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NUMERICAL ALGORITHM FOR THE MODEL DESCRIBING ANOMALOUS DIFFUSION IN EXPANDING MEDIA

Daxin Nie, Jing Sun and Weihua Deng*

Abstract. We provide a numerical algorithm for the model characterizing anomalous diffusion in expanding media, which is derived in Le Vot *et al.* [*Phys. Rev. E* **96** (2017) 032117]. The Sobolev regularity for the equation with variable coefficient is first established. Then we use the finite element method to discretize the Laplace operator and present error estimate of the spatial semi-discrete scheme based on the regularity of the solution; the backward Euler convolution quadrature is developed to approximate Riemann–Liouville fractional derivative and the error estimates for the fully discrete scheme are established by using the continuity of solution. Finally, the numerical experiments verify the effectiveness of the algorithm.

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1. Introduction

Currently, it is widely recognized that anomalous diffusions are ubiquitous in the natural world, and some important models are built, including the continuous time random walk (CTRW) model, *e.g.*, [3,4,20,21,30,32], and the Langevin picture, *e.g.*, [7]. Most of the CTRW models mimic anomalous diffusion processes in static media, while expanding media are typical in biology and cosmology. Recently, Le Vot *et al.* [19] builds the CTRW model for anomalous diffusion in expanding media, the Langevin picture of which is given in [8], and derives the corresponding Fokker–Planck equation

$$\begin{cases} \dfrac{\partial W(x,t)}{\partial t} = \dfrac{1}{a^2(t)}\Delta\left[{}_0D_t^{1-\alpha}W(x,t)\right] + f(x,t), & (x,t)\in\Omega\times(0,T], \\ W(x,0) = W_0(x), & x\in\Omega, \\ W(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T], \end{cases} \tag{1.1}$$

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School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, P.R. China.
*Corresponding author: dengwh@lzu.edu.cn

where Δ stands for Laplace operator; $f(x,t)$ is the source term; $\Omega \subset \mathbb{R}$ is a bounded domain; T is a fixed terminal time; ${}_0D_t^{1-\alpha}$ denotes the Riemann–Liouville fractional derivative, defined as [28]

$$ {}_0D_t^{1-\alpha}W(t) = \frac{\partial}{\partial t}\,{}_0I_t^{\alpha}W(t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{W(\xi)}{(t-\xi)^{1-\alpha}}\,\mathrm{d}\xi, \quad 0<\alpha<1, $$

and ${}_0I_t^{\alpha}$ denotes the Riemann–Liouville fractional integral; $a^2(t)$ is the variable coefficient satisfying $\frac{1}{a^2(t)} \neq 0$ for $t>0$ and

$$ \left|\frac{1}{a^2(t)} - \frac{1}{a^2(s)}\right| \le C|t-s|, \quad t,s \in [0,T] \tag{1.2} $$

with C being a positive constant.

So far, numerical methods for fractional differential equations have gained widespread concerns [5,6,9,10,13–16,22,23,31,33], and [17,27] also provide a complete numerical analysis for fractional differential equations with variable coefficients. Compared with them, the non-commutativity of the Riemann–Liouville fractional derivative and the variable coefficient, *i.e.*, $\frac{1}{a^2(t)}\,{}_0D_t^{1-\alpha} \neq {}_0D_t^{1-\alpha}\frac{1}{a^2(t)}$, brings new challenges in the priori estimate and numerical analysis. To obtain the priori estimate of the solution $W(x,t)$ of equation (1.1), the regularity of ${}_0D_t^{1-\alpha}W(x,t)$ is needed. As for the spatial discretization, we use finite element method to discretize Laplace operator Δ and get the optimal-order convergence rates. And then we use backward Euler convolution quadrature [24,25] to discretize Riemann–Liouville fractional derivative and derive error estimates for fully discrete scheme by using Hölder continuity.

The rest of the paper is organized as follows. We first provide some preliminaries and then give some priori estimates for the solution of equation (1.1) in Section 2. In Section 3, we use the finite element method to discretize the Laplace operator and get the error estimate of the spatial semi-discrete scheme. Section 4 approximates the Riemann–Liouville fractional derivative by backward Euler convolution quadrature and gives the error estimates of the fully discrete scheme for the homogeneous and inhomogeneous problems. Section 5 verifies the effectiveness of the algorithm by numerical experiments. In the last section, we summarize the conclusions.

2. Preliminaries

We first give some preliminaries. For $\kappa>0$ and $\pi/2<\theta<\pi$, we define sectors Σ_θ and $\Sigma_{\theta,\kappa}$ in the complex plane $\mathbb{C}$ as

$$ \Sigma_\theta = \{z \in \mathbb{C}\setminus\{0\}, |\arg z| \le \theta\}, \quad \Sigma_{\theta,\kappa} = \{z \in \mathbb{C} : |z| \ge \kappa, |\arg z| \le \theta\}, $$

and the contour $\Gamma_{\theta,\kappa}$ is defined by

$$ \Gamma_{\theta,\kappa} = \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \le \theta\} \cup \left\{z \in \mathbb{C} : z = r\mathrm{e}^{\pm \mathbf{i}\theta} : r \ge \kappa\right\}, $$

oriented with an increasing imaginary part, where $\mathbf{i}$ denotes the imaginary unit and $\mathbf{i}^2=-1$. We use $\|\cdot\|$ to denote the operator norm from $L^2(\Omega)$ to $L^2(\Omega)$.

Then we introduce $G(x,t) = {}_0D_t^{1-\alpha}W(x,t)$, $A=-\Delta$ with a zero Dirichlet boundary condition, and $A(t) = -\frac{1}{a^2(t)}\Delta$ with a zero Dirichlet boundary condition. For any $r \ge 0$, let

$$ A^r v = \sum_{j=1}^{\infty} \lambda_j^r (v,\varphi_j) $$

and denote the space $\dot{H}^r(\Omega) = \left\{v \in L^2(\Omega) : A^{\frac{r}{2}}v \in L^2(\Omega)\right\}$ with the norm [5]

$$ \|v\|^2_{\dot{H}^r(\Omega)} = \sum_{j=1}^{\infty} \lambda_j^r (v,\varphi_j)^2, $$

where (λ_j, φ_j) are the eigenvalues ordered non-decreasingly and the corresponding eigenfunctions normalized in the $L^2(\Omega)$ norm of operator A subject to the homogeneous Dirichlet boundary conditions on Ω. Thus $\dot{H}^0(\Omega) = L^2(\Omega)$, $\dot{H}^1(\Omega) = H_0^1(\Omega)$, and $\dot{H}^2(\Omega) = H^2(\Omega) \bigcap H_0^1(\Omega)$, where $H^k(\Omega)$ with $k = 1$ or 2 is Sobolev space and $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ [1]. For simplicity, we denote $G(t)$, $W(t)$, W_0, and $f(t)$ as $G(x,t)$, $W(x,t)$, $W_0(x)$, and $f(x,t)$ in the following. Throughout this paper, C denotes a generic positive constant, whose value may differ at each occurrence; $C(T)$ is a generic positive constant depending on time T.

According to (1.2), it holds

$$\|(A(t) - A(s))u\|_{L^2(\Omega)} \le C|t-s|\|u\|_{\dot{H}^2(\Omega)}. \tag{2.1}$$

Taking any fixed $t_0 \in (0,T]$ and the Laplace transforms on both sides of equation (1.1), one has

$$\left(z + A(t_0)z^{1-\alpha}\right)\widetilde{W} = W_0 + \widetilde{f} + \widetilde{\widetilde{G}}, \tag{2.2}$$

where $\widetilde{W}$, $\widetilde{f}$, and $\widetilde{\widetilde{G}}$ are the Laplace transforms of W, f, and $(A(t_0) - A(t))\,G$, respectively. Thus, the inverse Laplace transform of (2.2) leads to

$$W(t) = F(t,t_0)W_0 + \int_0^t F(t-s,t_0)f(s)\,\mathrm{d}s + \int_0^t F(t-s,t_0)\left(A(t_0) - A(s)\right)G(s)\,\mathrm{d}s, \tag{2.3}$$

where

$$F(t,t_0) := \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt}z^{\alpha-1}(z^\alpha + A(t_0))^{-1}\,\mathrm{d}z. \tag{2.4}$$

By means of the Laplace transform and the definition of $G(t)$, we get

$$G(t) = E(t,t_0)W_0 + \int_0^t E(t-s,t_0)f(s)\,\mathrm{d}s + \int_0^t E(t-s,t_0)(A(t_0) - A(s))G(s)\,\mathrm{d}s, \tag{2.5}$$

where

$$E(t,t_0) := \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt}(z^\alpha + A(t_0))^{-1}\,\mathrm{d}z. \tag{2.6}$$

As for the operators $F(t,t_0)$ and $E(t,t_0)$, there are the following estimates.

Lemma 2.1 ([17]). *The operators $F(t,t_0)$ and $E(t,t_0)$ defined in* (2.4) *and* (2.6) *satisfy*

$$\|E(t,t_0)\| \le Ct^{\alpha-1}, \quad \|F(t,t_0)\| \le C, \quad \|A^{1-\beta}E(t,t_0)\| \le Ct^{\alpha\beta-1},$$
$$\|A^{\beta}F(t,t_0)\| \le Ct^{-\alpha\beta}, \quad \|A^{-\beta}F'(t,t_0)\| \le Ct^{\alpha\beta-1},$$

where $F'(t,t_0)$ denotes the first derivative about t and $\beta \in [0,1]$.

Remark 2.2. The estimates in Lemma 2.1 are got by mainly using $\|(z+A)^{-1}\| \le C|z|^{-1}$ for $z \in \Sigma_\theta$. And the last estimate can be obtained by using the fact $z^\alpha(z^\alpha + A)^{-1} = \mathbf{I} - A(z^\alpha + A)^{-1}$, where $\mathbf{I}$ denotes the identity operator.

To get the priori estimate of $W(t)$, we first provide some estimates of $G(t)$.

Theorem 2.3. *If $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0,2]$, $f(0) \in L^2(\Omega)$ and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$ with $t \in (0,T]$, then $G(t)$ satisfies*

$$\|G(t)\|_{L^2(\Omega)} \le C(T)t^{\alpha-1}\|W_0\|_{L^2(\Omega)} + C(T)\|f(0)\|_{L^2(\Omega)} + C(T)\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s$$

and

$$\|G(t)\|_{\dot{H}^2(\Omega)} \le C(T)t^{\nu\alpha/2-1}\|W_0\|_{\dot{H}^\nu(\Omega)} + C(T)\|f(0)\|_{L^2(\Omega)} + C(T)\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

Proof. Applying $A(t_0)$ on both sides of (2.5) and taking L^2 norm on both sides yield

$$\|A(t_0)G(t_0)\|_{L^2(\Omega)} \leq \|A(t_0)E(t_0,t_0)W_0\|_{L^2(\Omega)} + \left\|\int_0^{t_0} A(t_0)E(t_0-s,t_0)f(s)\,\mathrm{d}s\right\|_{L^2(\Omega)} + \left\|\int_0^{t_0} A(t_0)E(t_0-s,t_0)(A(t_0)-A(s))G(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}.$$

According to Lemma 2.1, equation (2.1), and convolution properties, there holds

$$\|A(t_0)G(t_0)\|_{L^2(\Omega)} \leq Ct_0^{\nu\alpha/2-1}\|W_0\|_{\dot{H}^\nu(\Omega)} + C\|f(0)\|_{L^2(\Omega)} + C\int_0^{t_0}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^{t_0}\|G(s)\|_{\dot{H}^2(\Omega)}\,\mathrm{d}s, \quad \nu\in[0,2].$$

Taking $t_0 = t$ and using Grönwall's inequality [17,18] lead to

$$\|G(t)\|_{\dot{H}^2(\Omega)} \leq C(T)t^{\nu\alpha/2-1}\|W_0\|_{\dot{H}^\nu(\Omega)} + C(T)\|f(0)\|_{L^2(\Omega)} + C(T)\int_0^t\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s, \quad \nu\in(0,2].$$

Similarly we have

$$\|G(t)\|_{L^2(\Omega)} \leq C(T)t^{\alpha-1}\|W_0\|_{L^2(\Omega)} + C(T)\|f(0)\|_{L^2(\Omega)} + C(T)\int_0^t\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

□

Next we give the regularity estimate of $W(t)$.

Theorem 2.4. *If $W_0\in\dot{H}^\nu(\Omega)$, $\nu\in(0,2]$, $f(0)\in L^2(\Omega)$ and $\int_0^t\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s<\infty$ with $t\in(0,T]$, then the solution $W(t)$ of equation (1.1) satisfies*

$$\|W(t)\|_{\dot{H}^2(\Omega)} \leq C(T)t^{-\alpha}\|W_0\|_{\dot{H}^\nu(\Omega)} + C(T)\|f(0)\|_{L^2(\Omega)} + C(T)\int_0^t\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

Proof. Applying $A(t_0)$ on both sides of (2.3) and taking L^2 norm lead to

$$\|A(t_0)W(t_0)\|_{L^2(\Omega)} \leq \|A(t_0)F(t_0,t_0)W_0\|_{L^2(\Omega)} + \left\|\int_0^{t_0} A(t_0)F(t_0-s,t_0)f(s)\,\mathrm{d}s\right\|_{L^2(\Omega)} + \left\|\int_0^{t_0} A(t_0)F(t_0-s,t_0)(A(t_0)-A(s))G(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}.$$

According to (2.1), Lemma 2.1, and the fact $T/(t_0-s)>1$, there is

$$\|A(t_0)W(t_0)\|_{L^2(\Omega)} \leq Ct_0^{-\alpha}\|W_0\|_{L^2(\Omega)} + C\|f(0)\|_{L^2(\Omega)} + C\int_0^{t_0}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s + C\int_0^{t_0}(t_0-s)^{1-\alpha}\|G(s)\|_{\dot{H}^2(\Omega)}\,\mathrm{d}s.$$

Further combining Theorem 2.3 results in

$$\|W(t_0)\|_{\dot{H}^2(\Omega)} \leq C(T)t_0^{-\alpha}\|W_0\|_{\dot{H}^\nu(\Omega)} + C(T)\|f(0)\|_{L^2(\Omega)} + C(T)\int_0^{t_0}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s, \quad \nu\in(0,2],$$

which leads to the desired result after taking $t_0 = t$. □

3. Spatial discretization and error analysis

In this section, we discretize Laplace operator by the finite element method and provide the error estimates for the space semi-discrete scheme of equation (1.1). Let $\mathcal{T}_h$ be a shape regular quasi-uniform partitions of the domain Ω, where h is the maximum diameter. Denote X_h as piecewise linear finite element space

$$X_h = \left\{ v_h \in C(\bar{\Omega}) : v_h|_{\mathbf{T}} \in \mathcal{P}^1,\ \forall \mathbf{T} \in \mathcal{T}_h,\ v_h|_{\partial\Omega} = 0 \right\},$$

where $\mathcal{P}^1$ denotes the set of piecewise polynomials of degree 1 over $\mathcal{T}_h$. Then we define the L^2-orthogonal projection $P_h:\ L^2(\Omega) \to X_h$ and the Ritz projection $R_h:\ H_0^1(\Omega) \to X_h$ [5], respectively, by

$$\begin{aligned} (P_h u, v_h) &= (u, v_h) && \forall v_h \in X_h,\\ (\nabla R_h u, \nabla v_h) &= (\nabla u, \nabla v_h) && \forall v_h \in X_h. \end{aligned}$$

Lemma 3.1 ([5]). *The projections P_h and R_h satisfy*

$$\begin{aligned} \|P_h u - u\|_{L^2(\Omega)} + h\|\nabla(P_h u - u)\|_{L^2(\Omega)} &\le C h^q \|u\|_{\dot{H}^q(\Omega)} \quad \text{for } u \in \dot{H}^q(\Omega),\ q = 1, 2,\\ \|R_h u - u\|_{L^2(\Omega)} + h\|\nabla(R_h u - u)\|_{L^2(\Omega)} &\le C h^q \|u\|_{\dot{H}^q(\Omega)} \quad \text{for } u \in \dot{H}^q(\Omega),\ q = 1, 2. \end{aligned}$$

Denote $(\cdot,\cdot)$ as the L_2 inner product and A_h is defined by $(A_h u, v) = (\nabla u, \nabla v)$. The semi-discrete Galerkin scheme for equation (1.1) reads: For every $t \in (0, T]$, find $W_h \in X_h$ such that

$$\begin{cases} \left(\dfrac{\partial W_h}{\partial t}, v\right) + \left({}_0D_t^{1-\alpha} A_h(t_0) W_h, v\right) = (f, v) + \left((A_h(t_0) - A_h(t))\, {}_0D_t^{1-\alpha} W_h, v\right) \quad \text{for all } v \in X_h,\\ W_h(0) = W_{0,h}, \end{cases} \tag{3.1}$$

where

$$W_{0,h} = \begin{cases} P_h W_0, & W_0 \in L^2(\Omega),\\ R_h W_0, & W_0 \in \dot{H}^2(\Omega), \end{cases}$$

and

$$(A_h(t) u, v) = \frac{1}{a^2(t)} (\nabla u, \nabla v).$$

For convenience, we rewrite the spatial semi-discrete scheme as

$$\frac{\partial W_h}{\partial t} + {}_0D_t^{1-\alpha} A_h(t_0) W_h = f_h + (A_h(t_0) - A_h(t))\, {}_0D_t^{1-\alpha} W_h,$$

where $f_h = P_h f$. By means of Laplace transform, the solution of equation (3.1) can be rewritten as

$$W_h(t) = F_h(t, t_0) W_{0,h} + \int_0^t F_h(t-s, t_0) f_h(s)\,\mathrm{d}s + \int_0^t F_h(t-s, t_0)(A_h(t_0) - A_h(s))\, {}_0D_s^{1-\alpha} W_h(s)\,\mathrm{d}s, \tag{3.2}$$

where

$$F_h(t, t_0) := \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt} z^{\alpha-1} (z^\alpha + A_h(t_0))^{-1}\,\mathrm{d}z. \tag{3.3}$$

Introducing $G_h(t) = {}_0D_t^{1-\alpha} W_h(t)$, thus $G_h(t)$ can be represented by

$$G_h(t) = E_h(t, t_0) W_{0,h} + \int_0^t E_h(t-s, t_0) f_h(s)\,\mathrm{d}s + \int_0^t E_h(t-s, t_0)(A_h(t_0) - A_h(s)) G_h(s)\,\mathrm{d}s, \tag{3.4}$$

where

$$E_h(t, t_0) := \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt} (z^\alpha + A_h(t_0))^{-1}\,\mathrm{d}z. \tag{3.5}$$

Similar to Lemma 2.1, the following estimates about E_h and F_h hold.

Lemma 3.2 ([17]). *The operators $F_h(t,t_0)$ and $E_h(t,t_0)$ defined in (3.3) and (3.5) satisfy*

$$\|E_h(t,t_0)\| \le Ct^{\alpha-1}, \quad \|F_h(t,t_0)\| \le C, \quad \|A_h^{1-\beta}E_h(t,t_0)\| \le Ct^{\alpha\beta-1},$$
$$\|A_h^{\beta}F_h(t,t_0)\| \le Ct^{-\alpha\beta}, \quad \|A_h^{-\beta}F_h'(t,t_0)\| \le Ct^{\alpha\beta-1},$$

where $\beta \in [0,1]$.

Next, we provide the following lemma which helps us for the error estimate.

Lemma 3.3 ([5]). *Let $\phi \in L^2(\Omega)$, $z \in \Sigma_\theta$, $\omega = (z^\alpha \mathbf{I} + A)^{-1}\phi$, and $\omega_h = (z^\alpha \mathbf{I} + A_h)^{-1}P_h\phi$, where $\mathbf{I}$ denotes the identity operator. Then there holds*

$$\|\omega_h - \omega\|_{L^2(\Omega)} + h\|\nabla(\omega_h - \omega)\|_{L^2(\Omega)} \le Ch^2\|\phi\|_{L^2(\Omega)}.$$

To get the error estimate for the space semi-discrete scheme, we set $e_h(t) = P_hW(t) - W_h(t)$. From (2.3) and (3.2), there is

$$\begin{aligned}
e_h(t) = {}& (P_hF(t,t_0)W_0 - F_h(t,t_0)W_{0,h}) + \int_0^t (P_hF(t-s,t_0) - F_h(t-s,t_0)P_h)f(s)\,\mathrm{d}s \\
& + \int_0^t (P_hF(t-s,t_0) - F_h(t-s,t_0)P_h)(A(t_0) - A(s))\,{}_0D_s^{1-\alpha}W(s)\,\mathrm{d}s \\
& + \int_0^t F_h(t-s,t_0)((P_hA(t_0) - P_hA(s))\,{}_0D_s^{1-\alpha}W(s) - (A_h(t_0) - A_h(s))\,{}_0D_s^{1-\alpha}W_h(s))\,\mathrm{d}s \\
= {}& \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t) + \mathrm{IV}(t).
\end{aligned} \tag{3.6}$$

Then we need to provide the bounds of $\mathrm{I}(t)$, $\mathrm{II}(t)$, $\mathrm{III}(t)$, and $\mathrm{IV}(t)$ in (3.6).

Lemma 3.4. *If $W_0 \in L^2(\Omega)$, it holds*

$$\|\mathrm{I}(t)\|_{L^2(\Omega)} \le Ct^{-\alpha}h^2\|W_0\|_{L^2(\Omega)}.$$

Proof. According to Lemmas 3.1 and 3.3,

$$\begin{aligned}
\|\mathrm{I}(t)\|_{L^2(\Omega)} &\le \|(P_hF(t,t_0) - F_h(t,t_0)P_h)W_0\|_{L^2(\Omega)} \\
&\le \|(P_hF(t,t_0) - F(t,t_0))W_0\|_{L^2(\Omega)} + \|(F(t,t_0) - F_h(t,t_0)P_h)W_0\|_{L^2(\Omega)} \\
&\le Ct^{-\alpha}h^2\|W_0\|_{L^2(\Omega)},
\end{aligned}$$

which leads to the desired result. □

Similarly, we have the following estimate of $\mathrm{II}(t)$.

Lemma 3.5. *If $f(0) \in L^2(\Omega)$ and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$, then $\mathrm{II}(t)$ can be bounded by*

$$\|\mathrm{II}(t)\|_{L^2(\Omega)} \le Ch^2\left(\|f(0)\|_{L^2(\Omega)} + \int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s\right).$$

As for $\mathrm{III}(t)$, there is the estimate

Lemma 3.6. *If $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0,2]$, $f(0) \in L^2(\Omega)$ and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$, then*

$$\|\mathrm{III}(t)\|_{L^2(\Omega)} \le Ch^2\left(\|W_0\|_{\dot{H}^\nu(\Omega)} + \|f(0)\|_{L^2(\Omega)} + \int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s\right).$$

Proof. According to (2.1), Lemmas 3.1, 3.3, and Theorem 2.3, we have

$$\begin{aligned}\|\mathrm{III}(t_0)\|_{L^2(\Omega)} \leq & \int_0^{t_0} \|P_hF(t_0-s,t_0)-F(t_0-s,t_0)\|\|(A(t_0)-A(s))\,{}_0D_s^{1-\alpha}W(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &+\int_0^{t_0}\|F(t_0-s,t_0)-F_h(t_0-s,t_0)P_h\|\|(A(t_0)-A(s))\,{}_0D_s^{1-\alpha}W(s)\|_{L^2(\Omega)}\,\mathrm{d}s\\ &\leq Ch^2\int_0^{t_0}(t_0-s)^{1-\alpha}\|\,{}_0D_s^{1-\alpha}W(s)\|_{\dot H^2(\Omega)}\,\mathrm{d}s\\ &\leq Ch^2\left(\|W_0\|_{\dot H^\nu(\Omega)}+\|f(0)\|_{L^2(\Omega)}+\int_0^{t_0}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s\right),\quad \nu\in(0,2].\end{aligned}$$

Taking $t_0=t$ leads to the desired result. □

To estimate $\|\mathrm{IV}(t)\|_{L^2(\Omega)}$, introducing $\upsilon_h(t)={}_0D_t^{1-\alpha}e_h$ results in

$$\begin{aligned}\upsilon_h(t)=&\ (P_hE(t,t_0)W_0-E_h(t,t_0)W_{0,h})+\int_0^t(P_hE(t-s,t_0)-E_h(t-s,t_0)P_h)\,f(s)\,\mathrm{d}s\\ &+\int_0^t(P_hE(t-s,t_0)-E_h(t-s,t_0)P_h)\,(A(t_0)-A(s))G(s)\,\mathrm{d}s\\ &+\int_0^t E_h(t-s,t_0)\left((P_hA(t_0)-P_hA(s))G(s)-(A_h(t_0)-A_h(s))G_h(s)\right)\,\mathrm{d}s=\sum_{i=1}^4\upsilon_{i,h}(t).\end{aligned}$$

Next, we consider the estimate of $\|\upsilon_h(t)\|_{L^2(\Omega)}$, which helps to get the estimate of $\|\mathrm{IV}(t)\|_{L^2(\Omega)}$.

Lemma 3.7. *If $W_0\in\dot H^\nu(\Omega)$, $\nu\in(0,2]$, $f(0)\in L^2(\Omega)$ and $\int_0^t\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s<\infty$, then we have*

$$\|\upsilon_h(t)\|_{L^2(\Omega)}\leq Ch^2t^{\nu\alpha/2-1}\|W_0\|_{\dot H^\nu(\Omega)}+Ch^2\|f(0)\|_{L^2(\Omega)}+Ch^2\int_0^t\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

Proof. According to Lemma 3.1, we have the estimates

$$\|(P_hE(t,t_0)-E(t,t_0))W_0\|_{L^2(\Omega)}\leq\begin{cases}Ch^2t^{-1}\|W_0\|_{L^2(\Omega)}, & W_0\in L^2(\Omega),\\ Ch^2t^{\alpha-1}\|W_0\|_{\dot H^2(\Omega)}, & W_0\in\dot H^2(\Omega).\end{cases}$$

If $W_0\in L^2(\Omega)$, according to Lemma 3.3, the following estimate holds

$$\begin{aligned}\|(E(t,t_0)-E_h(t,t_0)P_h)W_0\|_{L^2(\Omega)}\leq&\left\|\int_{\Gamma_{\theta,\kappa}}\mathrm{e}^{zt}((z^\alpha+A(t_0))^{-1}-(z^\alpha+A_h(t_0))^{-1}P_h)W_0\,\mathrm{d}z\right\|_{L^2(\Omega)}\\ \leq&\ Ch^2t^{-1}\|W_0\|_{L^2(\Omega)}.\end{aligned}$$

If $W_0 \in \dot{H}^2(\Omega)$, then one has

$$
\begin{aligned}
&\|(E(t,t_0) - E_h(t,t_0)R_h)W_0\|_{L^2(\Omega)} \\
&\quad \le \left\| \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt} z^{-\alpha} (A(t_0)(z^\alpha + A(t_0))^{-1} - A_h(t_0)(z^\alpha + A_h(t_0))^{-1} R_h) W_0 \,\mathrm{d}z \right\|_{L^2(\Omega)} \\
&\qquad + \left\| \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt} z^{-\alpha} (\mathbf{I} - R_h) W_0 \,\mathrm{d}z \right\|_{L^2(\Omega)} \\
&\quad \le \left\| \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt} z^{-\alpha} ((z^\alpha + A(t_0))^{-1} - (z^\alpha + A_h(t_0))^{-1} P_h) A(t_0) W_0 \,\mathrm{d}z \right\|_{L^2(\Omega)} \\
&\qquad + Ch^2 t^{\alpha-1} \|W_0\|_{\dot{H}^2(\Omega)} \\
&\quad \le Ch^2 t^{\alpha-1} \|W_0\|_{\dot{H}^2(\Omega)},
\end{aligned}
$$

because of Lemma 3.3, the fact $A_h R_h = P_h A$ [5], and $(z^\alpha + A)^{-1} = z^{-\alpha}(\mathbf{I} - A(z^\alpha + A)^{-1})$; and here $\mathbf{I}$ is the identity operator. Thus we get

$$
\begin{aligned}
\|v_{1,h}(t_0)\|_{L^2(\Omega)} \le\ & \|(P_h E(t,t_0) - E(t,t_0))W_0\|_{L^2(\Omega)} \\
& + \|(E(t,t_0) - E_h(t,t_0)P_h)W_0\|_{L^2(\Omega)} \le Ch^2 t^{-1} \|W_0\|_{L^2(\Omega)} \text{ for } W_0 \in L^2(\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
\|v_{1,h}(t_0)\|_{L^2(\Omega)} \le\ & \|(P_h E(t,t_0) - E(t,t_0))W_0\|_{L^2(\Omega)} \\
& + \|(E(t,t_0) - E_h(t,t_0)R_h)W_0\|_{L^2(\Omega)} \le Ch^2 t^{\alpha-1} \|W_0\|_{\dot{H}^2(\Omega)} \text{ for } W_0 \in \dot{H}^2(\Omega).
\end{aligned}
$$

Taking $t_0 = t$ and using the interpolation property [1] lead to

$$
\|v_{1,h}(t)\|_{L^2(\Omega)} \le Ch^2 t^{\nu\alpha/2-1} \|W_0\|_{\dot{H}^\nu(\Omega)}, \quad \nu \in [0,2].
$$

Similarly, one has

$$
\|v_{2,h}(t)\|_{L^2(\Omega)} \le Ch^2 \|f(0)\|_{L^2(\Omega)} + Ch^2 \int_0^t \|f'(s)\|_{L^2(\Omega)} \,\mathrm{d}s,
$$

according to Lemmas 3.1, 3.3, and the convolution property $f(t) = f(0) + (1 * f')(t)$. As for $v_{3,h}(t)$, Theorem 2.3 gives

$$
\|v_{3,h}(t_0)\|_{L^2(\Omega)} \le Ch^2 \int_0^{t_0} \|G(s)\|_{\dot{H}^2(\Omega)} \,\mathrm{d}s \le Ch^2 \|W_0\|_{\dot{H}^\nu(\Omega)} + Ch^2 \|f(0)\|_{L^2(\Omega)} + Ch^2 \int_0^{t_0} \|f'(s)\|_{L^2(\Omega)} \,\mathrm{d}s.
$$

Combining Lemma 3.2, assumption (1.2), and $A_h R_h = P_h A$ results in

$$
\begin{aligned}
\|v_{4,h}(t_0)\|_{L^2(\Omega)} \le\ & \left\| \int_0^{t_0} E_h(t_0 - s, t_0)(A_h(t_0) - A_h(s)) v_h(s) \,\mathrm{d}s \right\|_{L^2(\Omega)} \\
& + \left\| \int_0^{t_0} E_h(t_0 - s, t_0) A_h(t_0)(1 - a^2(t_0)/a^2(s))(R_h - P_h) G(s) \,\mathrm{d}s \right\|_{L^2(\Omega)} \\
\le\ & C \int_0^{t_0} \|v_h(s)\|_{L^2(\Omega)} \,\mathrm{d}s + Ch^2 \int_0^{t_0} \|G(s)\|_{\dot{H}^2(\Omega)} \,\mathrm{d}s.
\end{aligned}
$$

Thus Gröwall's inequality and Theorem 2.3 imply the desired result where we need to require $\nu > 0$.

□

Lemma 3.8. *If $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0,2]$, $f(0) \in L^2(\Omega)$, and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$, then there holds*

$$\|\mathrm{IV}(t)\|_{L^2(\Omega)} \leq Ch^2 \left(\|W_0\|_{\dot{H}^\nu(\Omega)} + \|f(0)\|_{L^2(\Omega)} + \int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s \right).$$

Proof. According to (3.6), one can divide $\mathrm{IV}(t_0)$ into two parts, *i.e.*,

$$\begin{aligned}\|\mathrm{IV}(t_0)\|_{L^2(\Omega)} \leq & \left\| \int_0^{t_0} F_h(t_0-s,t_0)(A_h(t_0)(R_h-P_h) - A_h(s)(R_h-P_h))\,{}_0D_s^{1-\alpha}W(s)\,\mathrm{d}s \right\|_{L^2(\Omega)} \\ & + \left\| \int_0^{t_0} F_h(t_0-s,t_0)(A_h(t_0)-A_h(s))\,{}_0D_s^{1-\alpha}e_h(s)\,\mathrm{d}s \right\|_{L^2(\Omega)} \leq \mathrm{IV}_1(t_0) + \mathrm{IV}_2(t_0).\end{aligned}$$

By assumption (1.2), Lemma 2.1, and Theorem 2.3, one can derive

$$\begin{aligned}\mathrm{IV}_1(t_0) \leq & \int_0^{t_0} \|F_h(t_0-s,t_0)A_h(t_0)(1-a^2(t_0)/a^2(s))(R_h-P_h)\,{}_0D_s^{1-\alpha}W(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\ \leq & Ch^2 \int_0^{t_0} (t_0-s)^{1-\alpha} \|{}_0D_s^{1-\alpha}W(s)\|_{\dot{H}^2(\Omega)}\,\mathrm{d}s \\ \leq & Ch^2 \left(\|W_0\|_{\dot{H}^\nu(\Omega)} + \|f(0)\|_{L^2(\Omega)} + \int_0^{t_0} \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s \right).\end{aligned}$$

According to Lemmas 3.2, 3.7 and assumption (1.2), we have

$$\begin{aligned}\mathrm{IV}_2(t_0) \leq & \int_0^{t_0} \|A_h(t_0)F_h(t_0-s,t_0)\| \|1-a^2(t_0)/a^2(s)\| \|{}_0D_s^{1-\alpha}e_h(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\ \leq & \int_0^{t_0} (t_0-s)^{1-\alpha} \|{}_0D_s^{1-\alpha}e_h(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\ \leq & Ch^2 \left(\|W_0\|_{\dot{H}^\nu(\Omega)} + \|f(0)\|_{L^2(\Omega)} + \int_0^{t_0} \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s \right).\end{aligned}$$

Then the desired result is obtained after taking $t_0 = t$. ∎

Combining Theorem 2.4, Lemmas 3.1, 3.4–3.6, and 3.8 leads to the error estimate of spatial semi-discrete scheme.

Theorem 3.9. *Let $W(t)$ and $W_h(t)$ be the solutions of equations (1.1) and (3.1), respectively. If $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0,2]$, $f(0) \in L^2(\Omega)$, and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$, then there holds*

$$\|W(t) - W_h(t)\|_{L^2(\Omega)} \leq Ch^2 \left(t^{-\alpha}\|W_0\|_{\dot{H}^\nu(\Omega)} + \|f(0)\|_{L^2(\Omega)} + \int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s \right).$$

4. Temporal discretization and error analysis

In this section, we use backward Euler convolution quadrature [15, 24–26] to discretize the time fractional derivative and perform the error analyses of the fully discrete scheme for homogeneous and inhomogeneous problems. First, let the time step size $\tau = T/L$, $L \in \mathbb{N}$, $t_i = i\tau$, $i = 0, 1, \ldots, L$ and $0 = t_0 < t_1 < \cdots < t_L = T$. Taking $\delta_\tau(\zeta) = \frac{1-\zeta}{\tau}$ and using convolution quadrature for equation (3.1), we have the fully discrete scheme for any fixed integer $m \in [0, L]$,

$$\begin{cases} \dfrac{W_h^n - W_h^{n-1}}{\tau} + A_h(t_m) \displaystyle\sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i} = f_h^n + (A_h(t_m) - A_h(t_n)) \displaystyle\sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i}, \\ W_h^0 = W_{0,h}, \end{cases} \tag{4.1}$$

where

$$\sum_{i=0}^{\infty} d_i^{(1-\alpha)} \zeta^i = \delta_\tau(\zeta)^{1-\alpha}, \quad 0 < \alpha < 1,$$

and W_h^n denotes the numerical solution of (3.1) at $t = t_n$. Multiplying ζ^n on both sides of (4.1) and summing n from 1 to ∞ result in

$$\sum_{n=1}^{\infty} \frac{W_h^n - W_h^{n-1}}{\tau} \zeta^n + \sum_{n=1}^{\infty} A_h(t_m) \sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i} \zeta^n = \sum_{n=1}^{\infty} f_h^n \zeta^n + \sum_{n=1}^{\infty} (A_h(t_m) - A_h(t_n)) \sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i} \zeta^n;$$

after simple calculations, it holds

$$\left(\delta_\tau(\zeta) + A_h(t_m)\delta_\tau(\zeta)^{1-\alpha}\right) \sum_{n=1}^{\infty} W_h^n \zeta^n = \sum_{n=1}^{\infty} f_h^n \zeta^n + \sum_{n=1}^{\infty} (A_h(t_m) - A_h(t_n)) \sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i} \zeta^n + \frac{\zeta}{\tau} W_h^0.$$

Thus, choosing $\xi_\tau = \mathrm{e}^{-\tau(\kappa+1)}$, one has

$$\begin{aligned} W_h^n =& \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \sum_{j=1}^{\infty} f_h^j \zeta^j \,\mathrm{d}\zeta \\ &+ \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \frac{\zeta}{\tau} W_h^0 \,\mathrm{d}\zeta \\ &+ \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \left(\sum_{j=1}^{\infty} (A_h(t_m) - A_h(t_j)) \sum_{i=0}^{j-1} d_i^{(1-\alpha)} W_h^{j-i} \zeta^j \right) \mathrm{d}\zeta. \end{aligned} \tag{4.2}$$

Before providing the error estimates, we recall the following lemma.

Lemma 4.1 ([11]). *Let $0 < \alpha < 1$ and $\theta \in \left(\frac{\pi}{2}, \operatorname{arccot}\left(-\frac{2}{\pi}\right)\right)$ be given, where arccot means the inverse function of* cot, *and let $\rho \in (0,1)$ be fixed. Then, both $\delta_\tau(\mathrm{e}^{-z\tau})$ and $(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A)^{-1}$ are analytic with respect to z in the region enclosed by $\Gamma_\rho^\tau = \{z = -\ln(\rho)/\tau + \mathbf{i}y : y \in \mathbb{R}$ and $|y| \le \pi/\tau\}$, $\Gamma_{\theta,\kappa}^\tau = \left\{z \in \mathbb{C} : \kappa \le |z| \le \frac{\pi}{\tau \sin(\theta)}, |\arg z| = \theta\right\} \bigcup \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \le \theta\}$, and the two lines $\mathbb{R} \pm \mathbf{i}\pi/\tau$ whenever $0 < \kappa \le \min(1/T, -\ln(\rho)/\tau)$. Furthermore, there are the estimates*

$$\begin{aligned} &\delta_\tau\left(\mathrm{e}^{-z\tau}\right) \in \Sigma_\theta && \forall z \in \Gamma_{\theta,\kappa}^\tau, \\ &C_0|z| \le \left|\delta_\tau\left(\mathrm{e}^{-z\tau}\right)\right| \le C_1|z| && \forall z \in \Gamma_{\theta,\kappa}^\tau, \\ &\left|\delta_\tau\left(\mathrm{e}^{-z\tau}\right) - z\right| \le C\tau|z|^2 && \forall z \in \Gamma_{\theta,\kappa}^\tau, \\ &\left|\delta_\tau\left(\mathrm{e}^{-z\tau}\right)^\alpha - z^\alpha\right| \le C\tau|z|^{\alpha+1} && \forall z \in \Gamma_{\theta,\kappa}^\tau, \end{aligned}$$

where the constants C_0, C_1 and C are independent of τ and $\kappa \in (0, \min(1/T, -\ln(\rho)/\tau))$.

Below we provide the error estimates of the homogeneous and inhomogeneous problems separately.

4.1. Error estimate for the inhomogeneous problem

In this subsection, we consider the error estimate between $W_h(t_n)$ and W_h^n which are the solutions of equations (3.1) and (4.1) with the initial value $W_0 = 0$. Denote $\mathrm{e}_h^n = W_h(t_n) - W_h^n$. By (3.2) and (4.2), there holds

$$\|\mathrm{e}_h^n\|_{L^2(\Omega)} \le \mathrm{I} + \mathrm{II}, \tag{4.3}$$

where

$$
\begin{aligned}
\mathrm{I} \leq C & \left\| \int_0^{t_n} F_h(t_n - s, t_m) f_h(s)\,\mathrm{d}s - \frac{1}{2\pi\mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \sum_{j=1}^{\infty} f_h^j \zeta^j \,\mathrm{d}\zeta \right\|_{L^2(\Omega)}, \\
\mathrm{II} \leq C & \left\| \int_0^{t_n} F_h(t_n - s, t_m)(A_h(t_m) - A_h(s))\, {}_0D_s^{1-\alpha} W_h(s)\,\mathrm{d}s \right. \\
& \left. - \frac{1}{2\pi\mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \left(\sum_{j=1}^{\infty} (A_h(t_m) - A_h(t_j)) \sum_{i=0}^{j-1} d_i^{(1-\alpha)} W_h^{j-i} \zeta^j \right) \mathrm{d}\zeta \right\|_{L^2(\Omega)}.
\end{aligned}
$$

Like the proof in [15,26], one can get the following estimates of I and II defined in (4.3).

Theorem 4.2. *If $f(0) \in L^2(\Omega)$ and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$, then there holds*

$$
\mathrm{I} \leq C\tau \|f(0)\|_{L^2(\Omega)} + C\tau \int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.
$$

As for II, we introduce

$$
\tau \sum_{i=0}^{\infty} F_{\tau,m}^i \zeta^i = \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1},
$$

where

$$
F_{\tau,m}^n = \frac{1}{2\pi\tau\mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{\alpha-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \mathrm{d}\zeta
$$

and $\xi_\tau = \mathrm{e}^{-\tau(\kappa+1)}$. Taking $\zeta = \mathrm{e}^{-z\tau}$ and deforming the contour $\Gamma^\tau = \{z = \kappa + 1 + \mathbf{i}y : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau\}$ to $\Gamma_{\theta,\kappa}^\tau = \left\{ z \in \mathbb{C} : \kappa \leq |z| \leq \frac{\pi}{\tau \sin(\theta)}, |\arg z| = \theta \right\} \bigcup \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta\}$, one has

$$
F_{\tau,m}^n = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{zn\tau} \delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1} \left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m)\right)^{-1} \mathrm{d}z,
$$

and simple calculations lead to

$$
\|A_h(t_m) F_{\tau,m}^n\| = \left\| \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{zn\tau} A_h(t_m) \delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1} \left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m)\right)^{-1} \mathrm{d}z \right\| \leq C(t_n + \tau)^{-\alpha}. \tag{4.4}
$$

To get the estimates of II, we divide it into four parts, *i.e.*,

$$
\begin{aligned}
\mathrm{II} \leq C & \left\| \int_0^{t_n} F_h(t_n - s, t_m)(A_h(t_m) - A_h(s))\, {}_0D_s^{1-\alpha} W_h(s)\,\mathrm{d}s \right. \\
& \left. - \tau \sum_{k=1}^{n} F_{\tau,m}^{n-k} \left((A_h(t_m) - A_h(t_k)) \sum_{i=0}^{k-1} d_i^{(1-\alpha)} W_h^{k-i} \right) \right\|_{L^2(\Omega)} \leq \sum_{k=1}^{n} (\mathrm{II}_{1,k} + \mathrm{II}_{2,k} + \mathrm{II}_{3,k} + \mathrm{II}_{4,k}),
\end{aligned} \tag{4.5}
$$

where

$$\mathrm{II}_{1,k} \leq C \left\| \tau F_{\tau,m}^{n-k}(A_h(t_m) - A_h(t_k)) \left(\sum_{i=0}^{k-1} d_i^{(1-\alpha)} W_h^{k-i} - {}_0D_t^{1-\alpha} W_h(t_k) \right) \right\|_{L^2(\Omega)},$$

$$\mathrm{II}_{2,k} \leq C \left\| \left(\int_{t_{k-1}}^{t_k} F_h(t_n - s, t_m)\,\mathrm{d}s - \tau F_{\tau,m}^{n-k} \right) \left((A_h(t_m) - A_h(t_k))\,{}_0D_t^{1-\alpha} W_h(t_k) \right) \right\|_{L^2(\Omega)},$$

$$\mathrm{II}_{3,k} \leq C \left\| \int_{t_{k-1}}^{t_k} F_h(t_n - s, t_m) \left((A_h(t_m) - A_h(s)) - (A_h(t_m) - A_h(t_k)) \right) {}_0D_t^{1-\alpha} W_h(t_k)\,\mathrm{d}s \right\|_{L^2(\Omega)},$$

$$\mathrm{II}_{4,k} \leq C \left\| \int_{t_{k-1}}^{t_k} F_h(t_n - s, t_m)(A_h(t_m) - A_h(s)) \left({}_0D_t^{1-\alpha} W_h(s) - {}_0D_t^{1-\alpha} W_h(t_k) \right) \mathrm{d}s \right\|_{L^2(\Omega)}.$$

To get error estimates of II, the following estimates of G_h defined in (3.4) are also needed. Similar to Theorem 2.3, we have the following results.

Theorem 4.3. *If $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, $f(0) \in L^2(\Omega)$, and $\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s < \infty$, then $G_h(t)$ satisfies*

$$\|G_h(t)\|_{L^2(\Omega)} \leq Ct^{\alpha-1}\|W_0\|_{L^2(\Omega)} + C\|f(0)\|_{L^2(\Omega)} + C\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s$$

and

$$\|G_h(t)\|_{\dot{H}^2(\Omega)} \leq Ct^{\nu\alpha/2-1}\|W_0\|_{\dot{H}^\nu(\Omega)} + C\|f(0)\|_{L^2(\Omega)} + C\int_0^t \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

Theorem 4.4. *Let $G_h(t) = {}_0D_t^{1-\alpha} W_h(t)$. Assume $W_0 = 0$, $f(0) \in L^2(\Omega)$, and $f'(s) \in L^\infty(0, T, L^2(\Omega))$. There holds*

$$\left\| \frac{G_h(t) - G_h(t-\tau)}{\tau^\gamma} \right\|_{L^2(\Omega)} \leq Ct^{\alpha-\gamma} \left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))} \right),$$

where $\gamma < 1 + \alpha$.

Proof. According to (3.4), one has

$$\left\| \frac{G_h(t) - G_h(t-\tau)}{\tau^\gamma} \right\|_{L^2(\Omega)} \leq \upsilon_1 + \upsilon_2,$$

where

$$\upsilon_1 = \left\| \frac{\int_0^t E_h(t-s, t_0) f_h(s)\,\mathrm{d}s - \int_0^{t-\tau} E_h(t-\tau-s, t_0) f_h(s)\,\mathrm{d}s}{\tau^\gamma} \right\|_{L^2(\Omega)},$$

$$\upsilon_2 = \left\| \frac{\int_0^t E_h(t-s, t_0)(A_h(t_0) - A_h(s)) G_h(s)\,\mathrm{d}s - \int_0^{t-\tau} E_h(t-\tau-s, t_0)(A_h(t_0) - A_h(s)) G_h(s)\,\mathrm{d}s}{\tau^\gamma} \right\|_{L^2(\Omega)}.$$

As for υ_1, we split it into

$$\begin{aligned} \upsilon_1 \leq{}& C \left\| \frac{\int_0^t E_h(t-s, t_0) ds f_h(0) - \int_0^{t-\tau} E_h(t-\tau-s, t_0) ds f_h(0)}{\tau^\gamma} \right\|_{L^2(\Omega)} \\ &+ C \left\| \frac{\int_0^{t-\tau} \left(\int_0^{t-s} E_h(r, t_0) dr - \int_0^{t-\tau-s} E_h(r, t_0) dr \right) f_h'(s)\,\mathrm{d}s}{\tau^\gamma} \right\|_{L^2(\Omega)} \\ &+ C \left\| \frac{\int_{t-\tau}^t \int_0^{t-s} E_h(r, t_0) dr f_h'(s)\,\mathrm{d}s}{\tau^\gamma} \right\|_{L^2(\Omega)} \leq \upsilon_{1,1} + \upsilon_{1,2} + \upsilon_{1,3}. \end{aligned}$$

Using the fact $\left|\frac{1-\mathrm{e}^{-z\tau}}{\tau^\gamma}\right| \leq C|z|^\gamma$ on $\Gamma_{\theta,\kappa}$, there is

$$\begin{aligned}
\upsilon_{1,1} &\leq C\left\|\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt}\frac{1-\mathrm{e}^{-z\tau}}{\tau^\gamma}(z^\alpha + A_h(t_0))^{-1}z^{-1}\,\mathrm{d}z f_h(0)\right\|_{L^2(\Omega)}\\
&\leq C\int_{\Gamma_{\theta,\kappa}} |\mathrm{e}^{zt}||z|^{\gamma-\alpha-1}|\,\mathrm{d}z|\|f_h(0)\|_{L^2(\Omega)} \leq Ct^{\alpha-\gamma}\|f(0)\|_{L^2(\Omega)}.
\end{aligned}$$

Similarly, we can bound $\upsilon_{1,2}$ by

$$\begin{aligned}
\upsilon_{1,2} \leq C&\left\|\int_0^{t-\tau}\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t-s)}\frac{1-\mathrm{e}^{-z\tau}}{\tau^\gamma}(z^\alpha + A_h(t_0))^{-1}z^{-1}\,\mathrm{d}z f_h'(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\\
\leq& C\int_0^{t-\tau}\int_{\Gamma_{\theta,\kappa}} |\mathrm{e}^{z(t-s)}||z|^{\gamma-\alpha-1}|\,\mathrm{d}z|\|f_h'(s)\|_{L^2(\Omega)}\,\mathrm{d}s \leq C\int_0^{t-\tau}(t-s)^{\alpha-\gamma}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s,
\end{aligned}$$

where $\gamma < 1+\alpha$ is required to ensure $\upsilon_{1,2}$ convergent. Similarly one has

$$\begin{aligned}
\upsilon_{1,3} &\leq C\left\|\int_{t-\tau}^{t}\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t-s)}\tau^{-\gamma}(z^\alpha + A_h(t_0))^{-1}z^{-1}\,\mathrm{d}z f_h'(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\\
&\leq C\left\|\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z\tau}\frac{1-\mathrm{e}^{-z\tau}}{z\tau^\gamma}(z^\alpha + A_h(t_0))^{-1}z^{-1}\,\mathrm{d}z\right\|\|f_h'(s)\|_{L^\infty(0,T,L^2(\Omega))}\\
&\leq C\int_{\Gamma_{\theta,\kappa}} |z|^{\gamma-\alpha-2}|\,\mathrm{d}z|\|f_h'(s)\|_{L^\infty(0,T,L^2(\Omega))} \leq Ct^{1+\alpha-\gamma}\|f'(s)\|_{L^\infty(0,T,L^2(\Omega))},
\end{aligned}$$

where we take $\kappa = 1/t$ and require $\gamma < 1+\alpha$ to ensure $\upsilon_{1,3}$ convergent. Thus one has

$$\upsilon_1 \leq Ct^{\alpha-\gamma}\left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))}\right)$$

and $\gamma < 1+\alpha$. Similarly, when $t = t_0$, one can split υ_2 into

$$\begin{aligned}
\upsilon_2 \leq &\left\|\frac{\int_0^{t_0-\tau}\left(E_h(t_0-s,t_0) - E_h(t_0-\tau-s,t_0)\right)(A_h(t_0) - A_h(s))G_h(s)\,\mathrm{d}s}{\tau^\gamma}\right\|_{L^2(\Omega)}\\
&+\left\|\frac{\int_{t_0-\tau}^{t_0} E_h(t_0-s,t_0)(A_h(t_0)-A_h(s))G_h(s)\,\mathrm{d}s}{\tau^\gamma}\right\|_{L^2(\Omega)} \leq \upsilon_{2,1} + \upsilon_{2,2}.
\end{aligned}$$

Using Lemma 3.2 and (2.1), one has

$$\begin{aligned}
\upsilon_{2,1} \leq &\left\|\int_0^{t_0-\tau}\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_0-s)}\frac{1-\mathrm{e}^{-z\tau}}{\tau^\gamma}(z^\alpha + A_h(t_0))^{-1}\,\mathrm{d}z(A_h(t_0)-A_h(s))G_h(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\\
\leq &\int_0^{t_0-\tau}\int_{\Gamma_{\theta,\kappa}} |\mathrm{e}^{z(t_0-s)}||z|^{\gamma-\alpha}|\,\mathrm{d}z|\|(A_h(t_0)-A_h(s))G_h(s)\|_{L^2(\Omega)}\,\mathrm{d}s \leq \int_0^{t_0-\tau}(t_0-s)^{\alpha-\gamma}\|G_h(s)\|_{\dot H^2(\Omega)}\,\mathrm{d}s.
\end{aligned}$$

Similarly, when $\gamma \leq 1$, we have

$$\upsilon_{2,2} \leq C\left\|\frac{\int_{t_0-\tau}^{t_0} E_h(t_0-s,t_0)\,\mathrm{d}s}{\tau^{\gamma-1}}\right\|\|G_h(s)\|_{L^\infty(0,T,\dot H^2(\Omega))} \leq Ct_0^{1+\alpha-\gamma}\|G_h(s)\|_{L^\infty(0,T,\dot H^2(\Omega))};$$

when $\gamma > 1$, we obtain

$$\begin{aligned}
\upsilon_{2,2} &\leq C \left\| \frac{\int_{t_0-\tau}^{t_0} E_h(t_0-s,t_0)\,\mathrm{d}s}{\tau^{\gamma-1}} \right\| \|G_h(s)\|_{L^\infty(0,T,\dot{H}^2(\Omega))} \\
&\leq C \left\| \frac{\int_{t_0-\tau}^{t_0} \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_0-s)}(z^\alpha + A_h(t_0))^{-1}\,\mathrm{d}z\,\mathrm{d}s}{\tau^{\gamma-1}} \right\| \|G_h(s)\|_{L^\infty(0,T,\dot{H}^2(\Omega))} \\
&\leq C \left\| \int_{\Gamma_{\theta,\kappa}} \frac{1-\mathrm{e}^{z\tau}}{z\tau^{\gamma-1}} (z^\alpha + A_h(t_0))^{-1}\,\mathrm{d}z \right\| \|G_h(s)\|_{L^\infty(0,T,\dot{H}^2(\Omega))} \\
&\leq C \int_{\Gamma_{\theta,\kappa}} |z|^{\gamma-2-\alpha}|\,\mathrm{d}z| \|G_h(s)\|_{L^\infty(0,T,\dot{H}^2(\Omega))} \leq C t_0^{1+\alpha-\gamma} \|G_h(s)\|_{L^\infty(0,T,\dot{H}^2(\Omega))},
\end{aligned}$$

where we take $\kappa = 1/t_0$ and require $\gamma < 1+\alpha$ for the integral $\int_{\Gamma_{\theta,\kappa}} |z|^{\gamma-2-\alpha}|\,\mathrm{d}z|$ to be convergent. Taking $t_0 = t$ and using Theorem 4.3, the desired results can be obtained by the fact $T/t > 1$. □

Now we estimate II. First, we consider $\mathrm{II}_{1,k}$ defined in (4.5), which implies the difference between $\sum_{i=0}^{k-1} d_i^{(1-\alpha)} W_h^{k-i}$ and ${}_0D_t^{1-\alpha} W_h(t_k)$ needs to be obtained.

Lemma 4.5. *Let $G_h(t) = {}_0D_t^{1-\alpha} W_h(t)$ and $G_h^n = \sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i}$. Assume $W_0 = 0$, $f(0) \in L^2(\Omega)$ and $f'(s) \in L^\infty(0,T,L^2(\Omega))$. There is*

$$\|G_h^n - G_h(t_n)\|_{L^2(\Omega)} \leq C\tau t_n^{\alpha-1} \left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))} \right).$$

Proof. From the definition of G_h^n and equation (4.2), it has

$$\begin{aligned}
\sum_{n=1}^\infty G_h^n \zeta^n &= \sum_{n=1}^\infty \sum_{i=0}^{n-1} d_i^{(1-\alpha)} W_h^{n-i} \zeta^n = \delta_\tau(\zeta)^{1-\alpha} \sum_{n=1}^\infty W_h^n \zeta^n \\
&= (\delta_\tau(\zeta)^\alpha + A_h(t))^{-1} \sum_{n=1}^\infty f_h^n \zeta^n + (\delta_\tau(\zeta)^\alpha + A_h(t_m))^{-1} \sum_{n=1}^\infty (A_h(t_m) - A_h(t_n)) G_h^n \zeta^n.
\end{aligned}$$

Considering the error between G_h^m and $G_h(t_m)$, one has

$$\|G_h^m - G_h(t_m)\|_{L^2(\Omega)} \leq \sum_{k=1}^2 \upsilon_{k,h},$$

where

$$\begin{aligned}
\upsilon_{1,h} &\leq C \left\| \frac{1}{2\pi\mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-m-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \sum_{n=1}^\infty f_h^n \zeta^n\,\mathrm{d}\zeta - \int_0^{t_m} E_h(t_m-s,t_m) f_h(s)\,\mathrm{d}s \right\|_{L^2(\Omega)}, \\
\upsilon_{2,h} &\leq C \left\| \frac{1}{2\pi\mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-m-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1} \sum_{n=1}^\infty (A_h(t_m) - A_h(t_n)) G_h^n \zeta^n\,\mathrm{d}\zeta \right. \\
&\quad \left. - \int_0^{t_m} E_h(t_m-s,t_m)(A_h(t_m) - A_h(s)) G_h(s)\,\mathrm{d}s \right\|_{L^2(\Omega)}
\end{aligned}$$

with $\xi_\tau = \mathrm{e}^{-\tau(\kappa+1)}$. Similar to the proof in [15, 26], the following estimate of $\upsilon_{1,h}$ can be got

$$\upsilon_{1,h} \leq C\tau t_m^{\alpha-1} \|f(0)\|_{L^2(\Omega)} + C\tau \int_0^{t_m} (t_m-s)^{\alpha-1} \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

As for $\upsilon_{2,h}$, we introduce

$$\tau\sum_{i=0}^{\infty}E_{\tau,m}^{i}\zeta^{i}=\left(\delta_\tau(\zeta)^\alpha+A_h(t_m)\right)^{-1},$$

where

$$E_{\tau,m}^{n}=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{zn\tau}\left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m)\right)^{-1}\,\mathrm{d}z.$$

Thus

$$\|A_h(t_m)E_{\tau,m}^{n}\|=\left\|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{zn\tau}A_h(t_m)\left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m)\right)^{-1}\,\mathrm{d}z\right\|\leq C(t_n+\tau)^{-1}. \tag{4.6}$$

For convenience, we split $\upsilon_{2,h}$ into the following forms

$$\begin{aligned}
\upsilon_{2,h}\leq{}& C\left\|\tau\sum_{k=1}^{m}E_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k))G_h^k-\int_0^{t_m}E_h(t_m-s,t_m)(A_h(t_m)-A_h(s))G_h(s)\,\mathrm{d}s\right\|_{L^2(\Omega)}\\
\leq{}& C\sum_{k=1}^{m}\left\|\tau E_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k))(G_h^k-G_h(t_k))\right\|_{L^2(\Omega)}\\
&+C\sum_{k=1}^{m}\left\|\left(\tau E_{\tau,m}^{m-k}-\int_{t_{k-1}}^{t_k}E_h(t_m-s,t_m)\,\mathrm{d}s\right)(A_h(t_m)-A_h(t_k))G_h(t_k)\right\|_{L^2(\Omega)}\\
&+C\sum_{k=1}^{m}\left\|\int_{t_{k-1}}^{t_k}E_h(t_m-s,t_m)(A_h(t_k)-A_h(s))dsG_h(t_k)\right\|_{L^2(\Omega)}\\
&+C\sum_{k=1}^{m}\left\|\int_{t_{k-1}}^{t_k}E_h(t_m-s,t_m)(A_h(t_m)-A_h(s))(G_h(t_k)-G_h(s))\,\mathrm{d}s\right\|_{L^2(\Omega)}=\sum_{k=1}^{m}\sum_{j=1}^{4}\upsilon_{2,j,k,h}.
\end{aligned}$$

Assumption (1.2) and equation (4.6) lead to

$$\sum_{k=1}^{m}\upsilon_{2,1,k,h}\leq C\sum_{k=1}^{m}\tau\|G_h^k-G_h(t_k)\|_{L^2(\Omega)}.$$

As for $\upsilon_{2,2,k,h}$, there is

$$\begin{aligned}
&\left\|\tau E_{\tau,m}^{m-k}-\int_{t_{k-1}}^{t_k}E_h(t_m-s,t_m)\,\mathrm{d}s\right\|\leq\left\|\int_{t_{k-1}}^{t_k}E_{\tau,m}^{m-k}-E_h(t_m-s,t_m)\,\mathrm{d}s\right\|\\
&\quad\leq C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(m-k)\tau}\left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m)\right)^{-1}\,\mathrm{d}z-\int_{\Gamma_{\theta,\kappa}}\mathrm{e}^{z(t_m-s)}(z^\alpha+A_h(t_m))^{-1}\,\mathrm{d}z\,\mathrm{d}s\right\|\\
&\quad\leq C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-s)}(z^\alpha+A_h(t_m))^{-1}\,\mathrm{d}z\,\mathrm{d}s\right\|\\
&\qquad+C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-s)}(1-\mathrm{e}^{(s-k\tau)z})\left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m)\right)^{-1}\,\mathrm{d}z\,\mathrm{d}s\right\|\\
&\qquad+C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-s)}\left(\left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m)\right)^{-1}-(z^\alpha+A_h(t_m))^{-1}\right)\,\mathrm{d}z\,\mathrm{d}s\right\|
\end{aligned}$$

$$\leq C\tau \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha-2}\,\mathrm{d}s.$$

According to (2.1), it has

$$\sum_{k=1}^{m} \upsilon_{2,2,k,h} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha-1}\,\mathrm{d}s \|G_h(t_k)\|_{\dot{H}^2(\Omega)} \leq C\tau \|f(0)\|_{L^2(\Omega)} + C\tau \int_0^{t_m} \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s,$$

where the last inequality follows by Lemma 2.3. Combining Lemma 2.1 and (2.1), one obtains

$$\sum_{k=1}^{m} \upsilon_{2,3,k,h} \leq C \sum_{k=1}^{m} \tau \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha-1} \|G_h(t_k)\|_{\dot{H}^2(\Omega)}\,\mathrm{d}s \leq C\tau \|f(0)\|_{L^2(\Omega)} + C\tau \int_0^{t_m} \|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

From Lemma 4.4, it holds

$$\sum_{k=1}^{m} \upsilon_{2,4,k,h} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \left\| \frac{G_h(s) - G_h(t_k)}{\tau} \right\|_{L^2(\Omega)} \mathrm{d}s \leq C\tau \left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))} \right).$$

Since $m \in [0, L]$ is any fixed integer, taking $m = n$ results in

$$\|G_h^n - G_h(t_n)\|_{L^2(\Omega)} \leq C\tau t_n^{\alpha-1} \left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))} \right) + C \sum_{k=1}^{n} \tau \|G_h^k - G_h(t_k)\|_{L^2(\Omega)}.$$

Then the discrete Grönwall inequality [29] leads to the desired results. □

Theorem 4.6. *If* $W_0 = 0,\ f(0) \in L^2(\Omega)$ *and* $f'(s) \in L^\infty(0, T, L^2(\Omega))$, *then there holds*

$$\mathrm{II} \leq C\tau \left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))} \right),$$

where II *is defined in* (4.5).

Proof. According to Lemma 4.5 and equation (4.4), it has

$$\begin{aligned} \sum_{k=1}^{m} \mathrm{II}_{1,k} &\leq C\tau \sum_{k=1}^{m} (t_m - t_k)^{1-\alpha} \|G_h^k - G_h(t_k)\|_{L^2(\Omega)} \\ &\leq C\tau \|f(0)\|_{L^2(\Omega)} + C\tau \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))}. \end{aligned}$$

Next, consider the difference between $\tau F_{\tau,m}^{m-k}$ and $\int_{t_{k-1}}^{t_k} F_h(t_m - s, t_m)\,\mathrm{d}s$, *i.e.*,

$$\begin{aligned}
&\left\|\tau F_{\tau,m}^{m-k} - \int_{t_{k-1}}^{t_k} F_h(t_m - s, t_m)\,\mathrm{d}s\right\| \le \left\|\int_{t_{k-1}}^{t_k} F_{\tau,m}^{m-k} - F_h(t_m - s, t_m)\,\mathrm{d}s\right\| \\
&\quad \le C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}^{\tau}} \mathrm{e}^{z(m-k)\tau}\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1}\left(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha} + A_h(t_m)\right)^{-1}\mathrm{d}z\right. \\
&\qquad \left. - \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{(t_m-s)z} z^{\alpha-1}(z^\alpha + A_h(t_m))^{-1}\,\mathrm{d}z\,\mathrm{d}s\right\| \\
&\quad \le C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}\setminus\Gamma_{\theta,\kappa}^{\tau}} \mathrm{e}^{(t_m-s)z} z^{\alpha-1}(z^\alpha + A_h(t_m))^{-1}\,\mathrm{d}z\,\mathrm{d}s\right\| \\
&\qquad + C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}^{\tau}} \mathrm{e}^{z(t_m-s)}(1-\mathrm{e}^{(s-k\tau)z})\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1}\left(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha} + A_h(t_m)\right)^{-1}\mathrm{d}z\,\mathrm{d}s\right\| \\
&\qquad + C\left\|\int_{t_{k-1}}^{t_k}\int_{\Gamma_{\theta,\kappa}^{\tau}} \mathrm{e}^{z(t_m-s)}\left(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1}\left(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha} + A_h(t_m)\right)^{-1} - z^{\alpha-1}(z^\alpha + A_h(t_m))^{-1}\right)\mathrm{d}z\,\mathrm{d}s\right\| \\
&\quad \le C\tau\int_{t_{k-1}}^{t_k}(t_m - s)^{-1}\,\mathrm{d}s,
\end{aligned}$$

where Lemma 4.1 is used. According to (2.1) and $t_m - t_k \le t_m - s$ for $s \in [t_{k-1}, t_k]$, one has

$$\sum_{k=1}^{m}\mathrm{II}_{2,k} \le C\tau^2\sum_{k=1}^{m}\|G_h(t_k)\|_{\dot{H}^2(\Omega)} \le C\tau\|f(0)\|_{L^2(\Omega)} + C\tau\int_0^{t_m}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

Using Lemma 3.2, (2.1), and Theorem 4.3, one can get

$$\sum_{k=1}^{m}\mathrm{II}_{3,k} \le C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}\|G_h(s)\|_{\dot{H}^2(\Omega)}\,\mathrm{d}s \le C\tau\|f(0)\|_{L^2(\Omega)} + C\tau\int_0^{t_m}\|f'(s)\|_{L^2(\Omega)}\,\mathrm{d}s.$$

Combining Lemma 3.2, (2.1), and Theorem 4.4 results in

$$\sum_{k=1}^{m}\mathrm{II}_{4,k} \le C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_m - s)^{1-\alpha}\left\|\frac{G_h(s) - G_h(t_k)}{\tau}\right\|_{L^2(\Omega)}\mathrm{d}s \le C\tau\|f(0)\|_{L^2(\Omega)} + C\tau\|f'(s)\|_{L^\infty(0,T,L^2(\Omega))}.$$

Thus taking $m = n$ leads to

$$\mathrm{II} \le C\tau\left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))}\right).$$

□

Combining Theorems 4.2 and 4.6, one gets the result.

Theorem 4.7. *Let W_h and W_h^n be the solutions of equations (3.1) and (4.1), respectively. If $W_0 = 0$, $f(0) \in L^2(\Omega)$ and $f'(s) \in L^\infty(0,T,L^2(\Omega))$, then there holds*

$$\|W_h(t_n) - W_h^n\|_{L^2(\Omega)} \le C\tau\left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))}\right). \tag{4.7}$$

4.2. Error estimate for the homogeneous problem

In this subsection, we consider the error between $W_h(t_n)$ and W_h^n, which are the solutions of equations (3.1) and (4.1) with $f = 0$. Similarly, denote $\mathrm{e}_h^n = W_h(t_n) - W_h^n$. Thus

$$\|\mathrm{e}_h^n\|_{L^2(\Omega)} \leq \mathrm{I} + \mathrm{II},$$

where

$$\begin{aligned}
\mathrm{I} &\leq C\left\|F_h(t_n,t_m)W_{0,h} - \frac{1}{2\pi\mathbf{i}}\int_{\zeta=|\xi_\tau|}\zeta^{-n-1}\delta_\tau(\zeta)^{\alpha-1}\left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1}\frac{\zeta}{\tau}W_h^0\,\mathrm{d}\zeta\right\|_{L^2(\Omega)},\\
\mathrm{II} &\leq C\left\|\int_0^{t_n}F_h(t_n-s,t_m)(A_h(t_m)-A_h(s))\,{}_0D_s^{1-\alpha}W_h(s)\,\mathrm{d}s\right.\\
&\quad\left. - \frac{1}{2\pi\mathbf{i}}\int_{\zeta=|\xi_\tau|}\zeta^{-n-1}\delta_\tau(\zeta)^{\alpha-1}\left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1}\left(\sum_{j=1}^{\infty}(A_h(t_m)-A_h(t_j))\sum_{i=0}^{j-1}d_i^{(1-\alpha)}W_h^{j-i}\zeta^j\right)\mathrm{d}\zeta\right\|_{L^2(\Omega)}
\end{aligned}$$

with $\xi_\tau = \mathrm{e}^{-\tau(\kappa+1)}$. Similar to the proof in [15, 26], one can obtain the following estimates.

Theorem 4.8. *If $W_0 \in \dot{H}^\nu(\Omega)$ with $\nu \in (0,2]$, then we have*

$$\mathrm{I} \leq Ct^{\alpha\nu/2-1}\tau\|W_0\|_{\dot{H}^\nu(\Omega)}.$$

As for II, when $t_n = t_m$, it has

$$\begin{aligned}
\mathrm{II} &\leq C\left\|\int_0^{t_m}F_h(t_m-s,t_m)(A_h(t_m)-A_h(s))\,{}_0D_s^{1-\alpha}W_h(s)\,\mathrm{d}s\right.\\
&\quad\left. - \tau\sum_{k=1}^{m}F_{\tau,m}^{m-k}\left((A_h(t_m)-A_h(t_k))\sum_{i=0}^{k-1}d_i^{(1-\alpha)}W_h^{k-i}\right)\right\|_{L^2(\Omega)}\\
&\leq C\left\|-\int_0^{t_m}\frac{\partial}{\partial s}\left(F_h(t_m-s,t_m)(A_h(t_m)-A_h(s))\right){}_0I_s^\alpha W_h(s)\,\mathrm{d}s\right.\\
&\quad\left. - \tau\sum_{k=1}^{m}\left(F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k))\right)\left(\sum_{i=0}^{k-1}d_i^{(-\alpha)}W_h^{k-i} - \sum_{i=0}^{k-2}d_i^{(-\alpha)}W_h^{k-1-i}\right)/\tau\right\|_{L^2(\Omega)}\\
&\leq C\left\|\int_0^{t_m}\frac{\partial}{\partial(t_m-s)}\left(F_h(t_m-s,t_m)(A_h(t_m)-A_h(s))\right){}_0I_s^\alpha W_h(s)\,\mathrm{d}s\right.\\
&\quad\left. - \tau\sum_{k=1}^{m}\left(\sum\left(F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k)) - F_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_{k+1}))\right)/\tau\right)\sum_{i=0}^{k-1}d_i^{(-\alpha)}W_h^{k-i}\right\|_{L^2(\Omega)}\\
&\leq \sum_{k=1}^{m}(\mathrm{II}_{1,k}+\mathrm{II}_{2,k}+\mathrm{II}_{3,k}),
\end{aligned} \tag{4.8}$$

where

$$\mathrm{II}_{1,k} \leq \left\| \left(F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k)) - F_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_{k+1}))\right) \left({}_0I_{t_k}^{\alpha} W_h - \sum_{i=0}^{k-1} d_i^{(-\alpha)} W_h^{k-i}\right) \right\|_{L^2(\Omega)},$$

$$\begin{aligned}\mathrm{II}_{2,k} \leq \Bigg\| \Bigg(&\int_{t_{k-1}}^{t_k} \frac{\partial}{\partial(t_m-s)} \left(F_h(t_m-s,t_m)(A_h(t_m)-A_h(s))\right) \mathrm{d}s \\ &- \left(F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k)) - F_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_{k+1}))\right)\Bigg) {}_0I_{t_k}^{\alpha} W_h \Bigg\|_{L^2(\Omega)},\end{aligned}$$

$$\mathrm{II}_{3,k} \leq \left\| \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial(t_m-s)} \left(F_h(t_m-s,t_m)(A_h(t_m)-A_h(s))\right) \left({}_0I_s^{\alpha} W_h - {}_0I_{t_k}^{\alpha} W_h\right) \mathrm{d}s \right\|_{L^2(\Omega)}.$$

To estimate $\mathrm{II}_{1,k}$, denote $U_h(t_k) = {}_0I_{t_k}^{\alpha} W_h$ and $U_h^k = \sum_{i=0}^{k-1} d_i^{(-\alpha)} W_h^{k-i}$. By means of Laplace transform, U_h can be represented by

$$U_h(t) = H_h(t,t_m)W_h^0 + \int_0^t H_h(t-s,t_m)(A_h(t_m)-A_h(s)) \frac{\partial}{\partial s} U_h(s)\, \mathrm{d}s$$

and

$$\begin{aligned}U_h^n = \sum_{i=0}^{n-1} d_i^{(-\alpha)} W_h^{n-i} &= \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{-1} (\delta_\tau(\zeta)^{\alpha} + A_h(t_m))^{-1} \frac{\zeta}{\tau} W_h^0 \, \mathrm{d}\zeta \\ &+ \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-n-1} \delta_\tau(\zeta)^{-1} (\delta_\tau(\zeta)^{\alpha} + A_h(t_m))^{-1} \left(\sum_{j=1}^{\infty} (A_h(t_m)-A_h(t_j))(U_h^j - U_h^{j-1})\zeta^j/\tau \right) \mathrm{d}\zeta,\end{aligned}$$

where

$$H_h(t,t_m) = \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{zt} z^{-1} (z^{\alpha} + A_h(t_m))^{-1} \, \mathrm{d}z.$$

Similar to the proof of Theorems 2.3 and 4.4, one can get the following estimates of $U_h(t)$.

Theorem 4.9. *Assume $f=0$. If $W_0 \in L^2(\Omega)$, then $U_h(t)$ satisfies*

$$\|U_h(t)\|_{L^2(\Omega)} \leq C\|W_0\|_{L^2(\Omega)}, \quad \|U_h(t)\|_{\dot{H}^2(\Omega)} \leq C\|W_0\|_{L^2(\Omega)}.$$

And if $W_0 \in \dot{H}^{\eta}(\Omega)$, $\eta \in [0,2]$, then it holds

$$\|A_h^{\eta/2} U_h(t)\|_{\dot{H}^2(\Omega)} \leq C\|W_0\|_{\dot{H}^{\eta}(\Omega)}.$$

Theorem 4.10. *Let $U_h = {}_0I_t^{\alpha} W_h(t)$ and $f=0$. When $W_0 \in L^2(\Omega)$, one has*

$$\left\| \frac{U_h(t) - U_h(t-\tau)}{\tau^{\gamma_1}} \right\|_{L^2(\Omega)} \leq C t^{\alpha-\gamma_1} \|W_0\|_{L^2(\Omega)},$$

where $\gamma_1 < 1+\alpha$. And when $W_0 \in \dot{H}^{\nu}(\Omega)$ with $\nu \in (0,2]$, one has

$$\left\| A_h \frac{U_h(t) - U_h(t-\tau)}{\tau^{\gamma_2}} \right\|_{L^2(\Omega)} \leq C t^{\alpha\nu/2-\gamma_2} \|W_0\|_{\dot{H}^{\nu}(\Omega)},$$

where $\gamma_2 \leq 1$.

Then we consider the difference between $U_h(t_k)$ and U_h^k.

Theorem 4.11. *If $W_0 \in L^2(\Omega)$, $f = 0$ and $\frac{1}{a^2(t)} \in C^2[0,T]$, then we have*

$$\|U_h^n - U_h(t_n)\|_{L^2(\Omega)} \leq C t_n^{\alpha-1} \tau \|W_0\|_{L^2(\Omega)}.$$

Proof. Here, for $n = m$, we can split it into

$$\|U_h(t_m) - U_h^m\|_{L^2(\Omega)} \leq \upsilon_{1,h} + \upsilon_{2,h},$$

where

$$\upsilon_{1,h} \leq C \left\| H_h(t_m, t_m) W_{0,h} - \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-m-1} \delta_\tau(\zeta)^{-1} (\delta_\tau(\zeta)^\alpha + A_h(t_m))^{-1} \frac{\zeta}{\tau} W_h^0 \, \mathrm{d}\zeta \right\|_{L^2(\Omega)},$$

$$\begin{aligned} \upsilon_{2,h} \leq C \Bigg\| & \int_0^{t_m} H_h(t_m - s, t_m)(A_h(t_m) - A_h(s)) \frac{\partial}{\partial s} U_h(s) \, \mathrm{d}s \\ & - \frac{1}{2\pi \mathbf{i}} \int_{\zeta=|\xi_\tau|} \zeta^{-m-1} \delta_\tau(\zeta)^{-1} (\delta_\tau(\zeta)^\alpha + A_h(t_m))^{-1} \left(\sum_{j=1}^{\infty} (A_h(t_m) - A_h(t_j))(U_h^j - U_h^{j-1}) \zeta^j / \tau \right) \mathrm{d}\zeta \Bigg\|_{L^2(\Omega)}. \end{aligned}$$

Similar to the proof in [15, 26], the following estimate can be got

$$\upsilon_{1,h} \leq C \tau t_m^{\alpha-1} \|W_0\|_{L^2(\Omega)}.$$

To get the estimate of $\upsilon_{2,h}$, introduce

$$\tau \sum_{i=0}^{\infty} H_{\tau,m}^i \zeta^i = \delta_\tau(\zeta)^{-1} \left(\delta_\tau(\zeta)^\alpha + A_h(t_m)\right)^{-1},$$

where

$$H_{\tau,m}^n = \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{-zn\tau} \delta_\tau(\mathrm{e}^{-z\tau})^{-1} \left(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m)\right)^{-1} \mathrm{d}z.$$

The $\upsilon_{2,h}$ can be divided into the following parts, *i.e.*,

$$\begin{aligned} \upsilon_{2,h} \leq C \Bigg\| & \int_0^{t_m} \frac{\partial}{\partial(t_m - s)} (H_h(t_m - s, t_m)(A_h(t_m) - A_h(s))) U_h(s) \, \mathrm{d}s \\ & - \sum_{j=1}^{m} \left(H_{\tau,m}^{m-j}(A_h(t_m) - A_h(t_j)) - H_{\tau,m}^{m-j-1}(A_h(t_m) - A_h(t_{j+1}))\right) U_h^j \Bigg\|_{L^2(\Omega)} = \sum_{i=1}^{3} \sum_{k=1}^{m} \upsilon_{2,i,k,h}, \end{aligned}$$

where

$$\upsilon_{2,1,k,h} \leq \left\| \left(H_{\tau,m}^{m-k}(A_h(t_m) - A_h(t_k)) - H_{\tau,m}^{m-k-1}(A_h(t_m) - A_h(t_{k+1}))\right) \left(U_h(t_k) - U_h^k\right) \right\|_{L^2(\Omega)},$$

$$\begin{aligned} \upsilon_{2,2,k,h} \leq \Bigg\| \Bigg(& \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial(t_m - s)} (H_h(t_m - s, t_m)(A_h(t_m) - A_h(s))) \, \mathrm{d}s \\ & - \left(H_{\tau,m}^{m-k}(A_h(t_m) - A_h(t_k)) - H_{\tau,m}^{m-k-1}(A_h(t_m) - A_h(t_{k+1}))\right) \Bigg) U_h(t_k) \Bigg\|_{L^2(\Omega)}, \end{aligned}$$

$$\upsilon_{2,3,k,h} \leq \left\| \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial(t_m - s)} (H_h(t_m - s, t_m)(A_h(t_m) - A_h(s))) (U_h(s) - U_h(t_k)) \, \mathrm{d}s \right\|_{L^2(\Omega)}.$$

As for $\upsilon_{2,1,k,h}$, it has

$$\begin{aligned}&\|H_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k))-H_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_{k+1}))\|\\&\quad\le\|H_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k))-H_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_k))\|\\&\qquad+\|H_{\tau,m}^{m-k-1}(A_h(t_k)-A_h(t_{k+1}))\|\le\sigma_{1,k}+\sigma_{2,k}.\end{aligned}$$

Using Lemma 4.1, one can obtain

$$\sigma_{1,k}\le C\tau(t_m-t_k)\left\|\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{(m-k)\tau z}\frac{1-\mathrm{e}^{-z\tau}}{\tau}A_h(t_m)\delta_\tau(\mathrm{e}^{-z\tau})^{-1}(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z\right\|\le C\tau.$$

Similarly,

$$\sigma_{2,k}\le C\tau\left\|\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{(m-k-1)\tau z}A_h(t_m)\delta_\tau(\mathrm{e}^{-z\tau})^{-1}(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z\right\|\le C\tau.$$

Thus

$$\sum_{k=1}^{m}\upsilon_{2,1,k,h}\le C\tau\sum_{k=1}^{m}\|U_h(t_k)-U_h^k\|_{L^2(\Omega)}.$$

As for $\upsilon_{2,2,k,h}$, one can divide it into four parts, *i.e.*,

$$\begin{aligned}\upsilon_{2,2,k,h}\le{}&C\left\|\int_{t_{k-1}}^{t_k}(A_h(t_m)-A_h(s))\left(\frac{\partial}{\partial(t_m-s)}H_h(t_m-s,t_m)-\left(H_{\tau,m}^{m-k}-H_{\tau,m}^{m-k-1}\right)/\tau\right)\,\mathrm{d}s\,U_h(t_k)\right\|_{L^2(\Omega)}\\&+C\left\|\int_{t_{k-1}}^{t_k}(A_h(s)-A_h(t_{k+1}))\left(H_{\tau,m}^{m-k}-H_{\tau,m}^{m-k-1}\right)/\tau\,\mathrm{d}s\,U_h(t_k)\right\|_{L^2(\Omega)}\\&+C\left\|\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial s}A_h(s)\left(H_h(t_m-s,t_m)-H_{\tau,m}^{m-k}\right)\,\mathrm{d}s\,U_h(t_k)\right\|_{L^2(\Omega)}\\&+C\left\|\int_{t_{k-1}}^{t_k}\left(\frac{\partial}{\partial s}A_h(s)-\frac{A_h(t_{k+1})-A_h(t_k)}{\tau}\right)H_{\tau,m}^{m-k}\,\mathrm{d}s\,U_h(t_k)\right\|_{L^2(\Omega)}\le\sum_{i=1}^{4}\vartheta_{i,k}.\end{aligned}$$

For the first part $\vartheta_{1,k}$, using Lemma 4.1, one has the estimate of the difference between $\frac{\partial}{\partial(t_m-s)}H_h(t_m-s,t_m)$ and $\frac{H_{\tau,m}^{m-k}-H_{\tau,m}^{m-k-1}}{\tau}$, *i.e.*,

$$\begin{aligned}&\left\|\frac{\partial}{\partial(t_m-s)}H_h(t_m-s,t_m)-\frac{H_{\tau,m}^{m-k}-H_{\tau,m}^{m-k-1}}{\tau}\right\|\\&\quad\le C\left\|\int_{\Gamma_{\theta,\kappa}}\mathrm{e}^{z(t_m-s)}(z^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z-\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-t_k)}(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z\right\|\\&\quad\le C\tau(t_m-s)^{\alpha-2},\end{aligned}$$

which yields

$$\sum_{k=1}^{m}\vartheta_{1,k}\le C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_m-s)^{\alpha-1}\|U_h(t_k)\|_{\dot H^2(\Omega)}\,\mathrm{d}s.$$

Similarly, one has

$$\left\|\frac{H_{\tau,m}^{m-k}-H_{\tau,m}^{m-k-1}}{\tau}\right\|\le C\left\|\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-t_{k-1})}\mathrm{e}^{-z\tau}(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z\right\|\le C(t_m-s)^{\alpha-1}.$$

Therefore one has

$$\sum_{k=1}^{m}\vartheta_{2,k}\leq C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_m-s)^{\alpha-1}\|U_h(t_k)\|_{\dot H^2(\Omega)}\,\mathrm{d}s.$$

Moreover, according to Lemma 4.1, there is

$$\begin{aligned}&\left\|H_h(t_m-s,t_m)-H_{\tau,m}^{m-k}\right\|\\ &\leq C\left\|\int_{\Gamma_{\theta,\kappa}}\mathrm{e}^{z(t_m-s)}z^{-1}(z^\alpha+A_h(t_m))^{-1}\,\mathrm{d}z-\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-t_k)}\delta_\tau(\mathrm{e}^{-z\tau})^{-1}(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m))^{-1}\,\mathrm{d}z\right\|\\ &\leq C\tau(t_m-s)^{\alpha-1},\end{aligned}$$

which leads to

$$\sum_{k=1}^{m}\vartheta_{3,k}\leq C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_m-s)^{\alpha-1}\|U_h(t_k)\|_{\dot H^2(\Omega)}\,\mathrm{d}s.$$

On the other hand, according to Lemma 4.1, one has

$$\left\|H_{\tau,m}^{m-k}\right\|\leq C\left\|\int_{\Gamma_{\theta,\kappa}^{\tau}}\mathrm{e}^{z(t_m-t_k)}\delta_\tau(\mathrm{e}^{-z\tau})^{-1}(\delta_\tau(\mathrm{e}^{-z\tau})^\alpha+A_h(t_m))^{-1}\,\mathrm{d}z\right\|\leq C(t_m-t_k)^{\alpha}.$$

Combining $\frac{1}{a^2(t)}\in C^2[0,T]$ leads to

$$\sum_{k=1}^{m}\vartheta_{4,k}\leq C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_m-t_k)^{\alpha}\|U_h(t_k)\|_{\dot H^2(\Omega)}\,\mathrm{d}s. \tag{4.9}$$

The estimate (4.9) together with Theorem 4.9 yields that

$$\sum_{k=1}^{m}\upsilon_{2,2,k,h}\leq C\tau\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_m-s)^{\alpha-1}\|U_h(t_k)\|_{\dot H^2(\Omega)}\,\mathrm{d}s\leq C\tau\|W_0\|_{L^2(\Omega)}.$$

Using the condition $\frac{1}{a^2(t)}\in C^2[0,T]$, one can bound $\upsilon_{2,3,k,h}$ by

$$\begin{aligned}\upsilon_{2,3,k,h}\leq{}& C\left\|\int_{t_{k-1}}^{t_k}(A_h(t_m)-A_h(s))\frac{\partial}{\partial(t_m-s)}\left(H_h(t_m-s,t_m)\right)\left(U_h(s)-U_h(t_k)\right)\mathrm{d}s\right\|_{L^2(\Omega)}\\ &+C\left\|\int_{t_{k-1}}^{t_k}H_h(t_m-s,t_m)\frac{\partial}{\partial s}\left(A_h(t_m)-A_h(s)\right)\left(U_h(s)-U_h(t_k)\right)\mathrm{d}s\right\|_{L^2(\Omega)}\\ \leq{}& C\tau\int_{t_{k-1}}^{t_k}\left\|\frac{U_h(s)-U_h(t_k)}{\tau}\right\|_{L^2(\Omega)}\mathrm{d}s.\end{aligned}$$

According to Theorem 4.10, one has

$$\sum_{k=1}^{m}\upsilon_{2,3,k,h}\leq C\tau\|W_0\|_{L^2(\Omega)}.$$

Thus using discrete Grönwall inequality and taking $m=n$ result in

$$\|U_h^n-U_h(t_n)\|_{L^2(\Omega)}\leq Ct_n^{\alpha-1}\tau\|W_0\|_{L^2(\Omega)}.$$

□

Next consider the estimate of II defined in (4.8).

Theorem 4.12. *If $W_0 \in L^2(\Omega)$, $f=0$ and $\frac{1}{a^2(t)} \in C^2[0,T]$, then one has*

$$\sum_{k=1}^{m} \mathrm{II}_{1,k} \le C\tau \|W_0\|_{L^2(\Omega)},$$

where $\mathrm{II}_{1,k}$ is defined in (4.8).

Proof. By triangle inequality, we can divide it into two parts, *i.e.*,

$$\begin{aligned}&\|F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k)) - F_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_{k+1}))\|\\ &\quad\le \|F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k)) - F_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_k))\|\\ &\qquad+\|F_{\tau,m}^{m-k-1}(A_h(t_k)-A_h(t_{k+1}))\| \le \varrho_{1,k}+\varrho_{2,k}.\end{aligned}$$

The fact $|\frac{1-\mathrm{e}^{-z\tau}}{\tau}| \le C|z|$ and Lemma 4.1 show

$$\varrho_{1,k} \le C\tau(t_m-t_k)\left\|\int_{\Gamma_{\theta,\kappa}^{\tau}} \mathrm{e}^{(m-k)\tau z}\frac{\mathrm{e}^{-z\tau}-1}{\tau}A_h(t_m)\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1}(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z\right\| \le C(t_m-t_{k-1})^{-\alpha}\tau.$$

Similarly

$$\varrho_{2,k} \le C\tau\left\|\int_{\Gamma_{\theta,\kappa}^{\tau}} \mathrm{e}^{(m-k-1)\tau z}A_h(t_m)\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1}(\delta_\tau(\mathrm{e}^{-z\tau})^{\alpha}+A_h(t_m))^{-1}\,\mathrm{d}z\right\| \le C(t_m-t_{k-1})^{-\alpha}\tau.$$

Thus

$$\sum_{k=1}^{m} \mathrm{II}_{1,k} \le C\tau \|W_0\|_{L^2(\Omega)}.$$

□

Theorem 4.13. *If $W_0 \in \dot{H}^{\nu}(\Omega)$, $\nu \in (0,2]$, $f=0$ and $\frac{1}{a^2(t)} \in C^2[0,T]$, then one obtains*

$$\sum_{k=1}^{m} \mathrm{II}_{2,k} \le C\tau \|W_0\|_{\dot{H}^{\nu}(\Omega)},$$

where $\mathrm{II}_{2,k}$ is defined in (4.8).

Proof. By triangle inequality, it holds

$$\begin{aligned}\mathrm{II}_{2,k} \le{}& C\Bigg\|\Bigg(\int_{t_{k-1}}^{t_k}\frac{\partial}{\partial(t_m-s)}\left(F_h(t_m-s,t_m)(A_h(t_m)-A_h(s))\right)\mathrm{d}s\\ &-\left(F_{\tau,m}^{m-k}(A_h(t_m)-A_h(t_k))-F_{\tau,m}^{m-k-1}(A_h(t_m)-A_h(t_{k+1}))\right)\Bigg)U_h(t_k)\Bigg\|_{L^2(\Omega)}\\ \le{}& C\left\|\int_{t_{k-1}}^{t_k}(A_h(t_m)-A_h(s))\left(\frac{\partial}{\partial(t_m-s)}F_h(t_m-s,t_m)-\left(F_{\tau,m}^{m-k}-F_{\tau,m}^{m-k-1}\right)/\tau\right)\mathrm{d}s\,U_h(t_k)\right\|_{L^2(\Omega)}\\ &+C\left\|\int_{t_{k-1}}^{t_k}(A_h(s)-A_h(t_{k+1}))\left(F_{\tau,m}^{m-k}-F_{\tau,m}^{m-k-1}\right)/\tau\,\mathrm{d}s\,U_h(t_k)\right\|_{L^2(\Omega)}\end{aligned}$$

$$+ C \left\| \int_{t_{k-1}}^{t_k} \frac{\partial}{\partial s} A_h(s) \left(F_h(t_m - s, t_m) - F_{\tau,m}^{m-k} \right) \mathrm{d}s \, U_h(t_k) \right\|_{L^2(\Omega)}$$

$$+ C \left\| \int_{t_{k-1}}^{t_k} \left(\frac{\partial}{\partial s} A_h(s) - \frac{A_h(t_k) - A_h(t_{k+1})}{\tau} \right) F_{\tau,m}^{m-k} \, \mathrm{d}s \, U_h(t_k) \right\|_{L^2(\Omega)} \leq \sum_{i=1}^{4} \ell_{i,k}.$$

We can split $\ell_{1,k}$ into two parts

$$\begin{aligned}
\ell_{1,k} \leq C \Bigg\| & \int_{t_{k-1}}^{t_k} (A_h(t_m) - A_h(s)) \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_m - s)} A_h(t_m)^{1-\nu/2} (z^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z \, A_h(t_m)^{\nu/2} U_h(t_k) \, \mathrm{d}s \\
& - \int_{t_{k-1}}^{t_k} (A_h(t_m) - A_h(s)) \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_k)} A_h(t_m)^{1-\nu/2} (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z \, A_h(t_m)^{\nu/2} U_h(t_k) \, \mathrm{d}s \Bigg\|_{L^2(\Omega)} \\
& + C \left\| \int_{t_{k-1}}^{t_k} (A(t_m) - A(s)) \left(\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_m - s)} \mathbf{I} \, \mathrm{d}z - \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_k)} \mathbf{I} \, \mathrm{d}z \right) \mathrm{d}s \, U(t_k) \right\|_{L^2(\Omega)} \leq \ell_{1,k,1} + \ell_{1,k,2}.
\end{aligned}$$

Simple calculations lead to

$$\ell_{1,k,2} \leq C \left\| \int_{t_{k-1}}^{t_k} (A(t_m) - A(s)) \left(\boldsymbol{\delta}(t_m - s) - \bar{\boldsymbol{\delta}}(t_m - t_k)/\tau \right) \mathrm{d}s \, U(t_k) \right\|_{L^2(\Omega)} \leq C \bar{\boldsymbol{\delta}}(m-k)\tau \|U(t_m)\|_{\dot{H}^2(\Omega)},$$

where $\boldsymbol{\delta}$ denotes the Dirac function and

$$\bar{\boldsymbol{\delta}}(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 3.2, one has

$$\begin{aligned}
& \left\| \int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_m - s)} A_h(t_m)^{1-\nu/2} (z^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z - \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_k)} A_h(t_m)^{1-\nu/2} (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z \right\| \\
& \quad \leq C \left\| \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - s)} A_h(t_m)^{1-\nu/2} (z^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z \right\| \\
& \qquad + C \left\| \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - s)} (1 - \mathrm{e}^{z(s - t_k)}) A_h(t_m)^{1-\nu/2} (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z \right\| \\
& \qquad + C \left\| \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - s)} A_h(t_m)^{1-\nu/2} ((z^\alpha + A_h(t_m))^{-1} - (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1}) \, \mathrm{d}z \right\| \\
& \quad \leq C\tau (t_m - s)^{\alpha\nu/2 - 2},
\end{aligned}$$

which leads to

$$\sum_{k=1}^{m} \ell_{1,k} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha\nu/2 - 1} \|A_h(t_m)^{\nu/2} U_h(t_k)\|_{\dot{H}^2(\Omega)} \, \mathrm{d}s + C\tau \|U(t_m)\|_{\dot{H}^2(\Omega)}.$$

Similarly,

$$\begin{aligned}
\left\| A_h(t_m)^{-\nu/2} \frac{F_{\tau,m}^{m-k} - F_{\tau,m}^{m-k-1}}{\tau} \right\| & \leq \left\| \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_{k-1})} \mathrm{e}^{-z\tau} A_h(t_m)^{-\nu/2} \delta_\tau(\mathrm{e}^{-z\tau})^\alpha (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1} \, \mathrm{d}z \right\| \\
& \leq C(t_m - t_{k-1})^{\alpha\nu/2 - 1}.
\end{aligned}$$

Therefore $\sum_{k=1}^{m} \ell_{2,k}$ can be bounded as

$$\sum_{k=1}^{m} \ell_{2,k} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha\nu/2-1} \|A_h(t_m)^{\nu/2} U_h(t_k)\|_{\dot{H}^2(\Omega)} \,\mathrm{d}s.$$

On the other hand, one can get

$$\begin{aligned}
&\left\| \int_{t_{k-1}}^{t_k} A_h(t_m)^{-\nu/2} F_h(t_m - s, t_m) - A_h(t_m)^{-\nu/2} F_{\tau,m}^{m-k} \,\mathrm{d}s \right\| \\
&\quad \leq C \left\| \int_{t_{k-1}}^{t_k} \left(\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_m - s)} A_h(t_m)^{1-\nu/2} z^{-1} (z^\alpha + A_h(t_m))^{-1} \,\mathrm{d}z \right.\right. \\
&\qquad \left.\left. - \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_k)} A_h(t_m)^{1-\nu/2} \delta_\tau(\mathrm{e}^{-z\tau})^{-1} (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1} \,\mathrm{d}z \right) \mathrm{d}s \right\| \\
&\qquad + C \left\| \int_{t_{k-1}}^{t_k} A_h(t_m)^{-\nu/2} \left(\int_{\Gamma_{\theta,\kappa}} \mathrm{e}^{z(t_m - s)} z^{-1} \,\mathrm{d}z - \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_k)} \delta_\tau(\mathrm{e}^{-z\tau})^{-1} \,\mathrm{d}z \right) \mathrm{d}s \right\| \\
&\quad \leq C\tau \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha\nu/2-1} \,\mathrm{d}s,
\end{aligned}$$

which leads to

$$\sum_{k=1}^{m} \ell_{3,k} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha\nu/2-1} \|A_h(t_m)^{\nu/2} U_h(t_k)\|_{\dot{H}^2(\Omega)} \,\mathrm{d}s.$$

Next, using

$$\|F_{\tau,m}^{m-k}\| \leq C \left\| \int_{\Gamma_{\theta,\kappa}^\tau} \mathrm{e}^{z(t_m - t_k)} \delta_\tau(\mathrm{e}^{-z\tau})^{\alpha-1} (\delta_\tau(\mathrm{e}^{-z\tau})^\alpha + A_h(t_m))^{-1} \,\mathrm{d}z \right\| \leq C,$$

one has

$$\sum_{k=1}^{m} \ell_{4,k} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \|U_h(t_k)\|_{\dot{H}^2(\Omega)} \,\mathrm{d}s.$$

Thus, by Lemma 4.9, it has

$$\sum_{k=1}^{m} \mathrm{II}_{2,k} \leq C\tau \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_m - s)^{\alpha\nu/2-1} \|A_h(t_m)^{\nu/2} U_h(t_k)\|_{\dot{H}^2(\Omega)} \,\mathrm{d}s \leq C\tau \|W_0\|_{\dot{H}^\nu(\Omega)}.$$

□

Theorem 4.14. *If $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, $f = 0$ and $\frac{1}{a^2(t)} \in C^2[0, T]$, then it holds*

$$\sum_{k=1}^{m} \mathrm{II}_{3,k} \leq C\tau \|W_0\|_{\dot{H}^\nu(\Omega)},$$

where $\mathrm{II}_{3,k}$ is defined in (4.8).

Proof. According to Theorem 4.10, $\mathrm{II}_{3,k}$ can be bounded as

$$\begin{aligned}
\mathrm{II}_{3,k} \leq & \left\| \int_{t_{k-1}}^{t_k} (A_h(t_m) - A_h(s)) \frac{\partial}{\partial(t_m - s)} \left(F_h(t_m - s, t_m)\right) \left(U_h(s) - U_h(t_k)\right) \mathrm{d}s \right\|_{L^2(\Omega)} \\
& + \left\| \int_{t_{k-1}}^{t_k} F_h(t_m - s, t_m) \frac{\partial}{\partial(t_m - s)} \left(A_h(t_m) - A_h(s)\right) \left(U_h(s) - U_h(t_k)\right) \mathrm{d}s \right\|_{L^2(\Omega)} \\
\leq & \, C\tau \int_{t_{k-1}}^{t_k} \left\| A_h(t_m) \frac{U_h(s) - U_h(t_k)}{\tau} \right\|_{L^2(\Omega)} \mathrm{d}s \\
\leq & \, C\tau \int_{t_{k-1}}^{t_k} s^{\alpha\nu/2-1} \|W_0\|_{\dot{H}^\nu(\Omega)} \, \mathrm{d}s.
\end{aligned}$$

Summing k from 1 to n leads to the desired estimate. □

Thus the error estimate of the fully discrete scheme when $f = 0$ is obtained.

Theorem 4.15. *Let W_h and W_h^n be the solutions of equations (3.1) and (4.1), respectively. If $\frac{1}{a^2(t)} \in C^2[0, T]$, $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, and $f = 0$, then there holds*

$$\|W_h(t_n) - W_h^n\| \leq C\tau t_n^{\alpha\nu/2-1} \|W_0\|_{\dot{H}^\nu(\Omega)}. \tag{4.10}$$

According to Theorems 4.7 and 4.15, the following error estimate can be got.

Theorem 4.16. *Let W_h and W_h^n be the solutions of equations (3.1) and (4.1), respectively. If $\frac{1}{a^2(t)} \in C^2[0, T]$, $W_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$, $f(0) \in L^2(\Omega)$, and $f'(s) \in L^\infty(0, T, L^2(\Omega))$, then we have*

$$\|W_h(t_n) - W_h^n\| \leq C\tau t_n^{\alpha\nu/2-1} \|W_0\|_{\dot{H}^\nu(\Omega)} + C\tau \left(\|f(0)\|_{L^2(\Omega)} + \|f'(s)\|_{L^\infty(0,T,L^2(\Omega))} \right).$$

Remark 4.17. The stability of the scheme (4.1) can be got by changing the initial condition and source term as round-off errors of initial value and source term, respectively, in the proofs of Theorems 4.7 and 4.15.

5. Numerical experiments

In this section, we first perform a numerical experiment with known explicit solution to verify the effectiveness of our numerical scheme. Then three numerical experiments with unknown explicit solutions are presented to show the convergence with nonsmooth initial values and source terms. At last, we apply our numerical scheme to a specific physical model and calculate the second-order moment of the particles which are similar to the results in [19].

For the known explicit solution, the spatial and temporal errors can be measured by

$$E_h = \|W_h^n - W(t_n)\|_{L^2(\Omega)}, \quad E_\tau = \|W_\tau^n - W(t_n)\|_{L^2(\Omega)},$$

where W_h^n and W_τ^n mean the numerical solution of W at time t_n with mesh size h and time step size τ, respectively. As for the unknown explicit solutions, the spatial and temporal errors will be measured by

$$E_h = \|W_h^n - W_{h/2}^n\|_{L^2(\Omega)}, \quad E_\tau = \|W_\tau^n - W_{\tau/2}^n\|_{L^2(\Omega)}.$$

The corresponding convergence rates can be calculated by

$$\text{Rate} = \frac{\ln(E_h/E_{h/2})}{\ln(2)} \quad \text{and} \quad \text{Rate} = \frac{\ln(E_\tau/E_{\tau/2})}{\ln(2)}. \tag{5.1}$$

TABLE 1. Temporal errors and convergence rates at $T = 1$.

$\alpha \backslash L$	50	100	200	400	800
0.3	2.288E-04	1.140E-04	5.675E-05	2.815E-05	1.387E-05
	Rate	1.0049	1.0064	1.0112	1.0217
0.7	2.083E-04	1.040E-04	5.176E-05	2.565E-05	1.259E-05
	Rate	1.0024	1.0062	1.0131	1.0268

TABLE 2. Spatial errors and convergence rates at $T = 0.5$.

$\alpha \backslash h$	1/4	1/8	1/16	1/32	1/64
0.3	6.403E-04	1.672E-04	4.117E-05	8.799E-06	2.152E-06
	Rate	1.9369	2.0223	2.2260	2.0319
0.7	7.271E-04	1.885E-04	4.628E-05	9.922E-06	2.194E-06
	Rate	1.9476	2.0260	2.2218	2.1772

TABLE 3. Temporal errors and convergence rates for inhomogeneous problem at $T = 1$.

$\alpha \backslash L$	50	100	200	400	800
0.3	1.539E-04	7.809E-05	3.931E-05	1.972E-05	9.872E-06
	Rate	0.9790	0.9904	0.9955	0.9979
0.7	5.555E-05	2.816E-05	1.417E-05	7.105E-06	3.557E-06
	Rate	0.9801	0.9909	0.9958	0.9980

Example 5.1. Here, in (1.1), we take

$$\frac{1}{a^2(t)} = t^{2-\alpha}, \quad \Omega = (0,1),$$
$$W_0(x) = 0, \qquad f(x,t) = x(x-1) - \frac{2t^2}{\Gamma(1+\alpha)}.$$

Thus $W(x,t) = tx(x-1)$ solves (1.1). Table 1 shows the errors and convergence rates with $T = 1$, $h = 1/512$, $\alpha = 0.3$ and 0.7, which agree with Theorem 4.7. Table 2 shows the errors and convergence rates with $T = 0.5$, $\tau = 1/4000$, $\alpha = 0.3$ and 0.7, which validate Theorem 3.9.

Example 5.2. Here we consider temporal convergence rates for inhomogeneous problem (1.1), *i.e.*, Theorem 4.7. Let

$$\frac{1}{a^2(t)} = t^{1.01}, \quad f(x,t) = t^{1.1}\chi_{[0,1/2]}, \quad \Omega = (0,1),$$

where $\chi_{[a,b]}$ denotes the characteristic function on $[a,b]$. To investigate the convergence in time and eliminate the influence from spatial discretization, we set $h = 1/128$. Table 3 shows the errors and convergence rates at terminal time $T = 1$ with $\alpha = 0.3$ and 0.7. To verify that the constant C in estimate (4.7) is independent of the terminal time T, we also present the errors $\max_{0\le n\le T/\tau} \|W_h(t_n) - W_h^n\|_{L^2(\Omega)}$ and the corresponding convergence rates when $T = 0.1$, $\alpha = 0.1$ and 0.9 in Table 4. The results shown in Tables 3 and 4 validate Theorem 4.7.

TABLE 4. Maximum temporal errors and convergence rates for inhomogeneous problem at $T = 0.1$.

$\alpha\backslash L$	50	100	200	400	800
0.1	1.716E-06	8.595E-07	4.301E-07	2.151E-07	1.076E-07
	Rate	0.9973	0.9988	0.9995	0.9998
0.9	1.289E-04	6.451E-05	3.226E-05	1.613E-05	8.067E-06
	Rate	0.9990	0.9996	0.9998	0.9999

TABLE 5. Temporal errors and convergence rates for homogeneous problem at $T = 1$.

$\alpha\backslash L$	50	100	200	400	800
0.4	8.319E-03	4.193E-03	1.997E-03	9.421E-04	4.534E-04
	Rate	0.9885	1.0705	1.0835	1.0552
0.6	3.802E-03	1.873E-03	9.194E-04	4.542E-04	2.256E-04
	Rate	1.0217	1.0262	1.0172	1.0095

TABLE 6. Maximum errors and convergence rates for homogeneous problem at $T = 0.1$.

$\alpha\backslash L$	50	100	200	400	800
0.1	4.595E-04	1.639E-04	6.057E-05	2.461E-05	1.089E-05
	Rate	1.4869	1.4363	1.2992	1.1762
0.9	8.986E-05	4.500E-05	2.252E-05	1.126E-05	5.634E-06
	Rate	0.9976	0.9988	0.9994	0.9997

Example 5.3. Here we validate temporal convergence rates for homogeneous problem (1.1). To satisfy the condition provided in Theorem 4.15, we take

$$\frac{1}{a^2(t)} = t^{2.01}.$$

Set $T = 1$, $\Omega = (0, 1)$ and

$$W_0(x) = \chi_{(1/2,1]}. \tag{5.2}$$

We take small spatial mesh size $h = 1/128$ so that the spatial discretization error is relatively negligible. The corresponding results are shown in Table 5. According to (4.10) of Theorem 4.15, we have

$$t_n^{1-\alpha\nu/2}\|W_h(t_n) - W_h^n\| \le C\tau\|W_0\|_{\dot{H}^\nu(\Omega)}.$$

So, to verify that the constant C in the above estimate is independent of the terminal time T, we choose $\max_{0\le n\le T/\tau}(n\tau)^{1-\alpha\nu/2}\|W_h(t_n) - W_h^n\|_{L^2(\Omega)}$ to measure errors, where $\nu = 1/2$, $T = 0.1$, $\alpha = 0.1$ or $\alpha = 0.9$. According to (5.2), we have $W_0 \in H^{1/2-\varepsilon}(\Omega)$ with arbitrary small $\varepsilon > 0$. The numerical results are shown in Table 6. All results in Tables 5 and 6 agree with the predictions of Theorem 4.15.

Example 5.4. Here, we take

$$\frac{1}{a^2(t)} = 10t^{1.01}, \quad \Omega = (0, 1), \quad W_0(x) = \chi_{(1/2,1]}, \quad f(x, t) = t^{0.1}\chi_{[0,1/2]}$$

to verify the spatial convergence rates. Here we choose $T = 2$ and $\tau = 1/1000$. Table 7 shows the errors and convergence rates, which agree with the predictions of Theorem 3.9.

TABLE 7. Spatial errors and convergence rates at $T = 2$.

$\alpha \backslash h$	1/32	1/64	1/128	1/256	1/512
0.2	9.828E-04	2.483E-04	6.224E-05	1.557E-05	3.893E-06
	Rate	1.9848	1.9962	1.9990	1.9998
0.7	1.196E-04	3.341E-05	8.675E-06	2.192E-06	5.494E-07
	Rate	1.8395	1.9453	1.9849	1.9961

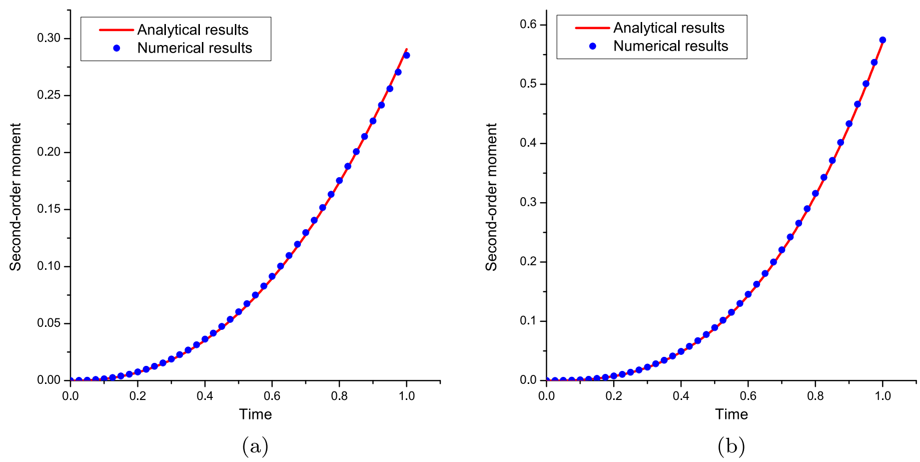

FIGURE 1. Evolution of the second-order moment with time. (a) $\alpha = 0.3$. (b) $\alpha = 0.7$.

Example 5.5. Second-order moment is one of the most important statistical observables to measure the diffusion velocity of particles. Here we simulate the time evolution of the second-moment of the stochastic process given in [8] by solving its Fokker–Planck equation (1.1) with $\alpha = 0.3$, 0.7, and

$$\begin{aligned}\frac{1}{a^2(t)} &= t^2,\\ W_0(x) &= \boldsymbol{\delta}(x),\\ f(x,t) &= 0.\end{aligned}$$

To ensure the accuracy of the simulation, we choose $\Omega = (-5, 5)$, $\tau = 1/100$, and $h = 1/200$. According to [19], the second-order moment has the analytical form

$$\langle x^2(t) \rangle = \frac{2}{2+\alpha} \frac{t^{2+\alpha}}{\Gamma(\alpha)}. \tag{5.3}$$

Both Figures 1a and 1b show that the simulation and analytical results are consistent.

6. CONCLUSIONS

The model describing anomalous diffusion in expanding media is with variable coefficient. The finite element method and backward Euler convolution quadrature are respectively used to approximate the Laplace operator and Riemann–Liouville fractional derivative. We first derive the priori estimate of the solution, and then present the error estimates of the space semi-discrete and the fully discrete schemes. The extensive numerical experiments validate the effectiveness of the numerical schemes.

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References

[1] A. Adams and J.F. Fournier, Sobolev Spaces. Academic Press (2003).
[2] A.A. Alikhanov, A new difference scheme for the time fractional diffusion equation. *J. Comput. Phys.* **280** (2015) 424–438.
[3] E. Barkai, Fractional Fokker–Planck equation, solution, and application. *Phys. Rev. E* **63** (2001) 046118.
[4] E. Barkai, R. Metzler and J. Klafter, From continuous time random walks to the fractional Fokker–Planck equation. *Phys. Rev. E* **61** (2000) 132–138.
[5] E. Bazhlekova, B.T. Jin, R. Lazarov and Z. Zhou, An analysis of the Rayleigh–Stokes problem for a generalized second-grade fluid. *Numer. Math.* **131** (2015) 1–31.
[6] S. Chen, F. Liu, P. Zhuang and V. Anh, Finite difference approximations for the fractional Fokker–Planck equation. *Appl. Math. Model.* **33** (2009) 256–273.
[7] Y. Chen, X.D. Wang and W.H. Deng, Langevin dynamics for Lévy walk with memory. *Phys. Rev. E* **99** (2019) 012135.
[8] Y. Chen, X.D. Wang and W.H. Deng, Subdiffusion in an external force field. *Phys. Rev. E* **99** (2019) 042125.
[9] W.H. Deng, Finite element method for the space and time fractional Fokker–Planck equation. *SIAM J. Numer. Anal.* **47** (2009) 204–226.
[10] W.H. Deng and J.S. Hesthaven, Local discontinuous Galerkin methods for fractional diffusion equations. *ESAIM: M2AN* **47** (2013) 1845–1864.
[11] M. Gunzburger, B.Y. Li and J.L. Wang, Sharp convergence rates of time discretization for stochastic time-fractional PDEs subject to additive space-time white noise. *Math. Comput.* **88** (2018) 1715–1741.
[12] B.T. Jin, R. Lazarov and Z. Zhou, Error estimates for a semidiscrete finite element method for fractional order parabolic equations. *SIAM J. Numer. Anal.* **51** (2013) 445–466.
[13] B.T. Jin, R. Lazarov, J. Pasciak and Z. Zhou, Error analysis of a finite element method for the space-fractional parabolic equation. *SIAM J. Numer. Anal.* **52** (2014) 2272–2294.
[14] B.T. Jin, R. Lazarov, J. Pasciak and Z. Zhou, Error analysis of semidiscrete finite element methods for inhomogeneous time-fractional diffusion. *IMA J. Numer. Anal.* **35** (2015) 561–582.
[15] B.T. Jin, R. Lazarov and Z. Zhou, Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. *SIAM J. Sci. Comput.* **38** (2016) A146–A170.
[16] B.T. Jin, B.Y. Li and Z. Zhou, Correction of high-order BDF convolution quadrature for fractional evolution equations. *SIAM J. Sci. Comput.* **39** (2017) A3129–A3152.
[17] B.T. Jin, B.Y. Li and Z. Zhou, Subdiffusion with a time-dependent coefficient: analysis and numerical solution. *Math. Comput.* **88** (2019) 2157–2186.
[18] C.E.S. Larsson, Error estimates with smooth and nonsmooth data for a finite element method for the Cahn–Hilliard equation. *Math. Comput.* **58** (1992) 603–630.
[19] F. Le Vot, E. Abad and S.B. Yuste, Continuous-time random-walk model for anomalous diffusion in expanding media. *Phys. Rev. E* **96** (2017) 032117.
[20] F. Le Vot and S.B. Yuste, Continuous-time random walks and Fokker–Planck equation in expanding media. *Phys. Rev. E* **98** (2018) 042117.
[21] F. Le Vot, S.B. Yuste and E. Abad, Standard and fractional Ornstein–Uhlenbeck process on a growing domain. *Phys. Rev. E* **100** (2019) 012142.
[22] X.J. Li and C.J. Xu, A space-time spectral method for the time fractional diffusion equation. *SIAM J. Numer. Anal.* **47** (2009) 2108–2131.
[23] Y.M. Lin and C.J. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation. *J. Comput. Phys.* **225** (2007) 1533–1552.
[24] C. Lubich, Convolution quadrature and discretized operational calculus I. *Numer. Math.* **52** (1988) 129–145.
[25] C. Lubich, Convolution quadrature and discretized operational calculus II. *Numer. Math.* **52** (1988) 413–425.
[26] C. Lubich, I.H. Sloan and V. Thome, Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comput.* **65** (1996) 1–17.
[27] K. Mustapha, FEM for time-fractional diffusion equations, novel optimal error analyses. *Math. Comput.* **87** (2018) 2259–2272.
[28] I. Podlubny, Fractional Differential Equations. Academic Press (1999).
[29] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag (2006).
[30] P.B. Xu and W.H. Deng, Fractional compound Poisson processes with multiple internal states. *MMNP* **13** (2018) 10.
[31] S.B. Yuste, Weighted average finite difference methods for fractional diffusion equations. *J. Comput. Phys.* **216** (2006) 264–274.
[32] S.B. Yuste, E. Abad and C. Escudero, Diffusion in an expanding medium: Fokker–Planck equation, Green's function, and first-passage properties. *Phys. Rev. E* **94** (2016) 032118.
[33] F.H. Zeng, I. Turner and K. Burrage, A stable fast time-stepping method for fractional integral and derivative operators. *J. Sci. Comput.* **77** (2018) 283–307.