 1, 2, 3$, denotes the i-th unit vector in the Cartesian coordinate system in $\mathbf{R}^3$. For every vector $W \in \mathbf{R}^3$ there exists an orthogonal rotation matrix R_W such that $W = |W| R_W e_1$.

The next result will allow us to introduce an efficient strategy for the approximate evaluation of the topological derivative $\mathrm{d}J(U_0, P_0)$ for any $U_0, P_0 \in \mathbf{R}^3$.

Main Theorem 3. *Let Assumption B hold and $U_0, P_0 \in \mathbf{R}^3$. Then it holds:*

(i) *the mapping $P \mapsto \mathrm{d}J(\Omega)(U_0, P)$ is linear on $\mathbf{R}^3$,*

(ii) *$\mathrm{d}J(\Omega)(R^\top U_0, R^\top P_0) = \mathrm{d}J(\Omega)(U_0, P_0)$ for all orthogonal matrices $R \in \mathbf{R}^{3\times 3}$.*

(iii) *Write $U_0 = |U_0| R_{U_0} e_1$ and $P_0 = |P_0| R_{P_0} e_1$ for some orthogonal matrices $R_{U_0}, R_{P_0} \in \mathbf{R}^{3\times 3}$ and set $(c_1, c_2, c_3)^\top := |P_0| R_{U_0}^\top R_{P_0} e_1$. Then we have*

$$\mathrm{d}J(\Omega)(U_0,P_0) = c_1 \mathrm{d}J(\Omega)(|U_0|e_1, e_1) + c_2 \mathrm{d}J(\Omega)(|U_0|e_1, e_2) + c_3 \mathrm{d}J(\Omega)(|U_0|e_1, e_3). \tag{5.5}$$

Corollary 5.2. *Let Assumption B hold. Suppose that the values $\mathrm{d}J(\Omega)(te_1, e_i)$, $i = 1, 2, 3$ are given for all $t \in [t_{\min}, t_{\max}]$ with $0 \le t_{\min} < t_{\max}$. Then, for all $U_0 \in \mathbf{R}^3$ with $t_{\min} \le |U_0| \le t_{\max}$ and all $P_0 \in \mathbf{R}^3$ it holds*

$$\mathrm{d}J(\Omega)(U_0,P_0) = c_1 \mathrm{d}J(\Omega)(|U_0|e_1, e_1) + c_2 \mathrm{d}J(\Omega)(|U_0|e_1, e_2) + c_3 \mathrm{d}J(\Omega)(|U_0|e_1, e_3). \tag{5.6}$$

with $(c_1, c_2, c_3)^\top = |P_0| R_{U_0}^\top R_{P_0} e_1$.

We first prove the following properties of the solution mapping $W \mapsto K_W$, where K_W denotes the unique solutions in $\dot{\mathrm{BLC}}(\mathbf{R}^3)$ to (3.5) with U_0 being replaced by $W \in \mathbf{R}^3$.

Lemma 5.3. *Let Assumption B hold. Let $W \in \mathbf{R}^3$, $R \in \mathbf{R}^{3\times 3}$ orthogonal. Then the following relations hold:*

$$K_{R^\top W}(x) = R^\top K_W(Rx) + \nabla \eta, \tag{5.7}$$

$$\operatorname{curl}(K_{R^\top W})(x) = R^\top (\operatorname{curl}(K_W))(Rx). \tag{5.8}$$

Proof. To see the first identity, we perform the change of variables $y = \Phi(x) = Rx$ in (3.5) with U_0 replaced by W. Noting that the chain rule yields

$$\operatorname{curl}_y(K) \circ \Phi = R \operatorname{curl}_x(R^T (K \circ \Phi)), \tag{5.9}$$

where we used $\det(R) = 1$, we get for (3.5)

$$
\begin{aligned}
&\int_{\mathbf{R}^3} \left(\mathcal{A}_{\Phi^{-1}(\omega)}(x, R\operatorname{curl}_x(\tilde{K}) + W) - \mathcal{A}_{\Phi^{-1}(\omega)}(x, W) \right) \cdot R \operatorname{curl}_x(\tilde{\varphi}) = \\
&\qquad - \int_{\phi^{-1}(\omega)} \left(a_1(W) - a_2(W) \right) \cdot R \operatorname{curl}_x(\tilde{\varphi}).
\end{aligned}
$$

Here we used the notation $\tilde{K} = R^\top (K_W \circ \Phi)$ and $\tilde{\varphi} = R^\top(\varphi \circ \Phi)$. Using Assumption B, this can be rewritten as

$$\int_{\mathbf{R}^3} \left(\mathcal{A}_\omega(x, \operatorname{curl}_x(\tilde{K}) + R^\top W) - \mathcal{A}_\omega(x, R^\top W) \right) \cdot \operatorname{curl}_x(\tilde{\varphi}) = - \int_\omega \left(a_1(R^\top W) - a_2(R^\top W) \right) \cdot \operatorname{curl}_x(\tilde{\varphi}).$$

Since $K_{R^\top W}$ is the unique solution in $\dot{\mathrm{BLC}}(\mathbf{R}^3)$ to the problem above, we conclude that $R^\top(K_W \circ \Phi) = \tilde{K} = K_{R^\top W}$ in $\dot{\mathrm{BLC}}(\mathbf{R}^3)$. Finally this relation together with (5.9) yields

$$\mathrm{curl}_x(K_{R^\top W}) = \mathrm{curl}_x(R^\top(K_W \circ \Phi)) = R^\top \mathrm{curl}_y(K_W) \circ \Phi. \tag{5.10}$$

□

Proof of Main Theorem 3. The first statement can be seen directly from (5.2) to (5.4). The second result follows immediately by Assumption B using (5.8) noting that Assumption B(i) implies $\partial_u a_i(Ry)(Rz) = R\partial_u a_i(y)(z)$ for $R \in \mathbf{R}^{3\times 3}$ orthogonal and $y, z \in \mathbf{R}^3$ and $i = 1, 2$.

Using the representations $U_0 = |U_0| R_{U_0} e_1$ and $P_0 = |P_0| R_{P_0} e_1$ and item (ii), we get

$$\mathrm{d}J(\Omega)(U_0, P_0) = \mathrm{d}J(\Omega)(|U_0|R_{U_0}e_1, |P_0|R_{P_0}e_1) = \mathrm{d}J(\Omega)(|U_0|e_1, |P_0|R_{U_0}^\top R_{P_0}e_1).$$

The result now follows from the definition $(c_1, c_2, c_3)^\top = |P_0| R_{U_0}^\top R_{P_0} e_1$ and the linearity of $\mathrm{d}J(\Omega)(\cdot,\cdot)$ in the second argument (*cf.* item (i)). □

Our proposed strategy now consists in first precomputing $\mathrm{d}J(\Omega)(te_1, e_i)$, $i = 1, 2, 3$ for a range of values of $t = |U_0| = |\operatorname{curl} u_0(z)|$ between a minimum value $t_{\min} = 0$ and a maximum value $t_{\max}$ in an offline stage. During the optimization, the values of $\mathrm{d}J(\Omega)(te_1, e_i)$ for any $t \in [t_{\min}, t_{\max}]$ can be approximated by interpolation and the topological derivative can be (approximately) evaluated with the help of Corollary 5.2. In practical applications, often reasonable values for $t_{\max}$ are known.

For the precomputation of the values $\mathrm{d}J(\Omega)(te_1, e_i)$ for a fixed $t \in [t_{\min}, t_{\max}]$, problem (3.5) has to be solved with $U_0 := te_1$. For the numerical solution of (3.5) recall that the solution H_ε to (3.14) is a good approximation of K for small $\varepsilon > 0$ due to Lemma 3.10. Moreover, it can be shown in an analogous way to Lemma 3.10 that for $B := B_R(0)$ with R such that $B \subset \mathsf{D}$, the solution $\tilde{H}_\varepsilon \in V(\varepsilon^{-1}B)^3$ to

$$\begin{aligned}\int_{\varepsilon^{-1}B} &(\mathcal{A}_\omega(x, \operatorname{curl}\tilde{H}_\varepsilon + U_0) - \mathcal{A}_\omega(x, U_0)) \cdot \operatorname{curl}\varphi \,\mathrm{d}x \\ &= -\int_\omega (a_1(U_0) - a_2(U_0)) \cdot \operatorname{curl}\varphi\,\mathrm{d}x \quad \text{for all} \quad \varphi \in V(\varepsilon^{-1}B)^3.\end{aligned} \tag{5.11}$$

satisfies $\operatorname{curl}\tilde{H}_\varepsilon \to \operatorname{curl} K$ strongly in $L_2(\mathbf{R}^3)$. Motivated by this observation, one may solve (5.11) with $B = B_1(0)$ and a comparatively small value for ε, *e.g.*, $\varepsilon = 1/1000$, as a good approximation to (3.5).

6. Application to electrical machines

In this section we show a real-world application where the setting of this paper applies. We consider the topology optimisation of an electrical machine in the setting of three-dimensional magnetostatics with nonlinear material behavior.

6.1. Physical modeling

The magnetostatic regime is a low frequency approximation to the full Maxwell equations where all quantities are assumed to be time-independent and where one only considers the magnetic equations

$$\operatorname{curl} H = J_i \quad \text{and} \quad \operatorname{div} B = 0. \tag{6.1}$$

Here, J_i denotes the impressed current density, and the magnetic field intensity H and the magnetic flux density B are related by the nonlinear material law

$$H = \nu(|B|)(B - M), \tag{6.2}$$

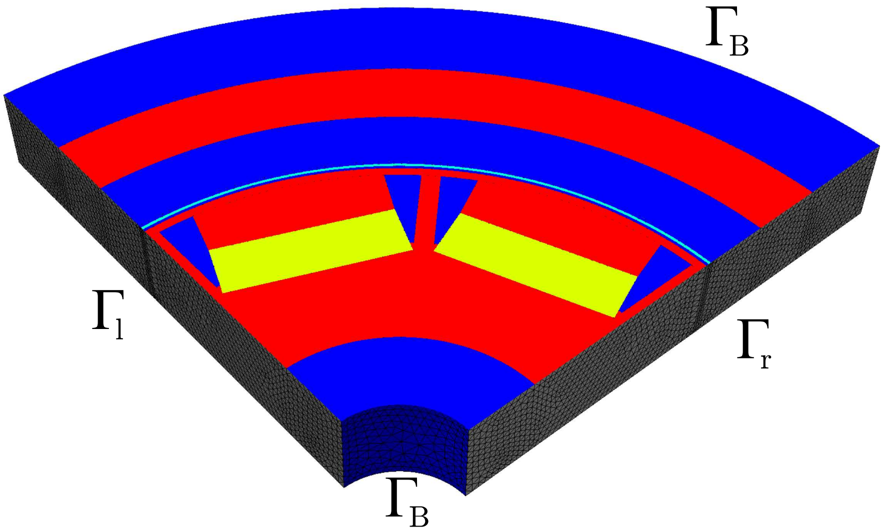

FIGURE 2. Three-dimensional model of electrical machine D with ferromagnetic subdomain Ω (red), permanent magnet regions Ω_2 (yellow), air subdomain $\mathsf{D} \setminus (\Omega \cup \Omega_2)$ (blue) with air gap region Ω_g (turquoise), lateral boundaries Γ_l, Γ_r and inner and outer boundaries Γ_B.

where $\nu : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the material-dependent magnetic relucitivity and M denotes the permanent magnetization. Due to symmetry, we consider only a quarter of the machine. Let $\mathsf{D} = \{(x, y, z) \in \mathbf{R}^3 : r_0 < \sqrt{x^2 + y^2} < r_3, -2.5 < z < 2.5\}$ with $r_0 = 4$mm and $r_3 = 33$mm the bounded, simply connected Lipschitz domain which contains the quarter of the electric motor as depicted in Figure 2. We set periodic boundary conditions on the lateral boundaries Γ_l and Γ_r, natural boundary conditions $\nu(|B|)B \times n = 0$ on the top and bottom and induction boundary conditions $B \cdot n = 0$ on the inner and outer parts Γ_B of $\partial\mathsf{D}$.

The motor consists of an inner, rotating part (the rotor) and an outer, fixed part (the stator), both containing ferromagnetic components. They are separated by a thin air gap $\Omega_g = \{(x, y, z) \in \mathsf{D} : r_1 < \sqrt{x^2 + y^2} < r_2\}$ with $r_1 = 19.67$ mm and $r_2 = 19.83$ mm, see the turquoise area in Figure 2. We denote the union of all ferromagnetic subdomains by Ω which we assume to be open. The current density J_i is in general supported in the coil regions $\Omega_1 \subset \mathsf{D} \setminus \overline{\Omega}$, which lie between the air gap and the stator core. The magnetization M is supported in the permanent magnets $\Omega_2 \subset \mathsf{D} \setminus \overline{\Omega}$. In this particular application, which was also treated in a two-dimensional setting in [4, 14], we assume the currents to be switched off, *i.e.*, $J_i = 0$ and therefore treat Ω_1 as air.

The magnetic reluctivity ν is equal to a constant $\nu_0 = 10^7/(4\pi)$ in the air and coil subdomains of the computational domain, a constant ν_m close to ν_0 in the permanent magnet regions Ω_2 and is given by a nonlinear function $\hat{\nu}$ in the ferromagnetic subdomain Ω. For more compact presentation we assume $\nu_m = \nu_0$. Moreover, we assume that $\hat{\nu}$ has the following properties:

Assumption C. *We assume that the magnetic reluctivity function $\hat{\nu} : \mathbf{R}_0^+ \to \mathbf{R}^+$ satisfies:*

(i) *The mapping $s \mapsto \hat{\nu}(s)s$, is strongly monotone, i.e., there is a constant $\underline{\nu}$ such that*

$$(\hat{\nu}(s)s - \hat{\nu}(t)t)(s - t) \geq \underline{\nu}(s - t)^2. \tag{6.3}$$

(ii) *The mapping $s \mapsto \hat{\nu}(s)s$ is Lipschitz continuous, i.e., there is a constant $\bar{\nu}$ such that*

$$|\hat{\nu}(s)s - \hat{\nu}(t)t| \leq \bar{\nu}|s - t|. \tag{6.4}$$

(iii) *We assume that for $\hat{\nu} \in C^2(\mathbf{R}_0^+)$, $\hat{\nu}'(0) = 0$, and that there is a constant c such that for all $s \in \mathbf{R}_0^+$, $\hat{\nu}'(s) \leq c$ and $\hat{\nu}''(s) \leq c$.*

The first two points of Assumption C follow from physical properties of $B{-}H$-curves, *i.e.*, of the relations between magnetic flux density B and magnetic field intensitiy H (*cf.* [25, 26]). In practice, the function $\hat{\nu}$ is obtained by interpolation of measured values [26], thus the smoothness assumption in Assumption C(iii) is justified. In our numerical experiments, we chose the analytic reluctivity function

$$\hat{\nu}(s) = \nu_0 - (\nu_0 - q_1)\exp(-q_2\, s^{q_3}), \tag{6.5}$$

which was also used in [34], with the values $q_1 = 200$, $q_2 = 0.001$ and $q_3 = 6$, which satisfies all of Assumption C.

Lemma 6.1. *Let Assumption C hold and define $a_1(y) := \hat{\nu}(|y|)y$ and $a_2(y) := \nu_0 y$ for $y \in \mathbf{R}^3$. Then Assumption A is satisfied.*

Proof. All properties of Assumption A are clear for the linear function a_2. For a_1, items (i) and (ii) of Assumption A follow immediately from items (i) and (ii) of Assumption C, respecitively (see *e.g.*, [25]). Moreover, it is shown in [4], Lemma 3.7 that Assumption C(iii) implies that a_2 is twice continuously differentiable, which is sufficient for Assumption A(iii). ☐

Using the ansatz $B = \operatorname{curl} u$ together with the Coulomb gauging condition $\operatorname{div} u = 0$, as well as the material law (6.2) and $a_1(y) := \hat{\nu}(|y|)y$ and $a_2(y) := \nu_0 y$, we get from (6.1) the boundary value problem

$$\text{find } u \in V : \int_{\mathsf{D}} \mathcal{A}_\Omega(x, \operatorname{curl} u) \cdot \operatorname{curl} v \,\mathrm{d}x = \int_{\Omega_2} M \cdot \operatorname{curl} v \,\mathrm{d}x \quad \text{for all} \quad v \in V, \tag{6.6}$$

with the function space $V = \{v \in H(\operatorname{curl}, \mathsf{D}) : u \times n = 0 \text{ on } \Gamma_B, u|_{\Gamma_l} = u|_{\Gamma_r}, \operatorname{div}(u) = 0 \text{ in } \mathsf{D}\}$ and the operator $\mathcal{A}_\Omega(x, y) = \chi_\Omega(x)a_1(y) + \chi_{D\setminus\Omega}(x)a_2(y)$. Note that we used the fact that $u \times n = 0$ on Γ_B is sufficient for $\operatorname{curl} u \cdot n = B \cdot n = 0$ on Γ_B, see [6, 21, 25].

As an objective function we consider

$$J(\Omega) = \int_{\Omega_g} |\operatorname{curl} u \cdot \hat{n} - B_d^n|^2 \,\mathrm{d}x \tag{6.7}$$

where Ω_g represents the air gap of the machine, $\hat{n} = (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2}, 0)^\top$ denotes an extension to the subdomain Ω_g of a unit normal vector field on a circular curve in the air gap and B_d^n denotes the desired distribution of the normal component of the magnetic flux density $B = \operatorname{curl} u$ in the air gap. In our experiments, B_d^n is given in cylindrical coordinates by

$$B_d^n(r, \varphi, z) = -\operatorname{amp}(z)\sin(4\varphi) \tag{6.8}$$

where $\operatorname{amp}(z)$ is given by the evaluation of $(\operatorname{curl} u_{init} \cdot \hat{n})$ at the point $(19.75, 22.5°, z)$ inside the air gap Ω_g. Here, u_{init} denotes the solution to the PDE constraint in the initial configuration. The left picture in Figure 3 shows $\operatorname{curl}(u) \cdot \hat{n}$ as a function of the angle $\varphi \in [0, 90°]$ and $z \in [-2.5, 2.5]$ for a fixed value of $r = 19.75$ (center of the air gap) for the initial configuration. The desired curve B_d^n is depicted in the center of Figure 3. We remark that the minimization of the objective function (6.7) yields a design of a machine which exhibits a smooth rotation pattern. Note the slight difference of objective function (6.7) to the functional (1.1) which was treated in the earlier sections. We remark, however, that all of the analysis can be performed for the given functional (6.7) in the exact same way. Note that the corresponding adjoint equation reads

$$\int_{\mathsf{D}} \partial_u \mathcal{A}_\Omega(x, \operatorname{curl} u)(\operatorname{curl} \varphi) \cdot \operatorname{curl} p \,\mathrm{d}x = -\int_{\mathsf{D}} 2(\operatorname{curl} u \cdot \hat{n} - B_d^n)(\operatorname{curl} \varphi \cdot \hat{n}) \,\mathrm{d}x, \tag{6.9}$$

for all $\varphi \in V(\mathsf{D})^3$ where u solves (6.6).

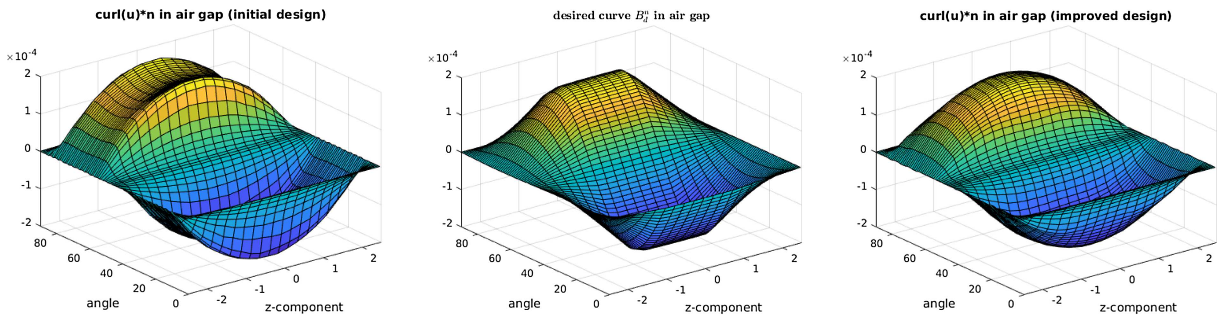

FIGURE 3. $\mathrm{curl}(u) \cdot n_{2D}$ along the air gap for (r, φ, z) with $r = 19.75$ and $\varphi \in [0, 90°]$, $z \in [-2.5, 2.5]$. *Left*: initial configuration. *Center*: desired curve B_d^n. *Right*: improved configuration.

6.2. Numerical results

In this section, we illustrate how the formula derived in Section 4 can be applied to the optimization of the electrical machine introduced in this section. The evaluation of the topological derivative (4.2) is done as described in Section 5. We precomputed the values $\mathrm{d}J(\Omega)(te_1, e_i)$ for $i = 1, 2, 3$ and $t \in \{j\, \delta_t\}_{j=0}^{40}$ with $\delta_t = 0.05$ and interpolated the obtained data using quadratic B-splines in an offline phase.

For the numerical solution of the state equation (6.6), we used second order Nédélec finite elements, see *e.g.*, [35], [28], Section 3, in the framework of the finite element software package `NETGEN/NGSolve` [29]. Problem (6.6) involves a divergence-free condition. In order to avoid solving a saddle point problem, we added an L^2-term $\int_{\mathsf{D}} \kappa u \cdot v \,\mathrm{d}x$ with a small constant $\kappa > 0$ as regularization to the bilinear form, yielding an elliptic problem on $H(\mathsf{D}, \mathrm{curl})$. We proceeded analogously in the numerical solution of the corresponding adjoint equation (6.9) and the problems for the approximation of the variation K (5.11) in the offline phase.

We started with the initial configuration shown in Figure 2, where all material data is constant in z-direction. Figures 4 and 5 show the application of a one-shot topology optimisation approach to (6.7) using a level set representation. The first row of Figure 4 shows the level set function in the two design subdomains of interest. We start with a constant level set function $\psi_0 = 1$ corresponding to ferromagnetic material in all of the two design subdomains. The left column in Figures 4 and 5 correspond to a horizontal cut at the bottom ($z \approx -2.5$), the central column shows a cut through the center of the machine ($z = 0$), and the right column a cut through the top of the machine ($z \approx 2.5$).

The second row of Figure 4 shows the absolute value of the magnetic flux density $|B| = |\,\mathrm{curl}\, u|$ for the three cross sections and the third row depicts the topological derivative. Note that the topological derivative attains its most negative values in the central cross section. For better visibility, we only show the negative part of the topological derivative in the central picture.

In order to change the material in the position where the topological derivative is most negative, we set

$$\psi_1 = (1 - s)\psi_0 + s \frac{\mathrm{d}J(\Omega)}{\|\mathrm{d}J(\Omega)\|_{L^2(\mathsf{D})}} \tag{6.10}$$

for an appropriately chosen value of s (here: $s \approx 0.14$).

The result can be seen in Figure 5 where the design in the top and bottom cross section remain unchanged and in total four holes of air are introduced in the center. The first row of Figure 5 shows the updated level set function ψ_1 and the second row the corresponding distribution of the magnetic flux density in the new design.

The third picture in Figure 3 shows the distribution of $\mathrm{curl}\, u \cdot \hat{n}$ for the new configuration. The objective value (6.7) has dropped from 2.33×10^{-8} to 4.68×10^{-9}.

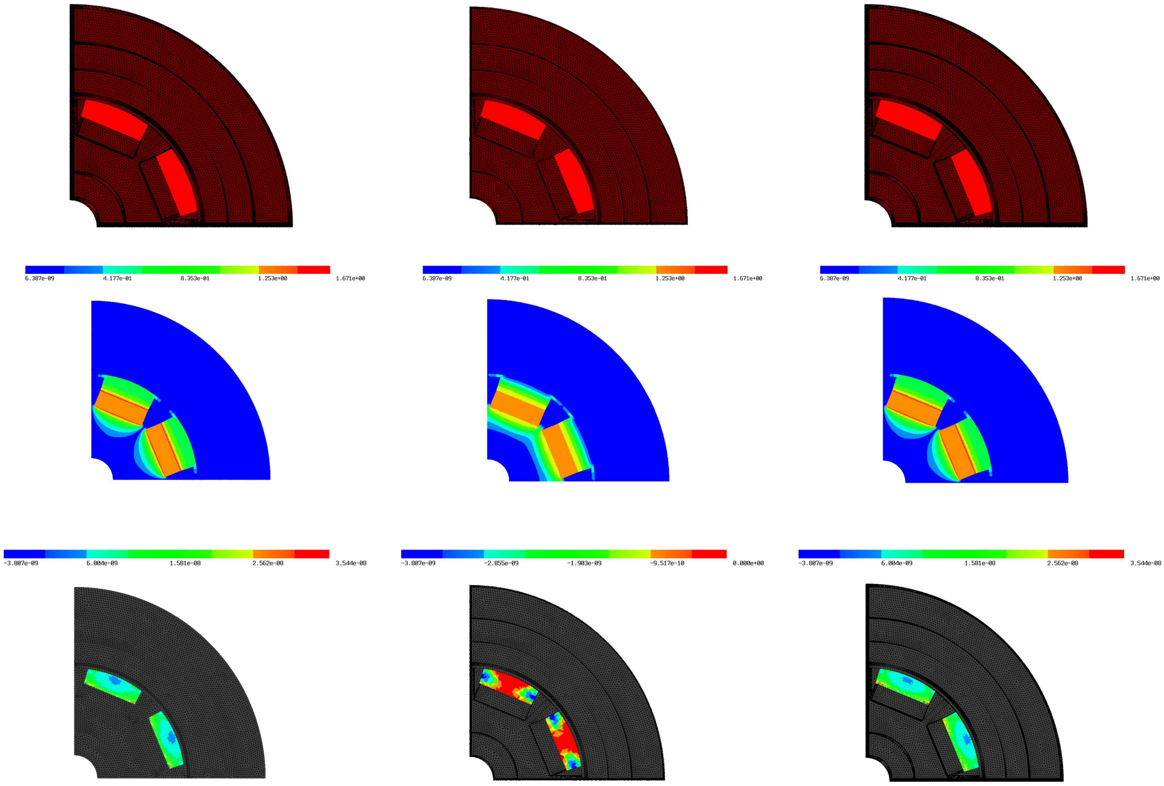

FIGURE 4. Initial configuration, objective value 2.33×10^{-8}. *1st row*: level set function. *2nd row*: B-field ($|B| = |\operatorname{curl} u|$). *3rd row*: topological derivative. *Left column*: bottom. *Central column*: center. *Right column*: top.

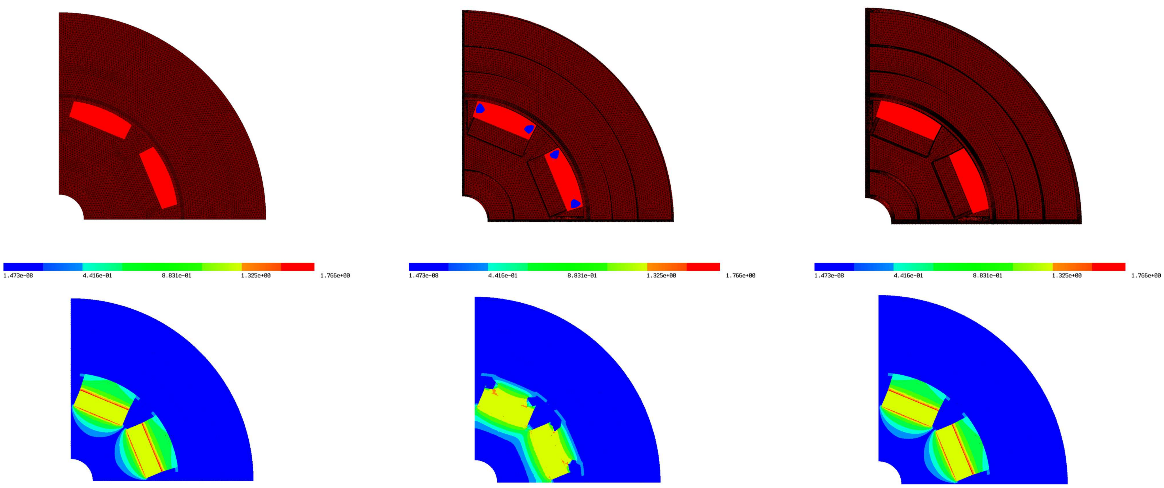

FIGURE 5. Improved configuration (after 1 iteration, objective value 4.68×10^{-9}). *1st row*: level set function. *2nd row*: B-field ($|B| = |\operatorname{curl} u|$). *Left column*: bottom. *Central column*: center. *Right column*: top.

7. Conclusion

In this work we presented the rigorous derivation of the topological derivative for a class of quasi-linear curl-curl problems under the assumption that $\operatorname{curl} u_0$ and $\operatorname{curl} p_0$ are (Hölder) continuous at the point where the topological perturbation takes place. We also discussed the efficient evaluation of the obtained formulas and applied our results to a physical model for an electrical machine. The results seem promising and show a significant improvement compared to the initial design.

The magnetostatic model does not capture eddy currents. Therefore in a future work it would be interesting to consider the time-dependent magnetoquasistatic problem rather than the magnetostatics case. This however requires a thorough analysis and new tools have to be developed.

Appendix A.

Lagrangian framework

In this section we recall results on a Lagrangian framework. This section is taken from [13], Section 2.

Definition A.1 (Parametrised Lagrangian)**.** Let X and Y be vector spaces and $\tau > 0$. A parametrised Lagrangian (or short Lagrangian) is a function

$$(\varepsilon, u, q) \mapsto G(\varepsilon, u, q) : [0,\tau] \times X \times Y \to \mathbf{R},$$

satisfying,

$$q \mapsto G(\varepsilon, u, q) \quad \text{ is affine on } Y. \tag{A.1}$$

Definition A.2 (State and adjoint state)**.** Let $\varepsilon \in [0,\tau]$. We define the state equation by: find $u_\varepsilon \in X$, such that

$$\partial_q G(\varepsilon, u_\varepsilon, 0)(\varphi) = 0 \quad \text{for all} \quad \varphi \in Y. \tag{A.2}$$

The set of states is denoted $E(\varepsilon)$. We define the adjoint state by: find $p_\varepsilon \in Y$, such that

$$\partial_u G(\varepsilon, u_\varepsilon, q_\varepsilon)(\varphi) = 0 \quad \text{for all} \quad \varphi \in X. \tag{A.3}$$

The set of adjoint states associated with $(\varepsilon, u_\varepsilon)$ is denoted $Y(\varepsilon, u_\varepsilon)$.

Definition A.3 (ℓ-differentiable Lagrangian)**.** Let X and Y be vector spaces and $\tau > 0$. Let $\ell : [0,\tau] \to \mathbf{R}$ be a given function satisfying $\ell(0) = 0$ and $\ell(\varepsilon) > 0$ for $\varepsilon \in (0,\tau]$. An ℓ-differentiable parametrised Lagrangian is a parametrised Lagrangian $G : [0,\tau] \times X \times Y \to \mathbf{R}$, satisfying,

(a) for all $v, w \in X$ and $p \in Y$,

$$s \mapsto G(\varepsilon, v + sw, p) \text{ is continuously differentiable on } [0,1]. \tag{A.4}$$

(b) for all $u_0 \in E(0)$ and $q_0 \in Y(0, u_0)$ the limit

$$\partial_\ell G(0, u_0, q_0) := \lim_{\varepsilon \searrow 0} \frac{G(\varepsilon, u_0, q_0) - G(0, u_0, q_0)}{\ell(\varepsilon)} \quad \text{ exists.} \tag{A.5}$$

Assumption D (H0)**.** (i) *We assume that for all* $\varepsilon \in [0,\tau]$*, the set* $E(\varepsilon) = \{u_\varepsilon\}$ *is a singleton.*

(ii) *We assume that the adjoint equation for* $\varepsilon = 0$*,* $\partial_u G(0, u_0, p_0)(\varphi) = 0$ *for all* $\varphi \in E$*, admits a unique solution.*

We now give sufficient conditions when the function

$$\begin{aligned} &\to \mathbf{R} \\ \varepsilon &\mapsto g(\varepsilon) := G(\varepsilon, u_\varepsilon, 0), \end{aligned} \tag{A.6}$$

is one sided ℓ-differentiable, that means, when the limit

$$d_\ell g(0) := \lim_{\varepsilon \searrow 0} \frac{g(\varepsilon) - g(0)}{\ell(\varepsilon)} \tag{A.7}$$

exists, where $\ell : [0, \tau] \to \mathbf{R}$ is a given function satisfying $\ell(0) = 0$ and $\ell(\varepsilon) > 0$ for $\varepsilon \in (0, \tau]$.

Theorem A.4 ([13], Thm. 3.4 and [10], Thm. 3.3)**.** *Let $G : [0, \tau] \times X \times Y \to \mathbf{R}$ be an ℓ-differentiable parametrised Lagrangian satisfying Hypothesis (H0). Define for $\varepsilon > 0$,*

$$R_1^\varepsilon(u_0, p_0) := \frac{1}{\ell(\varepsilon)} \int_0^1 \left(\partial_u G(\varepsilon, s u_\varepsilon + (1-s) u_0, p_0) - \partial_u G(\varepsilon, u_0, p_0)\right)(u_\varepsilon - u_0) \, \mathrm{d}s \tag{A.8}$$

and

$$R_2^\varepsilon(u, p) := \frac{1}{\ell(\varepsilon)} (\partial_u G(\varepsilon, u_0, p_0) - \partial_u G(0, u_0, p_0))(u_\varepsilon - u_0). \tag{A.9}$$

If $R_1(u_0, p_0) := \lim_{\varepsilon \searrow 0} R_1^\varepsilon(u_0, p_0)$ and $R_2(u_0, p_0) := \lim_{\varepsilon \searrow 0} R_2^\varepsilon(u_0, p_0)$ exist, then

$$d_\ell g(0) = \partial_\ell G(0, u_0, q_0) + R_1(u_0, p_0) + R_2(u_0, p_0).$$

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