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A CONFORMING MIXED FINITE ELEMENT METHOD FOR THE NAVIER–STOKES/DARCY–FORCHHEIMER COUPLED PROBLEM

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Abstract. In this work we present and analyse a mixed finite element method for the coupling of fluid flow with porous media flow. The flows are governed by the Navier–Stokes and the Darcy–Forchheimer equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law. We consider the standard mixed formulation in the Navier–Stokes domain and the dual-mixed one in the Darcy–Forchheimer region, which yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. The well-posedness of the problem is achieved by combining a fixed-point strategy, classical results on nonlinear monotone operators and the well-known Schauder and Banach fixed-point theorems. As for the associated Galerkin scheme we employ Bernardi–Raugel and Raviart–Thomas elements for the velocities, and piecewise constant elements for the pressures and the Lagrange multiplier, whereas its existence and uniqueness of solution is established similarly to its continuous counterpart, using in this case the Brouwer and Banach fixed-point theorems, respectively. We show stability, convergence, and *a priori* error estimates for the associated Galerkin scheme. Finally, we report some numerical examples confirming the predicted rates of convergence, and illustrating the performance of the method.

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1. Introduction

The modelling and numerical simulation of incompressible fluid flows in regions partially occupied by porous media has become a very active research area during the last decades, mostly due to its relevance in the fields of natural sciences and engineering branches. In particular, these kind of phenomena can be found in several applications such as in vuggy porous media appearing in petroleum extraction (see, *e.g.*, [3, 4]), groundwater system in karst aquifers (see, *e.g.*, [26,43]), reservoir wellbore (see, *e.g.*, [2,5]), internal ventilation of a motorcycle

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helmet (see, *e.g.*, [14,18]), and blood motion in tumors and microvessels (see, *e.g.*, [45,52]), to name a few. One of the most popular models utilised to describe the aforementioned interaction is the Navier–Stokes/Darcy–Forchheimer (or Navier–Stokes/Darcy, Stokes/Darcy) model, which consists in a set of differential equations where the Navier–Stokes (or Stokes) problem is coupled with the Darcy–Forchheimer (or Darcy) model through a set of coupling equations acting on a common interface, which are given by mass conservation, balance of normal forces, and the so called Beavers–Joseph–Saffman condition. In [7, 15, 16, 20–22, 28–31], and in the references therein, we can find a large list of contributions devoted to numerically approximate the solution of this interaction problem, including primal and mixed conforming formulations, as well as nonconforming methods. At this point we remark that the Navier–Stokes/Darcy–Forchheimer model is considered when the fluid velocity is higher and the porosity is nonuniform, which holds when the kinematic forces dominates over viscous forces. We refer the reader to [6, 34, 44, 48] for the derivation and analysis of the Darcy–Forchheimer equations.

Up to the authors' knowledge, one of the first works in analysing the coupling of Navier–Stokes and Darcy–Forchheimer equations is [2]. In that work, the authors study the coupling of a 2D reservoir model with a 1.5D vertical wellbore model, both written in axisymmetric form. The physical problems are described by the Darcy–Forchheimer and the compressible Navier–Stokes equations, respectively, together with an exhaustive energy equation. Later on, motivated by the study of the internal ventilation of a motorcycle helmet, a penalization approach was introduced and analysed in [18]. In particular, the authors consider the velocity and pressure in the whole domain as the main unknowns of the system, and the corresponding Galerkin approximation employs piecewise quadratic elements and piecewise linear for the velocity and pressure, respectively. Notice that this method is applied to both 2D and 3D domains. More recently, in [53] a 3D discrete dynamical system was derived from the generalized Navier–Stokes equations for incompressible flow with nonlinear drag forces (represented by Forchheimer terms) in porous media *via* a Galerkin procedure. We observe that this method can be employed in subgrid-scale models of synthetic-velocity form for large-eddy simulation of turbulent flow through porous media.

Furthermore, and concerning simpler related models, we highlight that a conforming mixed method for the Stokes–Darcy coupled problem has been introduced and analysed in [28]. In this work, the velocity-pressure formulation in the Stokes equation and the dual-mixed approach in the Darcy region is considered, which yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. Later on, it was shown in [29] that the use of any pair of stable Stokes and Darcy elements guarantees the well-posedness of the corresponding Stokes–Darcy Galerkin scheme. More recently, in [21] the authors extend the results from [28] to the Navier–Stokes/Darcy coupled problem. Since this coupled system is nonlinear (due to the convective term in the free fluid region), the analysis of the continuous problem begins with the linearisation of the Oseen problem in the free fluid domain. This simplified model is then studied by means of the classical Babuška–Brezzi theory, similarly as it was done for the Stokes–Darcy coupling in [28]. Then, a fixed-point strategy based on the aforementioned linearisation is associate to the nonlinear coupling, which allows to establish existence and uniqueness of solution thanks to Schauder's and Banach's fixed point theorems, respectively.

According to the above bibliographic discussion, in this paper we aim to extend the results obtained in [21, 28, 29] to the Navier–Stokes/Darcy–Forchheimer coupled problem. We consider the standard velocity-pressure formulation for the Navier–Stokes equation and unlike [21], in the porous medium we consider the Darcy–Forchheimer equation in its dual-mixed formulation. In this way, we obtain the velocity and the pressure of the fluid in both media as the main unknowns of the coupled system. Since one of the interface conditions becomes essential, we proceed similarly to [21,28] and incorporate the trace of the porous medium pressure as an additional unknown. The well-posedness of both the continuous and discrete formulations is proved, employing a fixed-point argument and clasical results on nonlinear monotone operators (see [50, 51]). In particular, for the continuous formulation, under a smallness data assumption, we prove existence and uniqueness of solution by means of a fixed-point strategy where the Schauder (for existence) and Banach (for uniqueness) fixed-point theorems are employed.

Using similar arguments (but applying Brower's fixed-point theorem instead of Schauder's for the existence result) we prove the well-posedness of the discrete problem for a specific choice of discrete space. More precisely, we consider Bernardi–Raugel elements for the velocity in the free fluid region, Raviart–Thomas elements of lowest order for the filtration velocity in the porous media, piecewise constants with null mean value for the pressures, and piecewise constant elements for the Lagrange multiplier on the interface.

The rest of this paper is organised as follows. In Section 2 we introduce the model problem and derive the variational formulation. Next, in Section 3, we establish that our variational formulation is well posed. The corresponding Galerkin scheme is introduced and analysed in Section 4. In Section 5 we derive the corresponding Céa's estimate and a sub-optimal rate of convergence. Finally, several numerical examples illustrating the performance of the method, confirming the theoretical sub-optimal order of convergence and suggesting an optimal rate of convergence, are reported in Section 6.

We end this section by introducing some definitions and fixing some notations. Let $\mathcal{O} \subseteq \mathbb{R}^n$, $n \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $s \geq 0$ and $p \in [1,+\infty]$, we denote by $\mathrm{L}^p(\mathcal{O})$ and $\mathrm{W}^{s,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^p(\mathcal{O})}$ and $\|\cdot\|_{s,p;\mathcal{O}}$, respectively. Note that $\mathrm{W}^{0,p}(\mathcal{O}) = \mathrm{L}^p(\mathcal{O})$. If $p=2$, we write $\mathrm{H}^s(\mathcal{O})$ in place of $\mathrm{W}^{s,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{s,\mathcal{O}}$. In addition, we denote by $\mathrm{W}^{\frac{1}{q},p}(\Gamma)$ the trace space of $\mathrm{W}^{1,p}(\mathcal{O})$ and $\mathrm{W}^{-\frac{1}{q},q}(\Gamma)$ the dual space of $\mathrm{W}^{\frac{1}{q},p}(\Gamma)$ endowed with the norms $\|\cdot\|_{\frac{1}{q},p;\Gamma}$ and $\|\cdot\|_{-\frac{1}{q},q;\Gamma}$, respectively, with $p,q \in (1,+\infty)$ satisfying $1/p + 1/q = 1$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. Additionally, we recall that $\mathbf{H}(\mathrm{div}\,;\mathcal{O}) := \Big\{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \mathrm{div}\,\mathbf{w} \in \mathrm{L}^2(\mathcal{O})\Big\}$, is a standard Hilbert space in the realm of mixed problems (see, *e.g.*, [12]). On the other hand, the following symbol for the $\mathrm{L}^2(\Gamma)$ inner product

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\lambda \quad \forall \xi, \lambda \in \mathrm{L}^2(\Gamma),$$

will also be employed for their respective extension as the duality parity between $\mathrm{W}^{-\frac{1}{q},q}(\Gamma)$ and $\mathrm{W}^{\frac{1}{q},p}(\Gamma)$. Hereafter, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$. Furthermore, given a non-negative integer k and a subset S of $\mathbb{R}^n$, $\mathbb{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

2. The continuous formulation

In this section we introduce the model problem and derive the corresponding weak formulation. For simplicity of exposition we set the problem in $\mathbb{R}^2$. However, our study can be extended to the 3D case with few modifications, which will be pointed out appropriately in the paper.

2.1. The model problem

In order to describe the geometry, we let Ω_S and Ω_D be two bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial\Omega_\mathrm{S} \cap \partial\Omega_\mathrm{D} = \Sigma \neq \emptyset$ and $\Omega_\mathrm{S} \cap \Omega_\mathrm{D} = \emptyset$. Then, let $\Gamma_\mathrm{S} := \partial\Omega_\mathrm{S} \setminus \overline{\Sigma}$, $\Gamma_\mathrm{D} := \partial\Omega_\mathrm{D} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_\mathrm{S} \cup \Sigma \cup \Omega_\mathrm{D}$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Figure 1). The problem we are interested in consists of the movement of an incompressible viscous fluid occupying Ω_S which flows towards and from a porous medium Ω_D through Σ, where Ω_D is saturated with the same fluid. The mathematical model is defined by two separate groups of equations and by a set of coupling terms. In the free

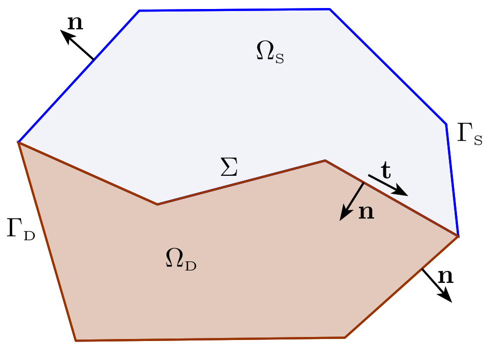

FIGURE 1. Sketch of a 2D geometry of our Navier–Stokes/Darcy–Forchheimer model.

fluid domain Ω_S, the motion of the fluid can be described by the incompressible Navier–Stokes equations:

$$\begin{gathered}\boldsymbol{\sigma}_S = -p_S\mathbb{I} + 2\mu\mathbf{e}(\mathbf{u}_S) \quad \text{in} \quad \Omega_S, \quad -\mathbf{div}\boldsymbol{\sigma}_S + \rho(\nabla\mathbf{u}_S)\mathbf{u}_S = \mathbf{f}_S \quad \text{in} \quad \Omega_S,\\ \operatorname{div}\mathbf{u}_S = 0 \quad \text{in} \quad \Omega_S, \quad \mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S,\end{gathered} \tag{2.1}$$

where the unknowns are the fluid velocity $\mathbf{u}_S$, the pressure p_S, and the Cauchy stress tensor $\boldsymbol{\sigma}_S$. In addition, $\mathbf{e}(\mathbf{u}_S) := \dfrac{1}{2}\left\{\nabla\mathbf{u}_S + (\nabla\mathbf{u}_S)^{\mathrm{t}}\right\}$ stands for the strain tensor of small deformations, μ is the viscosity of the fluid, ρ is the density, and $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ is a given external force.

In the porous medium Ω_D we consider a nonlinear version of the Darcy problem to approximate the velocity $\mathbf{u}_D$ and the pressure p_D, which is considered when the fluid velocity is higher and the porosity is nonuniform. More precisely, we consider the Darcy–Forchheimer equations [44,48]:

$$\frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_D + \frac{F}{\rho}|\mathbf{u}_D|\mathbf{u}_D + \nabla p_D = \mathbf{f}_D \quad \text{in} \quad \Omega_D, \quad \operatorname{div}\mathbf{u}_D = g_D \quad \text{in} \quad \Omega_D, \quad \mathbf{u}_D\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D, \tag{2.2}$$

where F represents the Forchheimer number of the porous medium, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_D)$ is a symmetric tensor in Ω_D representing the intrinsic permeability $\boldsymbol{\kappa}$ of the porous medium divided by the viscosity μ of the fluid. Throughout the paper we assume that there exists $C_{\mathbf{K}} > 0$ such that

$$\mathbf{w}\cdot\mathbf{K}^{-1}(\mathbf{x})\mathbf{w} \geq C_{\mathbf{K}}|\mathbf{w}|^2, \tag{2.3}$$

for almost all $\mathbf{x} \in \Omega_D$, and for all $\mathbf{w} \in \mathbb{R}^2$. In turn, as will be explained below, $\mathbf{f}_D$ and g_D are given functions in $\mathbf{L}^{3/2}(\Omega_D)$ and $\mathrm{L}^2(\Omega_D)$, respectively. In addition, according to the compressibility conditions, the boundary conditions on $\mathbf{u}_D$ and $\mathbf{u}_S$, and the principle of mass conservation (*cf.* (2.4)), g_D must satisfy the compatibility condition:

$$\int_{\Omega_D} g_D = 0.$$

Finally, the transmission conditions that couple the Navier–Stokes and the Darcy–Forchheimer models through the interface Σ are given by

$$\mathbf{u}_S\cdot\mathbf{n} = \mathbf{u}_D\cdot\mathbf{n} \quad \text{on} \quad \Sigma \quad \text{and} \quad \boldsymbol{\sigma}_S\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_S\cdot\mathbf{t})\,\mathbf{t} = -p_D\mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4}$$

where α_d is a dimensionless positive constant which depends only on the geometrical characteristics of the porous medium and usually assumes values between 0.8 and 1.2 (see [9,18]). The first condition in (2.4) is a consequence of the incompressibility of the fluid and of the conservation of mass across Σ. The second transmission condition on Σ can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_{\rm S}\mathbf{n})\cdot\mathbf{n} = -p_{\rm D} \quad \text{and} \quad (\boldsymbol{\sigma}_{\rm S}\mathbf{n})\cdot\mathbf{t} = -\frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_{\rm S}\cdot\mathbf{t}) \quad \text{on} \quad \Sigma. \tag{2.5}$$

The first equation in (2.5) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman condition, which establishes that the slip velocity along Σ is proportional to the shear stress long Σ. We refer the reader to Section 3.2 of [8] (see also [40, 49]) for further details on the choice of this interface condition.

2.2. The variational formulation

In this section we proceed analogously to Section 2 of [28] and derive a weak formulation of the coupled problem given by (2.1), (2.2), and (2.4). To this end, let us first introduce further notations and definitions. In what follows, given $\star\in\{\mathrm{S},\mathrm{D}\}$, we set

$$(p,q)_\star := \int_{\Omega_\star} p\,q, \quad (\mathbf{u},\mathbf{v})_\star := \int_{\Omega_\star}\mathbf{u}\cdot\mathbf{v}, \quad \text{and} \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_\star := \int_{\Omega_\star}\boldsymbol{\sigma}:\boldsymbol{\tau},$$

where, given two arbitrary tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, $\boldsymbol{\sigma}:\boldsymbol{\tau} = \mathrm{tr}\,(\boldsymbol{\sigma}^{\rm t}\boldsymbol{\tau}) = \sum_{i,j=1}^2 \sigma_{ij}\tau_{ij}$. Furthermore, in the sequel we will employ the following Banach space,

$$\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D}) := \Big\{\mathbf{v}_{\rm D}\in\mathbf{L}^3(\Omega_{\rm D}): \quad \mathrm{div}\,\mathbf{v}_{\rm D}\in\mathrm{L}^2(\Omega_{\rm D})\Big\},$$

endowed with the norm

$$\|\mathbf{v}_{\rm D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})} := \Big(\|\mathbf{v}_{\rm D}\|^3_{\mathbf{L}^3(\Omega_{\rm D})} + \|\mathrm{div}\,\mathbf{v}_{\rm D}\|^3_{0,\Omega_{\rm D}}\Big)^{1/3},$$

and the following subspaces of $\mathbf{H}^1(\Omega_{\rm S})$ and $\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})$, respectively

$$\begin{aligned}\mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}) &:= \Big\{\mathbf{v}_{\rm S}\in\mathbf{H}^1(\Omega_{\rm S}): \quad \mathbf{v}_{\rm S}=\mathbf{0} \quad \text{on} \quad \Gamma_{\rm S}\Big\},\\ \mathbf{H}^3_{\Gamma_{\rm D}}(\mathrm{div}\,;\Omega_{\rm D}) &:= \Big\{\mathbf{v}_{\rm D}\in\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D}): \quad \mathbf{v}_{\rm D}\cdot\mathbf{n}=0 \quad \text{on} \quad \Gamma_{\rm D}\Big\}.\end{aligned}$$

Notice that $\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D}) = \mathbf{H}(\mathrm{div}\,;\Omega_{\rm D})\cap\mathbf{L}^3(\Omega_{\rm D})$, which guarantees that $\mathbf{v}_{\rm D}\cdot\mathbf{n}$ is well defined for $\mathbf{v}_{\rm D}\in\mathbf{H}^3_{\Gamma_{\rm D}}(\mathrm{div}\,;\Omega_{\rm D})$.

To begin with the derivation of our variational formulation for the Navier–Stokes/Darcy–Forchheimer problem we first proceed similarly to [21, 28] and test the second equation of (2.1) by $\mathbf{v}_{\rm S}\in\mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$, integrate by parts and utilize the second equation of (2.4) to obtain

$$\begin{aligned}2\mu(\mathbf{e}(\mathbf{u}_{\rm S}),\mathbf{e}(\mathbf{v}_{\rm S}))_{\rm S} + \left\langle\frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\mathbf{u}_{\rm S}\cdot\mathbf{t},\mathbf{v}_{\rm S}\cdot\mathbf{t}\right\rangle_\Sigma + \rho((\nabla\mathbf{u}_{\rm S})\mathbf{u}_{\rm S},\mathbf{v}_{\rm S})_{\rm S}\\ - (p_{\rm S},\mathrm{div}\,\mathbf{v}_{\rm S})_{\rm S} + \langle\mathbf{v}_{\rm S}\cdot\mathbf{n},\lambda\rangle_\Sigma = (\mathbf{f},\mathbf{v}_{\rm S})_{\rm S},\end{aligned} \tag{2.6}$$

for all $\mathbf{v}_{\rm S}\in\mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$, where λ is a further unknown representing the trace of the porous medium pressure on Σ, that is $\lambda = p_{\rm D}|_\Sigma$. The corresponding space of λ will be specified next. In turn, we incorporate the incompressibility condition $\mathrm{div}\,\mathbf{u}_{\rm S}=0$ in $\Omega_{\rm S}$ weakly as follows

$$(q_{\rm S},\mathrm{div}\,\mathbf{u}_{\rm S})_{\rm S} = 0 \qquad \forall\,\mathrm{q}_{\rm S}\in\mathrm{L}^2(\Omega_{\rm S}). \tag{2.7}$$

Next, we multiply the first equation of (2.2) by $\mathbf{v}_{\rm D}\in\mathbf{H}^3_{\Gamma_{\rm D}}(\mathrm{div}\,;\Omega_{\rm D})$ and integrate by parts to obtain

$$\frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_{\rm D},\mathbf{v}_{\rm D})_{\rm D} + \frac{F}{\rho}(|\mathbf{u}_{\rm D}|\mathbf{u}_{\rm D},\mathbf{v}_{\rm D})_{\rm D} - (p_{\rm D},\mathrm{div}\,\mathbf{v}_{\rm D})_{\rm D} - \langle\mathbf{v}_{\rm D}\cdot\mathbf{n},\lambda\rangle_\Sigma = (\mathbf{f}_{\rm D},\mathbf{v}_{\rm D})_{\rm D}, \tag{2.8}$$

for all $\mathbf{v}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$. Observe that if $\mathbf{u}_\mathrm{D} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})$ and $p_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$, then $|\mathbf{u}_\mathrm{D}|\mathbf{u}_\mathrm{D}\cdot\mathbf{v}_\mathrm{D} \in \mathrm{L}^1(\Omega_\mathrm{D})$ and $p_\mathrm{D}\,\mathrm{div}\,\mathbf{v}_\mathrm{D} \in \mathrm{L}^1(\Omega_\mathrm{D})$, and hence the second and third terms of (2.8) are well defined, which justifies the introduction of the spaces $\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})$ for the derivation of our weak formulation. Moreover, for each $\mathbf{v}_\mathrm{D} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})$, the normal trace $\mathbf{v}_\mathrm{D}\cdot\mathbf{n} : \mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D}) \to \mathrm{W}^{-\frac{1}{3},3}(\partial\Omega_\mathrm{D})$ is well defined and continuous. In fact, since $\mathrm{W}^{1,\frac{3}{2}}(\Omega_\mathrm{D})$ is continuously embedded into $\mathrm{L}^2(\Omega_\mathrm{D})$ then for each $\xi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_\mathrm{D})$ the quantity

$$\langle \mathbf{v}_\mathrm{D}\cdot\mathbf{n}, \xi\rangle_{\partial\Omega_\mathrm{D}} := \int_{\Omega_\mathrm{D}} \mathbf{v}_\mathrm{D}\cdot\nabla\widetilde{\gamma}_0^{-1}(\xi) + \int_{\Omega_\mathrm{D}} \widetilde{\gamma}_0^{-1}(\xi)\mathrm{div}\,\mathbf{v}_\mathrm{D},$$

is well defined, where $\langle\cdot,\cdot\rangle_{\partial\Omega_\mathrm{D}}$ stands for the duality pairing between $\mathrm{W}^{-\frac{1}{3},3}(\partial\Omega_\mathrm{D})$ and $\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_\mathrm{D})$, and $\widetilde{\gamma}_0^{-1}$ is the right inverse of the well known trace operator $\gamma_0 : \mathrm{W}^{1,\frac{3}{2}}(\Omega_\mathrm{D}) \to \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_\mathrm{D})$. Furthermore, given $\mathbf{v}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$, the boundary condition $\mathbf{v}_\mathrm{D}\cdot\mathbf{n} = 0$ on Γ_D means (see *e.g.* [24], Appendix A and [21,31])

$$\langle \mathbf{v}_\mathrm{D}\cdot\mathbf{n}, E_{0,\mathrm{D}}(\xi)\rangle_{\partial\Omega_\mathrm{D}} = 0 \quad \forall\,\xi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma_\mathrm{D}),$$

where $E_{0,\mathrm{D}} : \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma_\mathrm{D}) \to \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_\mathrm{D})$ is the extension operator defined by

$$E_{0,\mathrm{D}}(\xi) := \begin{cases} \xi \text{ on } \Gamma_\mathrm{D} \\ 0 \text{ on } \Sigma \end{cases} \quad \forall\,\xi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma_\mathrm{D}),$$

We observe that according to Theorem 1.5.2.3 of [37], the operator $E_{0,\mathrm{D}}$ is well defined. In turn, similarly to equation (A.6) of [24] we can identify the restriction of $\mathbf{v}_\mathrm{D}\cdot\mathbf{n}$ to Σ with an element of $\mathrm{W}^{-\frac{1}{3},3}(\Sigma)$, namely

$$\langle \mathbf{v}_\mathrm{D}\cdot\mathbf{n}, \xi\rangle_\Sigma := \langle \mathbf{v}_\mathrm{D}\cdot\mathbf{n}, E_\Sigma(\xi)\rangle_{\partial\Omega_\mathrm{D}} \quad \forall \xi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma), \tag{2.9}$$

where $E_\Sigma : \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma) \to \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_\mathrm{D})$ is any bounded extension operator. In addition, analogously to the proof of Lemma A.2 from [24] one can show that for all $\psi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_\mathrm{D})$, there exist unique elements $\psi_\Sigma \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$ and $\psi_{\Gamma_\mathrm{D}} \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma_\mathrm{D})$ such that

$$\psi = E_\Sigma(\psi_\Sigma) + E_{0,\mathrm{D}}(\psi_{\Gamma_\mathrm{D}}), \tag{2.10}$$

and there exist $C_1, C_2 > 0$, such that

$$C_1\Big\{\|\psi_\Sigma\|_{\frac{1}{3},\frac{3}{2};\Sigma} + \|\psi_{\Gamma_\mathrm{D}}\|_{\frac{1}{3},\frac{3}{2};\Gamma_\mathrm{D}}\Big\} \le \|\psi\|_{\frac{1}{3},\frac{3}{2};\partial\Omega_\mathrm{D}} \le C_2\Big\{\|\psi_\Sigma\|_{\frac{1}{3},\frac{3}{2};\Sigma} + \|\psi_{\Gamma_\mathrm{D}}\|_{\frac{1}{3},\frac{3}{2};\Gamma_\mathrm{D}}\Big\}. \tag{2.11}$$

In fact, although Lemma A.2 of [24] is derived for $\mathrm{W}^{1-\frac{1}{p},p}(\partial\Omega_\mathrm{D})$ with $p \ge 2$, using a slight modification of Section 2 from [35] one can easily extend the analysis to the case $p > 1$. According to the above, for each $\mathbf{v}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$, $\mathbf{v}_\mathrm{D}\cdot\mathbf{n}|_\Sigma \in \mathrm{W}^{-\frac{1}{3},3}(\Sigma)$, which suggests to set $\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$ as the appropriate space for the unknown λ, that is

$$\lambda = p_\mathrm{D}|_\Sigma \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma).$$

Note that, in principle, the space for p_D does not allow enough regularity for the trace λ to exist. However, the solution of (2.2) has the pressure in $\mathrm{W}^{1,\frac{3}{2}}(\Omega_\mathrm{D}) \cap \mathrm{L}^2(\Omega_\mathrm{D})$.

Finally, we impose the second equation of (2.2) and the first equation of (2.4) weakly as follows

$$(q_\mathrm{D}, \mathrm{div}\,\mathbf{u}_\mathrm{D})_\mathrm{D} = (\mathrm{g}_\mathrm{D}, \mathrm{q}_\mathrm{D})_\mathrm{D} \qquad \forall\,\mathrm{q}_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D}), \tag{2.12}$$

and

$$\langle \mathbf{u}_\mathrm{S}\cdot\mathbf{n} - \mathbf{u}_\mathrm{D}\cdot\mathbf{n}, \xi\rangle_\Sigma = 0 \qquad \forall\,\xi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma). \tag{2.13}$$

As a consequence of the above, we write $\Omega := \Omega_\mathrm{S} \cup \Sigma \cup \Omega_\mathrm{D}$, and define $p := p_\mathrm{S}\chi_\mathrm{S} + p_\mathrm{D}\chi_\mathrm{D}$, with $\chi_\star$ being the characteristic function:

$$\chi_\star := \begin{cases} 1 & \text{in } \Omega_\star, \\ 0 & \text{in } \Omega\setminus\overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{S},\mathrm{D}\}$, to obtain the variational problem: Find $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, $p \in \mathrm{L}^2(\Omega)$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}})$ and $\lambda \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$ such that (2.6)–(2.13) hold.

Now, let us observe that if $(\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},p,\lambda)$ is a solution of the resulting variational problem, then for all $c\in\mathbb{R}$, $(\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}},p+c,\lambda+c)$ is also a solution. Then, we avoid the non-uniqueness of (2.6)–(2.13) by requiring from now on that $p\in\mathrm{L}^2_0(\Omega)$, where

$$\mathrm{L}^2_0(\Omega) := \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}.$$

In this way, we group the spaces and unknowns as follows:

$$\mathbf{H} := \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}}), \quad \mathbf{Q} := \mathrm{L}^2_0(\Omega) \times \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma),$$
$$\mathbf{u} := (\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}}) \in \mathbf{H}, \quad (p,\lambda) \in \mathbf{Q},$$

and propose the mixed variational formulation: Find $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$, such that

$$\begin{aligned} {[\mathbf{a}(\mathbf{u}_{\mathrm{S}})(\mathbf{u}),\mathbf{v}] + [\mathbf{b}(\mathbf{v}),(p,\lambda)]} &= [\mathbf{f},\mathbf{v}] && \forall\, \mathbf{v} := (\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H}, \\ {[\mathbf{b}(\mathbf{u}),(q,\xi)]} &= [\mathbf{g},(q,\xi)] && \forall\, (q,\xi) \in \mathbf{Q}, \end{aligned} \tag{2.14}$$

where, given $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$, the operator $\mathbf{a}(\mathbf{w}_{\mathrm{S}}) : \mathbf{H} \to \mathbf{H}'$ is defined by

$$[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}),\mathbf{v}] := [\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] + [\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] + [\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}], \tag{2.15}$$

with

$$\begin{aligned} {[\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]} &:= 2\mu(\mathbf{e}(\mathbf{u}_{\mathrm{S}}),\mathbf{e}(\mathbf{v}_{\mathrm{S}}))_{\mathrm{S}} + \left\langle \frac{\alpha_d \mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t}, \mathbf{v}_{\mathrm{S}}\cdot\mathbf{t} \right\rangle_{\Sigma}, \\ {[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]} &:= \rho((\nabla\mathbf{u}_{\mathrm{S}})\mathbf{w}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}}, \\ {[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}]} &:= \frac{\mu}{\rho}\left(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\right)_{\mathrm{D}} + \frac{F}{\rho}\left(|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\right)_{\mathrm{D}}, \end{aligned} \tag{2.16}$$

whereas the operator $\mathbf{b} : \mathbf{H} \to \mathbf{Q}'$ is given by

$$[\mathbf{b}(\mathbf{v}),(q,\xi)] := -(\mathrm{div}\,\mathbf{v}_{\mathrm{S}},\mathrm{q})_{\mathrm{S}} - (\mathrm{div}\,\mathbf{v}_{\mathrm{D}},\mathrm{q})_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma}. \tag{2.17}$$

In turn, the functionals $\mathbf{f}$ and $\mathbf{g}$ are defined by

$$[\mathbf{f},\mathbf{v}] := (\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}})_{\mathrm{S}} + (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \text{and} \quad [\mathbf{g},(q,\xi)] := -(g_{\mathrm{D}},q)_{\mathrm{D}}. \tag{2.18}$$

In all the terms above, $[\,\cdot,\cdot\,]$ denotes the duality pairing induced by the corresponding operators.

2.3. Stability properties

Let us now discuss the stability properties of the operators in (2.16) and (2.17). We begin by observing that the operators $\mathcal{A}_{\mathrm{S}}$, $\mathcal{B}_{\mathrm{S}}$ and $\mathbf{b}$ are continuous:

$$\begin{aligned} \Big|[\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]\Big| &\le C_{\mathcal{A}_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}, \\ \Big|[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]\Big| &\le \rho\, C^2(\Omega_{\mathrm{S}})\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}, \\ \Big|[\mathbf{b}(\mathbf{v}),(q,\xi)]\Big| &\le C_{\mathbf{b}}\|\mathbf{v}\|_{\mathbf{H}}\|(q,\xi)\|_{\mathbf{Q}}, \end{aligned} \tag{2.19}$$

where $C(\Omega_{\mathrm{S}})$ is the continuity constant of the Sobolev embedding from $\mathrm{H}^1(\Omega_{\mathrm{S}})$ into $\mathrm{L}^4(\Omega_{\mathrm{S}})$. In turn, from the definition of $\mathcal{A}_{\mathrm{D}}$ (*cf.* (2.16)), (2.3), and the triangle and Hölder inequalities, we obtain that there exists $L_{\mathcal{A}_{\mathrm{D}}} > 0$, depending only on $\mu,\rho,F,\mathbf{K}$, and Ω_{D}, such that

$$\begin{aligned} &\|\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}) - \mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})\|_{(\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}}))'} \\ &\quad \le L_{\mathcal{A}_{\mathrm{D}}}\Big\{\|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\Big(\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} + \|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\Big)\Big\}, \end{aligned} \tag{2.20}$$

for all $\mathbf{u}_{\rm D}, \mathbf{v}_{\rm D} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})$. In addition, using the Cauchy–Schwarz and Young inequalities, it is not difficult to see that $\mathbf{f}$ and $\mathbf{g}$ are bounded, that is, there exist constants $c_{\mathbf{f}}, c_{\mathbf{g}} > 0$, such that

$$\|\mathbf{f}\|_{\mathbf{H}'} \leq c_{\mathbf{f}} \Big\{ \|\mathbf{f}_{\rm S}\|_{0,\Omega_{\rm S}} + \|\mathbf{f}_{\rm D}\|_{\mathbf{L}^{3/2}(\Omega_{\rm D})} \Big\} \tag{2.21}$$

and

$$\|\mathbf{g}\|_{\mathbf{Q}'} \leq c_{\mathbf{g}} \|g_{\rm D}\|_{0,\Omega_{\rm D}}, \tag{2.22}$$

which confirm the announced smoothness of $\mathbf{f}_{\rm D}$. On the other hand, from the well known Korn and Poincaré inequalities (see, *e.g.*, [27]), we easily obtain that there exists a constant $\alpha_{\rm S} > 0$, depending only on $\Omega_{\rm S}$, such that

$$[\mathcal{A}_{\rm S}(\mathbf{v}_{\rm S}), \mathbf{v}_{\rm S}] \geq 2\mu\alpha_{\rm S} \|\mathbf{v}_{\rm S}\|^2_{1,\Omega_{\rm S}} \quad \forall \mathbf{v}_{\rm S} \in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}). \tag{2.23}$$

In turn, integrating by parts and assuming that $\mathrm{div}\, \mathbf{w}_{\rm S} = 0$ in $\Omega_{\rm S}$, similarly to equation (29) of [21], we obtain

$$[\mathcal{B}_{\rm S}(\mathbf{w}_{\rm S})(\mathbf{v}_{\rm S}), \mathbf{v}_{\rm S}] = \frac{\rho}{2} \int_\Sigma (\mathbf{w}_{\rm S} \cdot \mathbf{n}) |\mathbf{v}_{\rm S}|^2 \quad \forall \mathbf{w}_{\rm S}, \mathbf{v}_{\rm S} \in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}). \tag{2.24}$$

Finally, from the definition of $\mathcal{A}_{\rm D}$ (*cf.* (2.16)) and the inequality (2.3), we deduce that for a fixed $\mathbf{t}_{\rm D} \in \mathbf{L}^3(\Omega_{\rm D})$, there holds

$$\begin{aligned} &[\mathcal{A}_{\rm D}(\mathbf{u}_{\rm D} + \mathbf{t}_{\rm D}) - \mathcal{A}_{\rm D}(\mathbf{v}_{\rm D} + \mathbf{t}_{\rm D}), \mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}] \\ &\quad \geq \frac{\mu}{\rho} C_{\mathbf{K}} \|\mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}\|^2_{0,\Omega_{\rm D}} + \frac{F}{\rho} \left(|\mathbf{u}_{\rm D} + \mathbf{t}_{\rm D}|(\mathbf{u}_{\rm D} + \mathbf{t}_{\rm D}) - |\mathbf{v}_{\rm D} + \mathbf{t}_{\rm D}|(\mathbf{v}_{\rm D} + \mathbf{t}_{\rm D}), \mathbf{u}_{\rm D} - \mathbf{v}_{\rm D} \right)_{\rm D}, \end{aligned} \tag{2.25}$$

for all $\mathbf{u}_{\rm D}, \mathbf{v}_{\rm D} \in \mathbf{L}^3(\Omega_{\rm D})$. Then, thanks to Lemma 5.1 of [35], there exist $C_{\rm D} > 0$, depending only on $\Omega_{\rm D}$, such that

$$\left(|\mathbf{u}_{\rm D} + \mathbf{t}_{\rm D}|(\mathbf{u}_{\rm D} + \mathbf{t}_{\rm D}) - |\mathbf{v}_{\rm D} + \mathbf{t}_{\rm D}|(\mathbf{v}_{\rm D} + \mathbf{t}_{\rm D}), \mathbf{u}_{\rm D} - \mathbf{v}_{\rm D} \right)_{\rm D} \geq C_{\rm D} \|\mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}\|^3_{\mathbf{L}^3(\Omega_{\rm D})},$$

which, together with (2.25), and neglecting the first term on the right hand side of (2.25), yields

$$[\mathcal{A}_{\rm D}(\mathbf{u}_{\rm D} + \mathbf{t}_{\rm D}) - \mathcal{A}_{\rm D}(\mathbf{v}_{\rm D} + \mathbf{t}_{\rm D}), \mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}] \geq \alpha_{\rm D} \|\mathbf{u}_{\rm D} - \mathbf{v}_{\rm D}\|^3_{\mathbf{L}^3(\Omega_{\rm D})} \quad \forall \mathbf{u}_{\rm D}, \mathbf{v}_{\rm D} \in \mathbf{L}^3(\Omega_{\rm D}), \tag{2.26}$$

with $\alpha_{\rm D} = \dfrac{F C_{\rm D}}{\rho}$.

3. Analysis of the continuous formulation

In this section we analyse the well-posedness of problem (2.14) by means of a fixed-point argument and classical results on nonlinear monotone operators. We begin by introducing our fixed-point strategy.

3.1. The fixed-point operator

Let $\mathbf{T} : \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}) \to \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$ be the operator defined by

$$\mathbf{T}(\mathbf{w}_{\rm S}) := \hat{\mathbf{u}}_{\rm S} \quad \forall \mathbf{w}_{\rm S} \in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}), \tag{3.1}$$

where $\hat{\mathbf{u}} := (\hat{\mathbf{u}}_{\rm S}, \hat{\mathbf{u}}_{\rm D}) \in \mathbf{H}$ is the first component of the unique solution (to be confirmed below) of the nonlinear problem: Find $(\hat{\mathbf{u}}, (\hat{p}, \hat{\lambda})) \in \mathbf{H} \times \mathbf{Q}$, such that

$$\begin{aligned} [\mathbf{a}(\mathbf{w}_{\rm S})(\hat{\mathbf{u}}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (\hat{p}, \hat{\lambda})] &= [\mathbf{f}, \mathbf{v}] && \forall \mathbf{v} \in \mathbf{H}, \\ [\mathbf{b}(\hat{\mathbf{u}}), (q, \xi)] &= [\mathbf{g}, (q, \xi)] && \forall (q, \xi) \in \mathbf{Q}. \end{aligned} \tag{3.2}$$

It is not difficult to see that $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ is a solution of (2.14) if and only if $\mathbf{u}_{\rm S} \in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$ satisfies: $\mathbf{T}(\mathbf{u}_{\rm S}) = \mathbf{u}_{\rm S}$. In this way, in order to prove the well-posedness of (2.14), in what follows we equivalently show that $\mathbf{T}$ possesses a unique fixed-point in a closed ball of $\mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$. Before continuing with the solvability analysis of (2.14), we first provide the hypotheses under which operator $\mathbf{T}$ is well defined. To that end we first collect some preliminary results and notations that will serve for the forthcoming analysis.

3.2. Preliminary results

First we introduce some definitions that will be utilized next. To this end we let X and Y be reflexive Banach spaces. Then, we say that a nonlinear operator $T : X \to Y$ is bounded if $T(S)$ is bounded for each bounded set $S \subseteq X$. In addition, we say that a nonlinear operator $T : X \to X'$ is of *type M* if $u_n \rightharpoonup u$, $Tu_n \rightharpoonup f$ and $\limsup\,[Tu_n, u_n] \leq f(u)$ imply $Tu = f$. In turn, we say that T is coercive if

$$\frac{[Tu, u]}{\|u\|} \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Now, we establish the following abstract result taken from Proposition 2.3 of [50], which has been adapted to our context where the nonlinear operator is defined on a product space $X = X_1 \times X_2$, with X_1 and X_2 depending on parameters p_1 and p_2, respectively, in place of an space X depending on a parameter p.

Theorem 3.1. *Let X_1, X_2 and Y be separable and reflexive Banach spaces, being X_1 and X_2 uniformly convex, set $X = X_1 \times X_2$, and let X_1', X_2', Y', and $X' := X_1' \times X_2'$, be their respective duals. Let $a : X \to X'$ be a nonlinear operator and $b : X \to Y'$ be a linear bounded operator. In turn, we denote by V the kernel of b, that is,*

$$V := \Big\{ v \in X : \quad [b(v), q] = 0 \quad \forall q \in Y \Big\}.$$

Assume that

(i) *a is hemi-continuous, that is, for each $u, v \in X$, the real mapping*

$$J : \mathbb{R} \to \mathbb{R}, \quad t \to J(t) = [a(u + tv), v]$$

is continuous.

(ii) *there exist constants $\gamma > 0$ and $p_1, p_2 \geq 2$, such that*

$$\|a(u) - a(v)\|_{X'} \leq \gamma \sum_{j=1}^{2} \Big\{ \|u_j - v_j\|_{X_j} + \|u_j - v_j\|_{X_j} \Big(\|u_j\|_{X_j} + \|v_j\|_{X_j} \Big)^{p_j - 2} \Big\},$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in X$.

(iii) *for fixed $t \in X$, the operator $a(\cdot + t) : V \to V'$ is strictly monotone in the following sense: there exist $\alpha > 0$ and $p_1, p_2 \geq 2$, such that*

$$[a(u + t) - a(v + t), u - v] \geq \alpha \Big\{ \|u_1 - v_1\|_{X_1}^{p_1} + \|u_2 - v_2\|_{X_2}^{p_2} \Big\},$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in V$.

(iv) *there exists $\beta > 0$ such that*

$$\sup_{\substack{v \in X \\ v \neq 0}} \frac{[b(v), q]}{\|v\|_X} \geq \beta \|q\|_Y \quad \forall q \in Y.$$

Then, for each $(f, g) \in X' \times Y'$ there exists a unique $(u, p) \in X \times Y$ such that

$$\begin{aligned} &[a(u), v] + [b(v), p] = [f, v] \ \forall v \in X, \\ &[b(u), q] \qquad\qquad\quad\, = [g, q] \ \forall q \in Y. \end{aligned} \tag{3.3}$$

Moreover, there exists $C > 0$, depending only on $\alpha, \gamma, \beta, p_1$, and p_2, such that

$$\|(u, p)\|_{X \times Y} \leq C\, \mathcal{M}(f, g), \tag{3.4}$$

where

$$\mathcal{M}(f, g) := \max \Big\{ \mathcal{N}(f, g)^{\frac{1}{p_1 - 1}}, \mathcal{N}(f, g)^{\frac{1}{p_2 - 1}}, \mathcal{N}(f, g), \mathcal{N}(f, g)^{\frac{p_1 - 1}{p_2 - 1}}, \mathcal{N}(f, g)^{\frac{p_2 - 1}{p_1 - 1}} \Big\},$$

and

$$\mathcal{N}(f, g) := \|f\|_{X'} + \|g\|_{Y'} + \|g\|_{Y'}^{p_1 - 1} + \|g\|_{Y'}^{p_2 - 1} + \|a(0)\|_{X'}.$$

Proof. We begin by noting that hypothesis (iv) establishes, equivalently, that b is surjective.

Then, given $g \in Y'$ there exists $u_g \in X$, such that (see [23], Lems. A.36 and A.42 for details):

$$b(u_g) = g \quad \text{and} \quad \|u_g\|_X \leq \frac{1}{\beta}\|g\|_{Y'}. \tag{3.5}$$

Then, given this u_g in X satisfying (3.5), we observe that problem (3.3) with $v \in V$ leads to: find $\widetilde{u} \in V$, such that

$$[a_g(\widetilde{u}), v] := [a(\widetilde{u} + u_g), v] = [f, v] \quad \forall v \in V, \tag{3.6}$$

which suggests to define later on u as $\widetilde{u} + u_g$. In this way, since $f - a(u) \in {}^\circ V := \{G \in X' : G(v) = 0, \forall\, v \in V\}$ and hypothesis (iv) also guarantees that the adjoint operator b' is an isomorphism from Y into ${}^\circ V$, we deduce that there exists a unique $p \in Y$ such that $b'(p) = f - a(u)$ and

$$\|p\|_Y \leq \frac{1}{\beta}\|b'(p)\|_{X'} \leq \frac{1}{\beta}\Big\{\|f\|_{X'} + \|a(u)\|_{X'}\Big\}. \tag{3.7}$$

Therefore to prove that problem (3.3) is well posed, in what follows we prove equivalently that $a_g(\,\cdot\,) = a(\cdot + u_g)$ is bijective from V to V'. We begin by observing that the injectivity of the operator $a_g(\,\cdot\,)$ follows straightforwardly from hypothesis (iii). In addition, from hypotheses (i) and (iii) and Chapter II, Lemma 2.1 of [51] it can be readily seen that $a_g(\,\cdot\,)$ is an operator of *type M*. Now, given $v = (v_1, v_2) \in V$, and denoting by u_j^g, $j = 1, 2$, the components of u_g, we observe that, owing to (ii), (iii) and using the inequality $(a+b)^q \leq C(q)(a^q + b^q)$, with $C(q)$ depending only on q, which is valid for all $q \in [0, +\infty)$ and $a, b \geq 0$ Lemma 2.2 of [3], there hold

$$\begin{aligned}
\|a_g(v)\|_{X'} &\leq \|a_g(v) - a_g(0)\|_{X'} + \|a_g(0)\|_{X'} = \|a(v + u_g) - a(u_g)\|_{X'} + \|a(u_g)\|_{X'} \\
&\leq \gamma \sum_{j=1}^{2} \Big\{\|v_j\|_{X_j} + \|v_j\|_{X_j}\left(\|v_j + u_j^g\|_{X_j} + \|u_j^g\|_{X_j}\right)^{p_j - 2}\Big\} + \|a(u_g)\|_{X'} \\
&\leq C \sum_{j=1}^{2} \Big\{\|v_j\|_{X_j} + \|v_j\|_{X_j}^{p_j-1} + \|v_j\|_{X_j}\,\|u_j^g\|_{X_j}^{p_j-2}\Big\} + \|a(u_g)\|_{X'} \\
&\leq C\left(1 + \|v_1\|_{X_1}^{p_1-2} + \|v_2\|_{X_2}^{p_2-2} + \|u_1^g\|_{X_1}^{p_1-2} + \|u_2^g\|_{X_2}^{p_2-2}\right)\|v\|_X + \|a(u_g)\|_{X'},
\end{aligned}$$

and

$$\begin{aligned}
\frac{[a_g(v), v]}{\|v\|_X} &= \frac{[a(v + u_g) - a(0 + u_g), v]}{\|v\|_X} + \frac{[a(u_g), v]}{\|v\|_X} \geq \alpha \frac{\Big\{\|v_1\|_{X_1}^{p_1} + \|v_2\|_{X_2}^{p_2}\Big\}}{\|v\|_X} - \|a(u_g)\|_{X'} \\
&\geq C \min\Big\{\|v\|_X^{p_1-1}, \|v\|_X^{p_2-1}\Big\} - \|a(u_g)\|_{X'},
\end{aligned}$$

which clearly show that a_g is bounded and coercive on V, respectively. In this way, by applying Chapter II, Corollary 2.2 of [51] it can be readily seen that a_g is surjective on V. Having verified the bijectivity of a_g on V we deduce that problem (3.6) is well-posed, or equivalently (3.3) admits a unique solution $(u, p) = (\widetilde{u} + u_g, p) \in X \times Y$. Now, in order to obtain (3.4), we proceed similarly to Proposition 2.3 of [50]. In fact, taking $v = \widetilde{u} \in V$ in (3.6), we have

$$[a(\widetilde{u} + u_g) - a(0 + u_g), \widetilde{u}] = [f, \widetilde{u}] - [a(u_g), \widetilde{u}].$$

Then, combining hypothesis (ii), (iii) and (3.5), it is clear that

$$\begin{aligned}
\alpha\Big\{\|\widetilde{u}_1\|_{X_1}^{p_1} + \|\widetilde{u}_2\|_{X_2}^{p_2}\Big\} &\leq \Big\{\|f\|_{X'} + \|a(u_g)\|_{X'}\Big\}\|\widetilde{u}\|_X \\
&\leq c_1\Big\{\|f\|_{X'} + \|g\|_{Y'} + \|g\|_{Y'}^{p_1-1} + \|g\|_{Y'}^{p_2-1} + \|a(0)\|_{X'}\Big\}\|\widetilde{u}\|_X,
\end{aligned}$$

with $c_1 > 0$ depending only on γ, β, p_1, and p_2, which yields

$$\|\widetilde{u}\|_X \leq 2 \max \left\{ \left(\frac{2c_1}{\alpha} \mathcal{N}(f,g) \right)^{\frac{1}{p_1-1}}, \left(\frac{2c_1}{\alpha} \mathcal{N}(f,g) \right)^{\frac{1}{p_2-1}} \right\}, \tag{3.8}$$

where $\mathcal{N}(f,g) := \|f\|_{X'} + \|g\|_{Y'} + \|g\|_{Y'}^{p_1-1} + \|g\|_{Y'}^{p_2-1} + \|a(0)\|_{X'}$. In this way, due to $u = \widetilde{u} + u_g$, combining (3.5) and (3.8), we conclude that

$$\|u\|_X \leq \|\widetilde{u}\|_X + \|u_g\|_X \leq c_2 \max \left\{ \mathcal{N}(f,g)^{\frac{1}{p_1-1}}, \mathcal{N}(f,g)^{\frac{1}{p_2-1}} \right\}, \tag{3.9}$$

with $c_2 > 0$ depending only on $\alpha, \gamma, \beta, p_1$, and p_2. On the other hand, from (3.7) and using again (ii), we deduce that

$$\|p\|_Y \leq c_3 \left\{ \|f\|_{X'} + \|u\|_X + \|u_1\|_{X_1}^{p_1-1} + \|u_2\|_{X_2}^{p_2-1} + \|a(0)\|_{X'} \right\}, \tag{3.10}$$

with $c_3 > 0$ depending only on γ and β. Then, (3.9) and (3.10) conclude the proof. □

We remark that when $p_1 = p_2 = 2$ and $\|a(0)\|_{X'}$ is equal to zero, the previous analysis leads to the classical estimate

$$\|(u,p)\|_{X \times Y} \leq C \left\{ \|f\|_{X'} + \|g\|_{Y'} \right\},$$

with $C > 0$, depending only on α, γ, and β.

Finally, we observe that, since $\mathrm{H}^{1/2}(\partial\Omega_\mathrm{S})$ is continuously embedded into $\mathrm{L}^p(\partial\Omega_\mathrm{S})$, with $2 \leq p < \infty$ for the two dimensional case and $1 \leq p \leq 4$ for the three dimensional case (see [46], Thm. 1.3.4), and the trace operator is continuous, the following inequality holds:

$$\|\mathbf{v}_\mathrm{S}\|_{\mathbf{L}^p(\Sigma)} \leq \|\mathbf{v}_\mathrm{S}\|_{\mathbf{L}^p(\partial\Omega_\mathrm{S})} \leq C(\partial\Omega_\mathrm{S}) \|\mathbf{v}_\mathrm{S}\|_{1/2,\partial\Omega_\mathrm{S}} \leq C(\partial\Omega_\mathrm{S}) C_{\mathrm{tr}} \|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \quad \forall \mathbf{v}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}), \tag{3.11}$$

where $C(\partial\Omega_\mathrm{S})$ is the continuity constant of the Sobolev embedding from $\mathrm{H}^{1/2}(\partial\Omega_\mathrm{S})$ into $\mathrm{L}^p(\partial\Omega_\mathrm{S})$, and C_{tr} is the norm of the usual trace operator from $\mathrm{H}^1(\Omega_\mathrm{S})$ into $\mathrm{H}^{1/2}(\partial\Omega_\mathrm{S})$.

3.3. Well-definiteness of T

Given $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, it is clear that problem (3.2) has the same structure of the one in Theorem 3.1. Therefore, in what follows we apply this result to establish the well-posedness of (3.2), or equivalently, the well-definiteness of $\mathbf{T}$. We begin by observing that, thanks to the uniform convexity and separability of $\mathrm{L}^p(\Omega)$ for $p \in (1, +\infty)$, each space defining $\mathbf{H}$ and $\mathbf{Q}$ shares the same properties, which implies that $\mathbf{H}$ and $\mathbf{Q}$ are uniformly convex and separable as well.

We continue with the required continuity property of $\mathbf{a}(\mathbf{w}_\mathrm{S})$ for each $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$.

Lemma 3.2. *Given $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, the operator $\mathbf{a}(\mathbf{w}_\mathrm{S})$ is hemi-continuous in $\mathbf{H}$.*

Proof. For fixed $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, $\mathbf{u} = (\mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D})$, and $\mathbf{v} = (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{H}$, we introduce the real function $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}(t) := [\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u} + t\mathbf{v}), \mathbf{v}] &= [\mathcal{A}_\mathrm{S}(\mathbf{u}_\mathrm{S} + t\mathbf{v}_\mathrm{S}), \mathbf{v}_\mathrm{S}] \\ &+ [\mathcal{B}_\mathrm{S}(\mathbf{w}_\mathrm{S})(\mathbf{u}_\mathrm{S} + t\mathbf{v}_\mathrm{S}), \mathbf{v}_\mathrm{S}] + [\mathcal{A}_\mathrm{D}(\mathbf{u}_\mathrm{D} + t\mathbf{v}_\mathrm{D}), \mathbf{v}_\mathrm{D}]. \end{aligned}$$

Then, the hemi-continuity of $\mathbf{a}(\mathbf{w}_\mathrm{S})$, that is the continuity of $\mathcal{J}$, follows straightforwardly from the linearity and continuity of $\mathcal{A}_\mathrm{S}$ and $\mathcal{B}_\mathrm{S}(\mathbf{w}_\mathrm{S})$ and from Proposition 3 of [34]. We omit further details. □

We continue our analysis with the verification of hypothesis (ii) of Theorem 3.1.

Lemma 3.3. *Let* $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$. *Then, there exists* $\gamma > 0$, *depending on* $C_{\mathcal{A}_\mathrm{S}}$ *and* $L_{\mathcal{A}_\mathrm{D}}$ *(cf. (2.19) and (2.20)), such that*

$$\|\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}) - \mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{v})\|_{\mathbf{H}'} \leq \gamma\Big\{(1 + \|\mathbf{w}_\mathrm{S}\|_{1,\Omega_\mathrm{S}})\|\mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D} - \mathbf{v}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + \|\mathbf{u}_\mathrm{D} - \mathbf{v}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\Big(\|\mathbf{u}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + \|\mathbf{v}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\Big)\Big\},$$

for all $\mathbf{u} = (\mathbf{u}_\mathrm{S}, \mathbf{u}_\mathrm{D}), \mathbf{v} = (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{H}$.

Proof. The result follows straightforwardly from the definition of $\mathbf{a}(\mathbf{w}_\mathrm{S})$ (*cf.* (2.15)), the triangle inequality, and the stability properties (2.19) and (2.20). We omit further details. □

Now, let us look at the kernel of the operator $\mathbf{b}$, that is

$$\mathbf{V} := \Big\{\mathbf{v} \in \mathbf{H} : \quad [\mathbf{b}(\mathbf{v}), (q, \xi)] = 0 \quad \forall (q, \xi) \in \mathbf{Q}\Big\}. \tag{3.12}$$

According to the definition of $\mathbf{b}$ (*cf.* (2.17)), we observe that $\mathbf{v} = (\mathbf{v}_\mathrm{S}, \mathbf{v}_\mathrm{D}) \in \mathbf{V}$ if and only if

$$(\mathrm{div}\,\mathbf{v}_\mathrm{S}, \mathrm{q})_\mathrm{S} + (\mathrm{div}\,\mathbf{v}_\mathrm{D}, \mathrm{q})_\mathrm{D} = 0 \quad \forall \mathrm{q} \in \mathrm{L}^2_0(\Omega)$$

and

$$\langle \mathbf{v}_\mathrm{S} \cdot \mathbf{n} - \mathbf{v}_\mathrm{D} \cdot \mathbf{n}, \xi\rangle_\Sigma = 0 \quad \forall \xi \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma).$$

In this way, noting that $\mathrm{L}^2(\Omega) = \mathrm{L}^2_0(\Omega) \oplus \mathbb{R}$, and taking $\xi \in \mathbb{R}$ in the latter equation, we deduce that

$$(\mathrm{div}\,\mathbf{v}_\mathrm{S}, \mathrm{q})_\mathrm{S} + (\mathrm{div}\,\mathbf{v}_\mathrm{D}, \mathrm{q})_\mathrm{D} = 0 \quad \forall \mathrm{q} \in \mathrm{L}^2(\Omega),$$

which implies

$$\mathrm{div}\,\mathbf{v}_\mathrm{S} = 0 \quad \text{in} \quad \Omega_\mathrm{S} \quad \text{and} \quad \mathrm{div}\,\mathbf{v}_\mathrm{D} = 0 \quad \text{in} \quad \Omega_\mathrm{D}. \tag{3.13}$$

In the following result we provide the assumptions under which operator $\mathbf{a}(\mathbf{w}_\mathrm{S})$ satisfies hypothesis (iii) of Theorem 3.1.

Lemma 3.4. *Let* $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ *such that* $\mathrm{div}\,\mathbf{w}_\mathrm{S} = 0$ *in* Ω_S *and*

$$\|\mathbf{w}_\mathrm{S} \cdot \mathbf{n}\|_{0,\Sigma} \leq \frac{2\mu\alpha_\mathrm{S}}{\rho\, C^2_{\mathrm{tr}} C^2(\partial\Omega_\mathrm{S})}. \tag{3.14}$$

Then, for each $\mathbf{t} \in \mathbf{H} \setminus \mathbf{V}$, *the nonlinear operator* $\mathbf{a}(\mathbf{w}_\mathrm{S})(\,\cdot + \mathbf{t})$ *is strictly monotone on* $\mathbf{V}$ *(cf. (3.12)).*

Proof. Let $\mathbf{t} := (\mathbf{t}_\mathrm{S}, \mathbf{t}_\mathrm{D}) \in \mathbf{H} \setminus \mathbf{V}$ fixed, and let $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ as indicated. Then, according to (2.15), the linearity of $\mathcal{A}_\mathrm{S}$ and $\mathcal{B}_\mathrm{S}(\mathbf{w}_\mathrm{S})$, the identity (3.13) and the stabilities properties (2.23) and (2.26), we find that

$$[\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u} + \mathbf{t}) - \mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{v} + \mathbf{t}), \mathbf{u} - \mathbf{v}] \geq 2\mu\alpha_\mathrm{S}\|\mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}} + \alpha_\mathrm{D}\|\mathbf{u}_\mathrm{D} - \mathbf{v}_\mathrm{D}\|^3_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + [\mathcal{B}_\mathrm{S}(\mathbf{w}_\mathrm{S})(\mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}), \mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}],$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$. In addition, similarly to Lemma 2 of [21], we deduce from (2.24), applying Cauchy–Schwarz's inequality and (3.11) with $p = 4$, that

$$\Big|[\mathcal{B}_\mathrm{S}(\mathbf{w}_\mathrm{S})(\mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}), \mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}]\Big| \leq \frac{\rho\, C^2_{\mathrm{tr}} C^2(\partial\Omega_\mathrm{S})}{2}\|\mathbf{w}_\mathrm{S} \cdot \mathbf{n}\|_{0,\Sigma}\, \|\mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}},$$

which implies

$$[\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u} + \mathbf{t}) - \mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{v} + \mathbf{t}), \mathbf{u} - \mathbf{v}] \geq \left\{2\mu\alpha_\mathrm{S} - \frac{\rho\, C^2_{\mathrm{tr}} C^2(\partial\Omega_\mathrm{S})}{2}\|\mathbf{w}_\mathrm{S} \cdot \mathbf{n}\|_{0,\Sigma}\right\}\|\mathbf{u}_\mathrm{S} - \mathbf{v}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}} + \alpha_\mathrm{D}\|\mathbf{u}_\mathrm{D} - \mathbf{v}_\mathrm{D}\|^3_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}.$$

Consequently, the hypothesis (3.14) and the foregoing inequality imply

$$[\mathbf{a}(\mathbf{w}_{\rm S})(\mathbf{u}+\mathbf{t}) - \mathbf{a}(\mathbf{w}_{\rm S})(\mathbf{v}+\mathbf{t}), \mathbf{u}-\mathbf{v}] \geq \alpha(\Omega)\Big\{\|\mathbf{u}_{\rm S}-\mathbf{v}_{\rm S}\|^2_{1,\Omega_{\rm S}} + \|\mathbf{u}_{\rm D}-\mathbf{v}_{\rm D}\|^3_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})}\Big\},$$

for all $\mathbf{u},\mathbf{v}\in\mathbf{V}$, with $\alpha(\Omega) := \min\big\{\mu\alpha_{\rm S}, \alpha_{\rm D}\big\}$ independent of $\mathbf{w}_{\rm S}$. □

We remark that, similarly to the strict monotonicity of $\mathbf{a}(\mathbf{w}_{\rm S})(\,\cdot+\mathbf{t})$ on $\mathbf{V}$ with $\mathbf{t}\in\mathbf{H}\setminus\mathbf{V}$ fixed, using (2.26) with $\mathbf{t}_{\rm D} = \mathbf{0}\in\mathbf{L}^3(\Omega_{\rm D})$, we deduce that

$$[\mathbf{a}(\mathbf{w}_{\rm S})(\mathbf{u}) - \mathbf{a}(\mathbf{w}_{\rm S})(\mathbf{v}), \mathbf{u}-\mathbf{v}] \geq \alpha(\Omega)\Big\{\|\mathbf{u}_{\rm S}-\mathbf{v}_{\rm S}\|^2_{1,\Omega_{\rm S}} + \|\mathbf{u}_{\rm D}-\mathbf{v}_{\rm D}\|^3_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})}\Big\}, \tag{3.15}$$

for all $\mathbf{u},\mathbf{v}\in\mathbf{H}$ with $\mathrm{div}\,(\mathbf{u}_{\rm D}-\mathbf{v}_{\rm D}) = 0$ in $\Omega_{\rm D}$.

We end the verification of the hypotheses of Theorem 3.1 by proving the continuous inf-sup condition for $\mathbf{b}$. To that end, we adapt the proof of Lemma 2.1 from [28] to the present case, using similar results from Lemma 3.3 of [31] and Lemma 1 of [21] to handle the mixed boundary conditions on $\partial\Omega_{\rm D}$.

Lemma 3.5. *There exists $\beta > 0$ such that*

$$S(q,\xi) := \sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}),(q,\xi)]}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\,\|(q,\xi)\|_{\mathbf{Q}} \quad \forall (q,\xi)\in\mathbf{Q}. \tag{3.16}$$

Proof. Let $(q,\xi)\in\mathbf{Q}$. Since $q\in \mathrm{L}^2_0(\Omega)$, it is well known (see, *e.g.*, [33], Cor. 2.4) that there exists $\mathbf{z}\in\mathbf{H}^1_0(\Omega)$ such that $\mathrm{div}\,\mathbf{z} = -\mathrm{q}$ in Ω and $\|\mathbf{z}\|_{1,\Omega}\leq c\|q\|_{0,\Omega}$. Setting $\widehat{\mathbf{v}} := (\widehat{\mathbf{v}}_{\rm S}, \widehat{\mathbf{v}}_{\rm D})$ with $\widehat{\mathbf{v}}_\star = \mathbf{z}|_{\Omega_\star}$ for $\star\in\{\mathrm{S},\mathrm{D}\}$, we find that $\widehat{\mathbf{v}}_{\rm S}\cdot\mathbf{n} = \widehat{\mathbf{v}}_{\rm D}\cdot\mathbf{n}$ on Σ, and using the continuous embedding from $\mathrm{H}^1(\Omega_{\rm D})$ into $\mathrm{L}^3(\Omega_{\rm D})$, we obtain $\|\widehat{\mathbf{v}}\|_{\mathbf{H}}\leq\widehat{c}\|\mathbf{z}\|_{1,\Omega}\leq\widetilde{c}\|q\|_{0,\Omega}$, whence

$$S(q,\xi) \geq \frac{\Big|[\mathbf{b}(\widehat{\mathbf{v}}),(q,\xi)]\Big|}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} = \frac{\|q\|^2_{0,\Omega}}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} \geq c_1\|q\|_{0,\Omega}. \tag{3.17}$$

On the other hand, given $\phi\in \mathrm{W}^{-\frac{1}{3},3}(\Sigma)$, we define $\eta\in\mathrm{W}^{-\frac{1}{3},3}(\partial\Omega_{\rm D})$ as

$$\langle\eta,\mu\rangle_{\partial\Omega_{\rm D}} := \langle\phi,\mu_\Sigma\rangle_\Sigma \quad \forall\mu\in\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_{\rm D}),$$

where $\mu_\Sigma\in\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$ is given by the decomposition (2.10). It is not difficult to see that

$$\langle\eta, E_{0,\rm D}(\rho)\rangle_{\partial\Omega_{\rm D}} = 0 \quad \forall\rho\in\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma_{\rm D}), \tag{3.18}$$

$$\langle\eta, E_\Sigma(\varphi)\rangle_{\partial\Omega_{\rm D}} = \langle\phi,\varphi\rangle_\Sigma \quad \forall\varphi\in\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma), \tag{3.19}$$

and

$$\|\eta\|_{-\frac{1}{3},3;\partial\Omega_{\rm D}} \leq C\|\phi\|_{-\frac{1}{3},3;\Sigma}. \tag{3.20}$$

Next, we set $\widetilde{\mathbf{v}}_{\rm D} := \nabla z$ in $\Omega_{\rm D}$, with $z\in\mathrm{W}^{1,3}(\Omega_{\rm D})$ being the unique solution of the boundary value problem (see [32] for details):

$$-\Delta z = -\frac{1}{|\Omega_{\rm D}|}\langle\eta,1\rangle_{\partial\Omega_{\rm D}} \quad \text{in} \quad \Omega_{\rm D}, \quad \nabla z\cdot\mathbf{n} = \eta \quad \text{on} \quad \partial\Omega_{\rm D}, \quad (z,1)_{\rm D} = 0. \tag{3.21}$$

It follows that $\mathrm{div}\,\widetilde{\mathbf{v}}_{\rm D} = \frac{1}{|\Omega_{\rm D}|}\langle\eta,1\rangle_{\partial\Omega_{\rm D}}\in\mathbb{P}_0(\Omega_{\rm D})$, $\widetilde{\mathbf{v}}_{\rm D}\cdot\mathbf{n} = \eta$ on $\partial\Omega_{\rm D}$, and using (3.20) we find that

$$\|\widetilde{\mathbf{v}}_{\rm D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})} \leq c\|\eta\|_{-\frac{1}{3},3;\partial\Omega_{\rm D}} \leq C\|\phi\|_{-\frac{1}{3},3;\Sigma}. \tag{3.22}$$

Note that the first inequality here follows from the definition of the norm $\|\cdot\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}$, the continuous dependence result of the boundary value problem (3.21), and the fact that $\|\widetilde{\mathbf{v}}_\mathrm{D}\|_{\mathbf{L}^3(\Omega_\mathrm{D})} = \|\nabla z\|_{\mathbf{L}^3(\Omega_\mathrm{D})}$ and $\mathrm{div}\,\widetilde{\mathbf{v}}_\mathrm{D} = \frac{1}{|\Omega_\mathrm{D}|}\langle \eta, 1\rangle_{\partial\Omega_\mathrm{D}}$. In addition, using (2.9), (3.18) and (3.19), we deduce that

$$\langle \widetilde{\mathbf{v}}_\mathrm{D}\cdot\mathbf{n}, \xi\rangle_\Sigma = \langle \widetilde{\mathbf{v}}_\mathrm{D}\cdot\mathbf{n}, E_\Sigma(\xi)\rangle_{\partial\Omega_\mathrm{D}} = \langle \eta, E_\Sigma(\xi)\rangle_{\partial\Omega_\mathrm{D}} = \langle \phi, \xi\rangle_\Sigma\,,$$

and

$$\langle \widetilde{\mathbf{v}}_\mathrm{D}\cdot\mathbf{n}, E_{0,\mathrm{D}}(\rho)\rangle_{\partial\Omega_\mathrm{D}} = \langle \eta, E_{0,\mathrm{D}}(\rho)\rangle_{\partial\Omega_\mathrm{D}} = 0 \quad \forall \rho \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Gamma_\mathrm{D}).$$

The latter means that $\widetilde{\mathbf{v}}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$. In this way, defining $\widetilde{\mathbf{v}} := (\mathbf{0}, \widetilde{\mathbf{v}}_\mathrm{D}) \in \mathbf{H}$, we obtain, thanks to (3.20) and (3.22), that

$$\begin{aligned} S(q,\xi) &\geq \frac{\Big|[\mathbf{b}(\widetilde{\mathbf{v}}), (q,\xi)]\Big|}{\|\widetilde{\mathbf{v}}\|_\mathbf{H}} = \frac{\Big|\langle\phi,\xi\rangle_\Sigma + \frac{1}{|\Omega_\mathrm{D}|}\langle\eta,1\rangle_{\partial\Omega_\mathrm{D}}(q,1)_\mathrm{D}\Big|}{\|\widetilde{\mathbf{v}}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}} \\ &\geq c_2 \frac{\Big|\langle\phi,\xi\rangle_\Sigma\Big|}{\|\phi\|_{-\frac{1}{3},3;\Sigma}} - c_3\|q\|_{0,\Omega}, \end{aligned}$$

which, considering that $\phi \in \mathrm{W}^{-\frac{1}{3},3}(\Sigma)$ is arbitrary, yields

$$S(q,\xi) \geq c_2\|\xi\|_{\frac{1}{3},\frac{3}{2};\Sigma} - c_3\|q\|_{0,\Omega}. \tag{3.23}$$

Then, combining (3.17) and (3.23) we easily obtain that

$$S(q,\xi) \geq \frac{c_1c_2}{c_1+c_3}\|\xi\|_{\frac{1}{3},\frac{3}{2};\Sigma},$$

which, together with (3.17), completes the proof with β depending on c_1, c_2 and c_3. □

We are now in position of establishing the well-definiteness of $\mathbf{T}$. To that end, and in order to simplify the subsequent analysis, given $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ we first note that $\|\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{0})\|_{\mathbf{H}'} = 0$, and then, by considering $p_1 = 2$ and $p_2 = 3$ in Theorem 3.1, we introduce the following notation

$$\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) := \max\Big\{\mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^{1/2}, \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}), \mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D})^2\Big\}, \tag{3.24}$$

with

$$\mathcal{N}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) := \|\mathbf{f}_\mathrm{S}\|_{0,\Omega_\mathrm{S}} + \|\mathbf{f}_\mathrm{D}\|_{\mathbf{L}^{3/2}(\Omega_\mathrm{D})} + \|g_\mathrm{D}\|_{0,\Omega_\mathrm{D}} + \|g_\mathrm{D}\|^2_{0,\Omega_\mathrm{D}}.$$

The main result of this section is established now.

Theorem 3.6. *Let* $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ *such that* $\mathrm{div}\,\mathbf{w}_\mathrm{S} = 0$ *in* Ω_S *and*

$$\|\mathbf{w}_\mathrm{S}\cdot\mathbf{n}\|_{0,\Sigma} \leq \frac{2\mu\alpha_\mathrm{S}}{\rho\, C^2_{\mathrm{tr}} C^2(\partial\Omega_\mathrm{S})},$$

and let $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$, $\mathbf{f}_\mathrm{D} \in \mathbf{L}^{3/2}(\Omega_\mathrm{D})$ *and* $g_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$. *Then,* (3.2) *has a unique solution* $(\hat{\mathbf{u}}, (\hat{p}, \hat{\lambda})) \in \mathbf{H}\times\mathbf{Q}$, *with* $\hat{\mathbf{u}} := (\hat{\mathbf{u}}_\mathrm{S}, \hat{\mathbf{u}}_\mathrm{D})$, *which allows to define* $\mathbf{T}(\mathbf{w}_\mathrm{S}) := \hat{\mathbf{u}}_\mathrm{S}$. *Moreover, there exists a constant* $c_\mathbf{T} > 0$, *independent of the solution, such that*

$$\|\mathbf{T}(\mathbf{w}_\mathrm{S})\|_{1,\Omega_\mathrm{S}} = \|\hat{\mathbf{u}}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq \|(\hat{\mathbf{u}}, (\hat{p}, \hat{\lambda}))\|_{\mathbf{H}\times\mathbf{Q}} \leq c_\mathbf{T}\,\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}). \tag{3.25}$$

Proof. It follows from Lemmas 3.4 and 3.5 and a straightforward application of Theorem 3.1. In turn, estimate (3.25) is a direct consequence of (3.4) (*cf.* Thm. 3.1) and (2.21) and (2.22). □

3.4. Solvability analysis of the fixed-point equation

In this section we proceed analogously to Section 2.4 of [21] (see also [13, 15]) and establish the existence of a fixed-point of operator $\mathbf{T}$ (*cf.* (3.1)) by means of the well known Schauder fixed-point theorem and a sufficiently small data assumption. In addition, under a more restrictive small data assumption, the uniqueness of solution is also established by means of the Banach fixed-point theorem. We begin by recalling the first of the aforementioned results (see, *e.g.*, [17], Thm. 9.12-1(b)).

Theorem 3.7. *Let W be a closed and convex subset of a Banach space X, and let $T : W \to W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed-point.*

The verification of the hypotheses of Theorem 3.7 is provided in what follows. To this aim, we start by introducing the set

$$\mathbf{W} := \Big\{ \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \quad \operatorname{div} \mathbf{v}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \quad \text{and} \quad \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le \mathrm{c}_{\mathbf{T}} \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, \mathrm{g}_{\mathrm{D}}) \Big\}. \tag{3.26}$$

Then, assuming that (*cf.* (3.24)):

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \le \frac{2\mu\alpha_{\mathrm{S}}}{c_{\mathbf{T}}\, \rho\, C^3_{\mathrm{tr}} C^2(\partial\Omega_{\mathrm{S}})}, \tag{3.27}$$

with $c_{\mathbf{T}}$ the positive constant satisfying (3.25), it is not difficult to see that $\mathbf{T}$ is well defined from $\mathbf{W}$ to $\mathbf{W}$. In fact, given $\mathbf{w}_{\mathrm{S}} \in \mathbf{W}$, from (3.27) we deduce that

$$\|\mathbf{w}_{\mathrm{S}} \cdot \mathbf{n}\|_{0,\Sigma} \le C_{\mathrm{tr}} \|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le \frac{2\mu\alpha_{\mathrm{S}}}{\rho\, C^2_{\mathrm{tr}} C^2(\partial\Omega_{\mathrm{S}})}, \tag{3.28}$$

which together with Theorem 3.6 proves that $\mathbf{T}$ is well defined. In this way, we obtain the following result.

Lemma 3.8. *Let $\mathbf{W}$ be the closed ball defined by (3.26) and assume that the data $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$ satisfy (3.27). Then there holds $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$.*

We continue with the following result providing an estimate needed to derive next the required continuity and compactness properties of the operator $\mathbf{T}$ (*cf.* (3.1)).

Lemma 3.9. *Let $\mathbf{W}$ be the closed ball defined by (3.26) and assume that the data $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$ satisfy (3.27). Then,*

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{S}}) - \mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \le \frac{\rho\, C(\Omega_{\mathrm{S}})}{\mu\alpha_{\mathrm{S}}} \|\mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{w}_{\mathrm{S}} - \widetilde{\mathbf{w}}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})} \quad \forall\, \mathbf{w}_{\mathrm{S}}, \widetilde{\mathbf{w}}_{\mathrm{S}} \in \mathbf{W}. \tag{3.29}$$

Proof. Given $\mathbf{w}_{\mathrm{S}}, \widetilde{\mathbf{w}}_{\mathrm{S}} \in \mathbf{W}$, we let $\mathbf{u}_{\mathrm{S}} := \mathbf{T}(\mathbf{w}_{\mathrm{S}})$ and $\widetilde{\mathbf{u}}_{\mathrm{S}} := \mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})$. According to the definition of $\mathbf{T}$, it follows that

$$\begin{aligned} &[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p, \lambda)] = [\mathbf{f}, \mathbf{v}] && \forall \mathbf{v} \in \mathbf{H}, \\ &[\mathbf{b}(\mathbf{u}), (q, \xi)] = [\mathbf{g}, (q, \xi)] && \forall (q, \xi) \in \mathbf{Q}, \end{aligned}$$

and

$$\begin{aligned} &[\mathbf{a}(\widetilde{\mathbf{w}}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (\widetilde{p}, \widetilde{\lambda})] = [\mathbf{f}, \mathbf{v}] && \forall \mathbf{v} \in \mathbf{H}, \\ &[\mathbf{b}(\widetilde{\mathbf{u}}), (q, \xi)] = [\mathbf{g}, (q, \xi)] && \forall (q, \xi) \in \mathbf{Q}. \end{aligned}$$

Then, recalling the definition of $\mathbf{a}(\mathbf{w}_{\mathrm{S}})$ (*cf.* (2.15)) and subtracting both problems we obtain

$$\begin{aligned} &[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p - \widetilde{p}, \lambda - \widetilde{\lambda})] = 0 \\ &[\mathbf{b}(\mathbf{u} - \widetilde{\mathbf{u}}), (q, \xi)] = 0 \end{aligned}$$

for all $(\mathbf{v},(q,\xi)) \in \mathbf{H} \times \mathbf{Q}$. In particular, taking $\mathbf{v} = \mathbf{u} - \widetilde{\mathbf{u}}$, $q = p - \widetilde{p}$ and $\xi = \lambda - \widetilde{\lambda}$ in the latter system, the first equation becomes

$$[\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_\mathrm{S})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = 0. \tag{3.30}$$

Hence, adding and subtracting $\mathcal{B}_\mathrm{S}(\mathbf{w}_\mathrm{S})(\widetilde{\mathbf{u}}_\mathrm{S})$ in the second term of the left-hand side of (3.30), using the fact that $\mathbf{u} - \widetilde{\mathbf{u}} \in \mathbf{V}$ (*cf.* (3.13)), and the strict monotonicity of $\mathbf{a}(\mathbf{w}_\mathrm{S})$ (*cf.* (3.15)), it follows that

$$\mu\alpha_\mathrm{S}\|\mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}} \leq [\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}) - \mathbf{a}(\mathbf{w}_\mathrm{S})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = [\mathcal{B}_\mathrm{S}(\widetilde{\mathbf{w}}_\mathrm{S} - \mathbf{w}_\mathrm{S})(\widetilde{\mathbf{u}}_\mathrm{S}), \mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}].$$

In this way, the continuity of $\mathcal{B}_\mathrm{S}$ (*cf.* (2.19)) gives from the foregoing equation

$$\mu\alpha_\mathrm{S}\|\mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}} \leq \rho\, C(\Omega_\mathrm{S})\|\mathbf{w}_\mathrm{S} - \widetilde{\mathbf{w}}_\mathrm{S}\|_{\mathbf{L}^4(\Omega_\mathrm{S})}\,\|\widetilde{\mathbf{u}}_\mathrm{S}\|_{1,\Omega_\mathrm{S}}\,\|\mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}\|_{1,\Omega_\mathrm{S}},$$

which yields the result. □

Owing to the above analysis, we establish now the announced properties of the operator $\mathbf{T}$.

Lemma 3.10. *Assume that the data $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$, $\mathbf{f}_\mathrm{D} \in \mathbf{L}^{3/2}(\Omega_\mathrm{D})$ and $g_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$ satisfy* (3.27) *Then $\mathbf{T}$ has at least one fixed-point in $\mathbf{W}$.*

Proof. The required result follows straightforwardly from estimate (3.29), the continuity of the Sobolev embedding from $\mathrm{H}^1(\Omega_\mathrm{S})$ into $\mathrm{L}^4(\Omega_\mathrm{S})$, and the Schauder theorem. We omit further details and refer to Lemma 5 of [21]. □

Under a more restrictive assumption on the data, in what follows we prove that $\mathbf{T}$ has exactly one fixed-point by means of the well-known Banach fixed-point theorem.

Lemma 3.11. *Let $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$, $\mathbf{f}_\mathrm{D} \in \mathbf{L}^{3/2}(\Omega_\mathrm{D})$ and $g_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$, such that*

$$\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) < r, \tag{3.31}$$

where

$$r := \frac{\mu\alpha_\mathrm{S}}{c_\mathbf{T}\rho}\min\left\{\frac{1}{C^2(\Omega_\mathrm{S})}, \frac{2}{C^2(\partial\Omega_\mathrm{S})C^3_\mathrm{tr}}\right\}.$$

Then, $\mathbf{T}$ has a unique fixed-point.

Proof. The result follows straightforwardly from (3.29), the continuity of the compact injection from $\mathrm{H}^1(\Omega_\mathrm{S})$ into $\mathrm{L}^4(\Omega_\mathrm{S})$, the fact that $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$, and the constraint (3.31). □

We are now in position of establishing the main result of this section.

Theorem 3.12. *Assume that the data $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S})$, $\mathbf{f}_\mathrm{D} \in \mathbf{L}^{3/2}(\Omega_\mathrm{D})$ and $g_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$ satisfy* (3.27). *Then the problem* (2.14) *admits a solution $(\mathbf{u},(p,\lambda)) \in \mathbf{H} \times \mathbf{Q}$. In addition, if it is assumed that* (3.31) *holds, then the solution is unique. In any case, there exists a constant $c_\mathbf{T} > 0$ (cf.* (3.25)*), independent of the solution, such that*

$$\|(\mathbf{u},(p,\lambda))\|_{\mathbf{H}\times\mathbf{Q}} \leq c_\mathbf{T}\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}). \tag{3.32}$$

Proof. The existence and uniqueness of solution of problem (2.14) follows by recalling the definition of operator $\mathbf{T}$ and combining Lemmas 3.10 and 3.11. In addition, it is clear that the estimate (3.32) is consequence of (3.25). □

4. The Galerkin scheme

In this section we introduce the Galerkin scheme of problem (2.14) and analyse its well-posedness.

4.1. Discrete setting

Let $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ be respective triangulations of the domains Ω_{S} and Ω_{D} formed by shape-regular triangles of diameter h_T and denote by h_{S} and h_{D} their corresponding mesh sizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ is a triangulation of $\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}$. Hereafter $h := \max\{h_{\mathrm{S}}, h_{\mathrm{D}}\}$. For each $T \in \mathcal{T}_h^{\mathrm{D}}$ we consider the local Raviart–Thomas space of the lowest order [47]:

$$\mathrm{RT}_0(T) := \operatorname{span}\Big\{(1,0),(0,1),(x_1,x_2)\Big\}.$$

In addition, for each $T \in \mathcal{T}_h^{\mathrm{S}}$ we denote by $\mathrm{BR}(T)$ the local Bernardi–Raugel space (see [10,33]):

$$\mathrm{BR}(T) := [\mathbb{P}_1(T)]^2 \oplus \operatorname{span}\Big\{\eta_2\eta_3\mathbf{n}_1, \eta_1\eta_3\mathbf{n}_2, \eta_1\eta_2\mathbf{n}_3\Big\},$$

where $\{\eta_1,\eta_2,\eta_3\}$ are the baricentric coordinates of T, and $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ are the unit outward normals to the opposite sides of the corresponding vertices of T. Hence, we define the following finite element subspaces:

$$\begin{aligned}
\mathbf{H}_h(\Omega_{\mathrm{S}}) &:= \Big\{\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathrm{S}}) : \quad \mathbf{v}|_T \in \mathrm{BR}(T), \quad \forall\, T \in \mathcal{T}_h^{\mathrm{S}}\Big\},\\
\mathbf{H}_h(\Omega_{\mathrm{D}}) &:= \Big\{\mathbf{v} \in \mathbf{H}^3(\operatorname{div};\Omega_{\mathrm{D}}) : \quad \mathbf{v}|_{\mathrm{T}} \in \mathrm{RT}_0(\mathrm{T}), \quad \forall\, \mathrm{T} \in \mathcal{T}_{\mathrm{h}}^{\mathrm{D}}\Big\},\\
\mathrm{L}_h(\Omega) &:= \Big\{q \in \mathrm{L}^2(\Omega) : \quad q|_T \in \mathbb{P}_0(T), \quad \forall\, T \in \mathcal{T}_h\Big\}.
\end{aligned}$$

Then, the finite element subspaces for the velocities and pressure are, respectively,

$$\begin{aligned}
\mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) &:= \mathbf{H}_h(\Omega_{\mathrm{S}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}),\\
\mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}) &:= \mathbf{H}_h(\Omega_{\mathrm{D}}) \cap \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\operatorname{div};\Omega_{\mathrm{D}}),\\
\mathrm{L}_{h,0}(\Omega) &:= \mathrm{L}_h(\Omega) \cap \mathrm{L}^2_0(\Omega).
\end{aligned}$$

Next, for introducing the finite element subspace of $\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$, we denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{D}}$ (or $\mathcal{T}_h^{\mathrm{S}}$), which is formed by edges e of length h_e, and set $h_\Sigma := \max\{h_e : e \in \Sigma_h\}$. In turn, since the space $\prod_{e\in\Sigma_h} \mathrm{W}^{1-\frac{1}{p},p}(e)$ coincides with $\mathrm{W}^{1-\frac{1}{p},p}(\Sigma)$, without extra conditions when $1 < p < 2$ ([37], Thm. 1.5.2.3-(a); see also [38], Prop. 1.4.3 and [36], Sect. 2 for the 3D case), it can be readily seen that a conforming finite element subspace for $\mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$ can be defined by

$$\Lambda_h(\Sigma) := \Big\{\xi_h : \Sigma \to \mathbb{R} : \quad \xi_h|_e \in \mathbb{P}_0(e) \quad \forall \text{ edge } e \in \Sigma_h\Big\}.$$

Notice that this space coincides with the set of discrete normal traces on Σ of $\mathbf{H}_h(\Omega_{\mathrm{D}})$. Notice also that since $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ match on Σ, there holds $h_\Sigma \le \min\{h_{\mathrm{S}}, h_{\mathrm{D}}\}$.

In this way, grouping the unknowns and spaces as follows:

$$\begin{aligned}
&\mathbf{H}_h := \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}}), \quad \mathbf{Q}_h := \mathrm{L}_{h,0}(\Omega) \times \Lambda_h(\Sigma),\\
&\mathbf{u}_h := (\mathbf{u}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}) \in \mathbf{H}_h, \quad (p_h, \lambda_h) \in \mathbf{Q}_h,
\end{aligned}$$

where $p_h := p_{\mathrm{S},h}\chi_{\mathrm{S}} + p_{\mathrm{D},h}\chi_{\mathrm{D}}$, our Galerkin scheme for (2.14) reads: Find $(\mathbf{u}_h,(p_h,\lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, such that

$$\begin{aligned}
&[\mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}_h] + [\mathbf{b}(\mathbf{v}_h),(p_h,\lambda_h)] = [\mathbf{f}, \mathbf{v}_h] && \forall\, \mathbf{v}_h := (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{H}_h,\\
&[\mathbf{b}(\mathbf{u}_h),(q_h,\xi_h)] = [\mathbf{g},(q_h,\xi_h)] && \forall\, (q_h,\xi_h) \in \mathbf{Q}_h.
\end{aligned} \tag{4.1}$$

Here, $\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h}) : \mathbf{H}_h \to \mathbf{H}'_h$ is the discrete version of $\mathbf{a}(\mathbf{w}_\mathrm{S})$ (with $\mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ in place of $\mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$), which is defined by

$$[\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}_h] := [\mathcal{A}_\mathrm{S}(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] + [\mathcal{B}^h_\mathrm{S}(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] + [\mathcal{A}_\mathrm{D}(\mathbf{u}_{\mathrm{D},h}), \mathbf{v}_{\mathrm{D},h}], \tag{4.2}$$

where $\mathcal{B}^h_\mathrm{S}(\mathbf{w}_{\mathrm{S},h})$ is the well-known skew-symmetric convection form [54]:

$$[\mathcal{B}^h_\mathrm{S}(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] := \rho((\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{w}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_\mathrm{S} + \frac{\rho}{2}(\mathrm{div}\, \mathbf{w}_{\mathrm{S,h}} \mathbf{u}_{\mathrm{S,h}}, \mathbf{v}_{\mathrm{S,h}})_\mathrm{S},$$

for all $\mathbf{u}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h}, \mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$. Observe that integrating by parts, similarly to (2.24), there holds

$$[\mathcal{B}^h_\mathrm{S}(\mathbf{w}_{\mathrm{S},h})(\mathbf{v}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] = \frac{\rho}{2}\int_\Sigma (\mathbf{w}_{\mathrm{S},h}\cdot \mathbf{n})|\mathbf{v}_{\mathrm{S},h}|^2 \qquad \forall\, \mathbf{w}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}). \tag{4.3}$$

Moreover, proceeding as for $\mathcal{B}_\mathrm{S}$ (*cf.* (2.19)), it is easy to see that for all $\mathbf{w}_{\mathrm{S},h}$, $\mathbf{u}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, there holds

$$\left|[\mathcal{B}^h_\mathrm{S}(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}]\right| \le C_{\mathrm{sk}} \|\mathbf{w}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}} \|\mathbf{v}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}}, \tag{4.4}$$

with $C_{\mathrm{sk}} := \rho\, C^2(\Omega_\mathrm{S})\left(1 + \frac{\sqrt{2}}{2}\right)$.

Now, let $\Pi_\mathrm{S} : \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \to \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ be the Bernardi–Raugel interpolation operator [10], which is linear and bounded with respect to the $\mathbf{H}^1(\Omega_\mathrm{S})$-norm. In this regard, we recall that, given $\mathbf{v} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$, there holds

$$\int_e \Pi_\mathrm{S}(\mathbf{v})\cdot \mathbf{n} = \int_e \mathbf{v}\cdot\mathbf{n} \quad \text{for each edge } e \text{ of } \mathcal{T}^\mathrm{S}_h, \tag{4.5}$$

and hence

$$(\mathrm{div}\, \Pi_\mathrm{S}(\mathbf{v}), \mathrm{q_h})_\mathrm{S} = (\mathrm{div}\, \mathbf{v}, \mathrm{q_h})_\mathrm{S} \quad \forall \mathrm{q_h} \in \mathrm{L_h}(\Omega). \tag{4.6}$$

Equivalently, if $\mathcal{P}_\mathrm{S}$ denotes the $\mathrm{L}^2(\Omega_\mathrm{S})$-orthogonal projection onto the restriction of $\mathrm{L}_h(\Omega)$ to Ω_S, then the relation (4.6) can be written as

$$\mathcal{P}_\mathrm{S}(\mathrm{div}\,(\Pi_\mathrm{S}(\mathbf{v}))) = \mathcal{P}_\mathrm{S}(\mathrm{div}\, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}). \tag{4.7}$$

On the other hand, let $\Pi_\mathrm{D} : \mathbf{H}^1(\Omega_\mathrm{D}) \to \mathbf{H}_h(\Omega_\mathrm{D})$ be the well-known Raviart–Thomas interpolation operator. We recall that, given $\mathbf{v} \in \mathbf{H}^1(\Omega_\mathrm{D})$, this operator is characterized by

$$\int_e \Pi_\mathrm{D}(\mathbf{v})\cdot \mathbf{n} = \int_e \mathbf{v}\cdot\mathbf{n} \quad \text{for each edge } e \text{ of } \mathcal{T}^\mathrm{D}_h, \tag{4.8}$$

which implies that

$$(\mathrm{div}\, \Pi_\mathrm{D}(\mathbf{v}), \mathrm{q_h})_\mathrm{D} = (\mathrm{div}\, \mathbf{v}, \mathrm{q_h})_\mathrm{D} \quad \forall \mathrm{q_h} \in \mathrm{L_h}(\Omega). \tag{4.9}$$

Equivalently, if $\mathcal{P}_\mathrm{D}$ denotes the $\mathrm{L}^2(\Omega_\mathrm{D})$-orthogonal projection onto the restriction of $\mathrm{L}_h(\Omega)$ to Ω_D, then the relation (4.9) can be written as

$$\mathrm{div}\,(\Pi_\mathrm{D}(\mathbf{v})) = \mathcal{P}_\mathrm{D}(\mathrm{div}\, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega_\mathrm{D}). \tag{4.10}$$

At this point we recall, according to Sections 1.2.7 and 1.4.7 of [23] (see also [12], Chapt. III.3.3), that the Raviart–Thomas operator Π_D is also well defined for all $\mathbf{v} \in \mathbf{V}^{\mathrm{div}}(\Omega_\mathrm{D}) := \left\{\mathbf{v} \in \mathbf{L}^p(\Omega_\mathrm{D}) :\ \mathrm{div}\, \mathbf{v} \in \mathrm{L}^s(\Omega_\mathrm{D})\right\}$, with $p > 2$ and $s \ge q$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, since the local space $\mathbf{V}^{\mathrm{div}}(T)$ coincides with $\mathbf{W}^{1,t}(T)$ when $t > \frac{2n}{n+2}$, for each $T \in \mathcal{T}^\mathrm{D}_h$. In particular, considering $n = 2$, $p = 3$, and $s = 2$, we deduce that Π_D can be applied to functions in $\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})$. We will use this fact later on in the proof of the discrete inf-sup condition of $\mathbf{b}$.

4.2. Well-posedness of the discrete problem

In this section, analogously to the analysis of the continuous problem, we apply a fixed-point argument to prove the well-posedness of the Galerkin scheme (4.1). To that end, we now let $\mathbf{T}_h : \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \to \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ be the discrete operator defined by

$$\mathbf{T}_h(\mathbf{w}_{\mathrm{S},h}) := \hat{\mathbf{u}}_{\mathrm{S},h} \quad \forall \mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}), \tag{4.11}$$

where $\hat{\mathbf{u}}_h := (\hat{\mathbf{u}}_{\mathrm{S},h}, \hat{\mathbf{u}}_{\mathrm{D},h}) \in \mathbf{H}_h$ is the first component of the unique solution (to be confirmed below) of the discrete nonlinear problem: Find $(\hat{\mathbf{u}}_h, (\hat{p}_h, \hat{\lambda}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, such that

$$\begin{aligned} &[\mathbf{a}_h(\hat{\mathbf{w}}_{\mathrm{S},h})(\hat{\mathbf{u}}_h), \mathbf{v}_h] + [\mathbf{b}(\mathbf{v}_h), (\hat{p}_h, \hat{\lambda}_h)] = [\mathbf{f}, \mathbf{v}_h] && \forall \mathbf{v}_h \in \mathbf{H}_h, \\ &[\mathbf{b}(\hat{\mathbf{u}}_h), (q_h, \xi_h)] = [\mathbf{g}, (q_h, \xi_h)] && \forall (q_h, \xi_h) \in \mathbf{Q}_h. \end{aligned} \tag{4.12}$$

Then, similarly as for the continuous case, the Galerkin scheme (4.1) can be rewritten, equivalently, as the fixed-point problem: Find $\mathbf{u}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$ such that

$$\mathbf{T}_h(\mathbf{u}_{\mathrm{S},h}) = \mathbf{u}_{\mathrm{S},h}.$$

In this way, in what follows we focus on analysing the existence and uniqueness of such a fixed-point, for which we require the following discrete version of Theorem 3.1.

Theorem 4.1. *In addition to the spaces and operators defined in Theorem 3.1, let $X_{1,h}, X_{2,h}$ and Y_h be finite dimensional subspaces of X_1, X_2, and Y, respectively, and set $X_h = X_{1,h} \times X_{2,h} \subseteq X := X_1 \times X_2$. In addition, let V_h be the discrete kernel of b, that is,*

$$V_h := \Big\{ v_h \in X_h : \quad [b(v_h), q_h] = 0 \quad \forall q_h \in Y_h \Big\}.$$

Assume that

(i) a is hemi-continuous from X_h to X_h', that is, for each $u, v \in X_h$, the real mapping

$$J : \mathbb{R} \to \mathbb{R}, \quad t \to J(t) = [a(u + tv), v]$$

is continuous.

(ii) there exist constants $\widetilde{\gamma} > 0$ and $p_1, p_2 \geq 2$, such that

$$\|a(u_h) - a(v_h)\|_{X'} \leq \widetilde{\gamma} \sum_{j=1}^{2} \Big\{ \|u_{j,h} - v_{j,h}\|_{X_j} + \|u_{j,h} - v_{j,h}\|_{X_j} \Big(\|u_{j,h}\|_{X_j} + \|v_{j,h}\|_{X_j} \Big)^{p_j - 2} \Big\},$$

for all $u_h = (u_{1,h}, u_{2,h}), v_h = (v_{1,h}, v_{2,h}) \in X_h$.

(iii) for fixed $t_h \in X_h$, the operator $a(\cdot + t_h) : V_h \to V_h'$ is strictly monotone, that is, there exists $\widetilde{\alpha} > 0$ and $p_1, p_2 \geq 2$, such that

$$[a(u_h + t_h) - a(v_h + t_h), u_h - v_h] \geq \widetilde{\alpha} \Big\{ \|u_{1,h} - v_{1,h}\|_{X_1}^{p_1} + \|u_{2,h} - v_{2,h}\|_{X_2}^{p_2} \Big\},$$

for all $u_h = (u_{1,h}, u_{2,h}), v_h = (v_{1,h}, v_{2,h}) \in V_h$.

(iv) there exists $\widetilde{\beta} > 0$ such that

$$\sup_{\substack{v_h \in X_h \\ v_h \neq 0}} \frac{[b(v_h), q_h]}{\|v_h\|_X} \geq \widetilde{\beta} \|q_h\|_Y \quad \forall q_h \in Y_h.$$

Then, for each $(f,g)\in X'\times Y'$ *there exists a unique* $(u_h,p_h)\in X_h\times Y_h$*, such that*

$$\begin{aligned}&[a(u_h),v_h]+[b(v_h),p_h]=[f,v_h]\ \forall v_h\in X_h,\\&[b(u_h),q_h]\qquad\qquad\qquad\ \ =[g,q_h]\ \forall q_h\in Y_h.\end{aligned}$$

Moreover, there exists $\widetilde{C}>0$*, depending only on* $\widetilde{\alpha},\widetilde{\gamma},\widetilde{\beta},p_1$*, and* p_2*, such that*

$$\|(u_h,p_h)\|_{X\times Y}\le \widetilde{C}\,\mathcal{M}(f,g),$$

where

$$\mathcal{M}(f,g):=\max\left\{\mathcal{N}(f,g)^{\frac{1}{p_1-1}},\mathcal{N}(f,g)^{\frac{1}{p_2-1}},\mathcal{N}(f,g),\mathcal{N}(f,g)^{\frac{p_1-1}{p_2-1}},\mathcal{N}(f,g)^{\frac{p_2-1}{p_1-1}}\right\},$$

and

$$\mathcal{N}(f,g):=\|f\|_{X'}+\|g\|_{Y'}+\|g\|_{Y'}^{p_1-1}+\|g\|_{Y'}^{p_2-1}+\|a(0)\|_{X'}.$$

Proof. It reduces to a simple application of Theorem 3.1 to the present discrete setting. □

Similarly to the analysis developed in Section 3.3, in what follows we provide suitable assumptions under which problem (4.12) is well posed or equivalently $\mathbf{T}_h$ is well defined. For this purpose, we must verify that the operators defining the discrete problem (4.12) satisfy the hypotheses of Theorem 4.1. We begin with the hemi-continuity of $\mathbf{a}_h$.

Lemma 4.2. *Given* $\mathbf{w}_{\mathrm{S},h}\in\mathbf{H}^1_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$*, the operator* $\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})$ *is hemi-continuous in* $\mathbf{H}_h$*.*

Proof. The proof follows analogously to the proof of Lemma 3.2, by using now the linearity and continuity of $\mathcal{B}^h_\mathrm{S}(\mathbf{w}_{\mathrm{S},h})$ (in addition to those of $\mathcal{A}_\mathrm{S}$). □

Now we verify that hypothesis (ii) of Theorem 4.1 holds.

Lemma 4.3. *Let* $\mathbf{w}_{\mathrm{S},h}\in\mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$*. Then, there exists* $\tilde{\gamma}>0$*, depending on* $C_{\mathcal{A}_\mathrm{S}}$ *and* $L_{\mathcal{A}_\mathrm{D}}$ *(cf. (2.19), (2.20)), such that*

$$\begin{aligned}\|\mathbf{a}(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_h)-\mathbf{a}(\mathbf{w}_{\mathrm{S},h})(\mathbf{v}_h)\|_{\mathbf{H}'}\le\tilde{\gamma}\Big\{&(1+\|\mathbf{w}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}})\|\mathbf{u}_{\mathrm{S},h}-\mathbf{v}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}}+\|\mathbf{u}_{\mathrm{D},h}-\mathbf{v}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\\&+\|\mathbf{u}_{\mathrm{D},h}-\mathbf{v}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\Big(\|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}+\|\mathbf{v}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\Big)\Big\},\end{aligned}$$

for all $\mathbf{u}_h=(\mathbf{u}_{\mathrm{S},h},\mathbf{u}_{\mathrm{D},h}),\mathbf{v}_h=(\mathbf{v}_{\mathrm{S},h},\mathbf{v}_{\mathrm{D},h})\in\mathbf{H}_h$*.*

Proof. Similarly to the continuous case, the result follows straightforwardly from the definition of $\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})$ (*cf.* (4.2)), the triangle inequality, and the stability properties (2.19), (2.20) and (4.4). We omit further details. □

Now, we proceed to establish the strict monotonicity of $\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})$ on the discrete kernel of $\mathbf{b}$:

$$\mathbf{V}_h:=\Big\{\mathbf{v}_h:=(\mathbf{v}_{\mathrm{S},h},\mathbf{v}_{\mathrm{D},h})\in\mathbf{H}_h:\quad[\mathbf{b}(\mathbf{v}_h),(q_h,\xi_h)]=0\quad\forall(q_h,\xi_h)\in\mathbf{Q}_h\Big\},\tag{4.13}$$

for suitable $\mathbf{w}_{\mathrm{S},h}\in\mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S})$. Observe that, similarly to the continuous case, $\mathbf{v}_h\in\mathbf{V}_h$ if and only if

$$(\mathrm{div}\,\mathbf{v}_{\mathrm{S,h}},\mathrm{q_h})_\mathrm{S}+(\mathrm{div}\,\mathbf{v}_{\mathrm{D,h}},\mathrm{q_h})_\mathrm{D}=0\quad\forall\mathrm{q_h}\in\mathrm{L_{h,0}}(\Omega),$$

and

$$\langle\mathbf{v}_{\mathrm{S},h}\cdot\mathbf{n}-\mathbf{v}_{\mathrm{D},h}\cdot\mathbf{n},\xi_h\rangle_\Sigma=0\quad\forall\xi_h\in\Lambda_h(\Sigma),$$

which, in particular, imply that

$$(\mathrm{div}\,\mathbf{v}_{\mathrm{S,h}},\mathrm{q_h})_\mathrm{S}=0\quad\forall\mathrm{q_h}\in\mathrm{L_h}(\Omega_\mathrm{S})\quad\text{and}\quad\mathrm{div}\,\mathbf{v}_{\mathrm{D,h}}=0\quad\text{in}\quad\Omega_\mathrm{D},\tag{4.14}$$

where $\mathrm{L}_h(\Omega_\mathrm{S})$ is the set of functions of $\mathrm{L}_h(\Omega)$ restricted to Ω_S. Then, the announced result is as follows.

Lemma 4.4. *Let* $\mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ *such that*

$$\|\mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n}\|_{0,\Sigma} \le \frac{2\mu\alpha_{\mathrm{S}}}{\rho\, C_{\mathrm{tr}}^2 C^2(\partial\Omega_{\mathrm{S}})}. \tag{4.15}$$

Then, for fixed $\mathbf{t}_h \in \mathbf{H}_h \setminus \mathbf{V}_h$*, the nonlinear operator* $\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})(\,\cdot + \mathbf{t}_h)$ *is strictly monotone on* $\mathbf{V}_h$ *(cf. (4.13)).*

Proof. The proof follows analogously to the proof of Lemma 3.4. Further details are omitted. □

We continue by adapting the results provided in Section 4 of [28] to our domain and spaces configuration to prove that $\mathbf{b}$ satisfies the corresponding discrete inf-sup condition. We start by establishing the following two preliminary lemmas.

Lemma 4.5. *There exists* $\widetilde{C}_1 > 0$*, independent of* h*, such that for all* $(q_h, \xi_h) \in \mathbf{Q}_h$*, there holds*

$$S_h(q_h,\xi_h) := \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h),(q_h,\xi_h)]}{\|\mathbf{v}_h\|_{\mathbf{H}}} \ge \widetilde{C}_1 \|\xi_h\|_{\frac{1}{3},\frac{3}{2};\Sigma} - \|q_h\|_{0,\Omega}. \tag{4.16}$$

Proof. Let $\xi_h \in \Lambda_h(\Sigma) \subseteq \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$, $\xi_h \neq 0$. Since

$$\sup_{\substack{\widetilde{\phi} \in \mathrm{W}^{-\frac{1}{3},3}(\Sigma) \\ \widetilde{\phi} \neq 0}} \frac{\left\langle \widetilde{\phi}, \xi_h \right\rangle_\Sigma}{\|\widetilde{\phi}\|_{-\frac{1}{3},3;\Sigma}} = \|\xi_h\|_{\frac{1}{3},\frac{3}{2};\Sigma},$$

we deduce that there exists $\widetilde{\phi} \in \mathrm{W}^{-\frac{1}{3},3}(\Sigma)\backslash\{0\}$ such that

$$\left\langle \widetilde{\phi}, \xi_h \right\rangle_\Sigma \ge \frac{1}{2}\|\widetilde{\phi}\|_{-\frac{1}{3},3;\Sigma}\|\xi_h\|_{\frac{1}{3},\frac{3}{2};\Sigma}. \tag{4.17}$$

Next, exactly as we did in the proof of Lemma 3.5, we *"extend"* $\widetilde{\phi} \in \mathrm{W}^{-\frac{1}{3},3}(\Sigma)$ to $\eta \in \mathrm{W}^{-\frac{1}{3},3}(\partial\Omega_{\mathrm{D}})$ by

$$\langle \eta, \mu\rangle_{\partial\Omega_{\mathrm{D}}} := \left\langle \widetilde{\phi}, \mu_\Sigma \right\rangle_\Sigma \quad \forall \mu \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_{\mathrm{D}}),$$

where $\mu_\Sigma \in \mathrm{W}^{\frac{1}{3},\frac{3}{2}}(\Sigma)$ is given by the decomposition (2.10). Then, proceeding again as in the second part of the proof of Lemma 3.5, we find $\widetilde{\mathbf{v}}_{\mathrm{D}} \in \mathbf{H}^3_{\Gamma_{\mathrm{D}}}(\mathrm{div}\,;\Omega_{\mathrm{D}})$ satisfying $\widetilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n} = \eta$ on $\partial\Omega_{\mathrm{D}}$, and (*cf.* (3.22))

$$\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} \le C\|\eta\|_{-\frac{1}{3},3;\partial\Omega_{\mathrm{D}}} \le C\|\widetilde{\phi}\|_{-\frac{1}{3},3;\Sigma},$$

which, combined with (4.17), implies

$$\begin{aligned} \langle \widetilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}, \xi_h\rangle_\Sigma &:= \langle \widetilde{\mathbf{v}}_{\mathrm{D}} \cdot \mathbf{n}, E_\Sigma(\xi_h)\rangle_{\partial\Omega_{\mathrm{D}}} = \langle \eta, E_\Sigma(\xi_h)\rangle_{\partial\Omega_{\mathrm{D}}} = \left\langle \widetilde{\phi}, \xi_h \right\rangle_\Sigma \\ &\ge \frac{1}{2C}\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}\|\xi_h\|_{\frac{1}{3},\frac{3}{2};\Sigma}. \end{aligned} \tag{4.18}$$

On the other hand, given $\mathbf{v}_{\mathrm{D}} \in \mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})$, the properties of Π_{D} (*cf.* (4.8), (4.9)) and Lemma 3.2 of [25] allow to establish that

$$\langle \mathbf{v}_{\mathrm{D}} \cdot \mathbf{n}, \xi_h\rangle_\Sigma = \int_\Sigma (\Pi_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}}) \cdot \mathbf{n})\xi_h \quad \forall \xi_h \in \Lambda_h(\Sigma), \tag{4.19}$$

and

$$\|\Pi_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})} \le C_{\mathrm{D}}\|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}. \tag{4.20}$$

Thus, defining $\widetilde{\mathbf{v}}_{\mathrm{D},h} := \Pi_{\mathrm{D}}(\widetilde{\mathbf{v}}_{\mathrm{D}}) \in \mathbf{H}_{h,\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{D}})$, and then using (4.18), (4.19), and (4.20), we obtain

$$\frac{\left|\langle \widetilde{\mathbf{v}}_{\mathrm{D},h}\cdot\mathbf{n},\xi_h\rangle_\Sigma\right|}{\|\widetilde{\mathbf{v}}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}} \geq \frac{1}{C_{\mathrm{D}}}\frac{\left|\langle \widetilde{\mathbf{v}}_{\mathrm{D}}\cdot\mathbf{n},\xi_h\rangle_\Sigma\right|}{\|\widetilde{\mathbf{v}}_{\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}} \geq \widetilde{C}_1\|\xi_h\|_{\frac{1}{3},\frac{3}{2};\Sigma}. \tag{4.21}$$

Finally, setting $\widetilde{\mathbf{v}}_h := (\mathbf{0},\widetilde{\mathbf{v}}_{\mathrm{D},h}) \in \mathbf{H}_h$, we deduce that

$$\begin{aligned} S_h(q_h,\xi_h) := \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h),(q_h,\xi_h)]}{\|\mathbf{v}_h\|_{\mathbf{H}}} &\geq \frac{\left|[\mathbf{b}(\widetilde{\mathbf{v}}_h),(q_h,\xi_h)]\right|}{\|\widetilde{\mathbf{v}}_h\|_{\mathbf{H}}} \\ = \frac{\left|\langle \widetilde{\mathbf{v}}_{\mathrm{D},h}\cdot\mathbf{n},\xi_h\rangle_\Sigma - (\mathrm{div}\,\widetilde{\mathbf{v}}_{\mathrm{D,h}},\mathrm{q_h})_{\mathrm{D}}\right|}{\|\widetilde{\mathbf{v}}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}} &\geq \frac{\left|\langle \widetilde{\mathbf{v}}_{\mathrm{D},h}\cdot\mathbf{n},\xi_h\rangle_\Sigma\right|}{\|\widetilde{\mathbf{v}}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\mathrm{D}})}} - \|q_h\|_{0,\Omega}, \end{aligned}$$

which, together with (4.21), imply (4.16) and complete the proof. □

Lemma 4.6. *There exists $\widetilde{C}_2 > 0$, independent of h, such that for all $(q_h,\xi_h) \in \mathbf{Q}_h$, there holds*

$$S_h(q_h,\xi_h) := \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h),(q_h,\xi_h)]}{\|\mathbf{v}_h\|_{\mathbf{H}}} \geq \widetilde{C}_2\|q_h\|_{0,\Omega}. \tag{4.22}$$

Proof. The proof follows similarly to the first part of the proof of Lemma 3.5. In fact, given $(q_h,\xi_h) \in \mathbf{Q}_h$ we recall that $q_h \in \mathrm{L}^2_0(\Omega)$ and apply again ([33], Cor. 2.4) to deduce that there exists $\mathbf{z} \in \mathbf{H}^1_0(\Omega)$ such that

$$\mathrm{div}\,\mathbf{z} = -\mathrm{q_h} \quad \text{in} \quad \Omega \quad \text{and} \quad \|\mathbf{z}\|_{1,\Omega} \leq \mathrm{c}\|\mathrm{q_h}\|_{0,\Omega}. \tag{4.23}$$

Then, we let $\mathbf{z}_\star := \mathbf{z}|_{\Omega_\star}$ for $\star \in \{\mathrm{S},\mathrm{D}\}$ and observe that $\mathbf{z}_{\mathrm{S}} = \mathbf{z}_{\mathrm{D}}$ on Σ, which implies that

$$(\mathbf{z}_{\mathrm{S}} - \mathbf{z}_{\mathrm{D}})\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma.$$

Hence, defining $\mathbf{z}_h := (\mathbf{z}_{\mathrm{S},h},\mathbf{z}_{\mathrm{D},h})$, with $\mathbf{z}_{\mathrm{S},h} = \Pi_{\mathrm{S}}(\mathbf{z}_{\mathrm{S}})$ and $\mathbf{z}_{\mathrm{D},h} = \Pi_{\mathrm{D}}(\mathbf{z}_{\mathrm{D}})$, we observe from (4.5), (4.8), and the fact that $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ match on Σ, that

$$\langle(\mathbf{z}_{\mathrm{S},h} - \mathbf{z}_{\mathrm{D},h})\cdot\mathbf{n},\xi_h\rangle_\Sigma = \langle(\mathbf{z}_{\mathrm{S}} - \mathbf{z}_{\mathrm{D}})\cdot\mathbf{n},\xi_h\rangle_\Sigma = 0. \tag{4.24}$$

In addition, since $\mathbf{z} = \mathbf{0}$ on $\partial\Omega := \Gamma_{\mathrm{S}} \cup \Gamma_{\mathrm{D}}$, it is clear that $\mathbf{z}_h \in \mathbf{H}_h$, and therefore, thanks to the continuity of Π_{S} and the estimate (4.20), we obtain that

$$\|\mathbf{z}_h\|_{\mathbf{H}} \leq C\|q_h\|_{0,\Omega}, \tag{4.25}$$

with $C > 0$ independent of h. Finally, from the identities (4.7) and (4.10), it can be readily seen that

$$\mathrm{div}\,\mathbf{z}_{\mathrm{h}} = -\mathrm{q_h} \quad \text{in} \quad \Omega, \tag{4.26}$$

which, together with (4.24) and (4.25), yield

$$\sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h),(q_h,\xi_h)]}{\|\mathbf{v}_h\|_{\mathbf{H}}} \geq \frac{[\mathbf{b}(\mathbf{z}_h),(q_h,\xi_h)]}{\|\mathbf{z}_h\|_{\mathbf{H}}} \geq \frac{1}{C}\|q_h\|_{0,\Omega},$$

which concludes the proof. □

Owing to Lemmas 4.5 and 4.6, now we are in position of establishing the full discrete inf-sup condition of $\mathbf{b}$.

Lemma 4.7. *There exists $\widetilde{\beta} > 0$, independent of h, such that for all $(q_h, \xi_h) \in \mathbf{Q}_h$ there holds*

$$S_h(q_h, \xi_h) := \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h), (q_h, \xi_h)]}{\|\mathbf{v}_h\|_{\mathbf{H}}} \geq \widetilde{\beta} \|(q_h, \xi_h)\|_{\mathbf{Q}}. \tag{4.27}$$

Proof. It follows straightforwardly from the estimates (4.16) and (4.22). □

The following result establishes the well-definiteness of operator $\mathbf{T}_h$.

Theorem 4.8. *Let $\mathbf{w}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that*

$$\|\mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n}\|_{0,\Sigma} \leq \frac{2\mu\alpha_{\mathrm{S}}}{\rho\, C_{\mathrm{tr}}^2 C^2(\partial\Omega_{\mathrm{S}})}, \tag{4.28}$$

and let $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$. Then, (4.12) has a unique solution $(\hat{\mathbf{u}}_h, (\hat{p}_h, \hat{\lambda}_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, with $\hat{\mathbf{u}}_h := (\hat{\mathbf{u}}_{\mathrm{S},h}, \hat{\mathbf{u}}_{\mathrm{D},h})$, which allows to define $\mathbf{T}_h(\mathbf{w}_{\mathrm{S},h}) = \hat{\mathbf{u}}_{\mathrm{S},h}$. Moreover, there exists a constant $\widetilde{c}_{\mathbf{T}} > 0$, independent of the solution, such that

$$\|\mathbf{T}_h(\mathbf{w}_{\mathrm{S},h})\|_{1,\Omega_{\mathrm{S}}} = \|\hat{\mathbf{u}}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \leq \|(\hat{\mathbf{u}}_h, (\hat{p}_h, \hat{\lambda}_h))\|_{\mathbf{H}\times\mathbf{Q}} \leq \widetilde{c}_{\mathbf{T}} \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}). \tag{4.29}$$

Proof. Similarly to the continuous case, the result is a direct consequence of Lemmas 4.2–4.4, 4.7 and Theorem 4.1. □

Having verified the well-definiteness of operator $\mathbf{T}_h$, now we are in position of establishing the main result of this section, namely, the well-posedness of problem (4.1).

Theorem 4.9. *Let $\mathbf{W}_h$ be the compact convex subset of $\mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ defined by*

$$\mathbf{W}_h := \Big\{ \mathbf{v}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \quad \|\mathbf{v}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \leq \widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \Big\}. \tag{4.30}$$

Assume that the data $\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}$, and g_{D} satisfy

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) < \widetilde{r}, \tag{4.31}$$

where

$$\widetilde{r} := \frac{2\mu\alpha_{\mathrm{S}}}{\widetilde{c}_{\mathbf{T}}\rho} \min\left\{ \frac{1}{C^2(\Omega_{\mathrm{S}})(2+\sqrt{2})}, \frac{1}{C^2(\partial\Omega_{\mathrm{S}}) C_{\mathrm{tr}}^3} \right\},$$

and $\widetilde{c}_{\mathbf{T}} > 0$ is the constant in (4.29). Then, there exists a unique $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution to (4.1), which satisfies $\mathbf{u}_{\mathrm{S},h} \in \mathbf{W}_h$ and

$$\|(\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \leq \widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}). \tag{4.32}$$

Proof. We first observe thanks to (4.29), that assumption (4.31) guarantees that $\mathbf{T}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$. Next, proceeding analogously to the proof of Lemma 3.9, the assumption (4.31) implies the estimate

$$\begin{aligned} \mu\alpha_{\mathrm{S}} \|\mathbf{T}_h(\mathbf{w}_{\mathrm{S},h}) - \mathbf{T}_h(\widetilde{\mathbf{w}}_{\mathrm{S},h})\|^2_{1,\Omega_{\mathrm{S}}} &\leq [\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_h) - \mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})(\widetilde{\mathbf{u}}_h), \mathbf{u}_h - \widetilde{\mathbf{u}}_h] \\ &= [\mathcal{B}^h_{\mathrm{S}}(\widetilde{\mathbf{w}}_{\mathrm{S},h} - \mathbf{w}_{\mathrm{S},h})(\widetilde{\mathbf{u}}_{\mathrm{S},h}), \mathbf{u}_{\mathrm{S},h} - \widetilde{\mathbf{u}}_{\mathrm{S},h}], \end{aligned}$$

which, together with the continuity of $\mathcal{B}^h_{\mathrm{S}}$ (see (4.4)) leads to

$$\|\mathbf{T}_h(\mathbf{w}_{\mathrm{S},h}) - \mathbf{T}_h(\widetilde{\mathbf{w}}_{\mathrm{S},h})\|_{1,\Omega_{\mathrm{S}}} \leq \frac{\rho\, C^2(\Omega_{\mathrm{S}})(2+\sqrt{2})}{2\mu\alpha_{\mathrm{S}}} \|\mathbf{T}_h(\widetilde{\mathbf{w}}_{\mathrm{S},h})\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{w}_{\mathrm{S},h} - \widetilde{\mathbf{w}}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}, \tag{4.33}$$

thus proving the continuity of $\mathbf{T}_h$. Then, the existence result follows from the Brower fixed-point theorem. Moreover, from (4.33) and the fact that $\mathbf{T}_h(\widetilde{\mathbf{w}}_{\mathrm{S},h})$ belongs to $\mathbf{W}_h$, it is easy to see that $\mathbf{T}_h$ is a contraction mapping if and only if (4.31) holds, which due to the Banach fixed-point theorem, implies the uniqueness of solution. In turn, the *a priori* estimate (4.32) follows directly from (4.29). □

5. A PRIORI ERROR ANALYSIS

Now we establish the corresponding Céa estimate and the theoretical rate of convergence of the Galerkin scheme (4.1). To that end, we first introduce some notations and state some previous results. We begin by defining the set

$$\mathbf{H}_h^{\mathbf{g}} := \Big\{ \mathbf{v}_h := (\mathbf{v}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{H}_h : \quad [\mathbf{b}(\mathbf{v}_h), (q_h, \xi_h)] = [\mathbf{g}, (q_h, \xi_h)] \quad \forall (q_h, \xi_h) \in \mathbf{Q}_h \Big\},$$

which is clearly noempty, since (4.27) holds. Also, it is not difficult to see that, due to the inf-sup condition (4.27), the following inequality holds (*cf.* [27], Thm. 2.6, [50], Thm. 2.1):

$$\inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{g}}} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} \leq \left(1 + \frac{C_{\mathbf{b}}}{\widetilde{\beta}}\right) \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}}. \tag{5.1}$$

In turn, in order to simplify the subsequent analysis, we write $\mathbf{e}_{\mathbf{u}_S} = \mathbf{u}_S - \mathbf{u}_{S,h}$, $\mathbf{e}_{\mathbf{u}_D} = \mathbf{u}_D - \mathbf{u}_{D,h}$, $e_p = p - p_h$, and $e_\lambda = \lambda - \lambda_h$. As usual, for a given $\overline{\mathbf{v}}_h = (\overline{\mathbf{v}}_{S,h}, \overline{\mathbf{v}}_{D,h}) \in \mathbf{H}_h^{\mathbf{g}}$ and $(\overline{q}_h, \overline{\xi}_h) \in \mathbf{Q}_h$, we shall then decompose these errors into

$$\mathbf{e}_{\mathbf{u}_S} = \boldsymbol{\delta}_{\mathbf{u}_S} + \boldsymbol{\eta}_{\mathbf{u}_S}, \quad \mathbf{e}_{\mathbf{u}_D} = \boldsymbol{\delta}_{\mathbf{u}_D} + \boldsymbol{\eta}_{\mathbf{u}_D}, \quad e_p = \delta_p + \eta_p, \quad e_\lambda = \delta_\lambda + \eta_\lambda, \tag{5.2}$$

with

$$\begin{gathered} \boldsymbol{\delta}_{\mathbf{u}_S} = \mathbf{u}_S - \overline{\mathbf{v}}_{S,h}, \quad \boldsymbol{\eta}_{\mathbf{u}_S} = \overline{\mathbf{v}}_{S,h} - \mathbf{u}_{S,h}, \quad \boldsymbol{\delta}_{\mathbf{u}_D} = \mathbf{u}_D - \overline{\mathbf{v}}_{D,h}, \quad \boldsymbol{\eta}_{\mathbf{u}_D} = \overline{\mathbf{v}}_{D,h} - \mathbf{u}_{D,h}, \\ \delta_p = p - \overline{q}_h, \quad \eta_p = \overline{q}_h - p_h, \quad \delta_\lambda = \lambda - \overline{\xi}_h, \quad \eta_\lambda = \overline{\xi}_h - \lambda_h. \end{gathered} \tag{5.3}$$

Finally, since the exact solution $\mathbf{u}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ satisfies $\operatorname{div} \mathbf{u}_S = 0$ in Ω_S, we have

$$[\mathcal{B}_S^h(\mathbf{u}_S)(\mathbf{u}_S), \mathbf{v}_{S,h}] = [\mathcal{B}_S(\mathbf{u}_S)(\mathbf{u}_S), \mathbf{v}_{S,h}] \quad \forall \mathbf{v}_{S,h} \in \mathbf{H}_{h,\Gamma_S}(\Omega_S).$$

Consequently, the following Galerkin orthogonality property holds:

$$\begin{aligned} [\mathcal{A}_S(\mathbf{e}_{\mathbf{u}_S}), \mathbf{v}_{S,h}] + [\mathcal{B}_S^h(\mathbf{u}_S)(\mathbf{u}_S), \mathbf{v}_{S,h}] - [\mathcal{B}_S^h(\mathbf{u}_{S,h})(\mathbf{u}_{S,h}), \mathbf{v}_{S,h}] & \\ + [\mathcal{A}_D(\mathbf{u}_D) - \mathcal{A}_D(\mathbf{u}_{D,h}), \mathbf{v}_{D,h}] + [\mathbf{b}(\mathbf{v}_h), (e_p, e_\lambda)] &= 0 \\ [\mathbf{b}(\mathbf{e}_{\mathbf{u}_S}, \mathbf{e}_{\mathbf{u}_D}), (q_h, \xi_h)] &= 0 \end{aligned} \tag{5.4}$$

for all $\mathbf{v}_h := (\mathbf{v}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{H}_h$ and $(q_h, \xi_h) \in \mathbf{Q}_h$.

We now establish the main result of this section.

Theorem 5.1. *Let $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$, $\mathbf{f}_D \in \mathbf{L}^{3/2}(\Omega_D)$ and $g_D \in \mathrm{L}^2(\Omega_D)$, such that*

$$\mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D) < \frac{1}{2} \min\{r, \widetilde{r}\}, \tag{5.5}$$

where r and $\widetilde{r}$ are the constants defined in Lemma 3.11 and Theorem 4.9, respectively. Let $(\mathbf{u}, (p, \lambda)) := ((\mathbf{u}_S, \mathbf{u}_D), (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ and $(\mathbf{u}_h, (p_h, \lambda_h)) := ((\mathbf{u}_{S,h}, \mathbf{u}_{D,h}), (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ be the unique solutions of the continuous and discrete problems (2.14) and (4.1), respectively. Then there exists $C > 0$, independent of h and the continuous and discrete solutions, such that

$$\|(\mathbf{u}, (p, \lambda)) - (\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H} \times \mathbf{Q}} \leq C \max_{i \in \{2,3\}} \left\{ \left(\inf_{\mathbf{v}_h \in \mathbf{H}_h} \left(\|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} + \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}}^2 \right) + \inf_{(q_h, \xi_h) \in \mathbf{Q}_h} \|(p, \lambda) - (q_h, \xi_h)\|_{\mathbf{Q}} \right)^{\frac{1}{i-1}} \right\}. \tag{5.6}$$

Proof. In what follows we adapt the proof of Theorem 5 of [21] to the present case. To do that, we let $\overline{\mathbf{v}}_h = (\overline{\mathbf{v}}_{S,h}, \overline{\mathbf{v}}_{D,h}) \in \mathbf{H}_h^{\mathbf{g}}$ and $(\overline{q}_h, \overline{\xi}_h) \in \mathbf{Q}_h$, and define $\boldsymbol{\delta}_{\mathbf{u}_S}, \boldsymbol{\delta}_{\mathbf{u}_D}, \delta_p, \delta_\lambda, \boldsymbol{\eta}_{\mathbf{u}_S}, \boldsymbol{\eta}_{\mathbf{u}_D}, \eta_p$, and η_λ, as in (5.3). In addition, we recall that thanks to assumption (5.5), it follows that $\mathbf{u}_{\rm S} \in \mathbf{W}$ and $\mathbf{u}_{{\rm S},h} \in \mathbf{W}_h$ (*cf.* (3.26) and (4.30)), which implies (*cf.* Thm. 3.12 and 4.9):

$$\begin{aligned} \|\mathbf{u}_{\rm D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})},\ \|\mathbf{u}_{\rm S}\|_{1,\Omega_{\rm S}} &\leq c_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\rm S},\mathbf{f}_{\rm D},g_{\rm D}),\\ \|\mathbf{u}_{{\rm D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})},\ \|\mathbf{u}_{{\rm S},h}\|_{1,\Omega_{\rm S}} &\leq \widetilde{c}_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\rm S},\mathbf{f}_{\rm D},g_{\rm D}). \end{aligned} \tag{5.7}$$

In turn, since $\mathbf{u}_h, \overline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{g}}$, we observe that

$$(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}) := \overline{\mathbf{v}}_h - \mathbf{u}_h \in \mathbf{V}_h. \tag{5.8}$$

According to the above, we first note that for all $\mathbf{v}_{{\rm S},h} \in \mathbf{H}_{h,\Gamma_{\rm S}}(\Omega_{\rm S})$, there holds

$$\begin{aligned} \big[\mathcal{B}_{\rm S}^h(\mathbf{u}_{\rm S})(\mathbf{u}_{\rm S}),\mathbf{v}_{{\rm S},h}\big] - \big[\mathcal{B}_{\rm S}^h(\mathbf{u}_{{\rm S},h})(\mathbf{u}_{{\rm S},h}),\mathbf{v}_{{\rm S},h}\big] &= \big[\mathcal{B}_{\rm S}^h(\mathbf{e}_{\mathbf{u}_{\rm S}})(\mathbf{u}_{\rm S}),\mathbf{v}_{{\rm S},h}\big] + \big[\mathcal{B}_{\rm S}^h(\mathbf{u}_{{\rm S},h})(\mathbf{e}_{\mathbf{u}_{\rm S}}),\mathbf{v}_{{\rm S},h}\big]\\ &= \big[\mathcal{B}_{\rm S}^h(\mathbf{u}_{{\rm S},h})(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}),\mathbf{v}_{{\rm S},h}\big] + \mathcal{R}(\mathbf{v}_{{\rm S},h}), \end{aligned} \tag{5.9}$$

with

$$\mathcal{R}(\mathbf{v}_{{\rm S},h}) = \big[\mathcal{B}_{\rm S}^h(\mathbf{u}_{{\rm S},h})(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}),\mathbf{v}_{{\rm S},h}\big] + \big[\mathcal{B}_{\rm S}^h(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}})(\mathbf{u}_{\rm S}),\mathbf{v}_{{\rm S},h}\big] + \big[\mathcal{B}_{\rm S}^h(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}})(\mathbf{u}_{\rm S}),\mathbf{v}_{{\rm S},h}\big]\,.$$

Then, adding and subtracting suitable terms in the first equation of (5.4) with $\mathbf{v}_h = (\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}) \in \mathbf{V}_h$ (*cf.* (5.8)), and observing that $[\mathbf{b}(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}),(\eta_p,\eta_\lambda)] = 0$, we obtain

$$\begin{aligned} &[\mathbf{a}_h(\mathbf{u}_{{\rm S},h})(\overline{\mathbf{v}}_h) - \mathbf{a}_h(\mathbf{u}_{{\rm S},h})(\mathbf{u}_h), \overline{\mathbf{v}}_h - \mathbf{u}_h]\\ &\quad = -\,\big[\mathcal{A}_{\rm S}(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}),\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}\big] - \mathcal{R}(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}) - \big[\mathcal{A}_{\rm D}(\mathbf{u}_{\rm D}) - \mathcal{A}_{\rm D}(\overline{\mathbf{v}}_{{\rm D},h}),\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}\big] - \big[\mathbf{b}(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}),(\delta_p,\delta_\lambda)\big]\,. \end{aligned}$$

Hence, proceeding analogously to the proof of Lemma 3.4, using the continuity of $\mathcal{A}_{\rm S}$, $\mathcal{B}_{\rm S}^h$ and $\mathbf{b}$ (*cf.* (2.19) and (4.4)), and inequality (2.20), we deduce that

$$\begin{aligned} &\mu\alpha_{\rm S}\|\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}\|_{1,\Omega_{\rm S}}^2 + \alpha_{\rm D}\|\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})}^3\\ &\quad\leq \Big\{C_{\mathcal{A}_{\rm S}} + C_{\rm sk}\Big(\|\mathbf{u}_{{\rm S},h}\|_{1,\Omega_{\rm S}} + \|\mathbf{u}_{\rm S}\|_{1,\Omega_{\rm S}}\Big)\Big\}\|\boldsymbol{\delta}_{\mathbf{u}_{\rm S}}\|_{1,\Omega_{\rm S}}\|\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}\|_{1,\Omega_{\rm S}} + C_{\rm sk}\|\mathbf{u}_{\rm S}\|_{1,\Omega_{\rm S}}\|\boldsymbol{\eta}_{\mathbf{u}_{\rm S}}\|_{1,\Omega_{\rm S}}^2\\ &\qquad + L_{\mathcal{A}_{\rm D}}\Big\{\Big(1 + 2\|\mathbf{u}_{\rm D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})}\Big)\|\boldsymbol{\delta}_{\mathbf{u}_{\rm D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})} + \|\boldsymbol{\delta}_{\mathbf{u}_{\rm D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})}^2\Big\}\|\boldsymbol{\eta}_{\mathbf{u}_{\rm D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_{\rm D})}\\ &\qquad + C_{\mathbf{b}}\|(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}}\|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}}, \end{aligned}$$

which, together with (5.7) and assumption (5.5), implies that there exists $C > 0$, depending only on parameters, data and other constants, all of them independent of h, such that

$$\|(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} \leq C \max_{i\in\{2,3\}}\left\{\Big(\|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}}^2 + \|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}}\Big)^{\frac{1}{i-1}}\right\}. \tag{5.10}$$

In this way, from (5.2), (5.10), and the triangle inequality, we obtain

$$\begin{aligned} \|(\mathbf{e}_{\mathbf{u}_{\rm S}},\mathbf{e}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} &\leq \|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\boldsymbol{\eta}_{\mathbf{u}_{\rm S}},\boldsymbol{\eta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}}\\ &\leq \widetilde{C} \max_{i\in\{2,3\}}\left\{\Big(\|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}} + \|(\boldsymbol{\delta}_{\mathbf{u}_{\rm S}},\boldsymbol{\delta}_{\mathbf{u}_{\rm D}})\|_{\mathbf{H}}^2 + \|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}}\Big)^{\frac{1}{i-1}}\right\}. \end{aligned} \tag{5.11}$$

In turn, to estimate e_p and e_λ we observe that from the discrete inf-sup condition (4.27), the first equation of (5.4), and the first equation of (5.9), there holds

$$\begin{aligned} \widetilde{\beta}\|(\eta_p,\eta_\lambda)\|_{\mathbf{Q}} &\leq \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h),(\eta_p,\eta_\lambda)]}{\|\mathbf{v}_h\|_{\mathbf{H}}} = \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}_h),(e_p,e_\lambda)] - [\mathbf{b}(\mathbf{v}_h),(\delta_p,\delta_\lambda)]}{\|\mathbf{v}_h\|_{\mathbf{H}}}\\ &= \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h\\ \mathbf{v}_h\neq\mathbf{0}}} -\left\{\frac{[\mathcal{A}_{\rm S}(\mathbf{e}_{\mathbf{u}_{\rm S}}),\mathbf{v}_{{\rm S},h}] + [\mathcal{B}_{\rm S}^h(\mathbf{e}_{\mathbf{u}_{\rm S}})(\mathbf{u}_{\rm S}),\mathbf{v}_{{\rm S},h}] + [\mathcal{B}_{\rm S}^h(\mathbf{u}_{{\rm S},h})(\mathbf{e}_{\mathbf{u}_{\rm S}}),\mathbf{v}_{{\rm S},h}]}{\|\mathbf{v}_h\|_{\mathbf{H}}}\right.\\ &\qquad\left. + \frac{[\mathcal{A}_{\rm D}(\mathbf{u}_{\rm D}) - \mathcal{A}_{\rm D}(\mathbf{u}_{{\rm D},h}),\mathbf{v}_{{\rm D},h}] + [\mathbf{b}(\mathbf{v}_h),(\delta_p,\delta_\lambda)]}{\|\mathbf{v}_h\|_{\mathbf{H}}}\right\}. \end{aligned}$$

Then, the continuity of $\mathcal{A}_\mathrm{S}$, $\mathcal{B}_\mathrm{S}^h$, and $\mathbf{b}$ (*cf.* (2.19) and (4.4)), and the inequality (2.20), imply

$$\widetilde{\beta}\|(\eta_p,\eta_\lambda)\|_{\mathbf{Q}} \leq \Big\{C_{\mathcal{A}_\mathrm{S}} + C_{\mathrm{sk}}\Big(\|\mathbf{u}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} + \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_\mathrm{S}}\Big)\Big\}\|\mathbf{e}_{\mathbf{u}_\mathrm{S}}\|_{1,\Omega_\mathrm{S}} + L_{\mathcal{A}_\mathrm{D}}\Big\{1 + \|\mathbf{u}_\mathrm{D}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + \|\mathbf{u}_{\mathrm{D},h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})}\Big\}\|\mathbf{e}_{\mathbf{u}_\mathrm{D}}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} + C_{\mathbf{b}}\|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}},$$

which, together with assumption (5.5), inequalities (5.7) and (5.11), yield

$$\|(\eta_p,\eta_\lambda)\|_{\mathbf{Q}} \leq c \max_{i\in\{2,3\}}\left\{\Big(\|(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}},\boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}})\|_{\mathbf{H}} + \|(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}},\boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}})\|_{\mathbf{H}}^2 + \|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}}\Big)^{\frac{1}{i-1}}\right\}.$$

Thus, from (5.2), the triangle inequality, and the foregoing bound, we obtain

$$\begin{aligned}\|(e_p,e_\lambda)\|_{\mathbf{Q}} &\leq \|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}} + \|(\eta_p,\eta_\lambda)\|_{\mathbf{Q}}\\ &\leq \widetilde{c}\max_{i\in\{2,3\}}\left\{\Big(\|(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}},\boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}})\|_{\mathbf{H}} + \|(\boldsymbol{\delta}_{\mathbf{u}_\mathrm{S}},\boldsymbol{\delta}_{\mathbf{u}_\mathrm{D}})\|_{\mathbf{H}}^2 + \|(\delta_p,\delta_\lambda)\|_{\mathbf{Q}}\Big)^{\frac{1}{i-1}}\right\},\end{aligned}\tag{5.12}$$

where $\widetilde{c} > 0$ is independent of h. Therefore, recalling that $\overline{\mathbf{v}}_h \in \mathbf{H}_h^{\mathbf{g}}$ and $(\overline{q}_h,\overline{\lambda}_h) \in \mathbf{Q}_h$ are arbitrary, (5.11) and (5.12) give

$$\|((\mathbf{e}_{\mathbf{u}_\mathrm{S}},\mathbf{e}_{\mathbf{u}_\mathrm{D}}),(e_p,e_\lambda))\|_{\mathbf{H}\times\mathbf{Q}} \leq C\max_{i\in\{2,3\}}\left\{\left(\inf_{\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{g}}}\Big(\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}} + \|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}}^2\Big) + \inf_{(q_h,\xi_h)\in\mathbf{Q}_h}\|(p,\lambda)-(q_h,\xi_h)\|_{\mathbf{Q}}\right)^{\frac{1}{i-1}}\right\},$$

which, together with (5.1), concludes the proof. □

Now, in order to provide the theoretical rate of convergence of the Galerkin scheme (4.1), we recall the approximation properties of the subspaces involved (see, *e.g.*, [10, 23, 25, 27]). Note that each one of them is named after the unknown to which it is applied later on.

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{S}})$ For each $\mathbf{v}_\mathrm{S} \in \mathbf{H}^2(\Omega_\mathrm{S})$, there holds

$$\|\mathbf{v}_\mathrm{S} - \Pi_\mathrm{S}(\mathbf{v}_\mathrm{S})\|_{1,\Omega_\mathrm{S}} \leq Ch\|\mathbf{v}_\mathrm{S}\|_{2,\Omega_\mathrm{S}}.$$

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{D}})$ For each $\mathbf{v}_\mathrm{D} \in \mathbf{W}^{1,3}(\Omega_\mathrm{D})$ with $\mathrm{div}\,\mathbf{v}_\mathrm{D} \in \mathrm{H}^1(\Omega_\mathrm{D})$, there holds

$$\|\mathbf{v}_\mathrm{D} - \Pi_\mathrm{D}(\mathbf{v}_\mathrm{D})\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_\mathrm{D})} \leq Ch\Big\{\|\mathbf{v}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{v}_\mathrm{D}\|_{1,\Omega_\mathrm{D}}\Big\}.$$

$(\mathbf{AP}_h^{p})$ For each $q \in \mathrm{H}^1(\Omega)\cap\mathrm{L}_0^2(\Omega)$, there exists $q_h \in \mathrm{L}_{h,0}(\Omega)$ such that

$$\|q - q_h\|_{0,\Omega} \leq Ch\|q\|_{1,\Omega}.$$

$(\mathbf{AP}_h^{\lambda})$ For each $\xi \in \mathrm{W}^{1,\frac{3}{2}}(\Sigma)$, there exists $\xi_h \in \Lambda_h(\Sigma)$ such that

$$\|\xi - \xi_h\|_{\frac{1}{3},\frac{3}{2};\Sigma} \leq Ch^{2/3}\|\xi\|_{1,\frac{3}{2};\Sigma}.$$

The following theorem provides the theoretical sub-optimal rate of convergence of the Galerkin scheme (4.1), under suitable regularity assumptions on the exact solution.

Theorem 5.2. *Let* $\mathbf{f}_\mathrm{S} \in \mathbf{L}^2(\Omega_\mathrm{S}), \mathbf{f}_\mathrm{D} \in \mathbf{L}^{3/2}(\Omega_\mathrm{D})$ *and* $g_\mathrm{D} \in \mathrm{L}^2(\Omega_\mathrm{D})$*, such that* (5.5) *holds. Let* $(\mathbf{u},(p,\lambda)) := ((\mathbf{u}_\mathrm{S},\mathbf{u}_\mathrm{D}),(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ *and* $(\mathbf{u}_h,(p_h,\lambda_h)) := ((\mathbf{u}_{\mathrm{S},h},\mathbf{u}_{\mathrm{D},h}),(p_h,\lambda_h)) \in \mathbf{H}_h\times\mathbf{Q}_h$ *be the unique solutions of the continuous and discrete problems* (2.14) *and* (4.1)*, respectively, and assume that* $\mathbf{u}_\mathrm{S} \in \mathbf{H}^2(\Omega_\mathrm{S})$*,* $\mathbf{u}_\mathrm{D} \in \mathbf{W}^{1,3}(\Omega_\mathrm{D})$*,*

$\mathrm{div}\,\mathbf{u}_\mathrm{D} \in \mathrm{H}^1(\Omega_\mathrm{D})$, $p \in \mathrm{H}^1(\Omega)$, *and* $\lambda \in \mathrm{W}^{1,\frac{3}{2}}(\Sigma)$. *Then, there exists* $C > 0$, *independent of* h *and the continuous and discrete solutions, such that*

$$\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \le C h^{1/3} \max_{i\in\{2,3\}} \Bigg\{ \Big(\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}} + \|p\|_{1,\Omega} + \|\lambda\|_{1,\frac{3}{2};\Sigma} \Big)^{\frac{1}{i-1}} \Bigg\}. \tag{5.13}$$

Proof. From $(\mathbf{AP}_h^{\mathbf{u}_\mathrm{S}})$ and $(\mathbf{AP}_h^{\mathbf{u}_\mathrm{D}})$, it is not difficult to see that

$$\begin{aligned}\inf_{\mathbf{v}_h\in\mathbf{H}_h} \Big(\|\mathbf{u}-\mathbf{v}_h\|_\mathbf{H} + \|\mathbf{u}-\mathbf{v}_h\|_\mathbf{H}^2 \Big) &\le Ch\Big(\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}} \Big) \\ &\quad + 2C^2 h^2 \Big(\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}}^2 + (\|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}})^2 \Big) \\ &= ChA\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}} + ChB(\|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}}),\end{aligned}$$

with $A = 1+2Ch\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}}$ and $B = 1+2Ch(\|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}})$. Then, using the fact that for sufficiently small values of h there hold $A \le c$ and $B \le c$, with $c > 0$ independent of h, from the above inequality it follows that

$$\inf_{\mathbf{v}_h\in\mathbf{H}_h} \Big(\|\mathbf{u}-\mathbf{v}_h\|_\mathbf{H} + \|\mathbf{u}-\mathbf{v}_h\|_\mathbf{H}^2 \Big) \le Ch\Big(\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}} \Big). \tag{5.14}$$

In turn, from $(\mathbf{AP}_h^p)$ and $(\mathbf{AP}_h^\lambda)$, we have that

$$\inf_{(q_h,\xi_h)\in\mathbf{Q}_h} \|(p,\lambda) - (q_h,\xi_h)\|_\mathbf{Q} \le C\Big(h\|p\|_{1,\Omega} + h^{2/3}\|\lambda\|_{1,\frac{3}{2};\Sigma} \Big),$$

which together with (5.14), implies

$$\begin{aligned}\inf_{\mathbf{v}_h\in\mathbf{H}_h} \Big(\|\mathbf{u}-\mathbf{v}_h\|_\mathbf{H} + \|\mathbf{u}-\mathbf{v}_h\|_\mathbf{H}^2 \Big) + \inf_{(q_h,\xi_h)\in\mathbf{Q}_h} \|(p,\lambda) - (q_h,\xi_h)\|_\mathbf{Q} &\le Ch\Big(\|\mathbf{u}_\mathrm{S}\|_{2,\Omega_\mathrm{S}} + \|\mathbf{u}_\mathrm{D}\|_{1,3;\Omega_\mathrm{D}} \\ &\quad + \|\mathrm{div}\,\mathbf{u}_\mathrm{D}\|_{1,\Omega_\mathrm{D}} + \|\mathrm{p}\|_{1,\Omega} \Big) + \mathrm{Ch}^{2/3}\|\lambda\|_{1,\frac{3}{2};\Sigma}.\end{aligned}$$

In this way, from the latter and (5.6) we obtain the desired result. ☐

6. Numerical results

In this section we present some examples illustrating the performance of our mixed finite element scheme (4.1) on a set of quasi-uniform triangulations of the corresponding domains. Our implementation is based on a *FreeFem++* code [39], in conjunction with the direct linear solver UMFPACK [19].

In order to solve the nonlinear problem (4.1), given $\mathbf{w}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$ we introduce the Gâteaux derivative associated to $\mathcal{A}_\mathrm{D}$ (*cf.* (2.16)), *i.e.*,

$$\mathcal{D}\mathcal{A}_\mathrm{D}(\mathbf{w}_\mathrm{D})(\mathbf{u}_\mathrm{D},\mathbf{v}_\mathrm{D}) := \frac{\mu}{\rho}\left(\mathbf{K}^{-1}\mathbf{u}_\mathrm{D},\mathbf{v}_\mathrm{D}\right)_\mathrm{D} + \frac{F}{\rho}\left(|\mathbf{w}_\mathrm{D}|\mathbf{u}_\mathrm{D},\mathbf{v}_\mathrm{D}\right)_\mathrm{D} + \frac{F}{\rho}\left(\frac{\mathbf{w}_\mathrm{D}\cdot\mathbf{u}_\mathrm{D}}{|\mathbf{w}_\mathrm{D}|}, \mathbf{w}_\mathrm{D}\cdot\mathbf{v}_\mathrm{D}\right)_\mathrm{D},$$

for all $\mathbf{u}_\mathrm{D},\mathbf{v}_\mathrm{D} \in \mathbf{H}^3_{\Gamma_\mathrm{D}}(\mathrm{div}\,;\Omega_\mathrm{D})$. In this way, we propose the Newton-type strategy: Given $\mathbf{u}_h^0 = (\mathbf{u}_{\mathrm{S},h}^0, \mathbf{u}_{\mathrm{D},h}^0) \in \mathbf{H}_h$, $p_h^0 \in \mathrm{L}_{h,0}(\Omega)$ and $\lambda_h^0 \in \Lambda_h(\Sigma)$, for $m \ge 1$, find $\mathbf{u}_h^m = (\mathbf{u}_{\mathrm{S},h}^m, \mathbf{u}_{\mathrm{D},h}^m) \in \mathbf{H}_h$, $p_h^m \in \mathrm{L}_{h,0}(\Omega)$ and $\lambda_h^m \in \Lambda_h(\Sigma)$, such that

$$\begin{aligned}&[\mathcal{A}_\mathrm{S}(\mathbf{u}_{\mathrm{S},h}^m),\mathbf{v}_{\mathrm{S},h}] + [\mathcal{B}_\mathrm{S}^h(\mathbf{u}_{\mathrm{S},h}^{m-1})(\mathbf{u}_{\mathrm{S},h}^m),\mathbf{v}_{\mathrm{S},h}] + [\mathcal{B}_\mathrm{S}^h(\mathbf{u}_{\mathrm{S},h}^m)(\mathbf{u}_{\mathrm{S},h}^{m-1}),\mathbf{v}_{\mathrm{S},h}] + \mathcal{D}\mathcal{A}_\mathrm{D}(\mathbf{u}_{\mathrm{D},h}^{m-1})(\mathbf{u}_{\mathrm{D},h}^m,\mathbf{v}_{\mathrm{D},h}) \\ &\quad + [\mathbf{b}(\mathbf{v}_h),(p_h^m,\lambda_h^m)] = [\mathcal{B}_\mathrm{S}^h(\mathbf{u}_{\mathrm{S},h}^{m-1})(\mathbf{u}_{\mathrm{S},h}^{m-1}),\mathbf{v}_{\mathrm{S},h}] + \frac{F}{\rho}\left(|\mathbf{u}_{\mathrm{D},h}^{m-1}|\mathbf{u}_{\mathrm{D},h}^{m-1},\mathbf{v}_{\mathrm{D},h}\right)_\mathrm{D} + [\mathbf{f},\mathbf{v}_h] \\ &\quad [\mathbf{b}(\mathbf{u}_h^m),(q_h,\xi_h)] = [\mathbf{g},(q_h,\xi_h)]\end{aligned} \tag{6.1}$$

for all $\mathbf{v}_h = (\mathbf{v}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{H}_h$ and $(q_h, \xi_h) \in \mathbf{Q}_h$.

In all the numerical experiments below, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, *i.e.*,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2-norm in $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h$ and $\mathbf{Q}_h$, and tol is a fixed tolerance chosen as $tol = 1E-06$. For each example shown below we simply take $\mathbf{u}_h^0 = (\mathbf{0}, (0.1, 0))$ and $(p_h^0, \lambda_h^0) = \mathbf{0}$ as initial guess. As usual, the individual errors are denoted by:

$$\mathrm{e}(\mathbf{u}_S) := \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{1,\Omega_S}, \quad \mathrm{e}(\mathbf{u}_D) := \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathbf{H}^3(\mathrm{div}\,;\Omega_D)},$$
$$\mathrm{e}(p_S) := \|p_S - p_{S,h}\|_{0,\Omega_S}, \quad \mathrm{e}(p_D) := \|p_D - p_{D,h}\|_{0,\Omega_D}, \quad \mathrm{e}(\lambda) := \|\lambda - \lambda_h\|_{\mathrm{L}^{3/2}(\Sigma)}.$$

Notice that we considered $\|\lambda - \lambda_h\|_{\mathrm{L}^{3/2}(\Sigma)}$ in place of $\|\lambda - \lambda_h\|_{\frac{1}{3},\frac{3}{2};\Sigma}$ because of the last norm is not computable. Notice also that $\|\lambda - \lambda_h\|_{\mathrm{L}^{3/2}(\Sigma)}$ satisfies the sub-optimal rate of convergence (5.13). Next, we define the experimental rates of convergence

$$r(\mathbf{u}_S) := \frac{\log(e(\mathbf{u}_S)/e'(\mathbf{u}_S))}{\log(h_S/h'_S)}, \quad r(\mathbf{u}_D) := \frac{\log(e(\mathbf{u}_D)/e'(\mathbf{u}_D))}{\log(h_D/h'_D)},$$
$$r(p_S) := \frac{\log(e(p_S)/e'(p_S))}{\log(h_S/h'_S)}, \quad r(p_D) := \frac{\log(e(p_D)/e'(p_D))}{\log(h_D/h'_D)}, \quad r(\lambda) := \frac{\log(e(\lambda)/e'(\lambda))}{\log(h_\Sigma/h'_\Sigma)},$$

where $h_\star$ and $h'_\star$ ($\star \in \{S, D, \Sigma\}$) denote two consecutive mesh sizes with their respective errors e and e', respectively.

The examples to be considered in this section are described next. In all of them, for the sake of simplicity, we choose the parameters $\mu = 1$, $\rho = 1$, $\alpha_d = 1$, $\boldsymbol{\kappa} = \mathbb{I}$, and $\mathbf{K} = \mathbb{I}$. In addition, the condition $\int_\Omega p_h = 0$ is imposed *via* a penalization strategy.

6.1. Example 1: Tombstone-shaped domain without source in the porous media

In our first example we consider a semi-disk-shaped fluid domain coupled with a porous unit square, *i.e.*, $\Omega_S := \Big\{(x_1, x_2) : x_1^2 + (x_2 - 0.5)^2 < 0.5^2, x_2 > 0.5\Big\}$ and $\Omega_D := (-0.5, 0.5)^2$. We consider the Forchheimer number $F = 1$ and the data $\mathbf{f}_S, \mathbf{f}_D$, and g_D, are adjusted so that the exact solution in the tombstone-shaped domain $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$ is given by the smooth functions

$$\mathbf{u}_S(x_1, x_2) = \begin{pmatrix} \pi\cos(\pi x_1)\sin(\pi x_2) \\ -\pi\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_S,$$
$$\mathbf{u}_D(x_1, x_2) = \begin{pmatrix} \pi\sin(\pi x_2)\exp(x_1) \\ \cos(\pi x_2)\exp(x_1) \end{pmatrix} \quad \text{in} \quad \Omega_D,$$
$$p_\star(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2) \quad \text{in} \quad \Omega_\star, \quad \text{with } \star \in \{S, D\}.$$

Notice that the source of the porous media is $g_D = 0$. Notice also that this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ. However, the Beavers–Joseph–Saffman condition (*cf.* (2.4)) is not satisfied, the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_S and the Neumann boundary condition for the Darcy–Forchheimer velocity on Γ_D are both non-homogeneous. In this way, the right-hand side of the resulting system must be modified accordingly.

TABLE 1. Example 1: Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $F = 1$.

N	h_S	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(p_S)$	$r(p_S)$
691	0.1915	0.4439	–	0.1588	–
2491	0.0911	0.2293	0.8896	0.0725	1.0561
9562	0.0486	0.1188	1.0441	0.0382	1.0179
37 815	0.0242	0.0531	1.1558	0.0175	1.1214
149 693	0.0134	0.0288	1.0380	0.0094	1.0474
588 445	0.0078	0.0147	1.2290	0.0048	1.2231
N	h_D	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$
691	0.1901	0.3481	–	0.0643	–
2491	0.0978	0.1678	1.0974	0.0305	1.1202
9562	0.0535	0.0856	1.1169	0.0151	1.1629
37 815	0.0249	0.0427	0.9122	0.0075	0.9206
149 693	0.0145	0.0214	1.2713	0.0037	1.2840
588 445	0.0068	0.0107	0.9140	0.0019	0.9087
N	h_Σ	$e(\lambda)$	$r(\lambda)$	iter	
691	0.1250	0.0718	–	7	
2491	0.0625	0.0352	1.0308	7	
9562	0.0313	0.0175	1.0084	8	
37 815	0.0156	0.0087	1.0060	8	
149 693	0.0078	0.0043	1.0012	8	
588 445	0.0039	0.0022	1.0004	8	

6.2. Example 2: Rectangle domain with a Kovasznay solution

In our second example we consider a rectangular domain $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$, with $\Omega_S := (-0.5, 1.5) \times (0, 0.5)$ and $\Omega_D := (-0.5, 1.5) \times (-0.5, 0)$. We consider the Forchheimer number $F = 1$ and the data $\mathbf{f}_S, \mathbf{f}_D$, and g_D, are adjusted so that the exact solution in the rectangle domain Ω is given by the smooth functions

$$\mathbf{u}_S(x_1, x_2) = \begin{pmatrix} 1 - \exp(\omega x_1)\cos(2\pi x_2) \\ \dfrac{\omega}{2\pi}\exp(\omega x_1)\sin(2\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_S,$$

$$\mathbf{u}_D(x_1, x_2) = \begin{pmatrix} (x_1 + 0.5)(x_1 - 1.5)\exp(x_2) \\ (x_2 + 2)(2x_2 + 1)\exp(x_1) \end{pmatrix} \quad \text{in} \quad \Omega_D,$$

$$p_\star(x_1, x_2) = -\frac{1}{2}\exp(2\omega x_1) + p_0 \quad \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{S, D\},$$

and

$$\omega = \frac{-8\pi^2}{\mu^{-1} + \sqrt{\mu^{-1} + 16\pi^2}}.$$

The constant p_0 is such that $\int_\Omega p = 0$. Notice that $(\mathbf{u}_S, p_S)$ is the well known analytical solution for the Navier–Stokes problem obtained by Kovasznay in [41], which presents a boundary layer at $\{-0.5\} \times (-0.5, 0.5)$. Notice also that in this example both the conservation of mass and the Beavers–Joseph–Saffman boundary conditions (*cf.* (2.4)) are not satisfied and the right-hand side of the resulting system must be modified accordingly.

6.3. Example 3: 2D helmet-shaped domain with different Forchheimer numbers

In our last example we focus on the performance of the iterative method (6.1) with respect to the Forchheimer number F. To that end, and motivated by Section 2 of [14], we consider a 2D helmet-shaped domain. More

Table 2. Example 2: Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $F = 1$.

N	h_{S}	$\mathrm{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathrm{e}(p_{\mathrm{S}})$	$r(p_{\mathrm{S}})$
989	0.2001	10.3170	–	8.2614	–
3880	0.0966	4.5495	1.1249	3.9855	1.0015
13 888	0.0492	2.2051	1.0713	1.8753	1.1151
55 727	0.0270	1.1168	1.1342	0.9489	1.1357
213 833	0.0161	0.5456	1.3877	0.4746	1.3423
858 658	0.0078	0.2769	0.9419	0.2404	0.9444
N	h_{D}	$\mathrm{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathrm{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$
989	0.2001	0.4678	–	7.2964	–
3880	0.0950	0.2249	0.9835	3.3197	1.0578
13 888	0.0500	0.1145	1.0518	1.7322	1.0135
55 727	0.0254	0.0569	1.0326	0.9133	0.9457
213 833	0.0160	0.0278	1.5453	0.4353	1.5956
858 658	0.0066	0.0141	0.7674	0.2295	0.7283
N	h_{Σ}	$\mathrm{e}(\lambda)$	$r(\lambda)$	iter	
989	0.1250	8.9940	–	6	
3880	0.0625	4.6538	0.9505	6	
13 888	0.0313	2.3459	0.9883	6	
55 727	0.0156	1.1788	0.9928	6	
213 833	0.0078	0.5962	0.9835	6	
858 658	0.0039	0.3078	0.9539	6	

Table 3. Example 3: Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $F = 10$.

N	h_{S}	$\mathrm{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathrm{e}(p_{\mathrm{S}})$	$r(p_{\mathrm{S}})$
1007	0.1881	1.0274	–	0.5355	–
3790	0.1088	0.5114	1.2753	0.2156	1.6636
14 014	0.0481	0.2472	0.8896	0.0978	0.9668
55 428	0.0254	0.1243	1.0742	0.0483	1.1028
214 828	0.0137	0.0620	1.1285	0.0237	1.1564
883 963	0.0077	0.0307	1.2174	0.0123	1.1392
N	h_{D}	$\mathrm{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathrm{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$
1007	0.2001	1.2760	–	0.1105	–
3790	0.0950	0.6135	0.9837	0.0385	1.4165
14 014	0.0494	0.3115	1.0366	0.0150	1.4375
55 428	0.0262	0.1566	1.0813	0.0067	1.2820
214 828	0.0146	0.0784	1.1839	0.0033	1.2215
883 963	0.0072	0.0393	0.9815	0.0016	0.9948
N	h_{Σ}	$\mathrm{e}(\lambda)$	$r(\lambda)$	iter	
1007	0.1250	0.1930	–	7	
3790	0.0625	0.0704	1.4545	8	
14 014	0.0313	0.0296	1.2527	9	
55 428	0.0156	0.0141	1.0638	9	
214 828	0.0078	0.0070	1.0217	9	
883 963	0.0039	0.0035	1.0093	9	

TABLE 4. Example 3: Convergence behavior of the iterative method (6.1) with respect to the Forchheimer number F.

F	$h = 0.2001$	$h = 0.1088$	$h = 0.0494$	$h = 0.0262$	$h = 0.0146$	$h = 0.0077$
0	4	4	4	4	4	4
1	5	5	5	6	6	6
10	7	8	9	9	9	9
100	8	9	10	10	11	11

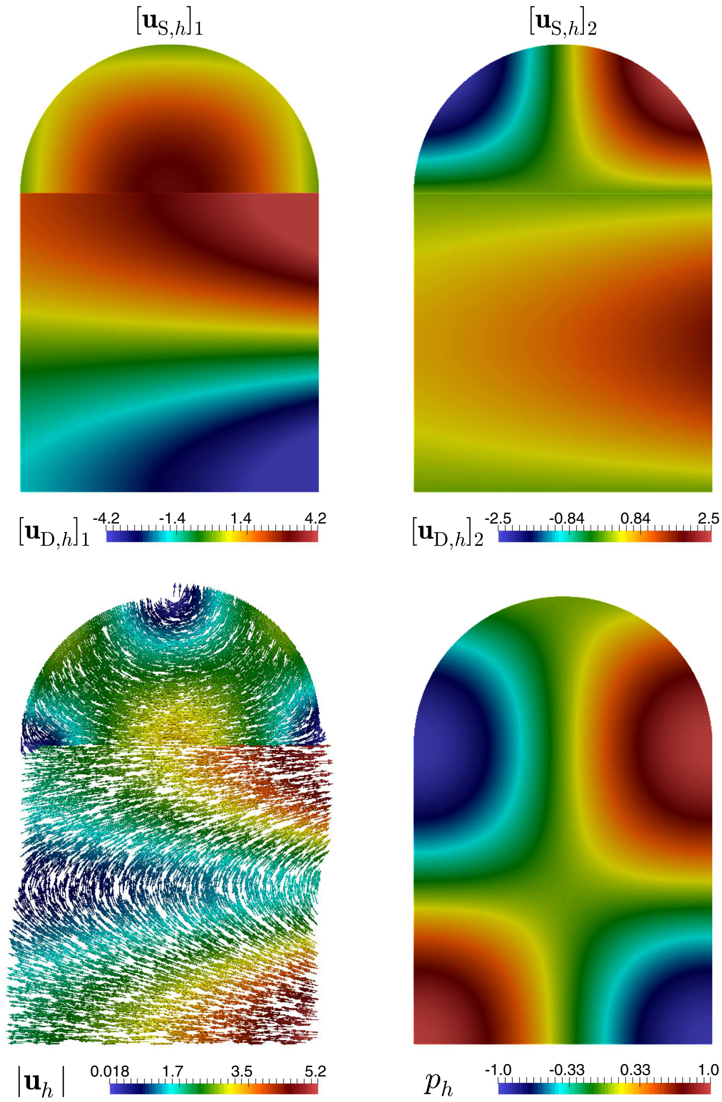

FIGURE 2. Example 1: Velocity components (*top panels*), velocity streamlines and pressure field in the whole domain (*bottom panels*).

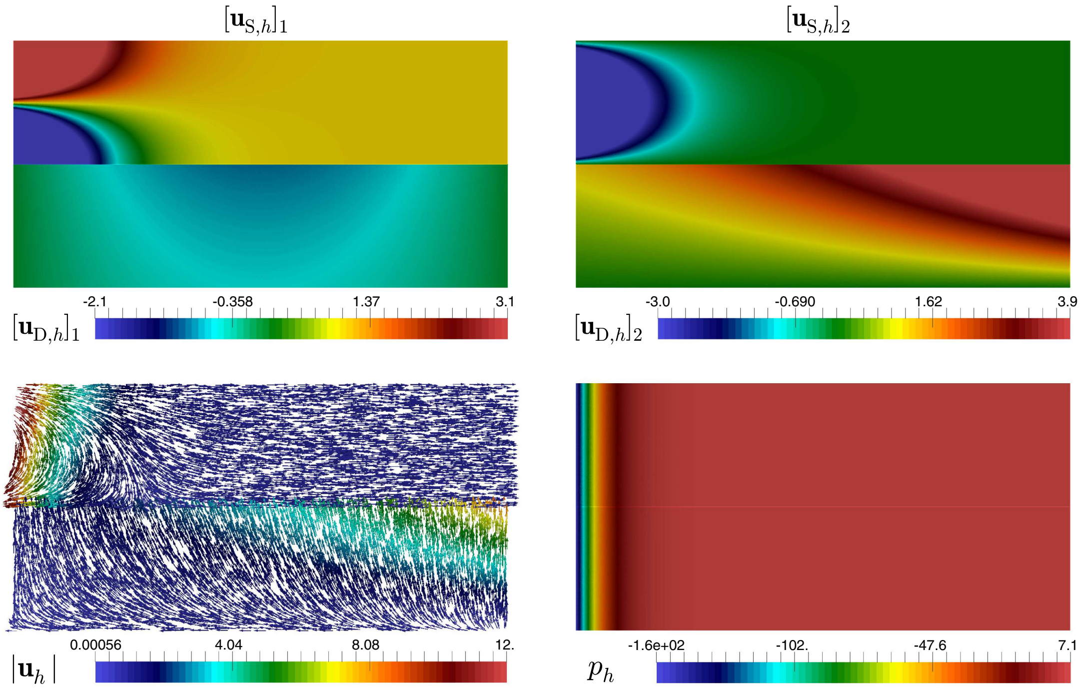

FIGURE 3. Example 2: Velocity components (*top panels*), velocity streamlines and pressure field in the whole domain (*bottom panels*).

precisely, we consider the domain $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$, where $\Omega_D := (-1,1) \times (-0.5,0)$ and $\Omega_S := (-1,-0.75) \times (0,1.25) \cup \Omega_{S,1} \cup (-0.5,0.5) \times (0,0.25) \cup \Omega_{S,2} \cup (0.75,1) \times (0,1.25)$, with

$$\Omega_{S,1} := \Big\{(x_1,x_2) : \, (x_1+0.5)^2 + (x_2-0.5)^2 > 0.25^2, \, -0.75 < x_1 < -0.5, \, x_2 > 0\Big\}$$

and

$$\Omega_{S,2} := \Big\{(x_1,x_2) : \, (x_1-0.5)^2 + (x_2-0.5)^2 > 0.25^2, \, 0.5 < x_1 < 0.75, \, x_2 > 0\Big\}.$$

The data $\mathbf{f}_S, \mathbf{f}_D$, and g_D, are chosen so that the exact solution in the 2D helmet-shaped domain Ω is given by the smooth functions

$$\begin{aligned}
\mathbf{u}_S(x_1,x_2) &= \begin{pmatrix} -\sin(2\pi x_1)\cos(2\pi x_2) \\ \cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix} && \text{in} \quad \Omega_S, \\
\mathbf{u}_D(x_1,x_2) &= \begin{pmatrix} \sin(2\pi x_1)\exp(x_2) \\ \sin(2\pi x_2)\exp(x_1) \end{pmatrix} && \text{in} \quad \Omega_D, \\
p_\star(x_1,x_2) &= \sin(\pi x_1)\exp(x_2) + p_0 && \text{in} \quad \Omega_\star, \qquad \text{with } \star \in \{S, D\}.
\end{aligned}$$

The constant p_0 is such that $\int_\Omega p = 0$. Notice that, this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ and $\mathbf{u}_D \cdot \mathbf{n} = 0$ on Γ_D. However, the Beavers–Joseph–Saffman condition (*cf.* (2.4)) is not satisfied and the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_S is non-homogeneous and therefore the right-hand side of the resulting system must be modified accordingly.

In Tables 1–3 we summarise the convergence history for a sequence of quasi-uniform triangulations, considering the finite element spaces introduced in Section 4.1, and solving the nonlinear problem (6.1), which require around eight, six and nine Newton iterations for the Examples 1, 2 and 3, respectively. We observe that the

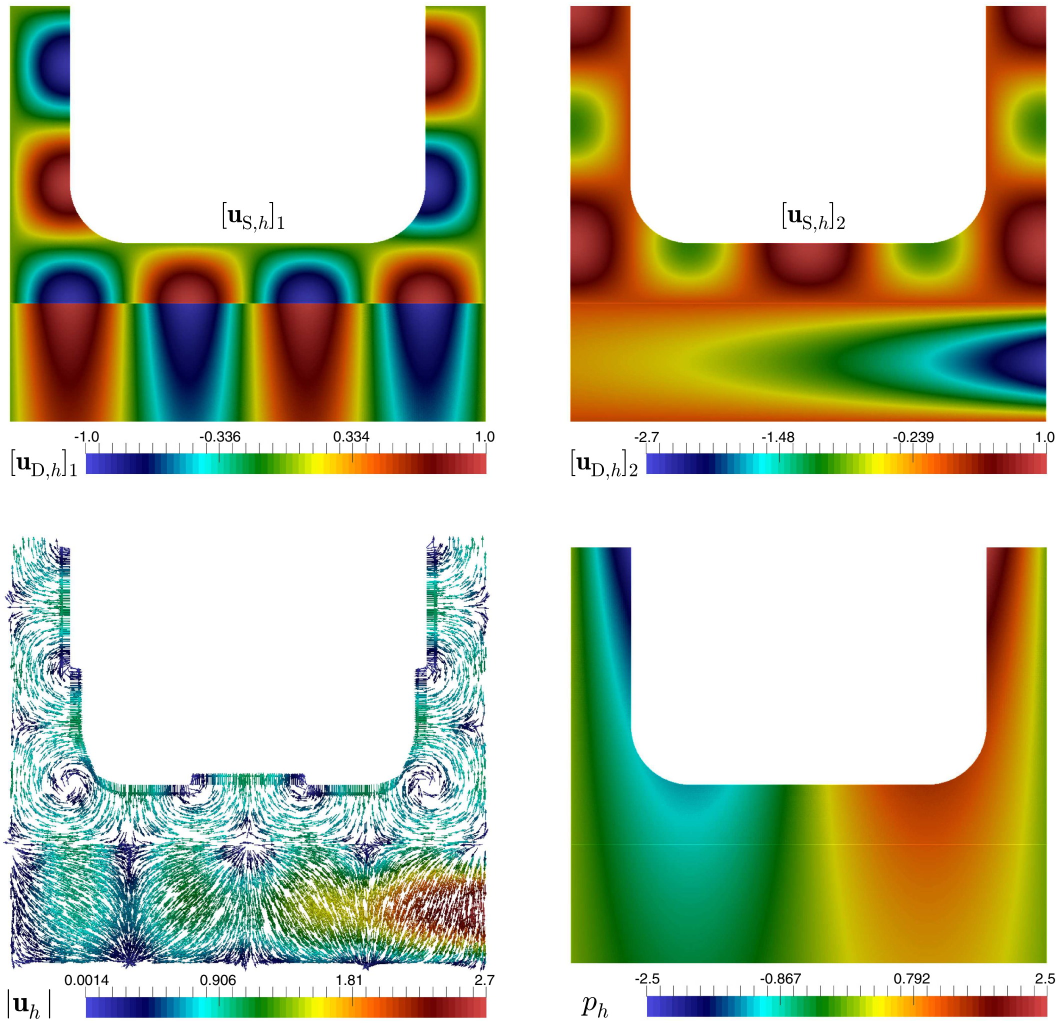

FIGURE 4. Example 3: Velocity components (*top panels*), velocity streamlines and pressure field in the whole domain (*bottom panels*).

sub-optimal rate of convergence $O(h^{1/3})$ provided by Theorem 5.2 is attained in all the cases. Even more, the numerical result suggest that there exist a way to prove optimal rate of convergence $O(h)$. In Table 4 we show the behaviour of the iterative method (6.1) as a function of the Forchheimer number F, considering different mesh sizes $h := \max\{h_S, h_D\}$, and a tolerance $tol = 1E-06$. Here we observe that the higher the parameter F the higher the number of iterations as it occurs also in the Newton method for the Navier–Stokes/Darcy–Forchheimer coupled problem. Notice also that when $F = 0$ the Darcy–Forchheimer equations reduce to the classical linear Darcy equations and as expected the iterative Newton method (6.1) is faster.

On the other hand, the velocity components, velocity streamlines and pressure field in the whole domain of the approximate solutions for the three examples are displayed in Figures 2–4. All the figures were obtained with 588 445, 858 658, and 883 963 degrees of freedom for the Examples 1, 2, and 3, respectively. In particular, we can observe in Figure 2 that the second components of $\mathbf{u}_S$ and $\mathbf{u}_D$ coincide on Σ as expected, and hence, the continuity of the normal components of the velocities on Σ is preserved. In turn, we can see that the velocity streamlines are higher in the Darcy–Forchheimer domain. Moreover, it can be seen that the pressure is continuous in the whole domain and preserves the sinusoidal behaviour. Next, in Figure 3 we observe that the pressure presents a boundary layer at $\{-0.5\} \times (-0.5, 0.5)$ as expected. Finally, similarly to Figure 2, in Figure 4 we can also observe the continuity of the normal components of the velocities on Σ since their second components coincide on the interface.

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