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ON TIME-SPLITTING METHODS FOR NONLINEAR SCHRÖDINGER EQUATION WITH HIGHLY OSCILLATORY POTENTIAL

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Abstract. In this work, we consider the numerical solution of the nonlinear Schrödinger equation with a highly oscillatory potential (NLSE-OP). The NLSE-OP is a model problem which frequently occurs in recent studies of some multiscale dynamical systems, where the potential introduces wide temporal oscillations to the solution and causes numerical difficulties. We aim to analyze rigorously the error bounds of the splitting schemes for solving the NLSE-OP to a fixed time. Our theoretical results show that the Lie–Trotter splitting scheme is uniformly and optimally accurate at the first order provided that the oscillatory potential is integrated exactly, while the Strang splitting scheme is not. Our results apply to general dispersive or wave equations with an oscillatory potential. The error estimates are confirmed by numerical results.

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1. Introduction

In quantum and plasma physics, to observe and address particular physical phenomenon, parameters of different scales are frequently introduced into the related dispersive or kinetic modelling equations. These parameters (often scaled to be small) cause multiscale behavior in the solution. For example, the non-relativistic limit regime of the Klein–Gordon equation [7, 9], the subsonic limit regime of the Zakharov system [3, 5] and the strong magnetic field regime of the Vlasov equation [23, 24] all induce high temporal oscillations of the corresponding solutions. This kind of problems, usually after some suitable change of variable which filters out the stiffest part in the equation, can be reformulated into a highly oscillatory problem:

$$\dot{u}(t) = f(t/\varepsilon, u(t)), \quad t > 0, \tag{1.1}$$

where $\varepsilon \in (0, 1]$. The common fact that $\ddot{u}(t) = O(1/\varepsilon)$ makes standard numerical integration schemes based on finite difference discretization suffering from low efficiency, since the time step has to be restricted by ε. Therefore, efforts have been made in recent research to design uniformly accurate methods aiming to overcome

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the time step size restriction [5,7,9,19,24]. To gain a uniform accuracy which means that the error is independent of ε, the proposed numerical schemes are equipped with multiscale techniques such as asymptotic expansion [5, 7, 22], two-scale formulation [19, 24] or iterative strategy [9]. However, these multiscale techniques would break the intrinsic structure such as symmetry and Hamiltonian in the original model, which lead to numerical schemes missing ideal long-time performance. Moreover, for some problems the multiscale methods may be over sophisticated to some extent.

In this paper, we shall investigate a simple case of (1.1) where the dependence on t/ε is explicit and linear, by considering the initial value problem of the periodic nonlinear Schrödinger equation with an external highly oscillatory potential (NLSE-OP):

$$i\partial_t u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + \left(V^\varepsilon(\mathbf{x},t) + f(|u(\mathbf{x},t)|^2)\right) u(\mathbf{x},t), \qquad t>0, \quad \mathbf{x}\in\mathbb{T}^d, \tag{1.2a}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{T}^d. \tag{1.2b}$$

Here $d = 1,2,3$, $\mathbb{T} = [-\pi,\pi]$, $u := u(\mathbf{x},t)$ is the complex-valued unknown, and u_0 is the given initial value, $V^\varepsilon(\mathbf{x},t)$ is a given real-valued smooth function with dependence on a parameter $0<\varepsilon\le 1$, periodic in $\mathbf{x}\in\mathbb{T}^d$ and $f\in C^\infty(\mathbb{R},\mathbb{R})$ is the given nonlinearity. The potential function $V^\varepsilon(\mathbf{x},t)$ is assumed to oscillate in time with frequency inversely proportional to ε and

$$\partial_t^k V^\varepsilon = O(\varepsilon^{-k}), \quad k\in\mathbb{N}, \quad 0<\varepsilon\ll 1. \tag{1.3}$$

The model problem (1.2) is essentially motivated from the recent numerical study of the subsonic limit of the Zakharov system [5], where the highly oscillatory potential represents the fast out-going initial layer. Problems with a similar form as (1.2) would also be encountered in situations such as the simultaneously high-plasma-frequency and subsonic limit regime of the Klein–Gordon–Zakharov system [6, 34], the NLS formulation of the nonlinear Klein–Gordon equation in the non-relativistic limit regime [9] and also the rapidly rotating regime of the Klein–Gordon equation in Lagrangian coordinates [35]. In case that $V^\varepsilon(\mathbf{x},t)$ could be integrated exactly in time, the operator splitting schemes which are undoubtedly one of the most popular numerical methods for solving Schrödinger type equations, can give promising numerical approximations to (1.2) in terms of accuracy for all $\varepsilon\in(0,1]$. In fact, a multiscale scheme based on a splitting solver for the NLSE-OP displays a uniform accuracy for solving the Zakharov system in the subsonic limit regime [5], where the fast out-going wave type potential V^ε in such case acts only for a very short time. For the NLSE under other important highly oscillatory situations, we refer to the work [18, 20, 31].

In this work, we are going to rigorously analyze the error bounds of the classical splitting schemes: the Lie–Trotter splitting and the Strang splitting for solving the NLSE-OP (1.2) with a general highly oscillatory potential (1.3). Note that the torus setup in (1.2) ensures that the potential V^ε keeps interacting with the solution even if it is a traveling wave. The convergence analysis of splitting methods for Schrödinger type problems has been widely carried out in the literature [11, 21, 27, 32, 37, 38]. By borrowing some of the state-of-the-art techniques, in this work we make special efforts to derive the dependence of the error bound on the wavelength ε in the potential V^ε up to a fixed time. We shall show through theoretical error estimates and numerical tests that the Lie–Trotter splitting scheme gives uniform first order accuracy in solving (1.2) for all $\varepsilon\in(0,1]$ if the integration of the oscillatory potential is exact, while the Strang splitting scheme fails to reach its optimal second order accuracy for all $\varepsilon\in(0,1]$. However, thanks to the exact integration of the oscillatory potential, the splitting schemes still give much more accurate approximations than the exponential integrators (or called as trigonometric integrators) [2, 40, 44] in the highly oscillatory regime. Though we focus on the NLSE-OP (1.2), our analysis also applies to general dispersive or wave equations with oscillatory potentials. The extension to the nonlinear Klein–Gordon equation with an oscillatory potential could be potentially used to design uniformly accurate schemes for the Klein–Gordon–Zakharov system in the simultaneously high-plasma-frequency and subsonic limit regime.

The rest of the paper is organized as follows. In Section 2, we present the splitting schemes for solving (1.2) and the corresponding convergence results. Sections 3 and 4 are devoted to proving the error estimates of the Lie–Trotter splitting scheme and the Strang splitting scheme, respectively. Numerical results are given in Section 5 to illustrate the theoretical results. Conclusions are drawn in Section 6.

2. Splitting schemes

In this section, we shall briefly present the Lie–Trotter splitting scheme and the Strang splitting scheme for solving the NLSE-OP, and then present the main results on the error estimates.

2.1. Schemes and notations

We denote in the following $\tau = \Delta t > 0$ and $t_n = n\tau$. As is widely used, the time splitting method splits the NLSE-OP (1.2) for some $t = s + t'$ into subproblems

$$\begin{cases} i\partial_s v(\mathbf{x}, s+t') = \Delta v(\mathbf{x}, s+t'), & \mathbf{x} \in \mathbb{T}^d,\ s > 0, \\ v(\mathbf{x}, t') = v_0(\mathbf{x}), & \mathbf{x} \in \mathbb{T}^d, \end{cases} \tag{2.1}$$

and

$$\begin{cases} i\partial_s w(\mathbf{x}, s+t') = \left(V^\varepsilon(\mathbf{x}, s+t') + f(|w(\mathbf{x}, s+t')|^2)\right) w(\mathbf{x}, s+t'), & \mathbf{x} \in \mathbb{T}^d,\ s > 0, \\ w(\mathbf{x}, t') = w_0(\mathbf{x}), & \mathbf{x} \in \mathbb{T}^d. \end{cases} \tag{2.2}$$

Note in (2.2) $|w(\mathbf{x}, s+t')| \equiv |w_0(\mathbf{x})|$ for all $s \geq 0$, since V^ε and f are real-valued. The exact solutions of the subproblems (2.1) and (2.2) can be written explicitly as

$$v(\mathbf{x}, s+t') = \varphi_T^s(v_0) := \mathrm{e}^{-is\Delta} v_0(\mathbf{x}),$$

and

$$w(\mathbf{x}, s+t') = \varphi_V^{s,t'}(w_0) := \mathrm{e}^{-i\int_0^s V^\varepsilon(\mathbf{x}, t'+y)\,\mathrm{d}y - isf(|w_0(\mathbf{x})|^2)} w_0(\mathbf{x}),$$

respectively. Setting $u^0 = u_0$. The first order Lie–Trotter splitting scheme reads as

$$u^n = \Phi_L^{\tau, t_{n-1}}(u^{n-1}), \quad n \geq 1, \quad \Phi_L^{\tau,t'}(\eta) := \varphi_T^\tau \circ \varphi_V^{\tau,t'}(\eta), \tag{2.3}$$

and the second order Strang splitting scheme reads as

$$u^n = \Phi_S^{\tau, t_{n-1}}(u^{n-1}), \quad n \geq 1, \quad \Phi_S^{\tau,t'}(\eta) := \varphi_T^{\tau/2} \circ \varphi_V^{\tau,t'} \circ \varphi_T^{\tau/2}(\eta). \tag{2.4}$$

We assume that $\int_0^s V^\varepsilon(\mathbf{x}, t'+y)\,\mathrm{d}y$ can be evaluated exactly in (2.3) and (2.4). For $m \in \mathbb{R}$, we denote by $\|\cdot\|_m$ the standard $H^m = H^m(\mathbb{T}^d)$ Sobolev norm, which reads as

$$\|u\|_m^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^m |\widehat{u}_k|^2, \quad \text{where} \quad \widehat{u}_k = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^d} u(\mathbf{x}) \mathrm{e}^{-ik\mathbf{x}}\,\mathrm{d}\mathbf{x}.$$

For $m = 0$, the space is exactly L^2 and the corresponding norm is denoted as $\|\cdot\|$ for simplicity. Throughout the paper we assume $\sigma > d/2$ so that the well-known bilinear estimate holds [1]:

$$\|fg\|_\sigma \leq C_{\sigma,d} \|f\|_\sigma \|g\|_\sigma, \tag{2.5}$$

where $C_{\sigma,d}$ represents a positive constant depending on σ and d. We consider the problem (1.2) till some time $0 < T_0 < T^*$, where $T^* > 0$ denotes the maximum existence time of the solution. We give the error bounds of the splitting schemes for solving (1.2) till $t = T_0$ without discretizing the space variable, while the full discretization case can be analyzed similarly. In practice, the spatial discretization can be done by either Fourier pseudo-spectral method [38] or Hermite collocation method [27,37] regarding the consistent boundary conditions. As we are considering the periodic setup in (1.2), the Fourier pseudo-spectral method can be easily applied. Throughout the rest of the paper, to simplify the notations and simultaneously address the dependence of the error on ε, we adopt the notation $A \lesssim B$ to represent that there exists a generic constant $C > 0$, which is independent of the time step τ (or n) and the parameter ε, such that $|A| \leq CB$.

2.2. Main results

For the Lie–Trotter splitting scheme (2.3), assume that the potential and the solution of the NLSE-OP (1.2) satisfy

$$V^\varepsilon \in C([0,T_0];H^{\sigma+2}), \quad u \in C([0,T_0];H^{\sigma+2}) \cap C^1([0,T_0];H^\sigma), \tag{2.6a}$$
$$\|V^\varepsilon\|_{L^\infty([0,T_0];H^{\sigma+2})} + \|u\|_{L^\infty([0,T_0];H^{\sigma+2})} + \|\partial_t u\|_{L^\infty([0,T_0];H^\sigma)} \lesssim 1, \tag{2.6b}$$

where $\sigma > d/2$ is a real number. Then we have the following error estimate which shows uniform accuracy of the Lie–Trotter splitting scheme.

Theorem 2.1 (Lie–Trotter)**.** *Under the regularity assumption* (2.6)*, there exists $\tau_0 > 0$ independent of ε and τ (or n), such that when $0 < \tau \le \tau_0$, the error of the Lie–Trotter scheme satisfies:*

$$\|u^n - u(\cdot,t_n)\|_\sigma \lesssim \tau, \quad 0 \le n \le T_0/\tau.$$

For the Strang splitting scheme, assume that the given potential and the solution of the NLSE-OP (1.2) satisfy

$$V^\varepsilon \in C([0,T_0];H^{\sigma+4}) \cap C^1([0,T_0];H^{\sigma+2}), \tag{2.7a}$$
$$u \in C([0,T_0];H^{\sigma+4}) \cap C^1([0,T_0];H^{\sigma+2}) \cap C^2([0,T_0];H^\sigma), \tag{2.7b}$$
$$\|V^\varepsilon\|_{L^\infty([0,T_0];H^{\sigma+4})} + \varepsilon\|\partial_t V^\varepsilon\|_{L^\infty([0,T_0];H^{\sigma+2})} \lesssim 1, \tag{2.7c}$$
$$\|u\|_{L^\infty([0,T_0];H^{\sigma+4})} + \|\partial_t u\|_{L^\infty([0,T_0];H^{\sigma+2})} + \varepsilon\|\partial_{tt} u\|_{L^\infty([0,T_0];H^\sigma)} \lesssim 1, \tag{2.7d}$$

with $\sigma > d/2$. The error bound of the Strang splitting scheme is stated as the follows.

Theorem 2.2 (Strang)**.** *Under the regularity assumption (2.7), there exists $\tau_0 > 0$ independent of ε and τ (or n), such that when $0 < \tau \le \tau_0$, the following error estimate of the Strang splitting scheme holds:*

$$\|u^n - u(\cdot,t_n)\|_\sigma \lesssim \min\left\{\tau, \frac{\tau^2}{\varepsilon}\right\}, \quad 0 \le n \le T_0/\tau.$$

The proofs of Theorems 2.1 and 2.2 will be given in Sections 3 and 4, respectively. As shall be seen from the numerical results, our error estimate for the Lie–Trotter splitting scheme in Theorem 2.1 is optimal. While for the Strang splitting scheme, the numerical results in Section 5 indicate that the scheme has some super-convergence effect when τ is either small enough or large enough under a given ε. Although our theoretical result in Theorem 2.2 might be refined, it together with the numerical results shows that Strang splitting scheme is not uniformly second order accurate for solving the NLSE-OP (1.2).

Remark 2.3. The local/global well-posedness of the Schrödinger equations without time-dependent potential on $\mathbb{T}^d$ have been widely studied [12–14, 29, 30]. For the Schrödinger equations with an external time-dependent potential case, we refer to [10, 15, 17, 25, 39] for the growth of high Sobolev norms. Based on the existing analytical results and in order to obtain a rather sharp error estimate, we make the assumptions that there exists a maximum existence time of solutions $T^* > 0$ independent of ε and the regularities (2.6) or (2.7) are satisfied. The requirements on the initial data, V^ε and f which ensure these assumptions are beyond the topics of this work.

2.3. Extension to Klein–Gordon equation

The splitting schemes for the NLSE-OP (1.2) and corresponding error estimates can be extended to other wave or dispersive equations with a highly oscillatory potential. In the following, we present the extension to the nonlinear Klein–Gordon equation.

For describing cosmic superfluid, the rotating nonlinear Klein–Gordon equation was introduced, see *e.g.* [42] and the references therein. After applying the rotating Lagrangian coordinates transform, the following nonlinear Klein–Gordon equation with a highly oscillatory potential (KGE-OP) occurs [35]:

$$\partial_{tt}u(\mathbf{x},t) - \Delta u(\mathbf{x},t) + u(\mathbf{x},t) + V^\varepsilon(\mathbf{x},t)u(\mathbf{x},t) + f(u(\mathbf{x},t)) = 0, \qquad \mathbf{x} \in \mathbb{T}^d, \quad t > 0, \tag{2.8a}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ \partial_t u(\mathbf{x},0) = u_1(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{T}^d, \tag{2.8b}$$

where u is the unknown, u_0 and u_1 are given real-valued initial data, $f : \mathbb{R} \to \mathbb{R}$ is the given nonlinearity and V^ε is the highly oscillatory potential. In [35], the highly oscillatory V^ε is an anisotropic trapping potential that composites with rapid rotation. The KGE-OP (2.8) also shows up in the counterpart of the study for the subsonic limit of the Klein–Gordon–Zakharov system [4, 33]. To apply the splitting schemes, we firstly rewrite (2.8) into a first order system by introducing

$$v(\mathbf{x},t) = \langle\nabla\rangle^{-1}\partial_t u(\mathbf{x},t), \quad \langle\nabla\rangle = \sqrt{1-\Delta}.$$

Then the vectorised equation reads

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ \langle\nabla\rangle^{-1}(V^\varepsilon u + f(u)) \end{pmatrix}, \quad t > 0. \tag{2.9}$$

Note that the linear operator

$$\begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & 0 \end{pmatrix}$$

is skew symmetric and $0 < \langle\nabla\rangle^{-1} \le 1$, then (2.9) is analogous to the NLSE-OP (1.2). One can thus split (2.9) into

$$\partial_t \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & \langle\nabla\rangle \\ -\langle\nabla\rangle & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \quad \text{and} \quad \partial_t \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = -\begin{pmatrix} 0 \\ \langle\nabla\rangle^{-1}(V^\varepsilon u_2 + f(u_2)) \end{pmatrix}.$$

The subproblems can be exactly integrated respectively as

$$\begin{pmatrix} u_1(\mathbf{x},t) \\ v_1(\mathbf{x},t) \end{pmatrix} = \begin{pmatrix} \cos(\langle\nabla\rangle t) & \sin(\langle\nabla\rangle t) \\ -\sin(\langle\nabla\rangle t) & \cos(\langle\nabla\rangle t) \end{pmatrix} \begin{pmatrix} u_1(\mathbf{x},0) \\ v_1(\mathbf{x},0) \end{pmatrix},$$

and

$$\begin{pmatrix} u_2(\mathbf{x},t) \\ v_2(\mathbf{x},t) \end{pmatrix} = \begin{pmatrix} u_2(\mathbf{x},0) \\ v_2(\mathbf{x},0) - \langle\nabla\rangle^{-1}\left[\int_0^t V^\varepsilon(\mathbf{x},s)\,\mathrm{d}s\, u_2(\mathbf{x},0) + t f(u_2(\mathbf{x},0))\right] \end{pmatrix}.$$

It is straightforward to apply the Lie–Trotter or Strang splitting scheme, and the theoretical error estimates hold as well. Compared to the strategy introduced in [5] that integrates the oscillation in phase space with convolution, the splitting schemes could surely improve the efficiency.

3. Error estimate of the Lie–Trotter splitting

In this section, we aim to prove Theorem 2.1 for the error bound of the Lie–Trotter splitting scheme (2.3) for solving the NLSE-OP (1.2). Denote

$$R = \|u\|_{L^\infty([0,T];H^\sigma)}, \quad B_R^\sigma = \{v \in H^\sigma : \ \|v\|_\sigma \le R\}.$$

Firstly we introduce some auxiliary results which will be used in our proof.

Proposition 3.1 ([21])**.** *For any function $g \in C^\infty(\mathbb{C},\mathbb{C})$, there exists a nondecreasing function $\chi_g : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|g(u)\|_\sigma \le \|g(0)\|_\sigma + \chi_g(\|u\|_{L^\infty})\|u\|_\sigma, \quad \forall u \in H^\sigma. \tag{3.1}$$

For all $v, w \in B_R^\sigma$, we have

$$\|g(v) - g(w)\|_\sigma \le \alpha(g,R)\|v-w\|_\sigma, \tag{3.2}$$

where $\alpha(g,R) = \|g'(0)\|_\sigma + R\chi_{g'}(cR)$, with $c>0$ being the constant for the Sobolev imbedding $\|\cdot\|_{L^\infty} \le c\|\cdot\|_\sigma$.

Applying the triangle inequality, (3.1) and (3.2), we have

$$\|f(|v|^2)v - f(|w|^2)w\|_\sigma \le M_0\|v-w\|_\sigma, \quad v,w \in B_R^\sigma, \tag{3.3}$$

with $M_0 = C_{\sigma,d}\|f(0)\|_\sigma + C_{\sigma,d}^2 R^2\left[\chi_f(c^2R^2) + 2\alpha(f, C_{\sigma,d}R^2)\right]$.

Next we establish the stability result and the local truncation error.

Lemma 3.2 (Stability)**.** *For $v,w \in B_R^\sigma$ and $0 \le t' \le T_0 - \tau$, the propagator of the Lie–Trotter splitting scheme (2.3) satisfies:*

$$\left\|\Phi_L^{\tau,t'}(v) - \Phi_L^{\tau,t'}(w)\right\|_\sigma \le \mathrm{e}^{M\tau}\|v-w\|_\sigma,$$

where $M>0$ depends on σ, d, R, f and $\|V^\varepsilon\|_{L^\infty([0,T_0];H^\sigma)}$.

Proof. Noticing φ_T^τ preserves H^σ-norm, we get

$$\left\|\Phi_L^{\tau,t'}(v) - \Phi_L^{\tau,t'}(w)\right\|_\sigma = \left\|\varphi_V^{\tau,t'}(v) - \varphi_V^{\tau,t'}(w)\right\|_\sigma.$$

Denote $\tilde{v}(\mathbf{x},t) = \varphi_V^{t,t'}(v)$ and $\tilde{w}(\mathbf{x},t) = \varphi_V^{t,t'}(w)$. Define $g(x) := \mathrm{e}^{-ix}$. It is obvious that $g \in C^\infty(\mathbb{C},\mathbb{C})$. It follows from (3.1) that

$$\begin{aligned}\|\mathrm{e}^{-i\int_0^t V^\varepsilon(t'+s)\,\mathrm{d}s}\|_\sigma &\le \|g(0)\|_\sigma + \chi_g\Big(\Big\|\int_0^t V^\varepsilon(t'+s)\,\mathrm{d}s\Big\|_{L^\infty}\Big)\Big\|\int_0^t V^\varepsilon(t'+s)\,\mathrm{d}s\Big\|_\sigma \\ &\le \|1\|_\sigma + \chi_g\left(c\,t\sup_{0\le s\le t}\|V^\varepsilon(t'+s)\|_\sigma\right) t\sup_{0\le s\le t}\|V^\varepsilon(t'+s)\|_\sigma. \end{aligned}\tag{3.4}$$

Similarly, we get

$$\begin{aligned}\|\mathrm{e}^{-itf(|v|^2)}\|_\sigma &\le \|1\|_\sigma + t\chi_g\left(t\|f\|_{L^\infty}\right)\|f(|v|^2)\|_\sigma \\ &\le \|1\|_\sigma + t\chi_g\left(t\|f\|_{L^\infty}\right)\left[\|f(0)\|_\sigma + C_{\sigma,d}\chi_f(c^2\|v\|_\sigma^2)\|v\|_\sigma^2\right].\end{aligned}$$

Combining the above inequalities and using the bilinear inequality (2.5), we obtain

$$\|\tilde{v}(\cdot,t)\|_\sigma = \|\varphi_V^{t,t'}(v)\|_\sigma \le C_{\sigma,d}^2\|v\|_\sigma\|\mathrm{e}^{-i\int_0^t V^\varepsilon(t'+s)\,\mathrm{d}s}\|_\sigma\|\mathrm{e}^{-itf(|v|^2)}\|_\sigma \le M_1\|v\|_\sigma,$$

where M_1 depends on σ, d, t, f, $\|v\|_\sigma$ and $\sup_{0\le s\le t}\|V^\varepsilon(t'+s)\|_\sigma$. This implies that $\tilde{v}(\mathbf{x},t), \tilde{w}(\mathbf{x},t) \in H^\sigma$.

By Duhamel's formula, Minkovski's inequality, (3.3) and the bilinear inequality (2.5), we have

$$\begin{aligned}\|\tilde{v}(\cdot,\tau) - \tilde{w}(\cdot,\tau)\|_\sigma &\le \|v-w\|_\sigma + \int_0^\tau \|V^\varepsilon(\cdot,t+t')\left(\tilde{v}(\cdot,t) - \tilde{w}(\cdot,t)\right)\|_\sigma\,\mathrm{d}t \\ &\quad + \int_0^\tau \left\|f(|\tilde{v}(\cdot,t)|^2)\tilde{v}(\mathbf{x},t) - f(|\tilde{w}(\cdot,t)|^2)\tilde{w}(\cdot,t)\right\|_\sigma\,\mathrm{d}t \\ &\le \|v-w\|_\sigma + \Big(M_0 + C_{\sigma,d}\sup_{t\in[0,T_0]}\|V^\varepsilon\|_\sigma\Big)\int_0^\tau \|\tilde{v}(\cdot,t) - \tilde{w}(\cdot,t)\|_\sigma\,\mathrm{d}t.\end{aligned}$$

By Gronwall's inequality, we get

$$\|\tilde{v}(\cdot,\tau)-\tilde{w}(\cdot,\tau)\|_\sigma \le \mathrm{e}^{M\tau}\|v-w\|_\sigma,$$

where $M = M_0 + C_{\sigma,d}\|V^\varepsilon\|_{L^\infty([0,T_0];H^\sigma)}$ depends on σ, d, R, f and $\|V^\varepsilon\|_{L^\infty([0,T_0];H^\sigma)}$. $\square$

Omit the space variable for simplicity and denote the local truncation error for $n \ge 0$ as

$$\xi^n := u(t_{n+1}) - \Phi_L^{\tau,t_n}(u(t_n)) = u(t_{n+1}) - \mathrm{e}^{-i\tau\Delta}\left(\mathrm{e}^{-i\int_0^\tau V^\varepsilon(s+t_n)\,\mathrm{d}s - i\tau f(|u(t_n)|^2)}u(t_n)\right). \tag{3.5}$$

For the local truncation error, we have the following estimate.

Lemma 3.3 (Local error)**.** *Under the regularity assumption (2.6), the local truncation error of the Lie–Trotter splitting scheme* (2.3) *satisfies*

$$\|\xi^n\|_\sigma \lesssim \tau^2, \quad 0 \le n < T_0/\tau.$$

Proof. For simplicity of notation, we denote $F_n(s) = f(|u(t_n+s)|^2)$, $V_n^\varepsilon(s) = V^\varepsilon(t_n+s)$. By Taylor expansion, we have

$$\begin{aligned}\xi^n = u(t_{n+1}) - \mathrm{e}^{-i\tau\Delta}\Big[&u(t_n) - i\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s\, u(t_n) - i\tau f(|u(t_n)|^2)u(t_n)\\ &- \int_0^1 (1-\theta)\mathrm{e}^{-i\theta\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s - i\theta\tau f(|u(t_n)|^2)}\,\mathrm{d}\theta\left(\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s + \tau f(|u(t_n)|^2)\right)^2 u(t_n)\Big].\end{aligned}$$

Duhamel's principle gives that

$$\begin{aligned}u(t_{n+1}) &= \mathrm{e}^{-i\tau\Delta}u(t_n) + \int_0^\tau \mathrm{e}^{-i(\tau-s)\Delta}\left[-iV_n^\varepsilon(s)u(t_n+s) - iF_n(s)u(t_n+s)\right]\mathrm{d}s\\ &= \mathrm{e}^{-i\tau\Delta}u(t_n) - i\mathrm{e}^{-i\tau\Delta}\int_0^\tau \left[V_n^\varepsilon(s)u(t_n) + f(|u(t_n)|^2)u(t_n)\right]\mathrm{d}s\\ &\quad + \mathrm{e}^{-i\tau\Delta}\int_0^\tau\int_0^1 \mathrm{e}^{is\theta\Delta}\,\mathrm{d}\theta(s\Delta)\left[V_n^\varepsilon(s)u(t_n+s) + F_n(s)u(t_n+s)\right]\mathrm{d}s\\ &\quad - i\mathrm{e}^{-i\tau\Delta}\int_0^\tau \left[V_n^\varepsilon(s)(u(t_n+s)-u(t_n)) + F_n(s)u(t_n+s) - F_n(0)u(t_n)\right]\mathrm{d}s.\end{aligned}$$

Thus the local error can be written as

$$\begin{aligned}\xi^n = \mathrm{e}^{-i\tau\Delta}&\left[\int_0^1 (1-\theta)\mathrm{e}^{-i\theta\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s - i\theta\tau f(|u(t_n)|^2)}\,\mathrm{d}\theta\left(\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s + \tau f(|u(t_n)|^2)\right)^2 u(t_n)\right]\\ &+ \mathrm{e}^{-i\tau\Delta}\int_0^\tau\int_0^1 \mathrm{e}^{is\theta\Delta}\,\mathrm{d}\theta(s\Delta)\left[V_n^\varepsilon(s)u(t_n+s) + F_n(s)u(t_n+s)\right]\mathrm{d}s\\ &- i\mathrm{e}^{-i\tau\Delta}\int_0^\tau \left[V_n^\varepsilon(s)(u(t_n+s)-u(t_n)) + F_n(s)u(t_n+s) - F_n(0)u(t_n)\right]\mathrm{d}s.\end{aligned}$$

By using (2.5) and (3.2), we arrive at

$$\begin{aligned}\|\xi^n\|_\sigma \le \tau^2 C_{\sigma,d}^3\|u(t_n)\|_\sigma &\left[\|f(|u(t_n)|^2)\|_\sigma^2 + \sup_{0\le s\le\tau}\|V_n^\varepsilon(s)\|_\sigma\right]\sup_{0\le\theta\le 1}\left\|\mathrm{e}^{-i\theta\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s - i\theta\tau f(|u(t_n)|^2)}\right\|_\sigma\\ &+ \tau^2 C_{\sigma+2,d}\sup_{0\le s\le\tau}\|u(t_n+s)\|_{\sigma+2}\left(\|V_n^\varepsilon(s)\|_{\sigma+2} + \|F_n(s)\|_{\sigma+2}\right)\\ &+ \left(M_0 + \sup_{0\le s\le\tau}\|V_n^\varepsilon(s)\|_\sigma\right)\int_0^\tau \|u(t_n+s)-u(t_n)\|_\sigma\,\mathrm{d}s.\end{aligned}$$

Applying similar arguments as in the proof of Lemma 3.2, we have

$$\left\| \mathrm{e}^{-i\theta \int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s - i\theta\tau f(|u(t_n)|^2)} \right\|_\sigma \le M_2,$$

where M_2 depends on σ, d, τ, f, $\|u(t_n)\|_\sigma$ and $\sup\limits_{0\le s\le\tau} \|V_n^\varepsilon(s)\|_\sigma$. Owing to (3.1), we obtain

$$\|F_n(s)\|_{\sigma+2} \le \|f(0)\|_{\sigma+2} + C_{\sigma+2,d}\,\chi_f(c^2\|u(t_n+s)\|_\sigma^2)\|u(t_n+s)\|_{\sigma+2}^2,$$

and

$$\begin{aligned}\int_0^\tau \|u(t_n+s)-u(t_n)\|_\sigma\,\mathrm{d}s &= \int_0^\tau \left\| \int_0^s \partial_t u(t_n+y)\,\mathrm{d}y \right\|_\sigma \mathrm{d}s \\ &\le \int_0^\tau \int_0^s \|\partial_t u(t_n+y)\|_\sigma\,\mathrm{d}y\,\mathrm{d}s \le \tau^2 \sup_{0\le s\le\tau} \|\partial_t u(t_n+s)\|_\sigma.\end{aligned}$$

Hence we conclude that there exists a constant $C>0$ such that

$$\|\xi^n\|_\sigma \le C\tau^2, \quad 0\le n < T_0/\tau,$$

where C depends on σ, d, f, $\|V^\varepsilon\|_{L^\infty([0,T_0];H^{\sigma+2})}$, $\|u\|_{L^\infty([0,T_0];H^{\sigma+2})}$ and $\|\partial_t u\|_{L^\infty([0,T_0];H^\sigma)}$. □

With the above two lemmas, we give the proof of the global error estimate for the Lie–Trotter splitting (2.3) which is stated as Theorem 2.1.

Proof of Theorem 2.1. We use an induction argument for the boundedness of the numerical solution. Denote $R = \|u\|_{L^\infty([0,T_0];H^\sigma)}$. We next show the numerical solution $u^n \in B_{R+1}^\sigma$. Firstly, it is obvious for $n=0$ since $u^0 = u_0 \in B_R^\sigma$. Assume $u^l \in B_{R+1}^\sigma$ for $0\le l\le n < T_0/\tau$. Denote $\mathrm{e}^n = u(t_n) - u^n$. Taking the difference between (2.3) and (3.5), we get

$$\mathrm{e}^{n+1} = \xi^n + \Phi_L^{\tau,t_n}(u(t_n)) - \Phi_L^{\tau,t_n}(u^n).$$

Using Lemmas 3.2 and 3.3, we get

$$\begin{aligned}\|\mathrm{e}^{n+1}\|_\sigma \le\ & \|\xi^n\|_\sigma + \left\|\Phi_L^{\tau,t_n}(u(t_n)) - \Phi_L^{\tau,t_n}(u^n)\right\|_\sigma \\ &\le \mathrm{e}^{M\tau}\|\mathrm{e}^n\|_\sigma + C\tau^2 \le \mathrm{e}^{M(n+1)\tau}\|\mathrm{e}^0\|_\sigma + C\tau^2\sum_{l=0}^n \mathrm{e}^{Ml\tau}.\end{aligned}$$

Then we get

$$\|\mathrm{e}^{n+1}\|_\sigma \le C\frac{\mathrm{e}^{MT_0}-1}{M}\tau, \quad 0\le n < T_0/\tau,$$

and when $0<\tau\le\tau_0 := \frac{M}{C(\mathrm{e}^{MT_0}-1)}$, we have

$$\|u^{n+1}\|_\sigma \le \|u(t_{n+1})\|_\sigma + 1.$$

Hence, we have $u^{n+1}\in B_{R+1}^\sigma$ and the induction proof is completed. □

Remark 3.4. For the other Lie–Trotter splitting

$$u^n = \prod_{m=1}^n \Psi_L^{\tau,t_{m-1}}(u_0), \quad \Psi_L^{\tau,t'}(\xi) := \varphi_V^{\tau,t'}\circ\varphi_T^\tau(\xi), \quad n\ge 0,$$

we can get the first-order convergence under the same regularity assumptions by noticing that

$$\begin{aligned}\|\Psi_L^{\tau,t'}(v) - \Psi_L^{\tau,t'}(w)\|_\sigma &\le C_{\sigma,d}\left\|\mathrm{e}^{-i\int_0^\tau V^\varepsilon(t'+y)\,\mathrm{d}y}\right\|_\sigma \left\|\mathrm{e}^{-i\tau f(|\mathrm{e}^{-i\tau\Delta}v|^2)}\mathrm{e}^{-i\tau\Delta}v - \mathrm{e}^{-i\tau f(|\mathrm{e}^{-i\tau\Delta}w|^2)}\mathrm{e}^{-i\tau\Delta}w\right\|_\sigma \\ &\le M\|v-w\|_\sigma,\end{aligned}$$

where we have used (3.2) and (3.4).

4. Error estamte of Strang splitting

In this section, we establish the error bound of the Strang splitting scheme (2.4) by proving Theorem 2.2. We adopt the same notations as in the previous section.

Lemma 4.1 (Stability)**.** *For $v, w \in B_R^\sigma$ and $0 \le t' \le T_0 - \tau$, the propagator of the Strang splitting scheme (2.4) satisfies:*

$$\left\| \Phi_S^{\tau,t'}(v) - \Phi_S^{\tau,t'}(w) \right\|_\sigma \le \mathrm{e}^{M\tau} \|v - w\|_\sigma,$$

for some $M > 0$ depending on σ, d, R, f and $\|V^\varepsilon\|_{L^\infty([0,T_0];H^\sigma)}$.

Proof. Directly by the fact that φ_T^τ preserves the H^s-norm and Lemma 3.2, we have

$$\begin{aligned} \left\| \Phi_S^{\tau,t'}(v) - \Phi_S^{\tau,t'}(w) \right\|_\sigma &= \left\| \varphi_V^{\tau,t'} \circ \varphi_T^{\tau/2}(v) - \varphi_V^{\tau,t'} \circ \varphi_T^{\tau/2}(w) \right\|_{H^{\sigma-4}} \\ &\le \mathrm{e}^{M\tau} \left\| \varphi_T^{\tau/2}(v) - \varphi_T^{\tau/2}(w) \right\|_\sigma = \mathrm{e}^{M\tau} \|v - w\|_\sigma, \end{aligned}$$

which completes the proof. □

Denote the local truncation error of the Strang splitting for $n \ge 0$ as

$$\eta^n := u(t_{n+1}) - \Phi_S^{\tau,t_n}(u(t_n)) = u(t_{n+1}) - \mathrm{e}^{-i\frac{\tau}{2}\Delta} \left(\mathrm{e}^{-i\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s - i\tau f(|\mathrm{e}^{-i\frac{\tau}{2}\Delta} u(t_n)|^2)} \mathrm{e}^{-i\frac{\tau}{2}\Delta} u(t_n) \right). \tag{4.1}$$

To do so, we further introduce some notations. Let $R = R(v, t, s)$ be a term that depends on the function values $v(t + t')$ for $0 \le t' \le s$. We say that $R \in \mathcal{R}_\beta(v, s^\alpha)$ if and only if

$$\|R(v, t, s)\|_\sigma \le C s^\alpha,$$

where C depends on $\sup_{0 \le t' \le s} \|v(t + t')\|_{\sigma+\beta}$. Similarly, we denote by $R = R(v, w, t, s) \in \mathcal{R}_{\beta,\gamma}(v, w, s^\alpha)$ if and only if

$$\|R(v, w, t, s)\|_\sigma \le C s^\alpha,$$

where C depends on $\sup_{0 \le t' \le s} \|v(t + t')\|_{\sigma+\beta}$ and $\sup_{0 \le t' \le s} \|w(t + t')\|_{\sigma+\gamma}$. For simplicity, we write $\phi = \psi + \mathcal{R}_\beta(v, s^\alpha)$ whenever $\phi = \psi + R$ with $R \in \mathcal{R}_\beta(v, s^\alpha)$ and similarly $\phi = \psi + \mathcal{R}_{\beta,\gamma}(v, w, s^\alpha)$ whenever $\phi = \psi + R$ with $R \in \mathcal{R}_{\beta,\gamma}(v, w, s^\alpha)$. Next we introduce a lemma on expansion which will be used frequently afterwards.

Lemma 4.2. *For all $s \ge 0$, we have*

$$\|\mathrm{e}^{\pm is\Delta} v - v\|_\sigma \le s\|v\|_{\sigma+2} \quad \text{and} \quad \|\mathrm{e}^{\pm is\Delta} v - v \mp is\Delta v\|_\sigma \le \frac{s^2}{2}\|v\|_{\sigma+4}. \tag{4.2}$$

Proof. For $v = \sum_{k\in\mathbb{Z}^d} \widehat{v}_k \mathrm{e}^{ik\mathbf{x}}$, we have

$$\mathrm{e}^{\pm is\Delta} v - v = \sum_{k\in\mathbb{Z}^d} \widehat{v}_k \left(\mathrm{e}^{\mp is|k|^2} - 1 \right) \mathrm{e}^{ik\mathbf{x}},$$

which yields that

$$\begin{aligned} \|\mathrm{e}^{\pm is\Delta} v - v\|_\sigma^2 &= \sum_{k\in\mathbb{Z}^d} (1 + |k|^2)^\sigma |\widehat{v}_k|^2 |\mathrm{e}^{\mp is|k|^2} - 1|^2 \le s^2 \sum_{k\in\mathbb{Z}^d} (1 + |k|^2)^\sigma |\widehat{v}_k|^2 |k|^4 \\ &\le s^2 \sum_{k\in\mathbb{Z}^d} (1 + |k|^2)^{\sigma+2} |\widehat{v}_k|^2 \le s^2 \|v\|_{\sigma+2}^2. \end{aligned}$$

Similarly, we have

$$\|\mathrm{e}^{\pm is\Delta}v - v \mp is\Delta v\|_\sigma^2 = \sum_{k\in\mathbb{Z}^d}(1+|k|^2)^\sigma|\widehat{v}_k|^2|\mathrm{e}^{\mp is|k|^2}-1\pm is|k|^2|^2 \le \frac{s^4}{4}\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^\sigma|\widehat{v}_k|^2|k|^8$$
$$\le \frac{s^4}{4}\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^{\sigma+4}|\widehat{v}_k|^2 \le \frac{s^4}{4}\|v\|_{\sigma+4}^2,$$

which completes the proof. □

Lemma 4.3 (Local error)**.** *Under the regularity assumption (2.7), the local truncation error of the Strang splitting scheme* (2.4) *satisfies*

$$\|\eta^n\|_\sigma \lesssim \min\left\{\tau^2, \frac{\tau^3}{\varepsilon}\right\}, \quad 0 \le n < T_0/\tau.$$

Proof. Denote $u_{\frac{1}{2}} = \mathrm{e}^{-i\tau\Delta/2}u(t_n)$. An application of the Duhamel's formula and Lemma 4.2 leads to the following representation for $0 \le s \le \tau$:

$$\begin{aligned}
u(t_n+s) &= \mathrm{e}^{-is\Delta}u(t_n) - i\int_0^s \mathrm{e}^{-i(s-y)\Delta}\left[(V_n^\varepsilon(y)+F_n(y))u(t_n+y)\right]\mathrm{d}y\\
&= u_{\frac{1}{2}} + \left(\mathrm{e}^{-i(s-\tau/2)\Delta}-1\right)u_{\frac{1}{2}} - i\int_0^s \mathrm{e}^{-i(s-y)\Delta}\left[(V_n^\varepsilon(y)+F_n(y))u(t_n+y)\right]\mathrm{d}y\\
&= u_{\frac{1}{2}} + \mathcal{R}_2(u,\tau) + \mathcal{R}_{0,0}(V_n^\varepsilon,u,\tau) + \mathcal{R}_{0,0}(F_n,u,\tau)\\
&= u_{\frac{1}{2}} + \mathcal{R}_2(u,\tau) + \mathcal{R}_{0,0}(V_n^\varepsilon,u,\tau),
\end{aligned}$$

where we have used the inequality (3.1) that $\|F_n(y)\|_\sigma \le \|f(0)\|_\sigma + C_{\sigma,d}\|u\|_\sigma^2\chi_f(c^2\|u\|_\sigma^2)$. Plugging this approximation into the Duhamel's formula and applying Lemma 4.2, we obtain

$$\begin{aligned}
u(t_n+s) &= \mathrm{e}^{-is\Delta}u(t_n) - i\int_0^s \mathrm{e}^{-i(s-y)\Delta}\left[\left(V_n^\varepsilon(y)+f\left(|u_{\frac{1}{2}}|^2\right)\right)u_{\frac{1}{2}}\right]\mathrm{d}y + \mathcal{R}_2(u,\tau^2) + \mathcal{R}_{0,2}(V_n^\varepsilon,u,\tau^2)\\
&= \mathrm{e}^{i(\tau/2-s)\Delta}u_{\frac{1}{2}} - i\int_0^s V_n^\varepsilon(y)u_{\frac{1}{2}}\,\mathrm{d}y - i\int_0^s f\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}}\,\mathrm{d}y + \mathcal{R}_2\left(u,\tau^2\right) + \mathcal{R}_{2,2}\left(V_n^\varepsilon,u,\tau^2\right)\\
&= \left[1+i\left(\frac{\tau}{2}-s\right)\Delta\right]u_{\frac{1}{2}} - iG_n^\varepsilon(s)u_{\frac{1}{2}} - isf\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}} + \mathcal{R}_4(u,\tau^2) + \mathcal{R}_{2,2}(V_n^\varepsilon,u,\tau^2),
\end{aligned} \tag{4.3}$$

where

$$G_n^\varepsilon(s) = \int_0^s V_n^\varepsilon(y)\,\mathrm{d}y.$$

With this approximation, setting $s=\tau$ in the Duhamel's formula, one gets

$$\begin{aligned}
u(t_n+\tau) &= \mathrm{e}^{-i\tau\Delta}u(t_n) - i\int_0^\tau \mathrm{e}^{-i(\tau-s)\Delta}\left[(V_n^\varepsilon(s)+F_n(s))u(t_n+s)\right]\mathrm{d}s\\
&= \mathrm{e}^{-i\tau\Delta}u(t_n) - i\mathrm{e}^{-i\tau\Delta}(Q_1+Q_2) + \mathcal{R}_{0,4}(V_n^\varepsilon,u,\tau^3) + \mathcal{R}_{2,2}(V_n^\varepsilon,u,\tau^3),
\end{aligned}$$

where

$$\begin{aligned}
Q_1 &= \int_0^\tau \mathrm{e}^{is\Delta}\left[V_n^\varepsilon(s)\left(1+i\left(\frac{\tau}{2}-s\right)\Delta - iG_n^\varepsilon(s) - isf\left(|u_{\frac{1}{2}}|^2\right)\right)u_{\frac{1}{2}}\right]\mathrm{d}s,\\
Q_2 &= \int_0^\tau \mathrm{e}^{is\Delta}[F_n(s)u(t_n+s)]\,\mathrm{d}s.
\end{aligned}$$

Applying Lemma 4.2, Q_1 can be expanded as

$$\begin{aligned}
Q_1 &= \int_0^\tau (1+is\Delta)\left(V_n^\varepsilon(s)u_{\frac{1}{2}}\right)\,\mathrm{d}s + i\int_0^\tau V_n^\varepsilon(s)\left[\left(\frac{\tau}{2}-s\right)\Delta - G_n^\varepsilon(s) - sf\left(|u_{\frac{1}{2}}|^2\right)\right]u_{\frac{1}{2}}\,\mathrm{d}s + \mathcal{R}_{4,4}(V_n^\varepsilon,u,\tau^3)\\
&= G_n^\varepsilon(\tau)u_{\frac{1}{2}} + i\Delta\left(H_n^\varepsilon(\tau)u_{\frac{1}{2}}\right) + i\left[\frac{\tau}{2}G_n^\varepsilon(\tau) - H_n^\varepsilon(\tau)\right]\Delta u_{\frac{1}{2}}\\
&\quad - i\left[\frac{G_n^\varepsilon(\tau)^2}{2} + H_n^\varepsilon(\tau)f\left(|u_{\frac{1}{2}}|^2\right)\right]u_{\frac{1}{2}} + \mathcal{R}_{4,4}(V_n^\varepsilon,u,\tau^3),
\end{aligned}$$

where $H_n^\varepsilon(\tau) = \int_0^\tau sV_n^\varepsilon(s)\,\mathrm{d}s$. Moreover, by Taylor expansion, we have

$$\begin{aligned}
Q_2 &= \int_0^\tau \mathrm{e}^{is\Delta}\left[F_n\left(\frac{\tau}{2}\right)u(t_n+s)\right]\,\mathrm{d}s + \int_0^\tau \mathrm{e}^{is\Delta}\left[\int_{\frac{\tau}{2}}^s F_n'(y)\,\mathrm{d}y\,u(t_n+s)\right]\,\mathrm{d}s\\
&= \int_0^\tau \mathrm{e}^{is\Delta}\left[F_n\left(\frac{\tau}{2}\right)u(t_n+s)\right]\,\mathrm{d}s + \int_0^\tau\int_{\frac{\tau}{2}}^s F_n'(y)\,\mathrm{d}y\,u(t_n+s)\,\mathrm{d}s + \mathcal{R}_{2,2}(u,\partial_t u,\tau^3)\\
&= \int_0^\tau \mathrm{e}^{is\Delta}\left[F_n\left(\frac{\tau}{2}\right)u(t_n+s)\right]\,\mathrm{d}s + \int_0^\tau\int_{\frac{\tau}{2}}^s F_n'(y)\,\mathrm{d}y\,u\left(t_n+\frac{\tau}{2}\right)\,\mathrm{d}s + \mathcal{R}_{2,2}(u,\partial_t u,\tau^3)\\
&= \int_0^\tau \mathrm{e}^{is\Delta}\left[F_n\left(\frac{\tau}{2}\right)u(t_n+s)\right]\,\mathrm{d}s + u\left(t_n+\frac{\tau}{2}\right)\int_0^{\frac{\tau}{2}} y\left[F_n'(\tau-y)-F_n'(y)\right]\,\mathrm{d}y + \mathcal{R}_{2,2}(u,\partial_t u,\tau^3)\\
&= \int_0^\tau \mathrm{e}^{is\Delta}\left[F_n\left(\frac{\tau}{2}\right)u(t_n+s)\right]\,\mathrm{d}s + \mathcal{R}_{2,2}(u,\partial_t u,\tau^3) + \mathcal{R}_{0,0}(u,\partial_t u,\tau^2).
\end{aligned}\tag{4.4}$$

Noticing that

$$\int_0^{\frac{\tau}{2}} y\left[F_n'(\tau-y)-F_n'(y)\right]\,\mathrm{d}y = \int_0^{\frac{\tau}{2}} F_n''(s)\frac{s^2}{2}\,\mathrm{d}s + \int_{\frac{\tau}{2}}^\tau F_n''(s)\frac{(\tau-s)^2}{2}\,\mathrm{d}s,$$

this implies

$$Q_2 = \int_0^\tau \mathrm{e}^{is\Delta}\left[F_n\left(\frac{\tau}{2}\right)u(t_n+s)\right]\,\mathrm{d}s + \mathcal{R}_{2,2}(u,\partial_t u,\tau^3) + \mathcal{R}_{0,0}(u,\partial_{tt}u,\tau^3),\tag{4.5}$$

where we have used (3.1) repeatedly for f, f' and f''. Setting $s=\frac{\tau}{2}$ in (4.3), we have

$$u\left(t_n+\frac{\tau}{2}\right) = u_{\frac{1}{2}} - iG_n^\varepsilon\left(\frac{\tau}{2}\right)u_{\frac{1}{2}} - \frac{i\tau}{2}f\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}} + \mathcal{R}_4(u,\tau^2) + \mathcal{R}_{2,2}(V_n^\varepsilon,u,\tau^2),$$

which yields that

$$\begin{aligned}
\left|u\left(t_n+\frac{\tau}{2}\right)\right|^2 &= |u_{\frac{1}{2}}|^2 + \mathcal{R}_4(u,\tau^2) + \mathcal{R}_{2,2}(V_n^\varepsilon,u,\tau^2),\\
F_n\left(\frac{\tau}{2}\right) &= f\left(|u_{\frac{1}{2}}|^2\right) + \mathcal{R}_4(u,\tau^2) + \mathcal{R}_{2,2}(V_n^\varepsilon,u,\tau^2).
\end{aligned}$$

Plugging this approximation and (4.3) into Q_2, using Lemma 4.2, we get

$$\begin{aligned}
Q_2 &= \int_0^\tau \mathrm{e}^{is\Delta}\left[f\left(|u_{\frac{1}{2}}|^2\right)u(t_n+s)\right]\,\mathrm{d}s + Q_3\\
&= \int_0^\tau \mathrm{e}^{is\Delta}\left[f\left(|u_{\frac{1}{2}}|^2\right)\left(1+i\left(\frac{\tau}{2}-s\right)\Delta - iG_n^\varepsilon(s) - isf\left(|u_{\frac{1}{2}}|^2\right)\right)u_{\frac{1}{2}}\right]\,\mathrm{d}s + Q_4\\
&= \int_0^\tau (1+is\Delta)\left(f\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}}\right)\,\mathrm{d}s + if\left(|u_{\frac{1}{2}}|^2\right)\int_0^\tau\left[\left(\frac{\tau}{2}-s\right)\Delta - G_n^\varepsilon(s) - sf\left(|u_{\frac{1}{2}}|^2\right)\right]u_{\frac{1}{2}}\,\mathrm{d}s + Q_5\\
&= \tau f\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}} + \frac{i\tau^2}{2}\Delta\left(f\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}}\right) - i\int_0^\tau G_n^\varepsilon(s)\,\mathrm{d}s\,f\left(|u_{\frac{1}{2}}|^2\right)u_{\frac{1}{2}} - \frac{i\tau^2}{2}\left[f\left(|u_{\frac{1}{2}}|^2\right)\right]^2 u_{\frac{1}{2}} + Q_5,
\end{aligned}$$

where by (4.4) and (4.5),

$$Q_3, Q_4, Q_5 \in \mathcal{R}_4(u,\tau^3) + \mathcal{R}_{2,2}(V_n^\varepsilon, u, \tau^3) + \mathcal{R}_{2,2}(u, \partial_t u, \tau^3) + \mathcal{R}_{0,0}(u, \partial_t u, \tau^2) \cap \mathcal{R}_{0,0}(u, \partial_{tt} u, \tau^3).$$

This concludes that

$$\begin{aligned} u(t_n+\tau) = {} & \mathrm{e}^{-i\tau\Delta} u(t_n) - iG_n^\varepsilon(\tau) u_{\frac12} - \tau\Delta\left(G_n^\varepsilon(\tau) u_{\frac12}\right) + \Delta\left(H_n^\varepsilon(\tau) u_{\frac12}\right) + \left(\frac{\tau}{2} G_n^\varepsilon(\tau) - H_n^\varepsilon(\tau)\right)\Delta u_{\frac12} \\ & - u_{\frac12}\left[\frac{G_n^\varepsilon(\tau)^2}{2} + (i\tau + H_n^\varepsilon(\tau)) f\left(|u_{\frac12}|^2\right)\right] - \frac{\tau^2}{2}\Delta\left(f\left(|u_{\frac12}|^2\right) u_{\frac12}\right) - \frac{\tau^2}{2} f\left(|u_{\frac12}|^2\right)^2 u_{\frac12} \\ & - \int_0^\tau G_n^\varepsilon(s)\,\mathrm{d}s\, f\left(|u_{\frac12}|^2\right) u_{\frac12} + \mathcal{R}_{4,4}(V_n^\varepsilon, u, \tau^3) + \mathcal{R}_{2,2}(u, \partial_t u, \tau^3) \\ & + \mathcal{R}_{0,0}(u, \partial_t u, \tau^2) \cap \mathcal{R}_{0,0}(u, \partial_{tt} u, \tau^3). \end{aligned}$$

On the other hand, an application of (4.2) yields

$$\begin{aligned} \Phi_S^{\tau,t_n}(u(t_n)) &= \mathrm{e}^{-i\tau\Delta/2}\left(\mathrm{e}^{-iG_n^\varepsilon(\tau) - i\tau f\left(|u_{\frac12}|^2\right)} u_{\frac12}\right) \\ &= \mathrm{e}^{-i\tau\Delta/2}\left(\left[1 - iG_n^\varepsilon(\tau) - i\tau f\left(|u_{\frac12}|^2\right) - \frac12\left(G_n^\varepsilon(\tau) + \tau f\left(|u_{\frac12}|^2\right)\right)^2\right] u_{\frac12}\right) + \mathcal{R}_{0,0}(V_n^\varepsilon, u, \tau^3) \\ &= \mathrm{e}^{-i\tau\Delta} u(t_n) - i\left(G_n^\varepsilon(\tau) + \tau f\left(|u_{\frac12}|^2\right)\right) u_{\frac12} - \frac{\tau}{2}\Delta\left[\left(G_n^\varepsilon(\tau) + \tau f\left(|u_{\frac12}|^2\right)\right) u_{\frac12}\right] \\ &\quad - \frac12\left(G_n^\varepsilon(\tau) + \tau f\left(|u_{\frac12}|^2\right)\right)^2 u_{\frac12} + \mathcal{R}_{4,4}(V_n^\varepsilon, u, \tau^3). \end{aligned}$$

By subtraction, we get

$$\begin{aligned} \eta^n &= u(t_{n+1}) - \Phi_S^{\tau,t_n}(u(t_n)) \\ &= -\frac{\tau}{2}\Delta\left(G_n^\varepsilon(\tau) u_{\frac12}\right) + \Delta\left(H_n^\varepsilon(\tau) u_{\frac12}\right) + \left[\frac{\tau}{2} G_n^\varepsilon(\tau) - H_n^\varepsilon(\tau)\right]\Delta u_{\frac12} + \mathcal{R}_{0,0}(u, \partial_{tt} u, \tau^3) \\ &\quad - \left[H_n^\varepsilon(\tau) + \int_0^\tau G_n^\varepsilon(s)\,\mathrm{d}s - \tau G_n^\varepsilon(\tau)\right] f\left(|u_{\frac12}|^2\right) u_{\frac12} + \mathcal{R}_{4,4}(V_n^\varepsilon, u, \tau^3) + \mathcal{R}_{2,2}(u, \partial_t u, \tau^3) \\ &= 2\nabla u_{\frac12}\cdot\nabla\left[H_n^\varepsilon(\tau) - \frac{\tau}{2} G_n^\varepsilon(\tau)\right] + u_{\frac12}\Delta\left[H_n^\varepsilon(\tau) - \frac{\tau}{2} G_n^\varepsilon(\tau)\right] \\ &\quad + \mathcal{R}_{4,4}(V_n^\varepsilon, u, \tau^3) + \mathcal{R}_{2,2}(u, \partial_t u, \tau^3) + \mathcal{R}_{0,0}(u, \partial_t u, \tau^2) \cap \mathcal{R}_{0,0}(u, \partial_{tt} u, \tau^3), \end{aligned}$$

where we have used the property that

$$\begin{aligned} H_n^\varepsilon(\tau) + \int_0^\tau G_n^\varepsilon(s)\,\mathrm{d}s - \tau G_n^\varepsilon(\tau) &= \int_0^\tau s V_n^\varepsilon(s)\,\mathrm{d}s + \int_0^\tau \int_0^s V_n^\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}s - \tau\int_0^\tau V_n^\varepsilon(s)\,\mathrm{d}s \\ &= \int_0^\tau (s-\tau) V_n^\varepsilon(s)\,\mathrm{d}s + \int_0^\tau V_n^\varepsilon(y)\int_y^\tau \mathrm{d}s\,\mathrm{d}y = 0. \end{aligned}$$

Noticing that

$$H_n^\varepsilon(\tau) - \frac{\tau}{2} G_n^\varepsilon(\tau) = \int_0^\tau \left(s - \frac{\tau}{2}\right) V_n^\varepsilon(s)\,\mathrm{d}s = \frac12\int_0^\tau s(\tau - s)\partial_t V_n^\varepsilon(s)\,\mathrm{d}s,$$

which together with assumption (2.7) yields that

$$\begin{aligned} \eta^n &= \mathcal{R}_{4,4}(V_n^\varepsilon, u, \tau^3) + \mathcal{R}_{2,2}(u, \partial_t u, \tau^3) + \mathcal{R}_{0,0}(u, \partial_t u, \tau^2) \cap \mathcal{R}_{0,0}(u, \partial_{tt} u, \tau^3) \\ &\quad + \mathcal{R}_{1,2}(u, V_n^\varepsilon, \tau^2) \cap \mathcal{R}_{1,2}(u, \partial_t V_n^\varepsilon, \tau^3) \\ &\lesssim \min\left\{\tau^2, \frac{\tau^3}{\varepsilon}\right\}, \end{aligned}$$

and the proof is completed. ∎

Combining the local error bound in Lemma 4.3 and the stability estimate (*cf.* Lem. 4.1), and applying a similar argument as in the proof of Theorem 2.1, we can get the error bound of the second-order scheme (2.4) as shown in Theorem 2.2.

Remark 4.4. Although our presentation stops at the Strang splitting scheme, the higher-order splitting schemes whose local truncation error certainly involves higher-order time derivatives of V^ε, are not expected to be uniformly accurate at their optimal convergence rate.

5. Numerical result

In this section, we present the convergence tests results of the splitting schemes for solving the NLSE-OP (1.2) and the KGE-OP (2.8) for a wide range of $\varepsilon \in (0,1]$. We implement the spatial discretization of the splitting schemes by Fourier pseudo-spectral method [36].

Example 5.1 (Test for NLSE-OP)**.** We take one-dimensional example of NLSE-OP for test, *i.e.*, $d = 1, \mathbf{x} = x$ in (1.2). We consider the cubic nonlinearity $f(\rho) = -\rho$ and choose the initial condition for (1.2) as

$$u_0(x) = \frac{\sin(2x)}{2+\cos(x)}, \quad x \in [-\pi, \pi].$$

We take the oscillatory potential $V^\varepsilon(x,t)$ as

$$V^\varepsilon(x,t) = U^\varepsilon(x,t) + 2\cos(t)\cos^2(x), \tag{5.1}$$

where $U^\varepsilon(x,t)$ is the solution of the following wave equation:

$$\begin{cases} \varepsilon^2 \partial_{tt} U^\varepsilon(x,t) - \partial_{xx} U^\varepsilon(x,t) = 0, \quad x \in (-\pi,\pi),\ t > 0, \\ U^\varepsilon(x,0) = \dfrac{\sin(x)}{2+\cos(2x)} + \sin(2x),\ \partial_t U^\varepsilon(x,0) = \dfrac{1}{\varepsilon}\left[\cos(3x) + \sin(2x)\right],\ x \in [-\pi,\pi], \\ U^\varepsilon(\pi,t) = U^\varepsilon(-\pi,t), \quad \partial_x U^\varepsilon(\pi,t) = \partial_x U^\varepsilon(-\pi,t), \quad t \geq 0. \end{cases}$$

We solve the NLSE-OP (1.2) till $t = 2$ and compute the error of splitting schemes:

$$\|u(\cdot,t_n) - u^n\|_{H^1}.$$

The reference solution is given by a 4-th order Yoshida splitting scheme [43] with Fourier pseudo-spectral method under very small time step size $\tau = 10^{-5}$ and fine mesh size $\Delta x = 2\pi/128$. Furthermore, the free wave equation for U^ε is solved by Fourier pseudo-spectral discretization in space under the same mesh size and integrated exactly in time. We plot the temporal error of the Lie–Trotter splitting scheme and the Strang splitting scheme under different $0 < \varepsilon < 1$ in Figure 1, where the spatial mesh size $\Delta x = 2\pi/128$ is kept so that the spatial discretization error is negligible. The spatial discretization error of Strang splitting (Lie–Trotter is similar) is given in Figure 2 for different ε and Δx under fixed $\tau = 10^{-5}$.

From the numerical results in Figures 1 and 2, we can see that

(1) The Lie–Trotter splitting scheme (2.3) converges uniformly for all $\varepsilon \in (0,1]$ at the first order rate for solving the NLSE-OP (1.2).
(2) It seems that the Strang splitting scheme (2.4) converges quadratically when the time step $\tau \lesssim \varepsilon^\beta$ or $\varepsilon^\alpha \lesssim \tau$ with $\alpha \geq \beta > 0$, which suggests that our estimate in Theorem 2.2 might not be optimal. Refined estimates might be obtained by similar ideas for splitting methods in [21] with some restriction on the time step size. This will be considered in a future work. Nevertheless the bump part in the lower left one of Figure 1 shows that the Strang splitting scheme (2.4) is **not** uniformly accurate at the second order for $\varepsilon \in (0,1]$.
(3) The spatial discretizations of the splitting schemes have uniformly spectral accuracy for $\varepsilon \in (0,1]$.

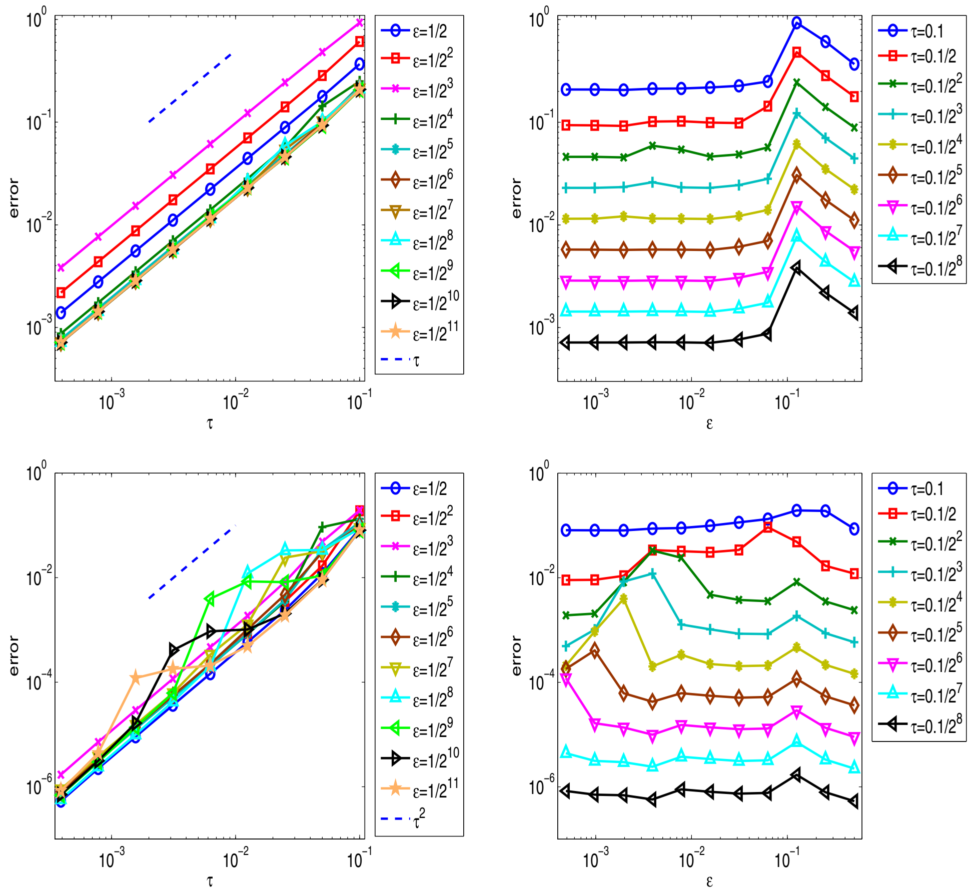

FIGURE 1. Error $\|u(\cdot,t_n)-u^n\|_{H^1}$ at $t=2$ of the Lie–Trotter splitting (*above*) and Strang splitting (*below*) for the NLSE-OP (1.2) under different ε and τ.

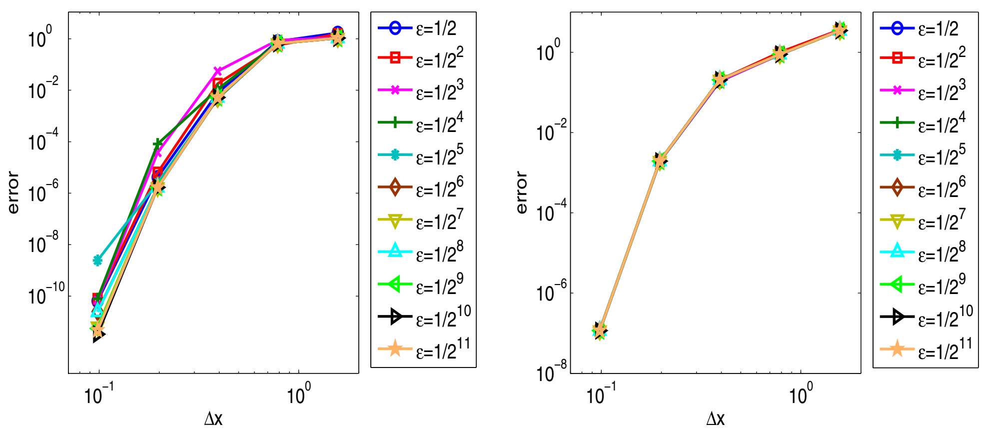

FIGURE 2. Spatial discretization error of Strang splitting for NLSE-OP (1.2) (*left*) and KGE-OP (2.8) (*right*): $\|u(\cdot,t_n)-u^n\|_{H^1}$ at $t=2$ under different ε and Δx.

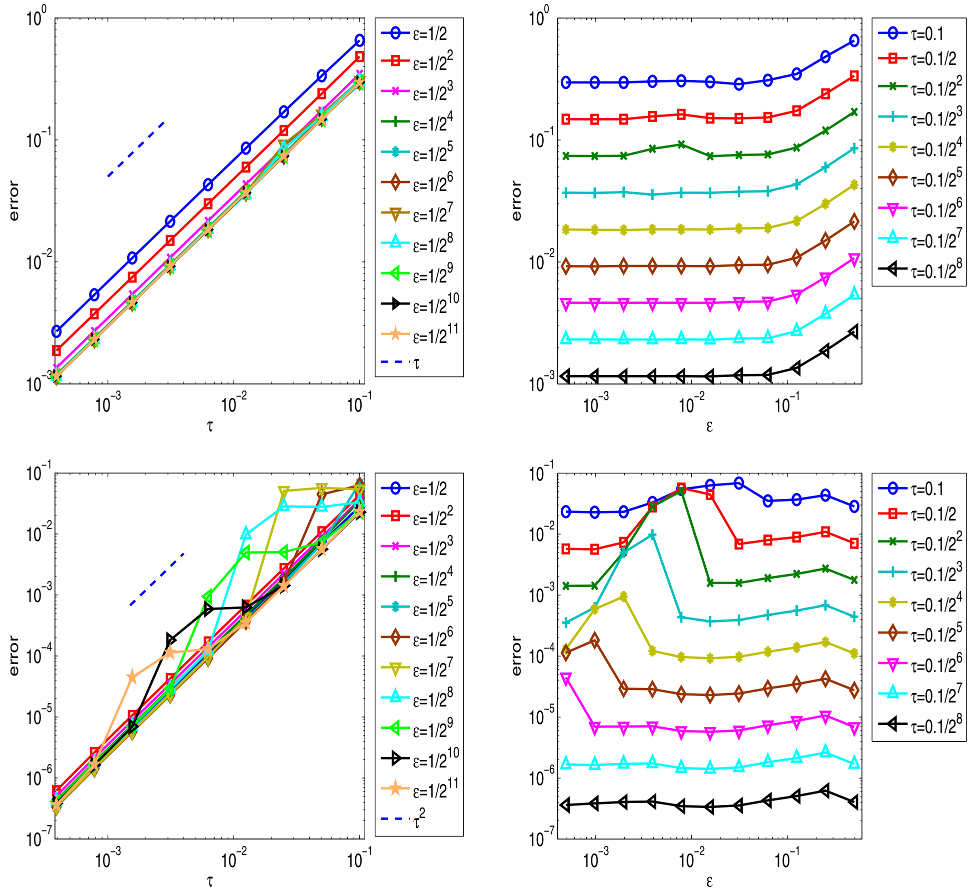

FIGURE 3. Error $\|u(\cdot,t_n)-u^n\|_{H^1}+\|\partial_t u(\cdot,t_n)-\dot{u}^n\|_{H^1}$ at $t=2$ of the Lie–Trotter splitting (*above*) and Strang splitting (*below*) for the KGE-OP (2.8) under different ε and τ.

Example 5.2 (Test for KGE-OP)**.** As another example, we solve the one-dimensional KGE-OP (2.8). We take a cubic nonlinearity $f(u)=u^3$ and initial values in (2.8) as:

$$u_0(x)=\frac{\sin(2x)}{2+\cos(x)},\quad u_1(x)=\frac{1+\cos(x)}{1-\sin(2x)/2},\quad x\in[-\pi,\pi].$$

The potential $V^\varepsilon(x,t)$ is chosen as same as in (5.1).

We solve the KGE-OP (2.8) till $t=2$ and compute the error:

$$\|u(\cdot,t_n)-u^n\|_{H^1}+\|\partial_t u(\cdot,t_n)-\dot{u}^n\|_{H^1}.$$

The reference solution is obtained similarly as before. Figure 3 shows the temporal error (fixed $\Delta x=2\pi/128$) of the Lie–Trotter splitting scheme and the Strang splitting scheme under different $\varepsilon\in(0,1]$. The spatial error is plotted in Figure 2. As can be seen, the numerical results in Figure 3 are similar as before and hence illustrates once again the same conclusions on the convergence result of the splitting schemes. As for comparisons, we show the corresponding error of the Deuflhard-type exponential integrator [44] in Figure 4. Although the splitting methods and the exponential integrators (or known as trigonometric integrators) are closely related [16,32], it is

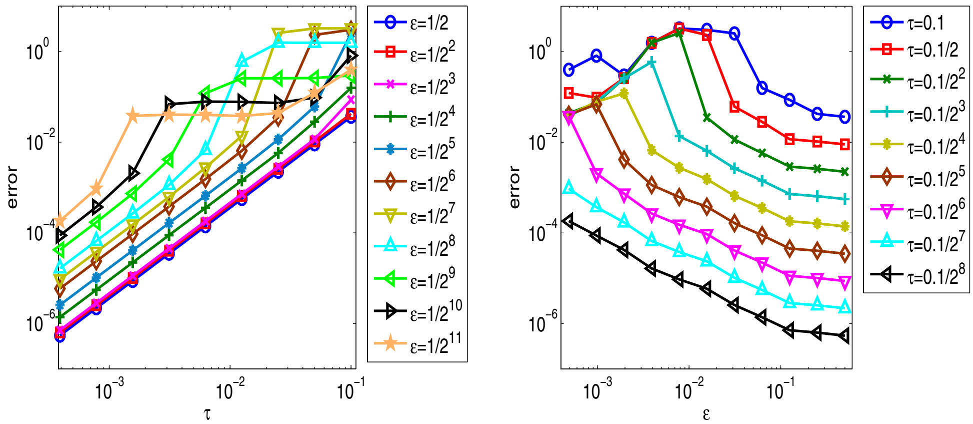

FIGURE 4. Error $\|u(\cdot,t)-u^n\|_{H^1}+\|\partial_t(\cdot,t)-\dot{u}^n\|_{H^1}$ at $t=2$ of the Deuflhard-type exponential integrator for the KGE-OP (2.8) under different ε and τ.

clear that the exponential integrator gives much worse approximations than the splitting scheme in our model case, where the error in Figure 4 increases dramatically as ε becomes small. We comment that the error of the Gautschi-type exponential integrator [2, 40] is even worse when ε is small.

6. Conclusion

In this work, we consider the nonlinear Schrödinger equation with a highly oscillatory potential (NLSE-OP) as our model problem, where the potential introduces fast temporal oscillations to the solution. This model problem is motivated from recent studies on multiscale problems such as the subsonic limit of the Zakharov system. The time-splitting schemes are applied to solve the NLSE-OP, where the sub-flows are integrated exactly. We rigorously analyze the error bounds of the splitting schemes, where the results show that the Lie–Trotter splitting scheme converges linearly and uniformly with respect to the oscillation frequency from the potential, while the Strang splitting scheme is not uniformly second order accurate. Due to the exact integration of the oscillatory potential, the splitting schemes still give much more accurate approximations than the exponential integrators in the highly oscillatory regime. Extensions are made to the nonlinear Klein–Gordon equation with an oscillatory potential. Numerical results justify the theoretical estimates.

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