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# TIME DISCRETIZATION OF AN INITIAL VALUE PROBLEM FOR A SIMULTANEOUS ABSTRACT EVOLUTION EQUATION APPLYING TO PARABOLIC-HYPERBOLIC PHASE-FIELD SYSTEMS

Shunsuke Kurima*

**Abstract.** This article deals with a simultaneous abstract evolution equation. This includes a parabolic-hyperbolic phase-field system as an example which consists of a parabolic equation for the relative temperature coupled with a semilinear damped wave equation for the order parameter (see *e.g.*, Grasselli and Pata [*Adv. Math. Sci. Appl.* **13** (2003) 443–459, *Comm. Pure Appl. Anal.* **3** (2004) 849–881], Grasselli *et al.* [*Comm. Pure Appl. Anal.* **5** (2006) 827–838], Wu *et al.* [*Math. Models Methods Appl. Sci.* **17** (2007) 125–153, *J. Math. Anal. Appl.* **329** (2007) 948–976]). On the other hand, a time discretization of an initial value problem for an abstract evolution equation has been studied (see *e.g.*, Colli and Favini [*Int. J. Math. Math. Sci.* **19** (1996) 481–494]) and Schimperna [*J. Differ. Equ.* **164** (2000) 395–430] has established existence of solutions to an abstract problem applying to a nonlinear phase-field system of Caginalp type on a bounded domain by employing a time discretization scheme. In this paper we focus on a time discretization of a simultaneous abstract evolution equation applying to parabolic-hyperbolic phase-field systems. Moreover, we can establish an error estimate for the difference between continuous and discrete solutions.

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## 1. Introduction

A time discretization of an initial value problem for an abstract evolution equation has been studied. For example, Colli and Favini [6] have proved existence of solutions to the nonlinear Cauchy problem

$$\begin{cases} L\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2} + B\dfrac{\mathrm{d}u}{\mathrm{d}t} + Au = g \qquad \text{in} \quad (0,T), \\ u(0) = u_0, \ \dfrac{\mathrm{d}u}{\mathrm{d}t}(0) = w_0 \end{cases}$$

by employing a time discretization scheme, where $T > 0$, $L : H \to H$ and $A : V \to V^*$ are linear positive selfadjoint operators, $H$ and $V$ are real Hilbert spaces, $V \subset H$, $V^*$ is the dual space of $V$, $B : V \to V^*$ is a

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Department of Mathematics, Tokyo University of Science, 1-3, Kagurazaka, 162-8601 Shinjuku-ku, Tokyo, Japan.
*Corresponding author: shunsuke.kurima@gmail.com

 

maximal monotone operator, $g : (0,T) \to V^*$ and $u_0, w_0 \in V$ are given. Moreover, they have derived an error estimate for the difference between continuous and discrete solutions.

The system

$$\begin{cases} (\theta+\lambda(\varphi))_t - \Delta\theta = f & \text{in} \quad \Omega\times(0,\infty), \\ \varepsilon\varphi_{tt} + \varphi_t - \Delta\varphi + \eta(\varphi) = \lambda'(\varphi)\theta & \text{in} \quad \Omega\times(0,\infty), \\ \theta(0)=\theta_0,\ \varphi(0)=\varphi_0,\ \varphi_t(0)=v_0 & \text{in} \quad \Omega \end{cases} \tag{E}$$

is a parabolic-hyperbolic phase-field system (see *e.g.*, [11–13, 20, 21]), where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $\lambda$ and $\eta$ are smooth functions, $\varepsilon > 0$, $f$ is a time dependent heat source, and $\theta_0$, $\varphi_0$, $v_0$ are given initial data defined in $\Omega$. The unknown function $\theta$ is the relative temperature. The unknown function $\varphi$ is the order parameter. The function $\lambda$ has a quadratic growth, *e.g.*, $\lambda(r) = ar^2 + br + c$ $(a,b,c \in \mathbb{R})$; while the function $\eta$ has a cubic growth, *e.g.*, $\eta(r) = d_1r^3 - d_2r$ $(d_1, d_2 > 0)$. The second time derivative $\varepsilon\varphi_{tt}$ is the inertial term which characterizes the hyperbolic dynamics. In the case that $\varepsilon = 0$ the system (E) is the classical phase-field model proposed by Caginalp (*cf.* [5, 9]; one may also see the monographs [4, 10, 19]). The system (E) endowed with homogeneous Dirichlet–Neumann boundary conditions has been analyzed by *e.g.*, Grasselli and Pata [11, 12] and Grasselli *et al.* [13]. The paper Wu *et al.* [20] has studied the system (E) with homogeneous Neumann boundary conditions for both $\theta$ and $\varphi$. In the case that $\lambda(r) = r$ for all $r \in \mathbb{R}$, the system (E) with dynamical boundary condition has been analyzed by *e.g.*, Wu *et al.* [21]. In the case that $\varepsilon = 0$ and $\lambda(r) = r$ for all $r \in \mathbb{R}$, Schimperna [16] has derived existence of solutions to an initial value problem for a simultaneous abstract evolution equation applying to the system (E) under homogeneous Neumann–Neumann boundary conditions by employing a time discretization scheme (but Schimperna [16] did not obtain error estimates for the difference between continuous and discrete solutions). Also, in the case that $\varepsilon = 0$, $\lambda(r) = r$ for all $r \in \mathbb{R}$ and $\Omega$ is a bounded or an unbounded domain, Colli and Kurima [7] have employed a time discretization scheme to prove existence of solutions to the system (E) under homogeneous Neumann–Neumann boundary conditions and established an error estimate for the difference between continuous and discrete solutions. However, time discretizations of parabolic-hyperbolic phase-field systems seem to be not studied yet.

In this paper we consider the initial value problem for the simultaneous abstract evolution equation

$$\begin{cases} \dfrac{d\theta}{dt} + \dfrac{d\varphi}{dt} + A_1\theta = f & \text{in} \quad (0,T), \\ L\dfrac{d^2\varphi}{dt^2} + B\dfrac{d\varphi}{dt} + A_2\varphi + \Phi\varphi + \mathcal{L}\varphi = \theta & \text{in} \quad (0,T), \\ \theta(0)=\theta_0,\ \varphi(0)=\varphi_0,\ \dfrac{d\varphi}{dt}(0) = v_0, \end{cases} \tag{P}$$

where $T > 0$, $L : H \to H$ is a linear positive selfadjoint operator, $B : D(B) \subset H \to H$, $A_j : D(A_j) \subset H \to H$ $(j = 1, 2)$ are linear maximal monotone selfadjoint operators, $V_j$ $(j = 1, 2)$ are linear subspaces of $V$ satisfying $D(A_j) \subset V_j$ $(j = 1, 2)$, $\Phi : D(\Phi) \subset H \to H$ is a maximal monotone operator, $\mathcal{L} : H \to H$ is a Lipschitz continuous operator, $f : (0,T) \to H$ and $\theta_0 \in V_1$, $\varphi_0, v_0 \in V_2$ are given. Moreover, in reference to [6, 7], we deal with the problem

$$\begin{cases} \delta_h\theta_n + \delta_h\varphi_n + A_1\theta_{n+1} = f_{n+1}, \\ Lz_{n+1} + Bv_{n+1} + A_2\varphi_{n+1} + \Phi\varphi_{n+1} + \mathcal{L}\varphi_{n+1} = \theta_{n+1}, \\ z_0 = z_1,\ z_{n+1} = \delta_h v_n, \\ v_{n+1} = \delta_h\varphi_n \end{cases} \tag{P$_n$}$$

for $n = 0, \ldots, N-1$, where $h = \frac{T}{N}$, $N \in \mathbb{N}$,

$$\delta_h\theta_n := \frac{\theta_{n+1}-\theta_n}{h},\ \delta_h\varphi_n := \frac{\varphi_{n+1}-\varphi_n}{h},\ \delta_h v_n := \frac{v_{n+1}-v_n}{h}, \tag{1.1}$$

and $f_k := \frac{1}{h}\int_{(k-1)h}^{kh} f(s)\,\mathrm{d}s$ for $k = 1, \ldots, N$. Here, putting

$$\widehat{\theta}_h(0) := \theta_0, \ \frac{\mathrm{d}\widehat{\theta}_h}{\mathrm{d}t}(t) := \delta_h\theta_n, \quad \widehat{\varphi}_h(0) := \varphi_0, \ \frac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t}(t) := \delta_h\varphi_n, \tag{1.2}$$

$$\widehat{v}_h(0) := v_0, \ \frac{\mathrm{d}\widehat{v}_h}{\mathrm{d}t}(t) := \delta_h v_n, \tag{1.3}$$

$$\overline{\theta}_h(t) := \theta_{n+1}, \ \overline{z}_h(t) := z_{n+1}, \ \overline{\varphi}_h(t) := \varphi_{n+1}, \ \overline{v}_h(t) := v_{n+1}, \ \overline{f}_h(t) := f_{n+1} \tag{1.4}$$

for a.a. $t \in (nh, (n+1)h)$, $n = 0, \ldots, N-1$, we can rewrite (P)$_n$ as

$$\begin{cases} \dfrac{\mathrm{d}\widehat{\theta}_h}{\mathrm{d}t} + \dfrac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t} + A_1\overline{\theta}_h = \overline{f}_h & \text{in } (0,T), \\ L\overline{z}_h + B\overline{v}_h + A_2\overline{\varphi}_h + \Phi\overline{\varphi}_h + \mathcal{L}\overline{\varphi}_h = \overline{\theta}_h & \text{in } (0,T), \\ \overline{z}_h = \dfrac{\mathrm{d}\widehat{v}_h}{\mathrm{d}t}, \ \overline{v}_h = \dfrac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t} & \text{in } (0,T), \\ \widehat{\theta}_h(0) = \theta_0, \ \widehat{\varphi}_h(0) = \varphi_0, \ \widehat{v}_h(0) = v_0. \end{cases} \tag{P$_h$}$$

**Remark 1.1.** Owing to (1.2)–(1.4), the reader can check directly the following identities:

$$\|\widehat{\varphi}_h\|_{L^\infty(0,T;V_2)} = \max\{\|\varphi_0\|_{V_2}, \|\overline{\varphi}_h\|_{L^\infty(0,T;V_2)}\}, \tag{1.5}$$

$$\|\widehat{v}_h\|_{L^\infty(0,T;V_2)} = \max\{\|v_0\|_{V_2}, \|\overline{v}_h\|_{L^\infty(0,T;V_2)}\}, \tag{1.6}$$

$$\|\widehat{\theta}_h\|_{L^\infty(0,T;V_1)} = \max\{\|\theta_0\|_{V_1}, \|\overline{\theta}_h\|_{L^\infty(0,T;V_1)}\}, \tag{1.7}$$

$$\|\overline{\varphi}_h - \widehat{\varphi}_h\|_{L^\infty(0,T;V_2)} = h\left\|\frac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t}\right\|_{L^\infty(0,T;V_2)} = h\|\overline{v}_h\|_{L^\infty(0,T;V_2)}, \tag{1.8}$$

$$\|\overline{v}_h - \widehat{v}_h\|_{L^\infty(0,T;H)} = h\left\|\frac{\mathrm{d}\widehat{v}_h}{\mathrm{d}t}\right\|_{L^\infty(0,T;H)} = h\|\overline{z}_h\|_{L^\infty(0,T;H)}, \tag{1.9}$$

$$\|\overline{\theta}_h - \widehat{\theta}_h\|_{L^2(0,T;H)}^2 = \frac{h^2}{3}\left\|\frac{\mathrm{d}\widehat{\theta}_h}{\mathrm{d}t}\right\|_{L^2(0,T;H)}^2. \tag{1.10}$$

Moreover, we deal with the following conditions (C1)–(C14):

(C1) $V$ and $H$ are real Hilbert spaces satisfying $V \subset H$ with dense, continuous and compact embedding. Moreover, the inclusions $V \subset H \subset V^*$ hold by identifying $H$ with its dual space $H^*$, where $V^*$ is the dual space of $V$.

(C2) $V_j$ $(j = 1, 2)$ are closed linear subspaces of $V$, dense in $H$ and reflexive.

(C3) $L : H \to H$ is a bounded linear operator fulfilling

$$(Lw, z)_H = (w, Lz)_H \text{ for all } w, z \in H, \quad (Lw, w)_H \geq c_L\|w\|_H^2 \text{ for all } w \in H,$$

where $c_L > 0$ is a constant.

(C4) $A_1 : D(A_1) \subset H \to H$ is a linear maximal monotone selfadjoint operator, where $D(A_1)$ is a linear subspace of $H$ and $D(A_1) \subset V_1$. Moreover, there exists a bounded linear monotone operator $A_1^* : V_1 \to V_1^*$ such that

$$\begin{aligned} &\langle A_1^* w, z\rangle_{V_1^*,V_1} = \langle A_1^* z, w\rangle_{V_1^*,V_1} && \text{for all} && w, z \in V_1, \\ &A_1^* w = A_1 w && \text{for all} && w \in D(A_1). \end{aligned}$$

Moreover, for all $\alpha > 0$ there exists $\sigma_\alpha > 0$ such that

$$\langle A_1^* w, w\rangle_{V_1^*,V_1} + \alpha\|w\|_H^2 \geq \sigma_\alpha\|w\|_{V_1}^2 \quad \text{for all} \quad w \in V_1.$$

(C5) For all $g \in H$ and all $a > 0$, if there exists $\theta \in V_1$ such that $\theta + aA_1^*\theta = g$ in $V_1^*$, then it follows that $\theta \in D(A_1)$ and $\theta + aA_1\theta = g$ in $H$.

(C6) $B : D(B) \subset H \to H$, $A_2 : D(A_2) \subset H \to H$ are linear maximal monotone selfadjoint operators, where $D(B)$ and $D(A_2)$ are linear subspaces of $H$, satisfying

$$\begin{aligned} &D(B) \cap D(A_2) \neq \emptyset, && \\ &(Bw, A_2w)_H \geq 0 && \text{for all} \quad w \in D(B) \cap D(A_2), \\ &(Bw, A_2z)_H = (Bz, A_2w)_H && \text{for all} \quad w, z \in D(B) \cap D(A_2). \end{aligned}$$

Moreover, the inclusion $D(A_2) \subset V_2$ holds.

(C7) $\Phi : D(\Phi) \subset H \to H$ is a maximal monotone operator satisfying $\Phi(0) = 0$ and $V \subset D(\Phi)$. Moreover, there exist constants $p, q, C_\Phi > 0$ such that

$$\|\Phi w - \Phi z\|_H \leq C_\Phi (1 + \|w\|_V^p + \|z\|_V^q) \|w - z\|_V \quad \text{for all} \quad w, z \in V.$$

(C8) There exists a lower semicontinuous convex function $i : V \to \{x \in \mathbb{R} \mid x \geq 0\}$ such that $(\Phi w, w - z)_H \geq i(w) - i(z)$ for all $w, z \in V$.

(C9) $\Phi_\lambda(0) = 0$, $(\Phi_\lambda w, Bw)_H \geq 0$ for all $w \in D(B)$, $(\Phi_\lambda w, A_2 w)_H \geq 0$ for all $w \in D(A_2)$, where $\lambda > 0$ and $\Phi_\lambda : H \to H$ is the Yosida approximation of $\Phi$.

(C10) $B^* : V_2 \to V_2^*$ is a bounded linear monotone operator fulfilling

$$\begin{aligned} &\langle B^*w, z \rangle_{V_2^*, V_2} = \langle B^*z, w \rangle_{V_2^*, V_2} && \text{for all} \quad w, z \in V_2, \\ &B^*w = Bw && \text{for all} \quad w \in D(B) \cap V_2. \end{aligned}$$

(C11) $A_2^* : V_2 \to V_2^*$ is a bounded linear monotone operator fulfilling

$$\begin{aligned} &\langle A_2^*w, z \rangle_{V_2^*, V_2} = \langle A_2^*z, w \rangle_{V_2^*, V_2} && \text{for all} \quad w, z \in V_2, \\ &A_2^*w = A_2w && \text{for all} \quad w \in D(A_2). \end{aligned}$$

Moreover, for all $\alpha > 0$ there exists $\omega_\alpha > 0$ such that

$$\langle A_2^*w, w \rangle_{V_2^*, V_2} + \alpha \|w\|_H^2 \geq \omega_\alpha \|w\|_{V_2}^2 \quad \text{for all} \quad w \in V_2.$$

(C12) For all $g \in H$, $a, b, c, d > 0$, $\lambda > 0$, if there exists $\varphi_\lambda \in V_2$ such that

$$L\varphi_\lambda + aB^*\varphi_\lambda + bA_2^*\varphi_\lambda + c\Phi_\lambda\varphi_\lambda + d\mathcal{L}\varphi_\lambda = g \qquad \text{in} \quad V_2^*,$$

then it follows that $\varphi_\lambda \in D(B) \cap D(A_2)$ and

$$L\varphi_\lambda + aB\varphi_\lambda + bA_2\varphi_\lambda + c\Phi_\lambda\varphi_\lambda + d\mathcal{L}\varphi_\lambda = g \quad \text{in} \quad H.$$

(C13) $\mathcal{L} : H \to H$ is a Lipschitz continuous operator with Lipschitz constant $C_\mathcal{L} > 0$.

(C14) $\theta_0 \in V_1$, $\varphi_0 \in D(B) \cap D(A_2)$, $v_0 \in D(B) \cap V_2$, $f \in L^2(0, T; H)$.

**Remark 1.2.** We set the conditions (C3), (C4) and (C11) in reference to Section 2 of [6]. The conditions (C5) and (C12) are equivalent to the elliptic regularity theory under some cases (see Sect. 2). Moreover, we set the conditions (C7)–(C9) and (C13) by trying to keep a typical example (see Sect. 2) in reference to assumptions in [7, 11–13, 20, 21].

We define solutions of (P) as follows.

**Definition 1.3.** A pair $(\theta, \varphi)$ with

$$\begin{aligned}
&\theta \in H^1(0,T;H) \cap L^\infty(0,T;V_1) \cap L^2(0,T;D(A_1)),\\
&\varphi \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V_2) \cap L^2(0,T;D(A_2)),\\
&\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in L^2(0,T;D(B)),\ \Phi\varphi \in L^\infty(0,T;H)
\end{aligned}$$

is called a solution of (P) if $(\theta, \varphi)$ satisfies

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}\varphi}{\mathrm{d}t} + A_1\theta = f \quad \text{in} \quad H \quad \text{a.e. on} \quad (0,T), \tag{1.11}$$

$$L\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + B\frac{\mathrm{d}\varphi}{\mathrm{d}t} + A_2\varphi + \Phi\varphi + \mathcal{L}\varphi = \theta \quad \text{in} \quad H \quad \text{a.e. on} \quad (0,T), \tag{1.12}$$

$$\theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \frac{\mathrm{d}\varphi}{\mathrm{d}t}(0) = v_0 \quad \text{in} \quad H. \tag{1.13}$$

Now the main results read as follows.

**Theorem 1.4.** *Assume that (C1)–(C14) hold. Then there exists $h_0 \in (0,1)$ such that for all $h \in (0,h_0)$ there exists a unique solution $(\theta_{n+1}, \varphi_{n+1})$ of $(\mathrm{P})_n$ satisfying*

$$\theta_{n+1} \in D(A_1),\ \varphi_{n+1} \in D(B) \cap D(A_2) \quad \text{for} \quad n = 0, \dots, N-1.$$

**Theorem 1.5.** *Assume that (C1)–(C14) hold. Then there exists a unique solution $(\theta, \varphi)$ of (P).*

**Theorem 1.6.** *Let $h_0$ be as in Theorem 1.4. Assume that (C1)–(C14) hold. Assume further that $f \in W^{1,1}(0,T;H)$. Then there exist constants $h_{00} \in (0,h_0)$ and $M = M(T) > 0$ such that*

$$\begin{aligned}
&\|L^{1/2}(\widehat{v}_h - v)\|_{L^\infty(0,T;H)} + \|B^{1/2}(\overline{v}_h - v)\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;V_2)}\\
&\quad + \|\widehat{\theta}_h - \theta\|_{L^\infty(0,T;H)} + \|\overline{\theta}_h - \theta\|_{L^2(0,T;V_1)} \le Mh^{1/2}
\end{aligned}$$

*for all $h \in (0,h_{00})$, where $v = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$.*

This paper is organized as follows. Section 2 gives some examples. In Section 3 we establish existence of solutions to $(\mathrm{P})_n$ in reference to Section 4 of [8]. Section 4 contains the proof of existence for (P). In Section 5 we prove uniqueness for (P). In Section 6 we derive error estimates between solutions of (P) and solutions of $(\mathrm{P})_h$.

## 2. Examples

In this section we give the following examples.

**Example 2.1.** We consider the following homogeneous Dirichlet–Neumann problem

$$\begin{cases}
\theta_t + \varphi_t - \Delta\theta = f & \text{in} \quad \Omega \times (0,T),\\
\varphi_{tt} + \varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in} \quad \Omega \times (0,T),\\
\theta = \partial_\nu\varphi = 0 & \text{on} \quad \partial\Omega \times (0,T),\\
\theta(0) = \theta_0,\ \varphi(0) = \varphi_0,\ \varphi_t(0) = v_0 & \text{in} \quad \Omega,
\end{cases} \tag{P1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, under the following conditions:

(J1) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function and there exists a proper differentiable (lower semicontinuous) convex function $\widehat{\beta} : \mathbb{R} \to [0,+\infty)$ such that $\widehat{\beta}(0) = 0$ and $\beta(r) = \widehat{\beta}\,'(r) = \partial\widehat{\beta}(r)$ for all $r \in \mathbb{R}$, where $\widehat{\beta}\,'$ and $\partial\widehat{\beta}$, respectively, are the differential and subdifferential of $\widehat{\beta}$.

(J2) $\beta \in C^2(\mathbb{R})$. Moreover, there exists a constant $C_\beta > 0$ such that $|\beta''(r)| \le C_\beta(1+|r|)$ for all $r \in \mathbb{R}$.
(J3) $\pi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.
(J4) $\theta_0 \in H_0^1(\Omega)$, $\varphi_0 \in H^2(\Omega)$, $\partial_\nu \varphi_0 = 0$ a.e. on $\partial\Omega$, $v_0 \in H^1(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$.

Moreover, we put

$$\begin{aligned}
&V := H^1(\Omega),\ H := L^2(\Omega),\ V_1 := H_0^1(\Omega),\ V_2 := H^1(\Omega),\\
&L := I : H \to H,\\
&A_1 := -\Delta : D(A_1) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H,\\
&B := I : D(B) := H \to H,\\
&A_2 := -\Delta : D(A_2) := \{z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial\Omega\} \subset H \to H
\end{aligned}$$

and define the operators $A_1^* : V_1 \to V_1^*$, $B^* : V_2 \to V_2^*$, $A_2^* : V_2 \to V_2^*$, $\Phi : D(\Phi) \subset H \to H$, $\mathcal{L} : H \to H$ as

$$\begin{aligned}
&\langle A_1^* w, z\rangle_{V_1^*,V_1} := \int_\Omega \nabla w \cdot \nabla z && \text{for} \quad w, z \in V_1,\\
&\langle B^* w, z\rangle_{V_2^*,V_2} := (w,z)_H && \text{for} \quad w, z \in V_2,\\
&\langle A_2^* w, z\rangle_{V_2^*,V_2} := \int_\Omega \nabla w \cdot \nabla z && \text{for} \quad w, z \in V_2,\\
&\Phi(z) := \beta(z) && \text{for} \quad z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\},\\
&\mathcal{L}(z) := \pi(z) && \text{for} \quad z \in H.
\end{aligned}$$

Please note that the identity $\widehat{\beta}(0) = 0$ in (J1) entails $\beta(0) = 0$. We set (J1) in reference to an assumption in [7]. We assumed (J2) in reference to assumptions in [12,13,20,21]. Moreover, we set (J3) in reference to assumptions in [7,11]. Then the function $\mathbb{R} \ni r \mapsto d_1 r^3 - d_2 r \in \mathbb{R}$ ($d_1, d_2 > 0$) is a typical example of $\beta + \pi$. Now we verify that $\Phi : D(\Phi) \subset H \to H$ is maximal monotone. We define the function $\phi : H \to \overline{\mathbb{R}}$ as

$$\phi(z) = \begin{cases} \displaystyle\int_\Omega \hat{\beta}(z(x))\,\mathrm{d}x & \text{if } z \in D(\phi) := \{z \in H \mid \hat{\beta}(z) \in L^1(\Omega)\},\\ +\infty & \text{otherwise.}\end{cases}$$

Then $\phi : H \to \overline{\mathbb{R}}$ is proper lower semicontinuous convex, whence $\partial\phi : D(\partial\phi) \subset H \to H$ is maximal monotone (see *e.g.*, [2], Thm. 2.8). In addition, we have that

$$\begin{aligned}
&D(\partial\phi) = \{z \in H \mid \beta(z) \in H\} = D(\Phi),\\
&\partial\phi(z) = \beta(z) = \Phi(z) \quad \text{for all } z \in D(\Phi)
\end{aligned} \tag{2.1}$$

(see *e.g.*, [3], Example 2.8.3, [17], Example II.8.B). Thus $\Phi : D(\Phi) \subset H \to H$ is maximal monotone.

Next we show that $\Phi_\lambda w = \beta_\lambda(w)$ for all $w \in H$, where $\beta_\lambda$ is the Yosida approximation operator of $\beta$ on $\mathbb{R}$. Since it follows from (2.1) that $\Phi = \partial\phi$, the identities

$$\Phi_\lambda w = (\partial\phi)_\lambda w = \lambda^{-1}(w - J_\lambda^{\partial\phi} w) \tag{2.2}$$

hold for all $w \in H$, where $J_\lambda^{\partial\phi} : H \to H$ is the resolvent operator of $\partial\phi$, that is, $J_\lambda^{\partial\phi} w = (I + \lambda\partial\phi)^{-1} w$ for all $w \in H$. On the other hand, since we derive from (2.1) that $\partial\phi(z) = \beta(z)$ for all $z \in D(\Phi)$, we can check that

$$\lambda^{-1}(w - J_\lambda^{\partial\phi} w) = \lambda^{-1}(w - J_\lambda^{\beta}(w)) = \beta_\lambda(w) \tag{2.3}$$

for all $w \in H$, where $J_\lambda^\beta : \mathbb{R} \to \mathbb{R}$ is the resolvent operator of $\beta$ on $\mathbb{R}$, that is, $J_\lambda^\beta(r) = (I + \lambda\beta)^{-1}(r)$ for all $r \in \mathbb{R}$. Hence combining (2.2) and (2.3) leads to the identity $\Phi_\lambda w = \beta_\lambda(w)$ for all $w \in H$.

Next we prove that $V \subset D(\Phi)$ and there exist constants $p, q, C_\Phi > 0$ such that

$$\|\Phi w - \Phi z\|_H \le C_\Phi(1 + \|w\|_V^p + \|z\|_V^q)\|w - z\|_V$$

for all $w, z \in V$. The Taylor theorem and the condition (J2) mean that

$$\begin{aligned} |\beta(r) - \beta(s)| &= \left|\beta'(s)(r - s) + \frac{1}{2}\beta''(r_0)(r - s)^2\right| \\ &\le |\beta'(s)||r - s| + \frac{C_\beta}{2}(1 + |r| + |s|)(r - s)^2 \end{aligned} \tag{2.4}$$

for all $r, s \in \mathbb{R}$, where $r_0$ is a constant belonging to $[r, s]$ or $[s, r]$. Also, owing to the Taylor theorem and the condition (J2), it holds that

$$\begin{aligned} |\beta'(s)| = |\beta'(0) + \beta''(s_0)s| &\le |\beta'(0)| + C_\beta(1 + |s|)|s| \\ &= |\beta'(0)| + C_\beta(|s| + |s|^2) \end{aligned} \tag{2.5}$$

for all $s \in \mathbb{R}$, where $s_0 \in \mathbb{R}$ is a constant belonging to $[0, s]$ or $[s, 0]$. Thus we infer from (2.4), (2.5) and the Hölder inequality that

$$\begin{aligned} &\|\beta(w) - \beta(z)\|_H^2 \\ &\le C_1\|w - z\|_{L^2(\Omega)}^2 + C_1\|z(w - z)\|_{L^2(\Omega)}^2 + C_1\|z^2(w - z)\|_{L^2(\Omega)}^2 \\ &\quad + C_1\|w - z\|_{L^4(\Omega)}^4 + C_1\|w(w - z)^2\|_{L^2(\Omega)}^2 + C_1\|z(w - z)^2\|_{L^2(\Omega)}^2 \\ &\le C_1\|w - z\|_{L^2(\Omega)}^2 + C_1\|z\|_{L^4(\Omega)}^2\|w - z\|_{L^4(\Omega)}^2 + C_1\|z\|_{L^6(\Omega)}^4\|w - z\|_{L^6(\Omega)}^2 \\ &\quad + C_1\|w - z\|_{L^4(\Omega)}^4 + C_1\|w\|_{L^6(\Omega)}^2\|w - z\|_{L^6(\Omega)}^4 + C_1\|z\|_{L^6(\Omega)}^2\|w - z\|_{L^6(\Omega)}^4 \end{aligned} \tag{2.6}$$

for all $w, z \in V$, where $C_1 > 0$ is a constant. Here the continuity of the embedding $V \hookrightarrow L^6(\Omega)$ and the boundedness of $\Omega$ imply that

$$\begin{aligned} &C_1\|w - z\|_{L^2(\Omega)}^2 + C_1\|z\|_{L^4(\Omega)}^2\|w - z\|_{L^4(\Omega)}^2 + C_1\|z\|_{L^6(\Omega)}^4\|w - z\|_{L^6(\Omega)}^2 \\ &\quad + C_1\|w - z\|_{L^4(\Omega)}^4 + C_1\|w\|_{L^6(\Omega)}^2\|w - z\|_{L^6(\Omega)}^4 + C_1\|z\|_{L^6(\Omega)}^2\|w - z\|_{L^6(\Omega)}^4 \\ &\le C_2\|w - z\|_V^2 + C_2\|z\|_V^2\|w - z\|_V^2 + C_2\|z\|_V^4\|w - z\|_V^2 \\ &\quad + C_2\|w - z\|_V^4 + C_2\|w\|_V^2\|w - z\|_V^4 + C_2\|z\|_V^2\|w - z\|_V^4 \\ &\le C_3(1 + \|w\|_V^4 + \|z\|_V^4)\|w - z\|_V^2 \end{aligned} \tag{2.7}$$

for all $w, z \in V$, where $C_2 = C_2(\Omega), C_3 = C_3(\Omega) > 0$ are some constants. Hence we deduce from (2.6) and (2.7) that

$$\|\beta(w) - \beta(z)\|_H^2 \le C_3(1 + \|w\|_V^4 + \|z\|_V^4)\|w - z\|_V^2$$

for all $w, z \in V$. Then, thanks to the identity $\beta(0) = 0$, we have

$$\|\beta(w)\|_H^2 \le C_3(1 + \|w\|_V^4)\|w\|_V^2$$

for all $w \in V$. Therefore $V \subset D(\Phi)$ and there exist constants $p, q, C_\Phi > 0$ such that

$$\|\Phi w - \Phi z\|_H \le C_\Phi(1 + \|w\|_V^p + \|z\|_V^q)\|w - z\|_V$$

for all $w, z \in V$.

Next we confirm that there exists a function $i : V \to \{x \in \mathbb{R} \mid x \geq 0\}$ such that $(\Phi w, w - z)_H \geq i(w) - i(z)$ for all $w, z \in V$. We see from (J1) and the definition of the subdifferential that $\beta(r)(r - s) \geq \widehat{\beta}(r) - \widehat{\beta}(s)$ for all $r, s \in \mathbb{R}$. Thus, defining $i : V \to \{x \in \mathbb{R} \mid x \geq 0\}$ as

$$i(z) = \int_\Omega \widehat{\beta}(z) \quad \text{for} \quad z \in V \subset D(\Phi) \subset \{z \in H \mid \widehat{\beta}(z) \in L^1(\Omega)\},$$

we can obtain that $(\Phi w, w - z)_H \geq i(w) - i(z)$ for all $w, z \in V$.

Therefore the conditions (C1)–(C4), (C6)–(C11), (C13) and (C14) hold. Moreover, the elliptic regularity theory leads to (C5) and (C12). Similarly, we can check that the homogeneous Neumann–Neumann problem, the homogeneous Dirichlet–Dirichlet problem and the homogeneous Neumann–Dirichlet problem are examples.

**Example 2.2.** We can verify that the problem

$$\begin{cases} \theta_t + \varphi_t - \Delta\theta = f & \text{in} \quad \Omega \times (0, T), \\ \varphi_{tt} - \Delta\varphi_t - \Delta\varphi + \beta(\varphi) + \pi(\varphi) = \theta & \text{in} \quad \Omega \times (0, T), \\ \theta = \varphi = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ \theta(0) = \theta_0, \ \varphi(0) = \varphi_0, \ \varphi_t(0) = v_0 & \text{in} \quad \Omega, \end{cases} \tag{P2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, is an example under the three conditions (J1)–(J3) and the following condition

(J5) $\theta_0 \in H_0^1(\Omega), \varphi_0 \in H^2(\Omega) \cap H_0^1(\Omega), v_0 \in H^2(\Omega) \cap H_0^1(\Omega), f \in L^2(0, T; L^2(\Omega))$.

Indeed, putting

$$\begin{aligned} &V := H^1(\Omega), \ H := L^2(\Omega), \ V_1 := H_0^1(\Omega), \ V_2 := H_0^1(\Omega), \\ &L := I : H \to H, \\ &A_1 := -\Delta : D(A_1) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H, \\ &B := -\Delta : D(B) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H, \\ &A_2 := -\Delta : D(A_2) := H^2(\Omega) \cap H_0^1(\Omega) \subset H \to H \end{aligned}$$

and defining the operators $A_1^* : V_1 \to V_1^*$, $B^* : V_2 \to V_2^*$, $A_2^* : V_2 \to V_2^*$, $\Phi : D(\Phi) \subset H \to H$, $\mathcal{L} : H \to H$ as

$$\begin{aligned} &\langle A_1^* w, z \rangle_{V_1^*, V_1} := \int_\Omega \nabla w \cdot \nabla z && \text{for} && w, z \in V_1, \\ &\langle B^* w, z \rangle_{V_2^*, V_2} := \int_\Omega \nabla w \cdot \nabla z && \text{for} && w, z \in V_2, \\ &\langle A_2^* w, z \rangle_{V_2^*, V_2} := \int_\Omega \nabla w \cdot \nabla z && \text{for} && w, z \in V_2, \\ &\Phi(z) := \beta(z) && \text{for} && z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\}, \\ &\mathcal{L}(z) := \pi(z) && \text{for} && z \in H, \end{aligned}$$

we can confirm that (C1)–(C14) hold. Similarly, we can show that the homogeneous Dirichlet–Neumann problem, the homogeneous Neumann–Neumann problem and the homogeneous Neumann–Dirichlet problem are examples.

## 3. Existence of discrete solutions

In this section we will prove Theorem 1.4.

**Lemma 3.1.** *For all $g \in H$ and all $h > 0$ there exists a unique solution $\theta \in D(A_1)$ of the equation $\theta + hA_1\theta = g$ in $H$.*

*Proof.* We define the operator $\Psi : V_1 \to V_1^*$ as

$$\langle \Psi\theta, w\rangle_{V_1^*,V_1} := (\theta, w)_H + h\langle A_1^*\theta, w\rangle_{V_1^*,V_1} \quad \text{for} \quad \theta, w \in V_1.$$

Then, owing to (C4), this operator $\Psi : V_1 \to V_1^*$ is monotone, continuous and coercive, and then is surjective for all $h > 0$ (see *e.g.*, [2], p. 37). Therefore the condition (C5) leads to Lemma 3.1. □

**Lemma 3.2.** *There exists $h_1 \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$ such that for all $g \in H$ and all $h \in (0, h_1)$ there exists a unique solution $\varphi \in D(B) \cap D(A_2)$ of the equation*

$$L\varphi + hB\varphi + h^2A_2\varphi + h^2\Phi\varphi + h^2\mathcal{L}\varphi = g \quad \text{in} \quad H.$$

*Proof.* We define the operator $\Psi : V_2 \to V_2^*$ as

$$\begin{aligned}\langle \Psi\varphi, w\rangle_{V_2^*,V_2} &:= (L\varphi, w)_H + h\langle B^*\varphi, w\rangle_{V_2^*,V_2} \\ &\quad + h^2\langle A_2^*\varphi, w\rangle_{V_2^*,V_2} + h^2(\Phi_\lambda\varphi, w)_H + h^2(\mathcal{L}\varphi, w)_H \quad \text{for} \quad \varphi, w \in V_2.\end{aligned}$$

Then we see that this operator $\Psi : V_2 \to V_2^*$ is monotone, continuous and coercive for all $h \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$. Indeed, it follows from (C3), (C11), the monotonicity of $B^*$ and $\Phi_\lambda$, and (C13) that

$$\begin{aligned}&\langle \Psi\varphi - \Psi\overline{\varphi}, \varphi - \overline{\varphi}\rangle_{V_2^*,V_2} \\ &= (L(\varphi - \overline{\varphi}), \varphi - \overline{\varphi})_H + h\langle B^*(\varphi - \overline{\varphi}), \varphi - \overline{\varphi}\rangle_{V_2^*,V_2} + h^2\langle A_2^*(\varphi - \overline{\varphi}), \varphi - \overline{\varphi}\rangle_{V_2^*,V_2} \\ &\quad + h^2(\Phi_\lambda\varphi - \Phi_\lambda\overline{\varphi}, \varphi - \overline{\varphi})_H + h^2(\mathcal{L}\varphi - \mathcal{L}\overline{\varphi}, \varphi - \overline{\varphi})_H \\ &\geq c_L\|\varphi - \overline{\varphi}\|_H^2 + \omega_1h^2\|\varphi - \overline{\varphi}\|_{V_2}^2 - h^2\|\varphi - \overline{\varphi}\|_H^2 - C_{\mathcal{L}}h^2\|\varphi - \overline{\varphi}\|_H^2 \\ &\geq \omega_1h^2\|\varphi - \overline{\varphi}\|_{V_2}^2\end{aligned}$$

for all $\varphi, \overline{\varphi} \in V_2$ and all $h \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$. The boundedness of the operators $L : H \to H$, $B^* : V_2 \to V_2^*$, $A_2^* : V_2 \to V_2^*$, the Lipschitz continuity of $\Phi_\lambda$, the condition (C13) and the continuity of the embedding $V_2 \hookrightarrow H$ yield that there exists a constant $C_1 = C_1(\lambda) > 0$ such that

$$\begin{aligned}&|\langle \Psi\varphi - \Psi\overline{\varphi}, w\rangle_{V_2^*,V_2}| \\ &\leq |(L(\varphi - \overline{\varphi}), w)_H| + h|\langle B^*(\varphi - \overline{\varphi}), w\rangle_{V_2^*,V_2}| + h^2|\langle A_2^*(\varphi - \overline{\varphi}), w\rangle_{V_2^*,V_2}| \\ &\quad + h^2|(\Phi_\lambda\varphi - \Phi_\lambda\overline{\varphi}, w)_H| + h^2|(\mathcal{L}\varphi - \mathcal{L}\overline{\varphi}, w)_H| \\ &\leq C_1(1 + h + h^2)\|\varphi - \overline{\varphi}\|_{V_2}\|w\|_{V_2}\end{aligned}$$

for all $\varphi, \overline{\varphi}, w \in V_2$ and all $h > 0$. Also, we have that $\langle \Psi\varphi - \mathcal{L}0, \varphi\rangle_{V_2^*,V_2} \geq \omega_1h^2\|\varphi\|_{V_2}^2$ for all $\varphi \in V_2$ and all $h \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$. Thus the operator $\Psi : V_2 \to V_2^*$ is surjective for all $h \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$ (see *e.g.*, [2], p. 37), whence we can deduce from (C12) that for all $g \in H$ and all $h \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$ there exists a unique solution $\varphi_\lambda \in D(B) \cap D(A)$ of the equation

$$L\varphi_\lambda + hB\varphi_\lambda + h^2A_2\varphi_\lambda + h^2\Phi_\lambda\varphi_\lambda + h^2\mathcal{L}\varphi_\lambda = g \quad \text{in} \quad H. \tag{3.1}$$

Here we multiply (3.1) by $\varphi_\lambda$ and use the Young inequality, (C13) to infer that

$$\begin{aligned}
&(L\varphi_\lambda, \varphi_\lambda)_H + h(B\varphi_\lambda, \varphi_\lambda)_H + h^2\langle A_2^*\varphi_\lambda, \varphi_\lambda\rangle_{V_2^*, V_2} + h^2(\Phi_\lambda\varphi_\lambda, \varphi_\lambda)_H \\
&\quad = (g, \varphi_\lambda)_H - h^2(\mathcal{L}\varphi_\lambda - \mathcal{L}0, \varphi_\lambda)_H - h^2(\mathcal{L}0, \varphi_\lambda)_H \\
&\quad \le \frac{c_L}{2}\|\varphi_\lambda\|_H^2 + \frac{1}{2c_L}\|g\|_H^2 + C_{\mathcal{L}}h^2\|\varphi_\lambda\|_H^2 + \frac{\|\mathcal{L}0\|_H^2}{2}h^2 + \frac{1}{2}h^2\|\varphi_\lambda\|_H^2.
\end{aligned}$$

Then, by (C3), (C11), the monotonicity of $B$ and $\Phi_\lambda$, there exists $h_1 \in \left(0, \left(\frac{c_L}{1+C_{\mathcal{L}}}\right)^{1/2}\right)$ such that for all $h \in (0, h_1)$ there exists a constant $C_2 = C_2(h) > 0$ satisfying

$$\|\varphi_\lambda\|_{V_2}^2 \le C_2 \tag{3.2}$$

for all $\lambda > 0$. We derive from (3.1), (C9) and the Young inequality that

$$\begin{aligned}
h^2\|\Phi_\lambda\varphi_\lambda\|_H^2 &= (g, \Phi_\lambda\varphi_\lambda)_H - (L\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H - h(B\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H - h^2(A_2\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H \\
&\quad - h^2(\mathcal{L}\varphi_\lambda, \Phi_\lambda\varphi_\lambda)_H \\
&\le \frac{3}{2h^2}\|g\|_H^2 + \frac{3}{2h^2}\|L\varphi_\lambda\|_H^2 + \frac{3}{2}h^2\|\mathcal{L}\varphi_\lambda\|_H^2 + \frac{1}{2}h^2\|\Phi_\lambda\varphi_\lambda\|_H^2.
\end{aligned}$$

Hence, thanks to the boundedness of the operator $L : H \to H$, (C13) and (3.2), we can verify that for all $h \in (0, h_1)$ there exists a constant $C_3 = C_3(h) > 0$ such that

$$\|\Phi_\lambda\varphi_\lambda\|_H^2 \le C_3 \tag{3.3}$$

for all $\lambda > 0$. We can confirm that

$$\begin{aligned}
h\|B\varphi_\lambda\|_H^2 &= (g, B\varphi_\lambda)_H - (L\varphi_\lambda, B\varphi_\lambda)_H - h^2(A_2\varphi_\lambda, B\varphi_\lambda)_H - h^2(\Phi_\lambda\varphi_\lambda, B\varphi_\lambda)_H \\
&\quad - h^2(\mathcal{L}\varphi_\lambda, B\varphi_\lambda)_H
\end{aligned}$$

by (3.1) and then the boundedness of the operator $L : H \to H$, (C6), (C9), (C13), the Young inequality and (3.2) imply that for all $h \in (0, h_1)$ there exists a constant $C_4 = C_4(h) > 0$ satisfying

$$\|B\varphi_\lambda\|_H^2 \le C_4(h) \tag{3.4}$$

for all $\lambda > 0$. We see from (3.1) to (3.4) that for all $h \in (0, h_1)$ there exists a constant $C_5 = C_5(h) > 0$ such that

$$\|A_2\varphi_\lambda\|_H^2 \le C_5(h) \tag{3.5}$$

for all $\lambda > 0$. Thus by (3.2)–(3.5) there exist $\varphi \in D(B) \cap D(A_2)$ and $\xi \in H$ such that

$$\varphi_\lambda \to \varphi \quad \text{weakly in} \quad V_2, \tag{3.6}$$

$$L\varphi_\lambda \to L\varphi \quad \text{weakly in} \quad H, \tag{3.7}$$

$$\Phi_\lambda(\varphi_\lambda) \to \xi \quad \text{weakly in} \quad H, \tag{3.8}$$

$$B\varphi_\lambda \to B\varphi \quad \text{weakly in} \quad H, \tag{3.9}$$

$$A_2\varphi_\lambda \to A_2\varphi \quad \text{weakly in} \quad H \tag{3.10}$$

as $\lambda = \lambda_j \to +0$. Here the inequality (3.2), the convergence (3.6) and the compactness of the embedding $V_2 \hookrightarrow H$ yield that

$$\varphi_\lambda \to \varphi \quad \text{strongly in} \quad H \tag{3.11}$$

as $\lambda = \lambda_j \to +0$. Moreover, we have from (3.8) and (3.11) that $(\Phi_\lambda\varphi_\lambda, \varphi_\lambda)_H \to (\xi, \varphi)_H$ as $\lambda = \lambda_j \to +0$. Hence the inclusion and the identity

$$\varphi \in D(\Phi),\ \xi = \Phi\varphi \tag{3.12}$$

hold (see *e.g.*, [1], Lem. 1.3, p. 42).

Therefore, by virtue of (3.1), (3.7)–(3.12) and (C13), we can check that there exists a solution $\varphi \in D(B) \cap D(A_2)$ of the equation

$$L\varphi + hB\varphi + h^2A_2\varphi + h^2\Phi\varphi + h^2\mathcal{L}\varphi = g \quad \text{in} \quad H.$$

Moreover, owing to (C3), (C11), the monotonicity of $B$ and $\Phi$, and (C13), the solution $\varphi$ of this problem is unique. □

*Proof of Theorem 1.4.* Let $h_1$ be as in Lemma 3.2 and let $h \in (0, h_1)$. Then we infer from (1.1), the linearity of the operators $A_1$, $L$, $B$ and $A_2$ that the problem (P)$_n$ can be written as

$$\begin{cases} \theta_{n+1} + hA_1\theta_{n+1} = \theta_n + \varphi_n + hf_{n+1} - \varphi_{n+1}, \\ L\varphi_{n+1} + hB\varphi_{n+1} + h^2A_2\varphi_{n+1} + h^2\Phi\varphi_{n+1} + h^2\mathcal{L}\varphi_{n+1} \\ = L\varphi_n + hLv_n + hB\varphi_n + h^2\theta_{n+1}, \end{cases} \tag{Q$_n$}$$

whence proving Theorem 1.4 is equivalent to establish existence and uniqueness of solutions to (Q)$_n$ for $n = 0, \ldots, N-1$. It suffices to consider the case that $n = 0$. Thanks to Lemma 3.1, we can verify that for all $\varphi \in H$ there exists a unique solution $\overline{\theta} \in H$ of the equation

$$\overline{\theta} + hA_1\overline{\theta} = \theta_0 + \varphi_0 + hf_1 - \varphi. \tag{3.13}$$

Also, Lemma 3.2 means that for all $\theta \in H$ there exists a unique solution $\overline{\varphi} \in H$ of the equation

$$L\overline{\varphi} + hB\overline{\varphi} + h^2A_2\overline{\varphi} + h^2\Phi\overline{\varphi} + h^2\mathcal{L}\overline{\varphi} = L\varphi_0 + hLv_0 + hB\varphi_0 + h^2\theta. \tag{3.14}$$

Therefore we can define the operators $\mathcal{A} : H \to H$, $\mathcal{B} : H \to H$ and $\mathcal{S} : H \to H$ as

$$\mathcal{A}(\varphi) = \overline{\theta},\ \mathcal{B}(\theta) = \overline{\varphi} \quad \text{for } \varphi, \theta \in H$$

and

$$\mathcal{S} = \mathcal{B} \circ \mathcal{A},$$

respectively. Then we see from (3.13) and the Young inequality that

$$\begin{aligned} &\|\mathcal{A}\varphi - \mathcal{A}\zeta\|_H^2 + h(A_1(\mathcal{A}\varphi - \mathcal{A}\zeta), \mathcal{A}\varphi - \mathcal{A}\zeta)_H \\ &\quad = -(\varphi - \zeta, \mathcal{A}\varphi - \mathcal{A}\zeta)_H \le \frac{1}{2}\|\varphi - \zeta\|_H^2 + \frac{1}{2}\|\mathcal{A}\varphi - \mathcal{A}\zeta\|_H^2 \end{aligned}$$

for all $\varphi \in H$ and all $\zeta \in H$, and hence the inequality

$$\|\mathcal{A}\varphi - \mathcal{A}\zeta\|_H \le \|\varphi - \zeta\|_H \tag{3.15}$$

holds for all $\varphi \in H$ and all $\zeta \in H$ by the monotonicity of $A_1$. On the other hand, since we derive from (3.14), (C13) and the Young inequality that

$$\begin{aligned} &(L(\mathcal{S}\varphi - \mathcal{S}\zeta), \mathcal{S}\varphi - \mathcal{S}\zeta)_H + h(B(\mathcal{S}\varphi - \mathcal{S}\zeta), \mathcal{S}\varphi - \mathcal{S}\zeta)_H \\ &\quad + h^2(A_2(\mathcal{S}\varphi - \mathcal{S}\zeta), \mathcal{S}\varphi - \mathcal{S}\zeta)_H + h^2(\Phi\mathcal{S}\varphi - \Phi\mathcal{S}\zeta, \mathcal{S}\varphi - \mathcal{S}\zeta)_H \\ &= h^2(\mathcal{A}\varphi - \mathcal{A}\zeta, \mathcal{S}\varphi - \mathcal{S}\zeta)_H - h^2(\mathcal{L}\mathcal{S}\varphi - \mathcal{L}\mathcal{S}\zeta, \mathcal{S}\varphi - \mathcal{S}\zeta)_H \\ &\le \frac{h^2}{4}\|\mathcal{A}\varphi - \mathcal{A}\zeta\|_H^2 + h^2\|\mathcal{S}\varphi - \mathcal{S}\zeta\|_H^2 + C_{\mathcal{L}}h^2\|\mathcal{S}\varphi - \mathcal{S}\zeta\|_H^2 \end{aligned}$$

for all $\varphi \in H$ and all $\zeta \in H$, it follows from (C3), the monotonicity of $B$, $A_2$ and $\Phi$ that

$$\|\mathcal{S}\varphi - \mathcal{S}\zeta\|_H \leq \frac{h}{2(c_L - h^2 - C_{\mathcal{L}}h^2)^{1/2}}\|\mathcal{A}\varphi - \mathcal{A}\zeta\|_H \tag{3.16}$$

for all $\varphi, \zeta \in H$ and all $h \in (0, h_1)$. Hence, combining (3.15) and (3.16), we have that

$$\|\mathcal{S}\varphi - \mathcal{S}\zeta\|_H \leq \frac{h}{2(c_L - h^2 - C_{\mathcal{L}}h^2)^{1/2}}\|\varphi - \zeta\|_H$$

for all $\varphi, \zeta \in H$ and all $h \in (0, h_1)$. Therefore there exists $h_0 \in (0, \min\{1, h_1\})$ such that the operator $\mathcal{S} : H \to H$ is a contraction mapping for all $h \in (0, h_0)$. Then the Banach fixed-point theorem yields that the operator $\mathcal{S} : H \to H$ has a unique fixed point, $\varphi_1 = \mathcal{S}\varphi_1 \in D(B) \cap D(A_2)$. Thus, putting $\theta_1 := \mathcal{A}\varphi_1 \in D(A_1)$, we can conclude that there exists a unique solution $(\theta_1, \varphi_1)$ of (Q)$_n$ in the case that $n = 0$. □

## 4. Uniform estimates for (P)$_h$ and passage to the limit

In this section we will establish a priori estimates for (P)$_h$ and will prove Theorem 1.5 by passing to the limit in (P)$_h$ as $h \to +0$.

**Lemma 4.1.** *Let $h_0$ be as in Theorem 1.4. Then there exist constants $h_2 \in (0, h_0)$ and $C = C(T) > 0$ such that*

$$\begin{aligned}&\|\overline{v}_h\|^2_{L^\infty(0,T;H)} + h\|\overline{z}_h\|^2_{L^2(0,T;H)} + \|B^{1/2}\overline{v}_h\|^2_{L^2(0,T;H)} + \|\overline{\varphi}_h\|^2_{L^\infty(0,T;V_2)} \\ &\quad + h\|\overline{v}_h\|^2_{L^2(0,T;V_2)} + \|\overline{\theta}_h\|^2_{L^\infty(0,T;H)} + h\left\|\frac{\mathrm{d}\widehat{\theta}_h}{\mathrm{d}t}\right\|^2_{L^2(0,T;H)} + \|\overline{\theta}_h\|^2_{L^2(0,T;V_1)} \leq C\end{aligned}$$

*for all $h \in (0, h_2)$.*

*Proof.* Multiplying the second equation in (P)$_n$ by $hv_{n+1}(= \varphi_{n+1} - \varphi_n)$ and recalling (1.1) lead to the identity

$$\begin{aligned}&(L(v_{n+1} - v_n), v_{n+1})_H + h(Bv_{n+1}, v_{n+1})_H + \langle A_2^*\varphi_{n+1}, \varphi_{n+1} - \varphi_n\rangle_{V_2^*,V_2} \\ &\quad + (\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H + (\Phi\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H \\ &\quad = h(\theta_{n+1}, v_{n+1})_H - h(\mathcal{L}\varphi_{n+1}, v_{n+1})_H + h(\varphi_{n+1}, v_{n+1})_H.\end{aligned} \tag{4.1}$$

Here we infer that

$$\begin{aligned}&(L(v_{n+1} - v_n), v_{n+1})_H \\ &\quad = (L^{1/2}(v_{n+1} - v_n), L^{1/2}v_{n+1})_H \\ &\quad = \frac{1}{2}\|L^{1/2}v_{n+1}\|^2_H - \frac{1}{2}\|L^{1/2}v_n\|^2_H + \frac{1}{2}\|L^{1/2}(v_{n+1} - v_n)\|^2_H\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\langle A_2^*\varphi_{n+1}, \varphi_{n+1} - \varphi_n\rangle_{V_2^*,V_2} + (\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H \\ &= \frac{1}{2}\langle A_2^*\varphi_{n+1}, \varphi_{n+1}\rangle_{V_2^*,V_2} - \frac{1}{2}\langle A_2^*\varphi_n, \varphi_n\rangle_{V_2^*,V_2} + \frac{1}{2}\langle A_2^*(\varphi_{n+1} - \varphi_n), \varphi_{n+1} - \varphi_n\rangle_{V_2^*,V_2} \\ &\quad + \frac{1}{2}\|\varphi_{n+1}\|^2_H - \frac{1}{2}\|\varphi_n\|^2_H + \frac{1}{2}\|\varphi_{n+1} - \varphi_n\|^2_H.\end{aligned} \tag{4.3}$$

Hence we deduce from (4.1)–(4.3), (C8), (C13), the continuity of the embedding $V_2 \hookrightarrow H$ and the Young inequality that there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|L^{1/2}v_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}v_n\|_H^2 + \frac{1}{2}\|L^{1/2}(v_{n+1}-v_n)\|_H^2 + h\|B^{1/2}v_{n+1}\|_H^2 \\
&\quad + \frac{1}{2}\langle A_2^*\varphi_{n+1}, \varphi_{n+1}\rangle_{V_2^*,V_2} - \frac{1}{2}\langle A_2^*\varphi_n, \varphi_n\rangle_{V_2^*,V_2} + \frac{1}{2}\langle A_2^*(\varphi_{n+1}-\varphi_n), \varphi_{n+1}-\varphi_n\rangle_{V_2^*,V_2} \\
&\quad + \frac{1}{2}\|\varphi_{n+1}\|_H^2 - \frac{1}{2}\|\varphi_n\|_H^2 + \frac{1}{2}\|\varphi_{n+1}-\varphi_n\|_H^2 + i(\varphi_{n+1}) - i(\varphi_n) \\
&\leq \frac{1}{2}h\|\theta_{n+1}\|_H^2 + \frac{3}{2}h\|v_{n+1}\|_H^2 + C_1 h\|\varphi_{n+1}\|_{V_2}^2 + C_2 h
\end{aligned} \tag{4.4}$$

for all $h \in (0, h_0)$. On the other hand, multiplying the first equation in (P)$_n$ by $h\theta_{n+1}$, we see from the Young inequality that

$$\begin{aligned}
&\frac{1}{2}\|\theta_{n+1}\|_H^2 - \frac{1}{2}\|\theta_n\|_H^2 + \frac{1}{2}\|\theta_{n+1}-\theta_n\|_H^2 + h(A_1\theta_{n+1}, \theta_{n+1})_H \\
&\quad = h(f_{n+1}, \theta_{n+1})_H - h(v_{n+1}, \theta_{n+1})_H \\
&\quad \leq \frac{1}{2}h\|f_{n+1}\|_H^2 + \frac{1}{2}h\|v_{n+1}\|_H^2 + h\|\theta_{n+1}\|_H^2.
\end{aligned} \tag{4.5}$$

Thus combining (4.4) and (4.5) implies that

$$\begin{aligned}
&\frac{1}{2}\|L^{1/2}v_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}v_n\|_H^2 + \frac{1}{2}\|L^{1/2}(v_{n+1}-v_n)\|_H^2 + h\|B^{1/2}v_{n+1}\|_H^2 \\
&\quad + \frac{1}{2}\langle A_2^*\varphi_{n+1}, \varphi_{n+1}\rangle_{V_2^*,V_2} - \frac{1}{2}\langle A_2^*\varphi_n, \varphi_n\rangle_{V_2^*,V_2} + \frac{1}{2}\langle A_2^*(\varphi_{n+1}-\varphi_n), \varphi_{n+1}-\varphi_n\rangle_{V_2^*,V_2} \\
&\quad + \frac{1}{2}\|\varphi_{n+1}\|_H^2 - \frac{1}{2}\|\varphi_n\|_H^2 + \frac{1}{2}\|\varphi_{n+1}-\varphi_n\|_H^2 + i(\varphi_{n+1}) - i(\varphi_n) \\
&\quad + \frac{1}{2}\|\theta_{n+1}\|_H^2 - \frac{1}{2}\|\theta_n\|_H^2 + \frac{1}{2}\|\theta_{n+1}-\theta_n\|_H^2 + h(A_1\theta_{n+1}, \theta_{n+1})_H \\
&\leq \frac{1}{2}h\|f_{n+1}\|_H^2 + \frac{3}{2}h\|\theta_{n+1}\|_H^2 + 2h\|v_{n+1}\|_H^2 + C_1 h\|\varphi_{n+1}\|_{V_2}^2 + C_2 h.
\end{aligned} \tag{4.6}$$

Moreover, we sum (4.6) over $n = 0, \ldots, m-1$ with $1 \leq m \leq N$ to obtain the inequality

$$\begin{aligned}
&\frac{1}{2}\|L^{1/2}v_m\|_H^2 + \frac{1}{2}\sum_{n=0}^{m-1}\|L^{1/2}(v_{n+1}-v_n)\|_H^2 + h\sum_{n=0}^{m-1}\|B^{1/2}v_{n+1}\|_H^2 + \frac{1}{2}\langle A_2^*\varphi_m, \varphi_m\rangle_{V_2^*,V_2} \\
&\quad + \frac{1}{2}\|\varphi_m\|_H^2 + \frac{1}{2}\sum_{n=0}^{m-1}\langle A_2^*(\varphi_{n+1}-\varphi_n), \varphi_{n+1}-\varphi_n\rangle_{V_2^*,V_2} + \frac{1}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|_H^2 \\
&\quad + i(\varphi_m) + \frac{1}{2}\|\theta_m\|_H^2 + \frac{1}{2}\sum_{n=0}^{m-1}\|\theta_{n+1}-\theta_n\|_H^2 + h\sum_{n=0}^{m-1}(A_1\theta_{n+1}, \theta_{n+1})_H \\
&\leq \frac{1}{2}\|L^{1/2}v_0\|_H^2 + \frac{1}{2}\langle A_2^*\varphi_0, \varphi_0\rangle_{V_2^*,V_2} + \frac{1}{2}\|\varphi_0\|_H^2 + i(\varphi_0) + \frac{1}{2}\|\theta_0\|_H^2 + \frac{1}{2}h\sum_{n=0}^{m-1}\|f_{n+1}\|_H^2 \\
&\quad + \frac{3}{2}h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_H^2 + 2h\sum_{n=0}^{m-1}\|v_{n+1}\|_H^2 + C_1 h\sum_{n=0}^{m-1}\|\varphi_{n+1}\|_{V_2}^2 + C_2 T.
\end{aligned} \tag{4.7}$$

Here, owing to (C11), it holds that

$$\frac{1}{2}\langle A_2^*\varphi_m, \varphi_m\rangle_{V_2^*,V_2} + \frac{1}{2}\|\varphi_m\|_H^2 \geq \frac{\omega_1}{2}\|\varphi_m\|_{V_2}^2 \tag{4.8}$$

and

$$\frac{1}{2}\sum_{n=0}^{m-1}\langle A_2^*(\varphi_{n+1}-\varphi_n),\varphi_{n+1}-\varphi_n\rangle_{V_2^*,V_2}+\frac{1}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|_H^2$$
$$\geq \frac{\omega_1}{2}\sum_{n=0}^{m-1}\|\varphi_{n+1}-\varphi_n\|_{V_2}^2=\frac{\omega_1}{2}h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_{V_2}^2. \tag{4.9}$$

Also, we see from (C4) that

$$h\sum_{n=0}^{m-1}(A_1\theta_{n+1},\theta_{n+1})_H=h\sum_{n=0}^{m-1}\langle A_1^*\theta_{n+1},\theta_{n+1}\rangle_{V_1^*,V_1}$$
$$\geq \sigma_1 h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_{V_1}^2-h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_H^2. \tag{4.10}$$

Hence it follows from (4.7) to (4.10) and (C3) that

$$\left(\frac{c_L}{2}-2h\right)\|v_m\|_H^2+\frac{c_L}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2+h\sum_{n=0}^{m-1}\|B^{1/2}v_{n+1}\|_H^2+\left(\frac{\omega_1}{2}-C_1h\right)\|\varphi_m\|_{V_2}^2$$
$$+\frac{\omega_1}{2}h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_{V_2}^2+\frac{1}{2}(1-5h)\|\theta_m\|_H^2+\frac{1}{2}h^2\sum_{n=0}^{m-1}\|\delta_h\theta_n\|_H^2+\sigma_1 h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_{V_1}^2$$
$$\leq \frac{1}{2}\|L^{1/2}v_0\|_H^2+\frac{1}{2}\langle A_2^*\varphi_0,\varphi_0\rangle_{V_2^*,V_2}+\frac{1}{2}\|\varphi_0\|_H^2+i(\varphi_0)+\frac{1}{2}\|\theta_0\|_H^2+\frac{1}{2}h\sum_{n=0}^{m-1}\|f_{n+1}\|_H^2$$
$$+\frac{5}{2}h\sum_{j=0}^{m-1}\|\theta_j\|_H^2+2h\sum_{j=0}^{m-1}\|v_j\|_H^2+C_1h\sum_{j=0}^{m-1}\|\varphi_j\|_{V_2}^2+C_2T$$

and then there exist constants $h_2\in(0,h_0)$ and $C_3=C_3(T)>0$ such that

$$\|v_m\|_H^2+h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2+h\sum_{n=0}^{m-1}\|B^{1/2}v_{n+1}\|_H^2$$
$$+\|\varphi_m\|_{V_2}^2+h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_{V_2}^2+\|\theta_m\|_H^2+h^2\sum_{n=0}^{m-1}\|\delta_h\theta_n\|_H^2+h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_{V_1}^2$$
$$\leq C_3h\sum_{j=0}^{m-1}\|\theta_j\|_H^2+C_3h\sum_{j=0}^{m-1}\|v_j\|_H^2+C_3h\sum_{j=0}^{m-1}\|\varphi_j\|_{V_2}^2+C_3 \tag{4.11}$$

for all $h\in(0,h_2)$. Therefore the inequality (4.11) and the discrete Gronwall lemma (see *e.g.*, [14], Prop. 2.2.1) imply that there exists a constant $C_4=C_4(T)>0$ such that

$$\|v_m\|_H^2+h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_H^2+h\sum_{n=0}^{m-1}\|B^{1/2}v_{n+1}\|_H^2$$
$$+\|\varphi_m\|_{V_2}^2+h^2\sum_{n=0}^{m-1}\|v_{n+1}\|_{V_2}^2+\|\theta_m\|_H^2+h^2\sum_{n=0}^{m-1}\|\delta_h\theta_n\|_H^2+h\sum_{n=0}^{m-1}\|\theta_{n+1}\|_{V_1}^2\leq C_4$$

for all $h\in(0,h_2)$ and $m=1,\dots,N$. $\square$

**Lemma 4.2.** *Let $h_2$ be as in Lemma 4.1. Then there exists a constant $C = C(T) > 0$ such that*

$$\left\|\frac{\mathrm{d}\widehat{\theta}_h}{\mathrm{d}t}\right\|^2_{L^2(0,T;H)} + h\left\|\frac{\mathrm{d}\widehat{\theta}_h}{\mathrm{d}t}\right\|^2_{L^2(0,T;V_1)} + \|\overline{\theta}_h\|^2_{L^\infty(0,T;V_1)} \le C$$

*for all $h \in (0, h_2)$.*

*Proof.* Multiplying the first equation in (P)$_n$ by $\theta_{n+1} - \theta_n$ and using the Young inequality mean that

$$\begin{aligned} h\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|^2_H + \langle A_1^*\theta_{n+1}, \theta_{n+1}-\theta_n\rangle_{V_1^*,V_1} &= h\left(f_{n+1} - v_{n+1}, \frac{\theta_{n+1}-\theta_n}{h}\right)_H \\ &\le h\|f_{n+1}\|^2_H + h\|v_{n+1}\|^2_H + \frac{1}{2}h\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|^2_H. \end{aligned} \tag{4.12}$$

Here we derive that

$$\begin{aligned} \langle A_1^*\theta_{n+1}, \theta_{n+1}-\theta_n\rangle_{V_1^*,V_1} &= \frac{1}{2}\langle A_1^*\theta_{n+1}, \theta_{n+1}\rangle_{V_1^*,V_1} - \frac{1}{2}\langle A_1^*\theta_n, \theta_n\rangle_{V_1^*,V_1} \\ &\quad + \frac{1}{2}\langle A_1^*(\theta_{n+1}-\theta_n), \theta_{n+1}-\theta_n\rangle_{V_1^*,V_1}. \end{aligned} \tag{4.13}$$

Thus, combining (4.12) and (4.13), we have

$$\begin{aligned} &\frac{1}{2}h\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|^2_H + \frac{1}{2}\langle A_1^*\theta_{n+1}, \theta_{n+1}\rangle_{V_1^*,V_1} - \frac{1}{2}\langle A_1^*\theta_n, \theta_n\rangle_{V_1^*,V_1} \\ &\quad + \frac{1}{2}\langle A_1^*(\theta_{n+1}-\theta_n), \theta_{n+1}-\theta_n\rangle_{V_1^*,V_1} \le h\|f_{n+1}\|^2_H + h\|v_{n+1}\|^2_H. \end{aligned} \tag{4.14}$$

Therefore summing (4.14) over $n = 0, \dots, m-1$ with $1 \le m \le N$, the condition (C4) and Lemma 4.1 lead to Lemma 4.2. □

**Lemma 4.3.** *Let $h_2$ be as in Lemma 4.1. Then there exists a constant $C = C(T) > 0$ such that*

$$\|A_1\overline{\theta}_h\|^2_{L^2(0,T;H)} \le C$$

*for all $h \in (0, h_2)$.*

*Proof.* This lemma holds by the first equation in (P)$_h$, Lemmas 4.1 and 4.2. □

**Lemma 4.4.** *Let $h_2$ be as in Lemma 4.1. Then there exists a constant $C = C(T) > 0$ such that*

$$\|z_1\|^2_H + h\|B^{1/2}z_1\|^2_H + \|v_1\|^2_{V_2} + h^2\|z_1\|^2_{V_2} \le C$$

*for all $h \in (0, h_2)$.*

*Proof.* Thanks to the second equation in (P)$_n$, the identities $v_1 = v_0 + hz_1$ and $\varphi_1 = \varphi_0 + hv_1$, we can obtain that

$$Lz_1 + Bv_0 + hBz_1 + A_2\varphi_0 + hA_2v_1 + \Phi\varphi_1 + \mathcal{L}\varphi_1 = \theta_1. \tag{4.15}$$

Then, multiplying (4.15) by $z_1$, we can check that

$$\begin{aligned} &\|L^{1/2}z_1\|^2_H + (Bv_0, z_1)_H + h(Bz_1, z_1)_H + (A_2\varphi_0, z_1)_H + h(A_2v_1, z_1)_H \\ &\quad + (\Phi\varphi_1, z_1)_H + (\mathcal{L}\varphi_1, z_1)_H = (\theta_1, z_1)_H. \end{aligned} \tag{4.16}$$

Here we see from (C11) that

$$\begin{aligned} h(A_2v_1, z_1)_H &= (A_2v_1, v_1 - v_0)_H = \langle A_2^* v_1, v_1 - v_0\rangle_{V_2^*,V_2} \\ &= \frac{1}{2}\langle A_2^* v_1, v_1\rangle_{V_2^*,V_2} - \frac{1}{2}\langle A_2^* v_0, v_0\rangle_{V_2^*,V_2} \\ &\quad + \frac{1}{2}\langle A_2^*(v_1 - v_0), v_1 - v_0\rangle_{V_2^*,V_2} \\ &\geq \frac{\omega_1}{2}\|v_1\|_{V_2}^2 - \frac{1}{2}\|v_1\|_H^2 - \frac{1}{2}\langle A_2^* v_0, v_0\rangle_{V_2^*,V_2} \\ &\quad + \frac{\omega_1}{2}\|v_1 - v_0\|_{V_2}^2 - \frac{1}{2}\|v_1 - v_0\|_H^2. \end{aligned} \tag{4.17}$$

The condition (C7) and Lemma 4.1 yield that there exists a constant $C_1 = C_1(T) > 0$ satisfying

$$|(\Phi\varphi_1, z_1)_H| \leq C_\Phi(1 + \|\varphi_1\|_V^p)\|\varphi_1\|_V\|z_1\|_H \leq C_1\|z_1\|_H. \tag{4.18}$$

Thus it follows from (4.16) to (4.18) and (C3) that

$$\begin{aligned} &c_L\|z_1\|_H^2 + h\|B^{1/2}z_1\|_H^2 + \frac{\omega_1}{2}\|v_1\|_{V_2}^2 + \frac{\omega_1}{2}h^2\|z_1\|_{V_2}^2 \\ &\leq -(Bv_0, z_1)_H - (A_2\varphi_0, z_1)_H + \frac{1}{2}\|v_1\|_H^2 + \frac{1}{2}\langle A_2^* v_0, v_0\rangle_{V_2^*,V_2} + \frac{1}{2}\|v_1 - v_0\|_H^2 \\ &\quad + C_1\|z_1\|_H - (\mathcal{L}\varphi_1, z_1)_H + (\theta_1, z_1)_H. \end{aligned} \tag{4.19}$$

Hence the inequality (4.19), the condition (C13), the Young inequality and Lemma 4.1 imply that Lemma 4.4 holds. ☐

**Lemma 4.5.** *Let $h_2$ be as in Lemma 4.1. Then there exist constants $h_3 \in (0, h_2)$ and $C = C(T) > 0$ such that*

$$\|\overline{z}_h\|_{L^\infty(0,T;H)}^2 + \|B^{1/2}\overline{z}_h\|_{L^2(0,T;H)}^2 + \|\overline{v}_h\|_{L^\infty(0,T;V_2)}^2 + h\|\overline{z}_h\|_{L^2(0,T;V_2)}^2 \leq C$$

*for all $h \in (0, h_3)$.*

*Proof.* Let $n \in \{1, \ldots, N-1\}$. Then the second equation in (P)$_n$ leads to the identity

$$\begin{aligned} &L(z_{n+1} - z_n) + B(v_{n+1} - v_n) + hA_2v_{n+1} + \Phi\varphi_{n+1} - \Phi\varphi_n + \mathcal{L}\varphi_{n+1} - \mathcal{L}\varphi_n \\ &\quad = \theta_{n+1} - \theta_n. \end{aligned}$$

Here it holds that

$$\begin{aligned} (L(z_{n+1} - z_n), z_{n+1})_H &= (L^{1/2}(z_{n+1} - z_n), L^{1/2}z_{n+1})_H \\ &= \frac{1}{2}\|L^{1/2}z_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}z_n\|_H^2 + \frac{1}{2}\|L^{1/2}(z_{n+1} - z_n)\|_H^2, \end{aligned}$$

and hence we have

$$\begin{aligned} &\frac{1}{2}\|L^{1/2}z_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2}z_n\|_H^2 + \frac{1}{2}\|L^{1/2}(z_{n+1} - z_n)\|_H^2 + h\|B^{1/2}z_{n+1}\|_H^2 \\ &\quad + \langle A_2^* v_{n+1}, v_{n+1} - v_n\rangle_{V_2^*,V_2} + (v_{n+1}, v_{n+1} - v_n)_H \\ &= -h\left(\frac{\Phi\varphi_{n+1} - \Phi\varphi_n}{h}, z_{n+1}\right)_H - h\left(\frac{\mathcal{L}\varphi_{n+1} - \mathcal{L}\varphi_n}{h}, z_{n+1}\right)_H \\ &\quad + h\left(\frac{\theta_{n+1} - \theta_n}{h}, z_{n+1}\right)_H + h(v_{n+1}, z_{n+1})_H. \end{aligned} \tag{4.20}$$

On the other hand, we derive that

$$
\begin{aligned}
&\langle A_2^* v_{n+1}, v_{n+1} - v_n \rangle_{V_2^*, V_2} + (v_{n+1}, v_{n+1} - v_n)_H \\
&= \frac{1}{2}\langle A_2^* v_{n+1}, v_{n+1} \rangle_{V_2^*, V_2} - \frac{1}{2}\langle A_2^* v_n, v_n \rangle_{V_2^*, V_2} + \frac{1}{2}\langle A_2^*(v_{n+1} - v_n), v_{n+1} - v_n \rangle_{V_2^*, V_2} \\
&\quad + \frac{1}{2}\|v_{n+1}\|_H^2 - \frac{1}{2}\|v_n\|_H^2 + \frac{1}{2}\|v_{n+1} - v_n\|_H^2. \qquad (4.21)
\end{aligned}
$$

We see from (C7) and Lemma 4.1 that there exists a constant $C_1 = C_1(T) > 0$ such that

$$
\begin{aligned}
-h\left(\frac{\Phi\varphi_{n+1} - \Phi\varphi_n}{h}, z_{n+1}\right)_H &\le C_\Phi h(1 + \|\varphi_{n+1}\|_V^p + \|\varphi_n\|_V^q)\|v_{n+1}\|_V \|z_{n+1}\|_H \\
&\le C_1 h \|v_{n+1}\|_V \|z_{n+1}\|_H \qquad (4.22)
\end{aligned}
$$

for all $h \in (0, h_2)$. Thus we combine (4.20)–(4.22) and (C13) to infer that there exists a constant $C_2 = C_2(T) > 0$ satisfying

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2} z_{n+1}\|_H^2 - \frac{1}{2}\|L^{1/2} z_n\|_H^2 + \frac{1}{2}\|L^{1/2}(z_{n+1} - z_n)\|_H^2 + h\|B^{1/2} z_{n+1}\|_H^2 \\
&\quad + \frac{1}{2}\langle A_2^* v_{n+1}, v_{n+1} \rangle_{V_2^*, V_2} - \frac{1}{2}\langle A_2^* v_n, v_n \rangle_{V_2^*, V_2} + \frac{1}{2}\langle A_2^*(v_{n+1} - v_n), v_{n+1} - v_n \rangle_{V_2^*, V_2} \\
&\quad + \frac{1}{2}\|v_{n+1}\|_H^2 - \frac{1}{2}\|v_n\|_H^2 + \frac{1}{2}\|v_{n+1} - v_n\|_H^2 \\
&\le C_2 h \|v_{n+1}\|_{V_2} \|z_{n+1}\|_H + h\left\|\frac{\theta_{n+1} - \theta_n}{h}\right\|_H \|z_{n+1}\|_H \qquad (4.23)
\end{aligned}
$$

for all $h \in (0, h_2)$. Then summing (4.23) over $n = 1, \ldots, \ell - 1$ with $2 \le \ell \le N$ means that

$$
\begin{aligned}
&\frac{1}{2}\|L^{1/2} z_\ell\|_H^2 + \frac{1}{2}\sum_{n=1}^{\ell-1}\|L^{1/2}(z_{n+1} - z_n)\|_H^2 + h\sum_{n=1}^{\ell-1}\|B^{1/2} z_{n+1}\|_H^2 \\
&\quad + \frac{1}{2}\langle A_2^* v_\ell, v_\ell \rangle_{V_2^*, V_2} + \frac{1}{2}\sum_{n=1}^{\ell-1}\langle A_2^*(v_{n+1} - v_n), v_{n+1} - v_n \rangle_{V_2^*, V_2} \\
&\quad + \frac{1}{2}\|v_\ell\|_H^2 + \frac{1}{2}\sum_{n=1}^{\ell-1}\|v_{n+1} - v_n\|_H^2 \\
&\le \frac{1}{2}\|L^{1/2} z_1\|_H^2 + \frac{1}{2}\langle A_2^* v_1, v_1 \rangle_{V_2^*, V_2} + \frac{1}{2}\|v_1\|_H^2 \\
&\quad + C_2 h \sum_{n=0}^{\ell-1}\|v_{n+1}\|_{V_2}\|z_{n+1}\|_H + h\sum_{n=0}^{\ell-1}\left\|\frac{\theta_{n+1} - \theta_n}{h}\right\|_H \|z_{n+1}\|_H,
\end{aligned}
$$

whence it follows from (C3) and (C11) that

$$
\begin{aligned}
&\frac{c_L}{2}\|z_\ell\|_H^2 + h\sum_{n=1}^{\ell-1}\|B^{1/2} z_{n+1}\|_H^2 + \frac{\omega_1}{2}\|v_\ell\|_{V_2}^2 + \frac{\omega_1}{2}h^2\sum_{n=1}^{\ell-1}\|z_{n+1}\|_{V_2}^2 \\
&\le \frac{1}{2}\|L^{1/2} z_1\|_H^2 + \frac{1}{2}\langle A_2^* v_1, v_1 \rangle_{V_2^*, V_2} + \frac{1}{2}\|v_1\|_H^2 \\
&\quad + C_2 h \sum_{n=0}^{\ell-1}\|v_{n+1}\|_{V_2}\|z_{n+1}\|_H + h\sum_{n=0}^{\ell-1}\left\|\frac{\theta_{n+1} - \theta_n}{h}\right\|_H \|z_{n+1}\|_H \qquad (4.24)
\end{aligned}
$$

for all $h \in (0, h_2)$ and $\ell = 2, \ldots, N$. Therefore we see from (4.24), the boundedness of the operators $L$ and $A_2^*$, and Lemma 4.4 that there exists a constant $C_3 = C_3(T) > 0$ such that

$$\begin{aligned}\frac{c_L}{2}\|z_m\|_H^2 + h\sum_{n=0}^{m-1}\|B^{1/2}z_{n+1}\|_H^2 + \frac{\omega_1}{2}\|v_m\|_{V_2}^2 + \frac{\omega_1}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_{V_2}^2 \\ \le C_3 + C_2 h\sum_{n=0}^{m-1}\|v_{n+1}\|_{V_2}\|z_{n+1}\|_H + h\sum_{n=0}^{m-1}\left\|\frac{\theta_{n+1}-\theta_n}{h}\right\|_H\|z_{n+1}\|_H \end{aligned} \tag{4.25}$$

for all $h \in (0, h_2)$ and $m = 1, \ldots, N$. Moreover, the inequality (4.25), the Young inequality and Lemma 4.2 yield that there exists a constant $C_4 = C_4(T) > 0$ such that

$$\begin{aligned}\frac{1}{2}(c_L - C_2 h - h)\|z_m\|_H^2 + h\sum_{n=0}^{m-1}\|B^{1/2}z_{n+1}\|_H^2 + \frac{1}{2}(\omega_1 - C_2 h)\|v_m\|_{V_2}^2 \\ + \frac{\omega_1}{2}h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_{V_2}^2 \le C_4 + \frac{C_2}{2}h\sum_{j=0}^{m-1}\|v_j\|_{V_2}^2 + \frac{1+C_2}{2}h\sum_{j=0}^{m-1}\|z_j\|_H^2\end{aligned} \tag{4.26}$$

for all $h \in (0, h_2)$ and $m = 1, \ldots, N$. Thus there exist constants $h_3 \in (0, h_2)$ and $C_5 = C_5(T) > 0$ such that

$$\begin{aligned}\|z_m\|_H^2 + h\sum_{n=0}^{m-1}\|B^{1/2}z_{n+1}\|_H^2 + \|v_m\|_{V_2}^2 + h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_{V_2}^2 \\ \le C_5 + C_5 h\sum_{j=0}^{m-1}\|v_j\|_{V_2}^2 + C_5 h\sum_{j=0}^{m-1}\|z_j\|_H^2\end{aligned}$$

for all $h \in (0, h_3)$ and $m = 1, \ldots, N$. Then we infer from the discrete Gronwall lemma (see *e.g.*, [14], Prop. 2.2.1) that there exists a constant $C_6 = C_6(T) > 0$ satisfying

$$\|z_m\|_H^2 + h\sum_{n=0}^{m-1}\|B^{1/2}z_{n+1}\|_H^2 + \|v_m\|_{V_2}^2 + h^2\sum_{n=0}^{m-1}\|z_{n+1}\|_{V_2}^2 \le C_6$$

for all $h \in (0, h_3)$ and $m = 1, \ldots, N$. □

**Lemma 4.6.** *Let $h_2$ be as in Lemma 4.1. Then there exists a constant $C = C(T) > 0$ such that*

$$\|\Phi\overline{\varphi}_h\|_{L^\infty(0,T;H)} \le C$$

*for all $h \in (0, h_2)$.*

*Proof.* This lemma can be proved by (C7) and Lemma 4.1. □

**Lemma 4.7.** *Let $h_3$ be as in Lemma 4.5. Then there exists a constant $C = C(T) > 0$ such that*

$$\|B\overline{v}_h\|_{L^2(0,T;H)}^2 + \|A_2\overline{\varphi}_h\|_{L^2(0,T;H)}^2 \le C$$

*for all $h \in (0, h_3)$.*

*Proof.* We derive from the second equation in (P)$_n$ that

$$\begin{aligned}h\|Bv_{n+1}\|_H^2 &= h(Bv_{n+1}, Bv_{n+1})_H \\ &= -h(Lz_{n+1}, Bv_{n+1})_H - h(A_2\varphi_{n+1}, Bv_{n+1})_H - h(\Phi\varphi_{n+1}, Bv_{n+1})_H \\ &\quad - h(\mathcal{L}\varphi_{n+1}, Bv_{n+1})_H + h(\theta_{n+1}, Bv_{n+1})_H,\end{aligned}$$

and hence it follows from the Young inequality, the boundedness of the operator $L$ and (C13) that there exists a constant $C_1 > 0$ satisfying

$$h\|Bv_{n+1}\|_H^2 \leq C_1 h\|z_{n+1}\|_H^2 - h(A_2\varphi_{n+1}, Bv_{n+1})_H + C_1 h\|\Phi\varphi_{n+1}\|_H^2 + C_1 h\|\theta_{n+1}\|_H^2 + C_1 h \tag{4.27}$$

for all $h \in (0, h_3)$. Here the condition (C6) implies that

$$\begin{aligned}
&- h(A_2\varphi_{n+1}, Bv_{n+1})_H \\
&\quad = -(A_2\varphi_{n+1}, B\varphi_{n+1} - B\varphi_n)_H \\
&\quad = -\frac{1}{2}(A_2\varphi_{n+1}, B\varphi_{n+1})_H + \frac{1}{2}(A_2\varphi_n, B\varphi_n)_H \\
&\qquad - \frac{1}{2}(A_2(\varphi_{n+1} - \varphi_n), B(\varphi_{n+1} - \varphi_n))_H.
\end{aligned} \tag{4.28}$$

Thus, summing (4.27) over $n = 0, \ldots, m-1$ with $1 \leq m \leq N$, we deduce from (4.28), Lemmas 4.1, 4.5 and 4.6 that there exists a constant $C_2 = C_2(T) > 0$ such that

$$\|B\overline{v}_h\|_{L^2(0,T;H)}^2 \leq C_2 \tag{4.29}$$

for all $h \in (0, h_3)$. Moreover, we see from the second equation in (P)$_h$, (4.29), Lemmas 4.1, 4.5 and 4.6 that there exists a constant $C_3 = C_3(T) > 0$ satisfying

$$\|A_2\overline{\varphi}_h\|_{L^2(0,T;H)}^2 \leq C_3$$

for all $h \in (0, h_3)$. □

**Lemma 4.8.** *Let $h_3$ be as in Lemma 4.5. Then there exists a constant $C = C(T) > 0$ such that*

$$\begin{aligned}
&\|\widehat{\varphi}_h\|_{W^{1,\infty}(0,T;V_2)} + \|\widehat{v}_h\|_{W^{1,\infty}(0,T;H)} \\
&\quad + \|\widehat{v}_h\|_{L^\infty(0,T;V_2)} + \|\widehat{\theta}_h\|_{H^1(0,T;H)} + \|\widehat{\theta}_h\|_{L^\infty(0,T;V_1)} \leq C
\end{aligned}$$

*for all $h \in (0, h_3)$.*

*Proof.* Thanks to (1.5)–(1.7), Lemmas 4.1, 4.2 and 4.5, we can obtain Lemma 4.8. □

*Proof of Theorem 1.5 (existence part).* Owing to Lemmas 4.1–4.3, 4.5–4.8, and (1.8)–(1.10), there exist some functions

$$\begin{aligned}
&\theta \in H^1(0,T;H) \cap L^\infty(0,T;V_1) \cap L^2(0,T;D(A_1)), \\
&\varphi \in L^\infty(0,T;V_2) \cap L^2(0,T;D(A_2)), \\
&\xi \in L^\infty(0,T;H)
\end{aligned}$$

such that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} \in L^\infty(0,T;V_2) \cap L^2(0,T;D(B)), \ \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} \in L^\infty(0,T;H)$$

and

$$\begin{aligned}
&\widehat{\varphi}_h \to \varphi && \text{weakly}^* \text{ in} && W^{1,\infty}(0,T;V_2), \\
&\overline{v}_h \to \frac{\mathrm{d}\varphi}{\mathrm{d}t} && \text{weakly}^* \text{ in} && L^\infty(0,T;V_2),
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&\widehat{v}_h \to \frac{d\varphi}{dt} && \text{weakly* in} \quad W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V_2), && (4.31)\\
&\overline{z}_h \to \frac{d^2\varphi}{dt^2} && \text{weakly* in} \quad L^\infty(0,T;H), && \\
&L\overline{z}_h \to L\frac{d^2\varphi}{dt^2} && \text{weakly* in} \quad L^\infty(0,T;H), && (4.32)\\
&\widehat{\theta}_h \to \theta && \text{weakly* in} \quad H^1(0,T;H) \cap L^\infty(0,T;V_1), && (4.33)\\
&\overline{\varphi}_h \to \varphi && \text{weakly* in} \quad L^\infty(0,T;V_2), && \\
&\overline{\theta}_h \to \theta && \text{weakly* in} \quad L^\infty(0,T;V_1), && \\
&A_1\overline{\theta}_h \to A_1\theta && \text{weakly in} \quad L^2(0,T;H), && (4.34)\\
&B\overline{v}_h \to B\frac{d\varphi}{dt} && \text{weakly in} \quad L^2(0,T;H), && (4.35)\\
&A_2\overline{\varphi}_h \to A_2\varphi && \text{weakly in} \quad L^2(0,T;H), && (4.36)\\
&\Phi\overline{\varphi}_h \to \xi && \text{weakly* in} \quad L^\infty(0,T;H) && (4.37)
\end{aligned}$$

as $h = h_j \to +0$. Here, since Lemma 4.8, the compactness of the embedding $V_2 \hookrightarrow H$ and the convergence (4.30) yield that

$$\widehat{\varphi}_h \to \varphi \quad \text{strongly in} \quad C([0,T];H) \tag{4.38}$$

as $h = h_j \to +0$ (see *e.g.*, [18], Sect. 8, Cor. 4), we infer from (1.8) and Lemma 4.5 that

$$\overline{\varphi}_h \to \varphi \quad \text{strongly in} \quad L^\infty(0,T;H) \tag{4.39}$$

as $h = h_j \to +0$. Thus it follows from (4.37) and (4.39) that

$$\int_0^T (\Phi\overline{\varphi}_h(t), \overline{\varphi}_h(t))_H \, dt \to \int_0^T (\xi(t), \varphi(t))_H \, dt$$

as $h = h_j \to +0$, whence we have

$$\xi = \Phi\varphi \quad \text{in} \quad H \quad \text{a.e. on} \quad (0,T) \tag{4.40}$$

(see e.g., [1], Lem. 1.3, p. 42). On the other hand, we derive from Lemma 4.8, the compactness of the embedding $V_1 \hookrightarrow H$ and (4.33) that

$$\widehat{\theta}_h \to \theta \quad \text{strongly in} \quad C([0,T];H) \tag{4.41}$$

as $h = h_j \to +0$. Similarly, we see from (4.31) that

$$\widehat{v}_h \to \frac{d\varphi}{dt} \quad \text{strongly in} \quad C([0,T];H) \tag{4.42}$$

as $h = h_j \to +0$. Therefore we can conclude that there exists a solution of (P) by combining (4.30), (4.32)–(4.42), (C13) and by observing that $\overline{f}_h \to f$ strongly in $L^2(0,T;H)$ as $h \to +0$ (see [7], Sect. 5). □

## 5. Uniqueness for (P)

In this section we establish uniqueness of solutions to (P).

*Proof of Theorem 1.5 (uniqueness part).* We let $(\theta, \varphi)$, $(\overline{\theta}, \overline{\varphi})$ be two solutions of (P) and put $\widetilde{\theta} := \theta - \overline{\theta}$, $\widetilde{\varphi} := \varphi - \overline{\varphi}$. Then the identity (1.11) means that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\theta}(t)\|_H^2 + \left(\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t), \widetilde{\theta}(t)\right)_H + (A_1\widetilde{\theta}(t), \widetilde{\theta}(t))_H = 0. \tag{5.1}$$

Here, by (1.12), the Young inequality, (C7), (C13), Lemma 4.1 and the continuity of the embedding $V_2 \hookrightarrow H$, we can verify that there exists a constant $C_1 = C_1(T) > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right\|_H^2 + \left(B\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t), \frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right)_H + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A_2^{1/2}\widetilde{\varphi}(t)\right\|_H^2 \\
&= \left(\widetilde{\theta}(t), \frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right)_H - \left(\Phi\varphi(t) - \Phi\overline{\varphi}(t), \frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right)_H - \left(\mathcal{L}\varphi(t) - \mathcal{L}\overline{\varphi}(t), \frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right)_H \\
&\leq \left(\widetilde{\theta}(t), \frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right)_H + \frac{C_\Phi^2}{2}(1 + \|\varphi(t)\|_V^p + \|\overline{\varphi}(t)\|_V^q)^2\|\widetilde{\varphi}(t)\|_V^2 \\
&\quad + \frac{C_{\mathcal{L}}^2}{2}\|\widetilde{\varphi}(t)\|_H^2 + \left\|\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right\|_H^2 \\
&\leq \left(\widetilde{\theta}(t), \frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right)_H + C_1\|\widetilde{\varphi}(t)\|_{V_2}^2 + \frac{1}{c_L}\left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right\|_H^2
\end{aligned} \tag{5.2}$$

for a.a. $t \in (0, T)$. Also, the Young inequality, (C3) and the continuity of the embedding $V_2 \hookrightarrow H$ imply that there exists a constant $C_2 > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\varphi}(t)\|_H^2 = \left(\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t), \widetilde{\varphi}(t)\right)_H \leq \frac{1}{2c_L}\left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right\|_H^2 + C_2\|\widetilde{\varphi}(t)\|_{V_2}^2 \tag{5.3}$$

for a.a. $t \in (0, T)$. Hence we deduce from (5.1) to (5.3), the integration over $(0, t)$, where $t \in [0, T]$, (1.13) and the monotonicity of $A_1$, $B$ that there exists a constant $C_3 = C_3(T) > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|\widetilde{\theta}(t)\|_H^2 + \frac{1}{2}\left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right\|_H^2 + \frac{1}{2}\left\|A_2^{1/2}\widetilde{\varphi}(t)\right\|_H^2 + \frac{1}{2}\|\widetilde{\varphi}\|_H^2 \\
&\quad \leq C_3\int_0^t \left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(s)\right\|_H^2 \mathrm{d}s + C_3\int_0^t \|\widetilde{\varphi}(s)\|_{V_2}^2 \,\mathrm{d}s
\end{aligned} \tag{5.4}$$

for all $t \in [0, T]$. Here, owing to (C11), it holds that

$$\begin{aligned}
&\frac{1}{2}\left\|A_2^{1/2}\widetilde{\varphi}(t)\right\|_H^2 + \frac{1}{2}\|\widetilde{\varphi}\|_H^2 \\
&\quad = \frac{1}{2}\langle A_2^*\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle_{V_2^*, V_2} + \frac{1}{2}\|\widetilde{\varphi}\|_H^2 \geq \frac{\omega_1}{2}\|\widetilde{\varphi}(t)\|_{V_2}^2.
\end{aligned} \tag{5.5}$$

Thus it follows from (5.4) and (5.5) that

$$\begin{aligned}
&\frac{1}{2}\|\widetilde{\theta}(t)\|_H^2 + \frac{1}{2}\left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(t)\right\|_H^2 + \frac{\omega_1}{2}\|\widetilde{\varphi}(t)\|_{V_2}^2 \\
&\quad \leq C_3\int_0^t \left\|L^{1/2}\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}t}(s)\right\|_H^2 \mathrm{d}s + C_3\int_0^t \|\widetilde{\varphi}(s)\|_{V_2}^2 \,\mathrm{d}s
\end{aligned}$$

and then applying the Gronwall lemma yields that $\widetilde{\theta} = \widetilde{\varphi} = 0$, which leads to the identities $\theta = \overline{\theta}$ and $\varphi = \overline{\varphi}$. $\square$

## 6. Error estimates

In this section we will prove Theorem 1.6.

**Lemma 6.1.** *Let $h_3$ be as in Lemma 4.5. Then there exists a constant $C = C(T) > 0$ such that*

$$\begin{aligned}&\|L^{1/2}(\widehat{v}_h - v)\|_{L^\infty(0,T;H)} + \|B^{1/2}(\overline{v}_h - v)\|_{L^2(0,T;H)} + \|\widehat{\varphi}_h - \varphi\|_{L^\infty(0,T;V_2)} \\ &\quad + \|\widehat{\theta}_h - \theta\|_{L^\infty(0,T;H)} + \|\overline{\theta}_h - \theta\|_{L^2(0,T;V_1)} \le Ch^{1/2} + C\|\overline{f}_h - f\|_{L^2(0,T;H)}\end{aligned}$$

*for all $h \in (0, h_3)$, where $v = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$.*

*Proof.* We infer from the first equations in (P)$_h$ and (1.11) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\theta}_h(t) - \theta(t)\|_H^2 \\ &\quad = -(\overline{v}_h(t) - v(t), \widehat{\theta}_h(t) - \theta(t))_H - (A_1(\overline{\theta}_h(t) - \theta(t)), \widehat{\theta}_h(t) - \overline{\theta}_h(t))_H \\ &\qquad - \langle A_1^*(\overline{\theta}_h(t) - \theta(t)), \overline{\theta}_h(t) - \theta(t)\rangle_{V_1^*,V_1} + (\overline{f}_h(t) - f(t), \widehat{\theta}_h(t) - \theta(t))_H. \end{aligned} \tag{6.1}$$

Here we derive from the Young inequality and (C3) that

$$\begin{aligned}&-(\overline{v}_h(t) - v(t), \widehat{\theta}_h(t) - \theta(t))_H \\ &\quad \le \frac{1}{2}\|\overline{v}_h(t) - v(t)\|_H^2 + \frac{1}{2}\|\widehat{\theta}_h(t) - \theta(t)\|_H^2 \\ &\quad \le \|\overline{v}_h(t) - \widehat{v}_h(t)\|_H^2 + \|\widehat{v}_h(t) - v(t)\|_H^2 + \frac{1}{2}\|\widehat{\theta}_h(t) - \theta(t)\|_H^2 \\ &\quad \le \|\overline{v}_h(t) - \widehat{v}_h(t)\|_H^2 + \frac{1}{c_L}\|L^{1/2}(\widehat{v}_h(t) - v(t))\|_H^2 + \frac{1}{2}\|\widehat{\theta}_h(t) - \theta(t)\|_H^2. \end{aligned} \tag{6.2}$$

It follows from (C4) that

$$\begin{aligned}&-\langle A_1^*(\overline{\theta}_h(t) - \theta(t)), \overline{\theta}_h(t) - \theta(t)\rangle_{V_1^*,V_1} \\ &\quad \le -\sigma_1\|\overline{\theta}_h(t) - \theta(t)\|_{V_1}^2 + \|\overline{\theta}_h(t) - \theta(t)\|_H^2 \\ &\quad \le -\sigma_1\|\overline{\theta}_h(t) - \theta(t)\|_{V_1}^2 + 2\|\overline{\theta}_h(t) - \widehat{\theta}_h(t)\|_H^2 + 2\|\widehat{\theta}_h(t) - \theta(t)\|_H^2. \end{aligned} \tag{6.3}$$

We have from the Young inequality that

$$(\overline{f}_h(t) - f(t), \widehat{\theta}_h(t) - \theta(t))_H \le \frac{1}{2}\|\overline{f}_h(t) - f(t)\|_H^2 + \frac{1}{2}\|\widehat{\theta}_h(t) - \theta(t)\|_H^2. \tag{6.4}$$

Thus we see from (6.1) to (6.4) and the integration over $(0, t)$, where $t \in [0, T]$, Lemma 4.3, (1.10), Lemma 4.2, (1.9) and Lemma 4.5 that there exists a constant $C_1 = C_1(T) > 0$ such that

$$\begin{aligned}&\frac{1}{2}\|\widehat{\theta}_h(t) - \theta(t)\|_H^2 + \sigma_1\int_0^t \|\overline{\theta}_h(s) - \theta(s)\|_{V_1}^2\,\mathrm{d}s \\ &\quad \le C_1 h + C_1\int_0^t \|L^{1/2}(\widehat{v}_h(s) - v(s))\|_H^2\,\mathrm{d}s + C_1\int_0^t \|\widehat{\theta}_h(s) - \theta(s)\|_H^2\,\mathrm{d}s \\ &\qquad + C_1\|\overline{f}_h - f\|_{L^2(0,T;H)}^2 \end{aligned} \tag{6.5}$$

for all $t \in [0, T]$ and all $h \in (0, h_3)$.

Next we observe that the identity $\frac{\mathrm{d}\widehat{v}_h}{\mathrm{d}t} = \overline{z}_h$, putting $z := \frac{dv}{\mathrm{d}t}$, the second equations in (P)$_h$ and (1.12) imply that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2\\
&\quad= (L(\overline{z}_h(t)-z(t)),\widehat{v}_h(t)-\overline{v}_h(t))_H + (L(\overline{z}_h(t)-z(t)),\overline{v}_h(t)-v(t))_H\\
&\quad= (L(\overline{z}_h(t)-z(t)),\widehat{v}_h(t)-\overline{v}_h(t))_H - (B(\overline{v}_h(t)-v(t)),\overline{v}_h(t)-v(t))_H\\
&\qquad - (A_2(\overline{\varphi}_h(t)-\varphi(t)),\overline{v}_h(t)-v(t))_H - (\Phi\overline{\varphi}_h(t)-\Phi\varphi(t),\overline{v}_h(t)-v(t))_H\\
&\qquad - (\mathcal{L}\overline{\varphi}_h(t)-\mathcal{L}\varphi(t),\overline{v}_h(t)-v(t))_H + (\overline{\theta}_h(t)-\theta(t),\overline{v}_h(t)-v(t))_H.
\end{aligned} \tag{6.6}$$

Here, recalling that the linear operator $L : H \to H$ is bounded, we can obtain that there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
(L(\overline{z}_h(t)-z(t)),\widehat{v}_h(t)-\overline{v}_h(t))_H &\le \|L(\overline{z}_h(t)-z(t))\|_H\|\widehat{v}_h(t)-\overline{v}_h(t)\|_H\\
&\le C_2\|\overline{z}_h(t)-z(t)\|_H\|\widehat{v}_h(t)-\overline{v}_h(t)\|_H
\end{aligned} \tag{6.7}$$

for a.a. $t \in (0,T)$ and all $h \in (0,h_3)$. Owing to the identities $\overline{v}_h = \frac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t}$, $v = \frac{\mathrm{d}\varphi}{\mathrm{d}t}$ and the boundedness of the operator $A_2^* : V_2 \to V_2^*$, it holds that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&-(A_2(\overline{\varphi}_h(t)-\varphi(t)),\overline{v}_h(t)-v(t))_H\\
&\quad= -\langle A_2^*(\overline{\varphi}_h(t)-\widehat{\varphi}_h(t)),\overline{v}_h(t)-v(t)\rangle_{V_2^*,V_2} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_2^{1/2}(\widehat{\varphi}_h(t)-\varphi(t))\|_H^2\\
&\quad\le C_3\|\overline{\varphi}_h(t)-\widehat{\varphi}_h(t)\|_{V_2}\|\overline{v}_h(t)-v(t)\|_{V_2} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_2^{1/2}(\widehat{\varphi}_h(t)-\varphi(t))\|_H^2
\end{aligned} \tag{6.8}$$

for a.a. $t \in (0,T)$ and all $h \in (0,h_3)$. We derive from (C7), Lemma 4.1, the Young inequality and (C3) that there exists a constant $C_4 = C_4(T) > 0$ such that

$$\begin{aligned}
&-(\Phi\overline{\varphi}_h(t)-\Phi\varphi(t),\overline{v}_h(t)-v(t))_H\\
&\quad\le C_\Phi(1+\|\overline{\varphi}_h(t)\|_V^p+\|\varphi(t)\|_V^q)\|\overline{\varphi}_h(t)-\varphi(t)\|_V\|\overline{v}_h(t)-v(t)\|_H\\
&\quad\le C_4\|\overline{\varphi}_h(t)-\varphi(t)\|_V\|\overline{v}_h(t)-v(t)\|_H\\
&\quad\le \frac{C_4}{2}\|\overline{\varphi}_h(t)-\varphi(t)\|_V^2 + \frac{C_4}{2}\|\overline{v}_h(t)-v(t)\|_H^2\\
&\quad\le C_4\|\overline{\varphi}_h(t)-\widehat{\varphi}_h(t)\|_V^2 + C_4\|\widehat{\varphi}_h(t)-\varphi(t)\|_V^2\\
&\qquad + C_4\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H^2 + \frac{C_4}{c_L}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2
\end{aligned} \tag{6.9}$$

for a.a. $t \in (0,T)$ and all $h \in (0,h_3)$. It follows from (C13), the continuity of the embedding $V \hookrightarrow H$, the Young inequality and (C3) that there exists a constant $C_5 > 0$ satisfying

$$\begin{aligned}
&-(\mathcal{L}\overline{\varphi}_h(t)-\mathcal{L}\varphi(t),\overline{v}_h(t)-v(t))_H\\
&\quad\le C_5\|\overline{\varphi}_h(t)-\varphi(t)\|_V\|\overline{v}_h(t)-v(t)\|_H\\
&\quad\le \frac{C_5}{2}\|\overline{\varphi}_h(t)-\varphi(t)\|_V^2 + \frac{C_5}{2}\|\overline{v}_h(t)-v(t)\|_H^2\\
&\quad\le C_5\|\overline{\varphi}_h(t)-\widehat{\varphi}_h(t)\|_V^2 + C_5\|\widehat{\varphi}_h(t)-\varphi(t)\|_V^2\\
&\qquad + C_5\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H^2 + \frac{C_5}{c_L}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2.
\end{aligned} \tag{6.10}$$

The Young inequality and (C3) yield that

$$\begin{aligned}
&(\overline{\theta}_h(t)-\theta(t),\overline{v}_h(t)-v(t))_H \\
&\quad=(\overline{\theta}_h(t)-\widehat{\theta}_h(t),\overline{v}_h(t)-v(t))_H+(\widehat{\theta}_h(t)-\theta(t),\overline{v}_h(t)-\widehat{v}_h(t))_H \\
&\qquad+(\widehat{\theta}_h(t)-\theta(t),\widehat{v}_h(t)-v(t))_H \\
&\quad\le\|\overline{\theta}_h(t)-\widehat{\theta}_h(t)\|_H\|\overline{v}_h(t)-v(t)\|_H+\|\widehat{\theta}_h(t)-\theta(t)\|_H\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H \\
&\qquad+\frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2+\frac{1}{2c_L}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2. 
\end{aligned}\tag{6.11}$$

Thus we infer from (6.6) to (6.11), the integration over $(0,t)$, where $t\in[0,T]$, (1.8)–(1.10), Lemmas 4.2 and 4.5 that there exists a constant $C_6=C_6(T)>0$ such that

$$\begin{aligned}
&\frac{1}{2}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+\frac{1}{2}\|A_2^{1/2}(\widehat{\varphi}_h(t)-\varphi(t))\|_H^2+\int_0^t\|B^{1/2}(\overline{v}_h(s)-v(s))\|_H^2\,\mathrm{d}s \\
&\le C_6h+C_6\int_0^t\|\widehat{\varphi}_h(s)-\varphi(s)\|_V^2\,\mathrm{d}s+C_6\int_0^t\|L^{1/2}(\widehat{v}_h(s)-v(s))\|_H^2\,\mathrm{d}s \\
&\quad+C_6\int_0^t\|\widehat{\theta}_h(s)-\theta(s)\|_H^2\,\mathrm{d}s
\end{aligned}\tag{6.12}$$

for all $t\in[0,T]$ and all $h\in(0,h_3)$. On the other hand, we have from the identities $\frac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t}=\overline{v}_h$, $\frac{\mathrm{d}\varphi}{\mathrm{d}t}=v$, the Young inequality, (C3) and the continuity of the embedding $V_2\hookrightarrow H$ that there exists a constant $C_7>0$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widehat{\varphi}_h(t)-\varphi(t)\|_H^2 \\
&\quad=(\overline{v}_h(t)-v(t),\widehat{\varphi}_h(t)-\varphi(t))_H \\
&\quad\le\frac{1}{2}\|\overline{v}_h(t)-v(t)\|_H^2+\frac{1}{2}\|\widehat{\varphi}_h(t)-\varphi(t)\|_H^2 \\
&\quad\le\|\overline{v}_h(t)-\widehat{v}_h(t)\|_H^2+\frac{1}{c_L}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+C_7\|\widehat{\varphi}_h(t)-\varphi(t)\|_{V_2}^2
\end{aligned}\tag{6.13}$$

for a.a. $t\in(0,T)$ and all $h\in(0,h_3)$. Hence we derive from (6.12), the integration (6.13) over $(0,t)$, where $t\in[0,T]$, and (C11) that there exists a constant $C_8=C_8(T)>0$ satisfying

$$\begin{aligned}
&\frac{1}{2}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+\frac{\omega_1}{2}\|\widehat{\varphi}_h(t)-\varphi(t)\|_{V_2}^2+\int_0^t\|B^{1/2}(\overline{v}_h(s)-v(s))\|_H^2\,\mathrm{d}s \\
&\quad\le C_8h+C_8\int_0^t\|\widehat{\varphi}_h(s)-\varphi(s)\|_{V_2}^2\,\mathrm{d}s+C_8\int_0^t\|L^{1/2}(\widehat{v}_h(s)-v(s))\|_H^2\,\mathrm{d}s \\
&\qquad+C_8\int_0^t\|\widehat{\theta}_h(s)-\theta(s)\|_H^2\,\mathrm{d}s.
\end{aligned}\tag{6.14}$$

Therefore combining (6.5) and (6.14) means that there exists a constant $C_9=C_9(T)>0$ such that

$$\begin{aligned}
&\frac{1}{2}\|L^{1/2}(\widehat{v}_h(t)-v(t))\|_H^2+\frac{\omega_1}{2}\|\widehat{\varphi}_h(t)-\varphi(t)\|_{V_2}^2+\int_0^t\|B^{1/2}(\overline{v}_h(s)-v(s))\|_H^2\,\mathrm{d}s \\
&\quad+\frac{1}{2}\|\widehat{\theta}_h(t)-\theta(t)\|_H^2+\sigma_1\int_0^t\|\overline{\theta}_h(s)-\theta(s)\|_{V_1}^2\,\mathrm{d}s
\end{aligned}$$

$$\leq C_9 h + C_9 \int_0^t \|\widehat{\varphi}_h(s) - \varphi(s)\|_{V_2}^2 \, \mathrm{d}s + C_9 \int_0^t \|L^{1/2}(\widehat{v}_h(s) - v(s))\|_H^2 \, \mathrm{d}s$$
$$+ C_9 \|\overline{f}_h - f\|_{L^2(0,T;H)}^2 + C_9 \int_0^t \|\widehat{\theta}_h(s) - \theta(s)\|_H^2 \, \mathrm{d}s$$

for all $t \in [0,T]$ and all $h \in (0, h_3)$. Then, applying the Gronwall lemma, we can obtain Lemma 6.1. □

*Proof of Theorem 1.6.* Observing that there exists a constant $C_1 > 0$ such that

$$\|\overline{f}_h - f\|_{L^2(0,T;H)} \leq C_1 h^{1/2}$$

for all $h > 0$ (see [7], Sect. 5), we can prove Theorem 1.6 by Lemma 6.1.

□

**Remark 6.2.** Excluding $\frac{\mathrm{d}\varphi}{\mathrm{d}t}$ in (P), $\frac{\mathrm{d}\widehat{\varphi}_h}{\mathrm{d}t}$ in $(\mathrm{P})_h$, the second equations in (P) and $(\mathrm{P})_h$, $\varphi(0) = \varphi_0$ and $\frac{\mathrm{d}\varphi}{\mathrm{d}t}(0) = v_0$ in (P), $\widehat{\varphi}_h(0) = \varphi_0$ and $\widehat{v}_h(0) = v_0$ in $(\mathrm{P})_h$, we can establish a $O(h)$ error estimate between solutions of (P) and solutions of $(\mathrm{P})_h$ for $\theta_0 \in D(A_1)$ and $f \in L^2(0,T;V_1) \cap H^1(0,T;H)$ (*cf.* [15]). However, in view of what happens for the hyperbolic equation alone (*cf.* [6]), the $O(h^{1/2})$ error estimate in Theorem 1.6 seems to be optimal for $\theta_0 \in V_1$, $\varphi_0 \in D(B) \cap D(A_2)$, $v_0 \in D(B) \cap V_2$, $f \in L^2(0,T;H) \cap W^{1,1}(0,T;H)$. Thus deriving a $O(h)$ error estimate between solutions of (P) and solutions of $(\mathrm{P})_h$ in the case that $\theta_0$, $\varphi_0$, $v_0$ and $f$ are regular enough remains as an open problem.

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