ESAIM: M2AN 54 (2020) 1465–1490
https://doi.org/10.1051/m2an/2019095

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

NONLINEAR ITERATION ACCELERATION SOLUTION FOR EQUILIBRIUM RADIATION DIFFUSION EQUATION

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Abstract. This paper discusses accelerating iterative methods for solving the fully implicit (FI) scheme of equilibrium radiation diffusion problem. Together with the FI Picard factorization (PF) iteration method, three new nonlinear iterative methods, namely, the FI Picard-Newton factorization (PNF), FI Picard-Newton (PN) and derivative free Picard-Newton factorization (DFPNF) iteration methods are studied, in which the resulting linear equations can preserve the parabolic feature of the original PDE. By using the induction reasoning technique to deal with the strong nonlinearity of the problem, rigorous theoretical analysis is performed on the fundamental properties of the four iteration methods. It shows that they all have first-order time and second-order space convergence, and moreover, can preserve the positivity of solutions. It is also proved that the iterative sequences of the PF iteration method and the three Newton-type iteration methods converge to the solution of the FI scheme with a linear and a quadratic speed respectively. Numerical tests are presented to confirm the theoretical results and highlight the high performance of these Newton acceleration methods.

Mathematics Subject Classification. 35K55, 65M06, 65M12, 65B99.

Received April 15, 2019. Accepted December 19, 2019.

1. Introduction

Radiation hydrodynamics problems appear in inertial confinement fusion, magnetic confinement fusion, astrophysics, combustion and many other fields. Radiation diffusion plays an important role in radiation hydrodynamics [2, 20, 23]. When the radiation field is in local thermodynamic equilibrium with the material, the system is referred to as equilibrium radiation diffusion, see [4, 6, 7, 14, 21, 23, 35]. In this case, for the radiant energy density E and material temperature T, there holds $E = aT^4$, where a is the radiation constant.

Much work has been done on numerical methods for solving radiation diffusion equations. In terms of spatial discretization, finite volume, finite difference and finite element methods have been studied [16, 19, 20, 24, 25, 27, 29, 33, 34, 36]. In terms of temporal discretization, due to the nonlinearity of the problem, fully implicit (FI) schemes with nonlinear iterations have been widely applied in practice [2, 13, 22, 32]. For example, Knoll *et al.* [12, 14, 17] discussed various time-discrete methods and nonlinear iterative methods. Kang [10] constructed a

Keywords and phrases. Equilibrium radiation diffusion equation, accelerated iteration method, convergence accuracy, convergence speed, positivity.

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nonconforming finite element method on unstructured grids, and then solved the discrete nonlinear equations with Picard iteration and inexact Newton iteration.

To solve the implicit scheme of nonlinear PDE, iterative methods are used which adopt the inner-outer iteration mode. The outer iteration, namely, the nonlinear iteration, refers to the algorithm to linearize the nonlinear PDE to a linear PDE, or a nonlinear discrete scheme to a system of linear algebraic equations (SLAE). For example, a Picard iteration or a Newton iteration is designed to derive an SLAE $\mathbb{A}(\mathbb{U}^{(s)})\mathbb{U}^{(s+1)} = \mathbf{b}^{(s)}$, where s is the index of iterations. The inner iteration, refers to the algorithm employed to solve the SLAE, *e.g.*, a GMRES solver, etc. To a great extent, the nonlinear iteration determines the performance (including the accuracy, the efficiency, and so on) of the entire solution procedure.

A key point in constructing an efficient nonlinear iteration is to preserve the characteristics of the original problem during the linearization procedure. An iteration method that preserves such characteristics during the iteration process is more valuable than that possesses them only at the end of the iteration procedure, not to mention those do not preserve them at all. Radiation diffusion equations are essentially parabolic equations with positivity solution. A lot of computational experiments tell us that keeping the positivity and parabolic features in the iterative procedure is not only a foundation of correct simulation of the physical problem, but also helpful for ensuring the efficiency of the computation [28].

Among the nonlinear iterative methods for radiation diffusion problems, Picard iteration is the most often used one. It retains the parabolic property and then is positivity-preserving or discrete-maximum-principle-satisfying, yet only has a linear convergence ratio. To accelerate the iteration convergence, Newton-like iterations are developed. Some earlier applications of Newton methods to radiative hydrodynamics can be found in [26], etc. For example, In [12, 13, 18], Newton–Krylov methods and Jacobian-free Newton–Krylov (JFNK) methods were developed and applied to non-equilibrium radiation diffusion problems. In [23], multigrid Newton–Krylov method was studied for equilibrium radiation diffusion simulation. In [1], Newton–Krylov methods were applied to the equilibrium radiation diffusion in low-energy-density regime.

This paper aims to design efficient iteration methods to accelerate the solution of equilibrium radiation diffusion problem as well as provide their rigorous theoretical analysis and demonstrate their good performance in convergence speed, accuracy and positivity.

For equilibrium radiation diffusion problem, the governing equation is the energy conservation equation. It is highly nonlinear due to the fourth-power dependence of the radiation energy density on temperature, and the radiation diffusion coefficient is nonlinearly dependent on the radiation temperature. So it is very difficult to design efficient iteration algorithms and analyze their fundamental properties.

First, we will derive some new Newton-type nonlinear iteration methods for equilibrium radiation diffusion, namely, the FI Picard-Newton factorization (PNF), FI Picard-Newton (PN) and derivative free Picard-Newton factorization (DFPNF) iteration methods. Instead of directly applying the Newton method or JFNK method to linearize the nonlinear system of algebraic equations given by the FI discrete scheme (namely, in a "discretization-linearization" way), we get these iterations in a "linearization-discretization" procedure [5, 28, 31]: first linearize the temporal discrete scheme of the original PDE, then carry out spatial discretization of the derived linear equation to get a linear system of algebraic equations. With such a strategy, the Newton iteration can be viewed as adding a small-quantity accelerative term on a Picard iteration, and is called a "Picard-Newton" iteration. For example, the PNF iteration can be viewed as established on the basis of the FI Picard factorization (PF) method [35]. And that makes it more convenient to acquire Newton acceleration based on existing Picard codes in practice. Moreover, with derivatives replaced by corresponding difference quotient approximations, a DFPNF iteration is put forward to handle the very likely case in applications wherein the diffusion coefficient is evaluated in tabular form and no analytic derivative formula is available. We'd like to mention that our PNF, PN and DFPNF iterations designed through such a way converge quadratically, which differs from the JFNK methods [11, 18] that usually have super-linear instead of quadratic convergence speed due to their outer inexact Newton iterations.

Furthermore, in contrast to many studies on time evolution focusing on demonstrating the accuracy and efficiency properties of the discrete or iterative algorithms by numerical tests, we will analyze theoretically the properties of these iterative methods. The positivity, convergence accuracy and convergence speed of the PF, PN, PNF and DFPNF iterations will be studied. This is accomplished by developing the discrete functional analysis techniques for diffusion equations [8,9,37]. There are some theoretical proofs on the convergence speed of Picard and PN iterations for diffusion problems, and less works on their detailed convergence accuracy analysis [5,28]. It is well-known that Picard iteration is positivity-preserving. However the positivity of Newton-type iterations is seldom revealed. In this work, by performing thorough deduction of the terms concerning the temporal difference quotients and developing induction reasoning techniques, we overcome the difficulties caused by the strong nonlinearity of the problem, and prove the convergence accuracy and convergence speed of the iterative methods and get the positivity at the same time.

The paper is organized as follows. First, some nonlinear iterative methods are put forward in Section 2, which include the PF, PN, PNF and DFPNF iteration methods. Then in Section 3 the convergence accuracy and positivity of the nonlinear iterations are proved. Their convergence speeds are verified in Section 4. In Section 5, numerical results are presented to show the performance of the methods. Finally, conclusions are made in Section 6.

2. Construction of iteration sequences

2.1. Problem and notations

Consider the one-dimensional equilibrium radiation diffusion problem as follows:

$$\frac{\partial e(u)}{\partial t} = \frac{\partial}{\partial x}\left(A(x,t,u)\frac{\partial u}{\partial x}\right), \quad \text{in } \Omega_T = \{0 < x < L, 0 < t \le T\}; \tag{2.1}$$

$$u(0,t) = \psi_0(t), \quad u(L,t) = \psi_1(t), \quad 0 < t \le T; \tag{2.2}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le L; \tag{2.3}$$

where the diffusion coefficient $A(x,t,u)$, boundary conditions $\psi_0(t)$, $\psi_1(t)$ and initial function $\varphi(x)$ are given functions, $e(u) = u + g(u)$, $g(u) = |u|^3 u$.

The study goes along under the following assumptions:

Assumption 2.1. *There exists a positive constant σ such that for all $(x,t,p) \in \bar{\Omega}_T \times R^+$, there is $A(x,t,p) \ge \sigma$.*

Assumption 2.2. *The partial derivatives A_{xxx}, A_{px}, A_{pt} and A_{pp} are continuous with respect to $(x,t,p) \in \bar{\Omega}_T \times R^+$.*

Assumption 2.3. *The initial-boundary values ψ_0, ψ_1 and φ have positive lower bound m and upper bound M.*

Assumption 2.4. *Problems (2.1)–(2.3) has a unique smooth solution satisfying $u \in C^{4,2}(\bar{\Omega}_T)$.*

Remark 2.5. $A(x,t,p)$ only makes sense for $p > 0$. So we need prove the solution positivity in various levels, including the continuum solution of original nonlinear PDE (2.1)–(2.3), the discrete solution of nonlinear discrete scheme and the iterative solutions of linearized schemes at each nonlinear iteration step. Thus, in the following we will prove Propositions 2.6, 2.9, 3.1, Theorems 3.4 and 3.5. It follows that $g(u) = |u|^3 u = u^4$ for $u \ge 0$.

Applying the maximum principle (see [15]), we can derive the following proposition and then $g(u) = u^4$ provided that $u = u(x,t)$ is the solution of (2.1)–(2.3).

Proposition 2.6. *Under the Assumptions 2.1–2.4, the solution of problem (2.1)–(2.3) is positive and bounded, namely*

$$0 < m \le u(x,t) \le M, \qquad \forall (x,t) \in \bar{\Omega}_T.$$

Divide Ω_T by using parallel lines $x = x_j (j = 0, 1, \ldots, J)$ and $t = t^n (n = 0, 1, \ldots, N)$, where $x_j = jh, t^n = n\tau$, and $Jh = L, N\tau = T$, J and N are positive integers, h and τ are the space and time step-lengths respectively. Denote $x_{j+\frac{1}{2}} = \frac{1}{2}(x_{j+1} + x_j)$, $\phi^{n+1}_{j+\frac{1}{2}} = \frac{1}{2}(\phi^{n+1}_{j+1} + \phi^{n+1}_j)$, $\delta\phi_{j+\frac{1}{2}} = \frac{1}{h}(\phi_{j+1} - \phi_j)$, and $d_t\phi^{n+1} = \frac{1}{\tau}(\phi^{n+1} - \phi^n)$.

For function ϕ, define

$$A^{n+1}_{j+\frac{1}{2}}(\phi) = A\left(x_{j+\frac{1}{2}}, t^{n+1}, \phi^{n+1}_{j+\frac{1}{2}}\right),$$
$$A^{n+1\,(s)}_{j+\frac{1}{2}}(\phi) = A\left(x_{j+\frac{1}{2}}, t^{n+1}, \phi^{n+1\,(s)}_{j+\frac{1}{2}}\right),$$
$$A'^{n+1\,(s)}_{j+\frac{1}{2}}(\phi) = A'_p\left(x_{j+\frac{1}{2}}, t^{n+1}, \phi^{n+1\,(s)}_{j+\frac{1}{2}}\right),$$

and so forth. Here and below, we will omit the superscript $n+1$ and use $\phi^{(s)}$ to represent $\phi^{n+1\,(s)}$ when no confusion occurs. Furthermore for functions ϕ and Φ, denote

$$\delta(A(\phi)\delta\Phi)_j = \frac{1}{h}\left[A_{j+\frac{1}{2}}(\phi)\delta\Phi_{j+\frac{1}{2}} - A_{j-\frac{1}{2}}(\phi)\delta\Phi_{j-\frac{1}{2}}\right]. \tag{2.4}$$

Throughout the article, we refer to a point x_j as a boundary point if $j = 0$ or $j = J$, as an interior point (IP) otherwise. Interior points and boundary points construct all points (APs).

Denote the following discrete L^p, L^∞ and H^1 spatial norms:

$$\|\phi\|_{L^p} = \|\phi\|_p = \left(\sum_{j=0}^{J} |\phi_j|^p h\right)^{\frac{1}{p}},$$
$$\|\phi\|_{L^\infty} = \|\phi\|_\infty = \max_{j=0,1,\ldots,J} |\phi_j|,$$
$$\|\phi\|_{H^1} = \|\delta\phi\| = \left(\sum_{j=0}^{J-1} |\delta\phi_{j+\frac{1}{2}}|^2 h\right)^{\frac{1}{2}}.$$

Denote $\|\phi\| = \|\phi\|_{L^2}$. And define the discrete time-space norms:

$$\|\phi\|_{L^\infty(L^2)} = \max_{n=0,1,\ldots,N} \|\phi^n\|,$$
$$\|\phi\|_{L^\infty(L^\infty)} = \max_{n=0,1,\ldots,N} \|\phi^n\|_\infty,$$
$$\|\phi\|_{L^\infty(H^1)} = \max_{n=0,1,\ldots,N} \|\delta\phi^n\|,$$
$$\|\phi\|_{L^2(L^2)} = \left(\tau\sum_{n=0}^{N} \|\phi^n\|^2\right)^{\frac{1}{2}}.$$

In the following, we take abbreviations $g^n_j = g^n_j(U) = \left((U^n_j)^+\right)^3 U^n_j$, $e^n_j = e^n_j(U) = U^n_j + g^n_j$, and so forth.

We need some lemmas as follows:

Lemma 2.7 (The discrete Sobolev inequality [37])**.** *For any discrete function $\phi = \{\phi_j | j = 0, 1, \ldots, J\}$ $(Jh = L)$, the following assertions hold.*

(i) *For all $\varepsilon > 0$, there is*

$$\|\phi\|^2_\infty \le \varepsilon\|\delta\phi\|^2 + \frac{C}{\varepsilon}\|\phi\|^2,$$

where C is a constant depending on L, and independent of ε, h and ϕ;

(ii) *If* $\phi_0 = \phi_J = 0$, *then*

$$\|\phi\| \le L\|\delta\phi\|,$$
$$\|\phi\|_\infty \le \|\delta\phi\|^{\frac{1}{2}}\|\phi\|^{\frac{1}{2}}.$$

Lemma 2.8 (Recursive inequality [30])**.** *Suppose* $\{\chi_s\}$ *is a nonnegative sequence, and satisfies*

$$\chi_{s+1} \le A_1\chi_s^2 + A_2\chi_s\chi_{s+1} + A_3, \quad \forall s \ge 0,$$

where A_i $(i = 1, 2, 3)$ *are nonnegative constants, then when* $4A_3(A_1 + A_2) \le 1$ *and* $\chi_0 \le 2A_3$*, there is*

$$\chi_s \le 2A_3, \quad \forall s \ge 1.$$

2.2. Fully implicit scheme (FIS)

A classical difference scheme for problem (2.1)–(2.3) is the following nonlinear fully implicit scheme

$$\frac{e_j^{n+1} - e_j^n}{\tau} = \delta(A(U)\delta U)_j^{n+1}, \quad IPs; \tag{2.5}$$
$$U_0^{n+1} = \psi_0^{n+1},\ U_J^{n+1} = \psi_1^{n+1}, \quad 0 \le n \le N-1; \tag{2.6}$$
$$U_j^0 = \varphi_j, \quad APs; \tag{2.7}$$

where $\psi_0^{n+1} = \psi_0(t^{n+1})$, $\psi_1^{n+1} = \psi_1(t^{n+1})$, and $\varphi_j = \varphi(x_j)$.

Denote the solution of (2.1)–(2.3) as $u = u(x, t)$. With Taylor's expansion, it is easy to get the truncation error equation

$$-R_j^{n+1} := \frac{e_j^{n+1}(u) - e_j^n(u)}{\tau} - \delta(A(u)\delta u)_j^{n+1} = O(h^2 + \tau), \tag{2.8}$$

where $e_j^n(u) = u_j^n + g_j^n(u)$ and $u_j^n = u(x_j, t^n)$.

From the Assumption 2.3, the initial-boundary values of scheme (2.5)–(2.7) have positive lower bound m and upper bound M, namely, $m \le U_j^0 \le M$, $\forall 0 \le j \le J$, and $m \le U_0^{n+1} \le M, m \le U_J^{n+1} \le M$, $\forall 0 \le n \le N-1$.

Proposition 2.9. *Under the Assumptions 2.1–2.3, for the discrete solution of (2.5)–(2.7), there holds*

$$0 < m \le \min_{0\le j\le J} U_j^{n+1} \le \max_{0\le j\le J} U_j^{n+1} \le M, \ \forall 0 \le n \le N-1.$$

Proof. By induction we assume $m \le U_j^n \le M$, $\forall 0 \le j \le J$ and $m \le U_0^{n+1} \le M, m \le U_J^{n+1} \le M$.

First, let's show $\min_{0\le j\le J} U_j^{n+1} \ge m$. Suppose $U_{j_0}^{n+1} = \min_{1\le j\le J-1} U_j^{n+1} \le 0$, then $1 \le j_0 \le J-1$. At point $j = j_0$, the value on the left hand side of (2.5) is strictly less than 0, and that on the right hand side is equal to or greater than 0, a contradiction occurs. So $\min_{1\le j\le J-1} U_j^{n+1} > 0$. If $\min_{0\le j\le J} U_j^{n+1} = U_{j_0}^{n+1} < m$, then $1 \le j_0 \le J-1$. At point $j = j_0$, the left hand side of (2.5) is strictly less than 0, and the right hand side is equal to or greater than 0, again a contradiction appears. Hence, $\min_{0\le j\le J} U_j^{n+1} \ge m$.

Similarly, if the initial-boundary values have positive upper bound M, then we can prove $\max_{0\le j\le J} U_j^{n+1} \le M$.

Proposition 2.9 is established. □

Taking into account Proposition 2.9, we can equivalently denote $g_j^n = g_j^n(U) = (U_j^n)^4$. In this paper, C refers to a positive constant independent of h, τ and s, and may be different each time it appears.

Assumption 2.10. $\frac{\tau}{h^2} \le C^*$, *where* C^* *is any fixed positive constant.*

Assumption 2.11. *Denote $\xi_j^n = U_j^n - u_j^n$. Suppose that the solution of scheme (2.5)–(2.7) satisfies*

$$\max_{0\le n\le N}(\|\xi^n\| + \|\delta\xi^n\|) \le C(h^2+\tau).$$

To emphasize the main character of iterative methods, in the following we omit the superscript $n+1$ and subscripts $j,\ j+\frac{1}{2}$ when no confusion occurs.

2.3. Picard factorization (PF) iteration (see [35])

For a fixed non-negative integer n $(0 \le n \le N-1)$, with initial iterative value $U_j^{(0)} = U_j^n$, the PF iteration is defined by linearizing $g(u)$ by factorization, namely

$$g_j^{(s+1)} - g_j^n = \left(U_j^{(s+1)} - U_j^n\right)\left(U_j^{(s)} + U_j^n\right)\left[\left(U_j^{(s)}\right)^2 + (U_j^n)^2\right] := I_1, \tag{2.9}$$

and then finding $\{U_j^{(s+1)} | j = 0,1,\ldots,J\}$ $(s = 0,1,\ldots)$ which satisfies the following equations:

$$\frac{e_j^{(s+1)} - e_j^n}{\tau} = \delta\left(A^{(s)}\delta U^{(s+1)}\right)_j, \quad IPs; \tag{2.10}$$

$$U_0^{(s+1)} = \psi_0^{n+1},\ U_J^{(s+1)} = \psi_1^{n+1}, \tag{2.11}$$

where $e_j^{(s+1)} - e_j^n = U_j^{(s+1)} + g_j^{(s+1)} - U_j^n - g_j^n,\ A^{(s)} = A(U^{(s)})$, hence $\delta\left(A^{(s)}\delta U^{(s+1)}\right)_j$ can be seen as in (2.4), *i.e.*, $\delta(A^{(s)}\delta U^{(s+1)})_j = \frac{1}{h}[A(U_{j+\frac{1}{2}}^{(s)})\delta U_{j+\frac{1}{2}}^{(s+1)} - A(U_{j-\frac{1}{2}}^{(s)})\delta U_{j-\frac{1}{2}}^{(s+1)}]$.

The construction of (2.9) is originally given in literature [35], but its motivation why do so is not explained. Anyway, when linearizing a nonlinear function such as $g(u) = u^4$, factorization should be a natural approach. Moreover, there is no result of theoretical analysis in [35]. The last three methods introduced in this paper are all Newton-type iterative methods, which are expected to be superior to Picard method theoretically and numerically in solving equilibrium radiation diffusion equations.

2.4. Picard-Newton factorization (PNF) iteration

Define the PNF iteration with (2.11) and the following equation:

$$\frac{e_j^{(s+1)} - e_j^n}{\tau} = \delta\left(A^{(s)}\delta U^{(s+1)}\right)_j + \delta\left[A'^{(s)}\left(U^{(s+1)} - U^{(s)}\right)\delta U^{(s)}\right]_j,\ IPs, \tag{2.12}$$

where $A'^{(s)} = A'^{(s)}(U)$, and

$$\begin{aligned}&\delta\left[A'^{(s)}\left(U^{(s+1)} - U^{(s)}\right)\delta U^{(s)}\right]_j\\ &\quad= \frac{1}{h}\left[A'^{(s)}_{p_{j+\frac{1}{2}}}(U)\left(U_{j+\frac{1}{2}}^{(s+1)} - U_{j+\frac{1}{2}}^{(s)}\right)\delta U_{j+\frac{1}{2}}^{(s)} - A'^{(s)}_{p_{j-\frac{1}{2}}}(U)\left(U_{j-\frac{1}{2}}^{(s+1)} - U_{j-\frac{1}{2}}^{(s)}\right)\delta U_{j-\frac{1}{2}}^{(s)}\right]\end{aligned}$$

as in (2.4), and

$$\begin{aligned} g_j^{(s+1)} - g_j^n &= \left(U_j^{(s+1)} - U_j^n\right)\left(U_j^{(s)} + U_j^n\right)\left[\left(U_j^{(s)}\right)^2 + (U_j^n)^2\right]\\ &\quad + \left(U_j^{(s)} - U_j^n\right)[3\left(U_j^{(s)}\right)^2 + (U_j^n)^2 + 2U_j^{(s)}U_j^n]\left(U_j^{(s+1)} - U_j^{(s)}\right)\\ &:= I_1 + I_2. \end{aligned} \tag{2.13}$$

This method can be obtained from PF (2.10) as follows: for the diffusion operator, add a small linear term $\delta\left[A'^{(s)}(U^{(s+1)}-U^{(s)})\delta U^{(s)}\right]$, for the time derivative term, when approximating $(g(u))_t$, we add a linear term I_2 on the basis of PF iteration. Note that I_2 is obtained by changing $U_j^{(s+1)}$ in I_1 to $U_j^{(s)}$, and replace $(U_j^{(s)}+U_j^n)\left[\left(U_j^{(s)}\right)^2+(U_j^n)^2\right]$ with the product of its partial derivative with respect to $U_j^{(s)}$ and the increment $U_j^{(s+1)}-U_j^{(s)}$. We observed that constructing linearization procedure in such a way is different from the usual Newton linearization (*e.g.*, see the next subsection). It facilitates the acceleration of existing PF programs. The resulting equations can preserve the parabolic property, and be solved quickly since they converge with a quadratic speed as will be proved in Section 4.

2.5. Picard-Newton (PN) iteration

Define

$$\begin{aligned} g_j^{(s+1)}-g_j^n &= g\left(U_j^{(s)}\right)+{g'_u}_j^{(s)}\left(U_j^{(s+1)}-U_j^{(s)}\right)-g_j^n(U)\\ &=\left(U_j^{(s)}\right)^4+4\left(U_j^{(s)}\right)^3\left(U_j^{(s+1)}-U_j^{(s)}\right)-\left(U_j^n\right)^4. \end{aligned} \tag{2.14}$$

The PN iteration method is constructed with (2.12), (2.14) and (2.11). Distinct from (2.13), to get (2.14), $g_j^{(s+1)}=g(U_j^{(s+1)})$ is directly evaluated as a first-order Taylor expansion at $U_j^{(s)}$. In the case of $g(u)=0$, existing literatures have studied the PN iterative method, see the literatures [3, 28]. This method requires less computational overhead to form linear algebraic equations than the standard Newton iteration method and can improve computational efficiency.

2.6. Derivative free Picard-Newton factorization (DFPNF) iteration

In PNF iteration (2.12), (2.13) and (2.11), we replace the derivative $A'^{(s)}_{j+\frac12}$ with the difference quotient

$$A'^{(s)}_{\varepsilon j+\frac12}=\frac{1}{\varepsilon^{(s)}_{j+\frac12}}\left[A\left(U^{(s)}_{j+\frac12}+\varepsilon^{(s)}_{j+\frac12}\right)-A\left(U^{(s)}_{j+\frac12}\right)\right],$$

where $\varepsilon^{(s)}_{j+\frac12}>0$ are some small parameters. And define the DFPNF iteration as follows:

$$\frac{e_j^{(s+1)}-e_j^n}{\tau}=\delta\left(A^{(s)}\delta U^{(s+1)}\right)_j+\delta\left[A'^{(s)}_{\varepsilon}\left(U^{(s+1)}-U^{(s)}\right)\delta U^{(s)}\right]_j,\ IPs. \tag{2.15}$$

where $\delta\left[A'^{(s)}_{\varepsilon}\left(U^{(s+1)}-U^{(s)}\right)\delta U^{(s)}\right]_j$ can be seen as in (2.4). This method can be used to solve nonlinear diffusion problems with general tabular diffusion coefficients.

3. Convergence accuracy and positivity preservation

In this section, we will prove the first-order time and second-order space L^2 and H^1 norm convergence of the four iterations. Also, the property of preserving physical bounds for Picard factorization iteration and that of preserving positivity (PP) for Newton-type iteration will be demonstrated. We assume that for a positive constant $\bar{M}$ there is

$$(S_1)\qquad \max_{0\le n\le N-1}\left(\|u^{n+1}\|_\infty+\|\delta u^{n+1}\|_\infty+\|d_t u^{n+1}\|_\infty\right)\le\bar{M}.$$

Denote $v_j^{(s)}=U_j^{(s)}-u_j^{n+1}$.

3.1. PF

With a similar reasoning procedure as for the FIS (see Prop. 2.9), we can obtain the property of preserving physical bounds for the PF iteration as follows.

Proposition 3.1. *Let the Assumptions 2.1–2.3 hold. Then for the solution of PF iteration (2.9)–(2.11), we have*

$$m \le \min_{0\le j\le J} U_j^{(s)} \le \max_{0\le j\le J} U_j^{(s)} \le M, \qquad \forall s \ge 0.$$

Theorem 3.2. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_1) *hold. If* τ *is small enough, then the solution of PF iteration (2.9)–(2.11) has first-order temporal and second-order* L^2 *and* H^1*-norm spatial convergence to the solution of problem (2.1)–(2.3), and such convergence is uniform with respect to* s*, that is,*

$$\|v^{(s)}\| + \|\delta v^{(s)}\| = O(h^2+\tau), \quad \forall s \ge 0. \tag{3.1}$$

Proof. From (2.10) and (2.8) we have

$$\frac{v_j^{(s+1)}-\xi_j^n+g_j^{(s+1)}-g_j^n-\left[g_j^{n+1}(u)-g_j^n(u)\right]}{\tau} = \delta(A^{(s)}\delta v^{(s+1)})_j + \delta[(A^{(s)}-A(u))\delta u]_j + R_j^{n+1}, \quad IPs, \tag{3.2}$$

where $v_0^{(s+1)} = v_j^{(s+1)} = 0$. The first term in (3.2) is determined by (2.9).

Multiplying (3.2) by $(v_j^{(s+1)}-\xi_j^n)h$, and summing up the products for $j=1,\dots,J-1$, we obtain

$$\begin{aligned}
&\frac{1}{\tau}\sum_{j=1}^{J-1}\left\{v_j^{(s+1)}-\xi_j^n+g_j^{(s+1)}-g_j^n-\left[g_j^{n+1}(u)-g_j^n(u)\right]\right\}\left(v_j^{(s+1)}-\xi_j^n\right)h + \sum_{j=0}^{J-1} A_{j+\frac12}^{(s)}|\delta v_{j+\frac12}^{(s+1)}|^2 h\\
&\quad= \sum_{j=0}^{J-1} A_{j+\frac12}^{(s)}\delta v_{j+\frac12}^{(s+1)}\delta\xi_{j+\frac12}^n h + \sum_{j=1}^{J-1} R_j^{n+1}\frac{v_j^{(s+1)}-\xi_j^n}{\tau}h\tau\\
&\qquad- \sum_{j=0}^{J-1}\left[A_{j+\frac12}^{(s)} - A_{j+\frac12}^{n+1}(u)\right]\delta u_{j+\frac12}^{n+1}\left(\delta v_{j+\frac12}^{(s+1)} - \delta\xi_{j+\frac12}^n\right)h.
\end{aligned}$$

Denote

$$Q^{s+1} =: \frac{1}{\tau}\sum_{j=1}^{J-1}\left\{v_j^{(s+1)}-\xi_j^n+g_j^{(s+1)}-g_j^n-\left[g_j^{n+1}(u)-g_j^n(u)\right]\right\}\left(v_j^{(s+1)}-\xi_j^n\right)h.$$

Note that

$$U^{(s+1)}-U^n = v^{(s+1)}-\xi^n+\tau d_t u^{n+1},\ (u^{n+1})^4-(u^n)^4 = \tau d_t u^{n+1}\left(u^{n+1}+u^n\right)\left[(u^{n+1})^2+(u^n)^2\right], \tag{3.3}$$

from (2.9), we derive

$$g_j^{(s+1)} - g_j^n - \left[g_j^{n+1}(u)-g_j^n(u)\right] = P_1\left(v_j^{(s+1)}-\xi_j^n\right) + \tau P_2 v_j^{(s)} + \tau P_3\xi_j^n,$$

where

$$\begin{aligned}
P_1 &= P_1\left(U_j^{(s)},U_j^n\right) = \left(U_j^{(s)}+U_j^n\right)\left[\left(U_j^{(s)}\right)^2+(U_j^n)^2\right],\\
P_2 &= P_2\left(U_j^{(s)},U_j^n,u_j^{n+1},u_j^n\right) = d_t u_j^{n+1}\left[\left(U_j^{(s)}+U_j^n\right)\left(U_j^{(s)}+u_j^{n+1}\right)+\left(u_j^{n+1}\right)^2+\left(u_j^n\right)^2\right],\\
P_3 &= P_3\left(U_j^{(s)},U_j^n,u_j^{n+1},u_j^n\right) = d_t u_j^{n+1}\left[\left(U_j^{(s)}+U_j^n\right)\left(U_j^n+u_j^n\right)+\left(u_j^{n+1}\right)^2+\left(u_j^n\right)^2\right].
\end{aligned}$$

Then

$$v_j^{(s+1)} - \xi_j^n + g_j^{(s+1)} - g_j^n - \left[g_j^{n+1}(u) - g_j^n(u)\right] = (1+P_1)\left(v_j^{(s+1)} - \xi_j^n\right) + \tau P_2 v_j^{(s)} + \tau P_3 \xi_j^n.$$

Note that

$$A_{j+\frac{1}{2}}^{(s)} - A_{j+\frac{1}{2}}^{n+1}(u) = A^{*(s)} v_{j+\frac{1}{2}}^{(s)},$$

where

$$A^{*(s)} = \int_0^1 A'\left(r v_{j+\frac{1}{2}}^{(s)} + u_{j+\frac{1}{2}}^{n+1}\right) \mathrm{d}r. \tag{3.4}$$

Let's suppose

$$|A_{j+\frac{1}{2}}^{(s)}| \le C_{A0},\ |A^{*(s)}| \le C_{A1},\ |P_2| \le C_P,\ |P_3| \le C_P, \tag{3.5}$$

where C_{A0}, C_{A1} and C_P are positive constants to be determined later.

According to Propositions 2.9 and 3.1, $P_1 \ge 0$, then we have

$$Q^{s+1} \ge \|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau - \frac{1}{4}\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau - C\|v^{(s)}\|^2\tau - C\|\xi^n\|^2\tau.$$

It's easy to show that

$$\sum_{j=0}^{J-1} A_{j+\frac{1}{2}}^{(s)} \delta v_{j+\frac{1}{2}}^{(s+1)} \delta \xi_j^n h \le \frac{\sigma}{4}\|\delta v^{(s+1)}\|^2 + C\|\delta\xi^n\|^2.$$

$$\sum_{j=1}^{J-1} R_j^{n+1} \frac{v_j^{(s+1)} - \xi_j^n}{\tau} h\tau \le \frac{1}{4}\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau + C\|R^{n+1}\|^2\tau.$$

From (S_1), there is $\|\delta u^{n+1}\|_\infty \le \bar{M}$, and then

$$\begin{aligned}
&\left|\sum_{j=0}^{J-1} \left[A_{j+\frac{1}{2}}^{(s)} - A_{j+\frac{1}{2}}^{n+1}(u)\right] \delta u_{j+\frac{1}{2}}^{n+1} \left(\delta v_{j+\frac{1}{2}}^{(s+1)} - \delta \xi_{j+\frac{1}{2}}^n\right) h\right| \\
&\qquad \le C_{A1}\bar{M} \sum_{j=0}^{J-1} |v_{j+\frac{1}{2}}^{(s)}||\delta v_{j+\frac{1}{2}}^{(s+1)} - \delta \xi_{j+\frac{1}{2}}^n| h \\
&\qquad \le \frac{\sigma}{2}\|\delta v^{(s+1)}\|^2 + \frac{C_{A1}^2 \bar{M}^2}{2\sigma}\|v^{(s)}\|^2 + C\|v^{(s)}\|^2 + C\|\delta\xi^n\|^2.
\end{aligned}$$

So we have

$$\frac{1}{2}\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau + \frac{\sigma}{4}\|\delta v^{(s+1)}\|^2 \le C\left(\|v^{(s)}\|^2 + \|\xi^n\|^2\tau + \|\delta\xi^n\|^2 + \|R^{n+1}\|^2\tau + \|v^{(s)}\|^2\tau\right),$$

where C only depends on the known data $\sigma,\ C_{A0},\ C_{A1},\ \bar{M}$ and M.

By noticing that for $l = s+1$,

$$\|v^{(l)}\|^2 \le 2\|\frac{v^{(l)} - \xi^n}{\tau}\|^2\tau^2 + 2\|\xi^n\|^2, \tag{3.6}$$

for sufficiently small τ, there stands

$$\|\frac{v^{(s+1)}-\xi^n}{\tau}\|^2\tau+\|v^{(s+1)}\|^2+\|\delta v^{(s+1)}\|^2\le C\|v^{(s)}-\xi^n\|^2+C\eta^{n+1}, \tag{3.7}$$

where we denote $\eta^{n+1}=\|R^{n+1}\|^2\tau+\|\xi^n\|^2+\|\delta\xi^n\|^2$.

So we have

$$\begin{aligned}\|v^{(s+1)}-\xi^n\|^2&\le C\tau\|v^{(s)}-\xi^n\|^2+C\tau\eta^{n+1}\le\cdots\\&\le(C\tau)^{s+1}\|v^{(0)}-\xi^n\|^2+\frac{C\tau[1-(C\tau)^{s+1}]}{1-C\tau}\eta^{n+1}.\end{aligned}$$

We assume $C\tau\le\frac{1}{2}$, then $\frac{C\tau[1-(C\tau)^{s+1}]}{1-C\tau}\le\frac{C\tau}{1-C\tau}\le 1$.

So we have

$$\|v^{(s+1)}-\xi^n\|^2\le(C\tau)^{s+1}\|v^{(0)}-\xi^n\|^2+\eta^{n+1}\le\frac{1}{2}\|v^{(0)}-\xi^n\|^2+\eta^{n+1}.$$

Then

$$\|v^{(s+1)}\|^2\le 2\|v^{(s+1)}-\xi^n\|^2+2\|\xi^n\|^2\le\|v^{(0)}-\xi^n\|^2+2\left(\|R^{n+1}\|^2\tau+\|\xi^n\|^2+\|\delta\xi^n\|^2\right)+2\|\xi^n\|^2. \tag{3.8}$$

We derive that

$$\|v^{(s+1)}\|\le\|v^{(0)}-\xi^n\|+C\left(\|R^{n+1}\|+\|\xi^n\|+\|\delta\xi^n\|\right)\le 2\sqrt{L}M+C\left(h^2+\tau\right)\le 2\sqrt{L}M+1, \tag{3.9}$$

where $\|U^{(0)}\|\le\sqrt{L}\|U^{(0)}\|_\infty\le\sqrt{L}M$, etc., has been used.

Furthermore, according to (3.7), there stands

$$\begin{aligned}\|v^{(s+1)}\|+\|\delta v^{(s+1)}\|&\le C_1\|v^{(s)}\|+C_2(\|R^{n+1}\|+\|\xi^n\|+\|\delta\xi^n\|)\\&\le C_1[2\sqrt{L}M+C(h^2+\tau)]+C_2C_3(h^2+\tau)\le\tilde{C}_0M+1,\end{aligned}$$

where $\tilde{C}_0\ge 2C_1\sqrt{L}$.

When we take initial iterative value $U_j^{(0)}=U^n$, the following estimation holds

$$\|v^{(0)}\|+\|\delta v^{(0)}\|\le C(h^2+\tau).$$

Hence

$$\|v^{(s+1)}\|+\|\delta v^{(s+1)}\|\le C(h^2+\tau),\quad s=0,1,2,\ldots.$$

Using the discrete Sobolev inequality, one gets

$$\|v^{(s+1)}\|_\infty\le\|v^{(s+1)}\|^{\frac{1}{2}}\|\delta v^{(s+1)}\|^{\frac{1}{2}}\le\|v^{(s+1)}\|+\|\delta v^{(s+1)}\|\le\tilde{C}_0M+1,$$

So by taking $C_0\ge\tilde{C}_0+1$ large enough and then taking

$$\begin{aligned}C_{A0}&=\max_{|v|\le C_0M+1}|A(v)|,\ C_{A1}=\max_{|v|\le C_0M+1}|A'(v)|,\\ C_P&=\max_{\substack{|v_1|\le C_0M+1\\|v_2|\le M,|v_3|\le M,|v_4|\le M}}\left\{|P_2(v_1,v_2,v_3,v_4)|,|P_3(v_1,v_2,v_3,v_4)|\right\},\end{aligned}$$

we see (3.5) is valid. Theorem 3.2 is established. □

Remark 3.3. From the convergence accuracy theorem we know that it is unnecessary to add new and severe restriction on the time step-length to assure the accuracy of iterative solutions; moreover, (3.1) is valid for any $s\ge 0$, where the O constant is independent of s, in particular, $s=0$ indicates that the semi-implicit scheme is unconditionally convergent.

3.2. PNF

Theorem 3.4. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and (S_1) hold. If τ is small enough, then the solution of PNF iteration (2.11)–(2.13) has first-order temporal and second-order L^2 and H^1-norm spatial convergence to the solution of problem (2.1)–(2.3), and such convergence is uniform in s, that is, (3.1) is valid. Furthermore,*

$$\min_{0\le j\le J} U_j^{(s)} \ge \frac{m}{2} > 0, \qquad \forall s \ge 0, \tag{3.10}$$

thus the PNF iteration is positivity-preserving.

Proof. From (2.12) and (2.8) we have

$$\begin{aligned}\frac{v_j^{(s+1)} - \xi_j^n + g_j^{(s+1)} - g_j^n - \left[g_j^{n+1}(u) - g_j^n(u)\right]}{\tau} &= \delta\left(A^{(s)}\delta v^{(s+1)}\right)_j + \delta\left\{\left[A^{(s)} - A(u) + A'^{(s)}\left(U^{(s+1)} - U^{(s)}\right)\right]\right.\\ &\quad \left.\times\, \delta u + A'^{(s)}\left(U^{(s+1)} - U^{(s)}\right)\delta\left(U^{(s)} - u\right)\right\}_j + R_j^{n+1},\end{aligned} \tag{3.11}$$

where $v_0^{(s+1)} = v_j^{(s+1)} = 0$, the first term in (3.11) is determined by (2.13). The term in {} on the right side of (3.11) is equal to the following expression:

$$\begin{aligned}&\left\{A^{*(s)}\left(U^{(s)} - u\right) + A'^{(s)}\left[U^{(s+1)} - u - \left(U^{(s)} - u\right)\right]\right\}\delta u + A'^{(s)}\left[U^{(s+1)} - u - \left(U^{(s)} - u\right)\right]\delta\left(U^{(s)} - u\right)\\ &\quad = \left[A''^{(s)}\left(v^{(s)}\right)^2 + A'^{(s)}v^{(s+1)}\right]\delta u + A'^{(s)}\left(v^{(s+1)} - v^{(s)}\right)\delta v^{(s)},\end{aligned}$$

where $A^{*(s)}$ is defined as in (3.4), and the following abbreviation is used:

$$A''^{(s)} = \int_0^1\int_0^1 A''\left(\bar r(r-1)v^{(s)}_{j+\frac12} + U^{(s)}_{j+\frac12}\right)\mathrm{d}\bar r(r-1)\,\mathrm{d}r.$$

Let's suppose

$$|A^{(s)}_{j+\frac12}| \le C_{A0}, \ \ |A'^{(s)}| \le C_{A1}, \ \ |A''^{(s)}| \le C_{A2}, \tag{3.12}$$

where C_{Ai} $(i = 0, 1, 2)$ are positive constants to be determined later.

Multiplying (3.11) by $(v_j^{(s+1)} - \xi_j^n)h$, and summing up the products for $j = 1, \dots, J-1$, we obtain

$$\begin{aligned}Q^{s+1} + \sigma\|\delta v^{(s+1)}\|^2 &\le C\sum_{j=0}^{J-1}|\delta v^{(s+1)}||\delta\xi^n|h + \sum_{j=1}^{J-1}|R^{n+1}||v^{(s+1)} - \xi^n|h\\ &\quad + C\sum_{j=0}^{J-1}\left[\left(|v^{(s)}|^2 + |v^{(s+1)}|\right)|\delta u| + |v^{(s+1)} - v^{(s)}||\delta v^{(s)}|\right]|\delta v^{(s+1)} - \delta\xi^n|h\\ &\le \frac{\sigma}{4}\|\delta v^{(s+1)}\|^2 + C\|\delta\xi^n\|^2 + C\|R^{n+1}\|^2 + C\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau^2 + \frac{\sigma}{4}\|\delta v^{(s+1)}\|^2\\ &\quad + C\|\delta\xi^n\|^2 + C\left(\|v^{(s)}\|_\infty^2\|v^{(s)}\|^2 + \|v^{(s+1)}\|^2 + \|v^{(s+1)} - v^{(s)}\|_\infty^2\|\delta v^{(s)}\|^2\right)\\ &\le \frac{\sigma}{2}\|\delta v^{(s+1)}\|^2 + C_1\left[\|\delta v^{(s)}\|^4 + \left(\|v^{(s+1)}\|_\infty^2 + \|v^{(s)}\|_\infty^2\right)\|\delta v^{(s)}\|^2\right]\\ &\quad + 2C_1\|\xi^n\|^2 + C_1\|\delta\xi^n\|^2 + C_1\|R^{n+1}\|^2 + 3C_1\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau^2,\end{aligned} \tag{3.13}$$

where $C_1 \geq 1$ depends on $\bar{M}$, and Lemma 2.7 and inequality (3.6) for $l = s+1$ have been used.

Note that (3.3) is valid, from (2.13), we derive

$$\begin{aligned} g_j^{(s+1)} - g_j^n - \left[g_j^{n+1}(u) - g_j^n(u)\right] &= P_1\left(v_j^{(s+1)} - \xi_j^n\right) + \tau P_2 v_j^{(s)} + \tau P_3 \xi_j^n + \left(v_j^{(s)} - \xi_j^n + \tau d_t u_j^{n+1}\right) \\ &\quad \times \left[3\left(U_j^{(s)}\right)^2 + \left(U_j^n\right)^2 + 2U_j^{(s)}U_j^n\right]\left(v_j^{(s+1)} - v_j^{(s)}\right) \\ &= P_1\left(v_j^{(s+1)} - \xi_j^n\right) + \tau P_6 v_j^{(s)} + \tau P_5 v_j^{(s+1)} + \tau P_3 \xi_j^n \\ &\quad + P_4\left(v_j^{(s)} - \xi_j^n\right)\left(v_j^{(s+1)} - v_j^{(s)}\right), \end{aligned}$$

where P_i $(i = 1,2,3)$ are defined as in Section 3.1, and

$$\begin{aligned} P_4 &= P_4\left(U_j^{(s)}, U_j^n\right) = 3\left(U_j^{(s)}\right)^2 + \left(U_j^n\right)^2 + 2U_j^{(s)}U_j^n. \\ P_5 &= P_5\left(U_j^{(s)}, U_j^n, u_j^{n+1}, u_j^n\right) = d_t u^{n+1} P_4. \\ P_6 &= P_6\left(U_j^{(s)}, U_j^n, u_j^{n+1}, u_j^n\right) = P_2 - P_5. \end{aligned}$$

Make the assumption that

$$U_j^{(s)} \geq 0, \quad \|U^{(s)}\|_\infty \leq \bar{K}. \tag{3.14}$$

So

$$P_1 \geq 0; \quad |P_i| \leq C_P, \quad i = 2,3,4,5.$$

Thus

$$\begin{aligned} Q^{s+1} &= \sum_{j=1}^{J-1}(1+P_1)\left(\frac{v_j^{(s+1)} - \xi_j^n}{\tau}\right)^2 h\tau + \sum_{j=1}^{J-1} P_6 v_j^{(s)}\left(v_j^{(s+1)} - \xi_j^n\right)h \\ &\quad + \sum_{j=1}^{J-1}\left(P_5 v_j^{(s+1)} + P_3\xi_j^n\right)\left(v_j^{(s+1)} - \xi_j^n\right)h + \sum_{j=1}^{J-1} P_4\left(v_j^{(s+1)} - v_j^{(s)}\right)\frac{v_j^{(s)} - \xi_j^n}{\tau}\frac{v_j^{(s+1)} - \xi_j^n}{\tau}h\tau \\ &\geq \left(\frac{3}{4} - 2C_2\tau\right)\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau - C_2\|v^{(s)}\|^2\tau - C_2\|\xi^n\|^2 - C_2\left(\|v^{(s+1)}\|_\infty^2 + \|v^{(s)}\|_\infty^2\right)\|\frac{v^{(s)} - \xi^n}{\tau}\|^2\tau, \end{aligned}$$

where C_2 is a positive constant depending on C_P.

Combining the above inequality with (3.13), and using Lemma 2.7 and Hölder's inequality, by a thorough derivation, we get

$$\begin{aligned} &\left(\frac{3}{4} - 5C_3\tau\right)\|\frac{v^{(s+1)} - \xi^n}{\tau}\|^2\tau + \frac{\sigma}{2}\|\delta v^{(s+1)}\|^2 \\ &\quad \leq C_3\left[\|\delta v^{(s)}\|^4 + \left(\|v^{(s+1)}\|_\infty^2 + \|v^{(s)}\|_\infty^2\right)\|\delta v^{(s)}\|^2 + \left(\|v^{(s+1)}\|_\infty^2 + \|v^{(s)}\|_\infty^2\right)\|\frac{v^{(s)} - \xi^n}{\tau}\|^2\tau \right. \\ &\qquad \left. + \|v^{(s)}\|^2\tau + 3\|\xi^n\|^2 + \|\delta\xi^n\|^2 + \|R^{n+1}\|^2\right] \\ &\quad \leq C_3\left[(1+L)\|\delta v^{(s)}\|^4 + \|v^{(s+1)}\|\|\delta v^{(s+1)}\|\|\delta v^{(s)}\|^2 + \left(\|v^{(s+1)}\|\|\delta v^{(s+1)}\| + L\|\delta v^{(s)}\|^2\right)\|\frac{v^{(s)} - \xi^n}{\tau}\|^2\tau \right. \\ &\qquad \left. + \|v^{(s)}\|^2\tau + 3\|\xi^n\|^2 + \|\delta\xi^n\|^2 + \|R^{n+1}\|^2\right] \end{aligned}$$

$$\leq C_4\left[\left(1+L+L^2\right)\|\delta v^{(s)}\|^4+\frac{2}{\sigma}\|v^{(s+1)}\|^2\|\delta v^{(s)}\|^4+\frac{2}{\sigma}\|v^{(s+1)}\|^2\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\right)^2\right.$$
$$\left.+\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\right)^2+\|v^{(s)}\|^2\tau+3\|\xi^n\|^2+C\|\delta\xi^n\|^2+\|R^{n+1}\|^2\right]+\frac{\sigma}{4}\|\delta v^{(s+1)}\|^2. \tag{3.15}$$

So when $10C_3\tau<1$, by using (3.6) for $l=s+1$, we have

$$\|\frac{v^{(s+1)}-\xi^n}{\tau}\|^2\tau+\|v^{(s+1)}\|^2+\|\delta v^{(s+1)}\|^2$$
$$\leq C_5\left[\|\delta v^{(s)}\|^4\left(1+\|v^{(s+1)}\|^2\right)+\|v^{(s+1)}\|^2\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\right)^2\right.$$
$$\left.+\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\right)^2+\|v^{(s)}\|^2\tau+\|\xi^n\|^2+\|\delta\xi^n\|^2+\|R^{n+1}\|^2\right],$$

where $C_5\geq 1$ is a positive constant depending on $C_{Ai}(i=0,1,2),\ C_P$ and σ.

Make induction assumption for $s'\leq s$,

$$C_5\left(\|\frac{v^{(s')}-\xi^n}{\tau}\|^2\tau+\|v^{(s')}\|^2+\|\delta v^{(s')}\|^2\right)<\frac{1}{2}.$$

We can derive that

$$\|\frac{v^{(s+1)}-\xi^n}{\tau}\|^2\tau+\|v^{(s+1)}\|^2+\|\delta v^{(s+1)}\|^2$$
$$\leq C_5\|\delta v^{(s)}\|^4+C_5^{-1}\tau\|\frac{v^{(s+1)}-\xi^n}{\tau}\|^2\tau+C_5\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\right)^2$$
$$+2C_5\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau^3+(C_5+C_5^{-1}+2C_5\tau)\|\xi^n\|^2+C_5\|\delta\xi^n\|^2+C_5\|R^{n+1}\|^2,$$

where (3.6) for $l=s,\ s+1$ has been used to get the above inequality.

If $2\tau\leq C_5$, then

$$\|\frac{v^{(s+1)}-\xi^n}{\tau}\|^2\tau+\|v^{(s+1)}\|^2+\|\delta v^{(s+1)}\|^2$$
$$\leq 2C_5\|\delta v^{(s)}\|^4+2C_5\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\right)^2$$
$$+4C_5\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau\tau^2+2(C_5+C_5^{-1}+2C_5\tau)\|\xi^n\|^2+2C_5\|\delta\xi^n\|^2+2C_5\|R^{n+1}\|^2.$$

Denote $y_s=2C_5\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau+\|v^{(s)}\|^2+\|\delta v^{(s)}\|^2\right)$, then we have

$$y_{s+1}\leq y_s^2+4C_5\tau^2y_s+4C_5(C_5+C_5^{-1}+2C_5\tau)\|\xi^n\|^2+4C_5^2\|\delta\xi^n\|^2+4C_5^2\|R^{n+1}\|^2.$$

With the Assumption 2.11 and (2.8), the last term of the above inequality can be upper bounded by $C_6(h^4+\tau^2)\leq\frac{1}{4}$. So we get

$$y_{s+1}\leq y_s^2+4C_5\tau^2y_s+C_6(h^4+\tau^2).$$

Denote $\chi_s = y_s + 4C_5\tau^2$, then

$$\begin{aligned}\chi_{s+1} &= y_{s+1} + 4C_5\tau^2 \le y_s^2 + 4C_5\tau^2 y_s + C_6(h^4+\tau^2) + 4C_5\tau^2 \\ &\le y_s(y_s + 4C_5\tau^2) + C_7(h^4+\tau^2) \le \chi_s^2 + C_7(h^4+\tau^2).\end{aligned}$$

Note that

$$v^{(0)} = U^{(0)} - u^{n+1} = U^{(0)} - U^n + \xi^n - \tau d_t u^{n+1}. \tag{3.16}$$

There is

$$\frac{v^{(0)} - \xi^n}{\tau} = \frac{U^{(0)} - U^n}{\tau} - d_t u^{n+1}.$$

So with $U^{(0)} = U^n$, we have

$$y_0 \le C(h^4 + \tau), \tag{3.17}$$

furthermore,

$$\begin{aligned}&\chi_0 = y_0 + 4C_5\tau^2 \le C(h^4+\tau).\\ &\chi_1 \le \chi_0^2 + C_7(h^4+\tau^2) \le C_8(h^4+\tau^2).\end{aligned}$$

Obviously, $C_8 > C_7$. Take $C_9 \ge C_8$, we have

$$\chi_{s+1} \le \chi_s^2 + C_9(h^4+\tau^2), \quad \forall s \ge 0,$$

and

$$\chi_1 \le 2C_9(h^4+\tau^2).$$

For h and τ small enough, there is

$$4C_9(h^4+\tau^2) \le 1.$$

Using Lemma 2.8, we have

$$\chi_s \le 2C_9(h^4+\tau^2), \quad \forall s \ge 2.$$

So

$$y_s \le \chi_s \le 2C_9(h^4+\tau^2), \quad \forall s \ge 1,$$

namely,

$$2C_5\left(\|\frac{v^{(s)} - \xi^n}{\tau}\|^2\tau + \|v^{(s)}\|^2 + \|\delta v^{(s)}\|^2\right) \le 2C_9(h^4+\tau^2), \quad \forall s \ge 1.$$

So $\forall s \ge 1$,

$$\|v^{(s)}\|^2 + \|\delta v^{(s)}\|^2 \le C(h^4+\tau^2). \tag{3.18}$$

By using (3.16) and the Assumption 2.11, we know (3.18) is valid for $s = 0$. Hence by Lemma 2.7, we find that for all $s \ge 0$, $\|v^{(s)}\|_\infty \le 1$. There exists a positive constant $\bar{K}$ such that the second inequality in (3.14) holds. Taking

$$C_{A0} = \max_{|v|\le M+1} |A(v)|, \;\; C_{A1} = \max_{|v|\le M+1} |A'(v)|, \;\; C_{A2} = \max_{|v|\le M+2} |A''(v)|,$$

we see that (3.12) is valid.

Using (3.17), we show

$$2C_5\left(\|\frac{v^{(0)}-\xi^n}{\tau}\|^2\tau+\|v^{(0)}\|^2+\|\delta v^{(0)}\|^2\right)\le C(h^4+\tau)<\min\left\{1,\left(\frac{m}{2}\right)^2\right\}.$$

Then using the Lemma 2.7 we have

$$\|v^{(s)}\|_\infty^2\le 2C_5\left(\|\frac{v^{(s)}-\xi^n}{\tau}\|^2\tau+\|v^{(s)}\|^2+\|\delta v^{(s)}\|^2\right)\le 2C_9(h^4+\tau^2)\le\left(\tfrac{m}{2}\right)^2,\ \ \forall s\ge 1.$$

Consequently,

$$\min_{0\le j\le J}U_j^{(s+1)}\ge\frac{m}{2}>0,\ \ \forall s\ge 0,$$

which means that the first inequality in (3.14) is also valid. The proof of Theorem 3.4 is accomplished. □

3.3. PN

Theorem 3.5. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_1) *hold. If* τ *is small enough, then the solution of PN iteration (2.11), (2.12), (2.14) has first-order temporal and second-order* L^2 *and* H^1*-norm spatial convergence to the exact solution of problem (2.1)–(2.3), and such convergence is uniform in* s*, that is, (3.1) is valid. Furthermore, the PN iteration is PP, that is, (3.10) is valid.*

3.4. DFPNF

Theorem 3.6. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_1) *hold. If*

$$\max_{0\le j\le J-1}|\varepsilon_{j+\frac12}^{(s)}|=O\left(\|v^{(s)}\|\right),$$

then for τ *small enough, the solution of DFPNF iteration (2.11), (2.13), (2.15) has first-order temporal and second-order* L^2 *and* H^1*-norm spatial convergence to the solution of problem (2.1)–(2.3), and such convergence is uniform in* s*, that is, (3.1) is valid. Furthermore, the DFPNF iteration is PP, that is, (3.10) is valid.*

Remark 3.7. The proofs of Theorems 3.5 and 3.6 are similar to that of Theorem 3.4, so we omit them.

Remark 3.8. Ensuring the numerical results at each step of nonlinear iteration being positive is also meaningful to the actual calculation, otherwise, when the nonlinear iteration has to terminate, one has to use certain processing method such as "enforcing the negative values to zero". Moreover, in the case of $g(u)=u^4$, we note that $\frac{\partial g(u)}{\partial t}=4u^3\frac{\partial u}{\partial t}$, and simple Picard iteration gives the term $4(U^{(s)})^3\frac{\partial U^{(s+1)}}{\partial t}$. So an anti-diffusion occurs when $U^{(s)}<0$, which doesn't conform to the physical meaning of the diffusion equation.

Remark 3.9. Rigorous convergence accuracy proof of the fully implicit scheme in Section 2 hasn't been presented, so our analysis of the various nonlinear iteration schemes in this section is necessary.

4. Convergence speed

In this section, we will prove the iterative sequence of the PF iteration converges to the solution of the FIS linearly, and those of the PNF, PN and DFPNF iterations converge quadratically.

We assume that

$$(S_2)\qquad \max_{0\le n\le N-1}\left(\|U^{n+1}\|_\infty+\|\delta U^{n+1}\|_\infty+\|d_tU^{n+1}\|_\infty\right)\le\bar{M}_1;$$

Denote $w_j^{(s)}=U_j^{(s)}-U_j^{n+1}$.

4.1. PF

Theorem 4.1. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_2) *hold. Then for the solution of PF iteration (2.9)–(2.11), for* τ *sufficiently small, there are*

$$\lim_{s\to\infty}\|w^{(s+1)}\|=0,\quad \|w^{(s+1)}\|\le C\tau^{\frac{1}{2}}\|w^{(s)}\|.$$

Proof. From (2.10) and (2.5) we have

$$\frac{w_j^{(s+1)}+g_j^{(s+1)}-g_j^n-\left(g_j^{n+1}-g_j^n\right)}{\tau}=\delta\left(A^{(s)}\delta w^{(s+1)}\right)_j-\delta\left[\left(A^{(s)}-A\right)\delta U\right]_j,\quad IPs, \tag{4.1}$$

where $w_0^{(s+1)}=w_j^{(s+1)}=0$, the first term in (4.1) is determined by (2.9), and

$$A_{j+\frac{1}{2}}^{(s)}-A_{j+\frac{1}{2}}^{n+1}=A\left(U_{j+\frac{1}{2}}^{(s)}\right)-A\left(U_{j+\frac{1}{2}}^{n+1}\right)=A^{(s)\star}w_{j+\frac{1}{2}}^{(s)},$$

where

$$A^{(s)\star}=\int_0^1 A'\left(rw_{j+\frac{1}{2}}^{(s)}+U_{j+\frac{1}{2}}^{n+1}\right)\,\mathrm{d}r. \tag{4.2}$$

Multiplying (4.1) by $w_j^{(s+1)}h$, and summing up the products for $j=1,\dots,J-1$, we obtain

$$\begin{aligned}&\frac{1}{\tau}\sum_{j=1}^{J-1}\left\{|w_j^{(s+1)}|^2+\left[g_j^{(s+1)}-g_j^n-\left(g_j^{n+1}-g_j^n\right)\right]w_j^{(s+1)}\right\}h\\&\quad+\sum_{j=0}^{J-1}A_{j+\frac{1}{2}}^{(s)}|\delta w_{j+\frac{1}{2}}^{(s+1)}|^2h+\sum_{j=0}^{J-1}\left(A_{j+\frac{1}{2}}^{(s)}-A_{j+\frac{1}{2}}^{n+1}\right)\delta U_{j+\frac{1}{2}}^{n+1}\delta w_{j+\frac{1}{2}}^{(s+1)}h=0.\end{aligned}$$

Note that from (2.9),

$$g_j^{(s+1)}-g_j^n-\left(g_j^{n+1}-g_j^n\right)=P_1w_j^{(s+1)}+\tau d_tU_j^{n+1}P_2w_j^{(s)},$$

where

$$\begin{aligned}P_1&=P_1\left(U_j^{(s)},U_j^n\right)=\left(U_j^{(s)}+U_j^n\right)\left[\left(U_j^{(s)}\right)^2+\left(U_j^n\right)^2\right],\\P_2&=P_2\left(U_j^{(s)},U_j^{n+1},U_j^n\right)=\left[\left(U_j^{(s)}\right)^2+(U_j^n)^2+\left(U_j^{n+1}+U_j^n\right)\left(U_j^{(s)}+U_j^{n+1}\right)\right].\end{aligned}$$

Let's assume

$$|A_{j+\frac{1}{2}}^{(s)}|\le C_{A0},\quad |A^{(s)\star}|\le C_{A1},\quad |P_2|\le C_P, \tag{4.3}$$

where C_{A0}, C_{A1} and C_P are positive constants to be determined later. Note that according to Propositions 2.9 and 3.1, $P_1>0$.

Assume (S_2) is established. According to Lemma 2.7, there stands

$$\begin{aligned}\left|\frac{1}{\tau}\sum_{j=1}^{J-1}\tau d_tU_j^{n+1}P_2w_j^{(s)}w_j^{(s+1)}h\right|&\le C_P\sum_{j=1}^{J-1}|d_tU_j^{n+1}||w_j^{(s)}||w_j^{(s+1)}|h\\&\le C\|d_tU^{n+1}\|\|w^{(s)}\|\|w^{(s+1)}\|_\infty\le C\|w^{(s)}\|^2+\frac{\sigma}{4}\|\delta w^{(s+1)}\|^2,\end{aligned}$$

where C is a positive constant which only depends on C_P and the known data.

Also, there is

$$\begin{aligned}\left|\sum_{j=0}^{J-1}\left(A^{(s)}_{j+\frac{1}{2}}-A^{n+1}_{j+\frac{1}{2}}\right)\delta U^{n+1}_{j+\frac{1}{2}}\delta w^{(s+1)}_{j+\frac{1}{2}}h\right| &\le C_{A1}\bar{M}_1\sum_{j=0}^{J-1}|w^{(s)}_{j+\frac{1}{2}}||\delta w^{(s+1)}_{j+\frac{1}{2}}|h\\ &\le \frac{\sigma}{2}\|\delta w^{(s+1)}\|^2+\frac{C_A^2\bar{M_1}^2}{2\sigma}\|w^{(s)}\|^2.\end{aligned}$$

In a word, we can derive

$$\frac{1}{\tau}\|w^{(s+1)}\|^2+\frac{\sigma}{4}\|\delta w^{(s+1)}\|^2\le C\|w^{(s)}\|^2, \tag{4.4}$$

where C only depends on $\sigma,\ C_{A0},\ C_{A1},\ C_P,\ M$ and $\bar{M}_1$.

So we have

$$\|w^{(s+1)}\|^2\le C\tau\|w^{(s)}\|^2\le\cdots\le(C\tau)^{s+1}\|w^{(0)}\|^2.$$

As long as $C\tau<1$, there is

$$\|w^{(s+1)}\|\le\|w^{(0)}\|\le 2\sqrt{L}M, \tag{4.5}$$

where the initial iterative value satisfies $0\le U^{(0)}_j\le M,\forall 1\le j\le J$.

Furthermore, according to Sobolev inequality and relations (4.4) and (4.5), there stands

$$\|w^{(s+1)}\|_\infty\le\|w^{(s+1)}\|^{\frac{1}{2}}\|\delta w^{(s+1)}\|^{\frac{1}{2}}\le\tilde{C}_0M,$$

where $\tilde{C}_0\ge 2(\frac{C}{\sigma})^{\frac{1}{4}}\sqrt{L}$. So by taking $C_0\ge\tilde{C}_0+1$ large enough and then taking

$$\begin{gathered}C_{A0}=\max_{|v|\le C_0M}|A(v)|,\quad C_{A1}=\max_{|v|\le C_0M}|A'(v)|,\\ C_P=\max_{|v_1|\le C_0M,|v_2|\le M,|v_3|\le M}|P_2(v_1,v_2,v_3)|,\end{gathered}$$

we show (4.3) is valid. The proof of Theorem 4.1 is completed. □

4.2. PNF

Theorem 4.2. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_2) *hold. Moreover,* $C\left(\frac{1}{\tau}\|w^{(0)}\|^2+\|\delta w^{(0)}\|^2\right)<1$ *holds for some positive constant C which depends only on the known data. Then for the solution of PNF iteration (2.11)–(2.13), there exists a positive constant τ_0 which depends only on the known data, for $\tau\le\tau_0$ there hold*

$$\lim_{s\to\infty}\left(\|w^{(s+1)}\|+\|\delta w^{(s+1)}\|\right)=0,\quad \overline{\lim_{s\to\infty}}\frac{\frac{1}{\tau}\|w^{(s+1)}\|^2+\|\delta w^{(s+1)}\|^2}{\left(\frac{1}{\tau}\|w^{(s)}\|^2+\|\delta w^{(s)}\|^2\right)^2}\le C. \tag{4.6}$$

Proof. From (2.12) and (2.5) we have

$$\begin{aligned}&\frac{w^{(s+1)}_j+g^{(s+1)}_j-g^n_j-\left(g^{n+1}_j-g^n_j\right)}{\tau}\\ &\quad=\delta\left(A^{(s)}\delta w^{(s+1)}\right)_j+\delta\left\{\left[A^{(s)}-A+{A'}^{(s)}\left(U^{(s+1)}-U^{(s)}\right)\right]\delta U+{A'}^{(s)}\left(U^{(s+1)}-U^{(s)}\right)\delta\left(U^{(s)}-U\right)\right\}_j,\end{aligned} \tag{4.7}$$

where $w_0^{(s+1)} = w_j^{(s+1)} = 0$, the first term in (4.7) is determined by (2.13). The term in {} on the right side of (4.7) is equal to the following formula:

$$\left\{A^{(s)\star}\left(U^{(s)}-U\right)+A'^{(s)}\left[U^{(s+1)}-U-\left(U^{(s)}-U\right)\right]\right\}\delta U+A'^{(s)}\left[U^{(s+1)}-U-\left(U^{(s)}-U\right)\right]\delta\left(U^{(s)}-U\right)$$
$$=\left[A''^{(s)\star}\left(w^{(s)}\right)^2+A'^{(s)}w^{(s+1)}\right]\delta U+A'^{(s)}\left(w^{(s+1)}-w^{(s)}\right)\delta w^{(s)},$$

where $A^{(s)\star}$ is defined as in (4.2), and

$$A''^{(s)\star}=\int_0^1\int_0^1 A''\left(\bar{r}(r-1)w_{j+\frac{1}{2}}^{(s)}+U_{j+\frac{1}{2}}^{(s)}\right)\mathrm{d}\bar{r}(r-1)\,\mathrm{d}r.$$

Let's suppose

$$\left|A_{j+\frac{1}{2}}^{(s)}\right|\le C_{A0},\quad \left|A'^{(s)}_{j+\frac{1}{2}}\right|\le C_{A1},\quad \left|A''^{(s)\star}\right|\le C_{A2}, \tag{4.8}$$

where C_{Ai} $(i=0,1,2)$ are positive constants to be determined later.

Multiplying (4.7) by $w_j^{(s+1)}h$, and summing up the products for $j=1,\ldots,J-1$, we obtain

$$\frac{1}{\tau}\|w^{(s+1)}\|^2+\frac{1}{\tau}\sum_{j=1}^{J-1}\left[g_j^{(s+1)}-g_j^n-\left(g_j^{n+1}+g_j^n\right)\right]w_j^{(s+1)}h+\sigma\|\delta w^{(s+1)}\|^2$$
$$\le C\sum_{j=0}^{J-1}\left[\left(|w^{(s)}|^2+|w^{(s+1)}|\right)|\delta U|+|w^{(s+1)}-w^{(s)}||\delta w^{(s)}|\right]|\delta w^{(s+1)}|h$$
$$\le\frac{\sigma}{2}\|\delta w^{(s+1)}\|^2+C\left(\|w^{(s)}\|_\infty^4+\|w^{(s+1)}\|^2+\|w^{(s+1)}-w^{(s)}\|_\infty^2\|\delta w^{(s)}\|^2\right),$$

where $C\ge 1$ depends on $\bar{M}_1$.

Note that from (2.13),

$$g_j^{(s+1)}-g_j^n-\left(g_j^{n+1}-g_j^n\right)=\left(P_1+\tau P_2+P_3w_j^{(s)}\right)w_j^{(s+1)}+P_4\left(w_j^{(s)}\right)^3+P_5\left(w_j^{(s)}\right)^2, \tag{4.9}$$

where

$$\begin{aligned}
P_1&=P_1\left(U_j^{(s)},U_j^n\right)=\left(U_j^{(s)}+U_j^n\right)\left[\left(U_j^{(s)}\right)^2+(U_j^n)^2\right],\\
P_2&=P_2\left(U_j^{(s)},U_j^{n+1},U_j^n\right)=d_tU_j^{n+1}\left[3\left(U_j^{(s)}\right)^2+2U_j^{(s)}U_j^n+(U_j^n)^2\right],\\
P_3&=P_3\left(U_j^{(s)},U_j^n\right)=3\left(U_j^{(s)}\right)^2+2U_j^{(s)}U_j^n+(U_j^n)^2,\\
P_4&=P_4\left(U_j^{(s)},U_j^{n+1}\right)=-\left(3U_j^{(s)}+5U_j^{n+1}\right),\\
P_5&=P_5\left(U_j^{n+1}\right)=-6\left(U_j^{n+1}\right)^2.
\end{aligned} \tag{4.10}$$

Make the assumption that

$$U_j^{(s)}\ge 0,\quad \|U^{(s)}\|_\infty\le\bar{K}. \tag{4.11}$$

Obviously, $P_1 \geq 0$. Hence

$$\frac{1}{\tau}\sum_{j=1}^{J-1}\left[g_j^{(s+1)} - g_j^n - \left(g_j^{n+1} - g_j^n\right)\right] w_j^{(s+1)} h$$
$$\geq -C_1\|w^{(s+1)}\|^2 - \frac{1}{\tau}\left(\frac{1}{4}\|w^{(s+1)}\|^2 + C_2\|w^{(s+1)}\|_\infty^2\|w^{(s)}\|^2 + C_3\|\delta w^{(s)}\|^2\|w^{(s)}\|^4 + C_4\|\delta w^{(s)}\|^2\|w^{(s)}\|^2\right),$$

where $C_i \geq 1$ $(i = 1, 2, 3, 4)$ are positive constants which depend on $\|U^{(s)}\|_\infty$, $\|U^{n+1}\|_\infty$, $\|U^n\|_\infty$ and $\|d_t U^{n+1}\|_\infty$, and the following inequalities have been used:

$$\|w^{(s)}\|_6^6 \leq C\|\delta w^{(s)}\|^2\|w^{(s)}\|^4, \quad \|w^{(s)}\|_4^4 \leq C\|\delta w^{(s)}\|^2\|w^{(s)}\|^2.$$

Then we have

$$\begin{aligned}
&\frac{3 - 4(C + C_1)\tau}{4\tau}\|w^{(s+1)}\|^2 + \frac{\sigma}{2}\|\delta w^{(s+1)}\|^2 \\
&\quad\leq C\left(\|\delta w^{(s)}\|^4 + \left(\|w^{(s+1)}\|_\infty^2 + \|w^{(s)}\|_\infty^2\right)\|\delta w^{(s)}\|^2\right) \\
&\qquad + \frac{1}{\tau}\left(C_2\|w^{(s+1)}\|_\infty^2\|w^{(s)}\|^2 + C_3\|\delta w^{(s)}\|^2\|w^{(s)}\|^4 + C_4\|\delta w^{(s)}\|^2\|w^{(s)}\|^2\right) \\
&\quad\leq C\left[(1 + L)\|\delta w^{(s)}\|^4 + \|w^{(s+1)}\|\|\delta w^{(s+1)}\|\|\delta w^{(s)}\|^2\right] \\
&\qquad + \frac{1}{\tau}\left(C_2\|\delta w^{(s+1)}\|^2\|w^{(s)}\|^2 + C_3\|\delta w^{(s)}\|^2\|w^{(s)}\|^4 + C_4\|\delta w^{(s)}\|^2\|w^{(s)}\|^2\right).
\end{aligned}$$

If $4(C + C_1)\tau < 1$, then

$$\begin{aligned}
\frac{1}{\tau}\|w^{(s+1)}\|^2 + \|\delta w^{(s+1)}\|^2 \leq C_5\Bigg[&\|\delta w^{(s)}\|^4\left(1 + \|w^{(s+1)}\|^2\right) + \frac{1}{\tau}\|w^{(s)}\|^2\|\delta w^{(s+1)}\|^2 + \frac{1}{\tau}\|w^{(s)}\|^4\|\delta w^{(s)}\|^2 \\
&+ \frac{1}{\tau}\|\delta w^{(s)}\|^2\|w^{(s)}\|^2\Bigg],
\end{aligned}$$

where $C_5 \geq 1$ is a positive constant depends on C_i $(i = 1, 2, 3, 4)$ and σ. Make induction assumption

$$C_5\left(\frac{1}{\tau}\|w^{(s')}\|^2 + \|\delta w^{(s')}\|^2\right) < \frac{1}{2}, \qquad \forall s' \leq s.$$

So we get

$$\begin{aligned}
\frac{1}{\tau}\|w^{(s+1)}\|^2 + \|\delta w^{(s+1)}\|^2 \leq C_5\|\delta w^{(s)}\|^4 + \frac{1}{2\tau}\|w^{(s+1)}\|^2 + \frac{1}{2}\|\delta w^{(s+1)}\|^2 + \|w^{(s)}\|^2\|\delta w^{(s)}\|^2 \\
+ \frac{C_5}{\tau}\|\delta w^{(s)}\|^2\|w^{(s)}\|^2.
\end{aligned}$$

Hence for $\tau \leq C_5$, there is

$$\begin{aligned}
2C_5\left(\frac{1}{\tau}\|w^{(s+1)}\|^2 + \|\delta w^{(s+1)}\|^2\right) &\leq \left[2C_5\left(\frac{1}{\tau}\|w^{(s)}\|^2 + \|\delta w^{(s)}\|^2\right)\right]^2 \\
&\leq \cdots \leq \left[2C_5\left(\frac{1}{\tau}\|w^{(0)}\|^2 + \|\delta w^{(0)}\|^2\right)\right]^{2^{s+1}}.
\end{aligned}$$

We observe that if only

$$2C_5\left(\frac{1}{\tau}\|w^{(0)}\|^2+\|\delta w^{(0)}\|^2\right)<1,$$

then

$$2C_5\left(\frac{1}{\tau}\|w^{(s)}\|^2+\|\delta w^{(s)}\|^2\right)<1,\quad \forall s\geq 0.$$

So $\|w^{(s)}\|_\infty\leq 1$ holds for all $s\geq 0$. Then $\|U^{(s)}\|_\infty\leq \bar{K}$ is satisfied. By taking

$$C_{A0}=\max_{|v|\leq M+1}|A(v)|,\ \ C_{A1}=\max_{|v|\leq M+1}|A'(v)|,\ \ C_{A2}=\max_{|v|\leq M+2}|A''(v)|,$$

we show (4.8) is satisfied.

Assume $2C_5\left(\frac{1}{\tau}\|w^{(0)}\|_2^2+\|\delta w^{(0)}\|_2^2\right)<\min\left(1,\frac{m}{2}\right)$. Then we have

$$\|w^{(s+1)}\|_\infty^2\leq 2C_5\left(\frac{1}{\tau}\|w^{(s+1)}\|_2^2+\|\delta w^{(s+1)}\|_2^2\right)\leq\left(\frac{m}{2}\right)^2,$$

so

$$\min_{0\leq j\leq J}U_j^{(s+1)}\geq\frac{m}{2}>0,\ \ \forall s\geq 0.$$

It follows that (4.11) is valid. The proof of Theorem 4.2 is completed. □

4.3. PN

Theorem 4.3. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_2) *hold. Moreover,* $C\left(\frac{1}{\tau}\|w^{(0)}\|^2+\|\delta w^{(0)}\|^2\right)<1$ *holds for some positive constant* C *which depends only on the known data. Then for the solution of PN iteration (2.11), (2.12), (2.14), there exists a positive constant* τ_0 *which depends only on the known data, for* $\tau\leq\tau_0$*, (4.6) holds.*

Proof. Note that from (2.14),

$$g_j^{(s+1)}-g_j^n-(g_j^{n+1}-g_j^n)=P_1w^{(s+1)}+P_2(w^{(s)})^2,$$

where

$$\begin{aligned}P_1&=P_1(U^{(s)})=4(U^{(s)})^3,\\ P_2&=P_2(U^{(s)},U^{n+1})=-[3(U^{(s)})^2+2U^{(s)}U^{n+1}+(U^{n+1})^2].\end{aligned}$$

Make assumption that

$$U_j^{(s)}\geq 0,\ \ \|U^{(s)}\|_\infty\leq\bar{K}, \tag{4.12}$$

then $P_1\geq 0$, hence

$$\begin{aligned}&\frac{1}{\tau}\sum_{j=1}^{J-1}\left[g_j^{(s+1)}-g_j^n-\left(g_j^{n+1}-g_j^n\right)\right]w_j^{(s+1)}h\\ &\quad\geq-\frac{1}{2}\frac{1}{\tau}\|w^{(s+1)}\|^2-C\frac{1}{\tau}\|w^{(s)}\|_4^4\geq-\frac{1}{2}\frac{1}{\tau}\|w^{(s+1)}\|^2-C_4\frac{1}{\tau}\|\delta w^{(s)}\|^2\|w^{(s)}\|^2,\end{aligned}$$

where $C_4\geq 1$ is a positive constant depends on the upper bound of $\|U^{(s)}\|_\infty$ and $\|U^{n+1}\|_\infty$.

Applying the derivation process similar to Section 4.2, we obtain

$$\begin{aligned}&\frac{1}{\tau}\|w^{(s+1)}\|^2+\sigma\|\delta w^{(s+1)}\|^2\\&\quad\le\frac{1}{2}\frac{1}{\tau}\|w^{(s+1)}\|^2+C_4\frac{1}{\tau}\|\delta w^{(s)}\|^2\|w^{(s)}\|^2+\frac{\sigma}{2}\|\delta w^{(s+1)}\|^2\\&\qquad+C\left(\|w^{(s)}\|_\infty^4+\|w^{(s+1)}\|^2+\|w^{(s+1)}-w^{(s)}\|_\infty^2\|\delta w^{(s)}\|^2\right),\end{aligned}$$

then

$$\frac{1-2C\tau}{2\tau}\|w^{(s+1)}\|^2+\frac{\sigma}{2}\|\delta w^{(s+1)}\|^2\le C\left(\|\delta w^{(s)}\|^4+\|w^{(s+1)}\|\|\delta w^{(s+1)}\|\|\delta w^{(s)}\|^2\right).$$

So we derive if $2C\tau<1$, then

$$\frac{1}{\tau}\|w^{(s+1)}\|^2+\|\delta w^{(s+1)}\|^2\le C_5\|\delta w^{(s)}\|^4\left(1+\|w^{(s+1)}\|^2\right),$$

where $C_5\ge 1$ is a positive constant depends on C and σ.

Then the proof is completed by following the same argument as that in Section 4.2. □

4.4. DFPNF

Similarly, we can prove the following theorem of the quadratic convergence speed of DFPNF method:

Theorem 4.4. *Let the Assumptions 2.1–2.4 and 2.10–2.11 and* (S_2) *hold. Moreover,* $C\left(\frac{1}{\tau}\|w^{(0)}\|^2+\|\delta w^{(0)}\|^2\right)<1$ *holds for some positive constant* C *which depends only on the known data. Then for the solution of DFPNF iteration (2.11), (2.13), (2.15),*

(i) *when* $\lim_{s\to\infty}\max_{0\le j\le J-1}|\varepsilon_{j+\frac{1}{2}}^{(s)}|=0$ *is satisfied, the first formula in (4.6) holds;*

(ii) *when* $\max_{0\le j\le J-1}|\varepsilon_{j+\frac{1}{2}}^{(s)}|=O(\|w^{(s)}\|)$ *is satisfied, the second formula in (4.6) holds.*

Remark 4.5. These theoretical results indicate that it is unnecessary to add new restriction on the time step-length to assure the convergence speed of these iteration methods.

Remark 4.6. In Theorem 4.4 (i) and (ii), the selection criteria of ε are given. In practical calculation, $\|w^{(s)}\|$ can be replaced by $\|U^{(s+1)}-U^{(s)}\|$, while in this paper, a fixed ε is taken.

Remark 4.7. In Sections 3 and 4, the convergence accuracy and convergence speed of the iterations are proved individually. Actually, we can make use of the conclusion of their counterpoints to simply their proofs.

Remark 4.8. So far, we illustrate the analysis on the nonlinear iterations corresponding to FIS with spatial finite different discrete method. Actually this performance is also available for finite volume (FV) schemes. For example, a vertex-centered FV scheme can be formulated through a derivation satisfying the discrete flux continuity requirement. On uniform meshes, with the new notation $A_{j+\frac{1}{2}}^{n+1}(U)=\frac{2A_j^{n+1}(U)A_{j+1}^{n+1}(U)}{A_j^{n+1}(U)+A_{j+1}^{n+1}(U)}$, where $A_j^{n+1}(U)=A(x_j,t^{n+1},U_j^{n+1})$, etc., in (2.5)–(2.7), it looks like the FD scheme, yet preserves local conservation. The same properties are gained after being analyzed alike with slight complexity.

Remark 4.9. The ideas in this paper can be extended to 2-dimensional problems, and similar conclusions are valid.

Remark 4.10. In this paper our theoretical results are proved under some assumptions on known data such as the diffusion coefficient $A(x,t,u)$. We will continue to research how to weaken these restrictions in future papers.

Table 1. Spatial accuracy of four iterative methods.

Mesh size		30×30	60×60	90×90	120×120
$L^\infty(L^2)$	Error	6.0400e-4	1.5104e-4	6.7142e-5	3.7775e-5
	Order	–	2.00	2.00	2.00
$L^\infty(H^1)$	Error	2.6962e-3	6.7440e-4	2.9974e-4	1.6859e-4
	Order	–	2.00	2.00	2.00

Table 2. Temporal accuracy of four iterative methods.

Mesh size		5	10	15	20
$L^\infty(L^2)$	Error	3.2492e-2	1.8674e-2	1.3002e-2	1.0012e-2
	Order	–	0.80	0.89	0.91
$L^\infty(H^1)$	Error	1.4454e-1	8.3090e-2	5.7859e-2	4.4548e-2
	Order	–	0.80	0.89	0.91

5. Numerical experiments

In this section, we present some numerical experiments on model problems to verify the theoretical results and compare the accuracy and efficiency of the four iterations, *i.e.*, PF, PNF, PN and DFPNF iterations. Let "DFPNF-ε" express the DFPNF iteration with small parameter ε. Take the nonlinear iteration convergent tolerance as $1.0e-8$ for accuracy tests and $1.0e-12$ for efficiency tests.

5.1. A manufactured solution problem

Consider a 2-dimensional version of equilibrium radiation diffusion problem (2.1)–(2.3) on $(0,1)\times(0,1)\times(0,2]$ with the following diffusion coefficient, additional source, initial and boundary conditions

$$\begin{aligned}
A(u) &= u+1,\\
f(x,y,t,u) &= (1+4u^3)[-e^{-t}\sin(\pi x)\sin(\pi y)] - u^2\\
&\quad -\pi^2(0.5+e^{-t})^2\left[\cos^2(\pi x)\sin^2(\pi y)+\sin^2(\pi x)\cos^2(\pi y)\right]\\
&\quad +(2\pi^2+1)\left[(0.5+e^{-t})\sin(\pi x)\sin(\pi y)+1\right]^2 - 2\pi^2,\\
\varphi(x,y) &= 1.5\sin(\pi x)\sin(\pi y)+1,\\
\psi(\partial\Omega,t) &= 1.
\end{aligned}$$

the solution of which is $u(x,y,t) = (0.5+e^{-t})\sin(\pi x)\sin(\pi y)+1$.

To demonstrate numerical accuracy, calculations are performed on four groups of meshes in each test. In the tests for spatial convergence, the spatial meshes are 30×30, 60×60, 90×90 and 120×120, while corresponding temporal meshes are 1800, 7200, 16 200 and 28 800. In the tests for temporal convergence, the spatial mesh is fixed to 145×145, while the temporal meshes are 5, 10, 15 and 20.

Spatial convergence errors between the iteration solutions and the exact solution are listed in Table 1. They include the results of PF, PNF, PN and DFPNF-$1.0e-4$, DFPNF-$1.0e-18$, with a precision of four decimal places. The results of these iterations are so similar that difference only occurs to/after the fifth digit, which cannot be seen in Table 1. It appears that all the convergence orders are about 2. Temporal convergence errors are listed in Table 2 similarly. Obviously the orders are all about 1 for the four schemes. Tables 1 and 2 verify the theorem of convergence accuracy in Section 3. The results are in accordance with theoretical expectation.

Table 3 compares the efficiency of the four iterative methods. Here, simulations proceed on a $60\times60\times7200$ mesh. "*Tol.nonl.*", "*Tol.lin.*", "*Avg.nonl.*" and "*Avg.lin.*" respectively represent the total number of nonlinear

TABLE 3. Efficiency comparison of different iterative methods.

	Tol.nonl.	*Tol.lin.*	*Avg.nonl.*	*Avg.lin.*
PF	29 440	243 651	4.09	33.84
PNF	15 276	144 382	2.12	20.05
PN	15 199	144 110	2.11	20.02
DFPNF-1.0$e-4$	15 278	144 377	2.12	20.05
DFPNF-1.0$e-18$	29 443	242 825	4.09	33.73

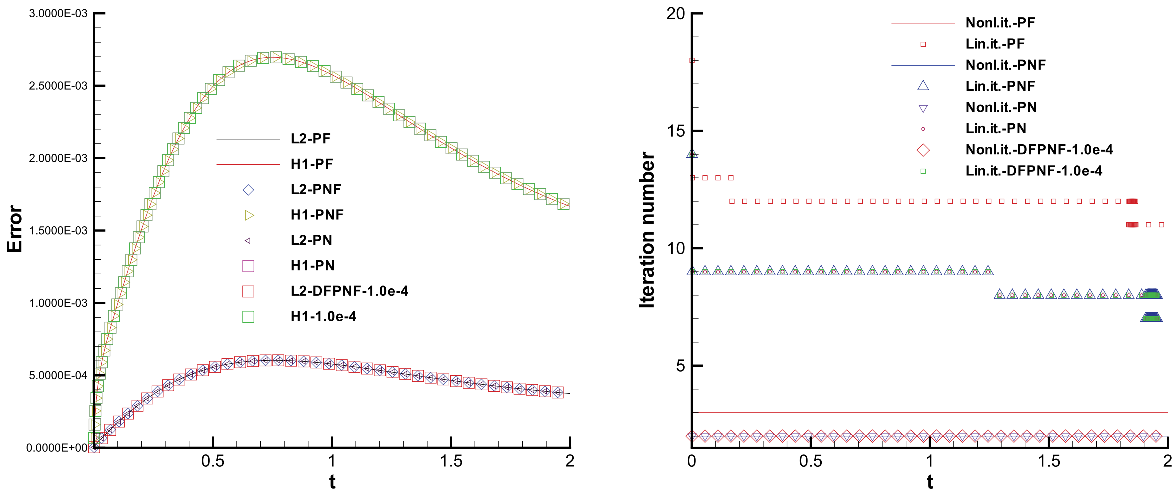

FIGURE 1. Comparison on convergence errors and iteration numbers of PF, PNF, PN and DFPNF iterations for Example 1.

and linear iterations needed for the entire computation and the number of nonlinear and linear iterations needed in each time step. It shows that PNF and PN are much more efficient than PF. From Tables 1 to 3, we observe that DFPNF can acquire similar accuracy as other iterations, and such result is not sensitive to the choice of small parameter, which makes it available to use fairly "larger" parameters in DFPNF to accelerate the computation. Table 3 verifies the theorem of convergence speed in Section 4.

Figure 1 compares the convergence errors and iteration numbers developing with time march of PF, PNF, PN and DFPNF iterations on a $30 \times 30 \times 1800$ mesh. Here "*Nonl.it.*", "*Lin.it.*" respectively recognize the number of nonlinear and linear iterations needed in each time step. It shows the four iterations have similar accuracy, while less iterations are needed for PNF, PN and DFPNF (with proper parameter) iterations than for PF iteration.

5.2. A more strongly nonlinear problem

Consider a 2-dimensional version of problem (2.1)–(2.3) on $(0,1) \times (0,1) \times (0,2]$ with the following diffusion coefficient, initial and boundary conditions

$$\begin{aligned} A(u) &= u^3 + 1, \\ \varphi(x,y) &= 1.5\sin(\pi x)\sin(\pi y) + 1, \\ \psi(\partial\Omega, t) &= 1. \end{aligned}$$

TABLE 4. Comparison of spatial accuracy of four iterative methods.

Mesh size			15 × 15	30 × 30	45 × 45	60 × 60
PF DFPNF-1.0$e-18$	L^2	Error	2.7705e-7	6.0502e-8	2.4944e-8	1.2811e-8
		Order	–	2.20	2.19	2.32
	H^1	Error	1.2286e-6	2.6868e-7	1.1081e-7	5.6954e-8
		Order	–	2.19	2.18	2.31
PNF PN DFPNF-1.0$e-4$	L^2	Error	2.7705e-7	6.0505e-8	2.4945e-8	1.2812e-8
		Order	–	2.20	2.19	2.32
	H^1	Error	1.2287e-6	2.6869e-7	1.1081e-7	5.6959e-8
		Order	–	2.19	2.18	2.31

TABLE 5. Comparison of temporal accuracy of four iterative methods.

Mesh size			6000	6500	7000	7500
PF DFPNF-1.0$e-18$	L^2	Error	1.3556e-8	1.2412e-8	1.1433e-8	1.0586e-8
		Order	–	1.10	1.11	1.12
	H^1	Error	6.0305e-8	5.5229e-8	5.0891e-8	4.7136e-8
		Order	–	1.10	1.10	1.11
PNF	L^2	Error	1.3556e-8	1.2412e-8	1.1434e-8	1.0587e-8
		Order	–	1.10	1.11	1.12
	H^1	Error	6.0308e-8	5.5232e-8	5.0893e-8	4.7138e-8
		Order	–	1.10	1.10	1.11
PN DFPNF-1.0$e-4$	L^2	Error	1.3556e-8	1.2412e-8	1.1434e-8	1.0587e-8
		Order	–	1.10	1.11	1.12
	H^1	Error	6.0307e-8	5.5232e-8	5.0893e-8	4.7138e-8
		Order	–	1.10	1.10	1.11

TABLE 6. Efficiency comparison of different iterative methods.

	$Tol.nonl.$	$Tol.lin.$	$Avg.nonl.$	$Avg.lin.$
PF	21 469	167 261	2.98	23.23
PNF	15 469	111 961	2.15	15.55
PN	15 158	110 987	2.11	15.41
DFPNF-1.0$e-4$	15 792	113 036	2.13	15.70
DFPNF-1.0$e-18$	21 558	167 686	2.99	23.29

Since the exact solution of this problem is unknown, the numerical solution on a fine $145 \times 145 \times 42\,050$ grid is taken as the reference solution. Then the errors between the numerical solutions on coarse meshes and the interpolation values of the reference solution (instead of the exact solution) on these meshes are measured to define the convergence errors.

Spatial and temporal convergence errors between the iteration solutions and the reference solution at the time $T = 2$ are listed in Tables 4 and 5 respectively. In the tests for spatial convergence, the spatial meshes are 15×15, 30×30, 45×45 and 60×60, while corresponding temporal meshes are 450, 1800, 4050 and 7200. In those for temporal convergence, the spatial mesh is fixed to 90×90, while the temporal meshes are 6000, 6500, 7000 and 7500. Table 6 compares the efficiency of the four iterative methods. Again the results are in accordance with theoretical expectation.

6. Conclusion

This article focuses on designing new efficient iteration methods for solving equilibrium radiation diffusion problem and performing theoretical analysis on their properties. The PF iteration and three Newton-type iterations, PNF, PN and DFPNF are studied. By thorough operating on the temporal difference quotient terms and developing the induction reasoning technique, we overcome the difficulties caused by the strong nonlinearity of the problem, and analyze the convergence accuracy, speed and positivity of these iterations rigorously. Our analysis shows that the four iteration methods all have first-order time and second-order space convergence in L^2 and H^1 norms. They are positivity-preserving as the time and space step lengths being sufficiently small. The iterative sequence of the PF iteration converges to the solution of FIS linearly and those of the PNF, PN and DFPNF iterations converge quadratically. Numerical experiments verify the theoretical results, and show that PN, PNF and DFPNF with suitable parameters are more efficient than PF iteration.

Acknowledgements. The authors are very grateful to the anonymous referees for their helpful suggestions to enhance the paper. This work is supported by the National Natural Science Foundation of China (11871112, 11971069, 11971071, U1630249), Yu Min Foundation and the Foundation of LCP.

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