ESAIM: M2AN 54 (2020) 1797–1820
https://doi.org/10.1051/m2an/2020023

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

ANALYSIS OF OPTIMAL SUPERCONVERGENCE OF AN ULTRAWEAK-LOCAL DISCONTINUOUS GALERKIN METHOD FOR A TIME DEPENDENT FOURTH-ORDER EQUATION

Yong Liu[1], Qi Tao[1] and Chi-Wang Shu[2,*]

Abstract. In this paper, we study superconvergence properties of the ultraweak-local discontinuous Galerkin (UWLDG) method in Tao *et al.* [To appear in *Math. Comput.* DOI: https://doi.org/10.1090/mcom/3562 (2020).] for an one-dimensional linear fourth-order equation. With special initial discretizations, we prove the numerical solution of the semi-discrete UWLDG scheme superconverges to a special projection of the exact solution. The order of this superconvergence is proved to be $k+\min(3,k)$ when piecewise $\mathbb{P}^k$ polynomials with $k \geq 2$ are used. We also prove a $2k$-th order superconvergence rate for the cell averages and for the function values and derivatives of the UWLDG approximation at cell boundaries. Moreover, we prove superconvergence of $(k+2)$-th and $(k+1)$-th order of the function values and the first order derivatives of the UWLDG solution at a class of special quadrature points, respectively. Our proof is valid for arbitrary non-uniform regular meshes and for arbitrary $k \geq 2$. Numerical experiments verify that all theoretical findings are sharp.

Mathematics Subject Classification. 65M60, 65M15.

Received October 24, 2019. Accepted April 4, 2020.

1. Introduction

Recently, Tao *et al.* [33] developed a new class of discontinuous Galerkin (DG) methods, termed ultraweak-local DG (UWLDG), for solving time dependent high order equations. In particular, for even order equations, Tao *et al.* [33] proved the UWLDG scheme achieves energy conserving stability without penalty terms, in comparison with the traditional ultra-weak DG method in [15] which would need penalty terms for stability. In this paper, we study the superconvergence properties of the UWLDG method in [33] for the linear fourth-order equation as follows,

$$u_t + u_{xxxx} = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

where Ω is an interval. For simplicity, we assume the boundary condition is periodic. Other types of boundary conditions can also be considered along the same lines for our analysis. The fourth order model has wide applications, such as thin beams and plates, strain gradient elasticity, and phase separation in binary mixtures [26].

Keywords and phrases. Ultraweak-local discontinuous Galerkin methods, superconvergence, fourth order derivatives.

[1] School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.

[2] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.

[*]Corresponding author: chi-wang_shu@brown.edu

The DG methods are a class of finite element methods devised to solve hyperbolic conservation laws and related equations, *e.g.* [18–20, 22, 23], using discontinuous piecewise polynomial function space for the test and trial functions in the spatial variables. For equations with higher-order spatial derivatives, such as the convection-diffusion equation, KdV equation etc., the DG method cannot be directly applied, due to the discontinuous finite element space which is not regular enough to handle higher order derivatives. There are several ways to solve this problem, including, for example, the local discontinuous Galerkin (LDG) method [21, 24, 27, 36–39], the interior penalty (IP) method [25,31], and the ultra-weak DG (UWDG) methods [15]. We focus on a new class of DG methods which combines the advantages of LDG and UWDG methodologies, to solve fourth-order partial differential equations (PDEs) [33]. It rewrites the fourth-order equation into a second order system and then applies the ultra-weak DG discretization to each of the second order equations. The key features of the UWLDG scheme are that they avoid too many auxiliary variables as in LDG to make the scheme more efficient, and they achieve energy stability without interior penalty terms for even-order equations.

It is important to study superconvergence, because *a posteriori* error estimates can be derived for designing trouble cell indicators in adaptive algorithms such as the KXRCF trouble cell indicator [28]. In the past few years, there have been many superconvergence results of the DG methods in the literature. We refer to [1,2] for ordinary differential equations, and to [14, 16, 40] for one-dimensional time dependent hyperbolic conservation laws and convection-diffusion equations. In [7], Cao *et al.* introduced an approach to study the superconvergence of the DG methods for linear hyperbolic equations by constructing a locally suitable correction function. They proved the $(2k + 1)$-th order superconvergence rate for the cell averages and the DG numerical fluxes when piecewise polynomials of degree k are used. Later, Cao *et al.* extended this technique to study upwind-biased numerical fluxes, degenerate variable coefficients, nonlinear hyperbolic conservation laws and two-dimensional hyperbolic equations [8, 9, 11, 12]. The correction function techniques also work well on other types of DG methods such as the ALE DG [32], the energy-conserving DG [30] etc. For higher-order equations, Cao *et al.* studied the superconvergence properties of the direct DG method for convection-diffusion equations in [10]. Cao and Huang gave a unified framework to study superconvergence results of the LDG method in [6]. More recently, Chen *et al.* studied the superconvergence of the ultra-weak DG methods for linear Schrödinger equations by using the correction function technique in [13].

In this paper, we continue to apply the correction function technique to design a special interpolation function to obtain superconvergence results for the UWLDG schemes in [33] for fourth order equations. For high-order equations, the current state of the art on using the correction function approach would lead to sub-optimal estimates of superconvergence in comparison with numerical results in certain cases. For example, for the DDG and UWDG methods for the second-order equations, the proof of the superconvergence rates will lose one order when k is even, as pointed out in [10, 13]. Thus, in order to obtain the optimal superconvergence estimates, we would need to introduce additional techniques. In [10], the authors improved the superconvergence estimates thanks to the diffusion terms in the DDG spatial operators. Chen *et al.* [13] used the superconvergence properties of the difference of projections in neighboring cells for uniform meshes in [3] to obtain the optimal superconvergence on uniform meshes. In this paper, our analysis of optimal superconvergence is valid for arbitrary regular nonuniform meshes. We use the important properties of the LDG operators, namely the derivative and the cell interface jump of the approximate solution can be bounded by the auxiliary variable [34, 35]. In [33], Tao *et al.* also proved similar properties of the second-order derivative DG operators. We first obtain estimates, for the derivative and the element interface jump, of the error between the special interpolation and the numerical solution by taking special test functions in the schemes. Then, under suitable conditions, the discrete Poincaré inequality [4, 5] implies that its own L^2 norm can be bounded by its derivative and the element interface jump. Thus, the desired superconvergence estimates can be obtained. The superconvergence of both the numerical solution and the auxiliary variable in the infinity norm in time can be obtained thanks to the special initial discretization.

The outline of this paper is as follows. We first recall the UWLDG method for the linear fourth-order equations in Section 2. Then we construct the special interpolation function and the superconvergence results are provided in Section 3. Numerical examples are provided to verify our theoretical findings in Section 4. The concluding remarks and plans for ongoing work are presented in Section 5. Finally, some technical proofs of the lemmas and theorems are collected in the appendix.

2. The UWLDG schemes

We consider the following one dimensional fourth-order equation

$$\begin{cases} u_t + u_{xxxx} = 0, & (x,t) \in [0,1] \times (0,T] \\ u(x,0) = u_0(x), & x \in [0,1] \end{cases} \tag{2.1}$$

with the periodic boundary condition. We first introduce the usual notation of the DG method. For a given interval $\Omega = [0,1]$ and the index set $\mathbb{Z}_N = \{1, 2, \dots, N\}$, the usual DG mesh $\mathcal{I}_N$ is defined as:

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N+\frac{1}{2}} = 1. \tag{2.2}$$

We denote

$$I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad x_j = \frac{1}{2}\left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right), \tag{2.3}$$

and

$$h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad \bar{h}_j = \frac{h_j}{2}, \quad h = \max_j h_j, \quad j \in \mathbb{Z}_N. \tag{2.4}$$

We also assume the mesh is regular, *i.e.* the ratio between the maximum and minimum mesh sizes shall stay bounded during mesh refinements. We define the finite element space as

$$V_h^k = \left\{ v_h : (v_h)|_{I_j} \in \mathbb{P}^k(I_j), j = 1, \dots, N \right\}. \tag{2.5}$$

Here $\mathbb{P}^k(I_j)$ denotes the set of all polynomials of degree at most k on I_j. For a function $v_h \in V_h^k$, we use $(v_h)^-_{j+\frac{1}{2}}$ and $(v_h)^+_{j+\frac{1}{2}}$ to refer to the value of v_h at $x_{j+\frac{1}{2}}$ form the left cell I_j and the right cell I_{j+1}, respectively. We use $[v_h] = v_h^+ - v_h^-$ and $\{v_h\} = \frac{1}{2}\left(v_h^- + v_h^+\right)$ to denote the jump and the average of v_h at element interfaces. The standard Sobolev space notations are introduced. For any integer $m > 0$, we let $W^{m,p}(D)$ be the standard Sobolev spaces on the sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$. If $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$ and we omit the index D, when $D = \Omega$.

To construct the UWLDG scheme for (2.1), we rewrite (2.1) into a second order system

$$u_t + v_{xx} = 0, \quad v = u_{xx}. \tag{2.6}$$

The semi-discrete UWLDG scheme formulated in [33] is to find u_h, $v_h \in V_h^k$ such that for all φ, $\psi \in V_h^k$,

$$a_j(u_h, v_h; \varphi) = 0, \tag{2.7a}$$

$$b_j(v_h, u_h; \psi) = 0, \quad \forall j \in \mathbb{Z}_N, \tag{2.7b}$$

where

$$a_j(u_h, v_h; \varphi) = \left((u_h)_t, \varphi\right)_j + A_j(v_h, \varphi), \tag{2.8}$$

$$b_j(v_h, u_h; \psi) = (v_h, \psi)_j - B_j(u_h, \psi), \tag{2.9}$$

with

$$A_j(v_h, \varphi) = (v_h, \varphi_{xx})_j + \widetilde{(v_h)_x}\varphi^-|_{j+\frac{1}{2}} - \widetilde{(v_h)_x}\varphi^+|_{j-\frac{1}{2}} - \widehat{v_h}\varphi_x^-|_{j+\frac{1}{2}} + \widehat{v_h}\varphi_x^+|_{j-\frac{1}{2}}, \tag{2.10}$$

$$B_j(u_h, \psi) = (u_h, \psi_{xx})_j + \widetilde{\widetilde{(u_h)_x}}\psi^-|_{j+\frac{1}{2}} - \widetilde{\widetilde{(u_h)_x}}\psi^+|_{j-\frac{1}{2}} - \widehat{\widehat{u_h}}\psi_x^-|_{j+\frac{1}{2}} + \widehat{\widehat{u_h}}\psi_x^+|_{j-\frac{1}{2}}, \tag{2.11}$$

being the UWDG spatial discretizations for the second order derivative terms. $(u,v)_j = \int_{I_j} uv \,\mathrm{d}x$, $v^-|_{j+\frac{1}{2}}$ and $v^+|_{j+\frac{1}{2}}$ denote the left and right limits of v at the point $x_{j+\frac{1}{2}}$, respectively, and $\widehat{\widehat{u_h}}, \widetilde{\widetilde{(u_h)_x}}, \widehat{v_h}, \widetilde{(v_h)_x}$ are the numerical fluxes. To ensure the stability and the local solvability of the intermediate variable v_h, we defined these four fluxes as follows:

$$\widehat{\widehat{u_h}} = \{u_h\} + \alpha_1 [u_h] + \beta_1 [(u_h)_x], \qquad \alpha_1, \beta_1 \in \mathbb{R}, \tag{2.12}$$

$$\widetilde{\widetilde{(u_h)_x}} = \{(u_h)_x\} + \alpha_2 [(u_h)_x] + \beta_2 [u_h], \quad \alpha_2, \beta_2 \in \mathbb{R}, \tag{2.13}$$

$$\widehat{v_h} = \{v_h\} - \alpha_2 [v_h] + \beta_1 [(v_h)_x], \tag{2.14}$$

$$\widetilde{(v_h)_x} = \{(v_h)_x\} - \alpha_1 [(v_h)_x] + \beta_2 [v_h], \tag{2.15}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ can be chosen as follows:

- central flux, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$;
- alternating flux, $\alpha_1 = \pm\frac{1}{2}, \alpha_2 = \pm\frac{1}{2}, \beta_1 = \beta_2 = 0$;
- IPDG like flux, $\alpha_1 = \alpha_2 = \beta_1 = 0, \beta_2 = \tilde{\beta}_2 h^{-1}$;
- DDG like flux, $\alpha_1 = \tilde{\alpha_1}, \alpha_2 = \tilde{\alpha_2}, \beta_1 = 0, \beta_2 = \tilde{\beta}_2 h^{-1}$;
- more generally, any scale invariant flux, $\alpha_1 = \tilde{\alpha_1}, \alpha_2 = \tilde{\alpha_2}, \beta_1 = \tilde{\beta}_1 h, \beta_2 = \tilde{\beta}_2 h^{-1}$;

where $\tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\beta}_1, \tilde{\beta}_2$ are prescribed constants independent of the mesh size. For simplicity, in this paper we will only consider the alternating flux choices, $\alpha_1 = -\frac{1}{2}, \alpha_2 = \frac{1}{2}, \beta_1 = \beta_2 = 0$, *i.e.*

$$\widehat{\widehat{u_h}} = u_h^-, \quad \widetilde{\widetilde{(u_h)_x}} = (u_h)_x^+, \quad \widehat{v_h} = v_h^-, \quad \widetilde{(v_h)_x} = (v_h)_x^+. \tag{2.16}$$

We now introduce

$$a(u_h, v_h; \varphi) = \sum_{j=1}^{N} a_j(u_h, v_h; \varphi), \quad b(v_h, u_h; \psi) = \sum_{j=1}^{N} b_j(v_h, u_h; \psi), \tag{2.17}$$

$$A(v_h, u_h) = \sum_{j=1}^{N} A_j(v_h, u_h) = \int_\Omega v_h (u_h)_{xx} \,\mathrm{d}x + \sum_{j=1}^{N} \left(\widehat{v_h} [(u_h)_x] - \widetilde{(v_h)_x} [u_h] \right) \Big|_{j+\frac{1}{2}}, \tag{2.18}$$

$$B(u_h, v_h) = \sum_{j=1}^{N} B_j(u_h, v_h) = \int_\Omega u_h (v_h)_{xx} \,\mathrm{d}x + \sum_{j=1}^{N} \left(\widehat{\widehat{u_h}} [(v_h)_x] - \widetilde{\widetilde{(u_h)_x}} [v_h] \right) \Big|_{j+\frac{1}{2}}. \tag{2.19}$$

By the same arguments as in [13], we have the following lemma.

Lemma 2.1. *For $u_h, v_h \in V_h^k$ satisfying periodic boundary condition, we have $A(v_h, u_h) = B(u_h, v_h)$.*

In [33], it was proved that the semi-discrete scheme is energy-conserving stable for the alternating flux, which is a direct result of the lemma above:

$$0 = a(u_h, v_h; u_h) + b(v_h, u_h; v_h) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2 + \|v_h\|^2. \tag{2.20}$$

Here and below, an unmarked norm $\|\cdot\|$ denotes the L^2 norm. Obviously, the exact solution u, v of (2.6) also satisfies

$$a_j(u, v; \varphi) = 0, \quad b_j(v, u; \psi) = 0, \quad \forall (\varphi, \psi) \in \left[V_h^k\right]^2. \tag{2.21}$$

Subtracting (2.7) from (2.21), we obtain the error equations

$$a_j(u - u_h, v - v_h; \varphi) = 0, \quad b_j(v - v_h, u - u_h; \psi) = 0, \quad \forall (\varphi, \psi) \in \left[V_h^k\right]^2. \tag{2.22}$$

In [33], for the error estimates, the special projection $P_h^\star u \in V_h^k$ of u has been defined by

$$\left(P_h^\star u, \varphi\right)_j = (u, \varphi)_j \qquad \forall \varphi \in \mathbb{P}^{k-2}\left(I_j\right) \quad \text{and} \tag{2.23a}$$

$$P_h^\star u\left(x_{j+\frac{1}{2}}^-\right) = u\left(x_{j+\frac{1}{2}}\right), \quad \left(P_h^\star u\right)_x\left(x_{j-\frac{1}{2}}^+\right) = u_x\left(x_{j-\frac{1}{2}}\right). \tag{2.23b}$$

For this projection, the following inequality holds [33]:

$$\|w^e\| + h\|w^e\|_\infty + h^{\frac{1}{2}}\|w^e\|_{\Gamma_h} \le Ch^{k+1}\|w\|_{k+1}, \tag{2.24}$$

where $w^e = P_h^\star w - w$, Γ_h denotes the set of boundary points of all elements I_j, $\|w^e\|_{\Gamma_h} = \left(\sum_{j=1}^N w^e\left(x_{j+\frac{1}{2}}^-\right)^2 + w^e\left(x_{j+\frac{1}{2}}^+\right)^2\right)^{\frac{1}{2}}$, and the constant C depends on k.

We also need the following basic facts. For any function $w_h \in V_h^k$, the following inequalities hold [17]:

$$\begin{aligned} &\text{(i)} \quad \|\left(w_h\right)_x\| \le Ch^{-1}\|w_h\|,\\ &\text{(ii)} \quad \|w_h\|_{\Gamma_h} \le Ch^{-\frac{1}{2}}\|w_h\|,\\ &\text{(iii)} \quad \|w_h\|_\infty \le Ch^{-\frac{1}{2}}\|w_h\|. \end{aligned} \tag{2.25}$$

We denote

$$\eta_u = u - P_h^\star u, \quad \xi_u = u_h - P_h^\star u, \tag{2.26}$$

$$\eta_v = v - P_h^\star v, \quad \xi_v = v_h - P_h^\star v. \tag{2.27}$$

Due to (2.22), we have

$$a\left(\xi_u, \xi_v; \xi_u\right) + b\left(\xi_v, \xi_u; \xi_v\right) = a\left(\eta_u, \eta_v; \xi_u\right) + b\left(\eta_v, \eta_u; \xi_v\right). \tag{2.28}$$

By using the definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ and Cauchy–Schwartz inequality we obtain

$$\left|a\left(\eta_u, \eta_v; \xi_u\right) + b\left(\eta_v, \eta_u; \xi_v\right)\right| = \left|\left(\left(\eta_u\right)_t, \xi_u\right) + \left(\eta_v, \xi_v\right)\right| \lesssim h^{k+1}\left(\|\xi_u\| + \|\xi_v\|\right),$$

then

$$\|\xi_u(\cdot,t)\| + \left(\int_0^t \|\xi_v(\cdot,\tau)\|^2 \,\mathrm{d}\tau\right)^{\frac{1}{2}} \lesssim h^{k+1}, \tag{2.29}$$

here and in the following, $A \lesssim B$ denotes that A can be bounded by B multiplied by a constant independent of the mesh size h. However, this estimate is not optimal and far from our superconvergence goal. We need to improve the analysis through constructing a series of correction functions $\left(\omega_u^{(i)}, \omega_v^{(i)}\right) \in \left[V_h^k\right]^2$, $1 \le i \le \left\lfloor \frac{k-1}{2} \right\rfloor$ such that

$$\begin{aligned} &\left|a\left(\eta_u + \sum_{i=1}^l \omega_u^{(i)}, \eta_v + \sum_{i=1}^l \omega_v^{(i)}; \varphi\right) + b\left(\eta_v + \sum_{i=1}^l \omega_v^{(i)}, \eta_u + \sum_{i=1}^l \omega_u^{(i)}; \psi\right)\right| \lesssim h^{k+1+2l}\left(\|\varphi\| + \|\psi\|\right),\\ &\qquad \forall (\varphi, \psi) \in \left[V_h^k\right]^2, 1 \le l \le \left\lfloor \frac{k-1}{2} \right\rfloor, \end{aligned} \tag{2.30}$$

where $\lfloor k \rfloor$ denotes the maximal integer no more than k. When k is even, $\left\lfloor \frac{k-1}{2} \right\rfloor = \frac{k-2}{2}$, which leads to one order lower than the optimal estimates. We will improve the estimates in such situation in the next subsections.

Remark 2.2. We note that the estimate of $\|\xi_v\|$ in (2.29) is in the L^2 norm of time. In fact, we can take time derivative for b_j in the error equation (2.22), then take test functions $\varphi = (\xi_u)_t$, $\psi = \xi_v$, to obtain

$$a\left(\xi_u, \xi_v; (\xi_u)_t\right) + b\left((\xi_v)_t, (\xi_u)_t; \xi_v\right) = a\left(\eta_u, \eta_v; (\xi_u)_t\right) + b\left((\eta_v)_t, (\eta_u)_t; \xi_v\right). \tag{2.31}$$

Thus, by Lemma 2.1, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_v\|^2 + \|(\xi_u)_t\|^2 \lesssim h^{k+1}\left(\|(\xi_u)_t\| + \|\xi_v\|\right). \tag{2.32}$$

Then, by Gronwall's inequality, we have

$$\|\xi_v(\cdot, t)\| \lesssim \|\xi_v(\cdot, 0)\| + h^{k+1}. \tag{2.33}$$

3. Construction of a special interpolation function

The correction functions $\omega_u^{(i)}$ and $\omega_v^{(i)}$, $1 \le i \le \lfloor \frac{k-1}{2} \rfloor$ are defined as follows. If we denote $\omega_u^{(0)} = u - P_h^\star u$, $\omega_v^{(0)} = v - P_h^\star v$, then

$$\left(\omega_u^{(i)}, \varphi_{xx}\right)_j = \left(\omega_v^{(i-1)}, \varphi\right)_j, \qquad \forall \varphi \in \mathbb{P}^k(I_j), \quad \omega_u^{(i)}\left(x^-_{j+\frac{1}{2}}\right) = 0, \ \left(\omega_u^{(i)}\right)_x\left(x^+_{j-\frac{1}{2}}\right) = 0, \tag{3.1}$$

$$\left(\omega_v^{(i)}, \varphi_{xx}\right)_j = -\left(\left(\omega_u^{(i-1)}\right)_t, \varphi\right)_j, \quad \forall \varphi \in \mathbb{P}^k(I_j), \quad \omega_v^{(i)}\left(x^-_{j+\frac{1}{2}}\right) = 0, \ \left(\omega_v^{(i)}\right)_x\left(x^+_{j-\frac{1}{2}}\right) = 0. \tag{3.2}$$

In [13], the authors defined similar correction functions for UWDG. By similar arguments, we have the following estimate for $\omega_u^{(i)}$ and $\omega_v^{(i)}$, $1 \le i \le \lfloor \frac{k-1}{2} \rfloor$.

Lemma 3.1. *For any $k \ge 3$, the functions $\omega_u^{(i)}$ and $\omega_v^{(i)}$, $1 \le i \le \lfloor \frac{k-1}{2} \rfloor$ are defined by (3.1) and (3.2). Then*

$$\omega_u^{(i)}|_{I_j} = \sum_{m=k-1-2i}^{k} c^i_{j,m} L_{j,m}, \quad \omega_v^{(i)}|_{I_j} = \sum_{m=k-1-2i}^{k} d^i_{j,m} L_{j,m}, \tag{3.3}$$

where $c^i_{j,m}$ and $d^i_{j,m}$ are some bounded constants, and $L_{j,m}$ are the standard Legendre polynomials with degree m on interval I_j. Furthermore, if $u \in H^{k+3+2i}$, there holds for $n = 0, 1$ and $q = u, v$

$$\|\partial_t^n \omega_q^{(i)}\| \lesssim h^{k+1+2i}\|\partial_t^n q\|_{k+1+2i}. \tag{3.4}$$

Proof. We provide the proof of this lemma in the Appendix A.1. From Lemma 3.1, we can obtain the corollary as follows. □

Corollary 3.2. *For any $k \ge 3$, if the exact solution of the equation (2.1), $u \in H^{k+5+2\lfloor \frac{k-1}{2} \rfloor}$, then*

$$\left| a\left(\eta_u + \sum_{i=1}^{l}\omega_u^{(i)}, \eta_v + \sum_{i=1}^{l}\omega_v^{(i)}; \varphi\right) + b\left(\eta_v + \sum_{i=1}^{l}\omega_v^{(i)}, \eta_u + \sum_{i=1}^{l}\omega_u^{(i)}; \psi\right)\right| \lesssim h^{k+1+2l}\left(\|\varphi\| + \|\psi\|\right)$$
$$\forall\ (\varphi, \psi) \in \left[V_h^k\right]^2, \ 1 \le l \le \left\lfloor \frac{k-1}{2} \right\rfloor. \tag{3.5}$$

Before studying the superconvergence properties of the UWLDG scheme, we need some lemmas which are also mentioned in [33].

Lemma 3.3. *Suppose $w \in L^2$ and $\xi \in V_h^k$ satisfy*

$$A_j(\xi, \eta) = (w, \eta)_j, \quad \forall \eta \in V_h^k, \tag{3.6}$$

or

$$B_j(\xi, \eta) = (w, \eta)_j, \quad \forall \eta \in V_h^k, \tag{3.7}$$

then

$$\|(\xi)_{xx}\|_{I_j} + h^{-\frac{1}{2}} | [(\xi)_x] |_{j+\frac{1}{2}} + h^{-\frac{3}{2}} |[\xi]|_{j-\frac{1}{2}} \lesssim \|w\|_{0,I_j}. \tag{3.8}$$

Proof. This proof is the same as Lemma 4.2 in [33]. We omit it here. □

Next, we shall study the superconvergence properties of the UWLDG solution, including superconvergence between a special interpolation of the exact solution and the numerical solution, the superconvergence of the cell averages, and the function and derivative values at some special quadrature points respectively.

3.1. Superconvergence of the interpolation

With (3.1) and (3.2), we define

$$(\omega_{u,l}, \omega_{v,l}) = \left(\sum_{i=1}^{l} \omega_u^{(i)}, \sum_{i=1}^{l} \omega_v^{(i)} \right), \tag{3.9}$$

$$\left(u_I^l, v_I^l\right) = \left(P_h^\star u - \omega_{u,l}, P_h^\star v - \omega_{v,l}\right), \quad 1 \le l \le \left\lfloor \frac{k-1}{2} \right\rfloor, \tag{3.10}$$

and we let

$$e_q = q - q_h = q - q_I^l - \left(q_h - q_I^l\right) = \epsilon_q - \bar{e}_q, \quad q = u, v. \tag{3.11}$$

As we know, the approximations of the initial condition are of great significance for superconvergence. In order to obtain our superconvergence rate, the initial error should be small enough to reach the same superconvergence rate. We have the following lemma.

Lemma 3.4. *For any $k \ge 2$, suppose the exact solution of the equation (2.1), $u \in H^{k+5+2\left\lfloor \frac{k-1}{2} \right\rfloor}$. If the initial data is taken such that*

$$v_h(x, 0) = P_h^\star v_0 - \omega_{v,l}(x, 0), \quad v_0 = \partial_x^2 u_0, \tag{3.12}$$

where $\omega_{v,l}$ is defined by (3.9), $l = \left\lfloor \frac{k-1}{2} \right\rfloor$. Then

$$\left(\|\bar{e}_u\| + \|\bar{e}_v\|\right)(0) \lesssim h^{2k}, \tag{3.13}$$

$$\left(\| (\bar{e}_u)_x \| + \| (\bar{e}_v)_x \|\right)(0) \lesssim h^{2k}. \tag{3.14}$$

Proof. The choice of the initial data $u_h(x, 0)$ and this proof are given in the Appendix A.2. □

With this initial solution, we have the following optimal superconvergence estimates which are stated as a theorem.

Theorem 3.5. *Suppose u_h and v_h are the approximate solutions of the semi-discrete scheme (2.7) with the initial data satisfying (3.12). Let u, v be the exact solutions of the system (2.6) satisfying $u \in H^{k+5+2l}$, and $\left(u_I^l, v_I^l\right) \in \left[V_h^k\right]^2$ is defined in (3.10), where $l = \left\lfloor \frac{k-1}{2} \right\rfloor$. For any $k \ge 2$, we have*

$$\left(\|\bar{e}_u\| + \|\bar{e}_v\|\right)(t) \lesssim h^{2k}, \tag{3.15}$$

$$\left(\| (\bar{e}_u)_x \| + \| (\bar{e}_v)_x \|\right)(t) \lesssim h^{2k}. \tag{3.16}$$

Especially, if $l = 0$*, or* 1*, we get*

$$\|P_h^\star u - u_h\| + \|P_h^\star v - v_h\| \lesssim \begin{cases} h^{2k} & \text{if } k = 2, \\ h^{k+3} & \text{if } k \geq 3, \end{cases} \tag{3.17}$$

and

$$\|\left(P_h^\star u - u_h\right)_x\| + \|\left(P_h^\star v - v_h\right)_x\| \lesssim h^{k+2}. \tag{3.18}$$

Proof. We give the proof of this theorem in the Appendix A.3. □

Remark 3.6. We note that when $k = 2$, the L^2 norm and H^1 semi-norm of the error between the numerical solutions and the special projections of the exact solution both have $2k$-th order superconvergence. Actually, thanks to (3.16), we obtain the optimal superconvergence rates of the derivative of the error between the special interpolation functions and approximate solutions. Numerical examples also verify this result in Section 4.

3.2. Superconvergence for the numerical fluxes and the cell averages

With Theorem 3.5, we can obtain the following superconvergence results.

Theorem 3.7. *We let*

$$e_{q,f} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(q - \widehat{q_h}\right)\left(x_{j+\frac{1}{2}}, t\right)^2\right)^{\frac{1}{2}}, \quad e_{q,fx} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(q_x - \widetilde{(q_h)_x}\right)\left(x_{j+\frac{1}{2}}, t\right)^2\right)^{\frac{1}{2}}, \tag{3.19}$$

$$e_{q,c} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{I_j} q - q_h \,\mathrm{d}x\right)^2\right)^{\frac{1}{2}}, \quad q = u, v, \tag{3.20}$$

be the errors of the two numerical fluxes and the cell averages, respectively. Suppose the exact solutions $u \in H^{2k+4}$*, and the initial discretizations satisfy (3.12). For* $k \geq 2$*,* $\forall t \in [0, T]$ *then*

$$e_{q,f} \lesssim h^{2k}, \quad e_{q,fx} \lesssim h^{2k}, \quad e_{q,c} \lesssim h^{2k}, \tag{3.21}$$

where $q = u, v$*.*

Proof. The proof of this theorem can be found in the Appendix A.4. □

3.3. Superconvergence at special quadrature points

We firstly denote D_j^s, $s = 0, 1$, as the the roots of $\frac{d^s}{\mathrm{d}x^s} R_{j,k+1}$, $D^s = \bigcup_{j=1}^{N} D_j^s$. Here $R_{j,k+1} = L_{j,k+1} - P_h^\star L_{j,k+1}$. We study the superconvergence rates at these points and state the results as a theorem.

Theorem 3.8. *If* D^s*,* $s = 0, 1$ *are not empty sets. Let*

$$e_{q,q} = \max_{x \in D^0} |\left(q - q_h\right)(x, t)|, \quad e_{q,q_x} = \max_{x \in D^1} |\left(q_x - (q_h)_x\right)(x, t)|, \quad q = u, v, \tag{3.22}$$

be the maximum point value error for the numerical solution, and for the derivative of the solution at the corresponding sets of points. If the exact solution of equation (2.1) $u \in W^{k+5,\infty}$*, and the initial data is given satisfying (3.12), then the DG solutions of (2.7),* (u_h, v_h)*, have*

$$e_{q,q} \lesssim h^{k+2}, \quad e_{q,q_x} \lesssim h^{k+1}, \quad q = u, v, \quad \forall k \geq 2. \tag{3.23}$$

Proof. The proof of this theorem can be found in the Appendix A.5. □

TABLE 1. Errors e_u, $e_{u,p}$, $e_{u,f}$, $e_{u,fx}$ and $e_{u,c}$ for $k=2,3,4$, $T=1$ on uniform mesh.

k	N	e_u	Order	$e_{u,p}$	Order	$e_{u,f}$	Order	$e_{u,fx}$	Order	$e_{u,c}$	Order
2	10	2.13E-03	–	6.15E-04	–	2.50E-04	–	1.24E-04	–	2.43E-04	–
	20	2.52E-04	3.08	3.94E-05	3.97	1.58E-05	3.99	7.88E-06	3.98	1.57E-05	3.95
	40	3.10E-05	3.02	2.48E-06	3.99	9.89E-07	4.00	4.94E-07	3.99	9.87E-07	3.99
	80	3.86E-06	3.01	1.55E-07	4.00	6.19E-08	4.00	3.09E-08	4.00	6.18E-08	4.00
3	10	5.50E-05	–	1.41E-06	–	2.76E-07	–	2.85E-07	–	5.54E-07	–
	20	3.44E-06	4.00	2.23E-08	5.98	4.39E-09	5.98	4.42E-09	6.01	8.78E-09	5.98
	40	2.15E-07	4.00	3.49E-10	6.00	6.88E-11	5.99	6.90E-11	6.00	1.38E-10	5.99
	80	1.35E-08	4.00	5.45E-12	6.00	1.08E-12	6.00	1.08E-12	6.00	2.15E-12	6.00
4	10	1.45E-06	–	6.12E-09	–	3.93E-10	–	3.46E-12	–	3.74E-10	–
	20	4.54E-08	5.00	4.63E-11	7.05	1.52E-12	8.01	3.20E-15	10.08	1.50E-12	7.96
	40	1.42E-09	5.00	3.59E-13	7.01	5.94E-15	8.00	3.08E-18	10.02	5.92E-15	7.99
	80	4.44E-11	5.00	2.80E-15	7.00	2.32E-17	8.00	3.00E-21	10.01	2.32E-17	8.00

4. NUMERICAL EXAMPLES

We provide some numerical experiments to confirm our theoretical results. Let us recall the definitions of the various errors as mentioned in previous sections.

$$e_q = \|q-q_h\|, \quad e_{q,p} = \|q_h - P_h^\star q\|, \qquad e_{q,f} = \left(\frac{1}{N}\sum_{j=1}^{N}(q-\widehat{q_h})\left(x_{j+\frac12},t\right)^2\right)^{\frac12}, \tag{4.1}$$

$$e_{q,fx} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(q_x - \widetilde{(q_h)_x}\right)\left(x_{j+\frac12},t\right)^2\right)^{\frac12}, \quad e_{q,c} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_j}\int_{I_j} q - q_h\,\mathrm{d}x\right)^2\right)^{\frac12}, \tag{4.2}$$

$$e_{q,q} = \max_{x\in D^0}|\left(q-q_h\right)(x,t)|, \quad e_{q,q_x} = \max_{x\in D^1}|\left(q_x-(q_h)_x\right)(x,t)|, \quad e_{q,px} = \|\left(q_h - P_h^\star q\right)_x\|, \tag{4.3}$$

where $q = u, v$.

Example 4.1. We consider the following linear fourth-order equation

$$\begin{cases} u_t + u_{xxxx} = 0, & (x,t)\in[0,2\pi]\times(0,T] \\ u(x,0) = \sin(x), & x\in[0,2\pi] \end{cases} \tag{4.4}$$

with periodic boundary condition. The exact solution is

$$u(x,t) = e^{-t}\sin(x), \quad v(x,t) = \partial_x^2 u(x,t) = -e^{-t}\sin(x). \tag{4.5}$$

We separately use the uniform mesh and nonuniform mesh with 10% random perturbation of N cells on $[0,2\pi]$, since after the UWLDG spatial discretization, we obtain a linear ODE system with constant matrix. We can exactly solve the this ODE system to reduce the temporal error. The initial discretization is taken to satisfy (3.12) and the numerical flux is the alternating flux (2.16). We set the terminal time $T=1.0$ and $T=10.$ and calculate various errors and numerical orders of convergence for $\mathbb{P}^k$ elements with $2\le k\le 4$ both on uniform and nonuniform meshes. The numerical results are listed in Tables 1–16. We find that all our theoretical results are optimal both on uniform and nonuniform meshes.

TABLE 2. Errors $e_{u,u}$, e_{u,u_x} and $e_{u,px}$ for $k = 2, 3, 4$, $T = 1$ on uniform mesh.

k	N	$e_{u,u}$	Order	e_{u,u_x}	Order	$e_{u,px}$	Order
2	10	5.64E-04	–	5.64E-04	–	5.29E-04	–
	20	3.73E-05	3.92	5.82E-05	3.28	3.37E-05	3.97
	40	2.35E-06	3.99	6.84E-06	3.09	2.12E-06	3.99
	80	1.47E-07	4.00	8.40E-07	3.02	1.33E-07	4.00
3	10	1.34E-06	–	1.60E-05	–	2.90E-06	–
	20	3.60E-08	5.22	1.01E-06	3.98	8.64E-08	5.07
	40	1.06E-09	5.08	6.35E-08	4.00	2.66E-09	5.02
	80	3.26E-11	5.02	3.97E-09	4.00	8.30E-11	5.00
4	10	4.93E-08	–	4.83E-07	–	4.93E-08	–
	20	7.71E-10	6.00	1.50E-08	5.01	7.65E-10	6.01
	40	1.22E-11	5.98	4.72E-10	4.99	1.19E-11	6.00
	80	1.91E-13	6.00	1.48E-11	5.00	1.86E-13	6.00

TABLE 3. Errors e_v, $e_{v,p}$, $e_{v,f}$, $e_{v,fx}$ and $e_{v,c}$ for $k = 2, 3, 4$, $T = 1$ on uniform mesh.

k	N	e_u	Order	$e_{v,p}$	Order	$e_{v,f}$	Order	$e_{v,fx}$	Order	$e_{v,c}$	Order
2	10	2.01E-03	–	3.08E-04	–	1.28E-04	–	1.33E-06	–	1.22E-04	–
	20	2.48E-04	3.02	1.97E-05	3.97	7.94E-06	4.01	2.15E-08	5.95	7.84E-06	3.96
	40	3.09E-05	3.01	1.24E-06	3.99	4.95E-07	4.00	3.38E-10	5.99	4.94E-07	3.99
	80	3.85E-06	3.00	7.75E-08	4.00	3.09E-08	4.00	5.30E-12	6.00	3.09E-08	4.00
3	10	5.49E-05	–	7.33E-07	–	1.22E-09	–	9.65E-09	–	2.81E-07	–
	20	3.44E-06	4.00	1.15E-08	5.99	4.25E-12	8.17	5.30E-11	7.51	4.40E-09	5.99
	40	2.15E-07	4.00	1.80E-10	6.00	1.61E-14	8.04	3.59E-13	7.21	6.89E-11	6.00
	80	1.35E-08	4.00	2.81E-12	6.00	6.24E-17	8.01	2.68E-15	7.06	1.08E-12	6.00
4	10	1.45E-06	–	5.89E-09	–	1.06E-11	–	3.88E-10	–	3.40E-12	–
	20	4.54E-08	5.00	4.59E-11	7.01	1.31E-14	9.65	1.52E-12	8.00	3.19E-15	10.06
	40	1.42E-09	5.00	3.58E-13	7.00	2.06E-17	9.32	5.93E-15	8.00	3.08E-18	10.02
	80	4.44E-11	5.00	2.80E-15	7.00	3.74E-20	9.11	2.32E-17	8.00	3.00E-21	10.00

TABLE 4. Errors $e_{v,v}$, e_{v,v_x} and $e_{v,px}$ for $k = 2, 3, 4$, $T = 1$ on uniform mesh.

k	N	$e_{v,v}$	Order	e_{v,v_x}	Order	$e_{v,px}$	Order
2	10	4.12E-04	–	4.42E-04	–	2.43E-04	–
	20	2.62E-05	3.97	5.43E-05	3.02	1.53E-05	3.99
	40	1.65E-06	3.99	6.71E-06	3.02	9.55E-07	4.00
	80	1.03E-07	4.00	8.36E-07	3.00	5.97E-08	4.00
3	10	1.14E-06	–	1.57E-05	–	2.71E-06	–
	20	3.38E-08	5.08	1.01E-06	3.96	8.48E-08	5.00
	40	1.04E-09	5.02	6.34E-08	3.99	2.65E-09	5.00
	80	3.24E-11	5.01	3.97E-09	4.00	8.29E-11	5.00
4	10	4.98E-08	–	4.83E-07	–	4.91E-08	–
	20	7.73E-10	6.01	1.50E-08	5.01	7.64E-10	6.01
	40	1.22E-11	5.99	4.72E-10	4.99	1.19E-11	6.00
	80	1.91E-13	6.00	1.48E-11	5.00	1.86E-13	6.00

TABLE 5. Errors e_u, $e_{u,p}$, $e_{u,f}$, $e_{u,fx}$ and $e_{u,c}$ for $k = 2, 3, 4$, $T = 1$ on nonuniform mesh.

k	N	e_u	Order	$e_{u,p}$	Order	$e_{u,f}$	Order	$e_{u,fx}$	Order	$e_{u,c}$	Order
2	10	2.13E-03	–	6.15E-04	–	2.50E-04	–	1.24E-04	–	2.42E-04	–
	20	2.55E-04	3.06	3.99E-05	3.94	1.60E-05	3.96	7.98E-06	3.96	1.59E-05	3.93
	40	3.14E-05	3.02	2.51E-06	3.99	1.00E-06	4.00	5.01E-07	3.99	9.99E-07	3.99
	80	3.94E-06	3.00	1.58E-07	3.99	6.32E-08	3.99	3.16E-08	3.99	6.31E-08	3.98
3	10	5.53E-05	–	1.42E-06	–	2.78E-07	–	2.88E-07	–	5.58E-07	–
	20	3.53E-06	3.97	2.31E-08	5.95	4.51E-09	5.95	4.55E-09	5.98	9.05E-09	5.95
	40	2.19E-07	4.01	3.59E-10	6.01	7.04E-11	6.00	7.05E-11	6.01	1.41E-10	6.00
	80	1.37E-08	4.00	5.62E-12	6.00	1.10E-12	6.00	1.10E-12	6.00	2.21E-12	6.00
4	10	1.56E-06	–	6.88E-09	–	4.42E-10	–	3.57E-11	–	4.19E-10	–
	20	4.88E-08	5.00	5.28E-11	7.02	1.68E-12	8.04	3.91E-14	9.84	1.66E-12	7.98
	40	1.47E-09	5.06	3.79E-13	7.12	6.20E-15	8.08	6.74E-17	9.18	6.18E-15	8.07
	80	4.58E-11	5.00	2.97E-15	7.00	2.41E-17	8.01	1.82E-19	8.53	2.41E-17	8.00

TABLE 6. Errors $e_{u,u}$, e_{u,u_x} and $e_{u,px}$ for $k = 2, 3, 4$, $T = 1$ on nonuniform mesh.

k	N	$e_{u,u}$	Order	e_{u,u_x}	Order	$e_{u,px}$	Order
2	10	6.28E-04	–	5.79E-04	–	5.30E-04	–
	20	4.21E-05	3.90	6.93E-05	3.06	3.42E-05	3.95
	40	2.51E-06	4.07	7.77E-06	3.16	2.15E-06	3.99
	80	1.67E-07	3.91	1.03E-06	2.92	1.36E-07	3.99
3	10	1.57E-06	–	1.74E-05	–	2.94E-06	–
	20	4.30E-08	5.19	1.19E-06	3.87	8.99E-08	5.03
	40	1.59E-09	4.76	8.34E-08	3.84	2.75E-09	5.03
	80	5.04E-11	4.98	5.64E-09	3.89	8.56E-11	5.00
4	10	4.99E-08	–	4.87E-07	–	5.41E-08	–
	20	1.05E-09	5.57	1.93E-08	4.66	8.45E-10	6.00
	40	1.74E-11	5.92	6.30E-10	4.94	1.25E-11	6.08
	80	2.93E-13	5.89	2.10E-11	4.91	1.95E-13	6.00

TABLE 7. Errors e_v, $e_{v,p}$, $e_{v,f}$, $e_{v,fx}$ and $e_{v,c}$ for $k = 2, 3, 4$, $T = 1$ on nonuniform mesh.

k	N	e_u	Order	$e_{v,p}$	Order	$e_{v,f}$	Order	$e_{v,fx}$	Order	$e_{v,c}$	Order
2	10	2.02E-03	–	3.08E-04	–	1.28E-04	–	2.84E-06	–	1.22E-04	–
	20	2.51E-04	3.01	2.00E-05	3.95	8.06E-06	3.99	1.48E-07	4.26	7.96E-06	3.94
	40	3.13E-05	3.00	1.25E-06	3.99	5.01E-07	4.01	2.27E-09	6.03	5.00E-07	3.99
	80	3.94E-06	2.99	7.91E-08	3.99	3.16E-08	3.99	6.82E-11	5.06	3.16E-08	3.98
3	10	5.53E-05	–	7.46E-07	–	3.80E-09	–	1.78E-08	–	2.85E-07	–
	20	3.53E-06	3.97	1.22E-08	5.93	9.64E-11	5.30	2.96E-10	5.91	4.63E-09	5.94
	40	2.19E-07	4.01	1.88E-10	6.02	3.55E-13	8.08	1.68E-12	7.46	7.15E-11	6.02
	80	1.37E-08	4.00	2.94E-12	6.00	7.25E-15	5.62	1.40E-14	6.90	1.12E-12	6.00
4	10	1.56E-06	–	6.62E-09	–	5.06E-11	–	4.31E-10	–	3.30E-11	–
	20	4.88E-08	5.00	5.23E-11	6.98	1.27E-13	8.64	1.67E-12	8.01	7.18E-14	8.84
	40	1.47E-09	5.06	3.79E-13	7.11	2.03E-16	9.28	6.18E-15	8.08	1.64E-16	8.77
	80	4.58E-11	5.00	2.97E-15	6.99	4.84E-19	8.71	2.41E-17	8.00	4.00E-19	8.68

TABLE 8. Errors $e_{v,v}$, e_{v,v_x} and $e_{v,px}$ for $k = 2, 3, 4$, $T = 1$ on nonuniform mesh.

k	N	$e_{v,v}$	Order	e_{v,v_x}	Order	$e_{v,px}$	Order
2	10	4.82E-04	–	5.17E-04	–	2.46E-04	–
	20	3.09E-05	3.96	6.68E-05	2.95	1.56E-05	3.98
	40	1.81E-06	4.09	7.58E-06	3.14	9.80E-07	3.99
	80	1.24E-07	3.87	1.04E-06	2.87	6.18E-08	3.99
3	10	1.28E-06	–	1.71E-05	–	2.75E-06	–
	20	4.14E-08	4.95	1.18E-06	3.85	8.83E-08	4.96
	40	1.55E-09	4.74	8.33E-08	3.83	2.73E-09	5.01
	80	5.03E-11	4.94	5.64E-09	3.88	8.55E-11	5.00
4	10	5.05E-08	–	4.87E-07	–	5.39E-08	–
	20	1.06E-09	5.58	1.93E-08	4.66	8.44E-10	6.00
	40	1.74E-11	5.92	6.30E-10	4.94	1.24E-11	6.08
	80	2.93E-13	5.89	2.10E-11	4.91	1.95E-13	6.00

TABLE 9. Errors e_u, $e_{u,p}$, $e_{u,f}$, $e_{u,fx}$ and $e_{u,c}$ for $k = 2, 3, 4$, $T = 10$ on uniform mesh.

k	N	e_u	Order	$e_{u,p}$	Order	$e_{u,f}$	Order	$e_{u,fx}$	Order	$e_{u,c}$	Order
2	10	8.28E-07	–	7.56E-07	–	3.02E-07	–	2.87E-07	–	2.97E-07	–
	20	5.93E-08	3.80	4.86E-08	3.96	1.94E-08	3.96	1.84E-08	3.96	1.93E-08	3.94
	40	4.97E-09	3.58	3.06E-09	3.99	1.22E-09	3.99	1.16E-09	3.99	1.22E-09	3.99
	80	5.16E-10	3.27	1.91E-10	4.00	7.63E-11	4.00	7.25E-11	4.00	7.63E-11	4.00
3	10	7.01E-09	–	1.72E-09	–	6.50E-10	–	6.52E-10	–	6.75E-10	–
	20	4.26E-10	4.04	2.72E-11	5.98	1.03E-11	5.98	1.03E-11	5.98	1.08E-11	5.97
	40	2.66E-11	4.00	4.26E-13	6.00	1.61E-13	6.00	1.61E-13	6.00	1.70E-13	5.99
	80	1.66E-12	4.00	6.66E-15	6.00	2.52E-15	6.00	2.52E-15	6.00	2.66E-15	6.00
4	10	1.79E-10	–	2.48E-12	–	9.02E-13	–	8.53E-13	–	8.85E-13	–
	20	5.61E-12	5.00	1.09E-14	7.83	3.55E-15	7.99	3.36E-15	7.99	3.53E-15	7.97
	40	1.75E-13	5.00	5.74E-17	7.57	1.39E-17	8.00	1.32E-17	8.00	1.39E-17	7.99
	80	5.48E-15	5.00	3.74E-19	7.26	5.43E-20	8.00	5.15E-20	8.00	5.43E-20	8.00

TABLE 10. Errors $e_{u,u}$, e_{u,u_x} and $e_{u,px}$ for $k = 2, 3, 4$, $T = 10$ on uniform mesh.

k	N	$e_{u,u}$	Order	e_{u,u_x}	Order	$e_{u,px}$	Order
2	10	4.47E-07	–	4.36E-07	–	7.38E-07	–
	20	2.92E-08	3.94	2.83E-08	3.95	4.77E-08	3.95
	40	1.84E-09	3.99	1.93E-09	3.87	3.00E-09	3.99
	80	1.15E-10	4.00	1.50E-10	3.68	1.88E-10	4.00
3	10	9.70E-10	–	2.81E-09	–	1.72E-09	–
	20	1.60E-11	5.92	1.39E-10	4.34	2.86E-11	5.91
	40	2.77E-13	5.85	8.05E-12	4.11	5.30E-13	5.75
	80	5.55E-15	5.64	4.94E-13	4.03	1.21E-14	5.45
4	10	4.96E-12	–	5.97E-11	–	6.97E-12	–
	20	9.07E-14	5.77	1.85E-12	5.01	9.69E-14	6.17
	40	1.49E-15	5.93	5.82E-14	4.99	1.48E-15	6.03
	80	2.35E-17	5.98	1.82E-15	5.00	2.30E-17	6.01

TABLE 11. Errors e_v, $e_{v,p}$, $e_{v,f}$, $e_{v,fx}$ and $e_{v,c}$ for $k = 2, 3, 4$, $T = 10$ on uniform mesh.

k	N	e_u	Order	$e_{v,p}$	Order	$e_{v,f}$	Order	$e_{v,fx}$	Order	$e_{v,c}$	Order
2	10	7.92E-07	–	7.18E-07	–	2.87E-07	–	2.72E-07	–	2.82E-07	–
	20	5.72E-08	3.79	4.62E-08	3.96	1.84E-08	3.96	1.74E-08	3.96	1.83E-08	3.94
	40	4.87E-09	3.55	2.90E-09	3.99	1.16E-09	3.99	1.10E-09	3.99	1.16E-09	3.99
	80	5.12E-10	3.25	1.82E-10	4.00	7.25E-11	4.00	6.87E-11	4.00	7.25E-11	4.00
3	10	6.99E-09	–	1.63E-09	–	6.16E-10	–	6.17E-10	–	6.41E-10	–
	20	4.26E-10	4.04	2.58E-11	5.98	9.76E-12	5.98	9.76E-12	5.98	1.03E-11	5.97
	40	2.66E-11	4.00	4.05E-13	6.00	1.53E-13	6.00	1.53E-13	6.00	1.61E-13	5.99
	80	1.66E-12	4.00	6.33E-15	6.00	2.39E-15	6.00	2.39E-15	6.00	2.52E-15	6.00
4	10	1.79E-10	–	2.37E-12	–	8.54E-13	–	8.05E-13	–	8.39E-13	–
	20	5.61E-12	5.00	1.05E-14	7.81	3.36E-15	7.99	3.18E-15	7.99	3.35E-15	7.97
	40	1.75E-13	5.00	5.62E-17	7.55	1.32E-17	8.00	1.24E-17	8.00	1.32E-17	7.99
	80	5.48E-15	5.00	3.71E-19	7.24	5.15E-20	8.00	4.86E-20	8.00	5.14E-20	8.00

TABLE 12. Errors $e_{v,v}$, e_{v,v_x} and $e_{v,px}$ for $k = 2, 3, 4$, $T = 10$ on uniform mesh.

k	N	$e_{v,v}$	Order	e_{v,v_x}	Order	$e_{v,px}$	Order
2	10	4.26E-07	–	4.15E-07	–	7.01E-07	–
	20	2.78E-08	3.94	2.70E-08	3.95	4.53E-08	3.95
	40	1.75E-09	3.99	1.86E-09	3.86	2.85E-09	3.99
	80	1.10E-10	4.00	1.47E-10	3.66	1.79E-10	4.00
3	10	9.22E-10	–	2.77E-09	–	1.63E-09	–
	20	1.53E-11	5.91	1.38E-10	4.33	2.74E-11	5.90
	40	2.67E-13	5.84	8.04E-12	4.10	5.14E-13	5.73
	80	5.42E-15	5.62	4.93E-13	4.03	1.20E-14	5.43
4	10	5.02E-12	–	5.97E-11	–	6.91E-12	–
	20	9.09E-14	5.79	1.85E-12	5.01	9.67E-14	6.16
	40	1.49E-15	5.93	5.82E-14	4.99	1.48E-15	6.03
	80	2.35E-17	5.98	1.82E-15	5.00	2.30E-17	6.01

TABLE 13. Errors e_u, $e_{u,p}$, $e_{u,f}$, $e_{u,fx}$ and $e_{u,c}$ for $k = 2, 3, 4$, $T = 10$ on nonuniform mesh.

k	N	e_u	Order	$e_{u,p}$	Order	$e_{u,f}$	Order	$e_{u,fx}$	Order	$e_{u,c}$	Order
2	10	8.52E-07	–	7.77E-07	–	3.12E-07	–	2.94E-07	–	3.06E-07	–
	20	6.02E-08	3.82	4.93E-08	3.98	1.96E-08	3.99	1.87E-08	3.97	1.95E-08	3.97
	40	5.06E-09	3.57	3.11E-09	3.99	1.24E-09	3.98	1.18E-09	3.99	1.24E-09	3.98
	80	5.27E-10	3.26	1.95E-10	4.00	7.78E-11	4.00	7.39E-11	3.99	7.78E-11	3.99
3	10	7.78E-09	–	1.98E-09	–	7.43E-10	–	7.60E-10	–	7.72E-10	–
	20	4.35E-10	4.16	2.78E-11	6.16	1.05E-11	6.14	1.05E-11	6.17	1.11E-11	6.12
	40	2.78E-11	3.97	4.51E-13	5.95	1.71E-13	5.95	1.71E-13	5.95	1.79E-13	5.95
	80	1.70E-12	4.04	6.84E-15	6.04	2.59E-15	6.04	2.59E-15	6.04	2.72E-15	6.04
4	10	1.77E-10	–	2.41E-12	–	8.69E-13	–	8.30E-13	–	8.53E-13	–
	20	5.79E-12	4.93	1.15E-14	7.72	3.71E-15	7.87	3.51E-15	7.89	3.69E-15	7.85
	40	1.79E-13	5.01	5.97E-17	7.59	1.43E-17	8.02	1.36E-17	8.01	1.43E-17	8.01
	80	5.70E-15	4.98	4.00E-19	7.22	5.72E-20	7.97	5.42E-20	7.97	5.72E-20	7.97

TABLE 14. Errors $e_{u,u}$, e_{u,u_x} and $e_{u,px}$ for $k = 2, 3, 4$, $T = 10$ on nonuniform mesh.

k	N	$e_{u,u}$	Order	e_{u,u_x}	Order	$e_{u,px}$	Order
2	10	4.68E-07	–	4.48E-07	–	7.60E-07	–
	20	3.00E-08	3.97	2.98E-08	3.91	4.84E-08	3.97
	40	1.87E-09	4.00	2.04E-09	3.87	3.06E-09	3.98
	80	1.20E-10	3.97	1.69E-10	3.59	1.92E-10	3.99
3	10	1.13E-09	–	3.09E-09	–	1.98E-09	–
	20	1.68E-11	6.08	1.82E-10	4.08	2.93E-11	6.08
	40	3.27E-13	5.68	1.10E-11	4.05	5.65E-13	5.70
	80	6.70E-15	5.61	6.74E-13	4.03	1.25E-14	5.50
4	10	6.35E-12	–	7.03E-11	–	6.88E-12	–
	20	1.56E-13	5.35	2.84E-12	4.63	1.01E-13	6.09
	40	2.46E-15	5.99	8.84E-14	5.01	1.53E-15	6.05
	80	2.91E-17	6.41	2.17E-15	5.35	2.43E-17	5.98

TABLE 15. Errors e_v, $e_{v,p}$, $e_{v,f}$, $e_{v,fx}$ and $e_{v,c}$ for $k = 2, 3, 4$, $T = 10$ on nonuniform mesh.

k	N	e_u	Order	$e_{v,p}$	Order	$e_{v,f}$	Order	$e_{v,fx}$	Order	$e_{v,c}$	Order
2	10	8.15E-07	–	7.39E-07	–	2.96E-07	–	2.79E-07	–	2.91E-07	–
	20	5.81E-08	3.81	4.68E-08	3.98	1.87E-08	3.99	1.77E-08	3.97	1.86E-08	3.97
	40	4.96E-09	3.55	2.96E-09	3.99	1.18E-09	3.98	1.12E-09	3.99	1.18E-09	3.98
	80	5.23E-10	3.25	1.85E-10	4.00	7.39E-11	4.00	7.00E-11	3.99	7.39E-11	3.99
3	10	7.75E-09	–	1.88E-09	–	7.04E-10	–	7.20E-10	–	7.33E-10	–
	20	4.35E-10	4.16	2.64E-11	6.16	9.99E-12	6.14	9.98E-12	6.17	1.05E-11	6.13
	40	2.78E-11	3.97	4.28E-13	5.95	1.62E-13	5.95	1.62E-13	5.95	1.70E-13	5.95
	80	1.70E-12	4.04	6.49E-15	6.04	2.45E-15	6.04	2.45E-15	6.04	2.59E-15	6.04
4	10	1.77E-10	–	2.30E-12	–	8.23E-13	–	7.84E-13	–	8.08E-13	–
	20	5.79E-12	4.93	1.10E-14	7.70	3.51E-15	7.87	3.31E-15	7.89	3.50E-15	7.85
	40	1.79E-13	5.01	5.85E-17	7.56	1.36E-17	8.02	1.28E-17	8.01	1.36E-17	8.01
	80	5.70E-15	4.98	3.98E-19	7.20	5.42E-20	7.97	5.12E-20	7.97	5.42E-20	7.97

TABLE 16. Errors $e_{v,v}$, e_{v,v_x} and $e_{v,px}$ for $k = 2, 3, 4$, $T = 10$ on nonuniform mesh.

k	N	$e_{v,v}$	Order	e_{v,v_x}	Order	$e_{v,px}$	Order
2	10	4.47E-07	–	4.27E-07	–	7.21E-07	–
	20	2.86E-08	3.97	2.86E-08	3.90	4.59E-08	3.97
	40	1.79E-09	4.00	1.96E-09	3.87	2.90E-09	3.98
	80	1.14E-10	3.97	1.67E-10	3.56	1.82E-10	3.99
3	10	1.08E-09	–	3.03E-09	–	1.88E-09	–
	20	1.60E-11	6.07	1.81E-10	4.07	2.80E-11	6.07
	40	3.16E-13	5.67	1.10E-11	4.04	5.47E-13	5.68
	80	6.58E-15	5.58	6.73E-13	4.03	1.23E-14	5.47
4	10	6.41E-12	–	7.03E-11	–	6.82E-12	–
	20	1.56E-13	5.36	2.84E-12	4.63	1.01E-13	6.08
	40	2.46E-15	5.99	8.84E-14	5.01	1.53E-15	6.05
	80	2.91E-17	6.41	2.17E-15	5.35	2.43E-17	5.98

5. Concluding remarks

We have studied the superconvergence properties of the UWLDG methods with alternating fluxes for linear fourth order derivatives equation in one dimension. Under suitable initial approximation, the error of the cell averages and the numerical fluxes of the function values and the derivatives converge with the rate of $(2k)$-th order when $k \geq 2$. Especially, for the superconvergence of the numerical fluxes of the derivatives, we obtain the optimal estimates which are confirmed by the numerical examples. Other superconvergence properties such as the numerical solution towards the special projection of the truth solution, the function values and first derivatives at a class of special quadrature points are also studied. A new technique in this paper leads to an improved estimate of superconvergence by using discrete Poincaré inequality when k is even. The superconvergence study of the UWLDG method for higher order equations is very interesting and challenging, when optimal estimates are desired. It is also intriguing to generalize our analysis to multi-dimensions and non-linear equations. These will be explored in the future.

Appendix A. Proof of a few technical lemmas and theorems

The proofs of some of the technical lemmas and theorems are provided in this appendix.

A.1. Proof of Lemma 3.1

Proof. We use induction to prove this lemma. Since $\omega_q^{(0)} \perp \mathbb{P}^{k-2}(I_j)$, $q = u, v$, from (3.1) to (3.2), we obtain

$$\omega_q^{(1)} \perp \mathbb{P}^{k-4}(I_j), \quad q = u, v. \tag{A.1}$$

Thus, we have

$$\omega_u^{(1)}|_{I_j} = \sum_{m=k-3}^{k} c_{j,m}^1 L_{j,m}(\xi), \quad \omega_v^{(1)}|_{I_j} = \sum_{m=k-3}^{k} d_{j,m}^1 L_{j,m}(\xi), \quad \xi = \frac{2(x - x_j)}{h_j} \in [-1, 1]. \tag{A.2}$$

Then, we define an integral operator D^{-1} by

$$D^{-1}w(x) = \frac{1}{\bar{h}_j} \int_{x_{j-\frac{1}{2}}}^{x} w(x)\,\mathrm{d}x = \int_{-1}^{s} r(s)\,\mathrm{d}s, \quad x \in I_j, \tag{A.3}$$

where $s = (x - x_j)/\bar{h}_j \in [-1, 1]$. We denote $D^{-l} = D^{-1}\left(D^{-(l-1)}\right)$, $l \geq 2$. Taking $\varphi = D^{-2}L_{j,m}$, $m = k-3, k-2$ respectively in (3.1), by Cauchy–Schwartz inequality, we have

$$\begin{aligned} \frac{4}{h_j}\frac{1}{2m+1} c_{j,m}^1 &= \int_{I_j} \omega_v^{(0)} D^{-2} L_{j,m}\,\mathrm{d}x \\ &\leq \|\omega_v^{(0)}\|_{I_j} \|D^{-2}L_{j,m}\|_{I_j} \\ &\lesssim h^{k+1+\frac{1}{2}} \|u\|_{k+3,I_j}. \end{aligned}$$

Thus,

$$|c_{j,m}^1| \lesssim h^{k+\frac{5}{2}} \|u\|_{k+3,I_j}, \quad m = k-3, k-2. \tag{A.4}$$

By the same arguments, we have

$$|d_{j,m}^1| \lesssim h^{k+\frac{5}{2}} \|v\|_{k+3,I_j}, \quad m = k-3, k-2. \tag{A.5}$$

Next, we estimate the $c^1_{j,m}$ and $d^1_{j,m}$, $m = k-1, k$. We obtain from (3.1) to (3.2),

$$\sum_{m=k-1}^{k} L_m(1)c^1_{j,m} = -\sum_{m=k-3}^{k-2} L_m(1)c^1_{j,m}, \tag{A.6}$$

$$\sum_{m=k-1}^{k} L'_m(-1)c^1_{j,m} = -\sum_{m=k-3}^{k-2} L'_m(-1)c^1_{j,m}, \tag{A.7}$$

where L_m denote the standard Legendre polynomial of degree m on the interval $[-1,1]$. Thus we have

$$A\vec{c} = \vec{b}, \tag{A.8}$$

where

$$A = \begin{pmatrix} 1 & 1 \\ \frac{(-1)^{k+1}}{2}k(k+1) & \frac{(-1)^k}{2}(k-1)k \end{pmatrix}, \tag{A.9}$$

$$\vec{b} = \left(-\sum_{m=k-3}^{k-2} L_m(1)c^1_{j,m}, -\sum_{m=k-3}^{k-2} L'_m(-1)c^1_{j,m}\right)^T, \tag{A.10}$$

$$\vec{c} = \left(c^1_{j,k}, c^1_{j,k-1}\right)^T. \tag{A.11}$$

We have used the following facts

$$L_m(\pm 1) = (\pm 1)^m, \quad L'_m(\pm 1) = \frac{1}{2}(\pm 1)^{m+1}m(m+1). \tag{A.12}$$

The determinate of A, $\mathrm{Det}(A) = (-1)^k k^2 \neq 0$ for $k \geq 1$. Therefore,

$$\sum_{j=1}^{N}\sum_{m=k-1}^{k}\left(c^1_{j,m}\right)^2 \lesssim \sum_{j=1}^{N}\sum_{m=k-3}^{k-2}\left(c^1_{j,m}\right)^2 \lesssim h^{2k+5}\|u\|_{k+3}. \tag{A.13}$$

Thus

$$\begin{aligned}\|\omega_u^{(1)}\| &\simeq \left(\sum_{j=1}^{N}\sum_{m=k-3}^{k} h\left(c^1_{j,m}\right)^2\right)^{\frac{1}{2}} \\ &\lesssim h^{k+3}\|u\|_{k+3}.\end{aligned} \tag{A.14}$$

Taking time derivative on the both sides of (3.1), the three identities still hold. Then following the same arguments as what we did for $\omega_u^{(1)}$, we get

$$\|\partial_t\omega_u^{(1)}\| \lesssim h^{k+3}\|\partial_t u\|_{k+3}. \tag{A.15}$$

By the same arguments, we have

$$\|\omega_v^{(1)}\| \lesssim h^{k+3}\|v\|_{k+3}, \quad \|\partial_t\omega_v^{(1)}\| \lesssim h^{k+3}\|\partial_t v\|_{k+3}. \tag{A.16}$$

By the recursion formula, (3.4) holds for all $1 \leq i \leq \lfloor \frac{k-2}{2} \rfloor$ and $n = 0, 1$. This finishes our proof. □

A.2. Proof of Lemma 3.4

Proof. The initial discretization $u_h(\cdot, 0)$ is the solution of the following equations,

$$(v_h, \varphi)_j = B_j(u_h, \varphi), \quad \text{for} \quad \forall j \in \mathbb{Z}_N, \quad \forall \varphi \in V_h^k, \tag{A.17}$$

and it also satisfies

$$\int_\Omega u_h \, \mathrm{d}x = \int_\Omega u_0 \, \mathrm{d}x. \tag{A.18}$$

As we know, $u_h + c$ also satisfies (A.17), here c is any constant. If there are two solutions u_{h1} and u_{h2} satisfying (A.17) and (A.18), then we denote $w_h = u_{h1} - u_{h2} \in V_h^k$ to obtain

$$B_j(w_h, \varphi) = 0, \quad \int_\Omega w_h \, \mathrm{d}x = 0, \quad \forall j \in \mathbb{Z}_N, \quad \forall \varphi \in V_h^k. \tag{A.19}$$

Applying Lemma 3.3, we have

$$(w_h)_{xx}|_{I_j} = 0, \quad [w_h]_{j-\frac{1}{2}} = 0, \quad [(w_h)_x]_{j-\frac{1}{2}} = 0, \quad \forall j \in \mathbb{Z}_N. \tag{A.20}$$

Together with $w_h \in V_h^k$, we have w_h is a constant function, and (A.19) implies $w_h \equiv 0$. Finally, we have proved u_h is well-defined. Since $\bar{e}_v(x, 0) = 0$ and (2.22),

$$a_j(\bar{e}_u, \bar{e}_v; \varphi) = a_j(\epsilon_u, \epsilon_v; \varphi) = \int_{I_j} \left(\omega_u^{(l)}\right)_t \varphi \, \mathrm{d}x, \tag{A.21}$$

$$b_j(\bar{e}_v, \bar{e}_u; \varphi) = b_j(\epsilon_v, \epsilon_u; \varphi) = \int_{I_j} \omega_v^{(l)} \varphi \, \mathrm{d}x \quad \forall j \in \mathbb{Z}_N, \quad \forall \varphi \in V_h^k. \tag{A.22}$$

Thus,

$$\left((\bar{e}_u)_t(0), \varphi\right)_j = \left(\left(\omega_u^{(l)}\right)_t(0), \varphi\right)_j, \tag{A.23}$$

$$B_j(\bar{e}_u(0), \varphi) = \int_{I_j} -\omega_v^{(l)}(0) \varphi \, \mathrm{d}x \quad \forall j \in \mathbb{Z}_N, \quad \forall \varphi \in V_h^k. \tag{A.24}$$

Therefore,

$$(\bar{e}_u)_t(0) = \left(\omega_u^{(l)}\right)_t(0), \tag{A.25}$$

which implies,

$$\|(\bar{e}_u)_t(0)\| \lesssim h^{k+1+2l}. \tag{A.26}$$

And by Lemma 3.3, we have

$$\|(\bar{e}_u)_{xx}\| + h^{-\frac{1}{2}} \left(\sum_{j=1}^N \left|[(\bar{e}_u)_x]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} + h^{-\frac{3}{2}} \left(\sum_{j=1}^N \left|[(\bar{e}_u)]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} \lesssim \left\|\omega_v^{(l)}(0)\right\| \lesssim h^{k+1+2l}. \tag{A.27}$$

By using the discrete Poincaré inequalities [4], we have

$$\begin{aligned}
\|(\bar{e}_u)_x\| &\leq \left\|(\bar{e}_u)_x - \int_\Omega (\bar{e}_u)_x \, \mathrm{d}x\right\| + \left|\int_\Omega (\bar{e}_u)_x \, \mathrm{d}x\right| \\
&\lesssim \|(\bar{e}_u)_{xx}\| + h^{-\frac{1}{2}} \left(\sum_{j=1}^N \left|[(\bar{e}_u)_x]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} + h^{-\frac{1}{2}} \left(\sum_{j=1}^N \left|[(\bar{e}_u)]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} \\
&\lesssim h^{k+1+2l}.
\end{aligned} \tag{A.28}$$

If k is odd, then $k+1+2l=2k$. We have,

$$\begin{aligned}\int_{I_j} \bar{e}_u \,\mathrm{d}x &= \int_{I_j} u_h - P_h^\star u + \sum_{i=1}^{l} \omega_u^{(i)} \,\mathrm{d}x\\ &= \int_{I_j} u_h - u \,\mathrm{d}x + \int_{I_j} \omega_u^{(l)} \,\mathrm{d}x,\end{aligned} \tag{A.29}$$

since $\omega_u^{(i)} \perp \mathbb{P}^0\left(I_j\right)$, $1 \le i \le l-1$. Then summing over j, and by (A.18)

$$\int_\Omega \bar{e}_u = \int_\Omega u_h - u \,\mathrm{d}x + \int_\Omega \omega_u^{(l)} \,\mathrm{d}x = \int_\Omega \omega_u^{(l)} \,\mathrm{d}x. \tag{A.30}$$

We apply the discrete Poincaré inequalities to obtain,

$$\begin{aligned}\|\bar{e}_u\| &\le \left\|\bar{e}_u - \int_\Omega \bar{e}_u \,\mathrm{d}x\right\| + \left|\int_\Omega \bar{e}_u \,\mathrm{d}x\right|\\ &\lesssim \|(\bar{e}_u)_x\| + h^{-\frac{1}{2}} \left(\sum_{j=1}^{N} \left|[(\bar{e}_u)]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} + \left|\int_\Omega \omega_u^{(l)} \,\mathrm{d}x\right|\\ &\lesssim h^{2k}.\end{aligned} \tag{A.31}$$

If k is even, $l=\frac{k-2}{2}$, we need to improve the estimates. From the conclusion of Lemma 3.1, $\omega_v^{(l)} \perp \mathbb{P}^0$, thus $D^{-1}\omega_v^{(l)}\left(x^{\mp}_{j\pm\frac{1}{2}}\right)=0$. Using the integration by parts on the right hand side of (A.24), we have

$$B_j\left(\bar{e}_u, \varphi\right) = \int_{I_j} \bar{h}_j D^{-1}\omega_v^{(l)} \varphi_x \,\mathrm{d}x, \quad \forall \varphi \in V_h^k. \tag{A.32}$$

We take the test function

$$\varphi|_{I_j} = \frac{(-1)^{k+1} h_j}{2k^2}\left(L_{j,k}(x) - L_{j,k-1}(x)\right), \tag{A.33}$$

such that

$$\varphi\left(x^-_{j+\frac{1}{2}}\right)=0, \varphi_x\left(x^+_{j-\frac{1}{2}}\right)=1, \quad \text{and} \quad \varphi(x)|_{I_j} \perp \mathbb{P}^{k-2}\left(I_j\right). \tag{A.34}$$

Thus,

$$|\left[\bar{e}_u\right]_{j-\frac{1}{2}}| = \left|\int_{I_j} \bar{h}_j D^{-1}\omega_v^{(l)} \varphi_x \,\mathrm{d}x\right| \tag{A.35}$$

$$\le \|\bar{h}_j D^{-1}\omega_v^{(l)}\|_{I_j} h^{\frac{1}{2}}. \tag{A.36}$$

Then, we take $\varphi=-\bar{e}_u$ in (A.32), after the integration by parts and using inverse inequality, we have

$$\begin{aligned}\|(\bar{e}_u)_x\|^2 &= \sum_{j=1}^{N} \int_{I_j} \bar{h}_j D^{-1}\omega_v^{(l)} \left(\bar{e}_u\right)_x \,\mathrm{d}x - 2\sum_{j=1}^{N} \left(\bar{e}_u\right)_x^+ \Big|_{j-\frac{1}{2}} \left[\bar{e}_u\right]_{j-\frac{1}{2}}\\ &\lesssim \left\|\bar{h}_j D^{-1}\omega_v^{(l)}\right\| \|(\bar{e}_u)_x\| + \|(\bar{e}_u)_x\| h^{-\frac{1}{2}} \left(\sum_{j=1}^{N} \left|\left[\bar{e}_u\right]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{A.37}$$

Together with (A.36), we have

$$\|(\bar{e}_u)_x\| + h^{-\frac{1}{2}} \left(\sum_{j=1}^{N} \left| [\bar{e}_u]_{j-\frac{1}{2}} \right|^2 \right)^{\frac{1}{2}} \lesssim h \left\| \omega_v^{(l)} \right\| \lesssim h^{2k}. \tag{A.38}$$

We note that

$$\begin{aligned} \int_{I_j} \bar{e}_u \,\mathrm{d}x &= \int_{I_j} u_h - P_h^\star u + \sum_{i=1}^{l} \omega_u^{(i)} \,\mathrm{d}x \\ &= \int_{I_j} u_h - u \,\mathrm{d}x, \end{aligned} \tag{A.39}$$

since $\omega_u^{(i)} \perp \mathbb{P}^0 (I_j)$, $1 \le i \le l$. Then summing over j, and by (A.18)

$$\int_\Omega \bar{e}_u = \int_\Omega u_h - u \,\mathrm{d}x = 0. \tag{A.40}$$

Therefore, by applying the discrete Poincaré inequalities [4], we have

$$\|\bar{e}_u\| \lesssim \| \left(\bar{e}_u\right)_x \| + h^{-\frac{1}{2}} \left(\sum_{j=1}^{N} \left| [\bar{e}_u]_{j-\frac{1}{2}} \right|^2 \right)^{\frac{1}{2}} \lesssim h^{2k}. \tag{A.41}$$

This finishes our proof. □

A.3. Proof of Theorem 3.5

Proof. We take the test functions $\varphi = \bar{e}_u$, $\psi = \bar{e}_v$ in (2.22). Summing over j and using periodic boundary condition we have

$$a\left(\bar{e}_u, \bar{e}_v; \bar{e}_u\right) + b\left(\bar{e}_v, \bar{e}_u; \bar{e}_v\right) = a\left(\epsilon_u, \epsilon_v; \bar{e}_u\right) + b\left(\epsilon_v, \epsilon_u; \bar{e}_v\right). \tag{A.42}$$

From Lemma 2.1 and Corollary 3.2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|^2 + \|\bar{e}_v\|^2 \lesssim h^{k+1+2\left\lfloor \frac{k-1}{2} \right\rfloor} \left(\|\bar{e}_u\| + \|\bar{e}_v\|\right). \tag{A.43}$$

Applying Gronwall's inequality, we obtain

$$\|\bar{e}_u\|(t) \lesssim h^{k+1+2\left\lfloor \frac{k-1}{2} \right\rfloor}. \tag{A.44}$$

□

Remark A.1. We note that, here, we use Gronwall's inequality to obtain the error bound for the error grows exponentially in time. In fact, we can obtain the error bound for the error grows linearly in time. From (A.43), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{e}_u\|^2 \le C_1 h^{k+1+2\left\lfloor \frac{k-1}{2} \right\rfloor} \left(\|\bar{e}_u\| + h^{k+1+2\left\lfloor \frac{k-1}{2} \right\rfloor}\right), \tag{A.45}$$

where C_1 is a constant depend on $\sup_{t\in[0,T]} \|u\|_{k+5+2\left\lfloor \frac{k-1}{2} \right\rfloor}$. Set $D = \frac{\|\bar{e}_u\|}{h^{k+1+2\left\lfloor \frac{k-1}{2} \right\rfloor}}$, then

$$D\frac{\mathrm{d}}{\mathrm{d}t}D \le C_1(D+1). \tag{A.46}$$

Then we refer to the proof of Theorem 2.2 in [29] and obtain

$$\|\bar{e}_u(\cdot,t)\| \le (C_2 + C_3 t)\, h^{k+1+2\left\lfloor \frac{k-1}{2} \right\rfloor}, \tag{A.47}$$

where C_2 is a constant of the error bound of the initial data in (A.41) and $C_3 = \frac{C_1 C_2}{G(C_2)}$, and $G(s) = s - \ln(s+1)$.

Then, by the same arguments in Remark 2.2, we also obtain

$$\|\bar{e}_v\|(t) \lesssim h^{k+1+2\left\lfloor\frac{k-1}{2}\right\rfloor}. \tag{A.48}$$

Taking time derivatives in (2.22) and with the special initial discretizations (A.26), we can obtain

$$\|\left(\bar{e}_u\right)_t\|(t) \lesssim h^{k+1+2\left\lfloor\frac{k-1}{2}\right\rfloor}. \tag{A.49}$$

If k is odd, then $k+1+2\left\lfloor\frac{k-1}{2}\right\rfloor = 2k$. From Lemma 3.3, we have

$$\|(\bar{e}_u)_{xx}\|_{I_j} + h^{-\frac{1}{2}}\left|[(\bar{e}_u)_x]_{j-\frac{1}{2}}\right| + h^{-\frac{3}{2}}\left|[(\bar{e}_u)]_{j+\frac{1}{2}}\right| \lesssim \left\|\bar{e}_v - \omega_v^{(l)}\right\|_{I_j}, \tag{A.50}$$

$$\|(\bar{e}_v)_{xx}\|_{I_j} + h^{-\frac{1}{2}}\left|[(\bar{e}_v)_x]_{j-\frac{1}{2}}\right| + h^{-\frac{3}{2}}\left|[(\bar{e}_v)]_{j+\frac{1}{2}}\right| \lesssim \left\|\left(\omega_u^{(l)}\right)_t - (\bar{e}_u)_t\right\|_{I_j}, \quad \forall j \in \mathbb{Z}_N. \tag{A.51}$$

By using the discrete Poincaré inequalities [4], we have

$$\begin{aligned}\|(\bar{e}_u)_x\| &\le \left\|(\bar{e}_u)_x - \int_\Omega (\bar{e}_u)_x \,\mathrm{d}x\right\| + \left|\int_\Omega (\bar{e}_u)_x \,\mathrm{d}x\right| \\ &\lesssim \|(\bar{e}_u)_{xx}\| + h^{-\frac{1}{2}}\left(\sum_{j=1}^N \left|[(\bar{e}_u)_x]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} + h^{-\frac{1}{2}}\left(\sum_{j=1}^N \left|[(\bar{e}_u)]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}\end{aligned} \tag{A.52}$$

with (A.50),

$$\|\left(\bar{e}_u\right)_x\| \lesssim h^{2k}. \tag{A.53}$$

By the same arguments for $\|\left(\bar{e}_v\right)_x\|$, we also have

$$\|\left(\bar{e}_v\right)_x\| \lesssim h^{2k}. \tag{A.54}$$

If k is even, then $k+1+2\left\lfloor\frac{k-1}{2}\right\rfloor = 2k-1$. We need to improve the estimation as following. First, we have

$$a_j\left(\epsilon_u, \epsilon_v; \varphi\right) = a_j\left(\bar{e}_u, \bar{e}_v; \varphi\right) = \int_{I_j} \left(\bar{e}_u\right)_t \varphi \,\mathrm{d}x + A_j\left(\bar{e}_v, \varphi\right) = \int_{I_j} \left(\omega_u^{(l)}\right)_t \varphi \,\mathrm{d}x, \tag{A.55}$$

$$b_j\left(\epsilon_v, \epsilon_u; \psi\right) = b_j\left(\bar{e}_v, \bar{e}_u; \psi\right) = \int_{I_j} \bar{e}_v \psi \,\mathrm{d}x - B_j\left(\bar{e}_u, \psi\right) = \int_{I_j} \omega_v^{(l)} \psi \,\mathrm{d}x, \quad \forall j \in \mathbb{Z}_N, \quad \forall (\varphi, \psi) \in \left[V_h^k\right]^2. \tag{A.56}$$

Since $l = \frac{k-2}{2}$, and $\left(\omega_u^l\right)_t$, $\omega_v^{(l)} \perp \mathbb{P}^0$, thus $D^{-1}\left(\omega_u^l\right)_t\left(x^{\mp}_{j\pm\frac{1}{2}}\right) = D^{-1}\omega_v^{(l)}\left(x^{\mp}_{j\pm\frac{1}{2}}\right) = 0$. By using the integration by parts on the right hand sides of (A.55) and (A.56), we have

$$A_j\left(\bar{e}_v, \varphi\right) = \int_{I_j} -\bar{h}_j D^{-1}\left(\omega_u^l\right)_t \varphi_x \,\mathrm{d}x - \int_{I_j} \left(\bar{e}_u\right)_t \varphi \,\mathrm{d}x, \tag{A.57}$$

$$B_j\left(\bar{e}_u, \psi\right) = \int_{I_j} \bar{h}_j D^{-1}\omega_v^{(l)} \psi_x \,\mathrm{d}x + \int_{I_j} \bar{e}_v \psi \,\mathrm{d}x. \tag{A.58}$$

By the similar arguments in the proof of Lemma 3.4, we take

$$\varphi|_{I_j} = \psi|_{I_j} = \frac{(-1)^{k+1}h_j}{2k^2}\left(L_{j,k}(x) - L_{j,k-1}(x)\right),$$

in (A.57) and (A.58) respectively, then

$$\begin{aligned}&\left|\left[\bar{e}_v\right]_{j-\frac{1}{2}}\right| \le \|\bar{h}_j D^{-1}\left(\omega_u^l\right)_t\|_{I_j} h^{\frac{1}{2}} + \|\left(\bar{e}_u\right)_t\|_{I_j} h^{\frac{3}{2}}, \\ &\left|\left[\bar{e}_u\right]_{j-\frac{1}{2}}\right| \le \|\bar{h}_j D^{-1}\omega_v^l\|_{I_j} h^{\frac{1}{2}} + \|\bar{e}_v\|_{I_j} h^{\frac{3}{2}},\end{aligned}$$

and

$$h^{-\frac{1}{2}}\sum_{j=1}^{N}\left(\left|\left[\bar{e}_v\right]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} \lesssim h^{2k}, \tag{A.59}$$

$$h^{-\frac{1}{2}}\sum_{j=1}^{N}\left(\left|\left[\bar{e}_u\right]_{j-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}} \lesssim h^{2k}. \tag{A.60}$$

Then, we take $\varphi=\psi=1$ in (A.57) and (A.58) and sum over j,

$$\int_\Omega \left(\bar{e}_u\right)_t \,\mathrm{d}x = 0, \qquad \int_\Omega \bar{e}_v \,\mathrm{d}x = 0. \tag{A.61}$$

Then, we take $\varphi=-\bar{e}_v$ in (A.57) and $\psi=-\left(\bar{e}_u\right)_t$ in (A.58) to obtain

$$\begin{aligned}\left\|\left(\bar{e}_v\right)_x\right\|^2 = {}& \sum_{j=1}^{N}\int_{I_j} -\bar{h}_j D^{-1}\left(\omega_u^l\right)_t \left(\bar{e}_v\right)_x \,\mathrm{d}x - 2\sum_{j=1}^{N}\left(\bar{e}_v\right)_x^+\Big|_{j-\frac{1}{2}}\left[\bar{e}_v\right]_{j-\frac{1}{2}} \\ & + \sum_{j=1}^{N}\int_{I_j}\left(\bar{e}_u\right)_t \bar{e}_v \,\mathrm{d}x,\end{aligned} \tag{A.62}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\bar{e}_u\right)_x\right\|^2 + \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left[\bar{e}_u\right]\left(\bar{e}_u\right)_x^+\big|_{j-\frac{1}{2}}\right) &= \sum_{j=1}^{N}\int_{I_j} -\bar{h}_j D^{-1}\omega_v^{(l)}\left(\bar{e}_u\right)_{tx}\,\mathrm{d}x - \sum_{j=1}^{N}\int_{I_j}\bar{e}_v\left(\bar{e}_u\right)_t\,\mathrm{d}x. \\ &= \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_j} -\bar{h}_j D^{-1}\omega_v^{(l)}\left(\bar{e}_u\right)_x\,\mathrm{d}x + \sum_{j=1}^{N}\int_{I_j}\bar{h}_j\left(D^{-1}\omega_v^{(l)}\right)_t\left(\bar{e}_u\right)_x\,\mathrm{d}x \\ &\quad - \sum_{j=1}^{N}\int_{I_j}\bar{e}_v\left(\bar{e}_u\right)_t\,\mathrm{d}x.\end{aligned} \tag{A.63}$$

We add (A.62) and (A.63) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\bar{e}_u\right)_x\right\|^2 + \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left[\bar{e}_u\right]\left(\bar{e}_u\right)_x^+\big|_{j-\frac{1}{2}}\right) + \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_j}\bar{h}_j D^{-1}\omega_v^{(l)}\left(\bar{e}_u\right)_x\,\mathrm{d}x + \left\|\left(\bar{e}_v\right)_x\right\|^2 \\ &\quad\lesssim h^{2k}\left(\left\|\left(\bar{e}_u\right)_x\right\| + \left\|\left(\bar{e}_v\right)_x\right\|\right),\end{aligned} \tag{A.64}$$

where we have used (A.59) and the inverse inequality. Furthermore, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\left(\bar{e}_u\right)_x\right\|^2 + \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left[\bar{e}_u\right]\left(\bar{e}_u\right)_x^+\big|_{j-\frac{1}{2}}\right) + \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_j}\bar{h}_j D^{-1}\omega_v^{(l)}\left(\bar{e}_u\right)_x\,\mathrm{d}x \\ &\quad\lesssim h^{2k}\left\|\left(\bar{e}_u\right)_x\right\| + h^{4k}.\end{aligned} \tag{A.65}$$

Suppose $\left\|\left(\bar{e}_u\right)_x\right\|(t^\star) := \sup_{t\in[0,T]}\left\|\left(\bar{e}_u\right)_x\right\|(t)$, then integrate (A.65) over $[0,t^\star]$ and using (A.60), the special initial solutions and inverse inequality, we have

$$\left\|\left(\bar{e}_u\right)_x\right\| \lesssim h^{2k}. \tag{A.66}$$

Thus, using the same arguments as in Remark 2.2, we can also obtain

$$\left\|\left(\bar{e}_v\right)_x\right\| \lesssim h^{2k}. \tag{A.67}$$

Finally, together with (A.60) and (A.59), and using the discrete Poincaré inequalities [4], we have

$$\|\bar{e}_u\| + \|\bar{e}_v\| \lesssim h^{2k}. \tag{A.68}$$

This finishes the proof of (3.15) and (3.16). Especially, if $k = 2$, then $q_h - P_h^\star q = \bar{e}_q$, $q = u, v$, thus (3.17) and (3.18) hold true. If $k \geq 3$, then we set $l = 1$,

$$\|u_h - P_h^\star u\| \leq \|\bar{e}_u\| + \|\omega_u^{(1)}\| \lesssim h^{k+3}, \tag{A.69}$$

$$\|\left(u_h - P_h^\star u\right)_x\| \leq \|\left(\bar{e}_u\right)_x\| + \|\left(\omega_u^{(1)}\right)_x\| \lesssim h^{k+2}. \tag{A.70}$$

Here we have used the inverse inequality for the last inequality. By the same steps as before, we can obtain the estimates for v_h. This finishes our proof.

A.4. Proof of Theorem 3.7

Proof. By (3.1) and the definition of $P_h^\star$, we have $u - u_h^-|_{j+\frac{1}{2}} = -\bar{e}_u^-|_{j+\frac{1}{2}}$ and $u_x - \left(u_h\right)_x^+|_{j+\frac{1}{2}} = -\left(\bar{e}_u\right)_x^+|_{j+\frac{1}{2}}$. Therefore, by the inverse inequality and (3.15) and (3.16), we have

$$e_{u,f} \lesssim \|\bar{e}_u\| \lesssim h^{2k}, \tag{A.71}$$

$$e_{u,f_x} \lesssim \|\left(\bar{e}_u\right)_x\| \lesssim h^{2k}. \tag{A.72}$$

Next, we give the estimates for $e_{u,c}$. If k is odd, then $\int_{I_j} \omega_u^{(i)}\,\mathrm{d}x = 0$, $1 \leq i \leq \frac{k-3}{2}$, thus,

$$\int_{I_j} u - u_h\,\mathrm{d}x = \int_{I_j} u - P_h^\star u + \sum_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} \omega_u^{(i)} - \bar{e}_u\,\mathrm{d}x = \int_{I_j} \omega_u^{\left(\lfloor\frac{k-1}{2}\rfloor\right)} - \int_{I_j} \bar{e}_u\,\mathrm{d}x. \tag{A.73}$$

Using the Cauchy–Schwartz inequality, we have

$$e_{u,c} \lesssim \left(\|\bar{e}_u\|^2 + \|\omega_u^{\left(\lfloor\frac{k-1}{2}\rfloor\right)}\|^2\right)^{\frac{1}{2}} \lesssim h^{2k}. \tag{A.74}$$

If k is even, then $\int_{I_j} \omega_u^{(i)}\,\mathrm{d}x = 0$, $1 \leq i \leq \frac{k-2}{2}$, thus, by similar arguments as before, we have

$$\int_{I_j} u - u_h\,\mathrm{d}x = \int_{I_j} \bar{e}_u\,\mathrm{d}x, \quad e_{u,c} \lesssim \|\bar{e}_u\| \lesssim h^{2k}. \tag{A.75}$$

Clearly, by the same steps as before, we can obtain the desired estimates for $e_{v,f}$, $e_{v,fx}$ and $e_{v,c}$. ☐

A.5. Proof of Theorem 3.8

Proof. By the standard approximation of the projection $P_h^\star$ [13], we have

$$\max_{x\in D^0} |u - P_h^\star u| \lesssim h^{k+2}, \quad \max_{x\in D^1} |\left(u - P_h^\star u\right)_x| \lesssim h^{k+1}. \tag{A.76}$$

By Theorem 3.5 and the inverse inequality, we have

$$e_{u,u} \lesssim \max_{x\in D^0} |u - P_h^\star u| + \|P_h^\star u - u_h\|_\infty \lesssim h^{k+2}\|u\|_{k+5,\infty}. \tag{A.77}$$

The estimates for e_{u,u_x}, $e_{v,v}$ and e_{v,v_x} can be proven following the same lines. ☐

Acknowledgements. Research of the first and second authors supported by the China Scholarship Council. Research of the third author supported by NSF grant DMS-1719410.

References

[1] S. Adjerid and T.C. Massey, Superconvergence of discontinuous Galerkin solutions for a nonlinear scalar hyperbolic problem. *Comput. Methods Appl. Mech. Eng.* **195** (2006) 3331–3346.

[2] S. Adjerid, K.D. Devine, J.E. Flaherty and L. Krivodonova, A posteriori error estimation for discontinuous Galerkin solutions of hyperbolic problems. *Comput. Methods Appl. Mech. Eng.* **191** (2002) 1097–1112.

[3] J. Bona, H. Chen, O. Karakashian and Y. Xing, Conservative, discontinuous Galerkin methods for the generalized Korteweg–de Vires equation. *Math. Comput.* **82** (2013) 1401–1432.

[4] S.C. Brenner, Poincaré–Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.* **41** (2003) 306–324.

[5] S.C. Brenner, Discrete Sobolev and Poincaré inequalities for piecewise polynomial functions. *Electron. Trans. Numer. Anal.* **18** (2004) 42–48.

[6] W. Cao and Q. Huang, Superconvergence of local discontinuous Galerkin methods for partial differential equations with higher order derivatives. *J. Sci. Comput.* **72** (2017) 761–791.

[7] W. Cao, Z. Zhang and Q. Zou, Superconvergence of discontinuous Galerkin methods for linear hyperbolic equations. *SIAM J. Numer. Anal.* **52** (2014) 2555–2573.

[8] W. Cao, C.-W. Shu, Y. Yang and Z. Zhang, Superconvergence of discontinuous Galerkin methods for two-dimensional hyperbolic equations. *SIAM J. Numer. Anal.* **53** (2015) 1651–1671.

[9] W. Cao, D. Li, Y. Yang and Z. Zhang, Superconvergence of discontinuous Galerkin methods based on upwind-biased fluxes for 1D linear hyperbolic equations. *ESAIM: M2AN* **51** (2017) 467–486.

[10] W. Cao, H. Liu and Z. Zhang, Superconvergence of the direct discontinuous Galerkin method for convection-diffusion equations. *Numer. Methods Part. Diff. Equ.* **33** (2017) 290–317.

[11] W. Cao, C.-W. Shu and Z. Zhang, Superconvergence of discontinuous Galerkin methods for 1-D linear hyperbolic equations with degenerate variable coefficients. *ESAIM: M2AN* **51** (2017) 2213–2235.

[12] W. Cao, C.-W. Shu, Y. Yang and Z. Zhang, Superconvergence of discontinuous Galerkin method for nonlinear hyperbolic equations. *SIAM J. Numer. Anal.* **56** (2018) 732–765.

[13] A. Chen, Y. Cheng, Y. Liu and M. Zhang, Superconvergence of ultra-weak discontinuous Galerkin methods for the linear Schrödinger equation in one dimension. *J. Sci. Comput.* **82** (2020) 1–44.

[14] Y. Cheng and C.-W. Shu, Superconvergence and time evolution of discontinuous Galerkin finite element solutions. *J. Comput. Phys.* **227** (2008) 9612–9627.

[15] Y. Cheng and C.-W. Shu, A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives. *Math. Comput.* **77** (2009) 699–730.

[16] Y. Cheng and C.-W. Shu, Superconvergence of discontinuous Galerkin and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension. *SIAM J. Numer. Anal.* **47** (2010) 4044–4072.

[17] P.G. Ciarlet, The Finite Element Method for Elliptic Problems. North Holland, Amsterdam, New York (1978).

[18] B. Cockburn and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws. II. General framework. *Math. Comput.* **52** (1989) 411–435.

[19] B. Cockburn and C.-W. Shu, The Runge–Kutta local projection P1-discontinuous Galerkin finite element method for scalar conservation laws. *ESAIM:M2AN* **25** (1991) 337–361.

[20] B. Cockburn and C.-W. Shu, The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems. *J. Comput. Phys.* **141** (1998) 199–224.

[21] B. Cockburn and C.-W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.* **35** (1998) 2440–2463.

[22] B. Cockburn, S.-Y. Lin and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one dimensional systems. *J. Comput. Phys.* **84** (1989) 90–113.

[23] B. Cockburn, S. Hou and C.-W. Shu, The Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case. *Math. Comput.* **54** (1990) 545–581.

[24] B. Dong and C.-W. Shu, Analysis of a local discontinuous Galerkin method for linear time-dependent fourth-order problems. *SIAM J. Numer. Anal.* **47** (2009) 3240–3268.

[25] J. Douglas and T. Dupont, Interior penalty procedures for elliptic and parabolic Galerkin methods. In: Computing Methods in Applied Sciences. Springer, Berlin, Heidelberg (1976) 207–216.

[26] S.M. Han, H. Benaroya and T. Wei, Dynamics of transversely vibrating beams using four engineering theories. *J. Sound Vibr.* **225** (1999) 935–988.

[27] L. Ji and Y. Xu, Optimal error estimates of the local discontinuous Galerkin method for Willmore flow of graphs on Cartesian meshes. *Int. J. Numer. Anal. Model.* **8** (2011) 252–283.

[28] L. Krivodonova, J. Xin, J.F. Remacle, N. Chevaugeon and J.E. Flaherty, Shock detection and limiting with discontinuous Galerkin methods for hyperbolic conservation laws. *Appl. Numer. Math.* **48** (2004) 323–338.

[29] H. Liu and P. Yin, A mixed discontinuous Galerkin method without interior penalty for time-dependent fourth order problems. *J. Sci. Comput.* **77** (2018) 467–501.

[30] Y. Liu, C.-W. Shu and M. Zhang, Superconvergence of energy-conserving discontinuous Galerkin methods for linear hyperbolic equations. *Commun. Appl. Math. Comput.* **1** (2019) 101–116.

[31] I. Mozolevski, E. Süli and P.R. Bösing, *hp*-Version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation. *J. Sci. Comput.* **30** (2007) 465–491.

[32] Q. Tao and Y. Xu, Superconvergence of arbitrary Lagrangian–Eulerian discontinuous Galerkin methods for linear hyperbolic equations. *SIAM J. Numer. Anal.* **57** (2019) 2141–2165.

[33] Q. Tao, Y. Xu and C.-W. Shu, An ultraweak-local discontinuous Galerkin method for PDEs with high order spatial derivatives. To appear in *Math. Comput.* DOI: https://doi.org/10.1090/mcom/3562 (2020).

[34] H. Wang, C.-W. Shu and Q. Zhang, Stability and error estimates of local discontinuous Galerkin methods with implicit-explicit time-marching for advection-diffusion problems. *SIAM J. Numer. Anal.* **53** (2015) 206–227.

[35] H. Wang, S. Wang, C.-W. Shu and Q. Zhang, Local discontinuous Galerkin methods with implicit-explicit time-marching for multi-dimensional convection-diffusion problems. *ESAIM: M2AN* **50** (2016) 1083–1105.

[36] Y. Xu and C.-W. Shu, Local discontinuous Galerkin methods for high-order time-dependent partial differential equations. *Commun. Comput. Phys.* **7** (2010) 1–46.

[37] Y. Xu and C.-W. Shu, Optimal error estimates of the semi-discrete local discontinuous Galerkin methods for high order wave equations. *SIAM J. Numer. Anal.* **50** (2012) 79–104.

[38] J. Yan and C.-W. Shu, A local discontinuous Galerkin method for KdV type equations. *SIAM J. Numer. Anal.* **40** (2002) 769–791.

[39] J. Yan and C.-W. Shu, Local discontinuous Galerkin methods for partial differential equations with higher order derivatives. *J. Sci. Comput.* **17** (2002) 27–47.

[40] Y. Yang and C.-W. Shu, Analysis of optimal superconvergence of discontinuous Galerkin method for linear hyperbolic equations. *SIAM J. Numer. Anal.* **50** (2012) 3110–3133.