ESAIM: M2AN 54 (2020) 1429–1463
https://doi.org/10.1051/m2an/2020001

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

T-JUNCTION OF FERROELECTRIC WIRES

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Abstract. In this paper, starting from a non-convex and nonlocal 3D variational mathematical model for the electric polarization in a ferroelectric material, and using an asymptotic process based on dimensional reduction, we analyze junction phenomena for two orthogonal joined ferroelectric wires. Depending on the initial boundary conditions, we get several different limit problems, sometimes uncoupled. We point out that all the limit problems remain non-convex, but the nonlocality disappears.

Mathematics Subject Classification. 35Q61, 78A25, 49J20.

Received December 2, 2019. Accepted December 23, 2019.

1. Introduction

In this paper, starting from a non-convex and nonlocal 3D variational mathematical model for the electric polarization in a ferroelectric material, and using an asymptotic process based on dimensional reduction, we analyze junction phenomena, from an energetic point of view, for two T-joined ferroelectric wires. Depending on the initial boundary conditions, we get several limit problems. We refer to [4,12,16,28,29,32,33] (see also the introduction in [7]) about general history, applications, and mathematical modeling of the electric polarization in ferroelectric structures.

Let $\{h_n\}_{n\in\mathbb{N}} \subset]0,1[$ be a sequence such that

$$\lim_n h_n = 0.$$

For every $n \in \mathbb{N}$, set (see Fig. 1)

$$\Omega_n^a = h_n \left]-\frac{1}{2},\frac{1}{2}\right[^2 \times [0,1[, \quad \Omega_n^b = \left]-\frac{1}{2},\frac{1}{2}\right[\times h_n \left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right), \quad \Omega_n = \Omega_n^a \cup \Omega_n^b.$$

The multidomain Ω_n models a ferroelectric structure consisting of two joined orthogonal parallelepipeds Ω_n^a and Ω_n^b. The first parallelepiped has constant height along the direction x_3, the second one has constant height along the direction x_1, while both of them have a small cross section of area h_n^2 and are joined by the small surface $h_n\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times \{0\}$. Several energetic approaches can be considered. We begin with a standard choice

Keywords and phrases. Electric polarization, nonlocal problems, optimal control, wire, junctions.

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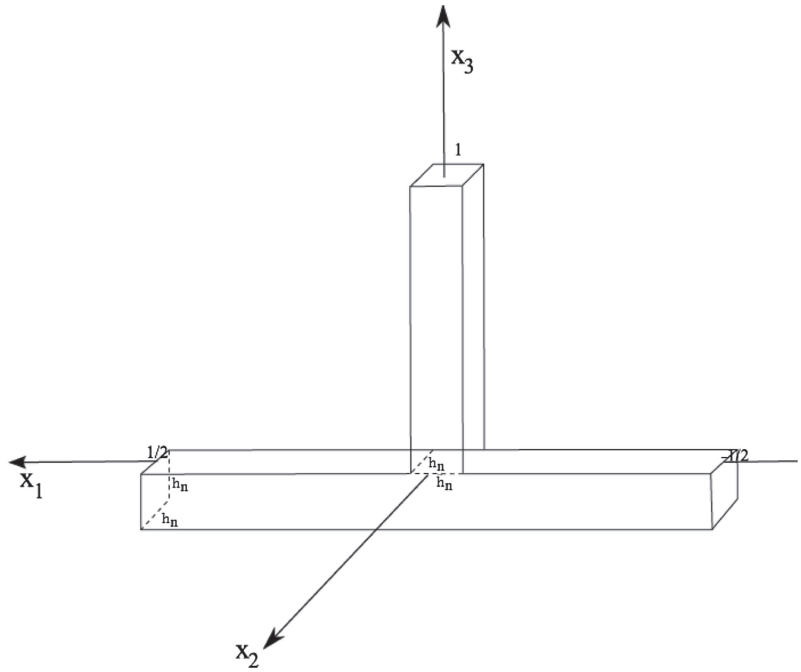

FIGURE 1. The set Ω_n.

for the functional representing the energy. Precisely, consider the following non-convex and non-local energy associated with Ω_n

$$\mathcal{E}_n : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 \to \int_{\Omega_n} \left(|\mathrm{D}\,\mathbf{P}|^2 + \alpha \left(|\mathbf{P}|^2 - 1 \right)^2 + |D\varphi_{\mathbf{P}}|^2 + \mathbf{F}_n \cdot \mathbf{P} \right) \mathrm{d}x, \tag{1.1}$$

where α is a positive constant, $\mathbf{F}_n \in \left(L^2(\Omega_n)\right)^3$, D denotes the gradient, $\cdot$ denotes the inner product in $\mathbb{R}^3$, and $\varphi_{\mathbf{P}}$ is the unique solution, up to an additive constant, to

$$\min \left\{ \int_{\Omega_n} \left(\frac{1}{2} |\mathrm{D}\varphi|^2 - \mathbf{P} \cdot D\varphi \right) \mathrm{d}x : \ \varphi \in H^1(\Omega_n) \right\}, \tag{1.2}$$

or alternatively $\varphi_{\mathbf{P}}$ is the unique solution to the following problem

$$\min \left\{ \int_{\Omega_n} \left(\frac{1}{2} |\mathrm{D}\varphi|^2 - \mathbf{P} \cdot D\varphi \right) \mathrm{d}x : \ \varphi \in H^1_0(\Omega_n) \right\}. \tag{1.3}$$

Note that in (1.2) and in (1.3) the vacuum permeability constant is assumed equal to 1.

The direct method of Calculus of Variations ensures that the following problems

$$\min \left\{ \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 \right\}, \tag{1.4}$$

$$\min \left\{ \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3, \ \mathbf{P} \cdot \nu = 0 \quad \text{on} \quad \partial\Omega_n \right\}, \tag{1.5}$$

and

$$\min \left\{ \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3, \ \mathbf{P} \, // \, e_3 \quad \text{on} \quad \partial\Omega_n \right\}, \tag{1.6}$$

admit solutions, where ν denotes the unit outer normal on $\partial\Omega_n$, $e_3 = (0,0,1)$, and $//$ is the symbol of parallelism.

Note that (1.4)–(1.6) are optimal control problems.

With respect to the conditions on $\varphi_{\mathbf{P}}$, we note that considering minimization problem (1.2) provides

$$\begin{cases} \mathrm{div}\,(-D\varphi_{\mathbf{P}} + \mathbf{P}) = 0, & \text{in} \quad \Omega_n, \\ (-D\varphi_{\mathbf{P}} + \mathbf{P}) \cdot \nu = 0, & \text{on} \quad \partial\Omega_n, \end{cases} \tag{1.7}$$

and so the classical boundary flow balance condition, while minimization problem (1.3) provides

$$\begin{cases} \mathrm{div}\,(-D\varphi_{\mathbf{P}} + \mathbf{P}) = 0, & \text{in} \quad \Omega_n, \\ \varphi_{\mathbf{P}} = 0, & \text{on} \quad \partial\Omega_n, \end{cases} \tag{1.8}$$

and so the classical boundary condition of "grounded domain".

We precise that our modeling is restricted to the cases where the external field $\mathbf{F}_n$ is weak with respect to the intensity of the intrinsic polarization $\mathbf{P}$. So, in our choice of energetic functional, we can omit to take into account formation of Weiss domains and walls, but we admit only transition regions. Moreover, considering minimization problem (1.5) entails the action of a very weak external field $\mathbf{F}_n$ on a body which was not previously polarized (see also introduction in [7]). Considering minimization problem (1.6) entails the action of an external field $\mathbf{F}_n$ on a body previously polarized along an assigned direction. The external field is not strong enough to change the orientation of the polarization on the boundary. Eventually, considering minimization problem (1.4) entails the action of a stronger electric field $\mathbf{F}_n$.

The goal of this paper is to study the asymptotic behavior, as n diverges, of these problems. To this aim, the external field $\mathbf{F}_n$ is rescaled on

$$\Omega^a \cup \Omega^b = \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times]0,1[\right) \cup \left(\left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times]-1,0[\right)$$

(see (2.2)) and the rescaled field is assumed to converge to (f^a, f^b) weakly in $\left(L^2\left(\Omega^a\right)\right)^3 \times \left(L^2\left(\Omega^b\right)\right)^3$ (see (2.14)). Moreover, let

$$\begin{aligned} E^a : q^a \in \left(H^1(]0,1[)\right)^3 \to & \int_0^1 \left(\left| \frac{\mathrm{d}q^a}{\mathrm{d}x_3} \right|^2 + \alpha \left(|q^a|^2 - 1\right)^2 + \eta \left(|q_1^a|^2 + |q_2^a|^2\right) + |q_3^a|^2 \right) \mathrm{d}x_3 \\ & + \int_0^1 \left(\int_{]-\frac{1}{2},\frac{1}{2}[^2} f^a \,\mathrm{d}x_1\,\mathrm{d}x_2 \cdot q^a \right) \mathrm{d}x_3, \\ E_l^b : q^b \in \left(H^1\left(\left]-\frac{1}{2},0\right[\right)\right)^3 \to & \int_{-\frac{1}{2}}^0 \left(\left| \frac{\mathrm{d}q^b}{\mathrm{d}x_1} \right|^2 + \alpha \left(|q^b|^2 - 1\right)^2 + |q_1^b|^2 + \eta \left(|q_2^b|^2 + |q_3^b|^2\right) \right) \mathrm{d}x_1 \\ & + \int_{-\frac{1}{2}}^0 \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} f^b \,\mathrm{d}x_2\,\mathrm{d}x_3 \cdot q^b \right) \mathrm{d}x_1, \end{aligned}$$

and

$$\begin{aligned} E_r^b : q^b \in \left(H^1\left(\left]0,\frac{1}{2}\right[\right)\right)^3 \to & \int_0^{\frac{1}{2}} \left(\left| \frac{\mathrm{d}q^b}{\mathrm{d}x_1} \right|^2 + \alpha \left(|q^b|^2 - 1\right)^2 + |q_1^b|^2 + \eta \left(|q_2^b|^2 + |q_3^b|^2\right) \right) \mathrm{d}x_1 \\ & + \int_0^{\frac{1}{2}} \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} f^b \,\mathrm{d}x_2\,\mathrm{d}x_3 \cdot q^b \right) \mathrm{d}x_1, \end{aligned}$$

where η is the constant defined by

$$\eta = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} |Dr|^2 \,\mathrm{d}y\,\mathrm{d}z, \tag{1.9}$$

r being the unique solution to a suitable variational problem (see (3.1)).

As the first problem with condition (1.2) is concerned, we prove that

$$\begin{aligned}&\lim_n \left(\min \left\{ \frac{1}{|\Omega_n|} \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 \right\} \right) = \frac{1}{2} \min \left\{ E^a(q^a) + E^b_l(q^b) + E^b_r(q^b) \right. : \\ &\left. (q^a, q^b) \in \left(H^1(]0,1[)\right)^3 \times \left(H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\right)\right)^3, \quad q^a(0) = q^b(0) \right\}.\end{aligned} \tag{1.10}$$

More precisely, (see Thm. 5.1) we study the asymptotic behavior of the rescaled polarization. On the vertical wire we obtain a limit polarization $p^a = (p^a_1, p^a_2, p^a_3)$ independent of (x_1, x_2). On the horizontal wire we obtain a limit polarization $p^b = (p^b_1, p^b_2, p^b_3)$, independent of (x_2, x_3). Moreover,

$$p^a(0) = p^b(0)$$

and (p^a, p^b) is a solution to the 1-dimensional vector valued problem in the right-hand side of (1.10).

As the second problem with condition (1.2) is concerned, we prove that

$$\begin{aligned}&\lim_n \left(\min \left\{ \frac{1}{|\Omega_n|} \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P} \cdot \nu = 0 \text{ on } \partial\Omega_n \right\} \right) \\ &= \frac{1}{2} \min \left\{ E^a(0,0,q^a_3) : q^a_3 \in H^1_0(]0,1[) \right\} \\ &\quad + \frac{1}{2} \min \left\{ E^b_l(q^b_1,0,0) : q^b_1 \in H^1_0\left(\left]-\tfrac{1}{2},0\right[\right) \right\} + \frac{1}{2} \min \left\{ E^b_r(q^b_1,0,0) : q^b_1 \in H^1_0\left(\left]0,\tfrac{1}{2}\right[\right) \right\}.\end{aligned} \tag{1.11}$$

More precisely, (see Thm. 6.1) on the vertical wire we obtain a limit polarization $(0, 0, p^a_3)$ where p^a_3 is independent of (x_1, x_2) and is a solution to the first 1-dimensional scalar problem in the right-hand side of (1.11). On the first half of the horizontal wire we obtain a limit polarization $(p^b_{1,l}, 0, 0)$ where $p^b_{1,l}$ is independent of (x_2, x_3) and it is a solution to the second 1-dimensional scalar problem in the right-hand side of (1.11). On the second half of the horizontal wire we obtain a limit polarization $(p^b_{1,r}, 0, 0)$ where $p^b_{1,r}$ is independent of (x_2, x_3) and is a solution to the third 1-dimensional scalar problem in the right-hand side of (1.11). Then, in this case we obtain three uncoupled problems.

As the third problem with condition (1.2) is concerned, we prove that

$$\begin{aligned}&\lim_n \left(\min \left\{ \frac{1}{|\Omega_n|} \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P} \,//\, e_3 \quad \text{on} \quad \partial\Omega_n \right\} \right) \\ &= \frac{1}{2} \min \left\{ E^a(0,0,q^a_3) + E^b_l(0,0,q^b_3) + E^b_r(0,0,q^b_3) \right. : \\ &\left. (q^a_3, q^b_3) \in H^1(]0,1[) \times H^1\left(\left]-\tfrac{1}{2},\tfrac{1}{2}\right[\right), \; q^a_3(0) = q^b_3(0) \right\}.\end{aligned} \tag{1.12}$$

More precisely, (see Thm. 7.1) on the vertical wire we obtain a limit polarization $(0, 0, p^a_3)$ where p^a_3 is independent of (x_1, x_2). On the horizontal wire we obtain a limit polarization $(0, 0, p^b_3)$ where p^b_3 is independent of (x_2, x_3). Moreover, $p^a_3(0) = p^b_3(0)$ and the couple (p^a_3, p^b_3) is a solution to the 1-dimensional scalar problem in the right-hand side of (1.12).

We point out that all the limit problems remained non-convex, but the nonlocal behavior disappeared, *i.e.* the limit problem is not longer an optimal control problem. Indeed, the nonlocal control term

$$\int_{\Omega_n} |D\varphi_{\mathbf{P}}|^2 \, \mathrm{d}x \tag{1.13}$$

produces the following weights in the limit minimization problems

$$\int_0^1 \left(\eta \left(|q_1^a|^2 + |q_2^a|^2\right) + |q_3^a|^2\right) \, \mathrm{d}x_3,$$

$$\int_{-\frac{1}{2}}^0 \left(|q_1^b|^2 + \eta \left(|q_2^b|^2 + |q_3^b|^2\right)\right) \, \mathrm{d}x_1, \quad \int_0^{\frac{1}{2}} \left(|q_1^b|^2 + \eta \left(|q_2^b|^2 + |q_3^b|^2\right)\right) \, \mathrm{d}x_1.$$

We just notice that the asymptotic behavior of problem (1.4) under the boundary condition $\mathbf{P} \,//\, (1,0,0)$ or $\mathbf{P} \,//\, (0,1,0)$ on $\partial\Omega_n$ can be treated similarly to the asymptotic behavior of problem (1.6). This easy task is left to an interested reader.

If we associate problems (1.4)–(1.6) with condition (1.3), we prove that nonlocal term (1.13) does not give any contribution to the limit problem.

Another very significant choice for an energetic approach consists in explicitly considering the energetic contribution for the polarization field given by the divergence term and the curl term. Consider the following non-convex and non-local energy associated with Ω_n

$$\begin{aligned} &\mathcal{S}_n : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 \to \\ &\int_{\Omega_n} \left(\beta |\mathrm{rot}\, \mathbf{P}|^2 + |\mathrm{div}\, \mathbf{P}|^2 + \alpha \left(|\mathbf{P}|^2 - 1\right)^2 + |D\varphi_{\mathbf{P}}|^2 + \mathbf{F}_n \cdot \mathbf{P}\right) \mathrm{d}x, \end{aligned} \tag{1.14}$$

where α and β are two positive constant, rot denotes the curl, and $\varphi_{\mathbf{P}}$ is a solution to (1.2) or alternatively to (1.3). For sake of simplicity, we limit ourselves to the cases where there is an equivalence between the term $||D\mathbf{P}||^2_{(L^2(\Omega_n))^9}$ and the term $||\mathrm{rot}\, \mathbf{P}||^2_{(L^2(\Omega_n))^3} + ||\mathrm{div}\, \mathbf{P}||^2_{L^2(\Omega_n)}$. This equivalence is assured by the boundary conditions on $\partial\Omega_n$

$$\mathbf{P} \cdot \nu = 0 \ \text{ or } \ \mathbf{P} \wedge \nu = 0,$$

with $\wedge$ denoting the cross product in $\mathbb{R}^3$ (for instance, see [15]). Then, the direct method of Calculus of Variations ensures that also the following problem

$$\min \left\{ \mathcal{S}_n(\mathbf{P}) : \ \mathbf{P} \in \left(H^1(\Omega_n)\right)^3, \ \mathbf{P} \cdot \nu = 0 \quad \text{on} \quad \partial\Omega_n \right\} \tag{1.15}$$

admits solution. In the case where $\varphi_{\mathbf{P}}$ is a solution to (1.2), in Theorem 8.1 we obtain the identity result

$$\begin{aligned} &\lim_n \left(\min \left\{ \frac{1}{|\Omega_n|} \mathcal{S}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P} \cdot \nu = 0 \quad \text{on} \quad \partial\Omega_n \right\} \right) \\ &= \lim_n \left(\min \left\{ \frac{1}{|\Omega_n|} \mathcal{E}_n(\mathbf{P}) : \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P} \cdot \nu = 0 \quad \text{on} \quad \partial\Omega_n \right\} \right) \end{aligned} \tag{1.16}$$

where the limit is given by (1.11). Moreover, (1.16) is true when $\varphi_{\mathbf{P}}$ is the solution to (1.3), too.

By considering this kind of results, we can explicitly note that the energetic curl term does not give any contribution to the limit problem and the constant β weighting this energetic term does not appear in the limit problem.

If in problems (1.5) and (1.15) the boundary condition $\mathbf{P} \cdot \nu = 0$ is replaced by $\mathbf{P} \wedge \nu = 0$ on $\partial\Omega_n$, it is easily seen that the limit of the energy is zero (for instance, compare [25]).

The paper is organized in the following way. In Section 2, previous problems are rescaled on a fixed domain independent of n. Section 3 is devoted to introduce the constant η defined in (1.9) which appears in the limit of the nonlocal term. Section 4 is the heart of the paper. According to the several boundary conditions on the polarization, different limit behaviors of the polarization are expected, and consequently also different behaviors of the nonlocal term could be produced. Indeed, in Proposition 4.2 we prove that if the potential generating the nonlocal term is solution to problem (1.2), then really the limit of the nonlocal term depends on boundary conditions on the polarization and we give a very general formula for the limit of the nonlocal term which covers all the possible cases coming from several boundary conditions on the polarization. If the potential generating the nonlocal term is the solution to problem (1.3), in Proposition 4.3 we prove that the limit of the nonlocal term is independent of the boundary conditions on the polarization and, precisely, it is always zero. Finally, using the main ideas of the Γ-convergence method introduced in [18] (see also [5,8,17]), in Sections 5–8 we study the asymptotic behavior of problems (1.4), (1.5), (1.6), and (1.15), respectively, obtaining L^2-strong convergences on the rescaled polarization p_n, on the rescaled potential φ_{p_n}, and on their rescaled gradients. In Section 9 we just sketch what happens when the potential generating the nonlocal term is the solution to problem (1.3).

Ferroelectric thin films and wires were studied in [24] and [25], respectively. The junction of ferroelectric thin films was examined in [7].

The 3D model of ferromagnetic microstructures is close to our model. For the limit behavior of ferromagnetic problems in thin structures involving wires we refer to [2,3,9,10,20,21,23,26,30,31], and the references therein. For other optimal control problems on a network of half-lines sharing an endpoint, we refer to [1] and the references therein. For other recent problems in a thin T-like shaped structure, we refer to [6,19], and the references therein.

2. The rescaled problems

In the sequel, suvently we omit the symbol $\cdot$ to denote the inner product in $\mathbb{R}^3$.

As in [14], Problems (1.4)–(1.6), and (1.15) are reformulated on a fixed domain through the maps

$$\begin{cases} x = (x_1, x_2, x_3) \in \Omega^a = \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times]0,1[\to (h_n x_1, h_n x_2, x_3) \in \mathrm{Int}(\Omega_n^a), \\ x = (x_1, x_2, x_3) \in \Omega^b = \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times \left]-1, 0\right[\to (x_1, h_n x_2, h_n x_3) \in \Omega_n^b, \end{cases} \tag{2.1}$$

where $\mathrm{Int}(\Omega_n^a)$ denotes the interior of Ω_n^a. Precisely, for every $n \in \mathbb{N}$ set

$$\begin{aligned}
&D_n^a : p^a \in \left(H^1(\Omega^a)\right)^k \to \left(\frac{1}{h_n}\frac{\partial p^a}{\partial x_1}, \frac{1}{h_n}\frac{\partial p^a}{\partial x_2}, \frac{\partial p^a}{\partial x_3}\right) \in \left(L^2(\Omega^a)\right)^{3k}, \quad k \in \{1,3\}, \\
&D_n^b : p^b \in \left(H^1(\Omega^b)\right)^k \to \left(\frac{\partial p^b}{\partial x_1}, \frac{1}{h_n}\frac{\partial p^b}{\partial x_2}, \frac{1}{h_n}\frac{\partial p^b}{\partial x_3}\right) \in \left(L^2(\Omega^b)\right)^{3k}, \quad k \in \{1,3\}, \\
&\mathrm{div}_n^a : p^a = (p_1^a, p_2^a, p_3^a) \in \left(H^1(\Omega^a)\right)^3 \to \frac{1}{h_n}\frac{\partial p_1^a}{\partial x_1} + \frac{1}{h_n}\frac{\partial p_2^a}{\partial x_2} + \frac{\partial p_3^a}{\partial x_3} \in L^2(\Omega^a), \\
&\mathrm{div}_n^b : p^b = (p_1^b, p_2^b, p_3^b) \in \left(H^1(\Omega^b)\right)^3 \to \frac{\partial p_1^b}{\partial x_1} + \frac{1}{h_n}\frac{\partial p_2^b}{\partial x_2} + \frac{1}{h_n}\frac{\partial p_3^b}{\partial x_3} \in L^2(\Omega^b), \\
&\mathrm{rot}_n^a : p^a = (p_1^a, p_2^a, p_3^a) \in \left(H^1(\Omega^a)\right)^3 \to \left(\frac{1}{h_n}\frac{\partial p_3^a}{\partial x_2} - \frac{\partial p_2^a}{\partial x_3}, \frac{\partial p_1^a}{\partial x_3} - \frac{1}{h_n}\frac{\partial p_3^a}{\partial x_1}, \frac{1}{h_n}\frac{\partial p_2^a}{\partial x_1} - \frac{1}{h_n}\frac{\partial p_1^a}{\partial x_2}\right) \in \left(L^2(\Omega^a)\right)^3, \\
&\mathrm{rot}_n^b : p^b = (p_1^b, p_2^b, p_3^b) \in \left(H^1(\Omega^a)\right)^3 \to \left(\frac{1}{h_n}\frac{\partial p_3^b}{\partial x_2} - \frac{1}{h_n}\frac{\partial p_2^b}{\partial x_3}, \frac{1}{h_n}\frac{\partial p_1^b}{\partial x_3} - \frac{\partial p_3^b}{\partial x_1}, \frac{\partial p_2^b}{\partial x_1} - \frac{1}{h_n}\frac{\partial p_1^b}{\partial x_2}\right) \in \left(L^2(\Omega^b)\right)^3,
\end{aligned}$$

and

$$\begin{cases} f_n^a : x = (x_1, x_2, x_3) \in \Omega^a \to \mathbf{F}_n(h_n x_1, h_n x_2, x_3), \\ f_n^b : x = (x_1, x_2, x_3) \in \Omega^b \to \mathbf{F}_n(x_1, h_n x_2, h_n x_3), \end{cases} \tag{2.2}$$

and define the sets

$$\begin{aligned}
P_n &= \left\{ (p^a, p^b) \in \left(H^1(\Omega^a)\right)^3 \times \left(H^1(\Omega^b)\right)^3 : p^a(x_1, x_2, 0) = p^b(h_n x_1, x_2, 0) \text{ on } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \right\}, \\
\widetilde{P}_n &= \Bigg\{ (p^a, p^b) \in \left(H^1(\Omega^a)\right)^3 \times \left(H^1(\Omega^b)\right)^3 : \, p^a \cdot \nu^a = 0 \text{ on } \partial\Omega^a \setminus \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right), \\
p^b \cdot \nu^b &= 0 \text{ on } \partial\Omega^b \setminus \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right), \\
p_3^b &= 0 \text{ on } \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\setminus \left]-\tfrac{h_n}{2}, \tfrac{h_n}{2}\right[\right) \times \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \{0\}, \\
p^a(x_1, x_2, 0) &= p^b(h_n x_1, x_2, 0) \text{ on } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \Bigg\}, \\
P_n^\star &= \Bigg\{ (p^a, p^b) \in \left(H^1(\Omega^a)\right)^3 \times \left(H^1(\Omega^b)\right)^3 : \, p^a \,//\, e_3 \text{ on } \partial\Omega^a \setminus \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right), \\
p^b &\,//\, e_3 \text{ on } \partial\Omega^b \setminus \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right), \\
p_1^b = p_2^b &= 0 \text{ on } \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\setminus \left]-\tfrac{h_n}{2}, \tfrac{h_n}{2}\right[\right) \times \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \{0\}, \\
p^a(x_1, x_2, 0) &= p^b(h_n x_1, x_2, 0) \text{ on } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \Bigg\}, \\
U_n &= \left\{ (\phi^a, \phi^b) \in H^1(\Omega^a) \times H^1(\Omega^b) : \phi^a(x_1, x_2, 0) = \phi^b(h_n x_1, x_2, 0) \text{ on } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \right\},
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
U_n^0 = \Bigg\{ &(\phi^a, \phi^b) \in H^1(\Omega^a) \times H^1(\Omega^b) : \, \phi^a = 0 \text{ on } \partial\Omega^a \setminus \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right), \\
&\phi^b = 0 \text{ on } \partial\Omega^b \setminus \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \times \{0\} \right), \\
&\phi^b = 0 \text{ on } \left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\setminus \left]-\tfrac{h_n}{2}, \tfrac{h_n}{2}\right[\right) \times \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\times \{0\}, \\
&\phi^a(x_1, x_2, 0) = \phi^b(h_n x_1, x_2, 0) \text{ on } \left]-\tfrac{1}{2}, \tfrac{1}{2}\right[^2 \Bigg\},
\end{aligned} \tag{2.4}$$

where ν^a and ν^b denote the unit outer normals on $\partial\Omega^a$ and $\partial\Omega^b$, respectively. Then $\mathcal{E}_n$ and $\mathcal{S}_n$, defined in (1.1) and (1.14), respectively, are rescaled by

$$\begin{aligned}
E_n : (p^a, p^b) \in P_n \to h_n^2 &\int_{\Omega^a} \left(|D_n^a \, p^a|^2 + \alpha \left(|p^a|^2 - 1\right)^2 + |D_n^a \phi^a_{(p^a, p^b)}|^2 + f_n^a p^a \right) \mathrm{d}x \\
&+ h_n^2 \int_{\Omega^b} \left(|D_n^b \, p^b|^2 + \alpha \left(|p^b|^2 - 1\right)^2 + |D_n^b \phi^b_{(p^a, p^b)}|^2 + f_n^b p^b \right) \mathrm{d}x,
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
S_n : (p^a, p^b) \in P_n \to h_n^2 &\int_{\Omega^a} \left(\beta |\mathrm{rot}_n^a \, p^a|^2 + |\mathrm{div}_n^a \, p^a|^2 + \alpha \left(|p^a|^2 - 1\right)^2 + |D_n^a \phi^a_{(p^a, p^b)}|^2 + f_n^a p^a \right) \mathrm{d}x \\
&+ h_n^2 \int_{\Omega^b} \left(\beta |\mathrm{rot}_n^b \, p^b|^2 + |\mathrm{div}_n^b \, p^b|^2 + \alpha \left(|p^b|^2 - 1\right)^2 + |D_n^b \phi^b_{(p^a, p^b)}|^2 + f_n^b p^b \right) \mathrm{d}x,
\end{aligned} \tag{2.6}$$

respectively, where $\left(\phi^a_{(p^a,p^b)},\phi^b_{(p^a,p^b)}\right)$ is the unique solution to

$$\begin{cases}\left(\phi^a_{(p^a,p^b)},\phi^b_{(p^a,p^b)}\right)\in U_n, \quad \displaystyle\int_{\Omega^a}\phi^a_{(p^a,p^b)}\,\mathrm{d}x=0,\\ \displaystyle\int_{\Omega^a}\left(-D^a_n\phi^a_{(p^a,p^b)}+p^a\right)D^a_n\phi^a\,\mathrm{d}x+\int_{\Omega^b}\left(-D^b_n\phi^b_{(p^a,p^b)}+p^b\right)D^b_n\phi^b\,\mathrm{d}x=0,\\ \forall(\phi^a,\phi^b)\in U_n,\end{cases}\tag{2.7}$$

which rescales the weak formulation of (1.7), *i.e.*

$$\varphi_{\mathbf{P}}\in H^1(\Omega_n),\quad \int_{\Omega^a_n}\varphi_{\mathbf{P}}\,\mathrm{d}x=0,\quad \int_{\Omega_n}\left(-D\varphi_{\mathbf{P}}+\mathbf{P}\right)D\varphi\,\mathrm{d}x=0,\quad \forall\varphi\in H^1(\Omega_n).\tag{2.8}$$

The Lax–Milgram Theorem provides that (2.8) admits solutions and it is unique.

The main goal of this paper becomes to study the asymptotic behavior, as n diverges, of the following problems

$$\min\left\{E_n((p^a,p^b)):(p^a,p^b)\in P_n\right\},\tag{2.9}$$

$$\min\left\{E_n((p^a,p^b)):(p^a,p^b)\in \widetilde{P}_n\right\},\tag{2.10}$$

$$\min\left\{E_n((p^a,p^b)):(p^a,p^b)\in P^\star_n\right\},\tag{2.11}$$

and

$$\min\left\{S_n((p^a,p^b)):(p^a,p^b)\in \widetilde{P}_n\right\}.\tag{2.12}$$

Moreover, we also study the asymptotic behavior, as n diverges, of previous problems when in (2.5) and in (2.6) $\left(\phi^a_{(p^a,p^b)},\phi^b_{(p^a,p^b)}\right)$ is the unique solution to

$$\begin{cases}\left(\phi^a_{(p^a,p^b)},\phi^b_{(p^a,p^b)}\right)\in U^0_n,\\ \displaystyle\int_{\Omega^a}\left(-D^a_n\phi^a_{(p^a,p^b)}+p^a\right)D^a_n\phi^a\,\mathrm{d}x+\int_{\Omega^b}\left(-D^b_n\phi^b_{(p^a,p^b)}+p^b\right)D^b_n\phi^b\,\mathrm{d}x=0,\\ \forall(\phi^a,\phi^b)\in U^0_n,\end{cases}\tag{2.13}$$

which rescales the weak formulation of (1.8), *i.e.*

$$\varphi_{\mathbf{P}}\in H^1_0(\Omega_n),\quad \int_{\Omega_n}\left(-D\varphi_{\mathbf{P}}+\mathbf{P}\right)D\varphi\,\mathrm{d}x=0,\quad \forall\varphi\in H^1_0(\Omega_n).$$

In the sequel, we assume

$$\begin{cases}f^a_n\rightharpoonup f^a \text{ weakly in } \left(L^2(\Omega^a)\right)^3,\\ f^b_n\rightharpoonup f^b \text{ weakly in } \left(L^2(\Omega^b)\right)^3.\end{cases}\tag{2.14}$$

3. Preliminaries

Let (y,z) denote the coordinates in $\mathbb{R}^2$. Obviously, each one of the following problems

$$\begin{cases} r \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} r\,\mathrm{d}y\,\mathrm{d}z = 0, \\ \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} DrD\phi\,\mathrm{d}y\,\mathrm{d}z = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} D_y\phi\,\mathrm{d}y\,\mathrm{d}z, \qquad \forall \phi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \end{cases} \tag{3.1}$$

$$\begin{cases} s \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} s\,\mathrm{d}y\,\mathrm{d}z = 0, \\ \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} DsD\phi\,\mathrm{d}y\,\mathrm{d}z = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} D_z\phi\,\mathrm{d}y\,\mathrm{d}z, \qquad \forall \phi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \end{cases} \tag{3.2}$$

$$\begin{cases} t_c \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} t_c\,\mathrm{d}y\,\mathrm{d}z = 0, \\ \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} Dt_cD\phi\,\mathrm{d}y\,\mathrm{d}z = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} cD\phi\,\mathrm{d}y\,\mathrm{d}z, \qquad \forall \phi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \end{cases} \tag{3.3}$$

with $c = (c_1, c_2) \in \mathbb{R}^2$, admits a unique solution.

Note that (compare also [11])

$$s(y,z) = r(z,-y), \text{ a.e. in } \left]-\frac{1}{2},\frac{1}{2}\right[^2,$$

consequently

$$Ds(y,z) = \left(-(D_z r)(z,-y), (D_y r)(z,-y)\right), \text{ a.e. in } \left]-\frac{1}{2},\frac{1}{2}\right[^2,$$

from which one obtains

$$\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} |Ds|^2\,\mathrm{d}y\,\mathrm{d}z = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} |Dr|^2\,\mathrm{d}y\,\mathrm{d}z \text{ and } \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} DrDs\,\mathrm{d}y\,\mathrm{d}z = 0. \tag{3.4}$$

Then, we set

$$\eta = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} |Ds|^2\,\mathrm{d}y\,\mathrm{d}z = \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} |Dr|^2\,\mathrm{d}y\,\mathrm{d}z. \tag{3.5}$$

In the sequel, we shall use the following result.

Lemma 3.1. *Let r and s be the unique solutions to (3.1) and (3.2), respectively. Then, for every $c = (c_1, c_2)$ in $\mathbb{R}^2$, the unique solution t_c to (3.3) is given by*

$$t_c = c_1 r + c_2 s.$$

We recall the Poincaré Lemma in an open bounded set (for instance, see [13], Thm. 6.17-4)

Lemma 3.2. *Let $\xi \in \left(L^2\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)\right)^2$ such that rot $\xi = 0$. Then, there exists $w \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$ such that $\xi = Dw$. Moreover, w is unique up to an additive constant.*

4. Two convergence results for the nonlocal term

This section is devoted to studying the asymptotic behavior of the nonlocal term generated by the potential solution to problem (2.7) or to problem (2.13).

Let

$$U = \left\{(\psi^a, \psi^b) \in H^1(]0,1[) \times H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\right) : \psi^a(0) = \psi^b(0)\right\} \tag{4.1}$$

and

$$U_{\mathrm{reg}} = \Bigg\{(\psi^a, \psi^b) \in C^1([0,1]) \times C([-\tfrac{1}{2}, \tfrac{1}{2}]) : \\ \psi^b|_{[-\frac{1}{2},0]} \in C^1([-\tfrac{1}{2}, 0]), \quad \psi^b|_{[0,\frac{1}{2}]} \in C^1([0, \tfrac{1}{2}]), \quad \psi^a(0) = \psi^b(0)\Bigg\}. \tag{4.2}$$

Proposition 4.1. *Let U and U_{reg} be defined in (4.1) and (4.2) respectively. Then U_{reg} is dense in U.*

Proof. Let $(\psi^a, \psi^b) \in U$. The goal is to find a sequence $\{(\psi^a_n, \psi^b_n)\}_{n\in\mathbb{N}} \subset U_{\mathrm{reg}}$ such that

$$(\psi^a_n, \psi^b_n) \to (\psi^a, \psi^b) \quad \text{strongly in} \quad H^1(]0,1[) \times H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\right). \tag{4.3}$$

To this end, split $\psi^b = \psi^e + \psi^o$ in the even part and in the odd part with respect to x_1 (compare [22] and [27]). Note that ψ^e and ψ^o belong to $H^1(\left]-\frac{1}{2}, \frac{1}{2}\right[)$, and

$$\psi^e(0) = \psi^b(0) = \psi^a(0), \quad \psi^o(0) = 0.$$

Consequently, a convolution argument allows us to build three sequences $\{\zeta^a_n\}_{n\in\mathbb{N}} \subset C^\infty(]0,1[)$, $\{\zeta^e_n\}_{n\in\mathbb{N}} \subset C(\left]-\frac{1}{2}, \frac{1}{2}\right[)$ and $\{\zeta^o_n\}_{n\in\mathbb{N}} \subset C^\infty(\left]-\frac{1}{2}, \frac{1}{2}\right[)$ such that

$$\begin{cases} \left\{\zeta^e_n|_{[-\frac{1}{2},0]}\right\}_{n\in\mathbb{N}} \subset C^\infty([-\tfrac{1}{2}, 0]), \\ \left\{\zeta^e_n|_{[0,\frac{1}{2}]}\right\}_{n\in\mathbb{N}} \subset C^\infty([0, \tfrac{1}{2}]), \\ \zeta^a_n \to \psi^a \quad \text{strongly in} \quad H^1(]0,1[), \\ \zeta^e_n \to \psi^e \quad \text{strongly in} \quad H^1(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[), \quad \zeta^0_n \to \zeta^0 \quad \text{strongly in} \quad H^1(]-\tfrac{1}{2}, \tfrac{1}{2}[), \\ \zeta^a_n(0) = \zeta^0_n(0), \quad \zeta^0_n(0) = 0, \quad \forall n \in \mathbb{N}. \end{cases} \tag{4.4}$$

This implies (4.3), setting $\psi^a_n = \zeta^a_n$ and $\psi^b_n = \zeta^e_n + \zeta^o_n$. ☐

Proposition 4.2. *Let $\{(q^a_n, q^b_n)\}_{n\in\mathbb{N}} \subset \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3$, and let $(q^a, q^b) = ((q^a_1, q^a_2, q^a_3), (q^b_1, q^b_2, q^b_3)) \in \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3$ be such that q^a is independent of (x_1, x_2), q^b is independent of (x_2, x_3) and*

$$(q^a_n, q^b_n) \to (q^a, q^b) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3. \tag{4.5}$$

Moreover, for $n \in \mathbb{N}$ let $(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)})$ be the unique solution to

$$\begin{cases} \left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right) \in U_n, \quad \displaystyle\int_{\Omega^a} \phi^a_{(q^a_n,q^b_n)}\,\mathrm{d}x = 0, \\ \displaystyle\int_{\Omega^a} \left(-D^a_n \phi^a_{(q^a_n,q^b_n)} + q^a_n\right) D^a_n \phi^a\,\mathrm{d}x + \int_{\Omega^b} \left(-D^b_n \phi^b_{(q^a_n,q^b_n)} + q^b_n\right) D^b_n \phi^b\,\mathrm{d}x = 0, \\ \forall (\phi^a, \phi^b) \in U_n, \end{cases} \tag{4.6}$$

where U_n is defined in (2.3). Then,

$$
\begin{cases}
\phi^a_{(q^a_n,q^b_n)} \to \displaystyle\int_0^{x_3} q^a_3(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q^a_3(t)\,\mathrm{d}t\right) \mathrm{d}x_3 & \text{strongly in} \quad \left(H^1(\Omega^a)\right), \\
\phi^b_{(q^a_n,q^b_n)} \to \displaystyle\int_{-\frac{1}{2}}^{x_1} q^b_1(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q^a_3(t)\,\mathrm{d}t\right) \mathrm{d}x_3 & \\
\qquad\qquad - \displaystyle\int_{-\frac{1}{2}}^{0} q^b_1(t)\,\mathrm{d}t & \text{strongly in} \quad \left(H^1(\Omega^b)\right), \\
\left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_2}\right) \to q^a_1 Dr + q^a_2 Ds & \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\
\left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_3}\right) \to q^b_2 D\overline{r} + q^b_3 D\overline{s} & \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2,
\end{cases}
\tag{4.7}
$$

and

$$
\begin{aligned}
\lim_n & \left(\int_{\Omega^a} \left|D^a_n \phi^a_{(q^a_n,q^b_n)}\right|^2 \mathrm{d}x + \int_{\Omega^b} \left|D^a_n \phi^b_{(q^a_n,q^b_n)}\right|^2 \mathrm{d}x\right) \\
&= \eta \int_0^1 \left(|q^a_1|^2 + |q^a_2|^2\right) \mathrm{d}x_3 + \int_0^1 |q^a_3|^2 \,\mathrm{d}x_3 \\
&\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q^b_1\right|^2 \mathrm{d}x_1 + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left|q^b_2\right|^2 + \left|q^b_3\right|^2\right) \mathrm{d}x_1.
\end{aligned}
\tag{4.8}
$$

where r and s are the unique solutions to (3.1) and (3.2), respectively, $\overline{r}$ and $\overline{s}$ are defined by

$$
\overline{r} = r\left(x_2, x_3 + \frac{1}{2}\right), \qquad \overline{s} = s\left(x_2, x_3 + \frac{1}{2}\right), \quad \text{a.e. in} \quad \left]-\frac{1}{2}, \frac{1}{2}\right[\times]0,1[,
$$

and η is defined in (3.5)*.*

Proof. In this proof, C denotes any positive constant independent of $n \in \mathbb{N}$.

Choosing $(\phi^a, \phi^b) = \left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right)$ as test function in (4.6), applying Young inequality, and using (4.5) give

$$
||D^a_n \phi^a_{(q^a_n,q^b_n)}||_{(L^2(\Omega^a))^3} \leq C, \quad ||D^b_n \phi^b_{(q^a_n,q^b_n)}||_{(L^2(\Omega^b))^3} \leq C, \quad \forall n \in \mathbb{N}. \tag{4.9}
$$

The first estimate in (4.9) implies

$$
||\phi^a_{(q^a_n,q^b_n)}||_{H^1(\Omega^a)} \leq C, \quad \forall n \in \mathbb{N}, \tag{4.10}
$$

since $\displaystyle\int_{\Omega^a} \phi^a_{(q^a_n,q^b_n)}\,\mathrm{d}x = 0$ and the Poincaré–Wirtinger inequality holds.

The next step is devoted to proving

$$
||\phi^b_{(q^a_n,q^b_n)}||_{H^1(\Omega^b)} \leq C, \quad \forall n \in \mathbb{N}. \tag{4.11}
$$

The junction condition in (2.3) gives

$$\begin{aligned}
&\int_{]-\frac{h_n}{2},\frac{h_n}{2}[\times]-\frac{1}{2},\frac{1}{2}[} \left|\phi^b_{(q^a_n,q^b_n)}(x_1,x_2,0)\right|^2 \mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad = h_n \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\phi^b_{(q^a_n,q^b_n)}(h_n x_1,x_2,0)\right|^2 \mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad = h_n \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left|\phi^a_{(q^a_n,q^b_n)}(x_1,x_2,0)\right|^2 \mathrm{d}x_1\,\mathrm{d}x_2, \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{4.12}$$

Then, (4.12), (4.10) and the trace theorem provide

$$||\phi^b_{(q^a_n,q^b_n)}||_{L^2\left(]-\frac{h_n}{2},\frac{h_n}{2}[\times]-\frac{1}{2},\frac{1}{2}[\times\{0\}\right)} \leq \sqrt{h_n}C, \quad \forall n\in\mathbb{N},$$

which implies

$$||\phi^b_{(q^a_n,q^b_n)}||_{H^1\left(]-\frac{h_n}{2},\frac{h_n}{2}[\times]-\frac{1}{2},\frac{1}{2}[\times]-1,0[\right)} \leq C, \quad \forall n\in\mathbb{N}, \tag{4.13}$$

by virtue of the second estimates in (4.9). Consequently, by virtue of trace theorem,

$$||\phi^b_{(q^a_n,q^b_n)}||_{L^2\left(\{0\}\times]-\frac{1}{2},\frac{1}{2}[\times]-1,0[\right)} \leq C, \quad \forall n\in\mathbb{N},$$

which combined again with the second estimates in (4.9) proves (4.11). Estimates (4.9)–(4.11) ensure the existence of a subsequence of $\mathbb{N}$, still denotes by $\{n\}$ and (in possible dependence on the subsequence) $(\tau^a,\tau^b)\in U$ defined in (4.1), $(\xi^a,\zeta^a)\in(L^2(\Omega^a))^2$ and $(\xi^b,\zeta^b)\in(L^2(\Omega^b))^2$ such that

$$\left(\phi^a_{(q^a_n,q^b_n)},\phi^b_{(q^a_n,q^b_n)}\right) \rightharpoonup (\tau^a,\tau^b) \text{ weakly in } H^1(\Omega^a)\times H^1(\Omega^b), \tag{4.14}$$

$$\left(\frac{1}{h_n}\frac{\partial\phi^a_{(q^a_n,q^b_n)}}{\partial x_1},\frac{1}{h_n}\frac{\partial\phi^a_{(q^a_n,q^b_n)}}{\partial x_2}\right) \rightharpoonup (\xi^a,\zeta^a) \text{ weakly in } \left(L^2(\Omega^a)\right)^2, \tag{4.15}$$

$$\left(\frac{1}{h_n}\frac{\partial\phi^b_{(q^a_n,q^b_n)}}{\partial x_2},\frac{1}{h_n}\frac{\partial\phi^b_{(q^a_n,q^b_n)}}{\partial x_3}\right) \rightharpoonup (\xi^b,\zeta^b) \text{ weakly in } \left(L^2(\Omega^b)\right)^2, \tag{4.16}$$

and

$$\int_0^1 \tau^a\,\mathrm{d}x_3 = 0. \tag{4.17}$$

Note that the junction condition $\tau^a(0)=\tau^b(0)$ can be obtained arguing as in [20], while (4.17) follows from $\int_{\Omega^a}\phi^a_{(q^a_n,q^b_n)}\,\mathrm{d}x = 0$.

The next step is devoted to identify (τ^a,τ^b). To this end, for every couple $(\psi^a,\psi^b)\in U_{\mathrm{reg}}$ where U_{reg} is defined in (4.2), consider a sequence $\{\mu_n\}_{n\in\mathbb{N}}\subset H^1(\Omega^a)$ (depending on (ψ^a,ψ^b)) such that

$$\begin{cases}
\left(\mu_n,\psi^b\right)\in U_n, & \forall n\in\mathbb{N}, \\
\mu_n\to\psi^a & \text{strongly in} \quad L^2(\Omega^a), \\
\left(\dfrac{1}{h_n}\dfrac{\partial\mu_n}{\partial x_1},\dfrac{1}{h_n}\dfrac{\partial\mu_n}{\partial x_2},\dfrac{\partial\mu_n}{\partial x_3}\right)\to\left(0,0,\dfrac{d\psi^a}{\mathrm{d}x_3}\right) & \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3.
\end{cases} \tag{4.18}$$

For instance, setting

$$\mu_n(x) = \begin{cases} \psi^a(x_3) & \text{if} \quad x = (x_1, x_2, x_3) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times]h_n, 1[, \\ \psi^a(h_n)\dfrac{x_3}{h_n} + \psi^b(h_n x_1)\dfrac{h_n - x_3}{h_n} & \text{if} \quad x = (x_1, x_2, x_3) \in \left]-\frac{1}{2}, \frac{1}{2}\right[^2 \times [0, h_n], \end{cases} \tag{4.19}$$

the first two properties in (4.18) can be immediately verified by the properties of U_{reg}, while the last ones follows from

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2 \times]0,h_n[} \left| \frac{1}{h_n} \frac{\partial \mu_n}{\partial x_1} \right|^2 \mathrm{d}x = \int_{]-\frac{1}{2},\frac{1}{2}[^2 \times]0,h_n[} \left| \frac{\mathrm{d}\psi^b}{\mathrm{d}x_1}(h_n x_1) \left(1 - \frac{x_3}{h_n}\right) \right|^2 \mathrm{d}x$$

$$= \int_{]-\frac{1}{2},\frac{1}{2}[^2} \left| \frac{\mathrm{d}\psi^b}{\mathrm{d}x_1}(h_n x_1) \right|^2 \mathrm{d}x_1\, \mathrm{d}x_2 \int_0^{h_n} \left(1 - \frac{x_3}{h_n}\right)^2 \mathrm{d}x_3 \leq ||\psi^b||^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[)} h_n, \quad \forall n \in \mathbb{N},$$

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2 \times]0,h_n[} \left| \frac{1}{h_n} \frac{\partial \mu_n}{\partial x_2} \right|^2 \mathrm{d}x = 0, \quad \forall n \in \mathbb{N},$$

$$\int_{]-\frac{1}{2},\frac{1}{2}[^2 \times]0,h_n[} \left| \frac{\partial \mu_n}{\partial x_3} \mu_n \right|^2 \mathrm{d}x = \int_{]-\frac{1}{2},\frac{1}{2}[^2 \times]0,h_n[} \left| \psi^a(h_n) \frac{1}{h_n} - \psi^b(h_n x_1) \frac{1}{h_n} \right|^2 \mathrm{d}x$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{h_n} \left| \psi^a(h_n) - \psi^b(h_n x_1) \right|^2 \mathrm{d}x_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{h_n} \left| \psi^a(h_n) - \psi^a(0) + \psi^b(0) - \psi^b(h_n x_1) \right|^2 \mathrm{d}x_1$$

$$\leq 2 \left(||\psi^a||^2_{W^{1,\infty}(]-0,1[)} + ||\psi^b||^2_{W^{1,\infty}(]-\frac{1}{2},\frac{1}{2}[)} \right) h_n, \quad \forall n \in \mathbb{N},$$

where again the properties of U_{reg} played a crucial role.

Now, fixing $(\psi^a, \psi^b) \in U_{\text{reg}}$, choosing (μ_n, ψ^b) as test function in (4.6) with μ_n satisfying (4.18), passing to the limit as n diverges, and using (4.5), (4.14)–(4.16), one obtains

$$\int_0^1 \left(-\frac{\mathrm{d}\tau^a}{\mathrm{d}x_3} + q_3^a \right) \frac{\mathrm{d}\psi^a}{\mathrm{d}x_3} \mathrm{d}x_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-\frac{\mathrm{d}\tau^b}{\mathrm{d}x_1} + q_1^b \right) \frac{\mathrm{d}\psi^b}{\mathrm{d}x_1} \mathrm{d}x_1 = 0. \tag{4.20}$$

By virtue of Proposition 4.1, equation (4.20) holds true also with any test function in U. The uniqueness of the solution of this problem is ensured by (4.17) and the junction condition $\tau^a(0) = \tau^b(0)$. Consequently, (τ^a, τ^b) is given by

$$\begin{cases} \tau^a = \displaystyle\int_0^{x_3} q_3^a(t)\, \mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q_3^a(t)\, \mathrm{d}t \right) \mathrm{d}x_3, \\ \tau^b = \displaystyle\int_{-\frac{1}{2}}^{x_1} q_1^b(t)\, \mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q_3^a(t)\, \mathrm{d}t \right) \mathrm{d}x_3 - \int_{-\frac{1}{2}}^0 q_1^b(t)\, \mathrm{d}t, \end{cases}$$

which combined with (4.14) proves that

$$\begin{cases} \phi^a_{(q_n^a, q_n^b)} \rightharpoonup \displaystyle\int_0^{x_3} q_3^a(t)\, \mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q_3^a(t)\, \mathrm{d}t \right) \mathrm{d}x_3 & \text{weakly in} \quad \left(H^1(\Omega^a)\right), \\ \phi^b_{(q_n^a, q_n^b)} \rightharpoonup \displaystyle\int_{-\frac{1}{2}}^{x_1} q_1^b(t)\, \mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q_3^a(t)\, \mathrm{d}t \right) \mathrm{d}x_3 & \\ \qquad - \displaystyle\int_{-\frac{1}{2}}^0 q_1^b(t)\, \mathrm{d}t & \text{weakly in} \quad \left(H^1(\Omega^b)\right). \end{cases} \tag{4.21}$$

Let us identify (ξ^a, ζ^a). To this aim, starting from the following evident relation

$$\frac{\partial}{\partial x_2}\left(\frac{1}{h_n}\frac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{1}{h_n}\frac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_2}\right) \quad \text{in} \quad \mathcal{D}'(\Omega^a), \quad \forall n \in \mathbb{N},$$

and using (4.15), one obtains that

$$\int_{\Omega^a} \xi^a \frac{\partial \varphi}{\partial x_2}\,\mathrm{d}x = \int_{\Omega_a} \zeta^a \frac{\partial \varphi}{\partial x_1}\,\mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega^a). \tag{4.22}$$

By taking $\varphi(x) = \phi(x_1,x_2)\chi(x_3)$ with $\phi \in C_0^\infty\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$ and $\chi \in C_0^\infty(]0,1[)$ and recalling that $C_0^\infty\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$ is separable, it follows from (4.22) that

$$\begin{cases} \text{for } x_3 \text{ a.e. in }]0,1[, \quad \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \xi^a(x_1,x_2,x_3)\frac{\partial \phi}{\partial x_2}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\[2ex] = \displaystyle\int_{\left(\left]-\frac{1}{2},\frac{1}{2}\right[\right)^2} \zeta^a(x_1,x_2,x_3)\frac{\partial \phi}{\partial x_1}(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad \forall \phi \in C_0^\infty\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right). \end{cases}$$

Consequently, by virtue of the Poincaré Lemma recalled in Lemma 3.2, it results that

$$\begin{cases} \text{for } x_3 \text{ a.e. in }]0,1[, \quad \exists! w^a(\cdot,\cdot,x_3) \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right): \\[2ex] \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} w^a(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0, \\[2ex] \xi^a(\cdot,\cdot,x_3) = \dfrac{\partial w^a(\cdot,\cdot,x_3)}{\partial x_1}, \quad \zeta^a(\cdot,\cdot,x_3) = \dfrac{\partial w^a(\cdot,\cdot,x_3)}{\partial x_2}, \quad \text{a.e. in } \left]-\frac{1}{2},\frac{1}{2}\right[^2. \end{cases} \tag{4.23}$$

Passing to the limit in (4.6) with $(\phi^a,\phi^b) = (h_n\varphi\chi, 0)$ where $\varphi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$ and $\chi \in C_0^\infty(]0,1[)$, and using (4.5), (4.21) and (4.15) give

$$\begin{aligned} &\int_0^1 \left(\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} (\xi^a,\zeta^a)\left(\frac{\partial \varphi}{\partial x_1},\frac{\partial \varphi}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2\right)\chi\,\mathrm{d}x_3 \\ &= \int_0^1 \left((q_1^a,q_2^a)\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\left(\frac{\partial \varphi}{\partial x_1},\frac{\partial \varphi}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2\right)\chi\,\mathrm{d}x_3, \\ &\forall \varphi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \quad \forall \chi \in C_0^\infty(]0,1[). \end{aligned} \tag{4.24}$$

Consequently, since $H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right)$ is separable, one obtains that

$$\begin{cases} \text{for } x_3 \text{ a.e. in }]0,1[, \\[2ex] \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} (\xi^a(x_1,x_2,x_3),\zeta^a(x_1,x_2,x_3))\left(\frac{\partial \varphi}{\partial x_1},\frac{\partial \varphi}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2 \\[2ex] = (q_1^a(x_3),q_2^a(x_3))\displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\left(\frac{\partial \varphi}{\partial x_1},\frac{\partial \varphi}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2, \quad \forall \phi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \end{cases} \tag{4.25}$$

from which, by virtue of (4.25), it follows that for x_3 a.e. in $]0,1[$, $w^a(\cdot,\cdot,x_3)$ solves the following problem

$$\begin{cases} w^a(\cdot,\cdot,x_3) \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right), \\ \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} w^a(x_1,x_2,x_3)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0, \\ \displaystyle\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} \left(\frac{\partial w^a}{\partial x_1},\frac{\partial w^a}{\partial x_2}\right)\left(\frac{\partial \varphi}{\partial x_1},\frac{\partial \varphi}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2 \\ \displaystyle = \left(q_1^a(x_3), q_2^a(x_3)\right)\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2}\left(\frac{\partial \varphi}{\partial x_1},\frac{\partial \varphi}{\partial x_2}\right)\mathrm{d}x_1\,\mathrm{d}x_2, \quad \forall \varphi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[^2\right). \end{cases} \tag{4.26}$$

Then, by virtue of Lemma 3.1, it results that, for x_3 a.e. in $]0,1[$,

$$w^a(\cdot,\cdot,x_3) = q_1^a(x_3)r(\cdot,\cdot) + q_2^a(x_3)s(\cdot,\cdot), \text{ a.e. in } \left]-\frac{1}{2},\frac{1}{2}\right[^2, \tag{4.27}$$

with r (resp. s) the unique solution to (3.1) (resp. (3.2)).

Finally, since Tonelli theorem assures that $q_1^a Dr_1 + q_2^a Ds_2$ belong to $(L^2(\Omega^a))^2$, using Fubini theorem with (4.25) and (4.27) one entails that

$$\begin{aligned} \int_{\Omega^a} (\xi^a,\zeta^a)\,\varphi\,\mathrm{d}x &= \int_0^1 \left(\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} (\xi^a,\zeta^a)\,\varphi\,\mathrm{d}x_1\,\mathrm{d}x_2\right)\mathrm{d}x_3 \\ &= \int_0^1 \left(\int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2} (q_1^a Dr + q_2^a Ds)\,\varphi\,\mathrm{d}x_1\,\mathrm{d}x_2\right)\mathrm{d}x_3 \\ &= \int_{\Omega^a} (q_1^a Dr + q_2^a Ds)\,\varphi\,\mathrm{d}x, \quad \forall \varphi \in C_0^\infty(\Omega^a), \end{aligned}$$

that is

$$(\xi^a,\zeta^a) = q_1^a(x_3)Dr(x_1,x_2) + q_2^a(x_3)Ds(x_1,x_2), \qquad \text{a.e. in } \Omega^a. \tag{4.28}$$

Then, combining (4.15) with (4.28) provides

$$\left(\frac{1}{h_n}\frac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_1}, \frac{1}{h_n}\frac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_2}\right) \rightharpoonup q_1^a Dr + q_2^a Ds \text{ weakly in } \left(L^2(\Omega^a)\right)^2. \tag{4.29}$$

Now, limits (4.5), (4.21), (4.29) imply

$$\lim_n \int_{\Omega^a} D_n^a \phi^a_{(q_n^a,q_n^b)} q_n^a\,\mathrm{d}x = \int_{\Omega^a} (q_1^a Dr + q_2^a Ds)(q_1^a,q_2^a)\,\mathrm{d}x + \int_0^1 |q_3^a|^2\,\mathrm{d}x_3. \tag{4.30}$$

On the other side, using equation (4.26) with test function $q_1^a(x_3)r(\cdot,\cdot) + q_2^a(x_3)s(\cdot,\cdot)$, for x_3 a.e. in $]0,1[$, and taking into account (4.27), (3.4), and (3.5) give

$$\int_{\Omega^a} (q_1^a Dr + q_2^a Ds)(q_1^a,q_2^a)\,\mathrm{d}x = \int_{\Omega^a} |q_1^a Dr + q_2^a Ds|^2\,\mathrm{d}x = \eta\int_0^1 \left(|q_1^a|^2 + |q_2^a|^2\right)\mathrm{d}x_3 \tag{4.31}$$

Then, combining (4.30) and (4.31) provides

$$\lim_n \int_{\Omega^a} D_n^a \phi^a_{(q_n^a,q_n^b)} q_n^a\,\mathrm{d}x = \eta\int_0^1 \left(|q_1^a|^2 + |q_2^a|^2\right)\mathrm{d}x_3 + \int_0^1 |q_3^a|^2\,\mathrm{d}x_3 \tag{4.32}$$

Similarly, to identify (ξ^b, ζ^b) one has

$$\begin{cases} \text{for } x_1 \text{ a.e. in } \left]-\dfrac{1}{2}, \dfrac{1}{2}\right[, \quad \exists ! w^b(x_1 \cdot, \cdot) \in H^1\left(\left]-\dfrac{1}{2}, \dfrac{1}{2}\right[\times]-1,0[\right): \\ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} w^b(x_1,x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 = 0, \\ \xi^b(x_1,\cdot,\cdot) = \dfrac{\partial w^b(x_1,\cdot,\cdot)}{\partial x_2}, \quad \zeta^b(x_1,\cdot,\cdot) = \dfrac{\partial w^b(x_1,\cdot,\cdot)}{\partial x_3}, \text{ a.e. in } \left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[. \end{cases} \tag{4.33}$$

Passing to the limit in (4.6) with $(\phi^a, \phi^b) = (0, h_n\varphi\chi)$ where $\varphi \in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right)$ and $\chi \in C_0^\infty\left(\left]-\frac{1}{2},0\right[\cup\left]0,\frac{1}{2}\right[\right)$ (note that $(0,h_n\varphi\chi)\in U_n$ for n large enough), and using (4.5), (4.21) and (4.16) give

$$\begin{aligned} &\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[}(\xi^b,\zeta^b)\left(\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_3}\right)\mathrm{d}x_2\,\mathrm{d}x_3\right)\chi\,\mathrm{d}x_1 \\ &\quad= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left((q_2^b,q_3^b)\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[}\left(\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_3}\right)\mathrm{d}x_2\,\mathrm{d}x_3\right)\chi\,\mathrm{d}x_1, \\ &\quad\forall\varphi\in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right), \quad \forall\chi\in C_0^\infty\left(\left]-\frac{1}{2},0\right[\cup\left]0,\frac{1}{2}\right[\right). \end{aligned} \tag{4.34}$$

Consequently, since $H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right)$ is separable, one obtains that

$$\begin{cases} \text{for } x_1 \text{ a.e. in } \left]-\frac{1}{2},\frac{1}{2}\right[, \\ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[}(\xi^b,\zeta^b)\left(\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_3}\right)\mathrm{d}x_2\,\mathrm{d}x_3 \\ \displaystyle= (q_2^b(x_1),q_3^b(x_1))\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[}\left(\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_3}\right)\mathrm{d}x_2\,\mathrm{d}x_3, \\ \forall\varphi\in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right), \end{cases} \tag{4.35}$$

from which, by virtue of (4.33), it follows that for x_1 a.e. in $\left]-\frac{1}{2},\frac{1}{2}\right[$, $w^b(x_1\cdot,\cdot)$ solves the following problem

$$\begin{cases} w^b(x_1\cdot,\cdot)\in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right), \\ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} w^b(x_1,x_2,x_3)\,\mathrm{d}x_2\,\mathrm{d}x_3 = 0, \\ \displaystyle\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[}\left(\frac{\partial w^b}{\partial x_2},\frac{\partial w^b}{\partial x_3}\right)\left(\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_3}\right)\mathrm{d}x_2\,\mathrm{d}x_3 \\ \displaystyle= (q_2^b(x_1),q_3^b(x_1))\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[}\left(\frac{\partial\varphi}{\partial x_2},\frac{\partial\varphi}{\partial x_3}\right)\mathrm{d}x_2\,\mathrm{d}x_3, \quad \forall\varphi\in H^1\left(\left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[\right), \end{cases}$$

Then, by virtue of Lemma 3.1, it results that, for x_1 a.e. in $\left]-\frac{1}{2},\frac{1}{2}\right[$,

$$w^b(x_1,\cdot,\cdot) = q_1^a(x_1)\overline{r}(\cdot,\cdot) + q_2^a(x_1)\overline{s}(\cdot,\cdot), \quad \text{a.e. in} \quad \left]-\frac{1}{2},\frac{1}{2}\right[\times]-1,0[,$$

where $\overline{r} = r\left(x_2, x_3 + \frac{1}{2}\right)$, $\overline{s} = s\left(x_2, x_3 + \frac{1}{2}\right)$.

Then, arguing as above, one obtains

$$\left(\frac{1}{h_n}\frac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_2}, \frac{1}{h_n}\frac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_3}\right) \rightharpoonup q^b_2 D\overline{r} + q^b_3 D\overline{s} \quad \text{weakly in} \quad \left(L^2(\Omega^b)\right)^2 \tag{4.36}$$

and

$$\begin{aligned} \lim_n \int_{\Omega^b} D^b_n \phi^a_{(q^a_n,q^b_n)} q^b_n \, \mathrm{d}x &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q^b_1\right|^2 \mathrm{d}x_1 + \int_{\Omega^b} |q^b_2 D\overline{r} + q^b_3 D\overline{s}|^2 \, \mathrm{d}x \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q^b_1\right|^2 \mathrm{d}x_1 + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left|q^b_2\right|^2 + \left|q^b_3\right|^2\right) \mathrm{d}x_1. \end{aligned} \tag{4.37}$$

Passing to the limit in (4.6) with $(\phi^a, \phi^b) = \left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right)$ and using (4.32) and (4.37) one obtains the convergence of the energies

$$\begin{aligned} &\lim_n \left(\int_{\Omega^a} \left|D^a_n \phi^a_{(q^a_n,q^b_n)}\right|^2 \mathrm{d}x + \int_{\Omega^b} \left|D^a_n \phi^b_{(q^a_n,q^b_n)}\right|^2 \mathrm{d}x\right) \\ &= \lim_n \left(\int_{\Omega^a} D^a_n \phi^a_{(q^a_n,q^b_n)} q^a_n \, \mathrm{d}x + \int_{\Omega^a} D^b_n \phi^b_{(q^a_n,q^b_n)} q^b_n \, \mathrm{d}x\right) \\ &= \int_{\Omega^a} |q^a_1 Dr + q^a_2 Ds|^2 \, \mathrm{d}x + \int_0^1 |q^a_3|^2 \, \mathrm{d}x_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q^b_1\right|^2 \mathrm{d}x_1 + \int_{\Omega^b} |q^b_2 D\overline{r} + q^b_3 D\overline{s}|^2 \, \mathrm{d}x \\ &= \eta \int_0^1 \left(|q^a_1|^2 + |q^a_2|^2\right) \mathrm{d}x_3 + \int_0^1 |q^a_3|^2 \, \mathrm{d}x_3 \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q^b_1\right|^2 \mathrm{d}x_1 + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left|q^b_2\right|^2 + \left|q^b_3\right|^2\right) \mathrm{d}x_1. \end{aligned} \tag{4.38}$$

Finally, (4.7) and (4.8) follow from (4.21), (4.29), (4.36), and(4.38). ☐

Proposition 4.3. *Let* $\{(q^a_n, q^b_n)\}_{n\in\mathbb{N}} \subset \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3$, *and let* $(q^a, q^b) = ((q^a_1, q^a_2, q^a_3), (q^b_1, q^b_2, q^b_3)) \in \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^b)\right)^3$ *be such that* q^a *is independent of* (x_1, x_2), q^b *is independent of* (x_2, x_3) *and*

$$(q^a_n, q^b_n) \to (q^a, q^b) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3. \tag{4.39}$$

Moreover, for $n \in \mathbb{N}$ *let* $(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)})$ *be the unique solution to*

$$\begin{cases} \left(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)}\right) \in U^0_n, \\ \displaystyle\int_{\Omega^a} \left(-D^a_n \phi^a_{(q^a_n,q^b_n)} + q^a_n\right) D^a_n \phi^a \, \mathrm{d}x \\ \displaystyle + \int_{\Omega^b} \left(-D^b_n \phi^b_{(q^a_n,q^b_n)} + q^b_n\right) D^b_n \phi^b \, \mathrm{d}x = 0, \quad \forall (\phi^a, \phi^b) \in U^0_n, \end{cases} \tag{4.40}$$

where U_n^0 is defined in (2.4). Then,

$$\begin{cases} \phi^a_{(q_n^a,q_n^b)} \to 0 & \text{strongly in} \quad \left(H^1(\Omega^a)\right), \\ \phi^b_{(q_n^a,q_n^b)} \to 0 & \text{strongly in} \quad \left(H^1(\Omega^b)\right), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_2}\right) \to (0,0) & \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_3}\right) \to (0,0) & \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2, \end{cases} \tag{4.41}$$

and

$$\lim_n \left(\int_{\Omega^a} \left|D_n^a \phi^a_{(q_n^a,q_n^b)}\right|^2 \,\mathrm{d}x + \int_{\Omega^b} \left|D_n^a \phi^b_{(q_n^a,q_n^b)}\right|^2 \,\mathrm{d}x\right) = 0. \tag{4.42}$$

Proof. In this proof, C denotes any positive constant independent of $n \in \mathbb{N}$.

Choosing $(\phi^a, \phi^b) = \left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right)$ as test function in (4.40), applying Young inequality, and using (4.39) give

$$||D_n^a \phi^a_{(q_n^a,q_n^b)}||_{(L^2(\Omega^a))^3} \le C, \quad ||D_n^b \phi^b_{(q_n^a,q_n^b)}||_{(L^2(\Omega^b))^3} \le C, \quad \forall n \in \mathbb{N}. \tag{4.43}$$

Consequently, taking into account the boundary conditions satisfied by $\left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right)$ and the trace theorem, one derives that

$$\left(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)}\right) \rightharpoonup (0,0) \text{ weakly in } H^1(\Omega^a) \times H^1(\Omega^b), \tag{4.44}$$

and the existence of a subsequence of $\mathbb{N}$, still denotes by $\{n\}$ and (in possible dependence on the subsequence) $(\xi^a, \zeta^a) \in (L^2(\Omega^a))^2$ and $(\xi^b, \zeta^b) \in (L^2(\Omega^b))^2$ such that

$$\left(\frac{1}{h_n}\frac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_1}, \frac{1}{h_n}\frac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_2}\right) \rightharpoonup (\xi^a, \zeta^a) \text{ weakly in } \left(L^2(\Omega^a)\right)^2, \tag{4.45}$$

$$\left(\frac{1}{h_n}\frac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_2}, \frac{1}{h_n}\frac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_3}\right) \rightharpoonup (\xi^b, \zeta^b) \text{ weakly in } \left(L^2(\Omega^b)\right)^2. \tag{4.46}$$

Let us prove that

$$\int_{\Omega^a} \xi^a q_1^a \,\mathrm{d}x = 0, \quad \int_{\Omega^a} \zeta^a q_2^a \,\mathrm{d}x = 0, \quad \int_{\Omega^b} \xi^b q_2^b \,\mathrm{d}x = 0, \quad \int_{\Omega^b} \zeta^b q_3^b \,\mathrm{d}x = 0. \tag{4.47}$$

Indeed, let

$$g_n(x_3) = \sum_{i=0}^{n-1} \left(n \int_{\frac{i}{n}}^{\frac{i+1}{n}} q_1^a(t)\,\mathrm{d}t \chi_{\left[\frac{i}{n},\frac{i+1}{n}\right[}(x_3)\right), \quad x_3 \quad \text{a.e. in} \quad]0,1[, \quad \forall n \in \mathbb{N}.$$

It is well known that

$$g_n \to q_1^a \quad \text{strongly in} \quad L^2(]0,1[),$$

as n diverges. Consequently, taking also into account (4.45), one has

$$\int_{\Omega^a} \xi^a q_1^a \, \mathrm{d}x = \lim_n \int_{\Omega^a} \frac{1}{h_n} \frac{\partial \phi^a_{(q_n^a, q_n^b)}}{\partial x_1}(x) g_n(x_3) \, \mathrm{d}x$$

$$= \lim_n \sum_{i=0}^{n-1} \left(n \int_{\frac{i}{n}}^{\frac{i+1}{n}} q_1^a(t) \, \mathrm{d}t \frac{1}{h_n} \int_{]-\frac{1}{2},\frac{1}{2}[^2 \times]\frac{i}{n},\frac{i+1}{n}[} \frac{\partial \phi^a_{(q_n^a, q_n^b)}}{\partial x_1}(x) \, \mathrm{d}x \right)$$

and the last integrals are zero due to the boundary condition on $\phi^a_{(q_n^a, q_n^b)}$. It is so proved the first equality in (4.47). Similarly, one proves the other ones.

Now (4.45), (4.46), and a l.s.c. argument provide

$$\lim_n \left(\int_{\Omega^a} \left| D_n^a \phi^a_{(q_n^a, q_n^b)} \right|^2 \mathrm{d}x + \int_{\Omega^b} \left| D_n^a \phi^b_{(q_n^a, q_n^b)} \right|^2 \mathrm{d}x \right) \tag{4.48}$$

$$\geq \int_{\Omega^a} |\xi^a|^2 \, \mathrm{d}x + \int_{\Omega^a} |\zeta^a|^2 \, \mathrm{d}x + \int_{\Omega^b} \left|\xi^b\right|^2 \mathrm{d}x + \int_{\Omega^b} \left|\zeta^b\right|^2 \mathrm{d}x, \tag{4.49}$$

while choosing $(\phi^a_{(q_n^a, q_n^b)}, \phi^b_{(q_n^a, q_n^b)})$ as test functions in (4.40) and using (4.39), (4.44), (4.45), (4.46), and (4.47) provide

$$\begin{aligned} &\int_{\Omega^a} \left| D_n^a \phi^a_{(q_n^a, q_n^b)} \right|^2 \mathrm{d}x + \int_{\Omega^b} \left| D_n^a \phi^b_{(q_n^a, q_n^b)} \right|^2 \mathrm{d}x = \\ &\int_{\Omega^a} D_n^a \phi^a_{(q_n^a, q_n^b)} q_n^a \, \mathrm{d}x + \int_{\Omega^a} D_n^b \phi^b_{(q_n^a, q_n^b)} q_n^b \, \mathrm{d}x \longrightarrow 0, \end{aligned} \tag{4.50}$$

as n diverges. Finally combining (4.49) and (4.50) implies

$$\xi^a = \zeta^a = 0 \text{ in } \Omega^a \text{ and } \xi^b = \zeta^b = 0 \text{ in } \Omega^b,$$

and convergences (4.44), (4.45), and (4.46) are strong. Note that also convergences in (4.45) and (4.46) hold true for the whole sequence, since the limits are uniquely identified.

5. The asymptotic behavior of problem (1.4)

5.1. The main result

Let

$$\begin{aligned} E : (q^a, q^b) = \left((q_1^a, q_2^a, q_3^a), (q_1^b, q_2^b, q_3^b)\right) \in \left(H^1(]0,1[)\right)^3 \times \left(H^1\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right)\right)^3 \to \\ \int_0^1 \left(\left| \frac{\mathrm{d}q^a}{\mathrm{d}x_3} \right|^2 + \alpha \left(|q^a|^2 - 1\right)^2 + \eta \left(|q_1^a|^2 + |q_2^a|^2\right) + |q_3^a|^2 \right) \mathrm{d}x_3 \\ + \int_0^1 \left(\int_{]-\frac{1}{2},\frac{1}{2}[^2} f^a \, \mathrm{d}x_1 \, \mathrm{d}x_2 q^a \right) \mathrm{d}x_3 \\ + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\left| \frac{\mathrm{d}q^b}{\mathrm{d}x_1} \right|^2 + \alpha \left(|q^b|^2 - 1\right)^2 + |q_1^b|^2 + \eta \left(|q_2^b|^2 + |q_3^b|^2\right) \right) \mathrm{d}x_1 \\ + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]-1,0[} f^b \, \mathrm{d}x_2 \, \mathrm{d}x_3 q^b \right) \mathrm{d}x_1, \end{aligned} \tag{5.1}$$

where f^a and f^b are defined in (2.14), and η in (3.5). Moreover, let

$$P = \left\{(q^a, q^b) \in \left(H^1(]0,1[)\right)^3 \times \left(H^1(]-\tfrac{1}{2}, \tfrac{1}{2}[)\right)^3 : q^a(0) = q^b(0)\right\}. \tag{5.2}$$

The main result of this section is the following one.

Theorem 5.1. *For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.9), and let $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution to (2.7) with $(p^a, p^b) = (p_n^a, p_n^b)$. Moreover, let E and P be defined by (5.1) and (5.2), respectively. Assume (2.14). Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $(p^a, p^b) \in P$ such that*

$$\begin{cases} p^a_{n_i} \to p^a \quad \text{strongly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p^b_{n_i} \to p^b \quad \text{strongly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{5.3}$$

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p_n^a}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial p_n^a}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^a)\right)^3, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p_n^b}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \end{cases} \tag{5.4}$$

$$\begin{cases} \phi^a_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_0^{x_3} p_3^a(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p_3^a(t)\,\mathrm{d}t\right) \mathrm{d}x_3 \quad \text{strongly in} \quad H^1(\Omega^a), \\ \phi^b_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_{-\frac{1}{2}}^{x_1} p_1^b(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p_3^a(t)\,\mathrm{d}t\right) \mathrm{d}x_3 \\ \qquad - \displaystyle\int_{-\frac{1}{2}}^{0} p_1^b(t)\,\mathrm{d}t \quad \text{strongly in} \quad H^1(\Omega^b), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}\right) \to p_1^a Dr + p_2^a Ds \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_3}\right) \to p_2^b D\overline{r} + p_3^b D\overline{s} \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2, \end{cases} \tag{5.5}$$

where r and s are the unique solutions to (3.1) and (3.2), respectively, $\overline{r}$ and $\overline{s}$ are defined by

$$\overline{r} = r\left(x_2, x_3 + \frac{1}{2}\right), \qquad \overline{s} = s\left(x_2, x_3 + \frac{1}{2}\right), \text{ a.e. } in \left]-\frac{1}{2}, \frac{1}{2}\right[\times]0,1[,$$

and (p^a, p^b) solves

$$E(p^a, p^b)) = \min\{E((q^a, q^b)) : (q^a, q^b) \in P\}. \tag{5.6}$$

Moreover

$$\lim_n \frac{E_n((p_n^a, p_n^b))}{h_n^2} = E((p^a, p^b)). \tag{5.7}$$

5.2. *A priori* estimates on polarization

Proposition 5.2. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.9). Then, there exists a constant c, independent of $n \in \mathbb{N}$, such that*

$$||p_n^a||_{(L^4(\Omega^a))^3} \le c, \quad ||p_n^b||_{(L^4(\Omega^b))^3} \le c, \qquad \forall n \in \mathbb{N}, \tag{5.8}$$

$$||D_n^a p_n^a||_{(L^2(\Omega^a))^9} \le c, \quad ||D_n^b p_n^b||_{(L^2(\Omega^b))^9} \le c, \quad \forall n \in \mathbb{N}. \tag{5.9}$$

Proof. Function 0 belonging to P_n gives

$$\begin{aligned} &\int_{\Omega^a} \left(|D_n^a\, p_n^a|^2 + \alpha \left(|p_n^a|^4 - 2|p_n^a|^2 \right) + |D_n^a \phi^a_{(p_n^a, p_n^b)}|^2 \right) \mathrm{d}x \\ &\quad + \int_{\Omega^b} \left(|D_n^b\, p_n^b|^2 + \alpha \left(|p_n^b|^4 - 2|p_n^b|^2 \right) + |D_n^b \phi^b_{(p_n^a, p_n^b)}|^2 \right) \mathrm{d}x \\ &\quad \le \frac{1}{2} \int_{\Omega^a} \left(|f_n^a|^2 + |p_n^a|^2 \right) \mathrm{d}x + \frac{1}{2} \int_{\Omega^b} \left(|f_n^b|^2 + |p_n^b|^2 \right) \mathrm{d}x, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.10}$$

Estimates (5.10) implies

$$\begin{aligned} &\int_{\Omega^a} \alpha \left(|p_n^a|^4 - \left(2 + \frac{1}{2\alpha} \right) |p_n^a|^2 \right) \mathrm{d}x + \int_{\Omega^b} \alpha \left(|p_n^b|^4 - \left(2 + \frac{1}{2\alpha} \right) |p_n^b|^2 \right) \mathrm{d}x \\ &\quad \le \frac{1}{2} \int_{\Omega^a} |f_n^a|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\Omega^b} |f_n^b|^2 \, \mathrm{d}x, \quad \forall n \in \mathbb{N}, \end{aligned}$$

which gives

$$\begin{aligned} &\int_{\Omega^a} \alpha \left(|p_n^a|^2 - \left(1 + \frac{1}{4\alpha} \right) \right)^2 \mathrm{d}x + \int_{\Omega^b} \alpha \left(|p_n^b|^2 - \left(1 + \frac{1}{4\alpha} \right) \right)^2 \mathrm{d}x \\ &\quad \le \alpha \left(1 + \frac{1}{4\alpha} \right)^2 \left(|\Omega^a| + |\Omega^b| \right) + \frac{1}{2} \int_{\Omega^a} |f_n^a|^2 \, \mathrm{d}x + \frac{1}{2} \int_{\Omega^b} |f_n^b|^2 \, \mathrm{d}x, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{5.11}$$

Then the estimates in (5.8) follow from (5.11) and (2.14). The estimatess in (5.9) follow from (5.10), (2.14), (5.8), and the continuous embedding of L^4 into L^2. □

By arguing as in [20], Proposition 5.2 provides the following result.

Corollary 5.3. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.9). Let P be defined in (5.2). Then there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p^a, p^b) \in P$ such that*

$$\begin{cases} p_n^a \rightharpoonup p^a & \text{weakly in } \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p_n^b \rightharpoonup p^b & \text{weakly in } \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3. \end{cases} \tag{5.12}$$

5.3. The proof of Theorem 5.1

Proposition 5.2 and Corollary 5.3 assert that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p^a, p^b) \in P$ and $(z^a, z^b) \in (L^2(\Omega^a))^6 \times (L^2(\Omega^b))^6$ satisfying (5.12) and

$$\begin{cases} \left(\dfrac{1}{h_n} \dfrac{\partial p_n^a}{\partial x_1}, \dfrac{1}{h_n} \dfrac{\partial p_n^a}{\partial x_2} \right) \rightharpoonup z^a & \text{weakly in} \quad (L^2(\Omega^a))^6, \\ \left(\dfrac{1}{h_n} \dfrac{\partial p_n^b}{\partial x_2}, \dfrac{1}{h_n} \dfrac{\partial p_n^b}{\partial x_3} \right) \rightharpoonup z^b & \text{weakly in} \quad (L^2(\Omega^b))^6. \end{cases} \tag{5.13}$$

Limits in (5.5) follow from (5.12), thanks to Proposition 4.2.

Let $(q^a, q^b) \in P$ be such that for each $i = 1, 2, 3$ $(q_i^a, q_i^b) \in U_{\text{reg}}$, where U_{reg} is defined in (4.2). As in (4.18) and (4.19), working on each couple (q_i^a, q_i^b), one can build a sequence $\{(q_n^a, q_n^b)\}_{n\in\mathbb{N}}$, with $(q_n^a, q_n^b) \in P_n$, such that, thanks also to (2.14) and Proposition 4.2,

$$\lim_n \frac{E_n\left((q_n^a, q_n^b)\right)}{h_n^2} = E\left((q^a, q^b)\right).$$

Consequently, recalling that (p_n^a, p_n^b) is a solution to (2.9) an using Proposition 4.1, one has

$$\limsup_n \frac{E_n\left((p_n^a, p_n^b)\right)}{h_n^2} \leq E\left((q^a, q^b)\right), \quad \forall (q^a, q^b) \in P. \tag{5.14}$$

On the other side, (2.14), (5.12), (5.13), a l.s.c. argument, and Proposition 4.2 ensure that

$$\int_{\Omega^a} |z^a|^2 \, \mathrm{d}x + \int_{\Omega^b} |z^b|^2 \, \mathrm{d}x + E\left((p^a, p^b)\right) \leq \liminf_n \frac{E_n\left((p_n^a, p_n^b)\right)}{h_n^2}. \tag{5.15}$$

Combining (5.15) and (5.14) with $(q^a, q^b) = (p^a, p^b)$ provides

$$z^a = 0 \text{ a.e. in } \Omega^a, \quad z^b = 0 \text{ a.e. in } \Omega^b. \tag{5.16}$$

Then, (5.6) and (5.7) follow again from (5.14) to (5.16).

To obtain (5.3) and (5.4), it remain to prove that convergences in (5.12) and (5.13) are strong. At first note that (5.7), (5.12), Proposition 4.2, and (2.14) give

$$\lim_n \left(\int_{\Omega^a} |D_n^a p_n^a|^2 \, \mathrm{d}x + \int_{\Omega^b} |D_n^b p_n^b|^2 \, \mathrm{d}x \right) = \int_{\Omega^a} \left| \frac{\mathrm{d}p^a}{\mathrm{d}x_3} \right|^2 \mathrm{d}x + \int_{\Omega^b} \left| \frac{\mathrm{d}p^b}{\mathrm{d}x_1} \right|^2 \mathrm{d}x,$$

which implies (5.4) and

$$\frac{\partial p_n^a}{\partial x_3} \to \frac{\mathrm{d}p^a}{\mathrm{d}x_3} \quad \text{strongly in} \quad L^2(\Omega^a), \quad \frac{\partial p_n^b}{\partial x_1} \to \frac{\mathrm{d}p^b}{\mathrm{d}x_1} \quad \text{strongly in} \quad L^2(\Omega^b), \tag{5.17}$$

thanks to (5.12), (5.13), and (5.16). Eventually, (5.3) follows from (5.4), (5.12), and (5.17).

□

6. The asymptotic behavior of problem (1.5)

6.1. The main result

Set

$$\begin{aligned} \widetilde{P} = \Big\{ & (q_3^a, q_1^b) \in H^1(]0,1[) \times H^1\left(\left]-\tfrac{1}{2}, \tfrac{1}{2}\right[\right) : q_3^a(1) = 0, \; q_1^b\left(\pm\tfrac{1}{2}\right) = 0, \\ & q_3^a(0) = q_1^b(0) = 0 \Big\}. \end{aligned} \tag{6.1}$$

The main result of this section is the following one.

Theorem 6.1. *For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.10), and let $\left(\phi^a_{(p_n^a, p_n^b)}, \phi^b_{(p_n^a, p_n^b)}\right)$ be the unique solution to (2.7) with $(p^a, p^b) = (p_n^a, p_n^b)$. Moreover, let E and $\widetilde{P}$ be defined by (5.1) and (6.1), respectively.*

Assume (2.14). Then there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $(p_3^a, p_1^b) \in \widetilde{P}$ such that

$$\begin{cases} p^a_{n_i} \to (0,0,p^a_3) \quad \text{strongly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p^b_{n_i} \to (p^b_1,0,0) \quad \text{strongly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{6.2}$$

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p^a_n}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial p^a_n}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^a)\right)^3, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p^b_n}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial p^b_n}{\partial x_3}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \end{cases} \tag{6.3}$$

$$\begin{cases} \phi^a_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_0^{x_3} p^a_3(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p^a_3(t)\,\mathrm{d}t\right) \mathrm{d}x_3 \quad \text{strongly in} \quad H^1(\Omega^a), \\ \phi^b_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_{-\frac12}^{x_1} p^b_1(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p^a_3(t)\,\mathrm{d}t\right) \mathrm{d}x_3 - \int_{-\frac12}^{0} p^b_1(t)\,\mathrm{d}t \quad \text{strongly in} \quad H^1(\Omega^b), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_3}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2, \end{cases}$$

where (p_3^a, p_1^b) solves

$$E(((0,0,p^a_3),(p^b_1,0,0))) = \min\left\{E(((0,0,q^a_3),(q^b_1,0,0))) : (q^a_3,q^b_1) \in \widetilde{P}\right\}, \tag{6.4}$$

Moreover

$$\lim_n \frac{E_n((p^a_n,p^b_n))}{h_n^2} = E(((0,0,p^a_3),(p^b_1,0,0))). \tag{6.5}$$

6.2. *A priori* estimates on polarization

Arguing as in the proof of Proposition 5.2 gives the following estimate result.

Proposition 6.2. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.10). Then, there exists a constant c, independent of n, such that*

$$||p^a_n||_{(L^4(\Omega^a))^3} \le c, \quad ||p^b_n||_{(L^4(\Omega^b))^3} \le c, \qquad \forall n \in \mathbb{N}, \tag{6.6}$$

$$||D^a_n p^a_n||_{(L^2(\Omega^a))^9} \le c, \quad ||D^b_n p^b_n||_{(L^2(\Omega^b))^9} \le c, \quad \forall n \in \mathbb{N}. \tag{6.7}$$

Corollary 6.3. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.10). Let $\widetilde{P}$ be defined in (6.1). Then there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p_3^a, p_1^b) \in \widetilde{P}$ such that*

$$\begin{cases} p^a_n \rightharpoonup (0,0,p^a_3) \quad \text{weakly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p^b_n \rightharpoonup (p^b_1,0,0) \quad \text{weakly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3. \end{cases} \tag{6.8}$$

Proof. Proposition 6.2 ensures that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p_1^a, p_2^a, p_3^a) \in \left(H^1(\Omega^a)\right)^3$ independent of x_1 and x_2, and $(p_1^b, p_2^b, p_3^b) \in \left(H^1(\Omega^b)\right)^3$ independent of x_2 and x_3 such that

$$\begin{cases} p_n^a \rightharpoonup (p_1^a, p_2^a, p_3^a) \quad \text{weakly in } \left(H^1(\Omega^a)\right)^3 \text{ and strongly in } \left(L^4(\Omega^a)\right)^3, \\ p_n^b \rightharpoonup (p_1^b, p_2^b, p_3^b) \quad \text{weakly in } \left(H^1(\Omega^b)\right)^3 \text{ and strongly in } \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{6.9}$$

and $(p_1^a, p_2^a, p_3^a)\nu^a = 0$ on $\partial\Omega^a \setminus \left(]-\frac{1}{2}, \frac{1}{2}[^2 \times \{0\}\right)$, $(p_1^b, p_2^b, p_3^b)\nu^b = 0$ on $\partial\Omega^b \setminus \left(]-\frac{1}{2}, \frac{1}{2}[^2 \times \{0\}\right)$. In particular, this implies

$$p_1^a = p_2^a = 0 \quad \text{in} \quad \Omega^a, \tag{6.10}$$
$$p_3^a(1) = 0,$$
$$p_2^b = p_3^b = 0 \quad \text{in} \quad \Omega^b, \tag{6.11}$$
$$p_1^b\left(\pm\tfrac{1}{2}\right) = 0.$$

By arguing as in [20], one proves that

$$(p_1^a(0), p_2^a(0), p_3^a(0)) = \left(p_1^b(0), p_2^b(0), p_3^b(0)\right).$$

Consequently, by virtue of (6.10) and (6.11), one has

$$p_3^a(0) = 0 = p_1^b(0). \tag{6.12}$$

□

6.3. A convergence result for problem (4.6)

Proposition 4.2 provides the following result.

Proposition 6.4. *Let $\{(q_n^a, q_n^b)\}_{n\in\mathbb{N}} \subset \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3$, and let $(q_3^a, q_1^b) \in L^2(\Omega^a) \times L^2(\Omega^b)$ be such that q_3^a is independent of (x_1, x_2), q_1^b is independent of (x_2, x_3) and*

$$(q_n^a, q_n^b) \to ((0,0,q_3^a), (q_1^b, 0, 0)) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3. \tag{6.13}$$

Moreover, for $n \in \mathbb{N}$ let $(\phi^a_{(q_n^a,q_n^b)}, \phi^b_{(q_n^a,q_n^b)})$ be the unique solution to (4.6). Then,

$$\begin{cases} \phi^a_{(q_n^a,q_n^b)} \to \displaystyle\int_0^{x_3} q_3^a(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q_3^a(t)\,\mathrm{d}t\right) \mathrm{d}x_3 \quad & \text{strongly in} \quad \left(H^1(\Omega^a)\right), \\ \phi^b_{(q_n^a,q_n^b)} \to \displaystyle\int_{-\frac{1}{2}}^{x_1} q_1^b(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q_3^a(t)\,\mathrm{d}t\right) \mathrm{d}x_3 & \\ \qquad\qquad - \displaystyle\int_{-\frac{1}{2}}^{0} q_1^b(t)\,\mathrm{d}t & \text{strongly in} \quad \left(H^1(\Omega^b)\right), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q_n^a,q_n^b)}}{\partial x_2}\right) \to (0,0) & \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q_n^a,q_n^b)}}{\partial x_3}\right) \to (0,0) & \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2. \end{cases} \tag{6.14}$$

6.4. The proof of Theorem 6.1

Before proving Theorem 6.1, let us recall an evident result. Let

$$P_{\rm reg} = C_0^1\left(]0,1[\right) \times C_0^1\left(]-\tfrac{1}{2},0[\cup]0,\tfrac{1}{2}[\right) \tag{6.15}$$

Then the following result holds true.

Proposition 6.5. *Let $\widetilde{P}$ and $P_{\rm reg}$ be defined in (6.1) and (6.15), respectively. Then $P_{\rm reg}$ is dense in $\widetilde{P}$.*

Now we have all tools to prove Theorem 6.1. In what follows, $p^a_{n,i}$ (resp. $p^b_{n,i}$) denotes the ith component, $i = 1, 2, 3$, of p^a_n (resp. p^b_n). Proposition 6.2 and Corollary 6.3 assert that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p^a_3, p^b_1) \in \widetilde{P}$ satisfying (6.8) and $(z^a, z^b) \in (L^2(\Omega^a))^6 \times (L^2(\Omega^b))^6$ satisfying

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p^a_n}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial p^a_n}{\partial x_2}\right) \rightharpoonup z^a \quad \text{weakly in} \quad (L^2(\Omega^a))^6, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p^b_n}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial p^b_n}{\partial x_3}\right) \rightharpoonup z^b \quad \text{weakly in} \quad (L^2(\Omega^b))^6. \end{cases} \tag{6.16}$$

The next step is devoted to identifying p^a_3, p^b_1, z^a, and z^b. To this end, let

$$v = \begin{cases} (0, 0, q^a_3), & \text{in} \quad \Omega^a, \\ (q^b_1, 0, 0), & \text{in} \quad \Omega^b, \end{cases}$$

with $(q^a_3, q^b_1) \in P_{\rm reg}$. Then v belongs to P_n, for n large enough. Consequently,

$$\frac{1}{h_n^2} E_n\left((p^a_n, p^b_n)\right) \le \frac{1}{h_n^2} E_n\left(((0,0,q^a),(q^b,0,0))\right), \text{ for } n \text{ large enough.} \tag{6.17}$$

Then, passing to the limit in (6.17), as n diverges, and using (2.14), (6.8), (6.16), Proposition 6.4, and a l.s.c. argument imply

$$\begin{aligned} &\int_{\Omega^a}\left(|z^a|^2 + \left|\frac{{\rm d}p^a_3}{{\rm d}x_3}\right|^2\right){\rm d}x + \int_0^1 \left(\alpha(|p^a_3|^2-1)^2 + |p^a_3|^2\right){\rm d}x_3 + \int_0^1\left(\int_{]-\frac12,\frac12[^2} f^a_3\,{\rm d}x_1\,{\rm d}x_2\,p^a_3\right){\rm d}x_3 \\ &\quad + \int_{\Omega^b}\left(|z^b|^2 + \left|\frac{{\rm d}p^b_1}{{\rm d}x_1}\right|^2\right){\rm d}x + \int_{-\frac12}^{\frac12}\left(\alpha(|p^b_1|^2-1)^2 + |p^b_1|^2\right){\rm d}x_1 + \int_{-\frac12}^{\frac12}\left(\int_{]-\frac12,\frac12[\times[-1,0]} f^b_1\,{\rm d}x_2\,{\rm d}x_3\,p^b_1\right){\rm d}x_1 \\ &\quad \le \liminf_n \frac{E_n\left((p^a_n,p^b_n)\right)}{h_n^2} \le \limsup_n \frac{E_n\left((p^a_n,p^b_n)\right)}{h_n^2} \le \liminf_n \frac{E_n\left(((0,0,q^a_3),(q^b_1,0,0))\right)}{h_n^2} \\ &\quad = E\left(((0,0,q^a_3),(q^b_1,0,0))\right). \end{aligned}$$

Then, by virtue of Proposition 6.5,

$$\begin{aligned} \int_{\Omega^a}|z^a|^2\,{\rm d}x + \int_{\Omega^b}|z^b|^2\,{\rm d}x + E\left(((0,0,p^a_3),(p^b_1,0,0))\right) &\le \liminf_n \frac{E_n\left((p^a_n,p^b_n)\right)}{h_n^2} \\ \le \limsup_n \frac{E_n\left((p^a_n,p^b_n)\right)}{h_n^2} &\le E\left(((0,0,q^a_3),(q^b_1,0,0))\right), \quad \forall (q^a_3,q^b_1) \in P, \end{aligned} \tag{6.18}$$

which implies that $z^a = 0$, $z^b = 0$, (p^a_3, p^b_1) solves (6.4), convergence (6.5) holds true, and convergences in (6.8) and (6.16) are strong. □

7. The asymptotic behavior of problem (1.6)

7.1. The main result

Set

$$P^\star = \left\{ (q_3^a, q_3^b) \in H^1(]0,1[) \times H^1(]-\tfrac{1}{2},\tfrac{1}{2}[) : q_3^a(0) = q_3^b(0) \right\}. \tag{7.1}$$

The main result of this section is the following one.

Theorem 7.1. *For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.11), and let $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution to (2.7) with $(p^a,p^b) = (p_n^a,p_n^b)$. Moreover, let E and $P^\star$ be defined by (5.1) and (7.1), respectively. Assume (2.14). Then there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $(p_3^a, p_3^b) \in P^\star$ such that*

$$\begin{cases} p^a_{n_i} \to (0,0,p_3^a) \quad \text{strongly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p^b_{n_i} \to (0,0,p_3^b) \quad \text{strongly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{7.2}$$

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p_n^a}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial p_n^a}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^a)\right)^3, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p_n^b}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \end{cases} \tag{7.3}$$

$$\begin{cases} \phi^a_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_0^{x_3} p_3^a(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p_3^a(t)\,\mathrm{d}t\right)\mathrm{d}x_3 \quad \text{strongly in} \quad H^1(\Omega^a), \\ \phi^b_{(p^a_{n_i},p^b_{n_i})} \to -\displaystyle\int_0^1 \left(\int_0^{x_3} p_3^a(t)\,\mathrm{d}t\right)\mathrm{d}x_3 \quad \text{strongly in} \quad H^1(\Omega^b), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_3}\right) \to p_3^b D\overline{s} \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2, \end{cases} \tag{7.4}$$

where s is the unique solutions to (3.2), $\overline{s}$ is defined by

$$\overline{s} = s\left(x_2, x_3 + \frac{1}{2}\right), \quad \text{a.e. in} \quad \left]-\frac{1}{2},\frac{1}{2}\right[\times]0,1[,$$

and (p_3^a, p_3^b) solves

$$E(((0,0,p_3^a),(0,0,p_3^b))) = \min\left\{E(((0,0,q_3^a),(0,0,q_3^b))) : (q_3^a,q_3^b) \in P^\star\right\}, \tag{7.5}$$

Moreover

$$\lim_n \frac{E_n((p_n^a,p_n^b))}{h_n^2} = E(((0,0,p_3^a),(0,0,p_3^b))). \tag{7.6}$$

7.2. *A priori* estimates on polarization

Arguing as in Proposition 5.2 provides that

Proposition 7.2. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.11). Then, there exists a constant c such that*

$$||p_n^a||_{(L^4(\Omega^a))^3} \le c, \quad ||p_n^b||_{(L^4(\Omega^b))^3} \le c, \qquad \forall n \in \mathbb{N}, \tag{7.7}$$

$$||D_n^a p_n^a||_{(L^2(\Omega^a))^9} \le c, \quad ||D_n^b p_n^b||_{(L^2(\Omega^b))^9} \le c, \quad \forall n \in \mathbb{N}. \tag{7.8}$$

Corollary 7.3. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.11). Let $P^\star$ be defined in (7.1). Then there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p_3^a, p_3^b) \in P^\star$ such that*

$$\begin{cases} p_n^a \rightharpoonup (0,0,p_3^a) & \text{weakly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p_n^b \rightharpoonup (0,0,p_3^b) & \text{weakly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3. \end{cases} \tag{7.9}$$

Proof. Proposition 7.2 ensures that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p_1^a, p_2^a, p_3^a) \in \left(H^1(\Omega^a)\right)^3$ independent of x_1 and x_2, and $(p_1^b, p_2^b, p_3^b) \in \left(H^1(\Omega^b)\right)^3$ independent of x_2 and x_3 such that

$$\begin{cases} p_n^a \rightharpoonup (p_1^a, p_2^a, p_3^a) & \text{weakly in} \ \left(H^1(\Omega^a)\right)^3 \ \text{and strongly in} \ \left(L^4(\Omega^a)\right)^3, \\ p_n^b \rightharpoonup (p_1^b, p_2^b, p_3^b) & \text{weakly in} \ \left(H^1(\Omega^b)\right)^3 \ \text{and strongly in} \ \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{7.10}$$

and $(p_1^a, p_2^a, p_3^a) \,//\, e_3$ on $\partial\Omega^a \setminus \left(]-\frac{1}{2}, \frac{1}{2}[^2 \times \{0\}\right)$, $(p_1^b, p_2^b, p_3^b) \,//\, e_3$ on $\partial\Omega^b \setminus \left(]-\frac{1}{2}, \frac{1}{2}[^2 \times \{0\}\right)$. In particular, this implies

$$p_1^a = p_2^a = 0 \quad \text{in} \quad \Omega^a, \tag{7.11}$$

$$p_1^b = p_2^b = 0 \quad \text{in} \quad \Omega^b. \tag{7.12}$$

By arguing as in [20], one proves that

$$(p_1^a(0), p_2^a(0), p_3^a(0)) = \left(p_1^b(0), p_2^b(0), p_3^b(0)\right).$$

Consequently, one has

$$p_3^a(0) = p_3^b(0). \tag{7.13}$$

□

7.3. A convergence result for problem (2.7)

Proposition 4.2 provides the following result.

Proposition 7.4. *Let $\{(q_n^a, q_n^b)\}_{n \in \mathbb{N}} \subset \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3$, and let $(q_3^a, q_3^b) \in L^2(\Omega^a) \times L^2(\Omega^b)$ be such that q_3^a is independent of (x_1, x_2), q_3^b is independent of (x_2, x_3) and*

$$(q_n^a, q_n^b) \to ((0,0,q_3^a),(0,0,q_3^b)) \quad \textit{strongly in} \ \left(L^2(\Omega^a)\right)^3 \times (L^2(\Omega^b))^3. \tag{7.14}$$

Moreover, for $n \in \mathbb{N}$ *let* $(\phi^a_{(q^a_n,q^b_n)}, \phi^b_{(q^a_n,q^b_n)})$ *be the unique solution to* (4.6) *Then,*

$$\begin{cases} \phi^a_{(q^a_n,q^b_n)} \to \displaystyle\int_0^{x_3} q^a_3(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} q^a_3(t)\,\mathrm{d}t\right) \mathrm{d}x_3 & \text{strongly in} \quad \left(H^1(\Omega^a)\right), \\ \phi^b_{(q^a_n,q^b_n)} \to -\displaystyle\int_0^1 \left(\int_0^{x_3} q^a_3(t)\,\mathrm{d}t\right) \mathrm{d}x_3 & \text{strongly in} \quad \left(H^1(\Omega^b)\right), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(q^a_n,q^b_n)}}{\partial x_2}\right) \to (0,0) & \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(q^a_n,q^b_n)}}{\partial x_3}\right) \to q^b_3 D\overline{s} & \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2, \end{cases} \tag{7.15}$$

and

$$\lim_n \left(\int_{\Omega^a} \left| D^a_n \phi^a_{(q^a_n,q^b_n)} \right|^2 \mathrm{d}x + \int_{\Omega^b} \left| D^a_n \phi^b_{(q^a_n,q^b_n)} \right|^2 \mathrm{d}x \right) = \int_0^1 |q^a_3|^2 \,\mathrm{d}x_3 + \eta \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|q^b_3\right|^2 \mathrm{d}x_1. \tag{7.16}$$

where s *is the unique solutions to (3.2),* $\overline{s}$ *is defined by*

$$\overline{s} = s\left(x_2, x_3 + \frac{1}{2}\right), \text{ a.e. } in \left]-\frac{1}{2}, \frac{1}{2}\right[\times]0,1[,$$

and η *is defined in* (3.5).

7.4. Proof of Theorem 7.1

We sketch the proof.

Proposition 7.2 and Corollary 7.3 assert that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p^a_3, p^b_3) \in P^\star$ and $(z^a, z^b) \in (L^2(\Omega^a))^6 \times (L^2(\Omega^b))^6$ satisfying (7.9) and

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p^a_n}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial p^a_n}{\partial x_2}\right) \rightharpoonup z^a & \text{weakly in} \quad (L^2(\Omega^a))^6, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p^b_n}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial p^b_n}{\partial x_3}\right) \rightharpoonup z^b & \text{weakly in} \quad (L^2(\Omega^b))^6. \end{cases} \tag{7.17}$$

Let U_{reg} be defined in (4.2) and let $(q^a_3, q^b_3) \in U_{\text{reg}}$. As in (4.18) and (4.19), one can build a sequence $\{(q^a_n, q^b_n)\}_{n\in\mathbb{N}}$, with $((0,0,q^a_n),(0,0,q^b_n)) \in P_n$, for each $n \in \mathbb{N}$, such that, thanks also to (2.14) and Proposition 7.4,

$$\limsup_n \frac{E_n\left(((0,0,q^a_n),(0,0,q^b_n))\right)}{h_n^2} = E\left(((0,0,q^a_3),(0,0,q^b_3))\right).$$

Consequently, by virtue of Proposition 4.1, one has

$$\limsup_n \frac{E_n\left((p^a_n, p^b_n)\right)}{h_n^2} \le E\left(((0,0,q^a_3),(0,0,q^b_3))\right), \quad \forall (q^a_3, q^b_3) \in \widetilde{P}. \tag{7.18}$$

On the other side, (2.14), (7.9), (7.17), a l.s.c. argument, and Proposition 7.4 ensure that

$$\int_{\Omega^a} |z^a|^2 \,\mathrm{d}x \int_{\Omega^b} |z^b|^2 \,\mathrm{d}x + E\left(((0,0,p^a_3),(0,0,p^b_3))\right) \le \liminf_n \frac{E_n\left((p^a_n, p^b_n)\right)}{h_n^2}. \tag{7.19}$$

Finally, combining (7.18) and (7.19) completes the proof, as ususual. $\square$

8. The asymptotic behavior of problem (1.15)

8.1. The main result

Theorem 8.1. *For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.12), and let $\left(\phi^a_{(p_n^a,p_n^b)}, \phi^b_{(p_n^a,p_n^b)}\right)$ be the unique solution to (2.7) with $(p^a, p^b) = (p_n^a, p_n^b)$. Moreover, let (5.1) and $\widetilde{P}$ be defined by (5.1) and (6.1), respectively. Assume (2.14). Then there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and (in possible dependence on the subsequence) $(p_3^a, p_1^b) \in \widetilde{P}$ such that*

$$\begin{cases} p^a_{n_i} \to (0,0,p_3^a) \quad \text{strongly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\ p^b_{n_i} \to (p_1^b,0,0) \quad \text{strongly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3, \end{cases} \tag{8.1}$$

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p_n^a}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial p_n^a}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3 \times \left(L^2(\Omega^a)\right)^3, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p_n^b}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial p_n^b}{\partial x_3}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^3 \times \left(L^2(\Omega^b)\right)^3, \end{cases} \tag{8.2}$$

$$\begin{cases} \phi^a_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_0^{x_3} p_3^a(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p_3^a(t)\,\mathrm{d}t\right)\mathrm{d}x_3 \quad \text{strongly in} \quad H^1(\Omega^a), \\ \phi^b_{(p^a_{n_i},p^b_{n_i})} \to \displaystyle\int_{-\frac12}^{x_1} p_1^b(t)\,\mathrm{d}t - \int_0^1 \left(\int_0^{x_3} p_3^a(t)\,\mathrm{d}t\right)\mathrm{d}x_3 \\ \qquad\qquad - \displaystyle\int_{-\frac12}^{0} p_1^b(t)\,\mathrm{d}t \quad \text{strongly in} \quad H^1(\Omega^b), \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_1}, \dfrac{1}{h_n}\dfrac{\partial \phi^a_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^2, \\ \left(\dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_2}, \dfrac{1}{h_n}\dfrac{\partial \phi^b_{(p^a_{n_i},p^b_{n_i})}}{\partial x_3}\right) \to (0,0) \quad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^2, \end{cases} \tag{8.3}$$

where (p_3^a, p_1^b) solves (6.4). Moreover

$$\lim_n \frac{S_n((p_n^a, p_n^b))}{h_n^2} = E(((0,0,p_3^a),(p_1^b,0,0))). \tag{8.4}$$

8.2. *A priori* estimates on polarization

At first note that (for instance see [15] and also Lem. 2.1 in [25])

$$\begin{aligned} &||D\mathbf{P}||^2_{(L^2(\Omega_n))^9} = ||\mathrm{rot}\,\mathbf{P}||^2_{(L^2(\Omega_n))^3} + ||\mathrm{div}\,\mathbf{P}||^2_{L^2(\Omega_n)}, \\ &\forall \mathbf{P} \in \left(H^1(\Omega_n)\right)^3 : \mathbf{P}\cdot\nu = 0 \text{ on } \partial\Omega_n, \end{aligned} \tag{8.5}$$

which by rescalings in (2.1) is transformed into

$$||D_n^a p^a||^2_{(L^2(\Omega^a))^9} + ||D_n^b p^b||^2_{(L^2(\Omega^b))^9} = ||\mathrm{rot}\,{}^a_n p^a||^2_{(L^2(\Omega^a))^3} + ||\mathrm{div}\,{}^a_n p^a||^2_{L^2(\Omega^a)} + ||\mathrm{rot}\,{}^b_n p^b||^2_{(L^2(\Omega^b))^3} + ||\mathrm{div}\,{}^b_n p^b||^2_{L^2(\Omega^b)}, \tag{8.6}$$

for all $(p^a, p^b) \in \widetilde{P}_n$ and all $n \in \mathbb{N}$.

We note that to our aim it is enough to have just an equivalence between the term $||D\mathbf{P}||^2_{(L^2(\Omega_n))^9}$ and the term $||\mathrm{rot}\,\mathbf{P}||^2_{(L^2(\Omega_n))^3} + ||\mathrm{div}\,\mathbf{P}||^2_{L^2(\Omega_n)}$ with a constant independent of n.

Proposition 8.2. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.12). Then, there exists a constant c such that*

$$||p_n^a||_{(L^4(\Omega^a))^3} \le c, \quad ||p_n^b||_{(L^4(\Omega^b))^3} \le c, \qquad \forall n \in \mathbb{N}, \tag{8.7}$$

$$||D_n^a p_n^a||_{(L^2(\Omega^a))^9} \le c, \quad ||D_n^b p_n^b||_{(L^2(\Omega^b))^9} \le c, \quad \forall n \in \mathbb{N}. \tag{8.8}$$

Proof. Function 0 belonging to $\widetilde{P}_n$ gives

$$\begin{aligned}
&\int_{\Omega^a} \left(\beta|\mathrm{rot}_n^a\, p_n^a|^2 + |\mathrm{div}_n^a\, p_n^a|^2 + \alpha\left(|p_n^a|^4 - 2|p_n^a|^2\right) + |D_n^a \phi^a_{(p_n^a,p_n^b)}|^2\right) \mathrm{d}x \\
&\quad + \int_{\Omega^b} \left(\beta|\mathrm{rot}_n^b\, p_n^b|^2 + |\mathrm{div}_n^b\, p_n^b|^2 + \alpha\left(|p_n^b|^4 - 2|p_n^b|^2\right) + |D_n^b \phi^b_{(p_n^a,p_n^b)}|^2\right) \mathrm{d}x \\
&\quad \le \frac{1}{2}\int_{\Omega^a} \left(|f_n^a|^2 + |p_n^a|^2\right) \mathrm{d}x + \frac{1}{2}\int_{\Omega^b} \left(|f_n^b|^2 + |p_n^b|^2\right) \mathrm{d}x, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{8.9}$$

Estimates (8.9) implies

$$\begin{aligned}
&\int_{\Omega^a} \alpha\left(|p_n^a|^4 - \left(2 + \frac{1}{2\alpha}\right)|p_n^a|^2\right) \mathrm{d}x + \int_{\Omega^b} \alpha\left(|p_n^b|^4 - \left(2 + \frac{1}{2\alpha}\right)|p_n^b|^2\right) \mathrm{d}x \\
&\quad \le \frac{1}{2}\int_{\Omega^a} |f_n^a|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^b} |f_n^b|^2\,\mathrm{d}x, \quad \forall n \in \mathbb{N},
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_{\Omega^a} \alpha\left(|p_n^a|^2 - \left(1 + \frac{1}{4\alpha}\right)\right)^2 \mathrm{d}x + \int_{\Omega^b} \alpha\left(|p_n^b|^2 - \left(1 + \frac{1}{4\alpha}\right)\right)^2 \mathrm{d}x \\
&\quad \le \alpha\left(1 + \frac{1}{4\alpha}\right)^2 \left(|\Omega^a| + |\Omega^b|\right) + \frac{1}{2}\int_{\Omega^a} |f_n^a|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega^b} |f_n^b|^2\,\mathrm{d}x, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{8.10}$$

Then the estimates in (8.7) follow from (8.10) and (2.14). The estimates in (8.8) follow from (8.9), (2.14), (8.7), the continuous embedding of L^4 into L^2, and (8.6). □

Proposition 8.2 with the same argument used in the proof of Corollary 6.3 provides the following result.

Corollary 8.3. *Assume (2.14). For every $n \in \mathbb{N}$, let (p_n^a, p_n^b) be a solution to (2.12). Let $\widetilde{P}$ be defined in (6.1). Then there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p_3^a, p_1^b) \in \widetilde{P}$ such that*

$$\begin{cases}
p_n^a \rightharpoonup (0, 0, p_3^a) \quad \text{weakly in} \quad \left(H^1(\Omega^a)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^a)\right)^3, \\
p_n^b \rightharpoonup (p_1^b, 0, 0) \quad \text{weakly in} \quad \left(H^1(\Omega^b)\right)^3 \quad \text{and strongly in} \quad \left(L^4(\Omega^b)\right)^3.
\end{cases} \tag{8.11}$$

8.3. The proof of Theorem 8.1

In what follows, $p^a_{n,i}$ (resp. $p^b_{n,i}$) denotes the ith component, $i = 1, 2, 3$, of p^a_n (resp. p^b_n). Proposition 8.2 and Corollary 8.3 assert that there exist a subsequence of $\mathbb{N}$, still denoted by $\{n\}$, and (in possible dependence on the subsequence) $(p^a_3, p^b_1) \in \widetilde{P}$ satisfying (8.11) and $(z^a, z^b) \in (L^2(\Omega^a))^{3\times 2} \times (L^2(\Omega^b))^{3\times 2}$ satisfying

$$\begin{cases} \left(\dfrac{1}{h_n}\dfrac{\partial p^a_{n,i}}{\partial x_j}\right)_{i=1,2,3,j=1,2} \rightharpoonup z^a \quad \text{weakly in} \quad (L^2(\Omega^a))^{3\times 2}, \\ \left(\dfrac{1}{h_n}\dfrac{\partial p^b_{n,i}}{\partial x_j}\right)_{i=1,2,3,j=1,2} \rightharpoonup z^b \quad \text{weakly in} \quad (L^2(\Omega^b))^{3\times 2}. \end{cases} \tag{8.12}$$

The next step is devoted to identifying p^a_3, p^b_1, z^a, and z^b. To this end, let

$$v = \begin{cases} (0, 0, q^a_3), & \text{in} \quad \Omega^a, \\ (q^b_1, 0, 0), & \text{in} \quad \Omega^b, \end{cases}$$

with $(q^a_3, q^b_1) \in P_{\mathrm{reg}}$ defined in (6.15). Then v belongs to $\widetilde{P}_n$, for n large enough. Consequently,

$$\frac{1}{h_n^2} S_n\left((p^a_n, p^b_n)\right) \leq \frac{1}{h_n^2} S_n\left(((0,0,q^a_3),(q^b_1,0,0))\right), \text{ for } n \text{ large enough.} \tag{8.13}$$

Then, passing to the limit in (8.13), as n diverges, and using (2.14), (8.11), (8.12), Proposition 6.4, and a l.s.c. argument imply

$$\begin{aligned} &\int_{\Omega^a} \left(\beta\left(\left|z^a_{3,2}\right|^2 + \left|z^a_{3,1}\right|^2 + \left|z^a_{2,1} - z^a_{1,2}\right|^2\right) + \left(\left|z^a_{1,1} + z^a_{2,2} + \frac{\mathrm{d}p^a_3}{\mathrm{d}x_3}\right|^2\right)\right) \mathrm{d}x \\ &\quad + \int_0^1 \left(\alpha(|p^a_3|^2 - 1)^2 + |p^a_3|^2\right) \mathrm{d}x_3 + \int_0^1 \left(\int_{]-\frac{1}{2},\frac{1}{2}[^2} f^a_3 \,\mathrm{d}x_1\,\mathrm{d}x_2\, p^a_3\right) \mathrm{d}x_3 \\ &\quad + \int_{\Omega^b} \left(\beta\left(\left|z^b_{3,2} - z^b_{2,3}\right|^2 + \left|z^b_{1,3}\right|^2 + \left|z^b_{1,2}\right|^2\right) + \left(\left|\frac{\mathrm{d}p^b_1}{\mathrm{d}x_1} + z^b_{2,2} + z^b_{3,3}\right|^2\right)\right) \mathrm{d}x \\ &\quad + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\alpha(|p^b_1|^2 - 1)^2 + |p^b_1|^2\right) \mathrm{d}x_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{]-\frac{1}{2},\frac{1}{2}[\times[-1,0]} f^b_1 \,\mathrm{d}x_2\,\mathrm{d}x_3\, p^b_1\right) \mathrm{d}x_1 \\ &\leq \liminf_n \frac{S_n\left((p^a_n, p^b_n)\right)}{h_n^2} \leq \limsup_n \frac{S_n\left((p^a_n, p^b_n)\right)}{h_n^2} \\ &\leq \lim_n \frac{S_n\left(((0,0,q^a_3),(q^b_1,0,0))\right)}{h_n^2} = E(((0,0,q^a_3),(q^b_1,0,0))). \end{aligned} \tag{8.14}$$

Now let us prove that

$$\int_{\Omega^a} \left(z^a_{1,1} + z^a_{2,2}\right) \frac{\mathrm{d}p^a_3}{\mathrm{d}x_3} \,\mathrm{d}x = 0, \quad \int_{\Omega^b} \left(z^b_{2,2} + z^b_{3,3}\right) \frac{\mathrm{d}p^b_1}{\mathrm{d}x_1} \,\mathrm{d}x = 0. \tag{8.15}$$

Indeed, let

$$g_n(x_3) = \sum_{i=0}^{n-1} \left(\frac{p_3^a\left(\frac{i+1}{n}\right) - p_3^a\left(\frac{i}{n}\right)}{\frac{1}{n}} \chi_{\left[\frac{i}{n}, \frac{i+1}{n}\right[}(x_3) \right), \quad x_3 \quad \text{a.e. in} \quad]0,1[, \quad \forall n \in \mathbb{N}.$$

Observing that $\dfrac{p_3^a\left(\frac{i+1}{n}\right) - p_3^a\left(\frac{i}{n}\right)}{\frac{1}{n}}$ is the average of $\dfrac{\mathrm{d}p_3^a}{\mathrm{d}x_3}$ on $\left]\frac{i}{n}, \frac{i+1}{n}\right[$, one easily has

$$g_n \to \frac{\mathrm{d}p_3^a}{\mathrm{d}x_3} \quad \text{strongly in} \quad L^2(]0,1[),$$

as n diverges. Consequently, taking also into account (8.12), one has

$$\begin{aligned}
\int_{\Omega^a} \left(z_{1,1}^a + z_{2,2}^a\right) \frac{\mathrm{d}p_3^a}{\mathrm{d}x_3}\,\mathrm{d}x &= \lim_n \int_{\Omega^a} \left(\frac{1}{h_n} \frac{\partial p_{n,1}^a}{\partial x_1}(x) - \frac{1}{h_n} \frac{\partial p_{n,2}^a}{\partial x_2}(x) \right) g_n(x_3)\,\mathrm{d}x \\
&= \lim_n \sum_{i=0}^{n-1} \left(\frac{p_3^a\left(\frac{i+1}{n}\right) - p_3^a\left(\frac{i}{n}\right)}{\frac{1}{n}} \frac{1}{h_n} \int_{\left]-\frac{1}{2},\frac{1}{2}\right[^2 \times \left]\frac{i}{n},\frac{i+1}{n}\right[} \left(\frac{\partial p_{n,1}^a}{\partial x_1}(x) - \frac{\partial p_{n,2}^a}{\partial x_2}(x) \right) \mathrm{d}x \right)
\end{aligned}$$

and the last integrals are zero due to the boundary condition on p_n^a. It is so proved the first equality in (8.15). Similarly, one proves the second one.

The properties of p_3^a, p_1^b and (8.15) give

$$\begin{cases}
\displaystyle\int_{\Omega^a} \left| z_{1,1}^a + z_{2,2}^a + \frac{\mathrm{d}p_3^a}{\mathrm{d}x_3} \right|^2 \mathrm{d}x = \int_0^1 \left| \frac{\mathrm{d}p_3^a}{\mathrm{d}x_3} \right|^2 \mathrm{d}x_3 + \int_{\Omega^a} \left| z_{1,1}^a + z_{2,2}^a \right|^2 \mathrm{d}x, \\
\displaystyle\int_{\Omega^b} \left| \frac{\mathrm{d}p_1^b}{\mathrm{d}x_1} + z_{2,2}^b + z_{3,3}^b \right|^2 \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\mathrm{d}p_1^b}{\mathrm{d}x_1} \right|^2 \mathrm{d}x_1 + \int_{\Omega^b} \left| z_{2,2}^b + z_{3,3}^b \right|^2 \mathrm{d}x.
\end{cases} \tag{8.16}$$

Then, inserting (8.16) in (8.14) provides

$$\begin{aligned}
&\int_{\Omega^a} \left[\beta \left(\left|z_{3,2}^a\right|^2 + \left|z_{3,1}^a\right|^2 + \left|z_{2,1}^a - z_{1,2}^a\right|^2 \right) + \left|z_{1,1}^a + z_{2,2}^a\right|^2 \right] \mathrm{d}x \\
&\qquad + \int_{\Omega^b} \left[\beta \left(\left|z_{3,2}^b - z_{2,3}^b\right|^2 + \left|z_{1,3}^b\right|^2 + \left|z_{1,2}^b\right|^2 \right) + \left|z_{2,2}^b + z_{3,3}^b\right|^2 \right] \mathrm{d}x \\
&\qquad + E(((0,0,p_3^a),(p_1^b,0,0))) \le \liminf_n \frac{S_n\left((p_n^a, p_n^b)\right)}{h_n^2} \le \limsup_n \frac{S_n\left((p_n^a, p_n^b)\right)}{h_n^2} \\
&\quad \le E(((0,0,q_3^a),(q_1^b,0,0))), \quad \forall (q_3^a, q_1^b) \in P_{\mathrm{reg}}.
\end{aligned} \tag{8.17}$$

By virtue of Proposition 6.5, inequality (8.17) also true for any $(q_3^a, q_1^b) \in \widetilde{P}$. Consequently, choosing $(q_3^a, q_1^b) = (p_3^a, p_1^b)$ in (8.17) one has

$$\begin{cases} z_{2,1}^a - z_{1,2}^a = 0 & \text{a.e. in} \quad \Omega^a, \\ z_{1,1}^a + z_{2,2}^a = 0 & \text{a.e. in} \quad \Omega^a, \\ z_{3,2}^a = z_{3,1}^a = 0 & \text{a.e. in} \quad \Omega^a, \\ z_{1,3}^b = z_{1,2}^b = 0 & \text{a.e. in} \quad \Omega^b, \\ z_{3,2}^b - z_{2,3}^b = 0 & \text{a.e. in} \quad \Omega^b, \\ z_{2,2}^b + z_{3,3}^b = 0 & \text{a.e. in} \quad \Omega^b. \end{cases} \tag{8.18}$$

Consequently, inserting (8.18) in (8.17), one obtains that (p_3^a, p_1^b) solves (6.4) and convergence (8.4) holds. We remark that convergence in (8.4) holds true for the whole sequence since the limit is uniquely identified. Moreover, (8.3) follows from (8.11) and Proposition 6.4.

The last step is devoted to proving (8.2) and that convergences in (8.11) are strong. To this aim, combining (8.4) with (2.14), (8.3) and (8.11) provides

$$\begin{aligned} &\lim_n \left(\int_{\Omega^a} \left(\beta |\mathrm{rot}\,{}_n^a p_n^a|^2 + |\mathrm{div}\,{}_n^a p_n^a|^2 \right) \mathrm{d}x + \int_{\Omega^b} \left(\beta |\mathrm{rot}\,{}_n^b p_n^b|^2 + |\mathrm{div}\,{}_n^b p_n^b|^2 \right) \mathrm{d}x \right) \\ &= \int_0^1 \left| \frac{\mathrm{d}p_3^a}{\mathrm{d}x_3} \right|^2 \mathrm{d}x_3 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\mathrm{d}p_1^b}{\mathrm{d}x_1} \right|^2 \mathrm{d}x_1. \end{aligned} \tag{8.19}$$

Moreover, from (8.11), (8.12) and (8.18) it follows that

$$\begin{cases} \mathrm{rot}\,{}_n^a p_n^a \rightharpoonup (0,0,0) = \mathrm{rot}\,(0,0,p_3^a) & \text{weakly in} \quad \left(L^2(\Omega^a)\right)^3, \\ \mathrm{rot}\,{}_n^b p_n^b \rightharpoonup (0,0,0) = \mathrm{rot}\,(p_1^b,0,0) & \text{weakly in} \quad \left(L^2(\Omega^b)\right)^3, \\ \mathrm{div}\,{}_n^a p_n^a \rightharpoonup \dfrac{\mathrm{d}p_3^a}{\mathrm{d}x_3} = \mathrm{div}\,(0,0,p_3^a) & \text{weakly in} \quad L^2(\Omega^a), \\ \mathrm{div}\,{}_n^b p_n^b \rightharpoonup \dfrac{\mathrm{d}p_1^b}{\mathrm{d}x_1} = \mathrm{div}\,(p_1^b,0,0) & \text{weakly in} \quad L^2(\Omega^b). \end{cases} \tag{8.20}$$

Consequently, combining convergence of the energies (8.19) with (8.20), one derives that

$$\begin{cases} \mathrm{rot}\,{}_n^a p_n^a \to \mathrm{rot}\,(0,0,p_3^a) & \text{strongly in} \quad \left(L^2(\Omega^a)\right)^3, \\ \mathrm{rot}\,{}_n^b p_n^b \to \mathrm{rot}\,(p_1^b,0,0) & \text{strongly in} \quad \left(L^2(\Omega^b)\right)^3, \\ \mathrm{div}\,{}_n^a p_n^a \to \mathrm{div}\,(0,0,p_3^a) & \text{strongly in} \quad L^2(\Omega^a), \\ \mathrm{div}\,{}_n^b p_n^b \to \mathrm{div}\,(p_1^b,0,0) & \text{strongly in} \quad L^2(\Omega^b). \end{cases} \tag{8.21}$$

Finally, taking into account that

$$\begin{cases} D(0,0,p_3^a) = D_n^a(0,0,p_3^a),\ \operatorname{rot}(0,0,p_3^a) = \operatorname{rot}_n^a(0,0,p_3^a),\ \operatorname{div}(0,0,p_3^a) = \operatorname{div}_n^a(0,0,p_3^a),\ \text{in } \Omega^a, \\ \\ D(p_1^b,0,0) = D_n^b(p_1^b,0,0),\ \operatorname{rot}(p_1^b,0,0) = \operatorname{rot}_n^b(p_1^b,0,0),\ \operatorname{div}(p_1^b,0,0) = \operatorname{div}_n^b(p_1^b,0,0),\ \text{in } \Omega^b, \end{cases}$$

from (8.6) and (8.21) one deduces that

$$\begin{cases} D_n^a p_n^a \to D(0,0,p_3^a) \qquad \text{strongly in} \quad \left(L^2(\Omega^a)\right)^9, \\ \\ D_n^b p_n^b \to D(p_1^b,0,0) \qquad \text{strongly in} \quad \left(L^2(\Omega^b)\right)^9, \end{cases}$$

i.e. (8.2) and that convergences in (8.11) are strong. We remark that convergences in (8.2) hold true for the whole sequence since the limits are uniquely identified. □

9. The asymptotic behavior of all previous problems when the control $\varphi_{\mathbf{P}}$ satisfies (1.8)

If $\left(\phi^a_{(p^a,p^b)}, \phi^b_{(p^a,p^b)}\right)$ is the unique solution to (2.13), thanks to Proposition 4.3, in the limit process there is no contribution of the nonlocal term. So, the limit functionals are obtained just eliminating the parts coming from the nonlocal term in the previous limit functionals and all previous convergences on the polarization hold true, while the potentials converge to zero.

Acknowledgements. The second author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The third author thanks CONICYT PFCHA/DOCTORADO NACIONAL/2015 – 21150001.

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