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HIGH-ORDER GALERKIN METHOD FOR HELMHOLTZ AND LAPLACE PROBLEMS ON MULTIPLE OPEN ARCS

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Abstract. We present a spectral Galerkin numerical scheme for solving Helmholtz and Laplace problems with Dirichlet boundary conditions on a finite collection of open arcs in two-dimensional space. A boundary integral method is employed, giving rise to a first kind Fredholm equation whose variational form is discretized using weighted Chebyshev polynomials. Well-posedness of the discrete problems is established as well as algebraic or even exponential convergence rates depending on the regularities of both arcs and excitations. Our numerical experiments show the robustness of the method with respect to number of arcs and large wavenumber range. Moreover, we present a suitable compression algorithm that further accelerates computational times.

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1. Introduction

We present a spectral Galerkin method for solving weakly singular boundary integral equations (BIEs) arising from Laplace or Helmholtz Dirichlet problems on unbounded domains with boundaries composed of finite collections of disjoints finite open arcs in $\mathbb{R}^2$. Such problems are of particular interest in multiple contexts: in structural and mechanical engineering, wherein fractures or cracks are represented as slits [5, 24, 35, 36]; in the detection of micro-fractures [1, 3] and even for the imaging of muscular strains due to sport injuries [38]. For these applications, one is interested in developing a numerical scheme that can robustly deal with large numbers of arcs – from tens to thousands – for a broad range of wavelengths – ranging from zero to several hundred times the length of the arcs.

For a single arc, Well-posedness of these problems was studied in [34]. Here, we only perform minor extensions to ensure uniqueness and existence of solutions for the multiple arcs case. In particular, volume solutions are shown to be constructed as superpositions of single layer potentials applied to surface densities over each arc; these layer densities are derived from solving a system of BIEs. Numerical approximations of these boundary unknowns are traditionally obtained *via* either variational methods such as the boundary element method (BEM) [32] or Nyström-type strategies [6, 9]. In this work, we opt for the former.

Keywords and phrases. Boundary integral equations, spectral methods, wave scattering problems, screens problems, non-Lipschitz domains.

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Still, for the type of applications considered, several issues hinder the standard low-order BEM performance. On one hand, solutions at the continuous level are well known to exhibit square-root singularities at the arcs' endpoints [8, 13, 23]. Consequently, convergence of low-order uniform-mesh discretizations is suboptimal with improvements relying on either graded [39] or adaptive mesh refinement [10], or on augmenting the approximation space [34]. Also, the Galerkin matrices derived from first kind Fredholm formulations are intrinsically ill-conditioned, thus heavily requiring preconditioning [15, 27]. Moreover, the minimal number of unknowns to ensure asymptotic convergence increases with the wavenumber [28] while the number of matrix entries grow quadratically with the number of arcs in order to account for cross-interactions. Hence, for the present problems of interest, one can expect extremely large numbers of degrees of freedom (dofs) when using mesh-dependent methods and alternative ones must be sought.

In [4, 21] a spectral Galerkin–Bubnov discretization for a single arc was shown to greatly reduce the number of dofs in comparison to the case of locally defined low-order bases. Specifically, the approximation basis employed is given by weighted first kind Chebyshev polynomials, where the weight mimics the singular behavior at the endpoints. Our work expands the use of such bases to multiple arcs and Helmholtz cases providing also a rigorous convergence analysis. The analysis presented here is based in the asymptotic decay of the Fourier–Chebyshev expansions coefficients of the solutions. With these tools, one can derive convergence rates for order p polynomial approximations that only depend on the smoothness of excitations and arcs, with constants that may depend on the wavenumber. In particular, one obtains super-algebraic convergence when both arcs and sources can be represented by analytic functions.

Alternatively, for two-dimensional problems, the BIEs for open arcs can be recasted as a problem of integral equations on closed boundaries for even functions. This is done using a cosine change of variables (*cf.* [4] or [31], Chap. 11). Using this property along with classical Fourier analysis, we retrieve convergence rates given in [4] for single arc. Thus, our proof of convergence can be seen as the Fourier–Chebyshev version of those results, with the additional extension to the Helmholtz case.

For implementation purposes, we follow the scheme introduced in [16] wherein all integral kernel singularities are subtracted. This gives rise to smooth and singular functions whose integrals are respectively computed *via* the Fast Fourier Transform (FFT) [20] and analytically using a Chebyshev polynomial expansion of the fundamental solution [11]. Recently, Slevinsky and Olver [33] devised a similar construction based on Chebyshev polynomials for more general integral equations, but limited to line segments and focused exclusively on the spectral properties of collocation methods. Though the authors also provide ideas on how to extend their method to more general arcs, the focus remains in solving a linear system. Hewett *et al.* [14] propose a different numerical method for which they also obtain super-convergence. Their discretization basis captures explicitly the oscillatory behavior on a segment while employing an adaptive low polynomial order bases for the slow but singular part. This splitting leads to impressive results especially for high-frequency, yet its use is restricted to collinear segments and not for general arcs. Still, our approach could be combined with this one but this would require significant work beyond the scope of the present manuscript.

The structure of a problem with multiple arcs implies that many of the interactions, in the BIE system, are characterized by a smooth kernel functions. Thus, one can generally compress these interactions by considering fewer functions than in the self-interaction case. This hints at a compression algorithm, in the same spirit of [22]. Here, the implementation is performed by a bisection algorithm which allows to reuse the integration routines of self-interactions terms. Moreover, we obtain bounds on how the introduction of this compression algorithm affects the accuracy of the numerical solution.

It is also well known that first kind formulations for open arc problems suffer from poor performance when solving the associated linear system *via* iterative methods. Many remedies for this issue have been proposed, among which the construction of preconditioners has received attention in recent years (*cf.* [15, 19, 25] for detailed reviews). These preconditioning techniques could be combined with our spectral solver. Indeed, as spectral methods entail significantly fewer dofs in comparison to low-order methods for a fixed accuracy, it is feasible to invert self-interaction parts of the matrix using a direct method and, by doing so, obtain a better preconditioner. Since the multiple scattering problem requires a large amount of memory to store the problem matrix, direct methods for the full matrix could only be used when the product of frequency and total length of the arcs is small. Moreover, and contrary to what one could expect, the direct method also suffers from numerical cancellation/round-off errors (see Sect. 7.1 for an ilustration). Hence, the need of iterative solvers is mandatory and effective use requires matrix-vector product acceleration.

The paper is organized as follows. Section 2 sets forward formal definitions and properties needed throughout. In Section 3, we formulate the problem as a system of BIEs and show that these are well posed. Section 4 gives details on the Galerkin discretization method; in particular, we establish error convergence rates for the discrete problem assuming regularity conditions on the data. Employed quadrature schemes are detailed in Section 5. Our proposed compression algorithm is given in Section 6. Numerical results illustrating the accuracy of the method as well as the performance of the compression algorithm are presented in Section 7. Finally, conclusions are drawn along with appendices for completeness.

2. Mathematical tools

2.1. General notation

We employ the standard $\mathcal{O}(\cdot)$ and $o(\cdot)$ notation for asymptotics. We also use the notation $a_n \lesssim b_n$ if there exists a positive constant C and an integer $N > 0$ such that $a_n \leq C b_n$ for all $n > N$.

Vectors are indicated by boldface symbols with Euclidean norm written as $\|\cdot\|_2$; other norms are signaled by subscripts. Quantities defined over volume domains will be written in capital case whereas those on boundaries in normal one, *e.g.* $U : G \to \mathbb{C}$ while $u : \partial G \to \mathbb{C}$.

Let $G \subseteq \mathbb{R}^d$, $d = 1, 2$, be an open domain. For $k \in \mathbb{N} \cup \{0\}$, $\mathcal{C}^k(G)$ denotes the set of k-times continuously differentiable functions over G. Compactly supported $\mathcal{C}^k(G)$-functions are designated by $\mathcal{C}^k_0(G)$. Denote by $\mathcal{D}(G) \equiv \mathcal{C}^\infty_0(G)$ the space of infinitely differentiable functions with compact support on a open set G. Duals are indicated by asterisks, *e.g.* the space of distributions is $\mathcal{D}^*(G)$. The class of p-integrable functions over G is written $L^p(G)$. Duality pairings and inner products are written as $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$, respectively, with subscripts declaring the domain involved, if not clear from the context.

We say that $g : (-1, 1) \to \mathbb{C}$ is in $\mathcal{C}^m_v(-1, 1)$, if g is in $\mathcal{C}^m(-1, 1)$ and its mth derivative has bounded variation, *i.e.* the distributional derivative $g^{(m+1)}$ is Lebesgue integrable. Notice that $\mathcal{C}^{m+1}(-1, 1) \subset \mathcal{C}^m_v(-1, 1)$. Also we say g (a function as before) is ρ-analytic, if there exists a Bernstein ellipse of parameter $\rho > 1$, such that g can be extended to an analytic function in the complex ellipse containing the interval $(-1, 1)$ (*cf.* [37], Chap. 8).

Lastly, throughout we will claim a sesquilinear form to be *coercive* if it is the addition of a positive definite form and a compact one; similarly for induced operators.

2.2. Arcs

We call $\Lambda \subset \mathbb{R}^2$ a regular Jordan arc of class $\mathcal{C}^m$ (resp. $\mathcal{C}^m_v$), for $m \in \mathbb{N}$, if there exists a bijective parametrization denoted by $\mathbf{r} : (-1, 1) \to \Lambda$, such that its components are $\mathcal{C}^m(-1, 1)$-functions (resp. $\mathcal{C}^m_v(-1, 1)$-functions) and $\inf_{t \in (-1,1)} \|\mathbf{r}'(t)\|_2 > 0$. Analogously, we say that Λ is ρ-analytic, if there is a corresponding parametrization that is ρ-analytic. Henceforth, we assume all arcs to be Jordan arcs of a given regularity and we will refer to them as open arcs or just arcs.

Assumption 2.1. *For any Λ open arc, there exists an extension to $\tilde{\Lambda}$ which is a simple closed curve containing and having the same regularity of Λ.*

We consider a finite number $M \in \mathbb{N}$ of open arcs $\{\Gamma_i\}_{i=1}^M$, such that under Assumption 2.1 their extensions are mutually disjoint. We define

$$\Gamma := \bigcup_{i=1}^{M} \Gamma_i \quad \text{and} \quad \Omega := \mathbb{R}^2 \setminus \overline{\Gamma}.$$

Assumption 2.2. *There are M domains Ω_i whose boundaries are given by $\partial\Omega_i = \widetilde{\Gamma}_i$, for $i = 1, \ldots, M$, and their closures $\overline{\Omega_i}$ are disjoints.*

For $m \in \mathbb{N}$, we say that the family of arcs Γ is of class $\mathcal{C}^m$ (resp. $\mathcal{C}^m_v$), if each arc Γ_i is of class $\mathcal{C}^m$ (resp. $\mathcal{C}^m_v$), and write $\Gamma \in \mathcal{C}^m$ (resp. $\Gamma \in \mathcal{C}^m_v$); similarly for ρ-analytic arcs. Denote by $\mathbf{r}_i$ a parametrization of the corresponding regularity mapping $(-1,1)$ to an arc Γ_i, $i \in \{1, \ldots, M\}$. For a vector function $\mathbf{g} = (g_1, \ldots, g_M)$ such that $g_i : \overline{\Gamma}_i \to \mathbb{C}$, for $i \in \{1, \ldots, M\}$, we state that $\mathbf{g}$ is of class $\mathcal{C}^m(\Gamma)$ (resp. $\mathcal{C}^m_v(\Gamma)$), if $g_i \circ \mathbf{r}_i \in \mathcal{C}^m(-1,1)$ (resp. $g_i \circ \mathbf{r}_i \in \mathcal{C}^m_v(-1,1)$), for $i \in \{1 \ldots M\}$, and denote $\mathbf{g} \in \mathcal{C}^m(\Gamma)$ (resp. $\mathbf{g} \in \mathcal{C}^m_v(\Gamma)$), and again the ρ-analytic case is defined analogously.

Finally, we will identify every open arc with a given parametrization so that for example $\Lambda_1 := \left\{(t^3, 1),\ t \in (-1,1)\right\}$ and $\Lambda_2 := \{(t,1),\ t \in (-1,1)\}$ are different arcs, even if they are the same set of points in $\mathbb{R}^2$. We will frequently refer to the canonical open arc:

$$\widehat{\Gamma} := \{(t,0),\ t \in (-1,1)\}.$$

2.3. Sobolev spaces and trace operators

Let $G \subseteq \mathbb{R}^d$, $d = 1, 2$, be an open domain. For $s \in \mathbb{R}$, we denote by $H^s(G)$ the standard Sobolev spaces in $L^2(G)$ and by $H^s_{\mathrm{loc}}(G)$ their locally integrable counterparts ([32], Sect. 2.3). We also use the following Hilbert space for $G \subset \mathbb{R}^2$:

$$W(G) := \left\{ U \in \mathcal{D}^*(G) : \frac{U(\mathbf{x})}{\sqrt{1 + \|\mathbf{x}\|_2^2}\log(2 + \|\mathbf{x}\|_2^2)} \in L^2(G), \nabla U \in L^2(G) \right\},$$

which is a subspace of $H^1_{\mathrm{loc}}(G)$. Under Assumption 2.1 for a open arc Λ, we define

$$\widetilde{H}^s(\Lambda) := \left\{ u \in \mathcal{D}^*(\Lambda) : \widetilde{u} \in H^s(\widetilde{\Lambda}) \right\}, \quad s > 0,$$

wherein $\widetilde{u}$ denotes the extension by zero of u to $\widetilde{\Lambda}$. For $s > 0$, we can identify

$$\widetilde{H}^{-s}(\Lambda) = (H^s(\Lambda))^* \quad \text{and} \quad H^{-s}(\Lambda) = (\widetilde{H}^s(\Lambda))^*.$$

We will also need the family of mean-zero Sobolev spaces:

$$\widetilde{H}^s_{\langle 0 \rangle}(\Lambda) = \left\{ u \in \widetilde{H}^s(\Lambda) : \langle u, 1 \rangle = 0 \right\}, \quad s \in \mathbb{R}.$$

The following result will be used to establish convergence rates and error computations in our numerical experiments (*cf.* Sect. 7) with proof given in Appendix B.

Lemma 2.3. *Let $\zeta \in H^{\frac{1}{2}}(\Gamma_i)$, $\psi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, and $\mathbf{r}_i : (-1,1) \to \Gamma_i$, the parametrization of Γ_i. Then, we have the norm equivalences:*

$$c \|\zeta\|_{H^{\frac{1}{2}}(\Gamma_i)} \leq \|\zeta \circ \mathbf{r}_i\|_{H^{\frac{1}{2}}(\widehat{\Gamma})} \leq C \|\zeta\|_{H^{\frac{1}{2}}(\Gamma_i)},$$
$$c \|\psi\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq \|\psi \circ \mathbf{r}_i\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \leq C \|\psi\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)},$$

where the pullbacks for negative order are defined by duality, with generic positive constants c and C depending on Γ_i.

For the finite union of disjoint open arcs Γ, we define piecewise spaces as

$$\mathbb{H}^s(\Gamma) := H^s(\Gamma_1) \times H^s(\Gamma_2) \times \cdots \times H^s(\Gamma_M).$$

Norms and dual products are naturally extended by the previous identification, similarly for spaces $\widetilde{\mathbb{H}}^s(\Gamma)$ and $\widetilde{\mathbb{H}}^s_{\langle 0\rangle}(\Gamma)$, while $\mathbb{H}^s(\widehat{\Gamma})$ is understood as the Cartesian product $\prod_{i=1}^M H^s(\widehat{\Gamma})$.

For $U \in \mathcal{C}^\infty(\overline{\Omega_i})$ (resp. $U \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \Omega_i)$), we can set the interior $(-)$ (resp. exterior $(+)$) Dirichlet traces:

$$\gamma_i^{\pm} U(\mathbf{x}) := \lim_{\epsilon \downarrow 0} U(\mathbf{x} \pm \epsilon \mathbf{n}_i) \quad \forall \mathbf{x} \in \ \Gamma_i,$$

where $\mathbf{n}_i$ denotes the unitary normal vector with direction $(r'_{i,2}, -r'_{i,1})$. If $\gamma_i^+ U = \gamma_i^- U$, we denote $\gamma_i U := \gamma_i^{\pm} U$. These definitions can be extended to more general Sobolev spaces by density, in particular, we have that $\gamma_i^{\pm} : H^1_{\text{loc}}(\Omega) \to H^{\frac{1}{2}}(\Gamma_i)$ as a bounded linear operator (see [26], Thm. 3.37). Neumann traces can be defined for smooth functions U as

$$\gamma_{\mathrm{N},i}^{\pm} U := \lim_{\epsilon \downarrow 0} \mathbf{n}_i \cdot \nabla U(\mathbf{x} \pm \epsilon \mathbf{n}_i), \quad \forall \mathbf{x} \in \Gamma_i.$$

In contrast to the Dirichlet trace, the extension to Sobolev spaces is carried out using Green's formula in Ω_i along with the restriction operator. For $U \in H^1_{\text{loc}}(\Omega_i)$ and $\Delta U \in L^2_{\text{loc}}(\Omega_i)$, then $\gamma_{\mathrm{N},i}^{\pm} U \in H^{-\frac{1}{2}}(\Gamma_i)$ (*cf.* [26], Lem. 4.3). Finally, traces taken with respect to the domains Ω_i, $i \in \{1, \ldots, M\}$ will be denoted $\widetilde{\gamma}_i^{\pm}$ and $\widetilde{\gamma}_{\mathrm{N},i}^{\pm}$ respectively.

3. Boundary integral problem formulation

As explained, we are interested in solving the families of boundary value problems in Ω below *via* suitable integral representations with unknowns densities over the boundaries Γ.

Problem 3.1 (Volume problem)**.** Let $\mathbf{g} = (g_1, \ldots, g_M) \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ and consider a bounded real wavenumber $\kappa \geq 0$. We seek $U \in H^1_{\text{loc}}(\Omega)$ such that

$$-\Delta U - \kappa^2 U = 0 \quad \text{in } \ \Omega, \tag{3.1}$$
$$\gamma_i^{\pm} U = g_i \quad \text{for } \ i = 1, \ldots, M, \tag{3.2}$$
$$\text{condition at infinity}(\kappa). \tag{3.3}$$

The case $\kappa = 0$ corresponds to the Laplace operator whereas $\kappa > 0$ to the Helmholtz one. The behavior at infinity (3.3) depends on κ in the following way: if $\kappa > 0$, we employ the classical Sommerfeld condition:

$$\lim_{R \to \infty} \int_{\|\mathbf{x}\| = R} \left| \frac{\partial U}{\partial r}(\mathbf{x}) - i\kappa U(\mathbf{x}) \right|^2 \mathrm{d}\Gamma_{\mathbf{x}} = 0,$$

where $R = \|\mathbf{x}\|_2$. If $\kappa = 0$, we seek solutions $U \in W(\Omega)$. For $\kappa > 0$ the existence and uniqueness of Problem 3.1 can be obtained from Lemma 1.2 of [34], while for $\kappa = 0$ although very similar to Lemma 1.1 of [34], the result is sightly different as we need to use the space $W(\Omega)$. For sake of completeness, uniqueness is addressed in Appendix A.

We can express the volume solution U as

$$U(\mathbf{x}) = \sum_{i=1}^{M} (\mathsf{SL}_i[\kappa]\lambda_i)(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega, \tag{3.4}$$

where

$$(\mathsf{SL}_i[\kappa]\lambda_i)(\mathbf{x}) := \int_{\Gamma_i} G_\kappa(\mathbf{x}, \mathbf{y}) \lambda_i(\mathbf{y}) \mathrm{d}\Gamma_i(\mathbf{y}),$$

denotes the single layer potential generated at a curve Γ_i with fundamental solution:

$$G_\kappa(\mathbf{x}, \mathbf{y}) = \begin{cases} \dfrac{-1}{2\pi} \log \|\mathbf{x} - \mathbf{y}\|_2 & k = 0, \\ \dfrac{i}{4} H_0^1(\kappa \|\mathbf{x} - \mathbf{y}\|_2) & k > 0. \end{cases} \tag{3.5}$$

Here, $H_0^1(\cdot)$ denotes the zeroth-order first kind Hankel function ([2], Chap. 9). From the properties of the single layer potential on closed domains ([26], Chap. 7) and the completion $\widetilde{\Gamma}_i$ for each arc, one can see that

$$\mathsf{SL}_i[\kappa] : H^{-\frac{1}{2}}(\Gamma_i) \to H^1_{\mathrm{loc}}(\mathbb{R}^2),$$

as a bounded linear map. Moreover, if U is expressed as in (3.4), then it solves (3.1). By Theorem 9.6 of [26] for $\kappa > 0$, the representation (3.4) satisfies the Sommerfeld condition. The case $\kappa = 0$ is given by the following result.

Lemma 3.2. *The single layer potential* $\mathsf{SL}_i[0]$ *is a bounded linear map between the spaces* $\widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$ *and* $W(\mathbb{R}^2 \setminus \overline{\Gamma}_i)$.

Proof. As $\widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i) \subset \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ we have that $\mathsf{SL}_i[0] : \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i) \to H^1_{\mathrm{loc}}(\mathbb{R}^2)$. Hence, we only need to verify the conditions:

$$\frac{(\mathsf{SL}_i[0]u)(\mathbf{x})}{\sqrt{1+\|\mathbf{x}\|_2^2}\log(2+\|\mathbf{x}\|_2^2)} \in L^2(\mathbb{R}^2 \setminus \overline{\Gamma}_i), \quad \text{and} \quad \nabla\left(\mathsf{SL}_i[0]u\right) \in L^2(\mathbb{R}^2 \setminus \overline{\Gamma}_i),$$

for every $u \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$. From Corollary 8.11 of [26], we know that the asymptotic behavior of the single layer potential for large arguments is

$$(\mathsf{SL}_i[0]u)(\mathbf{x}) = -\frac{1}{2\pi}\langle u, 1\rangle \log\|\mathbf{x}\|_2 + \mathcal{O}\left(\|\mathbf{x}\|_2^{-1}\right), \quad \text{for } \|\mathbf{x}\|_2 \to \infty.$$

Thus, if $u \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$ then

$$(\mathsf{SL}_i[0]u)(\mathbf{x}) = \mathcal{O}\left(\|\mathbf{x}\|_2^{-1}\right), \quad \text{for } \|\mathbf{x}\|_2 \to \infty. \tag{3.6}$$

Using polar coordinates and the above bound, we can verify the conditions directly. □

In order to find the boundary unknowns λ_i, we take Dirichlet traces of the single layers potentials and impose (3.2). This induces the definition of weakly singular boundary integral operators (BIOs) as

$$\mathcal{L}_{ij}[\kappa] := \frac{1}{2}\left(\gamma_i^+\mathsf{SL}_j[\kappa] + \gamma_i^-\mathsf{SL}_j[\kappa]\right) = \gamma_i\mathsf{SL}_j[\kappa],$$

the last equation resulting from the continuity properties of the SL_i across Γ_i for each $i = 1, \ldots, M$.

Problem 3.3. For $\kappa > 0$ and $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$, we seek $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ such that

$$\boldsymbol{\mathcal{L}}[\kappa]\boldsymbol{\lambda} = \mathbf{g},$$

or equivalently,

$$\langle \boldsymbol{\mathcal{L}}[\kappa]\boldsymbol{\lambda}, \boldsymbol{\phi}\rangle_\Gamma = \langle \mathbf{g}, \boldsymbol{\phi}\rangle_\Gamma, \quad \forall \boldsymbol{\phi} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma),$$

where

$$\boldsymbol{\mathcal{L}}[\kappa] := \begin{bmatrix} \mathcal{L}_{11}[\kappa] & \mathcal{L}_{12}[\kappa] & \ldots & \mathcal{L}_{1M}[\kappa] \\ \mathcal{L}_{21}[\kappa] & \mathcal{L}_{22}[\kappa] & \ldots & \mathcal{L}_{2M}[\kappa] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{M1}[\kappa] & \mathcal{L}_{M2}[\kappa] & \ldots & \mathcal{L}_{MM}[\kappa] \end{bmatrix} : \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{\frac{1}{2}}(\Gamma).$$

In the case $\kappa = 0$, we need $\mathbf{g} \in (\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma))^*$ and restrict $\boldsymbol{\lambda}$ to $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$.

Remark 3.4. Problem 3.3 can be recast in the reference space $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\widehat{\Gamma})$ ($\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\widehat{\Gamma})$ for $\kappa = 0$) so as to find $\widehat{\boldsymbol{\lambda}}$ such that

$$\widehat{\boldsymbol{\mathcal{L}}}[\kappa]\widehat{\boldsymbol{\lambda}} = \widehat{\mathbf{g}},$$

wherein $\widehat{g}_j := g_j \circ \mathbf{r}_j$, $\widehat{\mathcal{L}}_{ij}$ are the BIOs defined over the reference arc $\widehat{\Gamma}$ with integral kernel $G_\kappa(\mathbf{r}_i(t), \mathbf{r}_j(s))$ and the unknowns $\widehat{\lambda}_j := (\lambda_j \circ \mathbf{r}_j)/\|\mathbf{r}_j'\|_2$.

Remark 3.5. Later on we will use the operator $\mathcal{L}_{ii}[\kappa]$ for the choice $\Gamma_i = \widehat{\Gamma}$, which we denote by $\check{\mathcal{L}}[\kappa]$. The difference with respect to $\widehat{\mathcal{L}}_{ii}[\kappa]$ relies on the absence of parametrizations $\mathbf{r}_i$ involved in the kernel. In the case of a single open arc with parametrization $\mathbf{r}$, we will write $\widehat{\mathcal{L}}[\kappa] \equiv \widehat{\mathcal{L}}_{ii}[\kappa]$. In this case, and for $\kappa = 0$, one can deduce that the kernel function of the integral operator $\check{\mathcal{L}}[0] - \widehat{\mathcal{L}}[0]$ is given by

$$E_{\mathbf{r}}(t,s) := -\frac{1}{2\pi} \log\left(\frac{\|\mathbf{r}(t) - \mathbf{r}(s)\|_2}{|t-s|}\right)$$

for which we have the following result.

Lemma 3.6. *Let $m \in \mathbb{N}$ and Λ be a single $\mathcal{C}_v^m$-arc. Then, the function $E_{\mathbf{r}}(t,s)$ is a $\mathcal{C}_v^m(-1,1)$-function in each of its components. If Γ is ρ-analytic arc, $E_{\mathbf{r}}(t,s)$ is a bivariate ρ-analytic function.*

Proof. By performing a Taylor expansion in t, we can write

$$\Theta_{\mathbf{r}}(t,s) := \frac{\mathbf{r}(t) - \mathbf{r}(s)}{t-s} = \sum_{j=1}^{m-1} \frac{(t-s)^{j-1}\mathbf{r}^{(j)}(s)}{j!} + \frac{1}{t-s}\int_s^t \frac{(t-\xi)^m \mathbf{r}^{(m)}(\xi)}{m!}\,\mathrm{d}\xi.$$

This function admits m continuous derivatives in the t variable. As mentioned at the beginning of Section 2.2, open arc parametrizations are injective, and thus, the function can only be zero if $t = s$. However, as t approaches s, the above function behaves as $\mathbf{r}'(s)$, which is not zero. Hence, $\Theta_{\mathbf{r}}(t,s)$ does not vanish and so $E_{\mathbf{r}}(t,s)$ is the composition of $\mathcal{C}_v^m$-functions, despite there being an absolute value. The ρ-analytic case follows from the same argument. □

Remark 3.7. One should fully understand the differences between the cases $\kappa = 0$ and $\kappa > 0$. The first one is posed over the smaller space $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$, and the right-hand side must be in the dual of this space, which is bigger than $\mathbb{H}^{\frac{1}{2}}(\Gamma)$ under the identification of $L^2(\Gamma)$ with its own dual. However, one has to be careful with the identifications that occur as many elements of $\mathbb{H}^{\frac{1}{2}}(\Gamma)$ are identifiable with one element of $(\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma))^*$: for example, all constants are equivalent to the zero function. A more general formulation for the $\kappa = 0$ case can be found in [34].

Now, we show that Problem 3.3 is well posed. First, we prove that the diagonal operators $\mathcal{L}_{ii}[k]$ are coercive and use ideas from [34] to transform the problem into a closed domain one.

Lemma 3.8. *For $i \in \{1, \ldots, M\}$, $k \geq 0$, there exist a constant $c_{e,i}$ such that*

– *if $\kappa = 0$, it holds*

$$\langle \mathcal{L}_{ii}[0]u, u\rangle_{\Gamma_i} \geq c_{e,i}\,\|u\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \quad \forall u \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i);$$

– *if $\kappa > 0$, then there are compact BIOs $\mathcal{K}_{ii}[\kappa] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$, such that*

$$\langle (\mathcal{L}_{ii}[\kappa] + \mathcal{K}_{ii}[\kappa])u, u\rangle_{\Gamma_i} \geq c_{e,i}\,\|u\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \quad \forall u \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i).$$

Proof. Given u and v in $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, consider their respective zero extension $\widetilde{u}$ and $\widetilde{v}$ to $\partial\Omega_i$ (see Assumptions 2.1 and 2.2). Denote by $\mathcal{V}_{ii}[k]$ the weakly singular integral operator given by taking the trace over $\partial\Omega_i$ of the single layer potential in $\partial\Omega_i$. Then, we have that

$$\langle \mathcal{L}_{ii}[\kappa]u, u\rangle_{\Gamma_i} = \langle \mathcal{V}_{ii}[\kappa]\widetilde{u}, \widetilde{u}\rangle_{\partial\Omega_i}.$$

The results then follows from the closed curves case (*cf.* [7], Thm. 2). □

Remark 3.9. Continuity of operators $\mathcal{L}_{ij}$, $i, j \in \{1, \dots, M\}$, can be proved by using the same arguments as those for Lemma 3.8. Then, one can easily show that

$$\mathcal{L}_{ij}[\kappa] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_j) \to H^{\frac{1}{2}}(\Gamma_i)$$

as a bounded operator. Moreover, if $i \neq j$ the operator is compact as the kernel function is at least $\mathcal{C}^1$ in each component.

Theorem 3.10. *For $\kappa > 0$, Problem 3.3 has a unique solution $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, whereas for $\kappa = 0$ a unique solution exists in the subspace $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma)$. Also, we have the continuity estimate*

$$\|\boldsymbol{\lambda}\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}.$$

Proof. By compactness of the cross-interaction BIOs and the coercivity result of Lemma 3.8, the Fredholm alternative ([26], Thm. 2.33) indicates that we only need to prove injectivity to ensure existence. First, consider the case $M = 1$: for $\kappa = 0$, the result is obtained by applying the Lax–Milgram lemma while for $\kappa > 0$, we obtain the result from Theorem 1.7 of [34].

Now, we focus on in the case $M > 1$. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$ be such that

$$\sum_{j=1}^{M} \mathcal{L}_{ij}[\kappa]\lambda_j = 0 \quad \forall i = 1, \dots, M.$$

For $j \in \{1, \dots, M\}$, let us define volume potentials $U_j := \mathsf{SL}_j[k]\lambda_j$, solutions of individual homogenous Helmholtz problems over $\mathbb{R}^2 \setminus \overline{\Gamma}_j$ as well as the superposition $U_\sigma := \sum_{j=1}^{M} U_j$ defined over Ω. Then, it holds

$$\gamma_i U_\sigma = \gamma_i \sum_{j=1}^{M} U_j = \sum_{i=1}^{M} \mathcal{L}_{ij}[k]\lambda_j = 0, \quad \forall i = 1, \dots, M.$$

However, U_σ is also the solution of Problem 3.1, with zero Dirichlet boundary condition. Hence, as this problem has at most one solution we conclude that

$$U_\sigma = \sum_{j=1}^{M} \mathsf{SL}_j[k]\lambda_j = 0,$$

and consequently, for all $i = 1, \dots, M$, it holds that

$$U_i = \mathsf{SL}_i[k]\lambda_i = -\sum_{j \neq i} \mathsf{SL}_j[k]\lambda_j. \tag{3.7}$$

Let us now consider the closed curve $\widetilde{\Gamma}_i = \partial\Omega_i$, and denote by $\tilde{\lambda}_i \in \widetilde{H}^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$ the extension by zero of λ_i. Then, one derives,

$$U_i(\mathbf{x}) = \mathsf{SL}_i[k](\mathbf{x})\lambda_i = \mathsf{SL}_{\widetilde{\Gamma}_i}[k](\mathbf{x})\tilde{\lambda}_i \quad \forall \mathbf{x} \in \Omega,$$

where the last potential is defined on the closed curve $\widetilde{\Gamma}_i$. If we take normal jumps, defined as $[\gamma_N U] = \gamma_N^+ U - \gamma_N^- U$, by Theorem 3.3.1 of [32], we obtain

$$[\widetilde{\gamma}_{\mathrm{N},i} U_i]_{\widetilde{\Gamma}_i} = [\widetilde{\gamma}_{\mathrm{N},i} \mathsf{SL}_{\widetilde{\Gamma}_i}[k]\tilde{\lambda}_i]_{\widetilde{\Gamma}_i} = -\tilde{\lambda}_i.$$

Using (3.7) in the expression above yields

$$[\widetilde{\gamma}_{\mathrm{N},i} U_i]_{\widetilde{\Gamma}_i} = -\left[\widetilde{\gamma}_{\mathrm{N},i} \sum_{j \neq i} \mathsf{SL}_j[k]\lambda_j\right]_{\widetilde{\Gamma}_i} = 0$$

where the last equality comes from the smoothness of the integral kernel since $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j = \emptyset$, for $j \neq i$. Thus, we can conclude that $\lambda_j = 0$ and the same follows for all other components. □

Remark 3.11. Much of the ideas presented in this section can be used in a more general context. In a more abstract setting, the notion of open arcs Γ_i has to be changed Lipschitz subsets of the boundary of a domain $\Omega_i \in \mathbb{R}^d$, for $d = 2, 3$, and whose normal vector is continuous. Define Ω as the exterior of a finite set of generalized open arcs Γ. As in Chapter 4 of [26], consider any strongly elliptic second-order self-adjoint partial differential operator, denoted by $\mathcal{P}$, with smooth coefficients, acting on vector fields of $\mathbb{C}^m$. Thus, for a given Dirichlet or Neumann datum, $\mathbf{g} \in [\mathbb{H}^{\frac{1}{2}}(\Gamma)]^m$ or $\mathbf{h} \in [\mathbb{H}^{-\frac{1}{2}}(\Gamma)]^m$, respectively, we seek for $\mathbf{U} \in [H^1_{\text{loc}}(\Omega)]^m$ such that,

$$\begin{aligned} &\mathcal{P}\mathbf{U} = 0 \quad \text{in} \quad \Omega, \\ &\gamma\mathbf{U} = \mathbf{g} \quad \text{or} \quad B_{\mathbf{n}}\mathbf{U} = \mathbf{h} \quad \text{on} \quad \Gamma, \end{aligned}$$

with the conormal trace $B_{\mathbf{n}}$ defined as in Chapter 4 of [26]. The following points are needed in order to establish the existence and uniqueness of an equivalent boundary integral formulation for Cauchy data.

(i) An adequate condition at infinity that ensures the uniqueness of the boundary value problem.
(ii) A fundamental solution $G(\mathbf{x}, \mathbf{y})$, such that $\mathcal{P}_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}-\mathbf{y}} \mathsf{I}$, where I is the identity operator in $\mathbb{R}^{m \times m}$.
(iii) Layer potentials:

$$\begin{aligned} (\mathsf{SL}_i \boldsymbol{\lambda})(\mathbf{x}) &:= \int_{\Gamma_i} G(\mathbf{x}, \mathbf{y}) \boldsymbol{\lambda}(\mathbf{y}) \mathrm{d}\Gamma_i(\mathbf{y}) \qquad \text{(Dirichlet trace)}, \\ (\mathsf{DL}_i \boldsymbol{\lambda})(\mathbf{x}) &:= \int_{\Gamma_i} B_{\mathbf{n}(\mathbf{y})} G(\mathbf{x}, \mathbf{y}) \boldsymbol{\lambda}(\mathbf{y}) \mathrm{d}\Gamma_i(\mathbf{y}) \quad \text{(Conormal trace)}, \end{aligned}$$

that display the adequate behavior at infinity specified by the first point in the trace spaces. Specifically, $\boldsymbol{\lambda} \in \left[\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)\right]^m$ for the Dirichlet problem and $\boldsymbol{\lambda} \in \left[\widetilde{\mathbb{H}}^{\frac{1}{2}}(\Gamma)\right]^m$ for the conormal trace case.

With the above, the integral equation is constructed by imposing the boundary condition to the following representations:

$$\begin{aligned} \mathbf{U} &= \sum_{i=1}^{M} \mathsf{SL}_i \boldsymbol{\lambda}_i \quad \text{(Dirichlet trace)}, \\ \mathbf{U} &= \sum_{i=1}^{M} \mathsf{DL}_i \boldsymbol{\lambda}_i \quad \text{(Conormal trace)}. \end{aligned}$$

If the previously stated conditions are satisfied, then the construction of the arising BIEs as well as their well-posedness proofs is done as in the cases that we presented in detail. The 2D-Laplace case is slightly different as the condition at infinity of the potential only holds in a subspace.

4. Numerical analysis

We now describe a spectral Galerkin numerical scheme for solving Problem 3.3 and establish specific convergence rates.

4.1. Approximation spaces

Our aim is to construct a dense conforming discretization of the spaces $\widetilde{H}^{-1/2}(\Gamma_i)$ and $\widetilde{H}^{-1/2}_{\langle 0 \rangle}(\Gamma_i)$, for $i \in \{1, \ldots, M\}$. Certainly, one could use traditional low-order bases built on arc meshes for which approximation properties are well known. However, this would imply large numbers of dofs to solve problems with many arcs and/or large values of κ. Thus, we opt for high-order global polynomial bases such as weighted Chebyshev polynomials per arc.

4.1.1. Single arc approximation

We denote by $\{T_n\}_{n=0}^N$ the set of first $N+1$ first-kind Chebyshev polynomials, orthogonal under the weight w^{-1} with $w(t) := \sqrt{1-t^2}$. Consider the elements $p_n^i = T_n \circ \mathbf{r}_i^{-1}$ over each arc Γ_i, the space they span is denoted $\mathbb{T}_N(\Gamma_i)$, and define the normalized space:

$$\overline{\mathbb{T}}_N(\Gamma_i) := \left\{ \bar{p}^i \in C(\Gamma_i) : \ \bar{p}^i := \frac{p^i}{\left\| \mathbf{r}_i' \circ \mathbf{r}_i^{-1} \right\|_2}, \quad p^i \in \mathbb{T}_N(\Gamma_i) \right\}.$$

We account for edge singularities by multiplying by a suitable weight:

$$\mathbb{Q}_N(\Gamma_i) := \left\{ q^i := w_i^{-1} \bar{p}^i : \ \bar{p}^i \in \overline{\mathbb{T}}_N(\Gamma_i) \right\},$$

wherein $w_i := w \circ \mathbf{r}_i^{-1}$. The corresponding bases for $\mathbb{Q}_N(\Gamma_i)$ will be denoted $\left\{q_n^i\right\}_{n=0}^N$, and are characterized by $q_n^i = w_i^{-1} \|\mathbf{r}_i' \circ \mathbf{r}_i^{-1}\|_2^{-1} T_n \circ \mathbf{r}_i^{-1}$. By Chebyshev orthogonality, we can easily define the mean-zero subspace:

$$\mathbb{Q}_{N,\langle 0 \rangle}(\Gamma_i) := \mathbb{Q}_N(\Gamma_i) / \mathbb{Q}_0(\Gamma_i),$$

spanned by $\left\{q_n^i\right\}_{n=1}^N$. Basic approximation properties of the spaces $\mathbb{Q}_N(\Gamma_i)$ are detailed in Appendix C.

4.1.2. Multiple arcs approximation

Let us define the approximation product spaces:

$$\mathbb{H}^N := \prod_{i=1}^M \mathbb{Q}_N(\Gamma_i), \quad \mathbb{H}^N_{\langle 0 \rangle} := \prod_{i=1}^M \mathbb{Q}_{N,\langle 0 \rangle}(\Gamma_i).$$

With the previously defined discrete spaces, we can find an approximation to the solution of Problem 3.3 by solving the following linear system.

Problem 4.1 (Linear system)**.** Let $m, N \in \mathbb{N}$, $\Gamma \in \mathcal{C}_v^m$, $\kappa > 0$, and $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$, we seek coefficients $\mathfrak{u} = (\mathfrak{u}_1, \dots, \mathfrak{u}_M) \in \mathbb{C}^{M(N+1)}$, such that

$$\mathsf{L}[\kappa]\mathfrak{u} = \mathfrak{g},$$

wherein we have defined the Galerkin matrix $\mathsf{L}[\kappa] \in \mathbb{C}^{M(N+1) \times M(N+1)}$ with matrix blocks $\mathsf{L}_{ij} \in \mathbb{C}^{(N+1)\times(N+1)}$ whose entries are

$$(\mathsf{L}_{ij}[\kappa])_{lm} = \left\langle \mathcal{L}_{ij}[\kappa] q_m^j, q_l^i \right\rangle_{\Gamma_i}, \quad \forall i, j = 1, \dots, M, \quad \text{and} \quad \forall l, m = 0, \dots, N. \tag{4.1}$$

The right-hand $\mathfrak{g} = (\mathfrak{g}_1, \dots, \mathfrak{g}_M) \in \mathbb{C}^{M(N+1)}$ has components $(\mathfrak{g}_i)_l = \left\langle g_i, q_l^i \right\rangle_{\Gamma_i}$.

For $\kappa = 0$ we impose $g \in (\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma))^*$, and the spaces $\mathbb{Q}_N(\Gamma_j)$ have to be changed to $\mathbb{Q}_{N,\langle 0 \rangle}(\Gamma_j)$.

Approximations to solutions of Problem 3.3 are constructed using the solution $\mathfrak{u}$ of Problem 4.1 as follows

$$(\lambda_N)_i = \sum_{l=0}^N (\mathfrak{u}_i)_l q_l^i \ \text{ in } \Gamma_i, \quad \text{for all } i \in \{1, \dots, M\}.$$

Observe that the sum starts with $l = 1$ if $\kappa = 0$.

Remark 4.2. By performing a change of variables, we can recast Problem 4.1 on $\widehat{\Gamma}$ with matrix terms given by

$$(\mathsf{L}_{ij}[\kappa])_{lm} = \left\langle \widehat{\mathcal{L}}_{ij} w^{-1} T_m, w^{-1} T_l \right\rangle_{\widehat{\Gamma}}, \quad \forall i, j = 1, \dots, M, \quad \text{and} \quad \forall l, m = 0, \dots, N,$$

with $w(t) = \sqrt{1-t^2}$, and the right hand side $\mathfrak{g} = (\mathfrak{g}_1, \ldots, \mathfrak{g}_M) \in \mathbb{C}^{M(N+1)}$ with components $(\mathfrak{g}_i)_l = \left\langle g \circ \mathbf{r}_i, w^{-1} T_l \right\rangle_{\widehat{\Gamma}}$. We have the corresponding approximation of the pulled back solution $\widehat{\boldsymbol{\lambda}}$:

$$(\widehat{\lambda}_N)_i = \sum_{l=0}^{N} (\widehat{\mathfrak{u}}_i)_l w^{-1} T_l \quad \text{in } \widehat{\Gamma}, \quad \text{for all } i \in \{1, \ldots, M\}.$$

The following result is a direct consequence of the coercivity of $\boldsymbol{\mathcal{L}}[\kappa]$ and the basic approximation properties presented in Appendix C (see [32], Thm. 4.29 for a detailed proof).

Theorem 4.3. *For $\kappa > 0$, given $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$, there exist $N_0 \in \mathbb{N}$, and $C > 0$, both depending of Γ, $\mathbf{g}$, and κ such that for any $N \in \mathbb{N}$ with $N > N_0$, there exists only one solution $\mathfrak{u}$ of Problem 4.1. Moreover, for the approximation $\boldsymbol{\lambda}_N \in \mathbb{H}^N$ we can bound the error as*

$$\|\boldsymbol{\lambda}_N - \boldsymbol{\lambda}\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C \inf_{\mathbf{v}_N \in \mathbb{H}^N} \|\mathbf{v}_N - \boldsymbol{\lambda}\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)}.$$

For $\kappa = 0$, we need to take $\mathbf{g} \in (\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma))^*$ and $\mathbb{H}^N_{\langle 0 \rangle}$ as the discrete space for the result to hold.

4.2. Convergence results

The density of the family of spaces $\left\{\mathbb{H}^N\right\}_{N \in \mathbb{N}}$ in $\mathbb{H}^{-\frac{1}{2}}(\Gamma)$ $\left(\text{resp. } \left\{\mathbb{H}^N_{\langle 0 \rangle}\right\}_{N \in \mathbb{N}} \text{ in } \mathbb{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma)\right)$ shown in Appendix C combined with Theorem 4.3 allows to conclude that when N goes to infinity convergence occurs in the general context. However, this does not provide any insight on convergence rates.

In this section, we will bound the error in terms of the dimension N, the degree of polynomials used in each arc. Explicit convergence rates with respect to κ are not analyzed and we leave this as future work. Similar bounds for error convergence rates were established in [21] (for $\kappa = 0$ on an interval) and in [4]. This last work while only shows the Laplace case for one arc, could be extended for multiple arcs easily. The authors also consider the error introduced by the quadrature scheme. However, the extension to Helmholtz does not appear to be straightforward, as it is hard to argue data regularity is preserved. In fact, proving this last point takes significant effort. The effect of numeric integration will not be considered here but one can easily show that it introduces an extra error which decays as fast as the Fourier–Chebyshev coefficients of the (regular) right-hand side and the geometry (*cf.* Sect. 5).

Before carrying on, we outline the general ideas presented in this section. In Sections 4.2.1 and 4.2.2 we characterize the decay of Chebyshev coefficients $\{\lambda_n\}_{n \in \mathbb{N}}$ appearing in the solution of the single scatterer problem. This is done in a constructive way: we start with the most simple case ($\kappa = 0, \Gamma = \widehat{\Gamma}$) leading to Lemma 4.5, and finalize with a general arc for non-zero wavenumber in Lemma 4.16 (Lems. 4.9 and 4.13 are intermediate results). Once the coefficients' decay is characterized, we use it in conjunction with the quasi-optimality result to establish the error convergence of a single arc problem (Thm. 4.17). Finally, in Section 4.2.4 we generalize the results for multiple arcs. For this, we first establish the decay of the coefficients (Lem. 4.20) and conclude, as in the single arc case, with Theorem 4.21 which gives the rate of convergence for general multiple arcs and $\kappa \geq 0$.

We start by analyzing the most simple problem – $\kappa = 0$ and a single interval –, and from there we gradually consider more generalities until we arrive to the most complex case ($\kappa > 0$ for multiple arcs). Every function $\widehat{\lambda}$ in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$, can be expressed as a convergent series:

$$\widehat{\lambda}(s) = w^{-1} \sum_{n \geq 0} \lambda_n T_n(s), \quad s \in (-1, 1).$$

Furthermore, we have an explicit expression for the $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$-norm when such representation is used

$$\left\|\widehat{\lambda}\right\|^2_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} = \sum_{n\geq 0} |\lambda_n|^2 d_n, \tag{4.2}$$

where $d_0 = 1$, and $d_n = n^{-1}$ for $n > 0$ ([18], proof of Prop. 3.5).

4.2.1. Chebyshev coefficients behavior: Laplace case

We recall operators $\check{\mathcal{L}}[0]$ and $\widehat{\mathcal{L}}[0]$ defined over $\widehat{\Gamma}$ (*cf.* Rem. 3.5). In this section, we consider the pullback problem:

Problem 4.4. For $m \in \mathbb{N}$ given $\Gamma \in \mathcal{C}^m_v$, and $g \in \mathcal{C}^m_v(\Gamma) \cap (\widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma))^*$, we seek $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\widehat{\Gamma})$ such that

$$\widehat{\mathcal{L}}[0]\widehat{\lambda} = \widehat{g} \quad \text{on } \widehat{\Gamma},$$

which is equivalent to Problem 3.3 with $\kappa = 0$ and $M = 1$.

We aim to characterize the mapping properties of these weakly singular BIOs (defined as in Sect. 3) acting on weighted Chebyshev polynomials.

Lemma 4.5. *For n and l in $\mathbb{N}$, it holds*

$$\left\langle \check{\mathcal{L}}[0]\frac{T_n}{w}, \frac{T_l}{w} \right\rangle = \frac{\pi}{4n}\delta_{nl}.$$

Proof. Direct consequence of the kernel expansion ([19], Thm. 4.4):

$$G_0(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log|t-s| = \frac{1}{2\pi}\log 2 + \sum_{n\geq 1}\frac{1}{\pi n}T_n(t)T_n(s), \quad \forall s \neq t.$$

and the orthogonality property of Chebyshev polynomials. □

One can interpret this result as follows: given an element in $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$, its image by $\check{\mathcal{L}}[0]$ is a function whose Chebyshev coefficients decay as $\mathcal{O}(n^{-1})$. The rest of this section extends this idea to more general arcs.

Lemma 4.6. *For $m \in \mathbb{N}$, let $h : [-1,1]^2 \to \mathbb{C}$ be such that $h(t,\cdot)$ and $h(\cdot,s)$ are $\mathcal{C}^m_v(-1,1)$-functions as functions of s and t, respectively. Thus, we can write h as*

$$h(t,s) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{nk}T_n(t)T_k(s),$$

with coefficients decaying as follows:

$$b_{nk} = \mathcal{O}\left(\min\left\{n^{-m-1}, k^{-m-1}\right\}\right).$$

If h is ρ-analytic in both variables

$$b_{nk} = \mathcal{O}\left(\rho^{\min\{-n,-k\}}\right).$$

Proof. This is just the bivariate version of Theorem 7.1 from [37] and Theorem 8.1 from [37] (see Appendix B for a detailed proof). □

Lemma 4.7. *Let $m \in \mathbb{N}$ and $h : [-1,1]^2 \to \mathbb{C}$ be a $\mathcal{C}_v^m(-1,1)$-function in both arguments. Consider the integral operator taking as kernel the bivariate function h:*

$$(\mathcal{H}f)(s) := \int_{\widehat{\Gamma}} h(t,s) f(t)\, \mathrm{d}t,$$

Let $f \in \widetilde{H}^{-1/2}(\widehat{\Gamma})$, then for $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < 1$, we have that the Fourier–Chebyshev coefficients of $\mathcal{H}f$, denoted $\{v_l\}_{l \in \mathbb{N}_0}$, decay as

$$v_l = \mathcal{O}\left(l^{(-1+\epsilon)m}\right).$$

Moreover, if the kernel is ρ-analityc we have that

$$v_l = \mathcal{O}\left(\rho^{-l}\right).$$

Proof. See Appendix B. □

Remark 4.8. The previous result is by no means sharp. In the context of pseudo-differential operators using Fourier expansion for the norms one could obtain better bounds, see for example Chapter 7 of [31]. Results for open arcs in terms of Fourier–Chebyshev expansions can be obtained using the cosine change of variables.

We continue by estimating bounds for the Chebyshev coefficients of solutions of the BIE associated to the Laplace problem for any sufficiently smooth single arc.

Lemma 4.9. *Let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\widehat{\Gamma})$ be the unique solution of Problem 4.4, with $m \geq 2$. If we expand $\widehat{\lambda}$ as*

$$\widehat{\lambda} = w^{-1} \sum_{n=1}^{\infty} a_n T_n,$$

we obtain the following coefficient asymptotic behaviors:

$$a_n = \mathcal{O}\left(n^{-m}\right).$$

Moreover, if Γ is a ρ-analytic arc and g is also ρ-analytic, we obtain

$$a_n = \mathcal{O}\left(n\rho^{-n}\right).$$

Proof. Since $\widehat{g} = g \circ \mathbf{r}$, we can expand it as a Fourier–Chebyshev series with coefficients $\widehat{g}_l$ leading to

$$(\widehat{\mathcal{L}}[0]\widehat{\lambda})_l = \widehat{g}_l, \quad \forall l \in \mathbb{N}.$$

The coefficients of the left-hand side of the last equation can be computed by adding and subtracting the term $\check{\mathcal{L}}[0]\widehat{\lambda}$. By doing so and combining Lemmas 4.5–4.7 and 3.6, we obtain the following expression:

$$\frac{\pi^2}{4}\frac{a_l}{l} + v_l = \widehat{g}_l, \quad \forall l \in \mathbb{N},$$

where the coefficient v_l corresponds to that in the expansion of $(\widehat{\mathcal{L}}[0] - \check{\mathcal{L}}[0])\widehat{\lambda}$. By the regularity conditions, it holds that $\widehat{g}_l = \mathcal{O}(l^{-m-1})$, and therefore,

$$\frac{\pi^2}{4} a_l l^{-1} + v_l = \mathcal{O}\left(l^{-m-1}\right).$$

Hence, there are two alternatives: either (i) $a_l = \mathcal{O}(l^{-m})$ and $v_l = \mathcal{O}\left(l^{-m-1}\right)$, or (ii) both have the same decay order. As the first implies the result directly, we assume the second alternative in what follows.

Let $2 < m < \infty$. By Lemma 4.7 (i), we have that $v_l = \mathcal{O}\left(l^{(-1+\epsilon')m}\right)$, and under our current assumption, this implies that

$$a_l = \mathcal{O}\left(l^{(-1+\epsilon')m}\right).$$

Since $m > 2$, we can choose ϵ such that $\sum_{n=1}^{\infty} a_n$ is finite and a new estimate for v_l holds

$$v_l = \sum_{n=1}^{\infty} b_{nl} a_n \lesssim l^{-m-1}.$$

Here, b_{nl} are the coefficients detailed in Lemma 4.6 for the function $E_{\mathbf{r}}$ defined in Remark 3.5. This last equality implies the result directly. The case $m = 2$ is slightly more complicated as one can not directly ensure that the coefficients a_l are summable. However, by Lemma 4.7, for a small $\delta > 0$, then $v_l = \mathcal{O}\left(l^{-2+\delta}\right)$, which implies that $a_l = \mathcal{O}\left(l^{-1+\delta}\right)$. By re-estimating bounds on v_l, we now obtain that $v_l = \mathcal{O}\left(l^{-3+2\delta}\right)$. Hence, $a_l = \mathcal{O}\left(l^{-2+2\delta}\right)$ which are summable from where one can argue as before. For the ρ-analytic case, the result is direct as the v_l already has a decay that implies the corresponding behavior of the coefficients a_l. □

4.2.2. Chebyshev coefficients behavior: Helmholtz case

We now consider the following single arc problem:

Problem 4.10. For $m \in \mathbb{N}$, $\kappa > 0$, given $\Gamma \in \mathcal{C}_v^m$, and $g \in \mathcal{C}_v^m(\Gamma)$, we seek $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\widehat{\Gamma})$ such that

$$\widehat{\mathcal{L}}[\kappa]\widehat{\lambda} = \widehat{g} \quad \text{on} \quad \widehat{\Gamma}, \tag{4.3}$$

which is equivalent to Problem 3.3 with $\kappa > 0$ and $M = 1$.

One could see the Helmholtz case as a perturbation of the previous one, but this perturbation is not smooth as the operator difference $\widehat{\mathcal{L}}[\kappa] - \check{\mathcal{L}}[0]$ (*cf.* Rem. 3.5) only has a $\mathcal{C}^1$-kernel, even for smooth arcs. Thus, we can not replicate the previous arguments and need to examine in depth $\widehat{\mathcal{L}}[\kappa] - \check{\mathcal{L}}[0]$ in terms of Chebyshev coefficients.

Using Formula (9.1.13) of [2], the kernel of $\widehat{\mathcal{L}}[\kappa]$, given in (3.5), can be also be written as

$$\widehat{G}_k(t,s) = \frac{i}{4} H_0^1\left(k\left\|\mathbf{r}(t) - \mathbf{r}(s)\right\|_2\right) = \sum_{p=0}^{\infty} z_p R_p(t,s)|t-s|^{2p}\log|t-s| + \psi_R(t,s),$$

wherein $\mathbf{r} : (-1,1) \to \Gamma_i$ is a suitable parametrization,

$$z_p := \frac{1}{2\pi}(-1)^p\left(\frac{k}{2}\right)^{2p}(p!)^{-2}, \tag{4.4}$$

$$R_p(t,s) := \left(\frac{\|\mathbf{r}(t) - \mathbf{r}(s)\|_2}{|t-s|}\right)^{2p}, \tag{4.5}$$

and ψ_R is $C_v^m(-1,1)$-regular in each component. Notice that the term $|t-s|^{2p}\log|t-s|$ is a $\mathcal{C}^{2p-1}(-1,1)$-function in each component.

We begin by analyzing the Helmholtz case for $\widehat{\Gamma}$ following similar techniques to those in [11]. To simplify notation, we define kernels $\widehat{G}_k^p(t,s) := z_p R_p(t,s)|t-s|^{2p}\log|t-s|$ and their corresponding BIOs:

$$\widehat{\mathcal{L}}^p[\kappa]f := \int_{-1}^{1} \widehat{G}_k^p(t,s)f(t)\,\mathrm{d}t.$$

Extensive use will be given to the following lemma:

Lemma 4.11. *For $p \in \mathbb{N}_0$, we have*

$$|t-s|^{2p}\log|t-s| = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b_{nl}^p T_n(t)T_l(s)$$

where

$$b_{nl}^p = \begin{cases} \mathcal{O}(l^{-(2p+1)}) & n = l, l\pm 2, \dots, l \pm 2p \\ 0 & \text{any other case.} \end{cases}$$

Proof. We proceed by induction. As the case $p = 0$ was proven in Lemma 4.5, we start by setting $p = 1$. By Lemma D.2, it holds

$$|t-s|^2 \log|t-s| = \sum_{j\in\{-1,0,1\}}\sum_{n=0}^{\infty}\beta_n^{(j)} T_n(t) T_{|n+2j|}(s).$$

Moreover, bounds for coefficients $\beta_n^{(j)}$ are found by using Lemma D.2. Since in this case $a_n := b_n^0 = \mathcal{O}(\frac{1}{n})$ (*cf.* Lem. 4.5), we obtain the stated result.

Assuming now that the result holds for $p \geq 2$, we prove it for $p+1$. Indeed,

$$|t-s|^2(|t-s|^{2p}\log|t-s|) = |t-s|^2\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b_{nl}^p T_n(t)T_l(s) = |t-s|^2 \sum_{j\in\{-1,0,1\}^n}\sum_{n=0}^{\infty}\beta_n^{(j)} T_n(t) T_{|n+2\sum j|}(s)$$

and we proceed as in the proof of Lemma D.2 to obtain the expansion. The asymptotic behavior is obtained by a direct computation using expressions of Lemma D.2. ☐

Lemma 4.12. *Let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ with expansion*

$$\widehat{\lambda} = w^{-1}\sum_{n=0}^{\infty} a_n T_n.$$

Then, the Fourier–Chebyshev coefficients of $\widehat{\mathcal{L}}^p[\kappa]\widehat{\lambda}$, denoted $\{v_l^p\}_{l\in\mathbb{N}_0}$, are given by

$$v_l^p = z_p \sum_{n=0}^{\infty} b_{nl}^p a_n,$$

where the coefficients b_{nl}^p are given by Lemma 4.11, and terms z_p are defined in (4.4). Moreover, it holds that

$$v_l^p = \mathcal{O}\left(l^{-2p-\frac{1}{2}}\right).$$

Proof. The representation is a direct consequence of the Fourier–Chebyshev expansion of $\widehat{\lambda}$ and the kernel function given by Lemma 4.11. The asymptotic behavior is deduced as follows

$$|v_l^p| \sim \left|\sum_{n=0}^{\infty} b_{nl}^p a_n\right| \leq \|\widehat{\lambda}\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})}\left|\sum_{n=0}^{\infty}(b_{nl}^p)^2 d_n^{-1}\right|^{\frac{1}{2}} \lesssim l^{-2p-\frac{1}{2}},$$

with d_n coming from (4.2) and where the last inequality is obtained using Lemma 4.11. ☐

With the above results, we can estimate the asymptotic order of the Chebyshev coefficients of $\check{\mathcal{L}}[\kappa] - \check{\mathcal{L}}[0]$, where $\check{\mathcal{L}}[\kappa]$ is the weakly singular Helmholtz operator for the special case $\Gamma \equiv \widehat{\Gamma}$. This bound turns out to be crucial in proving the convergence of the proposed method.

Lemma 4.13. *Let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ be the only solution of Problem 4.10, with $\Gamma = \widehat{\Gamma}$, and expand it as*

$$\widehat{\lambda} = w^{-1} \sum_{n=0}^{\infty} a_n T_n.$$

Then, the coefficients a_n decay as

$$a_n = \mathcal{O}(n^{-m}).$$

Moreover, if g is ρ-analytic, we have that

$$a_n = \mathcal{O}(n\rho^{-n}).$$

Proof. By the regularity of g, we have

$$(\check{\mathcal{L}}[\kappa]\lambda)_l = g_l = \mathcal{O}\left(l^{-m-1}\right).$$

On the other hand, using the integral kernel expansion and Lemma 4.5, for any $Q \in \mathbb{N}$, with $Q > 1$, we derive

$$(\check{\mathcal{L}}[\kappa]\lambda)_l = \frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q-1} v_l^j + v_l^{R(Q)},$$

where coefficients v_l^j are given by Lemma 4.12 and $v_l^{R(Q)}$ is the remainder of order $\mathcal{O}\left(l^{-2Q-\frac{1}{2}}\right)$. Thus, if we choose Q as the upper integer part of $\frac{m+1}{2}$, we have that

$$\frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q-1} v_l^j = \mathcal{O}(l^{-m-1}).$$

From the last equation we need to deduce the behavior of the coefficients a_l given the value of m. We proceed by induction, if $m = 1$ we have that

$$\frac{\pi^2}{4}\frac{a_l}{l} = \mathcal{O}\left(l^{-2}\right),$$

which directly implies $a_l = \mathcal{O}(l^{-1})$. For the induction hypothesis we denote $Q(r)$ the corresponding value of Q given a natural number $r < m$. Then, the induction hypothesis reads as: if

$$\frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q(r)-1} v_l^j = \mathcal{O}\left(l^{-r-1}\right), \tag{4.6}$$

then $a_l = \mathcal{O}(l^{-r})$. Now, we prove for $r+1$, since we do not assume that r is even or odd we have two options: $Q(r+1) = Q(r)$ or $Q(r+1) = Q(r)+1$. If the latter is true, there is a new term of order $-r-1$. Thus, without loss of generality we can assume that

$$\frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q(r)-1} v_l^j = \mathcal{O}\left(l^{-r-1}\right).$$

By the induction hypothesis, $a_l = O(l^{-r})$. Then, by definition of coefficients v_l^j as in Lemmas 4.12 and 4.11 one has

$$\begin{aligned} v_l^1 &= \mathcal{O}\left(l^{-r-3}\right) \\ v_l^2 &= \mathcal{O}\left(l^{-r-5}\right) \\ \vdots &= \vdots \\ v_l^{Q(n)-1} &= \mathcal{O}\left(l^{-r-1-2(Q(r)-1)}\right), \end{aligned}$$

and so from (4.6) we obtain the desire order for a_l.

The ρ-analytic case employs the same argument. As $a_l l^{-1}$ and $\sum_{j=1}^{\infty} v_l^j$ cannot have the same decay order, the only option is for both terms to decay geometrically. □

To end this section, we consider the Helmholtz case for general arcs. Our main ingredients here are the bounds for Chebyshev coefficients of the product of two functions. For one-dimensional $\mathcal{C}^1$-functions, this can be done easily: let $f(t) = \sum_{k\in\mathbb{N}_0} f_k T_k(t)$ and $g(t) = \sum_{l\in\mathbb{N}_0} g_l T_l(t)$. One can write

$$f(t)g(t) = \sum_{n\in\mathbb{N}_0} e_n c_n T_n(t), \quad \text{where} \quad e_n = \int_{-1}^{1} f(t)g(t)\frac{T_n(t)}{w(t)}\,\mathrm{d}t,$$

and $c_0 = \pi^{-1}$, $c_n = 2\pi^{-1}$, for $n > 0$. By replacing the series expansion for f above, we derive

$$e_n = \sum_{k\in\mathbb{N}_0} f_k \int_{-1}^{1} g(t)T_k(t)\frac{T_n(t)}{w(t)}\,\mathrm{d}t,$$

Using now Lemma D.1 and Chebyshev orthogonality, holds that

$$e_n = \sum_{k\in\mathbb{N}_0} f_k \int_{-1}^{1} g(t)\frac{T_{k+n}(t) + T_{|k-n|}(t)}{2w(t)}\,\mathrm{d}t = \sum_{k\in\mathbb{N}_0} \frac{f_k}{2}\left(\frac{g_{k+n}}{c_{k+n}} + \frac{g_{|k-n|}}{c_{|k-n|}}\right).$$

Consequently, we can estimate the decay of e_n by the properties of f_n and g_n. In two dimensions we have a similar result.

Lemma 4.14. *Let $m \in \mathbb{N}$, $p \in \mathbb{N}$, and recall the definition of $R_p(t,s)$ given in (4.5). Then, the series*

$$R_p(t,s)|t-s|^{2p}\log|t-s| = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij}^p T_i(t)T_j(t), \quad \forall (t,s) \in [-1,1]^2,$$

holds, with coefficients

$$C_{ij}^p = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{b_{nl}^p}{4}\left(r_{n+i,l+j} + r_{n+i,|l-j|} + r_{|n-i|,l+j} + r_{|n-i|,|l-j|}\right)$$

with coefficients b_{nl}^p being those of Lemma 4.11 and $r_{i,j}$ the Chebyshev coefficients of $R_p(t,s)$. Moreover, the following asymptotic behavior hold

$$C_{ij}^p = \mathcal{O}\left(\min\left\{i^{-\min(m+1,2p+1)}, j^{-\min(m+1,2p+1)}\right\}\right).$$

If we consider a ρ-analytic arc we have

$$C_{ij}^p = \mathcal{O}\left(\min\left\{i^{-(2p+1)}, j^{-(2p+1)}\right\}\right).$$

Proof. See Appendix B. □

Lemma 4.15. *For $m \in \mathbb{N}$, let $\Gamma \in \mathcal{C}_v^m$ and $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ have the representation:*

$$\widehat{\lambda} = w^{-1}\sum_{n=0}^{\infty} a_n T_n.$$

Then, the Fourier–Chebyshev coefficients of $\widehat{\mathcal{L}}^p[\kappa]\widehat{\lambda}$, denoted $\{v_l^p\}_{l\in\mathbb{N}_0}$, satisfy

$$v_l^p = z_p \sum_{n=0}^{\infty} C_{nl}^p a_n,$$

where the coefficients C_{nl}^p are given in Lemma 4.14, z_p are defined in (4.4), and the asymptotic behaviors hold

(i) *If* $m \leq 2p$ *and for* $\epsilon \in \mathbb{R}$ *such that* $0 < \epsilon < 1 - \frac{1}{m+1}$, $v_l^p = \mathcal{O}\left(l^{-m+(m+1)\epsilon}\right)$.
(ii) *If* $m > 2p$ *and for* $\epsilon \in \mathbb{R}$ *such that* $0 < \epsilon < 1 - \frac{1}{2p+1}$, $v_l^p = \mathcal{O}\left(l^{-2p+(2p+1)\epsilon}\right)$.

Proof. The proof follows the steps of Lemma 4.7 but by using Lemma 4.14 instead of Lemma 4.11. □

Lemma 4.16. *For* $m \in \mathbb{N}$ *with* $m \geq 2$, *let* $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ *be the unique solution of Problem 4.10. Then, if the solution is expanded as* $\widehat{\lambda} = \sum_{n=0}^{\infty} a_n w^{-1} T_n$, *the following asymptotic behaviors for coefficients* a_n *holds*

$$a_n = \mathcal{O}(n^{-m}).$$

Moreover, if Γ *and* g *are* ρ*-analytic*

$$a_n = \mathcal{O}(n\rho^{-n}).$$

Proof. We follow similar steps of those for Lemmas 4.13 and 4.9, the integral equation reads as

$$(\widehat{\mathcal{L}}[\kappa]\widehat{\lambda})_l = \frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q} v_l^j + v_l^R = \mathcal{O}(l^{-m-1}),$$

where v_l^j are defined as in Lemma 4.15, and Q is fixed such that the remainder is given by a $\mathcal{C}_v^m(-1,1)$-function. Thus, for $\epsilon \in (0, 1-\frac{1}{m+1})$, $v_l^R = \mathcal{O}(l^{-m+(m+1)\epsilon})$. Moreover, we can assume that, for $\delta \in (0, 1-\frac{1}{3})$, by Lemma 4.15, it holds $v_l^j = \mathcal{O}(l^{-2j+(2j+1)\delta})$, for all $j = 1, \ldots, Q$. The rest of the proof is the same as in Lemma 4.9, and as before, the ρ-analytic case follows the same arguments. □

4.2.3. Convergence rates for a single arc

From the decay properties of Chebyshev coefficients, we can obtain bounds for the approximation error. First, notice that, by norm equivalences (*cf.* Lem. 2.3), we can do all the estimates in $\widehat{\Gamma}$ and transform $\lambda \mapsto \widehat{\lambda}$. On the other hand, we have the quasi-optimality result (*cf.* Thm. 4.3): there exists $N_0 > 0$ and a constant $C(\Gamma,\kappa) > 0$, such that for all $N > N_0$:

$$\|\lambda - \lambda_N\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma,\kappa) \inf_{q_N \in \mathbb{Q}_N(\widehat{\Gamma})} \left\|\widehat{\lambda} - q_N\right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})}.$$

For $\widehat{\lambda}$ we have an expansion of the form $\widehat{\lambda} = \sum a_n w^{-1} T_n$. Hence, we can choose $q_N = \sum_{n \leq N} a_n w^{-1} T_n$, and use the norm representation to estimate the error as

$$\left\|\widehat{\lambda} - q_N\right\|^2_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} = \sum_{n>N} \frac{|a_n|^2}{n}.$$

Finally, using the bounds from Lemmas 4.16 and 4.9 for the behavior of coefficients a_n, we can establish convergence rates.

Theorem 4.17. *Let* $\kappa > 0$, $m \in \mathbb{N}$ *with* $m \geq 2$, $\Gamma \in \mathcal{C}_v^m$. *For* $g \in \mathcal{C}_v^m(\Gamma)$, *let* λ *be the unique solution of Problem 3.3, and* λ_N *the approximation obtained from the solution of 4.1, with* $N > N_0$ *according to Theorem 4.3. Then there is a constant* $C(\Gamma,\kappa)$, *such that*

$$\|\lambda - \lambda_N\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma,\kappa) N^{-m}.$$

Moreover, if Γ *and* g *are* ρ*-analytic we have that*

$$\|\lambda - \lambda_N\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma,\kappa) \rho^{-N+2}\sqrt{N}.$$

If $\kappa = 0$ *we need also that* $g \in (\widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma))^*$, *for the result to hold true.*

Proof. Following the above discussion, we have to estimate $\sum_{n>N} \frac{|a_n|^2}{n}$, where the a_n are characterized in Lemmas 4.9 and 4.16. Since these are decreasing, the result follows from the following elementary estimation:

$$\sum_{n>N} \frac{|a_n|^2}{n} \leq \int_N^\infty \frac{a(\xi)^2}{\xi} \, \mathrm{d}\xi,$$

the result follows

where $a(\xi)$ is a monotonously continuous decreasing function such that $a(n) = |a_n|$. □

Remark 4.18. Though N_0 and $C(\Gamma, \kappa)$ depend on the geometry and wavenumber κ, the decay rates do not depend on any of these two.

4.2.4. Multiple arcs approximation

Since the existence of more than one arc translates into perturbations of the Chebyshev coefficients with decay rates given by arc regularity, convergence rates for the case of multiple arcs are given by those of the single arc case. To see this, let us recall Problem 3.3 for the case of two $\mathcal{C}_v^m$-arcs pullbacked onto $\widehat{\Gamma}$: for g_1, $g_2 \in C_v^m(\widehat{\Gamma})$, find $\widehat{\lambda}_1, \widehat{\lambda}_2 \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ such that

$$\widehat{\mathcal{L}}_{11}[\kappa]\widehat{\lambda}_1 + \widehat{\mathcal{L}}_{12}[\kappa]\widehat{\lambda}_2 = \widehat{g}_1,$$
$$\widehat{\mathcal{L}}_{21}[\kappa]\widehat{\lambda}_1 + \widehat{\mathcal{L}}_{22}[\kappa]\widehat{\lambda}_2 = \widehat{g}_2.$$

By Assumption 2.2, the arcs cannot touch nor intersect. Hence, there is always $d > 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in \Gamma_1 \times \Gamma_2$, $\|\mathbf{x} - \mathbf{y}\|_2 > d$. This leads to the next result.

Lemma 4.19. *Let $m \in \mathbb{N}$ consider two open $\mathcal{C}_v^m$-arcs fulfilling Assumption 2.2. Then, if we write the pulled back solutions as*

$$\widehat{\lambda}_i = \sum a_n^i \frac{T_n}{w},$$

for $i \neq j$, it holds

$$\left(\widehat{\mathcal{L}}_{ij}[\kappa]\widehat{\lambda}\right)_l = \sum_n b_{nl} a_n,$$

with asymptotic decay rate:

$$b_{nl} = \mathcal{O}\left(\min\left\{n^{-m-1}, l^{-m-1}\right\}\right).$$

Moreover, if the arcs Γ_1, Γ_2 and $\mathbf{g}$ are ρ-analytics we have that

$$b_{nl} = \mathcal{O}\left(\rho^{\min\{-m,-l\}}\right).$$

Proof. As the distance between two disjoint arcs is strictly positive, the kernel $G_\kappa(\mathbf{r}_i(t), \mathbf{r}_j(s))$ belongs to $\mathcal{C}_v^m$ and the proof follows verbatim that of Lemma 4.6. □

Lemma 4.20. *Let $m \in \mathbb{N}$ with $m \geq 2$, and $\boldsymbol{\lambda}$ be the only solution of Problem 3.3, whose pullback is expanded as $\sum_{n \geq \mathbb{N}_0} a_n^j w^{-1} T_n$, it holds*

$$a_n^j = \mathcal{O}(n^{-m}).$$

Moreover, for the ρ-analytic case we have that

$$a_n^j = \mathcal{O}(n\rho^{-n}).$$

Proof. The proof is similar to that of Lemma 4.16, now taking care of cross-interaction terms by Lemma 4.19 and using the same arguments from Lemma 4.9. □

Theorem 4.21. *Let $m \in \mathbb{N}$ with $m > 2$, $\kappa > 0$, $\Gamma \in \mathcal{C}_v^m$, $\mathbf{g} \in \mathcal{C}_v^m(\Gamma)$, $\boldsymbol{\lambda}$ the only solution of Problem 3.3 and $\boldsymbol{\lambda}_N$ approximation constructed from Problem 4.1, then we have the*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) N^{-m+1},$$

and for the ρ-analytic case

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) \rho^{-N+2}\sqrt{N}.$$

For $\kappa = 0$ we need to impose the condition $\mathbf{g} \in (\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma))^*$.

Proof. The proof follows that of Theorem 4.17 as the $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$-norm is equivalent to the Cartesian product of M times the space $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ with corresponding bounds for the coefficients established in Lemma 4.20. □

5. Matrix computations

We now explicitly describe numerically how to solve Problem 4.1 using the discrete spaces defined in Section 4.1. By definition (4.1), the matrix entries are

$$(\mathsf{L}_{ij}[\kappa])_{ln} = \left\langle \mathcal{L}_{ij}[\kappa] q_n^j, q_l^i \right\rangle_{\Gamma_i}.$$

In Remark 4.2, we showed that these can be computed as

$$(\mathsf{L}_{ij}[\kappa])_{ln} = \left\langle \widehat{\mathcal{L}}_{ij}[\kappa] w^{-1} T_n, w^{-1} T_l \right\rangle_{\widehat{\Gamma}}.$$

First, we review how the integrals involving the functions $w^{-1}T_n$ can be approximated.

5.1. Fourier–Chebyshev expansions

Every function in $\mathcal{C}^1([-1,1])$ can be expanded as a Chebyshev series (*cf.* [37], Thm. 3.1),

$$f(s) = \sum_{n=0}^{\infty} f_n T_n(s), \quad \forall s \in [-1,1] \quad \text{with} \quad f_n := c_n \left\langle f, w^{-1} T_n \right\rangle_{\widehat{\Gamma}},$$

with $c_0 = \pi$ and $c_n = \pi/2$ for $n > 0$. For a given $N \in \mathbb{N}$, the Fourier–Chebyshev coefficients $\{f_n\}_{n \in \mathbb{N}_0}$ can be approximated using the FFT as follows:

(i) Construct a vector $\mathbf{v}^N \in \mathbb{C}^{N+1}$ with entries $f(s_n^N)$, for $n = 0, \ldots, N$, and where the $s_n^N = \cos(n\pi/N)$ correspond to the Chebyshev points of order N.

(ii) Apply the FFT to a periodic extension of the vector $\mathbf{f}^N$,

$$\widetilde{\mathbf{f}^N} := \text{FFT}\left(v_N^N, v_{N-1}^N, \ldots, v_1^N, v_0^N, v_1^N, \ldots, v_N^N\right).$$

(iii) Define the approximations as

$$f_n^N := \widetilde{f^N}_n, \quad n = 1, \ldots, N-1, \quad f_0^N = \frac{1}{2}\widetilde{f^N}_0, \quad f_N^N = \frac{1}{2}\widetilde{f^N}_N.$$

Remark 5.1. Notice that Fourier–Chebyshev expansions correspond to the expansions of even functions in Fourier basis under a cosine change of variable.

Using aliasing properties of Chebyshev series, one can easily see that for $f \in \mathcal{C}_v^m(-1,1)$,

$$|f_n - f_n^N| = \mathcal{O}\left(N^{-m-1}\right),$$

while for ρ-analytic functions, it holds

$$|f_n - f_n^N| = \mathcal{O}\left(\rho^{-N}\right).$$

For more details see Chapter 4 of [37].

5.2. Kernel expansion

An expansion similar to the one above holds for the fundamental solution $G_0(\mathbf{x},\mathbf{y})$ when $\kappa = 0$ over $\widehat{\Gamma}$. Specifically, for collinear vectors, *i.e.* $\mathbf{x} = (t,0)$ and $\mathbf{y} = (s,0)$, $(s,t) \in [-1,1]^2$, it holds (*cf.* [29] and [19], Thm. 4.4):

$$G_0(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log|t-s| = \frac{1}{2\pi}\log 2 + \sum_{n\geq 1}\frac{1}{\pi n}T_n(t)T_n(s), \quad \forall s \neq t. \tag{5.1}$$

This series expansion converges point-wise for $t \neq s$ as the fundamental solution is then smooth.

5.3. Computations for $i \neq j$

We consider cross-interactions given by blocks $\mathsf{L}_{ij}[\kappa]$. The associated kernel is smooth, and consequently, we can expand it as a Chebyshev series using the FFT. To this end, we consider a bivariate version of the procedure presented in Section 5.1:

(i) Evaluate the function $F(t,s) := G_\kappa(\mathbf{r}_i(t),\mathbf{r}_j(s))$ in a grid of Chebyshev points (t_i^N, s_j^N), obtaining a matrix $\mathsf{F} \in \mathbb{C}^{(N+1)\times(N+1)}$.

(ii) For each row, we follow steps (i) and (ii) of the one-dimensional procedure detailed in Section 5.1. This leads to the following expansion:

$$F(t,s) = \sum_{n\geq 0} a_n(s)T_n(t),$$

where the coefficients of the matrix are approximations at the Chebyshev points, *i.e.* $\mathsf{F}_{jn} \approx a_n(x_j^N)$, $n = 0,\ldots,N$.

(iii) We repeat the last step but with the columns of the new matrix F, *i.e.* the same one-dimensional procedure for the functions $a_n(s)$, $n = 0,\ldots,N$. The matrix F is updated such that $\mathsf{F}_{ln} \approx a_{ln}$, where

$$F(t,s) = \sum_{l\geq 0}\sum_{n\geq 0} a_{ln}T_l(s)T_n(t).$$

Notice that this procedure requires $2(N+1)$ FFTs. Once the expansion is obtained, the integrals are computed directly using the orthogonality property of Chebyshev polynomials.

5.4. Computations for $i = j$

In this setting, we extract singularities by subtracting the purely logarithmic term:

$$R_k^i(t,s) := -\frac{1}{2\pi}\log|t-s|J_0(\kappa\,\|\mathbf{r}_i(t)-\mathbf{r}_i(s)\|_2),$$

and obtain two families of integrals:

$$I_{ln}^1 := \int_{-1}^{1}\int_{-1}^{1}(G_\kappa(\mathbf{r}_i(t),\mathbf{r}_i(s)) - R_k^i(t,s))w^{-1}T_n(t)w^{-1}T_l(s)\,\mathrm{d}t\,\mathrm{d}s,$$

$$I_{ln}^2 := \int_{-1}^{1}\int_{-1}^{1} R_k^i(t,s)w^{-1}T_n(t)w^{-1}T_l(s)\,\mathrm{d}t\,\mathrm{d}s.$$

Using the expansion ([2], 9.1.13), we find that $G_\kappa(\mathbf{r}_i(t),\mathbf{r}_i(s)) - R_k^i(t,s)$ has the same regularity of $\mathbf{r}_i$, and thus, we can compute I_{ln}^1 as in the case $i \neq j$. For I_{ln}^2, we notice that $R_k^i(t,s)$ is a product of two functions: $-\frac{1}{2\pi}\log|t-s|$, with known Chebyshev expansion (5.1) and $J_0(\kappa\,\|\mathbf{r}_i(t)-\mathbf{r}_i(s)\|_2)$, which by equation (9.1.12) of [2] has the same regularity of $\mathbf{r}_i$. Consequently, its Chebyshev expansion can be computed using the FFT. Finally, the Chebyshev expansion of $R_k^i(t,s)$ is computed using the technique shown in Lemma 4.14.

Remark 5.2. The evaluation of the Chebyshev expansion of $R_k^i(t,s)$ can be accelerated by extrapolation techniques like de-aliasing [16].

6. Compression algorithm

While the presented spectral algorithm reduces the number of dofs needed to obtain a desired accuracy with respect the most common low order h-refinement schemes, we lack any form of matrix compression such as Fast Multipole Method or Hierarchical Matrices ([32], Chap. 7). In what follows, we present a compression algorithm specially designed for problems with multiples arcs. The key idea is to recognize that the entries of the matrix $\mathbf{L}[\kappa]$ correspond to Fourier–Chebyshev coefficients of the kernel function. Hence, for smooth kernels, we observe fast decaying entries, and thus it can be approximated by just considering the first coefficients and setting others to zero. Specifically, the kernel function is smooth when we compute cross-interactions blocks.

Let the routine `Quadrature`$(l,\, m)$ compute the term (l, m) of this interaction matrix using a two-dimensional Gauss–Chebyshev quadrature[3]. Given a tolerance ϵ, we reduce the amount of computations needed by performing the following binary search:

Algorithm 6.1 (H)**.**

INPUT: Tolerance ϵ, Max Level of search $L_{\max}$
OUTPUT: Number of columns to use N_{cols}
INITIALIZE: $N_{\text{cols}} = N$, $level = 0$, $a = 0$, $b = N$
while** $level < L_{\max}$ **do
    $m = (a+b)/2$
    $T_{\text{left}} = m - 1$
    $T_{\text{center}} = m$
    $T_{\text{right}} = m + 1$
    $V_{\text{left}} = abs(Quadrature(0,\, T_{\text{left}}))$
    $V_{\text{center}} = abs(Quadrature(0,\, T_{\text{center}}))$
    $V_{\text{right}} = abs(Quadrature(0,\, T_{\text{right}}))$
    if** $\{V_{\text{right}} \& V_{\text{center}} < 0.5 * \epsilon\}$ or $\{V_{\text{left}} \& V_{\text{center}} < 0.5 * \epsilon\}$ **then
        $b = m$
    else
        $a = m$
    end if
    $level{+}{+}$
end while
$N_{\text{cols}} = b$

The algorithm returns the minimum number of columns N_{cols} to be used, by searching in the first row the minimum index such that the absolute value of the matrix entries is lower than ϵ. The binary search is restricted to a depth $L_{\max} \in \mathbb{N}$. The same procedure is used to estimate the number of rows, N_{rows}, by executing a binary search in the first column. Once N_{cols} and N_{rows} are selected, we define $N_\epsilon := \max\{N_{\text{rows}}, N_{\text{cols}}\}$ and compute the block of size $N_\epsilon \times N_\epsilon$ as in the full implementation.

Matrix compression also induces an extra error as it perturbs the original linear system in Problem 4.1. We can bound this error using the standard theory of perturbed linear systems. To that end, denote by $\mathbf{L}_\epsilon[\kappa]$ the matrix generated by the compression algorithm with tolerance ϵ, and define the matrix difference $\Delta\mathbf{L}_\epsilon[\kappa] := \mathbf{L}_\epsilon[\kappa] - \mathbf{L}[\kappa]$. We seek to control the solution $\mathfrak{u}^\epsilon = \mathfrak{u} + \Delta\mathfrak{u}$ of

$$(\mathbf{L}[\kappa] + \Delta\mathbf{L}_\epsilon[\kappa])\mathfrak{u}^\epsilon = \mathfrak{g},$$

where $\mathfrak{u}$ and $\mathfrak{g}$ are the same as in Problem 4.1. In order to bound this error, we will assume that, for every pair of indices (i, j) in the matrix $\mathbf{L}[\kappa]$, we have,

$$|(\Delta\mathbf{L}_\epsilon[\kappa])_{ij}| < \epsilon. \tag{6.1}$$

[3] We make the following approximation $\int_{\Gamma_i}\int_{\Gamma_j} G_\kappa(\mathbf{x},\mathbf{y}) q_m^j(\mathbf{x}) q_l^i(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \approx \sum_{p=1}^{N_q}\sum_{r=1}^{N_q} \omega_p\omega_r G_\kappa(\mathbf{r}_i(x_p), \mathbf{r}_j(x_r)) T_m(x_r) T_l(x_p)$, where ω_p, x_p denote the Gauss–Chebyshev quadrature weights and points respectively, and N_q is the number of points to use.

Theorem 6.2. *Let $N \in \mathbb{N}$ be such there is only one solution of Problem 4.1. Then, there is a constant $C(\Gamma,\kappa) > 0$ such that*

$$\frac{\|\Delta \mathbf{u}\|_2}{\|\mathbf{u}\|_2} \leq \left| \frac{N\epsilon}{C(\kappa,\Gamma) - N\epsilon} \right|.$$

Proof. By Section 1.13.2 of [30] we have that

$$\frac{\|\Delta \mathbf{u}\|_2}{\|\mathbf{u}\|_2} \leq \frac{\|\Delta \mathbf{L}_\epsilon[\kappa]\|_2}{\|(\mathbf{L}[\kappa])^{-1}\|_2 - \|\Delta \mathbf{L}_\epsilon[\kappa]\|_2},$$

and thus, we need to estimate $\|\Delta \mathbf{L}_\epsilon[\kappa]\|_2$ and $\left\|(\mathbf{L}[\kappa])^{-1}\right\|_2$. The bound for the first term can be obtained as

$$\|\Delta \mathbf{L}_\epsilon[\kappa]\|_2 = \sup_{\mathbf{x}\neq 0} \frac{\|\Delta \mathbf{L}_\epsilon[\kappa]\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \sup_{\mathbf{x}\neq 0} \frac{\left(\sum_{i=0}^N (\sum_{j=0}^N \Delta \mathbf{L}_\epsilon[\kappa]_{ij} x_j)^2\right)^{1/2}}{\|\mathbf{x}\|_2} \leq \sup_{\mathbf{x}\neq 0} \frac{\left(\sum_{i=0}^N \|\mathbf{x}\|_2^2 N\epsilon^2\right)^{1/2}}{\|\mathbf{x}\|_2} \leq N\epsilon.$$

To estimate $\left\|(\mathbf{L}[\kappa])^{-1}\right\|_2$, we have on one hand the classical result $\left\|(\mathbf{L}[\kappa])^{-1}\right\|_2 \geq \|(\mathbf{L}[\kappa])\|_2^{-1}$. On the other hand, by the operator continuity it is easy to see that

$$\|(\mathbf{L}[\kappa])\|_2 \leq C(\kappa,\Gamma),$$

the results follows directly from the latter estimation. For $\kappa = 0$, the proof is analogous with the corresponding change in the spaces. □

We can also estimate the error introduced by the compression algorithm in terms of the energy norm. In order to do so, define $(\boldsymbol{\lambda}_N^\epsilon)_i := \sum_{m=0}^N (\mathbf{u}_i^\epsilon)_m q_m^i$ in Γ_i. By the same arguments in the above proof, we obtain

$$\|\boldsymbol{\lambda}_N - \boldsymbol{\lambda}_N^\epsilon\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C_1(\kappa,\Gamma) \, \|\mathbf{g}\|_{\mathbb{H}}^{\frac{1}{2}}(\Gamma) \frac{\epsilon N^{\frac{3}{2}}}{C(\kappa,\Gamma) - \epsilon N},$$

where $\mathbf{g}$ is the same that in Problem 3.3, C_1 is the constant of Theorem 4.3, and an extra factor $N^{\frac{1}{2}}$ appears as $\|\mathbf{u}\|_2 \leq N^{\frac{1}{2}} \|\boldsymbol{\lambda}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C_1 N^{1/2} \|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}$.

Remark 6.3. We can use the compression algorithm to make a fast version of the matrix-vector product by splitting the product into blocks, and using the sparse representation for the cross interaction blocks.

Remark 6.4. For the Laplace case $\kappa = 0$, it is also possible to obtain sparse approximations of the self-interaction blocks. We refer to [22], for details, and also for a more complete analysis of similar the compression algorithm.

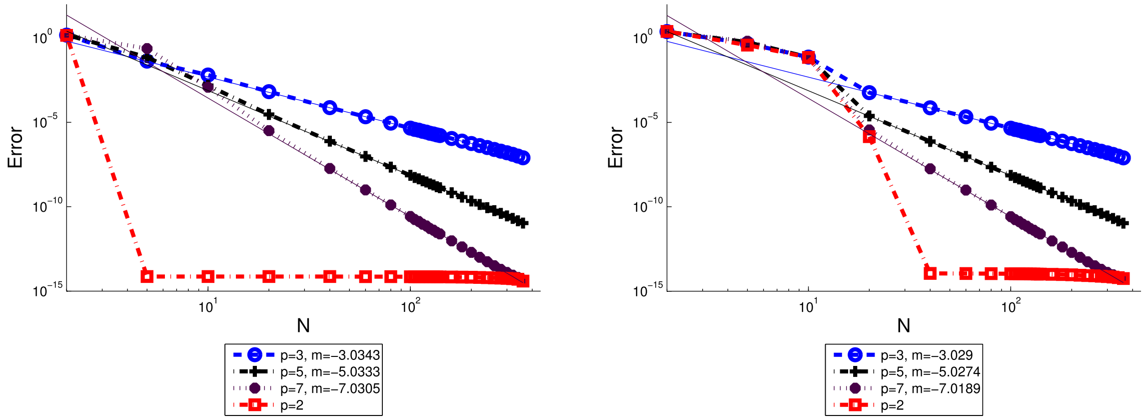
7. Numerical results

7.1. Convergence results

In what follows, we show experimental results confirming the convergence rates proven in Theorem 4.21. Let us first consider the case of a single arc $\widehat{\Gamma}$ and an excitation g with limited regularity. Figure 1 presents convergence results for different excitation functions. The first three are of the form $g(t) = |t|^p$, with $p = 3, 5, 7$. For these, g is in $\mathcal{C}_v^p(\widehat{\Gamma})$. Hence, by Theorem 4.21, we should observe the following error bounds:

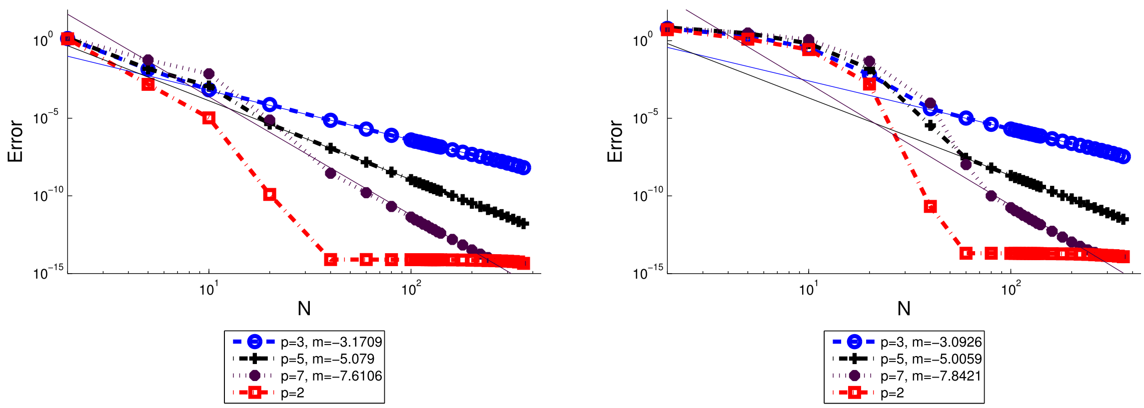
$$\text{Error} := \|\lambda - \lambda_N\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} = \mathcal{O}(N^{-p}).$$

Thus, we have that the error as a function of N has a slope of p in logarithmic scale. The fourth case has as right-hand side $g(t) = t^2$, and being an entire function, we observe the corresponding super-algebraic convergence.



(a) (b)

FIGURE 1. $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ errors, for $g(t) = |t|^p$. Values m are slopes of $\log_{10}(\text{Error})$ respect to $\log_{10} N$. Errors are computed with respect to an overkill solution with $N = 440$. (a) Laplace. (b) Helmholtz $\kappa = 10$.



(a) (b)

FIGURE 2. $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ errors, for Γ given by $\mathbf{r}(t) = (t, |t|^p)$ and $g(t) = t^2$. Values m are slopes of $\log_{10}(\text{Error})$ respect to $\log_{10} N$. Errors are computed with respect to an overkill solution with $N = 440$. (a) Laplace. (b) Helmholtz $\kappa = 10$.

Figure 2 shows convergence results for geometries with limited regularity and smooth excitation. Just as in the case of source terms of limited regularity, we obtain the convergence rates stated in Theorem 4.21.

Lastly, we consider the case of multiple arcs and where the excitation function and the geometry are smooth (see Fig. 3). We observe exponential error convergence in the polynomial degree, which is the same for each arc, as predicted. We also observe that, as a function of κ, the errors are increasingly bounded by below. Our experiments shows that this effect is caused by errors in the solution of the linear system, which is currently solved by a direct method (the residual $\|\mathsf{L}[\kappa]\lambda - b\|_2$ dominates the convergence error, see Fig. 3c). For the sake of brevity, we will not attempt to solve this anomaly, as it is a common issue when computing waves scattered by disjoint domains (*cf.* [12]). We remark that the $\widetilde{H}^{-\frac{1}{2}}$–norms are computed using expression (4.2).

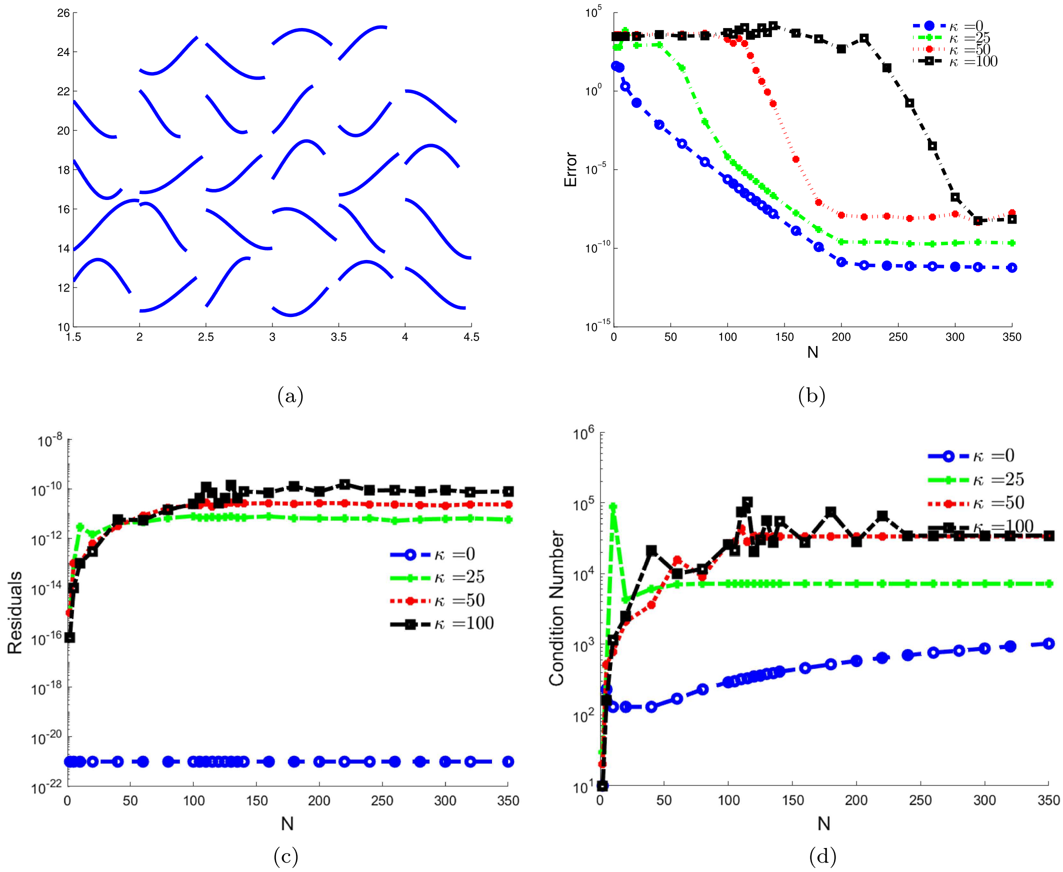


FIGURE 3. In (a), a smooth geometry with $M = 28$ open arcs, each with a parametrization $(at, c\sin(bt)) + \mathbf{d}$, where $a \in [0.45, 0.50]$, $b \in [1.0, 1.5]$, $c \in [1.0, 1.3]$, $\mathbf{d} \in [2, 3.5] \times [11, 25]$, and $t \in [-1, 1]$. In (b), convergence for the corresponding geometry and different wavenumbers using as right-hand side the trace of $g(\mathbf{x}) = \exp -i\hat{\kappa}\mathbf{x}\cdot\mathbf{y}$, where $\hat{\kappa} = \kappa$ for $k > 0$, $\hat{0} = 5$, $\mathbf{y} = (\cos\alpha, \sin\alpha)$, and $\alpha = \pi/4$. The x-axis denotes the number of polynomials used per arc. Errors are computed with respect to an overkill solution with $N = 500$ per arc. The mean arc lengths in terms of the wavelength are $8\lambda, 16\lambda, 32\lambda$ for $\kappa = 25, 50, 100$ respectively. In (c) the error in the solution of the linear system for the different cases, and in (d) the corresponding conditioning number (in norm 2) for the linear systems. (a) Geometry. (b) Convergence $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\widehat{\Gamma})$-norm. (c) Residuals: $\|\mathbf{L}\lambda - b\|_2$. (d) Condition number.

7.2. Compression results

We consider the test cases presented in Figure 3. Tables 1 and 2 showcase different measurements of the performance of the compression algorithm. We denote by: % NNZ, the percentage of non zero entries of the compressed matrix; Rel. Error, to the maximum absolute value between of the difference of uncompressed and compressed matrices; GMRES Full, the time (in seconds) that takes to solve the full linear system using GMRES with a tolerance of $1e{-8}$; and, GMRES Sparse, same as last but with compressed matrix and an optimized version of the matrix vector product. For the sake of completeness, we have also included the assembly times (in seconds) for the full matrix (Full Assembly), and the compressed one (Sparse Assembly), and observe that

Table 1. Compression performance $\epsilon = 1e{-}10$, $\kappa = 100$.

Order	% NNZ	Rel. Error	GMRES Full	GMRES Sparse	Full Assembly	Sparse Assembly
			$L_{\max} = 1$			
250	24	$1e{-}10$	25	12	109	96
300	24	$1e{-}10$	37	15	163	148
350	24	$1e{-}10$	48	19	215	198
400	24	$1e{-}10$	62	23	309	294
			$L_{\max} = 2$			
250	6	$1e{-}10$	25	8	109	95
300	6	$1e{-}10$	37	10	163	147
350	6	$1e{-}10$	48	12	215	198
400	6	$1e{-}10$	62	13	309	285
			$L_{\max} = 3$			
250	5	$1e{-}10$	25	7	109	95
300	3	$1e{-}10$	37	9	163	147
350	2	$1e{-}10$	48	9	215	196
400	1.7	$1e{-}10$	62	11	309	286

Table 2. Compression performance $\epsilon = 1e{-}14$, $\kappa = 100$.

Order	% NNZ	Rel. Error	GMRES Full	GMRES Sparse	Full Assembly	Sparse Assembly
			$L_{\max} = 1$			
250	24	$1e{-}14$	25	12	109	96
300	24	$1e{-}14$	37	16	163	149
350	25	$1e{-}14$	48	20	215	199
400	25	$1e{-}14$	62	24	309	294
			$L_{\max} = 2$			
250	6	$1e{-}14$	25	8	109	96
300	6	$1e{-}14$	37	10	163	147
350	7	$1e{-}14$	48	12	215	199
400	7	$1e{-}14$	62	14	309	284
			$L_{\max} = 3$			
250	5	$1e{-}14$	25	8	109	96
300	4	$1e{-}14$	37	10	163	148
350	5	$1e{-}14$	48	11	215	196
400	4	$1e{-}14$	62	12	309	283

they do not differ much as the most expensive part for this relative small problems is the computation of the self-interaction matrices.

8. Concluding remarks

The present work presents a high-order discretization method for the wave scattering by multiple disjoint arcs based on weighted polynomials bases with proven convergence rates similar to the classical interpolation theory of smooth functions. As an efficient solver for the forward problem, our method could be easily used for solving optimization or inverse problems, tasks which are currently under development. Still, for increasing frequencies and numbers of arcs, we remark that the solution of the resulting linear system can become a bottleneck, thus requiring further improvements.

Appendix A. Laplace uniqueness result

We define the energy space of homogeneous boundary condition as

$$W_0(\Omega) := \left\{ U \in W(\Omega) : \gamma_i^{\pm} U = 0, \quad \text{for } i = 1, \dots, M \right\}.$$

We also will need the traces over the complementary arcs $\Gamma_i^c := \partial\Omega_i \setminus \overline{\Gamma}_i$ that we denote them as $\gamma_{i^c}^{\pm}$ and $\gamma_{N,i^c}^{\pm}$ respectively. The following technical results will be needed, we omit the proofs as they can be found in the given references.

Lemma A.1 ([19], Lem. 2.2)**.** *The semi-norm $|U|_{W(\Omega)} := \|\nabla U\|_{L^2(\Omega)}$ bounds the $W(\Omega)$-norm for functions in $W_0(\Omega)$, i.e. there exists a constant $c > 0$ such that*

$$\|U\|_{W(\Omega)} \leq c\, |U|_{W(\Omega)}, \quad \forall U \in W_0(\Omega).$$

Lemma A.2 ([19], Prop. 2.6)**.** *Let U belong to $W(\Omega)$ such that $-\Delta U \in L^2_{\text{loc}}(\Omega)$. For $R > 0$, denote the ball of radius R centered at the origin by $B_R := \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < R \right\}$. Then,*

$$\lim_{R \to \infty} \left\langle \gamma_{N,R}^{-} U, \gamma_R^{-} V \right\rangle_{\partial B_R} = 0, \quad \forall V \in W(\Omega),$$

where γ_R^- and $\gamma_{N,R}^-$ denote interior Dirichlet and Neumann traces on ∂B_R, respectively, the latter being equivalent to the radial derivative on the boundary.

Lemma A.3 ([13], Thm. 1.7.1)**.** *Let $V \in W_0(\Omega)$. Then it holds*

$$\gamma_{i^c}^{+} V = \gamma_{i^c}^{-} V \quad i \, \{1, \dots, M\}.$$

Hence, we can denote indistinctly by γ_{i^c} the trace defined over Γ_i^c on $W_0(\Omega)$.

Lemma A.4 ([19], Sect. 2.6.1)**.** *Let a function $U \in W_0(\Omega)$ such that $-\Delta U = 0$ in Ω. Then, the normal jump on Γ_i^c is null, i.e. $\gamma_{N,i^c}^{+} U - \gamma_{N,i^c}^{-} U = 0$.*

Lemma A.5. *If $U \in W_0(\Omega)$, is such that $\Delta U = 0$, then $U = 0$.*

Proof. Let $\Omega_* := \bigcup_{j=1}^{M} \Omega_j$, where the collection is disjoint by Assumption 2.2, and choose $R > 0$ such that $\Omega_* \subset B_R$. Set $\Omega_0(R) := B_R \cap \overline{\Omega}_*^c$. We have that $\nabla U, \nabla V \in L^2(B_R)$ (as they are in $L^2(\Omega)$), hence

$$\langle \nabla U, \nabla V \rangle_{B_R} = \sum_{i=1}^{M} \langle \nabla U, \nabla V \rangle_{\Omega_i} + \langle \nabla U, \nabla V \rangle_{\Omega_0(R)}.$$

Using the Green formulas, and the null condition of V in Γ we obtain that

$$\langle \nabla U, \nabla V \rangle_{\Omega_i} = \langle -\Delta U, V \rangle_{\Omega_i} + \left\langle \gamma_{N,i^c}^{+} U, \gamma_{i^c} V \right\rangle_{\Gamma_i^c}$$

$$\langle \nabla U, \nabla V \rangle_{\Omega_0(R)} = \langle -\Delta U, V \rangle_{\Omega_0(R)} + \langle \gamma_{N,R} U, \gamma_R V \rangle_{\partial B_R} - \sum_{i=1}^{M} \left\langle \gamma_{N,i^c}^{-} U, \gamma_{i^c} V \right\rangle_{\Gamma_i^c}.$$

Finally adding the two terms and using Lemma A.4, and the condition $-\Delta U = 0$ in Ω we have that

$$\langle \nabla U, \nabla V \rangle_{B_R} = \langle \gamma_{N,R} U, \gamma_R V \rangle_{\partial B_R}.$$

The results follows directly from this last equation, and Lemmas A.1 and A.2. ∎

Appendix B. Technical lemmas

B.1. Proof of Lemma 2.3

We only need to proof for $H^{1/2}$ as the $\tilde{H}^{-1/2}$ case is obtained by duality arguments. By definition, it holds

$$\|\zeta\circ\mathbf{r}_i\|^2_{H^{\frac{1}{2}}(\hat{\Gamma})} = \int_{-1}^{1} |\zeta\circ\mathbf{r}_i(t)|^2\,\mathrm{d}t + \int_{-1}^{1}\int_{-1}^{1} \frac{|\zeta\circ\mathbf{r}_i(t)-\zeta\circ\mathbf{r}_i(s)|^2}{|t-s|^2}\,\mathrm{d}t\,\mathrm{d}s. \tag{B.1}$$

For the first integral on the right-hand side, we deduce

$$\begin{aligned}\int_{-1}^{1} |\zeta\circ\mathbf{r}_i(t)|^2\,\mathrm{d}t &= \int_{-1}^{1} |\zeta\circ\mathbf{r}_i|^2 \frac{\|\mathbf{r}_i'(t)\|_2}{\|\mathbf{r}_i'(t)\|_2}\,\mathrm{d}t = \int_{\Gamma_i} \frac{|\zeta|^2}{\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}\right\|_2}\mathrm{d}\Gamma_i \\ &\le \left\|\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}\right\|_2^{-1}\right\|_{L^\infty(\Gamma_i)} \int_{\Gamma_i} |\zeta|^2\mathrm{d}\Gamma_i. \end{aligned}\tag{B.2}$$

Similarly, by changing variables, the second term in (B.1) becomes

$$\int_{\Gamma_i}\int_{\Gamma_i} \frac{|\zeta(\mathbf{x})-\zeta(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|_2^2}\left(\frac{\|\mathbf{x}-\mathbf{y}\|_2^2}{\left\|\mathbf{r}_i^{-1}(\mathbf{x})-\mathbf{r}_i^{-1}(\mathbf{y})\right\|_2^2}\right)\frac{\mathrm{d}\Gamma_i(\mathbf{x})\mathrm{d}\Gamma_i(\mathbf{y})}{\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}(\mathbf{x})\right\|_2\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}(\mathbf{y})\right\|_2}.$$

Using the mean value theorem for $\mathbf{r}_i^{-1}$, we arrive at

$$\int_{-1}^{1}\int_{-1}^{1} \frac{|\zeta\circ\mathbf{r}_i(t)-\zeta\circ\mathbf{r}_i(s)|^2}{|t-s|^2}\,\mathrm{d}t\,\mathrm{d}s \le C_i \int_{\Gamma_i}\int_{\Gamma_i} \frac{|\zeta(\mathbf{x})-\zeta(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|_2^2}\mathrm{d}\Gamma_i(\mathbf{x})\mathrm{d}\Gamma_i(\mathbf{y}), \tag{B.3}$$

where

$$C_i = \left\|\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}\right\|_2^{-1}\right\|^4_{L^\infty(\Gamma_i)}.$$

Using (B.2) and (B.3) to define C we obtain the following inequality

$$\|\zeta\circ\mathbf{r}_i\|_{H^{\frac{1}{2}}(\hat{\Gamma})} \le C\,\|\zeta\|_{H^{\frac{1}{2}}(\hat{\Gamma}_i)}.$$

The second equivalence inequality is obtained using the same arguments.

B.2. Proof of Lemma 4.6

For any $s\in[-1,1]$, we can write the univariate Fourier–Chebyshev expansion in t:

$$h(t,s) = \sum_{n=0}^{\infty} a_n(s)T_n(t), \quad \forall t\in[-1,1].$$

In fact, the regularity of $h(t,\cdot)$ implies that the functions $a_n(s)$ belong to $\mathcal{C}^m(-1,1)$, and consequently, one can write down expansions:

$$a_n(s) = \sum_{k=0}^{\infty} b_{nk}T_k(s), \quad \forall s\in[-1,1], \quad \forall n\in\mathbb{N}_0.$$

If $m<\infty$, by Theorem 7.1 of [37], we have that $b_{nk}\lesssim k^{-m}$, where the constant depends on the m-th derivative of $a_n(s)$, which is bounded by the m-th derivative of h in s.

For the ρ-analytic case we have by Theorem 8.1 of [37] that $b_{nk}\lesssim \rho_n^{-k}$, with $\rho_n>1$. However, the coefficients $a_n(s)$ are given by

$$a_n(s) = c_n\int_{-1}^{1} h(t,s)w^{-1}(t)T_n(t)\,\mathrm{d}t,$$

where $c_0 = \pi^{-1}$, and $c_n = 2\pi^{-1}$, for $n \in \mathbb{N}$. Hence, since $h(t,\cdot)$ is ρ-analytic, we have that, for every z in the corresponding ellipse we can write

$$a_n(z) = \sum_{p\geq 0} z^p \int_{-1}^{1} A_p(t) w^{-1}(t) T_n(t)\,\mathrm{d}t,$$

where $A_p(t)$ are the coefficients of the power series of $h(t,\cdot)$. From this last expression, we have that a_n is analytic in the ellipse of parameter ρ for every n, and thus, we can take $\rho_n = \rho$ for every $n \in \mathbb{N}\cup\{0\}$.

The final result is obtained by repeating the above arguments inverting the roles of n and k.

B.3. Proof of Lemma 4.7

Consider $f = \sum_{n\geq 0} a_n w^{-1} T_n(t)$, by Lemma 4.6, we expand $h(t,s)$ as the series $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{nk} T_n(t) T_k(s)$. Hence, by the Chebyshev polynomials' orthogonality property, we can write

$$v_l = \frac{\pi^2}{4}\sum_{n=1}^{\infty} b_{nl} a_n + \frac{\pi^2}{2} b_{0l} a_0, \quad \forall l > 0.$$

Thus, by definition of constants d_n (4.2) and the series expression for $\widetilde{H}^{-1/2}(\widehat{\Gamma})$-norm, we obtain the following bound:

$$|v_l|^2 \lesssim \|f\|^2_{\widetilde{H}^{-1/2}(\widehat{\Gamma})} \sum_{n=0}^{\infty} |b_{nl}|^2 d_n^{-1}.$$

From here the result is direct if h is bivariate ρ-analytic function. For $m \in \mathbb{N}$, using Lemma 4.6, it holds

$$|b_{nl}|^2 \lesssim l^{-2(m+1)\mu} n^{-2(m+1)(1-\mu)}, \quad \forall \mu \in (0,1).$$

With the above bound and the estimate $d_n \sim n^{-1}$, we arrive to

$$|v_l|^2 \lesssim \|f\|^2_{\widetilde{H}^{-1/2}(\widehat{\Gamma})} l^{-2(m+1)\mu} \sum_{n=1}^{\infty} n^{-2(m+1)(1-\mu)+1},$$

by choosing $\mu = 1 - \frac{1}{m+1} - \epsilon$, the series in the right-hand side converges and we get the stated result.

Appendix C. Basic approximation properties

Lemma C.1. *The discretization is conforming, i.e.* $\mathbb{Q}_N(\Gamma_i) \subset \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ *(resp.* $\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i) \subset \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$*).*

Proof. For any $\zeta^i \in \mathbb{Q}_N(\Gamma_i)$ the representation:

$$\zeta^i = \frac{\widehat{p}\circ \mathbf{r}_i^{-1}}{w_i \left\|\mathbf{r}_i' \circ \mathbf{r}_i^{-1}\right\|_2},$$

holds, where $\widehat{p}$ is a polynomial in $(-1,1)$. By definition of dual norms, one can write

$$\left\|\zeta^i\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} = \sup_{\vartheta\in H^{\frac{1}{2}}(\Gamma_i)} \frac{\left\langle \zeta^i, \vartheta\right\rangle_{H^{\frac{1}{2}}(\Gamma_i)}}{\left\|\vartheta\right\|_{H^{\frac{1}{2}}(\Gamma_i)}}.$$

At the same time, it holds

$$\left\langle \zeta^i, \vartheta \right\rangle_{\Gamma_i} = \int_{-1}^{1} \frac{\widehat{p}(t)}{\sqrt{1-t^2}} (\vartheta \circ \mathbf{r}_i)(t)\, \mathrm{d}t \leq \|\widehat{p}\|_{L^\infty(-1,1)} \int_{-1}^{1} \frac{(\vartheta \circ \mathbf{r}_i)(t)}{w(t)}\, \mathrm{d}t$$
$$\leq \|\widehat{p}\|_{L^\infty(-1,1)} \left\| w^{-1} \right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \left\| \vartheta \circ \mathbf{r}_i \right\|_{H^{\frac{1}{2}}(\widehat{\Gamma})},$$

where $w(t) := \sqrt{1-t^2}$. Applying Lemma 2.3, we only need to check that the $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$-norm of w^{-1} is finite, which was already proved in Lemma 6.1.19 of [17]. The inclusion for the mean-zero spaces is immediate from the Chebyshev polynomials' orthogonality property. □

Lemma C.2. *The family* $\{\mathbb{Q}_N(\Gamma_i)\}_{N\in\mathbb{N}}$ *is dense in* $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$*, while the family* $\left\{\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)\right\}_{N\in\mathbb{N}}$ *is dense in* $\widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$.

Proof. We only need to prove that there is a fixed constant C such that, for a given $\epsilon > 0$ and $\phi \in \mathcal{D}(\Gamma_i)$, there exists $\zeta^i \in \mathbb{Q}_N(\Gamma_i)$ satisfying

$$\left\| \zeta^i - \phi \right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq C\epsilon.$$

By Lemma 6.1.20 of [17], there exists a polynomial $\widehat{p} \in \mathbb{P}_N(-1,1)$ satisfying

$$\left\| w^{-1}\widehat{p} - \|\mathbf{r}_i'\|_2 \,(\phi \circ \mathbf{r}_i) \right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} < \epsilon.$$

Let $\zeta^i = \dfrac{\widehat{p} \circ \mathbf{r}_i}{w_i \left\| \mathbf{r}_i' \circ \mathbf{r}_i^{-1} \right\|_2}$. Again, we take the dual norm

$$\left\| \zeta^i - \phi \right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} = \sup_{\vartheta \in H^{\frac{1}{2}}(\Gamma_i)} \frac{\left\langle \zeta^i - \phi, \vartheta \right\rangle_{\Gamma_i}}{\|\vartheta\|_{H^{\frac{1}{2}}(\Gamma_i)}}.$$

We can write

$$\left\langle \zeta^i - \phi, \vartheta \right\rangle_{\Gamma_i} = \int_{\Gamma_i} (\zeta^i - \phi)(\mathbf{x}) \vartheta(\mathbf{x}) \mathrm{d}\Gamma_i(\mathbf{x})$$
$$= \int_{-1}^{1} \left(w^{-1}(t)\widehat{p}(t) - \|\mathbf{r}_i'\|_2 (t) (\phi \circ \mathbf{r}_i)(t) \right) (\vartheta \circ \mathbf{r}_i)(t)\, \mathrm{d}t.$$

By Lemma 2.3, there exists a constant C independent of ϵ such that

$$\left\langle \zeta^i - \phi, \vartheta \right\rangle_{\Gamma_i} \leq C \|\vartheta\|_{H^{\frac{1}{2}}(\Gamma_i)} \left\| w^{-1}\widehat{p} - \|\mathbf{r}_i'\|_2 (\phi \circ \mathbf{r}_i) \right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \leq C\epsilon \|\vartheta\|_{H^{\frac{1}{2}}(\Gamma_i)},$$

and thus $\left\| \zeta^i - \phi \right\|_{\mathcal{H}^i} \leq C\epsilon$ as stated.

For the family $\left\{\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)\right\}_{N\in\mathbb{N}}$, by the previous result, we observe that, given $\phi \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$ and $\epsilon > 0$. there exists $N \in \mathbb{N}$ and $\zeta^i \in \mathbb{Q}_N(\Gamma_i)$, such that

$$\left\| \zeta^i - \phi \right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq \epsilon.$$

Thus, by the definition of the norm in $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, it holds

$$\left\langle \zeta^i, 1 \right\rangle_{\Gamma_i} = \left\langle \zeta^i - \phi, 1 \right\rangle_{\Gamma_i} \leq \left\| \zeta^i - \phi \right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}.$$

Hence, we can define $\zeta_0^i := \zeta^i - |\Gamma_i|^{-1} \left\langle \zeta^i, 1 \right\rangle_{\Gamma_i}$, where $|\Gamma_i|$ is the length of the arc Γ_i. Now, it is direct that $\zeta_0^i \in \mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)$ and

$$\left\| \zeta_0^i - \phi \right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq 2\epsilon,$$

which gives the desired density. □

C.1. Proof of Lemma 4.14

We proceed as in the one-dimensional case and assume, for simplicity, that the Chebyshev polynomials are normalized, thus omitting constants c_n. The coefficients C_{ij}^p are given by

$$\begin{aligned} C_{ij}^p &= \int_{-1}^{1}\int_{-1}^{1} R_p(t,s)|t-s|^{2p}\log|t-s|\frac{T_i(t)}{w(t)}\frac{T_j(s)}{w(s)}\,\mathrm{d}t\,\mathrm{d}s \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b_{nl}^p \int_{-1}^{1}\int_{-1}^{1} R_p(t,s)\frac{1}{4}\frac{T_{n+i}(t)+T_{|n-i|}(t)}{w(t)}\frac{T_{l+j}(s)+T_{|l-j|}(s)}{w(s)}\,\mathrm{d}t\,\mathrm{d}s \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{b_{nl}^p}{4}(r_{n+i,l+j}+r_{n+i,|l-j|}+r_{|n-i|,l+j}+r_{|n-i|,|l-j|}). \end{aligned}$$

Now, we have to find the decay order for the different terms. Define the index set $I_p(l) := \{l, l\pm 2, l\pm 4, \dots, l\pm 2p\}$. By Lemma 4.11, we have the estimate:

$$C_{ij}^p \sim \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1}\left(r_{n+i,l+j}+r_{n+i,|l-j|}+r_{|n-i|,l+j}+r_{|n-i|,|l-j|}\right). \tag{C.1}$$

By Lemma 4.6, it holds

$$r_{\nu,\mu} = \mathcal{O}\left(\min\left\{\nu^{-m-1},\mu^{-m-1}\right\}\right), \quad \text{for } \nu,\mu\in\mathbb{N},$$

and we can estimate each term in C_{ij}^p as follows, we provide details for the first two.

Define $K_1 := \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1} r_{n+i,l+j}$. Assume that $r_{n+i,l+j} = \mathcal{O}((l+j)^{-m-1})$, then

$$K_1 \lesssim 2p\sum_{l=1}^{\infty} l^{-2p-1}(l+j)^{-m-1} = \mathcal{O}(j^{-m-1}).$$

Alternatively, we can use that $r_{n+i,l+j} = \mathcal{O}((n+i)^{-m-1})$ so that

$$K_1 \lesssim \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1}(n+i)^{-m-1} = \mathcal{O}(i^{-m-1}).$$

Thus, we then conclude that

$$K_1 = \mathcal{O}\left(\min\left\{i^{-m-1}, j^{-m-1}\right\}\right).$$

Now set $K_2 := \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1} r_{n+i,|l-j|}$. Let $r_{n+i,|l-j|} = \mathcal{O}((|l-j|+1)^{-m-1})$, we obtain

$$K_2 \lesssim \sum_{l=1}^{\infty} l^{-2p-1}(|l-j|+1)^{-m-1},$$

where we added one to avoid infinity. Thus, we can split this last sum into two terms

$$K_2 \lesssim \sum_{l=1}^{j/2} l^{-2p-1}(j-l)^{-m-1} + \sum_{l>j/2} l^{-2p-1}(|l-j|+1)^{-m-1}.$$

The first one is bounded as

$$\sum_{l=0}^{j/2} l^{-2p-1}(j-l)^{-m-1} \lesssim j^{-m-1}\sum_{l=0}^{j/2} l^{-2p-1} \lesssim j^{-m-1},$$

whereas the second one

$$\sum_{l>j/2} l^{-2p-1}(|l-j|+1)^{-m-1} \lesssim j^{-2p-1}.$$

Hence, we have

$$K_2 = \mathcal{O}(j^{-m-1}) + \mathcal{O}(j^{-2p-1}) = \mathcal{O}\left(j^{-\min\{m,2p+1\}}\right).$$

If alternatively we use $r_{n+i,|l-j|} = \mathcal{O}((n+i)^{-m-1})$, then

$$K_2 \lesssim \sum_{l=0}^{\infty} l^{-2p-1}(n+i)^{-m-1} = \mathcal{O}(i^{-m-1}).$$

Combining both results yields

$$K_2 = \mathcal{O}\left(\min\left\{i^{-m-1}, j^{-\min\{m+1,2p+1\}}\right\}\right).$$

The remaining two terms in (C.1) are bounded in a similar manner so that

$$K_3 := \sum_{l=0}^{\infty} \sum_{n\in I_p(l)} l^{-2p-1} r_{|n-i|,l+j} = \mathcal{O}\left(\min\left\{j^{-m-1}, i^{-\min\{m+1,2p+1\}}\right\}\right)$$

$$K_4 := \sum_{l=0}^{\infty} \sum_{n\in I_p(l)} l^{-2p-1} r_{|n-i|,|l-j|} = \mathcal{O}\left(\min\left\{j^{-\min\{m+1,2p+1\}}, i^{-\min\{m+1,2p+1\}}\right\}\right).$$

Finally, considering all the bounds yields the stated result. The ρ-analytic case follows from the same arguments.

Appendix D. Some properties of Chebyshev polynomials

The next two identities follow directly from the explicit definition of Chebyshev polynomials as $T_n(t) = \cos(n \arccos(t))$.

Lemma D.1. *For $n, k \in \mathbb{N}_0$, let T_n and T_k denote two Chebyshev polynomials of first kind. Then,*

$$T_n T_k = \frac{1}{2}\left(T_{n+k} + T_{|n-k|}\right).$$

Moreover, for $(t,s) \in [-1,1]^2$, it holds

$$|t-s|^2 = 1 + \frac{1}{2}\left(T_2(t) + T_2(s)\right) - 2T_1(t)T_1(s).$$

Lemma D.2. *Consider a function of the form:*

$$U(t,s) = \sum_{n=0}^{\infty} a_n T_n(t) T_{|n-k|}(s).$$

Then,

$$|t-s|^2 U(t,s) = \sum_{j\in\{-1,0,1\}} \sum_{n=0}^{\infty} \beta_n^{(j)} T_n(t) T_{|n-k+2j|}(s),$$

wherein

$$\beta_n^{(1)} := \frac{1}{4}a_n - \frac{1}{2}a_{n+1} + \frac{1}{4}a_{n+2},$$

and coefficients $\beta_n^{(-1)}$ and $\beta_n^{(0)}$ are given in Table D.1 for $n \in \mathbb{N}_0$.

TABLE D.1. Coefficients used in Lemma D.1.

	$\beta_n^{(-1)}$	$\beta_n^{(0)}$
$n=0$	$\frac{1}{4}a_0$	$a_0 - \frac{1}{2}a_1$
$n=1$	$-a_0 + \frac{1}{4}a_1$	$-a_0 + \frac{5}{4}a_1 - \frac{1}{2}a_2$
$n=2$	$\frac{1}{2}a_0 - \frac{1}{2}a_1 + \frac{1}{4}a_2$	$-\frac{1}{2}a_1 + a_2 - \frac{1}{2}a_3$
$n\geq 3$	$\frac{1}{4}a_{n-2} - \frac{1}{2}a_{n-1} + \frac{1}{4}a_n$	$-\frac{1}{2}a_{n-1} + a_n - \frac{1}{2}a_{n+1}$

Proof. Using Lemma D.1, we have that

$$\begin{aligned}|t-s|^2U(t,s) = \sum_{n=0}^{\infty} a_n \Bigg(& T_n(t)T_{|n-k|}(s) + \frac{1}{4}T_{n+2}(t)T_{|n-k|}(s) + \frac{1}{4}T_{|n-2|}(t)T_{|n-k|}(s) \\ & + \frac{1}{4}T_n(t)T_{||n-k|+2|} + \frac{1}{4}T_n(t)T_{|n-k-2|} \\ & - \frac{1}{2}\left[T_{||n-k|+1|}(s) + T_{||n-k|-1|}(s)\right]\left[T_{|n-1|}(t) + T_{n+1}\right]\Bigg).\end{aligned}$$

Observe that, for $i \in \{1,2\}$, the index sums

$$|n-k|+i = \begin{cases}|n-k+i| & n \geq k, \\ |n-k-i| & n<k,\end{cases} \qquad ||n-k|-i| = \begin{cases}|n-k-i| & n\geq k, \\ |n-k+i| & n<k.\end{cases}$$

Employing this in writing $|t-s|^2U(t,s)$ as a series expansion, we find expressions for different $u_n(s)$:

$$\begin{aligned}u_0 &= \frac{a_0}{4}T_{|k+2|}(s) + \left(a_0 - \frac{a_1}{2}\right)T_{|k|}(s) + \left(\frac{a_0}{4} - \frac{a_1}{2} + \frac{a_2}{4}\right)T_{|k-2|}(s) \\ u_1 &= \left(-a_0 + \frac{a_1}{4}\right)T_{|k+1|}(s) - \left(a_0 + \frac{5a_1}{4} + \frac{a_2}{2}\right)T_{|1-k|}(s) + \left(\frac{a_1}{4} - \frac{a_2}{2} + \frac{a_3}{4}\right)T_{|k-3|}(s) \\ u_2 &= \left(\frac{a_0}{2} - \frac{a_1}{2} + \frac{a_2}{4}\right)T_{|k|}(s) - \left(\frac{a_1}{2} - a_2 + \frac{a_3}{2}\right)T_{|k-2|}(s) + \left(\frac{a_2}{4} - \frac{a_3}{2} + \frac{a_4}{4}\right)T_{|k-4|}(s) \\ u_n &= \left(\frac{a_{n-2}}{4} - \frac{a_{n-1}}{2} + \frac{a_n}{4}\right)T_{|n-k-2|}(s) + \left(-\frac{a_{n-1}}{2} + a_n - \frac{a_{n+1}}{2}\right)T_{|n-k|}(s) \\ &\quad + \left(\frac{a_n}{4} - \frac{a_{n+1}}{2} + \frac{a_{n+2}}{4}\right)T_{|n-k+2|}(s)\end{aligned}$$

for $n \geq 3$, yielding the stated result. □

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