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A Γ-CONVERGENCE RESULT FOR FLUID-FILLED FRACTURE PROPAGATION

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Abstract. In this paper we provide a rigorous asymptotic analysis of a phase-field model used to simulate pressure-driven fracture propagation in poro-elastic media. More precisely, assuming a given pressure $p \in W^{1,\infty}(\Omega)$ we show that functionals of the form

$$E(\mathbf{u}) = \int_\Omega e(\mathbf{u})\colon \mathbb{C}e(\mathbf{u}) + p\nabla\cdot\mathbf{u} + \langle \nabla p, \mathbf{u}\rangle \,\mathrm{d}x + \mathcal{H}^{n-1}(J_\mathbf{u}), \quad \mathbf{u} \in \mathrm{GSBD}(\Omega) \cap L^1(\Omega;\mathbb{R}^n)$$

can be approximated in terms of Γ-convergence by a sequence of phase-field functionals, which are suitable for numerical simulations. The Γ-convergence result is complemented by a numerical example where the phase-field model is implemented using a Discontinuous Galerkin Discretization.

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1. Introduction

In recent years the description of brittle fracture in elastic materials has drawn a lot of attention both from an analytical and from a numerical point of view (see, *e.g.*, [1,2,12,13,18,20,31,34–36,38], to mention just a few works in this area). Starting from the pioneering work of Griffith [33] and its later development by Francfort and Marigo [30] the propagation of a crack in an elastic material is modeled by minimizing a total energy consisting of two competing terms. A bulk term represents the stored energy in the elastic material and a surface term represents the energy needed to produce a crack. For brittle materials the surface energy is proportional to the length of the crack *via* a constant G_c, the toughness of the material. Moreover, in the small strain regime, the bulk energy can be expressed in terms of linear elasticity. In fact, representing by $\Omega \subset \mathbb{R}^n$ the region occupied by a homogeneous and isotropic elastic material and denoting by $\mathbf{u} : \Omega \to \mathbb{R}^n$ the displacement, a prototypical associated energy is given by

$$F(\mathbf{u}) = \frac{1}{2}\int_\Omega e(\mathbf{u})\colon \mathbb{C}e(\mathbf{u})\,\mathrm{d}x + G_c\mathcal{H}^{n-1}(J_\mathbf{u}). \tag{1.1}$$

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Here $\mathbf{u}$ belongs to the space of generalized special functions of bounded deformation $\mathrm{GSBD}(\Omega)$ introduced in [25]. The bulk energy depends on $\mathbf{u}$ through its approximate symmetric gradient $e(\mathbf{u})$ and is determined by the material-dependent fourth-order elasticity tensor $\mathbb{C}$ given by Hooke's law. Moreover, the crack is implicitly described through the jumpset $J_{\mathbf{u}}$ of $\mathbf{u}$.

In view of new technical developments during the last years (as for instance in the field of geothermal energy), there is an increasing interest to consider more general energies than in (1.1). An important issue is, for example, the modeling of pressure-driven crack propagation in a poro-elastic medium. In [14,38,42] the authors discuss a generalization of (1.1) to this framework. Following their argument and assuming a given pressure $p \in W^{1,\infty}(\Omega)$ in [27] the authors derive a total energy of the form

$$E(\mathbf{u}) = \frac{1}{2}\int_\Omega e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right)\mathrm{d}x + \int_\Omega \left(1-\alpha\right)p\nabla\cdot\mathbf{u} + \langle \nabla p, \mathbf{u}\rangle\,\mathrm{d}x + G_c\mathcal{H}^{n-1}(J_{\mathbf{u}}), \tag{1.2}$$

where $\alpha \in [0,1]$ is Biot's coefficient [11] and $\nabla\cdot\mathbf{u} = \operatorname{tr} e(\mathbf{u})$ denotes the divergence of $\mathbf{u}$ (see, *e.g.*, [27,38] and references therein for more details on the modeling).

The numerical minimization of functionals as in (1.1) and (1.2), however, bears several difficulties, since they contain the *a priori* unknown free-discontinuity set $J_{\mathbf{u}}$ that might exhibit complex topologies. In order to circumvent these difficulties it has proved successful to regularize the crack $J_{\mathbf{u}}$ using a phase-field approximation reminiscent of the Ambrosio–Tortorelli approximation [3,4] of the Mumford–Shah functional [39] (see Rem. 5.2). The main idea is to introduce an auxiliary variable φ, the so-called phase-field variable, which is close to 1 in large regions of Ω and approaches zero in a small region around the crack. Thus the set where φ is close to zero should provide a "regularization" of the crack. More precisely, a phase-field approximation of the functional F as in (1.1) is given by the sequence of functionals $F_\varepsilon : H^1\left(\Omega;\mathbb{R}^n\right)\times H^1(\Omega) \to [0,+\infty)$ defined as

$$F_\varepsilon(\mathbf{u},\varphi) = \frac{1}{2}\int_\Omega (\varphi^2+k_\varepsilon)e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right)\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{(\varphi-1)^2}{\varepsilon} + \varepsilon|\nabla\varphi|^2\,\mathrm{d}x \tag{1.3}$$

with $\varepsilon > 0$ and $0 < k_\varepsilon \ll \varepsilon$. Since the functionals F_ε are defined for Sobolev functions and thus do not contain any unknown free-discontinuity set, numerical schemes for the minimization of F_ε can be implemented with less difficulties. Moreover, in [18,19,21,34] it has been proved successively that for $\varepsilon \to 0$ the functionals F_ε approximate the functional F in the sense of Γ-convergence. Since Γ-convergence, when coupled with a suitable compactness result, implies convergence of minimizers, this justifies to use F_ε in place of F for numerical simulations.

A phase-field approach as described above has also been used to simulate pressure-driven crack propagation (see [14,27,37,38,42]). In fact, in [27] the authors replace the functional E in (1.2) by the functionals

$$\begin{aligned} E_\varepsilon(\mathbf{u},\varphi) = {} & \frac{1}{2}\int_\Omega (\varphi^2+k_\varepsilon)e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right)\mathrm{d}x + \int_\Omega \left(1-\alpha\right)\varphi p\nabla\cdot\mathbf{u} + \varphi\left\langle\nabla p,\mathbf{u}\right\rangle\mathrm{d}x \\ & + \frac{G_c}{2}\int_\Omega \frac{(\varphi-1)^2}{\varepsilon} + \varepsilon|\nabla\varphi|^2\,\mathrm{d}x \end{aligned} \tag{1.4}$$

for numerical simulations. Moreover, they show that for $n=1$ the sequence (E_ε) Γ-converges to E as $\varepsilon \to 0$. To our knowledge, despite the fact that a phase-field approximation as in (1.4) is widely used for numerical simulations, for $n > 1$ a rigorous asymptotic analysis for the functionals E_ε in terms of Γ-convergence is still missing.

The aim of this paper is to establish this missing analysis. In fact, our main result is Theorem 3.1 which shows that the functional E in (1.2) is the Γ-limit in the strong $\left(L^1\left(\Omega;\mathbb{R}^n\right)\times L^1\left(\Omega\right)\right)$-topology of the functionals E_ε as in (1.4). A key ingredient to prove this result is Lemma 4.1, which allows us to estimate the functionals E_ε from below in terms of F_ε. On account of Theorem 3.1 we also establish a convergence result for minimizers of a suitable perturbation of the functionals E_ε. Finally, having provided an analytical justification to use E_ε for numerical simulations, we also present a simulation of pressure-driven fracture propagation using a Discontinuous Galerkin discretization of the phase-field model as in [27].

More in detail, the plan of the paper is the following. After briefly recalling some preliminaries on Γ-convergence and on the spaces of bounded deformation, we will state the main result in Section 3. The proof of the result is carried out in Section 4. As usual it is divided into two steps. First, the Γ-liminf inequality is shown following the strategy of [28] and applying the compactness results in [25]. Second, to prove the Γ-limsup inequality we employ a recent density result of Chambolle and Crismale ([21], Thm. 1.1). In Section 5 we study the asymptotic behavior of a class of minimization problems associated to E_ε, and we conclude the paper with a numerical example exploring the relation between the mesh size and the parameters ε and k_ε in Section 6.

2. Notation and preliminaries

In this section we fix the notation and recall some preliminary results that we will employ in the sequel.

Main notation

Let $m, n \in \mathbb{N}$ with $m, n \geq 1$ and let $k \in \mathbb{N}$. Throughout this paper, if not specified otherwise, $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $\mathcal{L}^n$ denotes the n-dimensional Lebesgue-measure in $\mathbb{R}^n$ and $\mathcal{H}^k$ the k-dimensional Hausdorff-measure in $\mathbb{R}^n$. Moreover, $\#$ is the counting measure. Further, for $U, V \subset \mathbb{R}^n$ we denote by $U \Delta V$ the symmetric difference between U and V and $\mathcal{X}_U$ is the characteristic function of U. For $q \in [1, +\infty]$ we use standard notation for the Lebesgue spaces $L^q(\Omega; \mathbb{R}^m)$ and $W^{k,q}(\Omega; \mathbb{R}^m)$. If $k = 1, q = 2$ we write $H^1(\Omega; \mathbb{R}^m)$ in place of $W^{1,2}(\Omega; \mathbb{R}^m)$.

Let $\nu, \xi \in \mathbb{R}^n$; we write $\langle \nu, \xi \rangle$ for the scalar product between ν and ξ and $|\nu|$ for the Euclidean norm of ν. Moreover, if $A \in \mathbb{R}^{n\times n}$, then $|A| = \sqrt{\operatorname{tr}(A^T A)}$ denotes the Frobenius norm.

Suppose that $\mathbf{u} : \Omega \to \mathbb{R}^m$ is measurable. We say that $a \in \mathbb{R}^m$ is an approximate limit of $\mathbf{u}$ at $x \in \Omega$ and write

$$a = \operatorname*{ap-lim}_{y\to x} \mathbf{u}(y),$$

if for every $\delta > 0$ we have

$$\lim_{\rho\to 0^+} \frac{\mathcal{L}^n\left(\Omega \cap B_\rho(x) \cap \{|\mathbf{u} - a| > \delta\}\right)}{\rho^n} = 0.$$

In this case we also say that $\mathbf{u}$ is approximately continuous at x and we denote by $S_{\mathbf{u}}$ the set of all $x \in \Omega$ where $\mathbf{u}$ is not approximately continuous. Moreover, we say that $x \in S_{\mathbf{u}}$ is an approximate jump point of $\mathbf{u}$ and write $x \in J_{\mathbf{u}}$ if there exist $a, b \in \mathbb{R}^m$, $a \neq b$ and $\nu \in S^{n-1}$ satisfying

$$a = \operatorname*{ap-lim}_{\substack{y\to x\\ \langle \nu, y-x\rangle > 0}} \mathbf{u}(y) \qquad \text{and} \qquad b = \operatorname*{ap-lim}_{\substack{y\to x\\ \langle \nu, y-x\rangle < 0}} \mathbf{u}(y). \tag{2.1}$$

The triplet (a, b, ν) is uniquely determined by (2.1) up to a change of sign of ν and a simultaneous permutation of (a, b). We denote it by $(\mathbf{u}^+(x), \mathbf{u}^-(x), \nu_{\mathbf{u}}(x))$ and we set $[\mathbf{u}](x) := \mathbf{u}^+(x) - \mathbf{u}^-(x)$. Finally, $\mathbf{u}$ is called approximately differentiable at $x \in \Omega \setminus S_{\mathbf{u}}$ if there exists a matrix $L \in \mathbb{R}^{m\times n}$ such that

$$\operatorname*{ap-lim}_{y\to x} \frac{|\mathbf{u}(y) - \mathbf{u}(x) - L(y - x)|}{|y - x|} = 0 \tag{2.2}$$

The matrix L uniquely determined by (2.2) is called the approximate gradient of $\mathbf{u}$ at x and is denoted by $\nabla \mathbf{u}(x)$.

BV and BD functions

We assume that the reader is familiar with the spaces $\mathrm{BV}(\Omega;\mathbb{R}^m)$ and $\mathrm{SBV}(\Omega;\mathbb{R}^m)$. Here we just recall the main notation and refer to [6] (see also [16]) for more details. We say that a function $\mathbf{u}\in L^1(\Omega;\mathbb{R}^m)$ is a function of bounded variation and write $\mathbf{u}\in\mathrm{BV}(\Omega,\mathbb{R}^m)$ if its distributional derivative $D\mathbf{u}$ can be represented by a bounded (matrix-valued) Radon measure. If $m=1$ we simply write $\mathrm{BV}(\Omega)$. Recall that for every $\mathbf{u}\in\mathrm{BV}(\Omega;\mathbb{R}^m)$ the distributional derivative $D\mathbf{u}$ can be decomposed as

$$D\mathbf{u}=\nabla\mathbf{u}\mathcal{L}^n+[\mathbf{u}]\otimes\nu_{\mathbf{u}}\mathcal{H}^{n-1}\llcorner S_{\mathbf{u}}+D^c\mathbf{u}, \tag{2.3}$$

where $D^c\mathbf{u}$ is the so-called Cantor part of $D\mathbf{u}$ which vanishes on Borel sets $U\subset\Omega$ that are σ-finite with respect to $\mathcal{H}^{n-1}$. We call $\mathbf{u}$ a special function of bounded variation and write $\mathbf{u}\in\mathrm{SBV}(\Omega;\mathbb{R}^m)$, if $\mathbf{u}\in\mathrm{BV}(\Omega;\mathbb{R}^m)$ and $D^c\mathbf{u}=0$. We set

$$\mathrm{SBV}^2(\Omega;\mathbb{R}^m):=\left\{\mathbf{u}\in\mathrm{SBV}(\Omega;\mathbb{R}^m):\ \nabla\mathbf{u}\in L^2(\Omega;\mathbb{R}^{m\times n}),\ \mathcal{H}^{n-1}(S_{\mathbf{u}})<+\infty\right\}.$$

Moreover, we say that a measurable function $u:\Omega\to\mathbb{R}$ is a generalized special function of bounded variation and write $u\in\mathrm{GSBV}(\Omega)$ if for every $M\in\mathbb{N}$ the truncation of u at level M defined as $u^M:=-M\vee(u\wedge M)$ belongs to $\mathrm{SBV}(\Omega)$.

Further, we say that a function $u:\Omega\to\mathbb{R}$ belongs to $\mathrm{BV}_{\mathrm{loc}}(\Omega)$, if $u\in\mathrm{BV}(A)$ for every $A\subset\subset\Omega$ compactly embedded. Analogously, we write $u\in\mathrm{SBV}_{\mathrm{loc}}(\Omega)$ if $u\in\mathrm{SBV}(A)$ for every $A\subset\subset\Omega$.

In analogy to the definition of $\mathrm{BV}(\Omega;\mathbb{R}^n)$ one can define the space $\mathrm{BD}(\Omega)$ as the space of all vector-valued functions $\mathbf{u}\in L^1(\Omega;\mathbb{R}^n)$ such that its distributional symmetric gradient $E\mathbf{u}=\frac{1}{2}(D\mathbf{u}+(D\mathbf{u})^T)$ is representable as a bounded (matrix-valued) Radon measure (see, *e.g.*, [10, 41] for more details). Similar to (2.3) $E\mathbf{u}$ can be decomposed as

$$E\mathbf{u}=e\left(\mathbf{u}\right)\mathcal{L}^n+\left(\mathbf{u}^+-\mathbf{u}^-\right)\odot\nu_{\mathbf{u}}\mathcal{H}^{n-1}\llcorner J_{\mathbf{u}}+E^c\mathbf{u},$$

where for $\mathcal{L}^n$-*a.e.* $x\in\Omega$ the matrix $(e\left(\mathbf{u}\right))(x)$ is the approximate symmetric gradient of $\mathbf{u}$ at x satisfying

$$\operatorname*{ap-lim}_{y\to x}\frac{\langle\mathbf{u}(y)-\mathbf{u}(x)-(e\left(\mathbf{u}\right))(x)(y-x),y-x\rangle}{|y-x|^2}=0 \tag{2.4}$$

(see [5], Sect. 4). Moreover, every $\mathbf{u}\in\mathrm{BD}(\Omega)$ is approximately differentiable $\mathcal{L}^n$-a.e. in Ω and

$$e\left(\mathbf{u}\right)=\frac{1}{2}\left(\nabla\mathbf{u}+\left(\nabla\mathbf{u}\right)^T\right) \tag{2.5}$$

(see [5], Rem. 7.5). Finally, if $E^c\mathbf{u}=0$ we write $\mathbf{u}\in\mathrm{SBD}(\Omega)$. For more details on the spaces $\mathrm{BD}(\Omega)$ and $\mathrm{SBD}(\Omega)$ we refer the reader to [5, 10, 41].

We now recall the definition of the space $\mathrm{GSBD}(\Omega)$ introduced in [25]. To this end, we fix some notation. For every $\xi\in S^{n-1}$ we denote by

$$\Pi^\xi:=\{y\in\mathbb{R}^n:\ \langle y,\xi\rangle=0\}$$

the hyperplane orthogonal to ξ passing through the origin. For every $A\subset\Omega$ and every $y\in\Pi^\xi$ we set

$$A^\xi_y:=\{t\in\mathbb{R}:\ y+t\xi\in A\},$$

and for every $\mathbf{u}:A\to\mathbb{R}^m$ we define $\mathbf{u}^\xi_y:A^\xi_y\to\mathbb{R}^m$ by setting

$$\mathbf{u}^\xi_y(t):=\mathbf{u}(y+t\xi)\quad\forall\ t\in A^\xi_y.$$

Moreover, if $m=n$, we define the function $\hat{\mathbf{u}}^\xi_y:A^\xi_y\to\mathbb{R}$ as

$$\hat{\mathbf{u}}^\xi_y(t):=\langle\mathbf{u}(y+t\xi),\xi\rangle.$$

Then the space $\mathrm{GSBD}(\Omega)$ is defined as follows (see [25], Defs. 4.1 and 4.2)

Definition 2.1. Let $\mathbf{u} : \Omega \to \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then $\mathbf{u} \in \mathrm{GSBD}(\Omega)$ if there exists a positive finite Radon measure $\lambda_{\mathbf{u}}$ such that for all $\xi \in S^{n-1}$ there holds $\hat{\mathbf{u}}^\xi_y \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and

$$\int_{\Pi^\xi} |D\hat{\mathbf{u}}^\xi_y| \left(B^\xi_y \setminus J^1_{\hat{\mathbf{u}}^\xi_y}\right) + \# \left(B^\xi_y \cap J^1_{\hat{\mathbf{u}}^\xi_y}\right) d\mathcal{H}^{n-1}(y) \leq \lambda_{\mathbf{u}}(B)$$

for every Borel set $B \subset \Omega$, where

$$J^1_{\hat{\mathbf{u}}^\xi_y} := \left\{ y \in J_{\hat{\mathbf{u}}^\xi_y} : \ |[\hat{\mathbf{u}}^\xi_y(x)]| \geq 1 \right\}.$$

The structure theorem ([5], Thm. 4.5) together with Proposition 4.7 of [5] ensures that $\mathrm{SBD}(\Omega) \subset \mathrm{GSBD}(\Omega)$. Moreover, for $\mathbf{u} \in \mathrm{GSBD}(\Omega)$ and $\mathcal{L}^n$-a.e. $x \in \Omega$ there exists the approximate symmetric gradient $(e(\mathbf{u}))(x)$ satisfying (2.4). Further, $e(\mathbf{u}) \in L^1(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ and for every $\xi \in S^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ there holds

$$\langle (e(\mathbf{u}))^\xi_y \xi, \xi \rangle = \nabla \hat{\mathbf{u}}^\xi_y \qquad \mathcal{L}^1\text{-a.e. in } \Omega^\xi_y$$

(see [25], Thm. 9.1). This allows us to define a divergence for $\mathbf{u} \in \mathrm{GSBD}(\Omega)$. In fact, for every $\mathbf{u} \in \mathrm{GSBD}(\Omega)$ we set

$$\nabla \cdot \mathbf{u} := \mathrm{tr}(e(\mathbf{u})),$$

which is consistent with the usual definition of divergence thanks to (2.5).

Moreover, the space $\mathrm{GSBD}(\Omega)$ enjoys a compactness property ([25], Thm. 11.3). We will give a simplified version of the result therein, tailored to our needs.

Theorem 2.2 (Compactness in GSBD)**.** *Given a sequence $(\mathbf{u}_k) \subset \mathrm{GSBD}(\Omega)$ suppose that we can find $c > 0$ such that*

$$\int_\Omega |\mathbf{u}_k| \,\mathrm{d}x + \int_\Omega |e(\mathbf{u}_k)|^2 \,\mathrm{d}x + \mathcal{H}^{n-1}(J_{\mathbf{u}_k}) < c$$

for all $k \in \mathbb{N}$. Then there exists $\mathbf{u} \in \mathrm{GSBD}(\Omega)$ and a subsequence $\mathbf{u}_{k_j}$ such that

$$\begin{aligned} \mathbf{u}_{k_j} &\to \mathbf{u} \text{ pointwise } \mathcal{L}^n \text{a.e. on } \Omega, \\ e\left(\mathbf{u}_{k_j}\right) &\rightharpoonup e(\mathbf{u}) \text{ weakly in } L^1\left(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}}\right), \\ \mathcal{H}^{n-1}(J_{\mathbf{u}}) &\leq \liminf_{j\to 0} \mathcal{H}^{n-1}\left(J_{\mathbf{u}_{k_j}}\right). \end{aligned}$$

We finally recall the recent density result ([21], Thm. 1.1), adpated to the setting of $\mathrm{GSBD}^2(\Omega) \cap L^1(\Omega)$, where

$$\mathrm{GSBD}^2(\Omega) := \left\{ \mathbf{u} \in \mathrm{GSBD}(\Omega) : \ e(\mathbf{u}) \in L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}}), \ \mathcal{H}^{n-1}(J_{\mathbf{u}}) < +\infty \right\}.$$

Theorem 2.3 (Density in GSBD)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary $\partial\Omega$. Let $\mathbf{u} \in \mathrm{GSBD}^2(\Omega) \cap L^1(\Omega; \mathbb{R}^n)$. Then we can find a sequence $(\mathbf{u}_k) \subset \mathrm{SBV}^2(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ such that $J_{\mathbf{u}_k} \subset \Omega$ is closed and contained in a finite union of closed $\mathcal{C}^1$-hypersurfaces and $\mathbf{u}_k \in W^{1,\infty}(\Omega \setminus J_{\mathbf{u}_k}; \mathbb{R}^n)$ such that*

$$\mathbf{u}_k \to \mathbf{u} \text{ in } L^1(\Omega; \mathbb{R}^n), \tag{2.6}$$

$$e(\mathbf{u}_k) \to e(\mathbf{u}) \text{ in } L^2\left(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}}\right), \tag{2.7}$$

$$\mathcal{H}^{n-1}(J_{\mathbf{u}_k} \Delta J_{\mathbf{u}}) \to 0. \tag{2.8}$$

For further use we denote by $\mathcal{W}(\Omega; \mathbb{R}^n)$ the class of the approximating functions in Theorem 2.3, that is, the class of all functions $\mathbf{u} \in \mathrm{SBV}^2(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ such that $J_{\mathbf{u}} \subset \Omega$ is closed and contained in a finite union of closed $\mathcal{C}^1$-hypersurfaces and $\mathbf{u} \in W^{1,\infty}(\Omega \setminus J_{\mathbf{u}}; \mathbb{R}^n)$.

Remark 2.4. If the function $\mathbf{u}$ in Theorem 2.3 belongs to $\mathrm{GSBD}^2(\Omega) \cap L^2(\Omega; \mathbb{R}^n)$ the approximating sequence $(\mathbf{u}_k)$ converges to $\mathbf{u}$ in $L^2(\Omega; \mathbb{R}^n)$ (see [21], Thm. 1.1, Eq. (1.1e)).

Γ-convergence

Eventually, we also recall the definition of Γ-convergence [26] and its basic properties. For more details we refer the reader to the literature (see, *e.g.*, [17,24]). In all that follows $\varepsilon > 0$ is a positive parameter varying in a strictly decreasing sequence converging to zero. Let (X, d) be a metric space. We say that a sequence of functionals $F_\varepsilon : X \to [-\infty, +\infty]$ Γ-converges with respect to the metric d to a functional $F : X \to [-\infty, +\infty]$, if for all $x \in X$ the following two conditions are satisfied:

(1) (liminf-inequality) For every sequence $(x_\varepsilon) \subset X$ converging to x with respect to d there holds

$$F(x) \le \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon).$$

(2) (limsup-inequality) There exists a sequence $(x_\varepsilon) \subset X$ converging to x with respect to d and satisfying

$$\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \le F(x).$$

The function F uniquely determined by (1) and (2) is called the Γ-limit of F. We also consider the so-called Γ-lower and Γ-upper limits of (F_ε) defined by

$$F'(x) = \Gamma\text{-}\liminf_{\varepsilon \to 0} F_\varepsilon(x) = \inf\left\{\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \colon x_\varepsilon \to x \text{ with respect to } d\right\}$$

and

$$F''(x) = \Gamma\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon(x) = \inf\left\{\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon) \colon x_\varepsilon \to x \text{ with respect to } d\right\}.$$

We notice that F' and F'' are lower semicontinuous with respect to d ([24], Prop. 6.8) and conditions (1) and (2) are equivalent to

$$F(x) = F'(x) = F''(x) \quad \text{for every } x \in X.$$

In particular, the Γ-limit is always lower semicontinuous with respect to d.

A fundamental property is that a Γ-convergence result when coupled with a suitable compactness result ensures the convergence of minimizers and minimum values. Indeed, the following holds (see, *e.g.*, [17], Thm. 1.21, [24], Thm. 7.4).

Theorem 2.5 (Fundamental property of Γ-convergence)**.** *Let $F_\varepsilon : X \to [-\infty, +\infty]$ be equicoercive (i.e., for every $t \in \mathbb{R}$ there exists $K_t \subset \mathbb{R}$ compact such that for all $\varepsilon > 0$ there holds $\{F_\varepsilon < t\} \subset K_t$) and suppose that (F_ε) Γ-converges to some functional $F : X \to [-\infty, \infty]$ with $F \not\equiv +\infty$. Then there exists*

$$\min_X F = \lim_{\varepsilon \to 0} \inf_X F_\varepsilon.$$

Moreover, if $(x_\varepsilon) \subset X$ is a minimizing sequence for the functionals F_ε (that is, a sequence satisfying $\lim_\varepsilon (F_\varepsilon(x_\varepsilon) - \inf_X F_\varepsilon) = 0$) then (up to subsequences) (x_ε) converges to a minimizer $\bar{x}$ of F.

3. Setting of the problem and statement of the main result

We now introduce the functionals that we will consider in this paper. We define the functionals E, E_ε : $L^1(\Omega; \mathbb{R}^n) \times L^1(\Omega) \to (-\infty, +\infty]$ by setting

$$E(\mathbf{u}, \varphi) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega e(\mathbf{u}) : \mathbb{C} e(\mathbf{u}) \,\mathrm{d}x & + \displaystyle\int_\Omega \Big((1-\alpha)\, p \nabla \cdot \mathbf{u} + \langle \nabla p, \mathbf{u} \rangle \Big) \,\mathrm{d}x + G_c \mathcal{H}^{n-1}(J_{\mathbf{u}}) \\ & \text{if } \mathbf{u} \in \mathrm{GSBD}^2(\Omega) \cap L^1(\Omega; \mathbb{R}^n),\ \varphi = 1\ \mathcal{L}^n\text{-}a.e. \text{ in } \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^n) \times L^1(\Omega), \end{cases} \tag{3.1}$$

and

$$E_\varepsilon(\mathbf{u},\varphi) := \begin{cases} \frac{1}{2}\int_\Omega (\varphi^2 + k_\varepsilon) e(\mathbf{u}) : \mathbb{C} e(\mathbf{u})\,\mathrm{d}x & + \int_\Omega \Big((1-\alpha)\,\varphi p \nabla\cdot \mathbf{u} + \varphi\,\langle \nabla p, \mathbf{u}\rangle \Big)\,\mathrm{d}x \\ & + \frac{G_c}{2}\int_\Omega \Big(\frac{(\varphi-1)^2}{\varepsilon} + \varepsilon|\nabla\varphi|^2\Big)\,\mathrm{d}x \\ & \text{if } \mathbf{u}\in H^1(\Omega;\mathbb{R}^n),\ 0\le\varphi\le 1, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega). \end{cases} \tag{3.2}$$

Here $0 < k_\varepsilon \ll \varepsilon$, $G_c > 0$ is a positive constant and $\alpha \in [0,1]$ is the Biot's coefficient [11]. $\mathbb{C}$ denotes the fourth-order elasticity tensor for an isotropic material. It is given by

$$\mathbb{C}A = 2\mu A + \lambda(\operatorname{tr} A)I \quad \text{for every } A\in\mathbb{R}^{n\times n}_{\text{sym}}, \tag{3.3}$$

where $\mu > 0$ and $\lambda > 0$ are the material-dependent Lamé parameters and $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix. Moreover, $p \in W^{1,\infty}(\Omega)$ represents the pressure in the porous medium Ω. As $\mu > 0$, $\mathbb{C}$ is positive definite, since for every $A \in \mathbb{R}^{n\times n}_{\text{sym}}$ there holds

$$A\colon \mathbb{C}A = 2\mu|A|^2 + \lambda\,(\operatorname{tr} A)^2 \ge 2\mu|A|^2. \tag{3.4}$$

In addition, we have

$$A\colon \mathbb{C}A \le (2\mu + n\lambda)\,|A|^2.$$

We also consider the functionals $F, F_\varepsilon : L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega) \to [0,+\infty]$ without pressure terms defined as

$$F(\mathbf{u},\varphi) := \begin{cases} \frac{1}{2}\int\limits_\Omega e(\mathbf{u}) : \mathbb{C}e(\mathbf{u})\,\mathrm{d}x + G_c\mathcal{H}^{n-1}(J_{\mathbf{u}}) & \\ & \text{if } \mathbf{u}\in \mathrm{GSBD}^2(\Omega)\cap L^1(\Omega;\mathbb{R}^n),\ \varphi = 1\ \mathcal{L}^n\text{-}a.e. \text{ in } \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega), \end{cases} \tag{3.5}$$

$$F_\varepsilon(\mathbf{u},\varphi) := \begin{cases} \frac{1}{2}\int\limits_\Omega (\varphi^2 + k_\varepsilon)\, e(\mathbf{u}) : \mathbb{C}e(\mathbf{u})\,\mathrm{d}x + \frac{G_c}{2}\int_\Omega \Big(\frac{(\varphi-1)^2}{\varepsilon} + \varepsilon\,|\nabla\varphi|^2\Big)\,\mathrm{d}x & \\ & \text{if } \mathbf{u}\in H^1(\Omega;\mathbb{R}^n),\ 0\le\varphi\le 1, \\ +\infty & \text{otherwise in } L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega). \end{cases} \tag{3.6}$$

It is well-known that the functionals F_ε as in (3.6) Γ-converge to the functional F as in (3.5) (see [21,34] and also [18]). The aim of this paper is to extend this result to the functionals E_ε and E. More precisely, we prove the following Γ-convergence result.

Theorem 3.1. *Let E_ε be as in (3.2). Then the functionals E_ε Γ-converge in the strong $\big(L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)\big)$-topology to the functional E defined in (3.1).*

In the case $n = 1$ the result has already been proved in [27]. We notice that in this case the domain of the Γ-limit E is $\mathrm{SBV}^2(\Omega)$.

Remark 3.2. We mention here that on account of Theorem 3.1, following the approach of Theorem 1.2 in [21] one can also prove a Γ-convergence result for a suitable modification of E_ε that includes the prescription of Dirichlet boundary conditions on a subset Γ_D of $\partial\Omega$, satisfying some geometric condition. More precisely, suppose that $\Omega\subset\mathbb{R}^n$ is open, bounded, connected and with Lipschitz boundary $\partial\Omega$ and that $\Gamma_D, \Gamma_N \subset \partial\Omega$ are such that

$$\partial\Omega = \Gamma_D \cup \Gamma_N \cup N$$

with Γ_D, Γ_N relatively open, $\Gamma_D \cap \Gamma_N = \emptyset$, $\mathcal{H}^{n-1}(N) = 0$, $\Gamma_D \neq \emptyset$ and $\partial(\Gamma_D)$, $\partial(\Gamma_N)$ have finite $\mathcal{H}^{n-2}$-measure. Suppose moreover that there exists $\bar{\delta} > 0$ and $x_0 \in \mathbb{R}^n$ such that there holds

$$f_{\delta,x_0}(\Gamma_D) \subset \Omega \quad \text{for every } \delta \in (0, \bar{\delta}), \tag{3.7}$$

where $f_{\delta,x_0}(x) := x_0 + (1-\delta)(x - x_0)$. Finally, suppose that $\mathbf{u}_0 \in H^1(\mathbb{R}^n; \mathbb{R}^n)$. Then, applying Theorem 3.1 and following the arguments of Theorem 1.2 in [21] one can show that the functionals $\widetilde{E}_\varepsilon$ defined by

$$\widetilde{E}_\varepsilon(\mathbf{u}, \varphi) := \begin{cases} E_\varepsilon(\mathbf{u}, \varphi) & \text{if } \operatorname{tr}_\Omega \mathbf{u} = \operatorname{tr}_\Omega \mathbf{u}_0 \text{ on } \Gamma_D,\ \operatorname{tr}_\Omega \varphi = 1 \text{ on } \Gamma_D, \\ +\infty & \text{otherwise in } L^1(\Omega; \mathbb{R}^n) \times L^1(\Omega). \end{cases} \tag{3.8}$$

Γ-converge in the strong $\big(L^1(\Omega; \mathbb{R}^n) \times L^1(\Omega)\big)$-topology to the functional $\widetilde{E}$ defined as

$$\widetilde{E}(\mathbf{u}, \varphi) := E(\mathbf{u}, \varphi) + G_c \mathcal{H}^{n-1}(\Gamma_D \cap \{\operatorname{tr}_\Omega \mathbf{u} \neq \operatorname{tr}_\Omega \mathbf{u}_0\}).$$

The proof of Theorem 3.1 will be established in Section 4 below.

4. Proof of the main result

We prove Theorem 3.1 gathering Propositions 4.2 and 4.3 below, which establish the liminf-inequality and the limsup-inequality, respectively. The main difficulty in establishing the liminf-inequality consists in the fact that the new pressure-dependent terms might be negative. A key ingredient to bypass this difficulty is Young's inequality. Indeed, using Young's inequality together with the ellipticity condition (3.4) it is possible to estimate $E_\varepsilon(\mathbf{u}, \varphi)$ from below in terms of $F_\varepsilon(\mathbf{u}, \varphi)$. More precisely, we can prove the following auxiliary lemma.

Lemma 4.1. *Let E_ε be as in (3.2), F_ε as in (3.6). Then there exist constants $c_1, c_2, c_3 > 0$ depending only on n, p, Ω and μ such that*

$$E_\varepsilon(\mathbf{u}, \varphi) \geq c_1 F_\varepsilon(\mathbf{u}, \varphi) - c_2 \|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^n)} - c_3 \tag{4.1}$$

for all $(\mathbf{u}, \varphi) \in H^1(\Omega; \mathbb{R}^n) \times H^1(\Omega)$ with $0 \leq \varphi \leq 1$.

Proof. Let $(\mathbf{u}, \varphi) \in H^1(\Omega; \mathbb{R}^n) \times H^1(\Omega)$; we now apply Young's inequality in the form

$$ab \leq \frac{1}{\delta} a^2 + \frac{\delta}{4} b^2 \quad \text{for every } a, b \geq 0,\ \delta > 0,$$

and we choose $\delta \in (0, 2\mu/n)$, where μ is as in (3.3). Together with the fact that $|\varphi| \leq 1$ this yields

$$\begin{aligned}
\left| \int_\Omega \Big((1-\alpha)\varphi p \nabla \cdot \mathbf{u} + \varphi \langle \nabla p, \mathbf{u} \rangle \Big) \,\mathrm{d}x \right| &\leq \frac{(1-\alpha)^2}{\delta} \int_\Omega p^2 dx + \frac{\delta}{4} \int_\Omega \varphi^2 (\nabla \cdot \mathbf{u})^2 \,\mathrm{d}x + \int_\Omega |\nabla p||\mathbf{u}| \,\mathrm{d}x \\
&\leq \frac{(1-\alpha)^2}{\delta} \mathcal{L}^n(\Omega) \|p\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\delta n}{4} \int_\Omega \varphi^2 |e(\mathbf{u})|^2 \,\mathrm{d}x + \|\nabla p\|_{L^\infty(\Omega;\mathbb{R}^n)} \|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^n)} \\
&\leq \frac{(1-\alpha)^2}{\delta} \mathcal{L}^n(\Omega) \|p\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{\mu}{2} \int_\Omega \varphi^2 |e(\mathbf{u})|^2 \,\mathrm{d}x + \|\nabla p\|_{L^\infty(\Omega;\mathbb{R}^n)} \|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^n)} \\
&\leq \frac{(1-\alpha)^2}{\delta} \mathcal{L}^n(\Omega) \|p\|^2_{L^\infty(\Omega)} + \frac{1}{4} \int_\Omega (\varphi^2 + k_\varepsilon) e(\mathbf{u}) : \mathbb{C} e(\mathbf{u}) \,\mathrm{d}x \\
&\quad + \|\nabla p\|_{L^\infty(\Omega;\mathbb{R}^n)} \|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^n)}.
\end{aligned} \tag{4.2}$$

Thus, we obtain

$$\begin{aligned}E_\varepsilon(\mathbf{u},\varphi) &\geq F_\varepsilon(\mathbf{u},\varphi) - \left| \int_\Omega \Big((1-\alpha)\varphi p \nabla\cdot \mathbf{u} + \varphi \left\langle \nabla p, \mathbf{u}\right\rangle \Big) \,\mathrm{d}x \right| \\ &\geq F_\varepsilon(\mathbf{u},\varphi) - \frac{(1-\alpha)^2}{\delta}\mathcal{L}^n(\Omega)\|p\|^2_{L^\infty(\Omega)} - \frac{1}{4}\int_\Omega (\varphi^2 + k_\varepsilon) e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right)\mathrm{d}x \\ &\quad - \|\nabla p\|_{L^\infty(\Omega;\mathbb{R}^n)}\|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^n)} \\ &\geq \frac{1}{2}F_\varepsilon(\mathbf{u},\varphi) - \frac{(1-\alpha)^2}{\delta}\mathcal{L}^n(\Omega)\|p\|^2_{L^\infty(\Omega)} - \|\nabla p\|_{L^\infty(\Omega;\mathbb{R}^n)}\|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^n)}.\end{aligned}$$

Hence, (4.1) holds true with $c_1 = \frac{1}{2}$, $c_2 = \|\nabla p\|_{L^\infty(\Omega)}$ and $c_3 = \frac{(1-\alpha)^2}{\delta}\mathcal{L}^n(\Omega)\|p\|_{L^\infty(\Omega)}$. □

Lemma 4.1 enables us to prove the lower-bound inequality Proposition 4.2 below. Thereby and in all that follows $c > 0$ denotes a generic constant that may vary from line to line.

Proposition 4.2 (Γ-lim inf inequality)**.** *Let $(\mathbf{u},\varphi) \in L^1(\Omega;\mathbb{R}^n) \times L^1(\Omega)$; for every sequence $(\mathbf{u}_\varepsilon,\varphi_\varepsilon) \subset L^1(\Omega;\mathbb{R}^n) \times L^1(\Omega)$ that converges to $(\mathbf{u},\varphi)$ in $L^1(\Omega;\mathbb{R}^n) \times L^1(\Omega)$ it holds*

$$E\left(\mathbf{u},\varphi\right) \leq \liminf_{\varepsilon\to 0} E_\varepsilon\left(\mathbf{u}_\varepsilon,\varphi_\varepsilon\right).$$

Proof. Let $(\mathbf{u}_\varepsilon,\varphi_\varepsilon) \subset L^1(\Omega;\mathbb{R}^n) \times L^1(\Omega)$ and $(\mathbf{u},\varphi)$ be as in the statement. It is not restrictive to assume that

$$\sup_\varepsilon E_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty.$$

Then $(\mathbf{u}_\varepsilon,\varphi_\varepsilon) \subset H^1(\Omega;\mathbb{R}^n) \times H^1(\Omega)$, $0 \leq \varphi_\varepsilon \leq 1$ and in view of Lemma 4.1 for every $\varepsilon > 0$ there holds

$$E_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) \geq c_1 F_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) - c_2\|\mathbf{u}_\varepsilon\|_{L^1(\Omega;\mathbb{R}^n)} - c_3.$$

Since by hypotheses $\|\mathbf{u}_\varepsilon\|_{L^1(\Omega;\mathbb{R}^n)}$ is bounded uniformly with respect to ε, we deduce that

$$\sup_\varepsilon F_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty. \tag{4.3}$$

In particular,

$$\sup_\varepsilon \int_\Omega \frac{(\varphi_\varepsilon - 1)^2}{\varepsilon}\,\mathrm{d}x < +\infty$$

and thus we obtain that $\varphi_\varepsilon \to 1$ in $L^2(\Omega)$ and by the uniqueness of the limit $\varphi = 1$ $\mathcal{L}^n$-*a.e.* in Ω. Moreover, Theorem 8 of [34] ensures that $\mathbf{u} \in \mathrm{GSBD}^2(\Omega)$. It remains to prove that

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) \geq &\frac{1}{2}\int_\Omega e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right)\mathrm{d}x \\ &+ \int_\Omega \Big(\left(1-\alpha\right) p \nabla\cdot \mathbf{u} + \left\langle \nabla p, \mathbf{u}\right\rangle \Big)\,\mathrm{d}x \\ &+ G_c\mathcal{H}^{n-1}(J_\mathbf{u}).\end{aligned}$$

To this end, we show that

$$\liminf_{\varepsilon\to 0} \frac{1}{2}\int_\Omega (\varphi_\varepsilon^2 + k_\varepsilon)e(\mathbf{u}_\varepsilon) \colon \mathbb{C}e(\mathbf{u}_\varepsilon)\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega e(\mathbf{u}) \colon \mathbb{C}e(\mathbf{u})\,\mathrm{d}x, \tag{4.4}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \Big((1-\alpha)\varphi_\varepsilon p \nabla\cdot \mathbf{u}_\varepsilon + \varphi_\varepsilon \left\langle \nabla p, \mathbf{u}_\varepsilon\right\rangle \Big)\,\mathrm{d}x = \int_\Omega \Big((1-\alpha)p\nabla\cdot\mathbf{u} + \left\langle \nabla p, \mathbf{u}\right\rangle \Big)\,\mathrm{d}x, \tag{4.5}$$

$$\liminf_{\varepsilon\to 0} \frac{G_c}{2}\int_\Omega \frac{(\varphi_\varepsilon - 1)^2}{\varepsilon} + \varepsilon|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x \geq G_c\mathcal{H}^{n-1}(J_\mathbf{u}). \tag{4.6}$$

We first prove (4.4) and (4.5) following the lines of Lemma 3.3 from [28] (see also [29]). To this end, we first pass to a subsequence (not relabeled) such that the lower limit in (4.4) is a limit and such that $(\mathbf{u}_\varepsilon, \varphi_\varepsilon) \to (\mathbf{u}, 1)$ $\mathcal{L}^n$-*a.e.* in Ω. Now we define $\Phi : [0,1] \to [0,1]$ as

$$\Phi(t) := \int_0^t (1-s)\,\mathrm{d}s = \frac{t(2-t)}{2}$$

with $\Phi(0) = 0$ and $\Phi(1) = \frac{1}{2}$ and we notice that Φ and Φ^{-1} are non negative and strictly increasing on $[0,1]$. Moreover, the classical Modica–Mortola trick gives

$$|D\Phi(\varphi_\varepsilon)|(\Omega) = \int_\Omega |\nabla\varphi_\varepsilon|\,|\varphi_\varepsilon - 1|\,\mathrm{d}x \le \int_\Omega \frac{(\varphi_\varepsilon-1)^2}{2\varepsilon} + \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x\,. \tag{4.7}$$

Thus, thanks to (4.3) we deduce that $\sup_\varepsilon |D\Phi(\varphi_\varepsilon)|(\Omega) < +\infty$. For $t \in \mathbb{R}$ let us now define the superlevel set $U_{\varepsilon,t} := \{x \in \Omega \colon \Phi(\varphi_\varepsilon(x)) > t\}$ and let us denote by $P_\varepsilon(t) := \mathrm{Per}(U_{\varepsilon,t})$ the perimeter of $U_{\varepsilon,t}$ in Ω. By the Fleming-Rishel coarea formula ([6], Thm. 3.40) we have

$$\int_{-\infty}^{+\infty} P_\varepsilon(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \left|D\mathcal{X}_{U_{\varepsilon,t}}\right|(\Omega)\,\mathrm{d}x = |D\Phi(\varphi_\varepsilon)|(\Omega). \tag{4.8}$$

Thus, in view of (4.7) and (4.3), $U_{\varepsilon,t}$ has finite perimeter in Ω for almost every t in $\mathbb{R}$ independently of ε. Choose $0 < \gamma < \gamma' < \frac{1}{2} = \Phi(1)$ arbitrary; by the mean-value theorem for integrals we find a $t_\varepsilon \in (\gamma, \gamma')$ such that

$$(\gamma' - \gamma)P_\varepsilon(t_\varepsilon) \le \int_0^{\Phi(1)} P_\varepsilon(t)\,\mathrm{d}t \le |D\Phi(\varphi_\varepsilon)|(\Omega). \tag{4.9}$$

Let us now define $U_\varepsilon := U_{\varepsilon,t_\varepsilon}$ and $\mathbf{w}_\varepsilon := \mathbf{u}_\varepsilon \mathcal{X}_{U_\varepsilon}$. Since for every $\varepsilon > 0$ it holds $\mathbf{u}_\varepsilon \in H^1(\Omega;\mathbb{R}^n)$ and U_ε has finite perimeter in Ω, it can be easily seen that $\mathbf{w}_\varepsilon \in [\mathrm{GSBV}(\Omega)]^n \subset \mathrm{GSBD}(\Omega)$ for every $\varepsilon > 0$. Moreover, $\nabla\mathbf{w}_\varepsilon = \nabla\mathbf{u}_\varepsilon\mathcal{X}_{U_\varepsilon}$ and $e(\mathbf{w}_\varepsilon) = e(\mathbf{u}_\varepsilon)\mathcal{X}_{U_\varepsilon}$. Finally, $\mathbf{w}_\varepsilon \to \mathbf{u}$ $\mathcal{L}^n$-*a.e.* in Ω, since $\mathbf{u}_\varepsilon \to \mathbf{u}$ $\mathcal{L}^n$-*a.e.* in Ω and $\mathcal{X}_{U_\varepsilon} \to \mathcal{X}_\Omega$ $\mathcal{L}^n$-*a.e.* in Ω.

Now we show that the sequence $(\mathbf{w}_\varepsilon)$ satisfies all the properties of Theorem 2.2, which will then allow us to deduce (4.4) and (4.5). We first notice that

$$\sup_\varepsilon \|\mathbf{w}_\varepsilon\|_{L^1(\Omega;\mathbb{R}^n)} \le \sup_\varepsilon \|\mathbf{u}_\varepsilon\|_{L^1(\Omega;\mathbb{R}^n)} < +\infty.$$

Further, gathering (4.7) and (4.9) we obtain

$$\sup_\varepsilon \mathcal{H}^{n-1}(J_{\mathbf{w}_\varepsilon}) = \sup_\varepsilon P_\varepsilon(t_\varepsilon) < +\infty.$$

Finally, we have

$$\begin{aligned}\int_\Omega \left(\Phi^{-1}(\gamma)\right)^2 |e(\mathbf{w}_\varepsilon)|^2\,\mathrm{d}x &\le \int_{U_\varepsilon} \left(\Phi^{-1}(t_\varepsilon)\right)^2 |e(\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x \\ &\le \int_{U_\varepsilon} (\varphi_\varepsilon)^2 |e(\mathbf{u}_\varepsilon)|^2\,\mathrm{d}x \\ &\le \int_{U_\varepsilon} \left(\varphi_\varepsilon^2 + k_\varepsilon\right) e(\mathbf{u}_\varepsilon) : \mathbb{C}e(\mathbf{u}_\varepsilon)\,\mathrm{d}x\,. \end{aligned} \tag{4.10}$$

Thanks to (4.3) the right-hand side of (4.10) is bounded uniformly in ε and thus we can apply Theorem 2.2 to the sequence $(\mathbf{w}_\varepsilon) \subset \mathrm{GSBD}^2(\Omega)$. Then, the fact that $\mathbf{w}_\varepsilon \to \mathbf{u}$ $\mathcal{L}^n$-*a.e.* in Ω allows us to conclude that $e(\mathbf{w}_\varepsilon) \rightharpoonup e(\mathbf{u})$ weakly in $L^1(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and we immediately deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{1}{2}\int_\Omega \left(\varphi_\varepsilon^2 + k_\varepsilon\right) e\left(\mathbf{u}_\varepsilon\right) : \mathbb{C}e\left(\mathbf{u}_\varepsilon\right) \mathrm{d}x &\geq \liminf_{\varepsilon\to 0} \frac{1}{2}\left(\Phi^{-1}\left(\gamma\right)\right)^2 \int_{U_\varepsilon} e\left(\mathbf{u}_\varepsilon\right) : \mathbb{C}e\left(\mathbf{u}_\varepsilon\right) \mathrm{d}x \\ &= \liminf_{\varepsilon\to 0} \frac{1}{2}\left(\Phi^{-1}\left(\gamma\right)\right)^2 \int_{U_\varepsilon} e\left(\mathbf{w}_\varepsilon\right) : \mathbb{C}e\left(\mathbf{w}_\varepsilon\right) \mathrm{d}x \\ &= \liminf_{\varepsilon\to 0} \frac{1}{2}\left(\Phi^{-1}\left(\gamma\right)\right)^2 \int_\Omega e\left(\mathbf{w}_\varepsilon\right) : \mathbb{C}e\left(\mathbf{w}_\varepsilon\right) \mathrm{d}x \\ &\geq \frac{1}{2}\left(\Phi^{-1}\left(\gamma\right)\right)^2 \int_\Omega e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right) \mathrm{d}x\,. \end{aligned} \tag{4.11}$$

Furthermore, (4.10) yields $\sup_\varepsilon \|e\left(\mathbf{w}_\varepsilon\right)\|_{L^2\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)} < +\infty$, hence the weak convergence $e\left(\mathbf{w}_\varepsilon\right) \rightharpoonup e\left(\mathbf{u}\right)$ holds in $L^2\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)$. Since also $\varphi_\varepsilon \to 1$ in $L^2\left(\Omega\right)$, we have

$$\varphi_\varepsilon e(\mathbf{w}_\varepsilon) \rightharpoonup e(\mathbf{u}) \text{ weakly in } L^1\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right),$$

and the weak convergence holds componentwise. In particular,

$$\varphi_\varepsilon \left\langle e(\mathbf{w}_\varepsilon)e_k, e_k\right\rangle \rightharpoonup \left\langle e(\mathbf{u})e_k, e_k\right\rangle \text{ weakly in } L^1\left(\Omega\right),$$

for every $1 \leq k \leq n$. Thus, since $p \in L^\infty\left(\Omega\right)$, there holds

$$\lim_{\varepsilon\to 0}\int_\Omega \left(1-\alpha\right)\varphi_\varepsilon p \nabla\cdot \mathbf{w}_\varepsilon \,\mathrm{d}x = \int_\Omega \left(1-\alpha\right) p\nabla\cdot\mathbf{u}\,\mathrm{d}x\,.$$

Moreover, using Hölder's inequality we get

$$\left|\int_{\Omega\setminus U_\varepsilon} (1-\alpha)\varphi_\varepsilon p\nabla\cdot\mathbf{u}_\varepsilon \,\mathrm{d}x\right| \leq |1-\alpha|\|p\|_{L^\infty(\Omega)}(\mathcal{L}^n(\Omega\setminus U_\varepsilon))^{1/2}\sqrt{n}\|\varphi_\varepsilon e(\mathbf{u}_\varepsilon)\|_{L^2\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)}. \tag{4.12}$$

In view of (4.3) we have that $\|\varphi_\varepsilon e(\mathbf{u}_\varepsilon)\|_{L^2\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)}$ is bounded uniformly in ε. Further, $\mathcal{L}^n(\Omega\setminus U_\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, the right hand side in (4.12) converges to 0 as $\varepsilon \to 0$ and we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega \left(1-\alpha\right)\varphi_\varepsilon p\nabla\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x &= \lim_{\varepsilon\to 0}\left(\int_\Omega \left(1-\alpha\right)\varphi_\varepsilon p\nabla\cdot\mathbf{w}_\varepsilon\,\mathrm{d}x + \int_{\Omega\setminus U_\varepsilon}\left(1-\alpha\right)\varphi_\varepsilon p\nabla\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x\right) \\ &= \int_\Omega \left(1-\alpha\right)p\nabla\cdot\mathbf{u}\,\mathrm{d}x\,.\end{aligned} \tag{4.13}$$

Finally, as $(\mathbf{u}_\varepsilon, \varphi_\varepsilon) \to (\mathbf{u}, 1)$ in $L^1\left(\Omega;\mathbb{R}^n\right)\times L^1\left(\Omega\right)$ and $0 \leq \varphi_\varepsilon \leq 1$, Lebesgue's dominated convergence theorem yields $\varphi_\varepsilon\mathbf{u}_\varepsilon \to \mathbf{u}$ in $L^1\left(\Omega;\mathbb{R}^n\right)\times L^1\left(\Omega\right)$ and thus

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi_\varepsilon\left\langle\nabla p, \mathbf{u}_\varepsilon\right\rangle\mathrm{d}x = \int_\Omega \left\langle\nabla p, \mathbf{u}\right\rangle\mathrm{d}x\,. \tag{4.14}$$

Combining (4.13) and (4.14) we obtain (4.5). Eventually, (4.4) follows from (4.11) by letting $\gamma \to \Phi(1)$.

Finally, thanks to the uniform upper bound (4.3) the proof of (4.6) can be obtained, *e.g.*, by applying estimate (61) in ([34], Thm. 8) or (5.2b) in ([21], Thm. 5.1) (see also [18], Thm. 4), which follows *via* a slicing argument together with the structure theorem ([25], Thm. 8.1). □

It remains to prove the Γ-limsup inequality.

Proposition 4.3 (Γ-lim sup inequality)**.** *For all* $(\mathbf{u},\varphi)\in L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ *there exists a sequence* $(\mathbf{u}_\varepsilon,\varphi_\varepsilon)\subset L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ *that converges to* $(\mathbf{u},\varphi)$ *in* $L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ *and satisfies*

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon)\le E(\mathbf{u},\varphi). \tag{4.15}$$

Proof. It suffices to prove (4.15) for all pairs $(\mathbf{u},\varphi)$ with $\mathbf{u}\in \mathrm{GSBD}^2(\Omega;\mathbb{R}^n)\cap L^1(\Omega;\mathbb{R}^n)$ and $\varphi=1$ $\mathcal{L}^n$-*a.e.* in Ω. Moreover, thanks to Theorem 2.3 we can restrict ourselves to the case $\mathbf{u}\in\mathcal{W}(\Omega;\mathbb{R}^n)$ and deduce the general case by a density argument. In fact, suppose that we have proved (4.15) for $\mathbf{u}\in\mathcal{W}(\Omega;\mathbb{R}^n)$. Then, for $\mathbf{u}\in\mathrm{GSBD}^2(\Omega)\cap L^1(\Omega;\mathbb{R}^n)$ Theorem 2.3 provides a sequence $(\mathbf{u}_k)\subset\mathcal{W}(\Omega;\mathbb{R}^n)$ satisfying (2.6)–(2.8) and

$$E''(\mathbf{u}_k,1)\le E(\mathbf{u}_k,1)\quad\text{for all } k\in\mathbb{N}. \tag{4.16}$$

At this point it suffices to show that $E(\mathbf{u}_k,1)\to E(\mathbf{u},1)$ as $k\to+\infty$, then (4.16) and the L^1-lower semicontinuity of $E''(\cdot,1)$ allow us to deduce that

$$E''(\mathbf{u},1)\le\liminf_{k\to+\infty}E''(\mathbf{u}_k,1)\le\liminf_{k\to+\infty}E(\mathbf{u}_k,1)=\lim_{k\to+\infty}E(\mathbf{u}_k,1)=E(\mathbf{u},1).$$

The required convergence of $E(\mathbf{u}_k,1)$ follows from (2.6) to (2.8) as follows. (2.7) and (2.8) ensure that $F(\mathbf{u}_k,1)\to F(\mathbf{u},1)$ as $k\to+\infty$, while (2.6) yields $\int_\Omega\langle\nabla p,\mathbf{u}_k\rangle\,\mathrm{d}x\to\int_\Omega\langle\nabla p,\mathbf{u}\rangle\,\mathrm{d}x$. Eventually, applying once more (2.7) together with Hölder's inequality gives

$$\begin{aligned}&\Big|\int_\Omega(1-\alpha)p(\nabla\cdot\mathbf{u}_k-\nabla\cdot\mathbf{u})\,\mathrm{d}x\Big|\\&\qquad\le|1-\alpha|\|p\|_{L^\infty(\Omega)}(\mathcal{L}^n(\Omega))^{1/2}\sqrt{n}\|e(\mathbf{u}_k)-e(\mathbf{u})\|_{L^2\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right)}\ \to 0\quad\text{as } j\to+\infty.\end{aligned}$$

It thus remains to prove the limsup inequality for $\mathbf{u}\in\mathcal{W}(\Omega;\mathbb{R}^n)$. This can be done following the approach in Theorem 4 of [18]. We start observing that for $\mathbf{u}\in\mathcal{W}(\Omega;\mathbb{R}^n)$ the jumpset $J_\mathbf{u}$ admits a Minkowski content that coincides with $\mathcal{H}^{n-1}(J_\mathbf{u})$ (see [6], Thm. 2.104), that is, there exists

$$\lim_{\rho\to 0}\frac{\mathcal{L}^n(\{x\in\mathbb{R}^n\colon\ \mathrm{dist}\,(x,J_\mathbf{u}))<\rho\})}{2\rho}=\mathcal{H}^{n-1}(J_\mathbf{u}).$$

We denote by $d(x):=\mathrm{dist}\,(x,J_\mathbf{u})$ the distance from $J_\mathbf{u}$ and for every $\rho>0$ we define the set $A_\rho:=\{x\in\Omega\colon d(x)<\rho\}$, which satisfies

$$\limsup_{\rho\to 0}\frac{\mathcal{L}^n(A_\rho)}{2\rho}\le\mathcal{H}^{n-1}(J_\mathbf{u}). \tag{4.17}$$

Let $\xi_\varepsilon=\sqrt{k_\varepsilon\varepsilon}=o(\varepsilon)$, let ψ_ε be a smooth cut off between $A_{\xi_\varepsilon/2}$ and A_{ξ_ε} and set $\mathbf{u}_\varepsilon:=\mathbf{u}(1-\psi_\varepsilon)$. Then $\mathbf{u}_\varepsilon\subset H^1(\Omega;\mathbb{R}^n)$ and $\mathbf{u}_\varepsilon\to\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^n)$ thanks to the dominated convergence theorem. To define the phasefield variable of the recovery sequence we consider the solution to the so-called optimal-profile problem

$$\inf\left\{\int_0^\infty((f-1)^2+(f')^2)\,\mathrm{d}x\colon f\in H^1_{\mathrm{loc}}(0,+\infty),\ f(0)=0,\ \lim_{t\to+\infty}f(t)=+\infty\right\}$$

given by $1-\exp(-t)$. We then set

$$\varphi_\varepsilon(x):=\begin{cases}1-\exp\left(-\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right) & \text{if } d(x)>\xi_\varepsilon,\\ 0 & \text{otherwise,}\end{cases}$$

and we observe that $(\varphi_\varepsilon) \subset H^1(\Omega)$ and $0 \le \varphi_\varepsilon \le 1$. Moreover, $\varphi_\varepsilon \to 1$ in $L^1(\Omega)$ by the dominated convergence theorem (and then also in $L^2(\Omega)$ in view of the uniform L^∞-bound). It remains to estimate $E_\varepsilon(\mathbf{u}_\varepsilon, \varphi_\varepsilon)$. We start proving that

$$\limsup_{\varepsilon\to 0} \int_\Omega \left(\frac{1}{2}\left(\varphi_\varepsilon^2 + k_\varepsilon\right) e\left(\mathbf{u}_\varepsilon\right) : \mathbb{C}e\left(\mathbf{u}_\varepsilon\right) + (1-\alpha)\,\varphi_\varepsilon p \nabla\cdot \mathbf{u}_\varepsilon + \varphi_\varepsilon \left\langle \nabla p, \mathbf{u}_\varepsilon\right\rangle\right) \mathrm{d}x$$
$$\le \int_\Omega \left(\frac{1}{2}\left(e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right)\right) + (1-\alpha)\, p\nabla\cdot\mathbf{u} + \left\langle \nabla p, \mathbf{u}\right\rangle\right) \mathrm{d}x\,. \tag{4.18}$$

As in Theorem 4 of [18] we deduce that

$$\limsup_{\varepsilon\to 0} \int_\Omega \left(\varphi_\varepsilon^2 + k_\varepsilon\right) e\left(\mathbf{u}_\varepsilon\right) : \mathbb{C}e\left(\mathbf{u}_\varepsilon\right) \mathrm{d}x \le \limsup_{\varepsilon\to 0}\left(\int_{\Omega\setminus A_{\xi_\varepsilon}} (1+k_\varepsilon)\, e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right) \mathrm{d}x + c\int_{A_{\xi_\varepsilon}} k_\varepsilon |e(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x\right)$$
$$\le \int_\Omega e\left(\mathbf{u}\right) : \mathbb{C}e\left(\mathbf{u}\right) \mathrm{d}x, \tag{4.19}$$

where the last ineqality follows from (4.17) and our choice of ξ_ε together with the fact that $\mathbf{u} \in L^\infty(\Omega;\mathbb{R}^n)$, since

$$\int_{A_{\xi_\varepsilon}} k_\varepsilon |e(\mathbf{u}_\varepsilon)|^2 \,\mathrm{d}x \le c\|\mathbf{u}\|_{L^\infty} \frac{k_\varepsilon}{\xi_\varepsilon^2}\mathcal{L}^n(A_{\xi_\varepsilon}) = c\frac{\sqrt{k_\varepsilon}}{\sqrt{\varepsilon}}\frac{\mathcal{L}^n(A_{\xi_\varepsilon})}{\xi_\varepsilon} \quad \to 0 \text{ as } \varepsilon\to 0.$$

To estimate the pressure-dependent terms we observe that

$$\int_\Omega \Big((1-\alpha)\varphi_\varepsilon p\nabla\cdot\mathbf{u}_\varepsilon + \varphi_\varepsilon\langle\nabla p, \mathbf{u}_\varepsilon\rangle\Big)\,\mathrm{d}x = \int_\Omega \Big((1-\alpha)p\nabla\cdot\mathbf{u} + \langle p, \mathbf{u}\rangle\Big)\,\mathrm{d}x$$
$$+ \int_\Omega (1-\alpha)(\varphi_\varepsilon - 1)p\nabla\cdot\mathbf{u}\,\mathrm{d}x$$
$$+ \int_\Omega (\varphi_\varepsilon - 1)\langle\nabla p, \mathbf{u}\rangle\,\mathrm{d}x\,. \tag{4.20}$$

The last term in (4.20) vanishes as $\varepsilon \to 0$, since $\varphi_\varepsilon \to 1$ in $L^1(\Omega)$ and $\langle\nabla p, \mathbf{u}\rangle \in L^\infty(\Omega)$. Moreover, applying Hölder's inequality gives

$$\Big|\int_\Omega (1-\alpha)(\varphi_\varepsilon - 1)p\nabla\cdot\mathbf{u}\,\mathrm{d}x\Big| \le c|1-\alpha|\|\varphi_\varepsilon - 1\|_{L^2(\Omega)}\|p\|_{L^\infty(\Omega)}\|e(\mathbf{u})\|_{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}.$$

The right-hand side in the above inequality vanishes thanks to the L^2-convergence of φ_ε to 1, so that (4.18) follows from (4.19) and (4.20). Eventually, the upper bound for the surface part follows as in Theorem 4 of [18] by means of the coarea formula and (4.17). We briefly repeat the argument for the reader's convenience. A direct computation shows that

$$\frac{1}{2}\int_\Omega \left(\frac{(\varphi_\varepsilon - 1)^2}{\varepsilon} + \varepsilon|\nabla\varphi_\varepsilon|^2\right)\mathrm{d}x = \frac{1}{\varepsilon}\int_{\{d>\xi_\varepsilon\}} \exp\left(-2\frac{d(x)-\xi_\varepsilon}{\varepsilon}\right)\mathrm{d}x$$
$$= \frac{1}{\varepsilon}\int_{\xi_\varepsilon}^{+\infty} \exp\left(-2\frac{t-\xi_\varepsilon}{\varepsilon}\right)\mathcal{H}^{n-1}(\partial\{d>t\})\,\mathrm{d}t, \tag{4.21}$$

where the last equality is due to the coarea formula. Moreover, there holds $f(t) := \mathcal{L}^n(A_t) = \int_0^t \mathcal{H}^{n-1}(\partial\{d > s\})\,\mathrm{d}s$. Thus, integrating by parts allows to rewrite the right-hand side in (4.21) as

$$\begin{aligned}-\frac{1}{\varepsilon}f(\xi_\varepsilon) + \frac{2}{\varepsilon^2}\int_{\xi_\varepsilon}^{+\infty}\exp\left(-2\frac{t-\xi_\varepsilon}{\varepsilon}\right)f(t)\,\mathrm{d}t &\le \frac{1}{\varepsilon}\int_0^{+\infty}\exp(-s)f\left(\xi_\varepsilon + \frac{\varepsilon}{2}s\right)\mathrm{d}s\\ &= \int_0^{+\infty}\frac{f(\xi_\varepsilon + \frac{\varepsilon}{2}s)}{2\xi_\varepsilon + \varepsilon s}\left(2\frac{\xi_\varepsilon}{\varepsilon} + s\right)\exp(-s)\,\mathrm{d}s\,.\end{aligned}$$

Since $\xi_\varepsilon = o(\varepsilon)$ and $\int_0^{+\infty} s\exp(-s)\,\mathrm{d}s = 1$, thanks to (4.17) we deduce that

$$\limsup_{\varepsilon\to 0}\int_0^{+\infty}\frac{f(\xi_\varepsilon + \frac{\varepsilon}{2}s)}{2\xi_\varepsilon + \varepsilon s}(2\frac{\xi_\varepsilon}{\varepsilon} + s)\exp(-s)\,\mathrm{d}s \le \mathcal{H}^{n-1}(J_{\mathbf{u}}),$$

which in combination with (4.21) and (4.18) gives the reqired limsup inequality. □

5. Convergence of minimizers

In this section we study the asymptotic behavior of minimizers of a suitable modification of the functionals E_ε. Namely, we consider here functionals that satisfy the equicoercivity condition required in Theorem 2.5. More precisely, let $\mathbf{g} \in L^2(\Omega;\mathbb{R}^n)$ and consider the functionals

$$E_\varepsilon^{\mathbf{g}}(\mathbf{u},\varphi) := E_\varepsilon(\mathbf{u},\varphi) + \int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x \tag{5.1}$$

and

$$E^{\mathbf{g}}(\mathbf{u},\varphi) := E(\mathbf{u},\varphi) + \int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x\,. \tag{5.2}$$

Then Theorem 3.1 allows us to establish the following convergence result, which will be proved at the end of this section.

Corollary 5.1. *Let $\mathbf{g} \in L^2(\Omega;\mathbb{R}^n)$ and let $E_\varepsilon^{\mathbf{g}}$ and $E^{\mathbf{g}}$ be as in (5.1), (5.2). For every $\varepsilon > 0$ the minimization problem*

$$m_\varepsilon := \min\left\{E_\varepsilon^{\mathbf{g}}(\mathbf{u},\varphi)\colon (\mathbf{u},\varphi) \in L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)\right\}$$

admits a solution $(\hat{\mathbf{u}}_\varepsilon,\hat\varphi_\varepsilon)$. Moreover, up to subsequences, the sequence $(\hat{\mathbf{u}}_\varepsilon,\hat\varphi_\varepsilon)$ converges in $L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ to a pair $(\hat{\mathbf{u}},\hat\varphi)$ with $\hat{\mathbf{u}} \in \mathrm{GSBD}^2(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$ solution to the problem

$$m := \min\left\{E^{\mathbf{g}}(\mathbf{u},1)\colon \mathbf{u} \in \mathrm{GSBD}^2(\Omega)\right\}.$$

Finally, $m_\varepsilon \to m$ as $\varepsilon \to 0$.

Remark 5.2 (Addition of the fidelity term)**.** The additional term $\int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x$ has also been considered in [34] (see also [21] for a more general variant) for the functionals F_ε as in (3.6), which do not contain the pressure terms. Namely, Corollary 1 of [34] establishes the analogous result to Corollary 5.1 for the functionals $F_\varepsilon^{\mathbf{g}}, F^{\mathbf{g}}$ defined as

$$F_\varepsilon^{\mathbf{g}}(\mathbf{u},\varphi) := F_\varepsilon(\mathbf{u},\varphi) + \int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x \quad\text{and}\quad F^{\mathbf{g}}(\mathbf{u},\varphi) := F(\mathbf{u},\varphi) + \int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x\,.$$

The term $\int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x$, which is usually referred to as a fidelity term, has its origin in image-processing problems. In fact, in the scalar-valued case $u, g : \Omega \to \mathbb{R}$ the functionals F^g and F_ε^g coincide with the Mumford–Shah functional [39] for image segmentation and its Ambrosio–Tortorelli approximation [3, 4]. In this context the function g represents a possibly distorted digital image and a function u obtained by minimizing F^g can be interpreted as a restored version of the given input datum g. Thus the fidelity term has a clear physical meaning as it forces a minimizer u to stay close to the original image g.

Although a fidelity term has in general no physical meaning in the context of fracture mechanics (see, *e.g.*, the introduction to Theorems 5.1 and 5.2 from [21] and references therein), it is suitable to add this term, since it makes the functionals $F^{\mathbf{g}}_\varepsilon$ and also the functionals $E^{\mathbf{g}}_\varepsilon$ equicoercive in the strong $(L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega))$-topology. Indeed, it allows us to prove Proposition 5.4 below, which is a key ingredient for the proof of Corollary 5.1.

Remark 5.3 (The case of constant pressure and boundary data)**.** It is worth noticing that without any additional constraints on the functionals E_ε and E in general one cannot expect the limit functional E to admit a minimizer. In fact, in the case of a straight or penny-shaped crack under constant pressure p in an infinite medium it is known that above a critical pressure the sharp interface limit E does not admit a minimizer (see, *e.g.*, [40]). If the medium is restricted to a bounded domain Ω as in the present setting, a natural approach to obtain equicoercivity consists in adding Dirichlet boundary conditions, that is, in considering he functionals $\widetilde{E}_\varepsilon$ defined as in (3.8). From a physical point of view, this would be more feasible than adding a fidelity term. Indeed, in [20, 31] the authors obtained very general compactness results that apply to the functionals F_ε as in (3.6). However, compactness is obtained with respect to the convergence in measure and not with respect to strong L^1-convergence, and the results apply to our setting only if p is constant, *i.e.*, $\nabla p = 0$. To be more precise, in the case $\nabla p = 0$ the constant c_2 in Lemma 4.1 is zero, so that any sequence $(\mathbf{u}_\varepsilon,\varphi_\varepsilon)$ with $\sup_\varepsilon E_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty$ also satisfies $F_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty$, even if $\|\mathbf{u}_\varepsilon\|_{L^1}$ is unbounded. Moreover, in this case it should be possible to restate the Γ-convergence result Theorem 3.1 with respect to the convergence in measure as in [21]. In the general case, however, we need to consider strong L^1-convergence in order to deal with the term $\langle \mathbf{u}, p\rangle$. For this reason, we prefer to state the compactness results Corollary 5.1 and Proposition 5.4 in the present form.

As a preliminary step towards the proof of Corollary 5.1 we now prove the following compactness result.

Proposition 5.4. *Let* $\mathbf{g}\in L^2(\Omega;\mathbb{R}^n)$ *and let* $E^{\mathbf{g}}_\varepsilon$ *be as in* (5.1)*. Moreover, suppose that* $(\mathbf{u}_\varepsilon,\varphi_\varepsilon)\subset L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ *is a sequence satisfying* $\sup_\varepsilon E^{\mathbf{g}}_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty$*. Then* $\varphi_\varepsilon \to 1$ *in* $L^1(\Omega)$ *and there exists* $\mathbf{u}\in \mathrm{GSBD}^2(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$ *such that, up to subsequences,* $\mathbf{u}_\varepsilon\to\mathbf{u}$ *in* $L^1(\Omega;\mathbb{R}^n)$*.*

Proof. The statement follows by using Lemma 4.1 and applying Proposition 1 of [34]. Indeed, let $(\mathbf{u}_\varepsilon,\varphi_\varepsilon)\subset L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ be a sequence with

$$\sup_\varepsilon E^{\mathbf{g}}_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty.$$

Then Lemma 4.1 gives

$$\begin{aligned} E^{\mathbf{g}}_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) &\geq c_1 F_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) - c_2\|\mathbf{u}_\varepsilon\|_{L^1(\Omega;\mathbb{R}^n)} - c_3 + \int_\Omega |\mathbf{u}_\varepsilon - \mathbf{g}|^2\,\mathrm{d}x \\ &\geq c_1 F_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) + \frac{1}{2}\int_\Omega |\mathbf{u}_\varepsilon-\mathbf{g}|^2\,\mathrm{d}x - c, \end{aligned} \tag{5.3}$$

where in the second step we have used again Young's inequality. We thus obtain

$$\sup_\varepsilon F^{\mathbf{g}}_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) < +\infty,$$

and we can conclude thanks to Proposition 1 of [34]. □

Proposition 5.4 now allows us to prove Corollary 5.1.

Proof of Corollary 5.1. For fixed $\varepsilon > 0$ the existence of a minimizing pair for $E_\varepsilon^{\mathbf{g}}$ follows by applying the direct method of the calculus of variations upon observing that thanks to (5.3) a minimizing sequence $(\mathbf{u}_j, \varphi_j)$ for $E_\varepsilon^{\mathbf{g}}$ satisfies

$$\sup_j \left(\int_\Omega (\varphi_j + k_\varepsilon)^2 e(\mathbf{u}_j) \colon \mathbb{C}e(\mathbf{u}_j)\,\mathrm{d}x + \int_\Omega \frac{(1-\varphi_j)^2}{\varepsilon}\,\mathrm{d}x + \varepsilon|\nabla\varphi_j|^2\,\mathrm{d}x + \int_\Omega |\mathbf{u}_j - \mathbf{g}|^2\,\mathrm{d}x \right) < +\infty.$$

Thanks to Proposition 5.4 it then suffices to show that $E_\varepsilon^{\mathbf{g}}$ Γ-converges to $E^{\mathbf{g}}$ in the strong $(L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega))$-topology, then Corollary 5.1 follows from Theorem 2.5. To prove the required Γ-convergence result we first observe that the liminf-inequality is a direct consequence of Proposition 4.2 together with Fatou's lemma. Indeed, for any sequence $(\mathbf{u}_\varepsilon, \varphi_\varepsilon) \subset L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ converging $L^1(\Omega;\mathbb{R}^n)\times L^1(\Omega)$ to some $(\mathbf{u},\varphi)$ we have

$$\liminf_{\varepsilon\to 0} E_\varepsilon^{\mathbf{g}}(\mathbf{u}_\varepsilon,\varphi_\varepsilon) \geq \liminf_{\varepsilon\to 0} E_\varepsilon(\mathbf{u}_\varepsilon,\varphi_\varepsilon) + \liminf_{\varepsilon\to 0}\int_\Omega |\mathbf{u}_\varepsilon - \mathbf{g}|^2\,\mathrm{d}x \geq E^{\mathbf{g}}(\mathbf{u},\varphi).$$

It remains to prove the limsup-inequality. To this end we notice that using (4.2) together with an estimate as in (5.3) we also obtain $E_\varepsilon^{\mathbf{g}}(\mathbf{u},\varphi) \leq c(F_\varepsilon^{\mathbf{g}}(\mathbf{u},\varphi)+1)$. In particular, due to the addition of the term $\int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x$ the domain of the Γ-limit reduces to the space $\mathrm{GSBD}^2(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$. Moreover, for every $\mathbf{u}\in \mathrm{GSBD}^2(\Omega)\cap L^2(\Omega;\mathbb{R}^n)$ Theorem 2.3 together with Remark 2.4 provide us with a sequence $(\mathbf{u}_j)\subset \mathcal{W}(\Omega;\mathbb{R}^n)$ converging to $\mathbf{u}$ in $L^2(\Omega;\mathbb{R}^n)$ and satisfying $E(\mathbf{u}_j,1)\to E(\mathbf{u},1)$ as $j\to+\infty$. Thanks to the strong convergence in $L^2(\Omega;\mathbb{R}^n)$ we then obtain

$$\lim_{j\to+\infty}\int_\Omega |\mathbf{u}_j-\mathbf{g}|^2\,\mathrm{d}x = \int_\Omega |\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x,$$

which implies that also $E^{\mathbf{g}}(\mathbf{u}_j,1)\to E^{\mathbf{g}}(\mathbf{u},1)$ as $j\to+\infty$. Thus, it suffices to establish the upper bound $(E^{\mathbf{g}})''(\mathbf{u},1)\leq E^{\mathbf{g}}(\mathbf{u},1)$ for $\mathbf{u}\in\mathcal{W}(\Omega;\mathbb{R}^n)$. This can be done by taking the recovery sequence $(\mathbf{u}_\varepsilon,\varphi_\varepsilon)$ constructed in Step 2 in the proof of Proposition 4.3 upon noticing that the sequence $(\mathbf{u}_\varepsilon)$ defined therein actually converges to $\mathbf{u}$ in $L^2(\Omega;\mathbb{R}^n)$, which in its turn implies that $\int_\Omega |\mathbf{u}_\varepsilon-\mathbf{g}|^2\,\mathrm{d}x\to\int_\Omega|\mathbf{u}-\mathbf{g}|^2\,\mathrm{d}x$ as $\varepsilon\to 0$, hence $\limsup_\varepsilon E_\varepsilon^{\mathbf{g}}(\mathbf{u}_\varepsilon,\varphi_\varepsilon)\leq E^{\mathbf{g}}(\mathbf{u},1)$. □

6. Numerical example

We will conclude our paper with an example simulation of fracture propagation including pressure, that was also considered in [15,27,38,42]. It models a straight crack in 2D opened by a constant pressure. The advantage of the setting is that one can compare the crack opening displacement (COD) with an analytical solution by Sneddon [40]. We first introduce the setup of the example and briefly sketch the numerical approach, before we recall the analytical solution computed in [40] and compare it with the numerical solution. The setup is as follows. We consider the domain $\Omega = [0,4]\times[0,4]$ with an initial crack $\mathcal{C} = [1.8, 2.2]\times[2.0-h, 2.0+h]$. On $\partial\Omega$ we prescribe homogeneous Dirichlet boundary conditions for the displacement variable, while for the phase-field variable we prescribe the Dirichlet boundary condition $\varphi = 1$. Then, minimizing the functionals $\widetilde{E}_\varepsilon$ alternatingly in $\mathbf{u}$ and φ, the associated Euler–Lagrange equations become

$$0 = \int_\Omega (\varphi^2 + k_\varepsilon)e(\mathbf{u}) : \mathbb{C}e(\mathbf{v}) + (1-\alpha)\varphi p\nabla\cdot\mathbf{v} + \varphi\langle\nabla p, \mathbf{v}\rangle\,\mathrm{d}x \qquad \forall\ \mathbf{v}\in H_0^1(\Omega,\mathbb{R}^2), \tag{6.1}$$

$$\begin{aligned} 0 = &\int_\Omega \varphi\psi e(\mathbf{u}) : \mathbb{C}e(\mathbf{u}) + (1-\alpha)\psi p\nabla\cdot\mathbf{u} + \psi\langle\nabla p,\mathbf{u}\rangle\,\mathrm{d}x \\ &+ G_c\int_\Omega \frac{(\varphi-1)\psi}{\varepsilon} + \varepsilon\langle\nabla\varphi,\nabla\psi\rangle\,\mathrm{d}x \qquad \forall\ \psi\in H_0^1(\Omega). \end{aligned} \tag{6.2}$$

Eventually, (6.1) and (6.2) are discretized using the discretization scheme proposed by Engwer and the second author in [27]. We only briefly summarize here the main ideas for the reader's convenience and refer to Section 4 of [27] for more details. Fixing a triangular grid $\mathcal{T}_h$ with mesh-size $h > 0$ (*cf.* Fig. 1) the Euler–Lagrange equation for $\mathbf{u}$ is discretized using a discontinuous Galerkin scheme. Roughly speaking, this amounts to restricting (6.1) to functions $\mathbf{u}_h, \mathbf{v}_h$ belonging to the space of piecewise polynomials of degree at most 1 given by

$$V_h := \{\mathbf{w} = (w_1, w_2) \in L^1(\Omega;\mathbb{R}^2)\colon w_{1_{|T}},\, w_{2_{|T}} \in \mathbb{P}^1(\Omega)\ \forall\, T \in \mathcal{T}_h,\ \mathbf{w} = 0 \text{ on } \partial\Omega\},$$

and adding a penalization of possible jumps of $\mathbf{u}, \mathbf{v}$ across faces of the triangles T. Moreover, (6.2) is discretized using first-order Lagrange shape functions (*cf.*, *e.g.*, [22], Chap. 2.2), that is, by restricting it to φ_h, ψ_h belonging to the space

$$W_h := \{\phi \in C^0(\Omega)\colon \phi_{|T} \in \mathbb{P}^1(\Omega)\ \forall\, T \in \mathcal{T}_h\},$$

where in addition we require $\varphi_h = 1$ on $\partial\Omega$ and $\psi_h = 0$ on $\partial\Omega$. The constraint $0 \le \varphi \le 1$ is implemented using the Truncated Non-smooth Newton Multigrid Method (TNNMG) [32]. The corresponding discrete formulations of (6.1) and (6.2) are solved numerically using the DUNE framework [8], applying a fix-point iteration scheme, alternatingly solving for the displacement and the phasefield. The iteration scheme stops when the correction in the phasefield becomes smaller than a certain treshold.

Let us now fix the material parameters and recall the corresponding analytical solution in [40]. As Lamé constants we choose $\lambda = 0.27778\,\mathrm{N/m^2}$, $\mu = 0.41667\,\mathrm{N/m^2}$, while $G_c = 1.0\,\mathrm{N/m}$. A constant pressure $p \equiv 10^{-3}\,\mathrm{N/m^2}$ forces the crack to open. According to Sneddon in the described situation the crack opening displacement (COD) defined by

$$\mathrm{COD} := \int_{-\infty}^{\infty} \langle \mathbf{u}(x,y), \nabla\varphi(x,y)\rangle \,\mathrm{d}y$$

can be calculated analytically as

$$\mathrm{COD} = 2l_0 \frac{1-\nu^2}{E} p \sqrt{\left(1 - \frac{x^2}{l_0^2}\right)}.$$

Here $2l_0$ denotes the initial crack length, thus $l_0 = 0.2$. The material parameter E and ν stand for Young's modulus and Poisson's ratio, respectively. Figure 1 shows the mesh and the solution for the displacement in y-direction. We will compare the results obtained in the simulations with this COD, see Figure 2. We mention that Sneddon's benchmark example was already simulated in [27], here we will study the relation between the mesh size h and the parameters k_ε and ε in more detail. Recent studies of the discretization of the Ambrosio–Tortorelli functional suggest, that it is necessary to choose $h = o(\varepsilon)$ to keep Γ-convergence in the discrete setting (see [7] and also [9] and [23]). To study the convergence behavior in different parameter regimes we consider the following three cases:

- Case 1: Refine $\varepsilon, h, k_\varepsilon$ with the fixed relation $\varepsilon = 2h$, $k_\varepsilon = h^2$.
- Case 2: Refine $\varepsilon, h, k_\varepsilon$ with the fixed relation $\varepsilon = \sqrt{h}$, $k_\varepsilon = h$.
- Case 3: Refine $\varepsilon, h, k_\varepsilon$ with the fixed relation $\varepsilon = 2\sqrt{h}$, $k_\varepsilon = h^2$.

In the initial setup the crack is described implicitely by setting the phasefield to zero along the crack (see Fig. 3, where we show the initial and the final phasefiled for Case 3). The COD of the numerical simulations in the Cases 1–3 is shown in Figure 2. Notice that the numerical solution for the COD is non-zero at the boundary due to a smearing effect in x-direction resulting from the phase-field variable φ. This effect is quantified in Table 1, which shows the error in the COD both at the boundary of the crack (characterising the smearing effect) and in the middle of the crack (proportional to the error in the crack opening volume). Let us now briefly comment on the convergence behavior in the three different cases. In Case 1 the constraint $k_\varepsilon = o(\varepsilon)$ is satisfied, but the mesh size does not satisfy $h = o(\varepsilon)$, and indeed the numerically computed COD does not approximate well the analytical COD (see also the error in Tab. 1). In contrast, in Case 2 the mesh size is chosen sufficiently small,

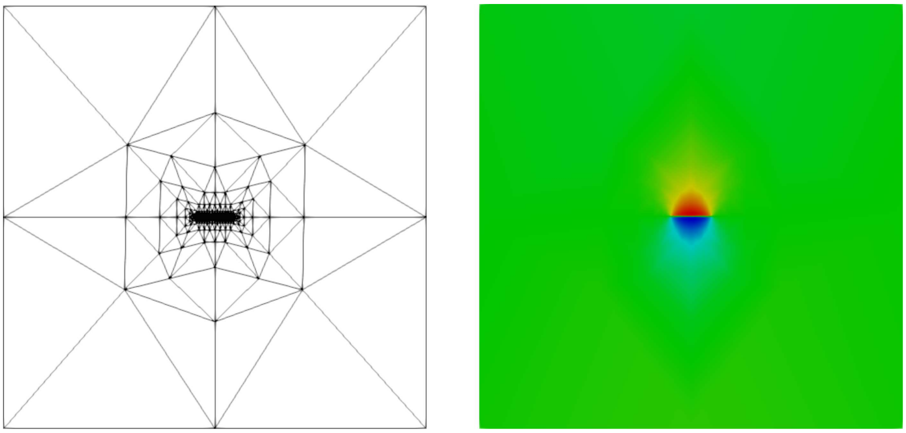

FIGURE 1. Simulation of a crack opening with an applied constant pressure. *Left*: grid for $h = 0.00625$ is shown. *Right*: corresponding solution for the displacement in y-direction is shown.

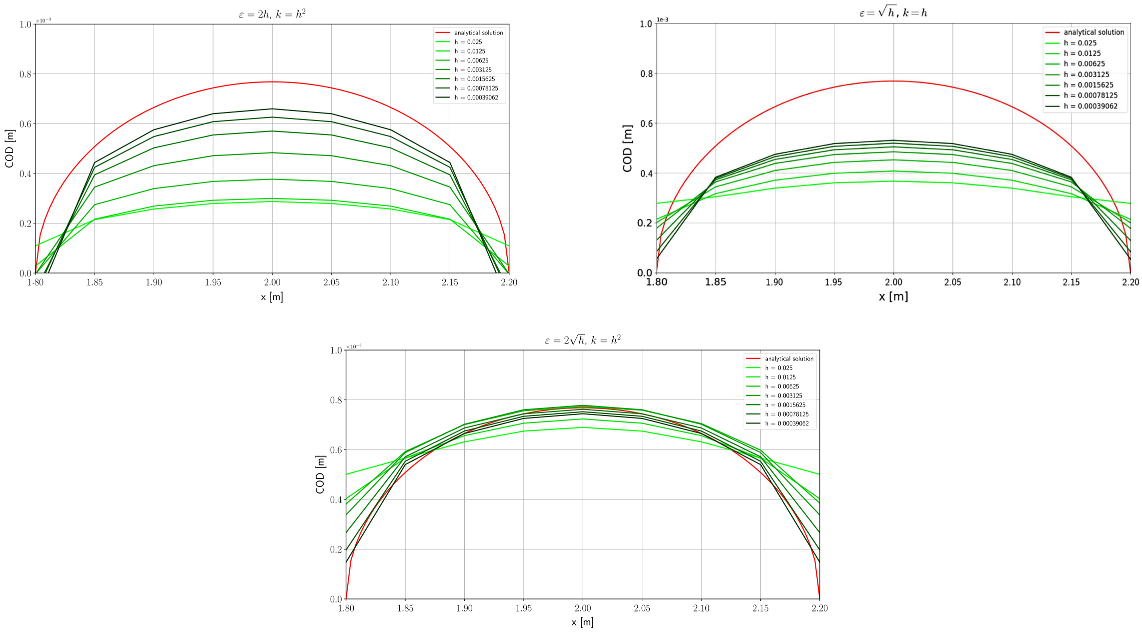

FIGURE 2. Comparison of the COD of the numerical simulation with Sneddon's analytical solution for the different parameter settings.

so that $h = o(\varepsilon)$ is satisfied, instead the constraint $k_\varepsilon = o(\varepsilon)$ is violated, which results in an even larger error in the COD. Finally, a good approximation of the COD (*cf.* again Tab. 1) is obtained in Case 3, where both the constraints $k_\varepsilon = o(\varepsilon)$ and $h = o(\varepsilon)$ are satisfied.

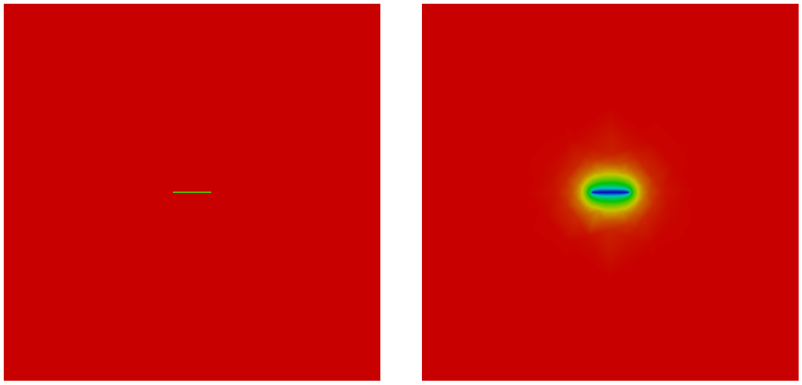

FIGURE 3. Initial and final phasefield in the Case 3.

TABLE 1. Error in the COD.

	$x = 1.8$ (boundary)		
	Case 1	Case 2	Case 3
$h = 0.025$	1.08884×10^{-4}	2.77893×10^{-4}	5.00649×10^{-4}
$h = 0.0125$	2.78525×10^{-5}	2.13652×10^{-4}	4.02968×10^{-4}
$h = 0.00625$	4.79258×10^{-6}	2.00319×10^{-4}	3.81487×10^{-4}
$h = 0.003125$	4.58196×10^{-6}	1.77422×10^{-4}	3.38051×10^{-4}
$h = 0.0015625$	6.92701×10^{-5}	1.29177×10^{-4}	2.66374×10^{-4}
$h = 0.00078125$	8.60155×10^{-5}	8.37184×10^{-5}	1.96768×10^{-4}
$h = 0.00039062$	1.23924×10^{-4}	5.40301×10^{-5}	1.47525×10^{-4}
	$x = 2.0$ (center)		
	Case 1	Case 2	Case 3
$h = 0.025$	4.80758×10^{-4}	4.01074×10^{-4}	7.8668×10^{-5}
$h = 0.0125$	4.68180×10^{-4}	3.60473×10^{-4}	4.4731×10^{-5}
$h = 0.00625$	3.91144×10^{-4}	3.15823×10^{-4}	5.1340×10^{-6}
$h = 0.003125$	2.84570×10^{-4}	2.83627×10^{-4}	9.8550×10^{-6}
$h = 0.0015625$	1.97725×10^{-4}	2.63613×10^{-4}	5.9020×10^{-6}
$h = 0.00078125$	1.41795×10^{-4}	2.48961×10^{-4}	1.5930×10^{-5}
$h = 0.00039062$	1.08338×10^{-4}	2.37518×10^{-4}	2.3898×10^{-5}

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