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ANALYSIS OF THE MORLEY ELEMENT FOR THE CAHN–HILLIARD EQUATION AND THE HELE-SHAW FLOW

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Abstract. The paper analyzes the Morley element method for the Cahn–Hilliard equation. The objective is to prove the numerical interfaces of the Morley element method approximate the Hele-Shaw flow. It is achieved by establishing the optimal error estimates which depend on $\frac{1}{\epsilon}$ polynomially, and the error estimates should be established from lower norms to higher norms progressively. If the higher norm error bound is derived by choosing test function directly, we cannot obtain the optimal error order, and we cannot establish the error bound which depends on $\frac{1}{\epsilon}$ polynomially either. Different from the discontinuous Galerkin (DG) space [Feng *et al. SIAM J. Numer. Anal.* **54** (2016) 825–847], the Morley element space does not contain the finite element space as a subspace such that the projection theory does not work. The enriching theory is used in this paper to overcome this difficulty, and some nonstandard techniques are combined in the process such as the *a priori* estimates of the exact solution u, integration by parts in space, summation by parts in time, and special properties of the Morley elements. If one of these techniques is lacked, either we can only obtain the sub-optimal piecewise $L^\infty(H^2)$ error order, or we can merely obtain the error bounds which are exponentially dependent on $\frac{1}{\epsilon}$. Numerical results are presented to validate the optimal $L^\infty(H^2)$ error order and the asymptotic behavior of the solutions of the Cahn–Hilliard equation.

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1. Introduction

Consider the following Cahn–Hilliard equation with Neumann boundary conditions:

$$u_t + \Delta(\epsilon\Delta u - \frac{1}{\epsilon}f(u)) = 0 \quad \text{in} \quad \Omega_T := \Omega \times (0,T], \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(\epsilon\Delta u - \frac{1}{\epsilon}f(u)) = 0 \quad \text{on} \quad \partial\Omega_T := \partial\Omega \times (0,T], \tag{1.2}$$

$$u = u_0 \quad \text{in} \quad \Omega \times \{t = 0\}, \tag{1.3}$$

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where $\Omega \subseteq \mathbf{R}^2$ is a bounded domain, $f(u) = u^3 - u$ is the derivative of a double well potential $F(u)$ which is defined by

$$F(u) = \frac{1}{4}(u^2 - 1)^2. \tag{1.4}$$

The Allen–Cahn equation [3,6,12,16,17,21,23,24] and the Cahn–Hilliard equation [2,12,25,28,31] are two basic phase field models to describe the phase transition process. They are also proved to be related to geometric flows. For example, the zero-level sets of the Allen–Cahn equation approximate the mean curvature flow [15,24] and the zero-level sets of the Cahn–Hilliard equation approximate the following Hele-Shaw flow [2,29]:

$$\Delta w = 0 \qquad \text{in} \qquad \Omega \setminus \Gamma_t, t \in [0,T], \tag{1.5}$$

$$\frac{\partial w}{\partial n} = 0 \qquad \text{on} \qquad \partial\Omega, t \in [0,T], \tag{1.6}$$

$$w = \sigma\kappa \qquad \text{on} \qquad \Gamma_t, t \in [0,T], \tag{1.7}$$

$$V = \frac{1}{2}\left[\frac{\partial w}{\partial n}\right]_{\Gamma_t} \qquad \text{on} \qquad \Gamma_t, t \in [0,T], \tag{1.8}$$

$$\Gamma_0 = \Gamma_{00} \qquad \text{when} \qquad t = 0, \tag{1.9}$$

where κ and V are, respectively, the mean curvature and the normal velocity of the interface Γ_t, Γ_{00} is the initial interface, n is the unit outward normal to either $\partial\Omega$ or Γ_t, $\sigma = \int_{-1}^{1} \sqrt{F(s)/2}\,\mathrm{d}s$, and $[\frac{\partial w}{\partial n}]_{\Gamma_t}$ denotes the jump of normal derivatives of w across the interface Γ_t.

The Cahn–Hilliard equation was introduced by J. Cahn and J. Hilliard in [11] to describe the process of phase separation, by which the two components of a binary fluid separate and form domains pure in each component. It can be interpreted as the H^{-1}-gradient flow [2] of the Cahn–Hilliard energy functional

$$J_\epsilon(v) := \int_\Omega \Big(\frac{\epsilon}{2}|\nabla v|^2 + \frac{1}{\epsilon}F(v)\Big)\,\mathrm{d}x. \tag{1.10}$$

There are a few papers [4,13,14,30] discussing the error bounds, which depend on the exponential power of $\frac{1}{\epsilon}$, of the numerical methods for the Cahn–Hilliard equation. Such estimates are clearly not useful for small ϵ, specifically, in addressing the issue whether the computed numerical interfaces converge to the original sharp interface of the Hele-Shaw flow. Instead, the polynomial dependence on $\frac{1}{\epsilon}$ is proved in [18,19] using the C^0-conforming finite element method and mixed formulation, and in [22,26] using the discontinuous Galerkin method. Compared to mixed finite element methods or C^1-conforming finite element methods, the Morley finite element method is used to derive the error bounds which depend on $\frac{1}{\epsilon}$ polynomially in this paper due to its high efficiency. The main idea of the Morley elements is different from the ideas of the C^0-conforming finite elements and discontinuous Galerkin elements, *i.e.*, the theory used to establish the discrete spectrum results. Specifically, the discrete spectrum result in the C^0-conforming finite element space holds [18,19] since it is a conforming space; the discrete spectrum result in the discontinuous Galerkin space is proved [22,26] by the projection theory using the C^0-conforming finite element space as a subspace. It is worth noting that the existing studies [18,19,22,26] hinge on the mixed formulation of Cahn–Hilliard equation. In the Morley element space, however, the discrete spectrum result is proved by the enriching theory using its enriched conforming space based on the primal formulation (1.1)–(1.3).

The highlights of this paper are fourfold. Firstly, it establishes the piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds which depend on $\frac{1}{\epsilon}$ polynomially. If the Gronwall's inequality is used, we can only prove that the error bounds depend on $\frac{1}{\epsilon}$ exponentially, which can not be used to prove our main theorem. To prove these bounds, special properties of the Morley elements are explored, *i.e.*, Lemma 2.3 in [14], and piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error bounds [27] are required. Secondly, by making use of the piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds above, it establishes the piecewise $L^\infty(H^2)$ error bound which depends on $\frac{1}{\epsilon}$ polynomially. The crux here is to employ the summation by parts in time and integration by parts in space techniques simultaneously, together

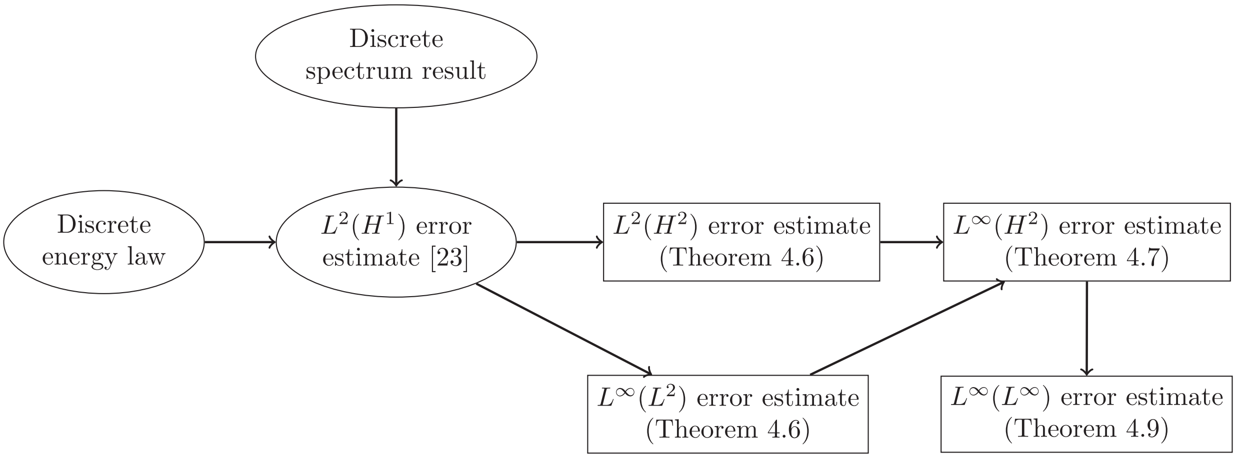

FIGURE 1. Outline of the analysis.

with the special properties of the Morley elements, to handle the nonlinear term. Otherwise, we can only get the error bound in Remark 4.8, which does not have an optimal order. Third, the minimal regularity of u is used, *i.e.*, $\|u_{tt}\|_{L^2(L^2)}$ regularity instead of $\|u_{tt}\|_{L^\infty(L^2)}$ regularity is used, and the *a priori* estimate is derived in Theorem 2.3. Fourth, the $L^\infty(L^\infty)$ error bound is established using the optimal piecewise $L^\infty(H^2)$ error, by which the main result that the zero-level sets of solutions of the Cahn–Hilliard equation approximate the Hele-Shaw flow is proved in Section 5. An outline of the analysis is depicted in Figure 1.

The organization of this paper is as follows. In Section 2, the standard Sobolev space notation is introduced, some useful lemmas are stated, and some new *a priori* estimates of the exact solution u are derived. In Section 3, the fully discrete approximation based on the Morley finite element space is presented. In Section 4, the polynomially dependent piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds are established first based on piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error bounds, and then the polynomially dependent piecewise $L^\infty(H^2)$ error bound is established based on piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds, by which the $L^\infty(L^\infty)$ error bound is proved. In Section 5, the approximation of the zero-level sets of the Cahn–Hilliard equation to the Hele-Shaw flow is proved. In Section 6, numerical tests are presented to validate our theoretical results, including the optimal error orders and the approximation of the Hele-Shaw flow.

2. Preliminaries

In this section, we present some results which will be used in the following sections. Throughout this paper, C denotes a generic positive constant which is independent of interfacial length ϵ, spacial size h, and time step size k, and it may have different values in different formulas. The standard Sobolev space notation below is used in this paper.

$$
\begin{aligned}
&\|v\|_{L^p,A} := \left(\int_A |v|^p \,\mathrm{d}x\right)^{1/p} \quad 1 \le p < \infty,\\
&\|v\|_{L^\infty,A} := \operatorname*{ess\,sup}_A |v|,\\
&|v|_{m,p,A} := \left(\sum_{|\alpha|=m} \|D^\alpha v\|^p_{L^p,A}\right)^{1/p}, \qquad \|v\|_{m,p,A} := \left(\sum_{j=0}^m |v|^p_{j,p,A}\right)^{1/p}, \qquad 1 \le p < \infty.
\end{aligned}
$$

Here A denotes some domain, *i.e.*, a single mesh element K or the whole domain Ω. When $A=\Omega$, $\|\cdot\|_{H^k}, \|\cdot\|_{L^p}$, and $|\cdot|_{m,p}$ are used to denote $\|\cdot\|_{H^k(\Omega)}, \|\cdot\|_{L^p(\Omega)}$, and $|\cdot|_{m,p,\Omega}$, respectively. Let $\mathcal{T}_h$ be a family of quasi-uniform triangulations of domain Ω with a mesh size h, and $\mathcal{E}_h$ be a collection of edges. Let $\mathcal{E}_h^i = \mathcal{E}_h \setminus \partial\Omega$ be the set of interior edges and $\mathcal{E}_h^\partial := \mathcal{E}_h \setminus \mathcal{E}_h^i$ be the set of boundary edges. The global mesh dependent semi-norm, norm and inner product are defined as below

$$|v|_{j,p,h} := \left(\sum_{K\in\mathcal{T}_h} |v|_{j,p,K}^p\right)^{1/p},$$

$$\|v\|_{j,p,h} := \left(\sum_{K\in\mathcal{T}_h} \|v\|_{j,p,K}^p\right)^{1/p},$$

$$(w,v)_h := \sum_{K\in\mathcal{T}_h} \int_K w(x)v(x)\,\mathrm{d}x.$$

Define $L_0^2(\Omega)$ as the mean zero functions in $L^2(\Omega)$. For $\Phi \in L_0^2(\Omega)$, let $u := (-\Delta)^{-1}\Phi \in H^1(\Omega)\cap L_0^2(\Omega)$ such that

$$\begin{aligned} -\Delta u &= \Phi \qquad \text{in} \quad \Omega,\\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on} \quad \partial\Omega. \end{aligned}$$

Then we have

$$-(\nabla\Delta^{-1}\Phi, \nabla v) = (\Phi, v) \quad \text{in} \quad \Omega \qquad \forall v \in H^1(\Omega)\cap L_0^2(\Omega). \tag{2.1}$$

For $v \in L_0^2(\Omega)$ and $\Phi \in L_0^2(\Omega)$, define the continuous H^{-1} inner product by

$$(\Phi, v)_{H^{-1}} := (\nabla\Delta^{-1}\Phi, \nabla\Delta^{-1}v) = (\Phi, -\Delta^{-1}v) = (v, -\Delta^{-1}\Phi). \tag{2.2}$$

As in [12, 18, 19, 22, 26, 27], we made the following assumptions on the initial condition. These assumptions were used to derive the *a priori* estimates for the solution of problem (1.1)–(1.4).

General Assumption (GA)

(1) Assume that $m_0 \in (-1,1)$ where

$$m_0 := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,\mathrm{d}x.$$

(2) There exists a nonnegative constant σ_1 such that

$$J_\epsilon(u_0) \le C\epsilon^{-2\sigma_1}.$$

(3) There exist nonnegative constants σ_2, σ_3 and σ_4 such that

$$\left\| -\epsilon\Delta u_0 + \epsilon^{-1} f(u_0)\right\|_{H^j} \le C\epsilon^{-\sigma_{2+j}} \qquad j = 0,1,2.$$

Under the above assumptions, the following *a priori* estimates of the solution were proved in [18, 19, 22, 26].

Theorem 2.1. *The solution u of problem (1.1)–(1.4) satisfies the following energy estimate:*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left(\frac{\epsilon}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{\epsilon}\|F(u)\|_{L^1}\right) + \int_0^T \|u_t(s)\|_{H^{-1}}^2\,\mathrm{d}s \le J_\epsilon(u_0). \tag{2.3}$$

Moreover, suppose that GA (1)–(3) hold, $u_0 \in H^4(\Omega)$ and $\partial\Omega \in C^{2,1}$, then u satisfies the additional estimates:

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x = m_0 \quad \forall t \ge 0, \tag{2.4}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\Delta u\|_{L^2} \le C\epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}, \tag{2.5}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla\Delta u\|_{L^2} \le C\epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}, \tag{2.6}$$

$$\epsilon\int_0^T \|\Delta u_t\|_{L^2}^2\,\mathrm{d}s + \operatorname*{ess\,sup}_{t\in[0,T]} \|u_t\|_{L^2}^2 \le C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}}. \tag{2.7}$$

Furthermore, if there exists $\sigma_5 > 0$ such that

$$\lim_{s\to 0^+} \|\nabla u_t(s)\|_{L^2} \le C\epsilon^{-\sigma_5}, \tag{2.8}$$

then there hold

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla u_t\|_{L^2}^2 + \epsilon\int_0^T \|\nabla\Delta u_t\|_{L^2}^2\,\mathrm{d}s \le C\rho_0(\epsilon), \tag{2.9}$$

$$\int_0^T \|u_{tt}\|_{H^{-1}}^2\,\mathrm{d}s \le C\rho_1(\epsilon), \tag{2.10}$$

where

$$\rho_0(\epsilon) := \epsilon^{-\frac{1}{2}\max\{2\sigma_1+5,2\sigma_3+2\}-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4\}} + \epsilon^{-2\sigma_5} + \epsilon^{-\max\{2\sigma_1+7,2\sigma_3+4\}},$$
$$\rho_1(\epsilon) := \epsilon\rho_0(\epsilon).$$

The next theorem shows the boundedness of the solution of the Cahn–Hilliard problem (1.1)–(1.4), provided that the Hele-Shaw flow (1.5)–(1.9) has a global (in time) classical solution. The proof was given in Lemma 2.2 [19] as a corollary of Theorems 2.1 and 2.3 in [2].

Theorem 2.2. *Suppose the Hele-Shaw problem (1.5)–(1.9) has a global (in time) classical solution. Then there exists a family of smooth initial datum functions $\{u_0^\epsilon(x)\}_{0<\epsilon\le 1}$, constants $\epsilon_0 \in (0,1]$, and C_0 such that for all $\epsilon \in (0,\epsilon_0)$ the solution u of the Cahn–Hilliard equation(1.1)–(1.4) with the above initial data u_0^ϵ satisfies*

$$\|u\|_{L^\infty} \le C_0. \tag{2.11}$$

We note that, as explained in the introduction of [2], this is not a serious restriction pertaining to the initial data $\{u_0^\epsilon(x)\}_{0<\epsilon\le 1}$. Hence, the above L^∞ bound of u is assumed in this paper. Besides, some extra *a priori* estimates of solution u are needed in this paper.

Theorem 2.3. *Under the assumptions of Theorem 2.1 and if there exists $\sigma_6 > 0$ such that*

$$\lim_{s\to 0^+} \|\Delta u_t(s)\|_{L^2} \le C\epsilon^{-\sigma_6}, \tag{2.12}$$

then there hold

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\Delta u_t\|_{L^2}^2 + \epsilon\int_0^T \|\Delta^2 u_t\|_{L^2}^2\,\mathrm{d}s \le C\rho_2(\epsilon), \tag{2.13}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \epsilon\|\Delta u_t\|_{L^2}^2 + \int_0^T \|u_{tt}\|_{L^2}^2\,\mathrm{d}s \le C\rho_3(\epsilon), \tag{2.14}$$

where

$$\rho_2(\epsilon) := \epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-\max\{2\sigma_1+5,2\sigma_3+2\}-4} + \epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}-3}\rho_0(\epsilon) + \epsilon^{-2\sigma_6},$$
$$\rho_3(\epsilon) := \epsilon\rho_2(\epsilon).$$

Proof. Using (2.11) and the Gagliardo–Nirenberg inequalities [1] in two-dimensional space, we have

$$\|\nabla u\|_{L^\infty} \le C\Big(\|\nabla\Delta u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^\infty}^{\frac{1}{2}} + \|u\|_{L^\infty}\Big) \le C\epsilon^{-\frac{1}{2}\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}. \tag{2.15}$$

Since $f'(u) = 3u^2 - 1$, we have

$$\begin{aligned}
&\int_0^T \|\Delta(f'(u)u_t)\|_{L^2}^2\,\mathrm{d}s \qquad (2.16)\\
&= \int_0^T \|6uu_t\Delta u + 12u\nabla u\cdot\nabla u_t + 6u_t\nabla u\cdot\nabla u + (3u^2-1)\Delta u_t\|_{L^2}^2\,\mathrm{d}s\\
&\le C\int_0^T \|\Delta u\|_{L^2}^2\|u_t\|_{L^\infty}^2\,\mathrm{d}s + C\int_0^T \|\nabla u\|_{L^\infty}^2\|\nabla u_t\|_{L^2}^2\,\mathrm{d}s + C\int_0^T \|\nabla u\|_{L^\infty}^4\|u_t\|_{L^2}^2\,\mathrm{d}s + C\int_0^T \|\Delta u_t\|_{L^2}^2\,\mathrm{d}s\\
&\le C\|\Delta u\|_{L^\infty(L^2)}^2\int_0^T \|u_t\|_{H^2}^2\,\mathrm{d}s + C\|\nabla u_t\|_{L^\infty(L^2)}^2\|\nabla u\|_{L^\infty(L^\infty)}^2\\
&\quad + C\|\nabla u\|_{L^\infty(L^\infty)}^4\|u_t\|_{L^\infty(L^2)}^2 + C\int_0^T \|\Delta u_t\|_{L^2}^2\,\mathrm{d}s\\
&\le C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-\max\{2\sigma_1+5,2\sigma_3+2\}-1}\\
&\quad + C\epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}\rho_0(\epsilon)\\
&\quad + C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-\max\{2\sigma_1+5,2\sigma_3+2\}}\\
&\quad + C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-1}\\
&\le C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-\max\{2\sigma_1+5,2\sigma_3+2\}-1} + C\epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}}\rho_0(\epsilon),
\end{aligned}$$

where the L^∞ bound (2.11) is used in the first inequality, the Sobolev embedding theorem [1] is used in the second inequality, and equations (2.5), (2.7), (2.9), (2.15) are used in the third inequality.

Taking the derivative with respect to t on both sides of (1.1), we get

$$u_{tt} + \epsilon\Delta^2 u_t - \frac{1}{\epsilon}\Delta(f'(u)u_t) = 0. \tag{2.17}$$

Testing (2.17) with $\Delta^2 u_t$, and taking the integral over $(0,t^*)$ for any $0 \le t^* \le T$, we obtain by (2.12) that

$$\begin{aligned}
\frac{1}{2}\|\Delta u_t(t^*)\|_{L^2}^2 + \epsilon\int_0^{t^*}\|\Delta^2 u_t\|_{L^2}^2\,\mathrm{d}s &= \frac{1}{\epsilon}\int_0^{t^*}(\Delta(f'(u)u_t),\Delta^2 u_t)\,\mathrm{d}s + \frac{1}{2}\|\Delta u_t(0)\|_{L^2}^2 \qquad (2.18)\\
&\le \frac{C}{\epsilon^3}\int_0^{t^*}\|\Delta(f'(u)u_t)\|_{L^2}^2\,\mathrm{d}s + \frac{\epsilon}{2}\int_0^{t^*}\|\Delta^2 u_t\|_{L^2}^2\,\mathrm{d}s + C\epsilon^{-2\sigma_6}.
\end{aligned}$$

Then (2.13) is obtained by (2.16) and the fact that (2.18) holds for any $0 \le t^* \le T$.

Next we bound (2.14). Testing (2.17) with u_{tt}, taking the integral over $(0,t^*)$ for any $0 \le t^* \le T$, and using (2.18), we obtain

$$\int_0^{t^*}\|u_{tt}\|_{L^2}^2\,\mathrm{d}s + \frac{\epsilon}{2}\|\Delta u_t(t^*)\|_{L^2}^2 \le \frac{\epsilon}{2}\|\Delta u_t(0)\|_{L^2}^2 + \frac{C}{\epsilon^2}\int_0^{t^*}\|\Delta(f'(u)u_t)\|_{L^2}^2\,\mathrm{d}s + \frac{1}{2}\int_0^{t^*}\|u_{tt}\|_{L^2}^2\,\mathrm{d}s. \tag{2.19}$$

Then (2.14) is obtained by (2.12), (2.16), and the fact that (2.19) holds for any $0 \le t^* \le T$. $\square$

Remark 2.4. The *a priori* estimates of the solution in Theorem 2.1 hold in 2D and 3D, whereas the *a priori* estimates of the solution in Theorem 2.3 can be proved only in 2D due to the Gagliardo–Nirenberg inequalities used in (2.15). We also note that the upper bounds in these estimates may not be sharp on ϵ.

The next lemma gives an ϵ-independent lower bound for the principal eigenvalue of the linearized Cahn–Hilliard operator $\mathcal{L}_{CH}$ which is defined as below. The proof of this lemma can be found in [12].

Lemma 2.5. *Suppose that GA (1)–(3) hold. Given a smooth initial curve/surface Γ_0, let u_0 be a smooth function which satisfies $\Gamma_0 = \{x \in \Omega; u_0(x) = 0\}$ and some profiles described in [12]. Let u be the solution to the problem* (1.1)–(1.4). *Define $\mathcal{L}_{CH}$ as*

$$\mathcal{L}_{CH} := \Delta\left(\epsilon\Delta - \frac{1}{\epsilon}f'(u)I\right).$$

Then there exist $0 < \epsilon_0 \ll 1$ and a positive constant C_0 such that the principal eigenvalue of the linearized Cahn–Hilliard operator $\mathcal{L}_{CH}$ satisfies

$$\lambda_{CH} := \inf_{\substack{0\neq\psi\in H^1(\Omega)\\ \Delta w=\psi}} \frac{\epsilon\|\nabla\psi\|_{L^2}^2 + \frac{1}{\epsilon}(f'(u)\psi,\psi)}{\|\nabla w\|_{L^2}^2} \geq -C_0$$

for $t \in [0,T]$ and $\epsilon \in (0,\epsilon_0)$.

3. Fully discrete approximation

In this section, the backward Euler method is used for the time stepping, and the Morley finite element method is used for the space discretization.

3.1. Morley finite element space

Define the Morley finite element spaces S^h as below [8, 10, 14]:

$$S^h := \Big\{v_h \in L^\infty(\Omega) : v_h \in P_2(K), v_h \text{ is continuous at the vertices of all triangles,}$$
$$\frac{\partial v_h}{\partial n} \text{ is continuous at the midpoints of interelement edges of triangles}\Big\}.$$

We use the following notation

$$H_E^j(\Omega) := \left\{v \in H^j(\Omega) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\right\} \qquad j = 1,2,3.$$

Corresponding to $H_E^j(\Omega)$, we define S_E^h as a subspace of S^h as below:

$$S_E^h := \left\{v_h \in S^h : \frac{\partial v_h}{\partial n} = 0 \text{ at the midpoints of the edges on } \partial\Omega\right\}.$$

We also define $\mathring{H}_E^j(\Omega) = H_E^j(\Omega) \cap L_0^2(\Omega), j = 1,2,3,$ and $\mathring{S}_E^h = S_E^h \cap L_0^2(\Omega)$, where $L_0^2(\Omega)$ denotes the set of mean zero functions in $L^2(\Omega)$.

The enriching operator $\widetilde{E}_h$ [7, 8, 10] is restated here. Let $\widetilde{S}_E^h$ be the Hsieh–Clough–Tocher macro element space, which is an enriched space of the Morley finite element space S_E^h. Let p and m be the internal vertices and midpoints of triangles $\mathcal{T}_h$. Define $\widetilde{E}_h : S_E^h \to \widetilde{S}_E^h$ by

$$\begin{aligned}(\widetilde{E}_h v)(p) &= v(p),\\ \frac{\partial(\widetilde{E}_h v)}{\partial n}(m) &= \frac{\partial v}{\partial n}(m),\\ (\partial^\beta(\widetilde{E}_h v))(p) &= \text{average of } (\partial^\beta v_i)(p) \qquad |\beta| = 1,\end{aligned}$$

where $v_i = v|_{T_i}$ and triangle T_i contains p as a vertex.

Define the interpolation operator $I_h : H_E^2(\Omega) \to S_E^h$ such that

$$\begin{aligned}(I_h v)(p) &= v(p),\\ \frac{\partial(I_h v)}{\partial n}(m) &= \frac{1}{|e|}\int_e \frac{\partial v}{\partial n}\,\mathrm{d}S,\end{aligned}$$

where p ranges over the internal vertices of all the triangles T, and m ranges over the midpoints of all the edges e. It can be proved that [7, 8, 10, 14]

$$|v - I_h v|_{j,p,K} \le Ch^{3-j}|v|_{3,p,K} \qquad \forall K \in \mathcal{T}_h, \quad \forall v \in H^3(K), \quad j = 0,1,2, \tag{3.1}$$

$$\|\widetilde{E}_h v - v\|_{j,2,h} \le Ch^{2-j}|v|_{2,2,h} \qquad \forall v \in S_E^h, \qquad j = 0,1,2. \tag{3.2}$$

Notice that $\widetilde{E}_h$ and I_h cannot preserve the mean zero functions. Let $\mathring{\widetilde{S}}{}_E^h := \widetilde{S}_E^h \cap L_0^2(\Omega)$. Define $\mathring{\widetilde{E}}_h : \mathring{S}_E^h \to \mathring{\widetilde{S}}{}_E^h$ such that

$$\mathring{\widetilde{E}}_h v = \widetilde{E}_h v - \frac{1}{|\Omega|}\int_\Omega \widetilde{E}_h v\,\mathrm{d}x \qquad \forall v \in \mathring{S}_E^h. \tag{3.3}$$

Using (3.2), we have

$$\int_\Omega \widetilde{E}_h v\,\mathrm{d}x = (\widetilde{E}_h v - v, 1) \le |\Omega|^{1/2}\|\widetilde{E}_h v - v\|_{L^2} \le Ch^2|v|_{2,2,h} \qquad \forall v \in \mathring{S}_E^h.$$

Then

$$\|\mathring{\widetilde{E}}_h v - v\|_{j,2,h} \le Ch^{2-j}|v|_{2,2,h} \qquad \forall v \in \mathring{S}_E^h, \quad j = 0,1,2. \tag{3.4}$$

Finally, the following spaces are needed

$$H^{i,h}(\Omega) = S^h \oplus H^i(\Omega), \qquad H_E^{i,h}(\Omega) = S_E^h \oplus H_E^i(\Omega), \quad i = 1,2,3,$$

where, for instance,

$$S_E^h \oplus H_E^i(\Omega) := \left\{u + v : u \in S_E^h \ \text{ and } \ v \in H_E^i(\Omega)\right\}, \quad i = 1,2,3.$$

3.2. Formulation

The weak form of (1.1)–(1.4) is to seek $u(\cdot,t) \in H_E^2(\Omega)$ such that

$$(u_t, v) + \epsilon a(u,v) + \frac{1}{\epsilon}(\nabla f(u), \nabla v) = 0 \quad \forall v \in H_E^2(\Omega), \tag{3.5}$$

$$u(\cdot,0) = u_0 \in H_E^2(\Omega), \tag{3.6}$$

where the bilinear form $a(\cdot,\cdot)$ is defined as

$$a(u,v) := \int_\Omega \Delta u \Delta v + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} \right) \mathrm{d}x\mathrm{d}y \tag{3.7}$$

with Poisson's ratio $\frac{1}{2}$. Next, we define the discrete bilinear form

$$a_h(u,v) := \sum_{K\in\mathcal{T}_h} \int_K \Delta u \Delta v + \left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2} \right) \mathrm{d}x\mathrm{d}y. \tag{3.8}$$

Denote k as the time step size and $\ell = T/k$ as the number of steps. Based on the bilinear form (3.8), a fully discrete Galerkin method is to seek $u_h^n \in S_E^h$ such that for $n = 1, 2, \cdots, \ell$,

$$(d_t u_h^n, v_h) + \epsilon a_h(u_h^n, v_h) + \frac{1}{\epsilon}(\nabla f(u_h^n), \nabla v_h)_h = 0 \quad \forall v_h \in S_E^h, \tag{3.9}$$

$$u_h^0 = u_0^h \in S_E^h, \tag{3.10}$$

where the difference operator $d_t u_h^n := \frac{u_h^n - u_h^{n-1}}{k}$ and $u_0^h := P_h u_0$. Here the operator P_h is defined as below.

3.3. Elliptic operator P_h

We define

$$R := \left\{ v \in H_E^2(\Omega) : \Delta v \in H_E^2(\Omega) \right\}.$$

Then for any $v \in R$, define the elliptic operator P_h (*cf.* [14]) by seeking $P_h v \in S_E^h$ such that

$$\tilde{b}_h(P_h v, w) := \left(\epsilon \Delta^2 v - \frac{1}{\epsilon}\nabla \cdot (f'(u)\nabla v) + \alpha v, w \right) \qquad \forall w \in S_E^h, \tag{3.11}$$

where

$$\tilde{b}_h(v,w) := \epsilon a_h(v,w) + \frac{1}{\epsilon}(f'(u)\nabla v, \nabla w)_h + \alpha(v,w), \tag{3.12}$$

and α should be chosen as $\alpha = \alpha_0 \epsilon^{-3}$ to guarantee the coercivity of $\tilde{b}_h(\cdot,\cdot)$. More precisely, we cite some lemmas in [14] first, which will be used in this paper.

Lemma 3.1 ([14], Lem. 2.3)**.** *Let $w, z \in H_E^{2,h}(\Omega)$, then*

$$\left| \sum_{K\in\mathcal{T}_h} \int_{\partial K} \frac{\partial w}{\partial n} z \,\mathrm{d}S \right| \le Ch(h\|w\|_{2,2,h}\|z\|_{2,2,h} + \|w\|_{1,2,h}\|z\|_{2,2,h} + \|w\|_{2,2,h}\|z\|_{1,2,h}).$$

Lemma 3.2 ([14], Lem. 2.5)**.** *Let $z \in H^{2,h}(\Omega)$ and $w \in H_E^2(\Omega) \cap H^3(\Omega)$, and define $B_h(w,z)$ by*

$$B_h(w,z) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\Delta w \frac{\partial z}{\partial n} + \frac{1}{2}\frac{\partial^2 w}{\partial n \partial s}\frac{\partial z}{\partial s} - \frac{1}{2}\frac{\partial^2 w}{\partial s^2}\frac{\partial z}{\partial n} \right) \mathrm{d}S,$$

then we have

$$|B_h(w,z)| \le Ch|w|_{3,2,h}|z|_{2,2,h}. \tag{3.13}$$

For any $w \in S_E^h$, using Lemma 3.1 and the inverse inequalities that $\|w\|_{2,2,h} \le Ch^{-2}\|w\|_{L^2}$ and $\|w\|_{1,2,h} \le Ch^{-1}\|w\|_{L^2}$, we have

$$\begin{aligned}|w|_{1,2,h}^2 &\le |w|_{2,2,h}\|w\|_{L^2} + \left|\sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\partial w}{\partial n} w\,\mathrm{d}S\right| \le C\|w\|_{2,2,h}\|w\|_{L^2}\\ &\le C\left(|w|_{2,2,h}\|w\|_{L^2} + |w|_{1,2,h}\|w\|_{L^2} + \|w\|_{L^2}^2\right)\\ &\le C\left(|w|_{2,2,h}\|w\|_{L^2} + \left(1+\frac{C}{2}\right)\|w\|_{L^2}^2\right) + \frac{1}{2}|w|_{1,2,h}^2,\end{aligned}$$

which gives

$$|w|_{1,2,h}^2 \le C(|w|_{2,2,h}\|w\|_{L^2} + \|w\|_{L^2}^2).$$

Since $f'(u) = 3u^2 - 1 \ge -1$ and $a_h(w,w) = \frac{1}{2}(\|\Delta w\|_{L^2}^2 + |w|_{2,2,h}^2)$, thanks to the Young's inequality, we have

$$\begin{aligned}\tilde{b}_h(w,w) &= \epsilon a_h(w,w) + \frac{1}{\epsilon}(f'(u)\nabla w, \nabla w) + \frac{\alpha_0}{\epsilon^3}(w,w)\\ &\ge \frac{1}{\epsilon^3}\left(\frac{\epsilon^4}{2}|w|_{2,2,h}^2 - C\epsilon^2|w|_{1,2,h}^2 + \alpha_0\|w\|_{L^2}^2\right)\\ &\ge \frac{1}{\epsilon^3}\left(\frac{\epsilon^4}{4}|w|_{2,2,h}^2 + (\alpha_0 - C)\|w\|_{L^2}^2\right),\end{aligned} \tag{3.14}$$

which implies the coercivity of $\tilde{b}_h(\cdot,\cdot)$ when α_0 is large enough but independent of ϵ.

Next, we give the properties of P_h. Define $b_h(\cdot,\cdot) := \epsilon^3\tilde{b}_h(\cdot,\cdot)$ and a norm

$$|||v|||_{2,2,h}^2 := \epsilon^4|v|_{2,2,h}^2 + \epsilon^2|v|_{1,2,h}^2 + \|v\|_{L^2}^2 \qquad \forall v \in H_E^{2,h}(\Omega).$$

Lemma 3.3. *Suppose $v \in H_E^2(\Omega)\cap H^3(\Omega)$ and $v_h \in S_E^h$ solve the following problems:*

$$b_h(v,\eta) = F_h(\eta) \quad \forall \eta \in H_E^2(\Omega), \tag{3.15}$$

$$b_h(v_h,\chi) = \widetilde{F}_h(\chi) \quad \forall \chi \in S_E^h, \tag{3.16}$$

where F_h and $\widetilde{F}_h$ are continuous linear functionals in $H_E^2(\Omega)$ and S_E^h, respectively. Then we have

$$|||v - v_h|||_{2,2,h} \le Ch\left\{(\epsilon+h)^2|v|_{3,2} + |v|_{1,2} + \sup_{\chi\in S_E^h}\frac{F_h(\widetilde{E}_h\chi) - \widetilde{F}_h(\chi) + \alpha_0(v, \chi - \widetilde{E}_h\chi)}{|||\chi|||_{2,2,h}}\right\}. \tag{3.17}$$

Proof. Using (3.14), (3.15) and the Strang Lemma, we have

$$\begin{aligned}&|||v - v_h|||_{2,2,h}\\ &\le C\left(\inf_{\psi\in S_E^h}|||v-\psi|||_{2,2,h} + \sup_{\chi\in S_E^h}\frac{b_h(v,\chi) - \widetilde{F}_h(\chi)}{|||\chi|||_{2,2,h}}\right)\\ &\le C\left(\inf_{\psi\in S_E^h}|||v-\psi|||_{2,2,h} + \sup_{\chi\in S_E^h}\frac{b_h(v,\chi-\widetilde{E}_h\chi) + b_h(v,\widetilde{E}_h\chi) - \widetilde{F}_h(\chi)}{|||\chi|||_{2,2,h}}\right)\\ &\le C\left(\inf_{\psi\in S_E^h}|||v-\psi|||_{2,2,h} + \sup_{\chi\in S_E^h}\frac{b_h(v,\chi-\widetilde{E}_h\chi) + F_h(\widetilde{E}_h\chi) - \widetilde{F}_h(\chi)}{|||\chi|||_{2,2,h}}\right).\end{aligned}$$

Since $v \in H^3(\Omega)$, using integration by parts yields

$$a_h(v,w) = (-\nabla\Delta v, \nabla w)_h + B_h(v,w) \qquad \forall w \in H^{2,h}(\Omega),$$

using Lemma 3.2 and (3.2), we have

$$\begin{aligned} b_h(v, \chi - \widetilde{E}_h\chi) &= \epsilon^4 a_h(v, \chi - \widetilde{E}_h\chi) + \epsilon^2 (f'(u)\nabla v, \nabla(\chi - \widetilde{E}_h\chi)) + (\alpha_0 v, \chi - \widetilde{E}_h\chi) \\ &\le Ch\left(\epsilon^4 |v|_{3,2}|\chi|_{2,2,h} + \epsilon^2 |v|_{1,2}|\chi|_{2,2,h}\right) + (\alpha_0 v, \chi - \widetilde{E}_h\chi) \\ &\le Ch\left(\epsilon^2 |v|_{3,2} + |v|_{1,2}\right) |||\chi|||_{2,2,h} + (\alpha_0 v, \chi - \widetilde{E}_h\chi). \end{aligned}$$

Then we obtain the desired bound (3.17) by taking $\psi = I_h v$ and using the approximation properties of Morley interpolation operator (3.1). □

Theorem 3.4. *Suppose u solves the Cahn–Hilliard problem (1.1)–(1.4), then we have*

$$\begin{aligned} &\epsilon^2 |u - P_h u|_{2,2,h} + \epsilon |u - P_h u|_{1,2,h} + \|u - P_h u\|_{L^2} \qquad (3.18) \\ &\qquad \le Ch\big((\epsilon + h)^2 |u|_{3,2} + |u|_{1,2} + \epsilon h^2 \|u_t\|_{L^2}\big), \\ &\epsilon^2 |u_t - (P_h u)_t|_{2,2,h} + \epsilon |u_t - (P_h u)_t|_{1,2,h} + \|u_t - (P_h u)_t\|_{L^2} \qquad (3.19) \\ &\qquad \le Ch\Big\{(\epsilon + h)^2 |u_t|_{3,2} + |u_t|_{1,2} + \epsilon h \|u_{tt}\|_{L^2} + \|u_t \nabla u\|_{L^2} \\ &\qquad\quad + \epsilon^{-1} |\ln h|^{1/2} \|u_t\|_{L^2} ((\epsilon + h)^2 |u|_{3,2} + |u|_{1,2} + \epsilon h^2 \|u_t\|_{L^2})\Big\}. \end{aligned}$$

Proof. Taking $v = u$ and $v_h = P_h u$ in Lemma 3.3, and noticing that for any $\psi \in H_E^{2,h}(\Omega)$,

$$F_h(\psi) = \tilde{F}_h(\psi) = (\epsilon^4 \Delta^2 u - \epsilon^2 \Delta f(u) + \alpha_0 u, \psi) = (-\epsilon^3 u_t + \alpha_0 u, \psi),$$

we obtain the bound (3.18) from (3.2) and (3.17).

Taking $v = u_t$ and $v_h = (P_h u)_t$, we have that for any $\psi \in H_E^{2,h}(\Omega)$,

$$\begin{aligned} F_h(\psi) &= (\epsilon^4 \Delta^2 u_t - \epsilon^2 \Delta f(u)_t + \alpha_0 u_t, \psi) - (\epsilon^2 f''(u) u_t \nabla u, \nabla\psi)_h, \\ \widetilde{F}_h(\psi) &= (\epsilon^4 \Delta^2 u_t - \epsilon^2 \Delta f(u)_t + \alpha_0 u_t, \psi) - (\epsilon^2 f''(u) u_t \nabla P_h u, \nabla\psi)_h. \end{aligned}$$

Then, using (3.2), (3.18), and the fact that $\epsilon^4 \Delta^2 u_t - \epsilon^2 \Delta f(u)_t = -\epsilon^3 u_{tt}$, we get

$$\begin{aligned} &F_h(\widetilde{E}_h\chi) - \widetilde{F}_h(\chi) + \alpha_0(u_t, \chi - \widetilde{E}_h\chi) \\ &\quad = (\epsilon^4 \Delta^2 u_t - \epsilon^2 \Delta f(u)_t, \widetilde{E}_h\chi - \chi) - (\epsilon^2 f''(u) u_t \nabla u, \nabla \widetilde{E}_h\chi - \nabla\chi) - (\epsilon^2 f''(u) u_t \nabla(u - P_h u), \nabla\chi) \\ &\quad \le \epsilon^3 h^2 \|u_{tt}\|_{L^2} |\chi|_{2,2,h} + C\epsilon^2 h \|u_t \nabla u\|_{L^2} |\chi|_{2,2,h} + C\epsilon^2 \|u_t\|_{L^2} \|\nabla\chi\|_{L^\infty} |u - P_h u|_{1,2,h} \\ &\quad \le Ch\Big\{\epsilon h \|u_{tt}\|_{L^2} + \|u_t \nabla u\|_{L^2} \\ &\qquad + \epsilon^{-1} |\ln h|^{1/2} \|u_t\|_{L^2} ((\epsilon + h)^2 |u|_{3,2} + |u|_{1,2} + \epsilon h^2 \|u_t\|_{L^2})\Big\} |||\chi|||_{2,2,h}, \end{aligned}$$

where we use the facts that $\nabla\chi$ belongs to the Crouzeix-Raviart finite element space and that the discrete Sobolev inequality $\|\nabla\chi\|_{L^\infty} \le C|\ln h|^{1/2} \|\chi\|_{2,2,h}$ (*cf.* [9], Sect. 2.4.2). This implies the bound (3.19). □

Combining with the *a priori* estimates given in Section 2, we have the following theorem.

Theorem 3.5. *Assume $h \le C\epsilon$, then there hold*

$$\epsilon^4 |u - P_h u|^2_{2,2,h} + \epsilon^2 |u - P_h u|^2_{1,2,h} + \|u - P_h u\|^2_{L^2} \le Ch^2 \rho_4(\epsilon), \qquad (3.20)$$

$$\int_0^T \epsilon^4 |u_t - (P_h u)_t|^2_{2,2,h} + \epsilon^2 |u_t - (P_h u)_t|^2_{1,2,h} + \|u_t - (P_h u)_t\|^2_{L^2} \, \mathrm{d}s \le Ch^2 \epsilon^4 \rho_3(\epsilon) + Ch^2 |\ln h| \rho_5(\epsilon), \qquad (3.21)$$

where

$$\begin{aligned} \rho_4(\epsilon) &:= \epsilon^{-\max\{2\sigma_1 + \frac{13}{2}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} + \frac{11}{2}}, \\ \rho_5(\epsilon) &:= \epsilon^{-2\max\{2\sigma_1 + \frac{13}{2}, 2\sigma_3 + \frac{7}{2}, 2\sigma_2 + 4, 2\sigma_4\} + 2}. \end{aligned}$$

Proof. Using (2.3), (2.6) and (2.7), we have

$$\begin{aligned}&(\epsilon+h)^4|u|_{3,2}^2+|u|_{1,2}^2+\epsilon^2h^4\|u_t\|_{L^2}^2\\&\le C\epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}+4}+C\epsilon^{-2\sigma_1-1}+C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}+6}\\&\le C\rho_4(\epsilon),\end{aligned}\tag{3.22}$$

which implies the bound (3.20) by (3.18).

Using (2.7), (2.9), (2.14) and (2.15), we obtain

$$\begin{aligned}&\int_0^T(\epsilon+h)^4|u_t|_{3,2}^2+|u_t|_{1,2}^2+\epsilon^2h^2\|u_{tt}\|_{L^2}^2+\|u_t\nabla u\|_{L^2}^2\,\mathrm{d}s\\&\le C\int_0^T\epsilon^4|u_t|_{3,2}^2+|u_t|_{1,2}^2+\epsilon^4\|u_{tt}\|_{L^2}^2+\|u_t\|_{L^2}^2\|\nabla u\|_{L^\infty}^2\,\mathrm{d}s\\&\le C\epsilon^3\rho_0(\epsilon)+C\rho_0(\epsilon)+C\epsilon^4\rho_3(\epsilon)+C\epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}}\\&\le C\epsilon^4\rho_3(\epsilon).\end{aligned}$$

Further, using (2.7) and (3.22), we obtain

$$\int_0^T\epsilon^{-2}\|u_t\|_{L^2}^2((\epsilon+h)^2|u|_{3,2}+|u|_{1,2}+\epsilon h^2\|u_t\|_{L^2})^2\,\mathrm{d}s\le C\rho_5(\epsilon).$$

This implies the bound (3.21). □

Corollary 3.6. *Under the conditions that*

$$h\le C\epsilon^2\rho_4^{-\frac{1}{2}}(\epsilon),\quad h\le C\rho_3^{-\frac{1}{2}}(\epsilon),\quad h|\ln h|^{\frac{1}{2}}\le C\epsilon^2\rho_5^{-\frac{1}{2}}(\epsilon),\tag{3.23}$$

there hold

$$|P_hu|_{j,2,h}^2\le C(1+|u|_{j,2,h}^2)\quad j=0,1,2,\tag{3.24}$$

$$\int_0^T|(P_hu)_t|_{j,2,h}^2\,\mathrm{d}s\le C(1+\int_0^T|u_t|_{j,2,h}^2)\quad j=0,1,2,\tag{3.25}$$

$$\|P_hu\|_{L^\infty}\le C.\tag{3.26}$$

Proof. By the Sobolev embedding and (3.20), we have

$$\|P_hu\|_{L^\infty}\le\|u\|_{L^\infty}+\|u-P_hu\|_{2,2,h}\le C+Ch\epsilon^{-2}\rho_4^{1/2}(\epsilon)\le C.$$

The first two bounds are the direct consequences of Theorem 3.5. □

Remark 3.7. We note that the conditions on h in Theorem 3.5 and Corollary 3.6 depend on ϵ polynomially, which is good enough to achieve the goal of this paper. See Theorem 4.9 and Remark 4.10 below.

4. Error estimates

In this section, first, we derive the piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds which depend on $\frac{1}{\epsilon}$ polynomially based on the generalized coercivity result shown in Theorem 4.4, and piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error bounds. Then we prove the piecewise $L^\infty(H^2)$ error bound based on the piecewise $L^\infty(L^2)$ and $L^2(H^2)$ error bounds. Finally, the $L^\infty(L^\infty)$ error bound is established.

Decompose the error at $t_n=nk$

$$u(t_n)-u_h^n=(u(t_n)-P_hu(t_n))+(P_hu(t_n)-u_h^n):=\phi^n+\theta^n.\tag{4.1}$$

The following two lemmas will be used in this section.

Lemma 4.1 (Summation by parts)**.** *Suppose* $\{a_n\}_{n=0}^{\ell}$ *and* $\{b_n\}_{n=0}^{\ell}$ *are two sequences, then*

$$\sum_{n=1}^{\ell}(a^n - a^{n-1}, b^n) = (a^\ell, b^\ell) - (a^0, b^0) - \sum_{n=1}^{\ell}(a^{n-1}, b^n - b^{n-1}).$$

Lemma 4.2. *Suppose* u *to be the solution of* (1.1)–(1.4)*, and* u_h^n *to be the solution of* (3.9)–(3.10)*, then*

$$\int_\Omega \phi^n \,\mathrm{d}x = 0, \quad \theta^n \in \mathring{S}_E^h.$$

Proof. Testing (1.1) with constant 1, and then taking the integration over $(0, t)$, we can obtain for any $t \geq 0$,

$$\int_\Omega u(t)\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x.$$

Then choosing $v = u(t), w = 1$ in (3.11), we have that for any $t \geq 0$,

$$\int_\Omega P_h u(t)\,\mathrm{d}x = \int_\Omega u(t)\,\mathrm{d}x.$$

Choosing $v_h = 1$ in (3.9), and then

$$\int_\Omega u_h^n\,\mathrm{d}x = \int_\Omega u_h^{n-1}\,\mathrm{d}x = \cdots = \int_\Omega u_h^0\,\mathrm{d}x.$$

Therefore, by choosing $u_h^0 = P_h u_0$, we have

$$\begin{aligned}\int_\Omega u_h^n\,\mathrm{d}x &= \int_\Omega u_h^0\,\mathrm{d}x = \int_\Omega P_h u_0\,\mathrm{d}x \\ &= \int_\Omega u_0\,\mathrm{d}x = \int_\Omega u(t_n)\,\mathrm{d}x = \int_\Omega P_h u(t_n)\,\mathrm{d}x.\end{aligned}$$

Hence, $\theta^n = P_h u(t_n) - u_h^n \in \mathring{S}_E^h$. ∎

4.1. Generalized coercivity result, piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error estimates

We first cite the generalized coercivity result, and piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error estimates established in [27]. These results are given under the following assumption: There exists $\gamma_1 \geq 0$ such that

$$\|u_h^n\|_{L^\infty} \leq C\epsilon^{-\gamma_1}. \tag{4.2}$$

We note that the assumption (4.2) has been numerically verified in [27]. This improves the assumption on $\|u_h^n\|_{1,\infty}$ in [14]. In the rest of this subsection, we state some main results in [27] under the mesh constraints

$$\begin{aligned} &h \leq C\epsilon^2 k, && k \leq C\epsilon^{3\sigma_1+13}, \\ &h \leq C\epsilon^{4\gamma_1+4}, && h \leq (C_1C_2)^{-1}\epsilon^{\gamma_3+3}, \end{aligned} \tag{4.3}$$

where the positive constant $\gamma_3 > 0$ is assumed to satisfy that, for the solution u of the Cahn–Hilliard problem (1.1)–(1.4) and the elliptic operator P_h,

$$\|u - P_h u\|_{L^\infty((0,T);L^\infty)} \leq C_1 h\epsilon^{-\gamma_3}. \tag{4.4}$$

Remark 4.3. Thanks to the Sobolev embedding theorem and (3.20), we have

$$\|u - P_h u\|_{L^\infty} \le \|u - P_h u\|_{2,2,h} \le Ch\epsilon^{-2}\rho_4^{\frac{1}{2}}(\epsilon), \tag{4.5}$$

which gives the explicit form of γ_3 in (4.4).

The first result that will be used in this paper is the following stability result ([27], Thm. 4) under the mesh conditions (4.3)

$$\|u_h^n\|_{2,2,h} \le C\epsilon^{-\gamma_2}, \quad \text{where } \gamma_2 := 2\gamma_1 + \sigma_1 + 6. \tag{4.6}$$

Theorem 4.4 (Generalized coercivity/Discrete spectrum [27], Thm. 5)**.** *Suppose there exists a positive number $\gamma_3 > 0$ such that (4.4) holds. Then, under the mesh constraints (4.3), there exists an ϵ-independent and h-independent constant $C > 0$ such that for $\epsilon \in (0, \epsilon_0)$, a.e. $t \in [0, T]$, and any $\psi \in \mathring{S}_E^h$,*

$$(\epsilon - \epsilon^4)(\nabla\psi, \nabla\psi)_h + \frac{1}{\epsilon}(f'(P_h u(t))\psi, \psi)_h \ge -C\|\nabla\Delta^{-1}\psi\|_{L^2}^2 - C\epsilon^{-2\gamma_2-4}h^2,$$

provided that h satisfies the constraint

$$h \le (C_1 C_2)^{-1}\epsilon^{\gamma_3+3}, \tag{4.7}$$

where $\gamma_2 := 2\gamma_1 + \sigma_1 + 6$ and C_2 is determined by

$$C_2 := \max_{|\xi| \le \|u\|_{L^\infty((0,T);L^\infty)}} |f''(\xi)|.$$

Theorem 4.5 (Piecewise $L^\infty(H^{-1})$ and $L^2(H^1)$ error estimates [27], Thm. 6)**.** *Assume u is the solution of (1.1)–(1.4), u_h^n is the numerical solution of scheme (3.9)–(3.10). Then, under the following mesh constraints (4.3), we have the following error estimate*

$$\frac{1}{4}\|\nabla\widetilde{\Delta}_h^{-1}\theta^\ell\|_{0,2,h}^2 + \frac{k^2}{4}\sum_{n=1}^{\ell}\|\nabla\widetilde{\Delta}_h^{-1}d_t\theta^n\|_{0,2,h}^2 + \frac{\epsilon^4 k}{16}\sum_{n=1}^{\ell}(\nabla\theta^n, \nabla\theta^n)_h$$
$$+ \frac{k}{\epsilon}\sum_{n=1}^{\ell}\|\theta^n\|_{0,4,h}^4 \le C(\tilde{\rho}_0(\epsilon)|\ln h|h^2 + \tilde{\rho}_1(\epsilon)k^2),$$

where $\widetilde{\Delta}_h^{-1}$ is a discrete inverse Laplace operator defined in Equation (36) of [27], and $\tilde{\rho}_0(\epsilon)$ and $\tilde{\rho}_1(\epsilon)$ are defined by

$$\begin{aligned}\tilde{\rho}_0(\epsilon) &:= \epsilon^4(\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-\max\{2\sigma_1+5,2\sigma_3+2\}-2} + \epsilon^{-\max\{\sigma_1+\frac{5}{2},\sigma_3+1\}-2}\rho_0(\epsilon) + \epsilon^{-2\sigma_6+1}) \\ &\quad + \epsilon^{-6}(\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}+4}) + \epsilon^{-2\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}+2}, \\ \tilde{\rho}_1(\epsilon) &:= \epsilon^{-\frac{1}{2}\max\{2\sigma_1+5,2\sigma_3+2\}-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4\}+1} + \epsilon^{-2\sigma_5+1} + \epsilon^{-\max\{2\sigma_1+7,2\sigma_3+4\}+1}.\end{aligned}$$

4.2. $L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates

Based on Theorem 4.5, the $L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates which depend on $\frac{1}{\epsilon}$ polynomially, instead of exponentially, are derived as below. Notice that the Theorem 4.5 is used to circumvent the use of interpolation of $\|\cdot\|_{1,2,h}$ between $\|\cdot\|_{L^2}$ and $\|\cdot\|_{2,2,h}$, by which only the exponential dependence can be derived.

Theorem 4.6. *Assume u is the solution of (1.1)–(1.4), u_h^n is the numerical solution of scheme (3.9)–(3.10). The initial condition is chosen to be $u_h^0 = P_h u_0$. Then, under the mesh constraints (3.23) and (4.3), the following $L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates hold*

$$\|\theta^\ell\|_{L^2}^2 + k^2 \sum_{n=1}^{\ell} \|d_t\theta^n\|_{L^2}^2 + \epsilon k \sum_{n=1}^{\ell} a_h(\theta^n, \theta^n) \le C\tilde{\rho}_2(\epsilon)|\ln h|^2 h^2 + C\tilde{\rho}_3(\epsilon)|\ln h| k^2, \tag{4.8}$$

where

$$\tilde{\rho}_2(\epsilon) := \epsilon^4 \rho_3(\epsilon) + \epsilon^{-2\sigma_1 - 6}\rho_4(\epsilon) + \rho_5(\epsilon) + \epsilon^{-5}\tilde{\rho}_0(\epsilon) + \epsilon^{-2\gamma_1 - 2\gamma_2 - 6}\tilde{\rho}_0(\epsilon),$$
$$\tilde{\rho}_3(\epsilon) := \rho_3(\epsilon) + \epsilon^{-5}\tilde{\rho}_1(\epsilon) + \epsilon^{-2\gamma_1 - 2\gamma_2 - 6}\tilde{\rho}_1(\epsilon).$$

Proof. It follows from (3.9), (3.11), and (3.12) that for any $v_h \in S_E^h$,

$$\begin{aligned}
&(d_t\theta^n, v_h) + \epsilon a_h(\theta^n, v_h) \\
&\quad = [(d_t P_h u(t_n), v_h) + \epsilon a_h(P_h u(t_n), v_h)] - [(d_t u_h^n, v_h) + \epsilon a_h(u_h^n, v_h)] \\
&\quad = -(d_t\phi^n, v_h) + (u_t(t_n) + \epsilon\Delta^2 u(t_n) - \frac{1}{\epsilon}\Delta f(u(t_n)) + \alpha u(t_n), v_h) + (R^n(u_{tt}), v_h) \\
&\qquad - \frac{1}{\epsilon}(f'(u(t_n))\nabla P_h u(t_n), \nabla v_h)_h - \alpha(P_h u(t_n), v_h) + \frac{1}{\epsilon}(\nabla f(u_h^n), \nabla v_h)_h \\
&\quad = \ (-d_t\phi^n + \alpha\phi^n, v_h) - \frac{1}{\epsilon}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(u_h^n), \nabla v_h)_h + (R^n(u_{tt}), v_h),
\end{aligned} \tag{4.9}$$

where the remainder

$$R^n(u_{tt}) := \frac{u(t_n) - u(t_{n-1})}{k} - u_t(t_n) = -\frac{1}{k}\int_{t_{n-1}}^{t_n} (s - t_{n-1}) u_{tt}(s)\,\mathrm{d}s. \tag{4.10}$$

Choosing $v_h = \theta^n$, taking summation over n from 1 to ℓ, multiplying k on both sides of (4.9), and choosing $u_h^0 = P_h u_0$, we have

$$\begin{aligned}
&\frac{1}{2}\|\theta^\ell\|_{L^2}^2 + \frac{k^2}{2}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 + \epsilon k \sum_{n=1}^{\ell} a_h(\theta^n, \theta^n) \\
&\quad = \underbrace{k\sum_{n=1}^{\ell}(-d_t\phi^n + \alpha\phi^n, \theta^n)}_{I_1} \underbrace{- \frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(u_h^n), \nabla\theta^n)_h}_{I_2} + \underbrace{k\sum_{n=1}^{\ell}(R^n(u_{tt}), \theta^n)}_{I_3}.
\end{aligned} \tag{4.11}$$

Estimate of I_1. The first term on the right hand side of (4.11) can be bounded by

$$\begin{aligned}
I_1 &= k\sum_{n=1}^{\ell}(-d_t\phi^n + \alpha\phi^n, \theta^n) \\
&\le Ck\sum_{n=1}^{\ell}\|d_t\phi^n\|_{L^2}^2 + Ck\sum_{n=1}^{\ell}\alpha^2\|\phi^n\|_{L^2}^2 + Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2 \\
&\le C(\epsilon^4\rho_3(\epsilon) + \epsilon^{-6}\rho_4(\epsilon))h^2 + C\rho_5(\epsilon)|\ln h| h^2 + Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2,
\end{aligned} \tag{4.12}$$

where by (3.20) and (3.21), letting $\tilde{\phi}(t) = u(t) - P_h u(t)$, and recalling $\alpha = \alpha_0\epsilon^{-3}$ in (3.12)

$$k\sum_{n=1}^{\ell}\|d_t\phi^n\|_{L^2}^2 = \frac{1}{k}\sum_{n=1}^{\ell}\|\int_{t_{n-1}}^{t_n}\tilde{\phi}_t(s)\,\mathrm{d}s\|_{L^2}^2 \le \sum_{n=1}^{\ell}\int_{t_{n-1}}^{t_n}\|\tilde{\phi}_t(s)\|_{L^2}^2\,\mathrm{d}s \tag{4.13}$$

$$= \int_0^T \|u_t(s) - (P_h u)_t(s)\|_{L^2}^2 \,\mathrm{d}s \le C\epsilon^4 \rho_3(\epsilon) h^2 + C\rho_5(\epsilon)|\ln h| h^2,$$

$$k\sum_{n=1}^{\ell} \alpha^2 \|\phi^n\|_{L^2}^2 \le C\epsilon^{-6} \sup_{1\le n\le \ell} \|\phi^n\|_{L^2}^2 \le C\epsilon^{-6}\rho_4(\epsilon) h^2. \tag{4.14}$$

Estimate of I_2. The second term on the right hand side of (4.11) can be written as

$$\begin{aligned}
&-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(u_h^n), \nabla\theta^n)_h \qquad (4.15)\\
&= \underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - f'(P_h u(t_n))\nabla P_h u(t_n), \nabla\theta^n)_h}_{J_1} \underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(\nabla f(P_h u(t_n)) - f'(P_h u(t_n))\nabla u_h^n, \nabla\theta^n)_h}_{J_2}\\
&\underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(P_h u(t_n))\nabla u_h^n - \nabla f(u_h^n), \nabla\theta^n)_h}_{J_3}.
\end{aligned}$$

By (2.3) and (3.24), we have

$$\|\nabla P_h u(t_n)\|_{0,2,h}^2 \le C\|\nabla u(t_n)\|_{L^2}^2 + C \le C\epsilon^{-2\sigma_1 - 1} \qquad \forall 1\le n\le \ell. \tag{4.16}$$

Using the fact that $f'(u) = 3u^2 - 1$ and $\phi^n = u(t_n) - P_h u(t_n)$, we have

$$f'(u(t_n))\nabla P_h u(t_n) - \nabla f(P_h u(t_n)) = 3\phi^n(u(t_n) + P_h u(t_n))\nabla P_h u(t_n).$$

Then, using (2.11), (3.26), (4.5), (4.16), and the piecewise $L^2(H^1)$ error estimate given in Theorem 4.5, the first term on the right-hand side of (4.15) can be bounded as below

$$\begin{aligned}
J_1 &= -\frac{3k}{\epsilon}\sum_{n=1}^{\ell}(\phi^n(u(t_n) + P_h u(t_n))\nabla P_h u(t_n), \nabla\theta^n)_h \qquad (4.17)\\
&\le \frac{Ck}{\epsilon}\sum_{n=1}^{\ell}\|u(t_n) + P_h u(t_n)\|_{L^\infty}^2 \|\phi^n\|_{L^\infty}^2 \|\nabla P_h u(t_n)\|_{0,2,h}^2 + \frac{Ck}{\epsilon}\sum_{n=1}^{\ell}(\nabla\theta^n, \nabla\theta^n)_h\\
&\le C\epsilon^{-2\sigma_1-6}\rho_4(\epsilon)h^2 + C\epsilon^{-5}\tilde\rho_0(\epsilon)|\ln h|h^2 + C\epsilon^{-5}\tilde\rho_1(\epsilon)k^2.
\end{aligned}$$

The second term on the right-hand side of (4.15) can be written as

$$\begin{aligned}
J_2 &= -\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(P_h u(t_n))\nabla\theta^n, \nabla\theta^n)_h \le \frac{Ck}{\epsilon}\sum_{n=1}^{\ell}(\nabla\theta^n, \nabla\theta^n)_h \qquad (4.18)\\
&\le C\epsilon^{-5}\tilde\rho_0(\epsilon)|\ln h|h^2 + C\epsilon^{-5}\tilde\rho_1(\epsilon)k^2,
\end{aligned}$$

where (3.26) and the piecewise $L^2(H^1)$ estimate in Theorem 4.5 are used in the last inequality.

Using the stability result (4.6) and the discrete Sobolev inequality (*cf.* [9], Sect. 2.4.2), we have for any $1\le n\le \ell$,

$$\|u_h^n\|_{1,\infty,h} \le C|\ln h|^{\frac12}\|u_h^n\|_{2,2,h} \le C\epsilon^{-\gamma_2}|\ln h|^{\frac12}. \tag{4.19}$$

Then, the third term on the right-hand side of (4.15) can be bounded by

$$J_3 = -\frac{3k}{\epsilon}\sum_{n=1}^{\ell}(\theta^n(P_h u(t_n) + u_h^n)\nabla u_h^n, \nabla\theta^n) \tag{4.20}$$

$$
\begin{aligned}
&\leq Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2+\frac{Ck}{\epsilon^2}\sum_{n=1}^{\ell}\|P_hu(t_n)+u_h^n\|_{L^\infty}^2\|u_h^n\|_{1,\infty,h}^2\|\nabla\theta^n\|_{0,2,h}^2\\
&\leq Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2+C\epsilon^{-2\gamma_1-2\gamma_2-2}|\ln h|k\sum_{n=1}^{\ell}\|\nabla\theta^n\|_{0,2,h}^2\\
&\leq Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2+C\epsilon^{-2\gamma_1-2\gamma_2-6}(\tilde\rho_0(\epsilon)|\ln h|^2h^2+\tilde\rho_1(\epsilon)|\ln h|k^2),
\end{aligned}
$$

where the Cauchy–Schwarz inequality is used in the first inequality, (3.26), (4.2) and (4.19) are used in the second inequality, and the piecewise $L^2(H^1)$ estimate in Theorem 4.5 is used in the third inequality.
Estimate of I_3. The third term on the right hand side of (4.11) can be bounded by

$$
\begin{aligned}
I_3=k\sum_{n=1}^{\ell}(R^n(u_{tt}),\theta^n)&\leq Ck\sum_{n=1}^{\ell}\|R^n(u_{tt})\|_{L^2}^2+Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2\\
&\leq C\rho_3(\epsilon)k^2+Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2,
\end{aligned}\tag{4.21}
$$

where by the Cauchy–Schwarz inequality, (2.14) and (4.10),

$$
\begin{aligned}
k\sum_{n=1}^{\ell}\|R^n(u_{tt})\|_{L^2}^2&\leq\frac{1}{k}\sum_{n=1}^{\ell}\left(\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2\,\mathrm{d}s\right)\left(\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|_{L^2}^2\,\mathrm{d}s\right)\\
&\leq C\rho_3(\epsilon)k^2.
\end{aligned}\tag{4.22}
$$

$L^\infty(L^2)$ and piecewise $L^2(H^2)$ error estimates. Taking (4.12), (4.17), (4.18), (4.20), (4.21) into (4.11), we have

$$
\begin{aligned}
&\frac12\|\theta^\ell\|_{L^2}^2+\frac{k^2}{2}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2+\epsilon k\sum_{n=1}^{\ell}a_h(\theta^n,\theta^n)\\
&\quad\leq Ck\sum_{n=1}^{\ell}\|\theta^n\|_{L^2}^2\\
&\qquad+C(\epsilon^4\rho_3(\epsilon)+\epsilon^{-2\sigma_1-6}\rho_4(\epsilon))h^2\\
&\qquad+C(\rho_5(\epsilon)+\epsilon^{-5}\tilde\rho_0(\epsilon))|\ln h|h^2+C\epsilon^{-2\gamma_1-2\gamma_2-6}\tilde\rho_0(\epsilon)|\ln h|^2h^2\\
&\qquad+C(\rho_3(\epsilon)+\epsilon^{-5}\tilde\rho_1(\epsilon))k^2+C\epsilon^{-2\gamma_1-2\gamma_2-6}\tilde\rho_1(\epsilon)|\ln h|k^2.
\end{aligned}\tag{4.23}
$$

The desired result (4.8) is therefore obtained by the Gronwall's inequality. □

4.3. Piecewise $L^\infty(H^2)$ and $L^\infty(L^\infty)$ error estimates

In this subsection, we give the $\|\theta^\ell\|_{2,2,h}^2$ estimate by taking the summation by parts in time and integration by parts in space, and using the special properties of the Morley element. The $\|\theta^\ell\|_{2,2,h}^2$ estimate below is "almost" optimal with respect to time and space.

Theorem 4.7. *Assume u is the solution of (1.1)–(1.4), u_h^n is the numerical solution of scheme (3.9) and (3.10). The initial condition is chosen to be $u_h^0=P_hu_0$. Then, under the mesh constraints (3.23) and (4.3), the following piecewise $L^\infty(H^2)$ error estimate holds*

$$
k\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2+\epsilon k^2\sum_{n=1}^{\ell}a_h(d_t\theta^n,d_t\theta^n)+\epsilon\|\theta^\ell\|_{2,2,h}^2\leq C\tilde\rho_4(\epsilon)|\ln h|^2h^2+C\tilde\rho_5(\epsilon)|\ln h|k^2,\tag{4.24}
$$

where

$$\tilde{\rho}_4(\epsilon) = \epsilon^{-2\sigma_1-1}\rho_3(\epsilon) + \epsilon^{-4}\rho_0(\epsilon)\rho_4(\epsilon) + \epsilon^{-2\sigma_1-5}\rho_5(\epsilon) \\ + \Big(\epsilon^{-4\gamma_1-3} + \epsilon^{-4\gamma_2-3} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}-3} + \epsilon^{2\gamma_1-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-1}\Big)\tilde{\rho}_2(\epsilon),$$

$$\tilde{\rho}_5(\epsilon) = \Big(\epsilon^{-4\gamma_1-3} + \epsilon^{-4\gamma_2-3} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}-3} + \epsilon^{2\gamma_1-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-1}\Big)\tilde{\rho}_3(\epsilon).$$

Proof. Choosing $v_h = \theta^n - \theta^{n-1} = kd_t\theta^n$ in (4.9), taking summation over n from 1 to ℓ, and choosing $u_h^0 = P_h u_0$, we get

$$k\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 + \frac{\epsilon}{2}a_h(\theta^\ell,\theta^\ell) + \frac{\epsilon k^2}{2}\sum_{n=1}^{\ell}a_h(d_t\theta^n, d_t\theta^n) \tag{4.25}$$
$$= \underbrace{k\sum_{n=1}^{\ell}(-d_t\phi^n + \alpha\phi^n, d_t\theta^n)}_{I_1} \underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(u_h^n), \nabla(d_t\theta^n))_h}_{I_2} + \underbrace{k\sum_{n=1}^{\ell}(R^n(u_{tt}), d_t\theta^n)}_{I_3}.$$

Here we use the fact that

$$\epsilon a_h(\theta^n, \theta^n - \theta^{n-1}) = \frac{\epsilon k^2}{2}a_h(d_t\theta^n, d_t\theta^n) + \frac{\epsilon}{2}a_h(\theta^n,\theta^n) - \frac{\epsilon}{2}a_h(\theta^{n-1},\theta^{n-1}).$$

Estimates of I_1 and I_3. Similar to (4.12), using (4.13) and (4.14), we have

$$I_1 \le Ck\sum_{n=1}^{\ell}\|d_t\phi^n\|_{L^2}^2 + Ck\sum_{n=1}^{\ell}\alpha^2\|\phi^n\|_{L^2}^2 + \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 \tag{4.26}$$
$$\le C(\epsilon^4\rho_3(\epsilon) + \epsilon^{-6}\rho_4(\epsilon))h^2 + C\rho_5(\epsilon)|\ln h|h^2 + \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2.$$

Similar to (4.21) and (4.22), we also obtain the estimate of I_3 below

$$I_3 = k\sum_{n=1}^{\ell}(R^n(u_{tt}), d_t\theta^n) \le Ck\sum_{n=1}^{\ell}\|R^n(u_{tt})\|_{L^2}^2 + \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 \tag{4.27}$$
$$\le C\rho_3(\epsilon)k^2 + \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2.$$

Estimate of I_2. Next, we bound the more complicated term I_2. We adopt the standard DG notation as follows (*cf.* [5]). Let $E \in \mathcal{E}_h^i$ shared by elements K^+ and K^-, and let n^i be the unit outward normal vector on ∂K^i, with $i = +, -$. For any scalar-valued function q and vector-valued function φ, we define $q^\pm$ and $\varphi^\pm$ by $q^\pm := q|_{\partial K^\pm}$, $\varphi^\pm := \varphi|_{\partial K^\pm}$. Then the average $\{\cdot\}$ and jump $[\![\cdot]\!]$ are defined as follows:

$$\{q\} := \frac{1}{2}(q^+ + q^-), \quad [\![q]\!] := q^+n^+ + q^-n^-, \quad \{\varphi\} := \frac{1}{2}(\varphi^+ + \varphi^-), \quad [\![\varphi]\!] := \varphi^+\cdot n^+ + \varphi^-\cdot n^-, \quad \text{on } E \in \mathcal{E}_h^i,$$
$$\{q\} := q, \quad [\![q]\!] := 0, \quad \{\varphi\} := \varphi, \quad [\![\varphi]\!] := \varphi\cdot n, \quad \text{on } E \in \mathcal{E}_h^\partial.$$

Using the integration by parts, we have

$$
\begin{aligned}
I_2 &= -\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(P_h u(t_n)), d_t\nabla\theta^n)_h - \frac{k}{\epsilon}\sum_{n=1}^{\ell}(\nabla(f(P_h u(t_n)) - f(u_h^n)), d_t\nabla\theta^n)_h \qquad (4.28)\\
&= \underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(P_h u(t_n)), d_t\nabla\theta^n)_h}_{J_1} + \underbrace{\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f(P_h u(t_n)) - f(u_h^n), d_t\Delta\theta^n)_h}_{J_2}\\
&\quad \underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}\sum_{E\in\mathcal{E}_h}(\{f(P_h u(t_n)) - f(u_h^n)\}, d_t[\![\nabla\theta^n]\!])_E}_{J_3} \underbrace{-\frac{k}{\epsilon}\sum_{n=1}^{\ell}\sum_{E\in\mathcal{E}_h}([\![f(P_h u(t_n)) - f(u_h^n)]\!], \{\nabla d_t\theta^n\})_E}_{J_4}.
\end{aligned}
$$

Here we adopt the DG identity, see [5, Equ. (3.3)]. We note here that the $[\![f(P_h u(t_n)) - f(u_h^n)]\!]$ vanishes on $\mathcal{E}_h^{\partial}$ due to the definition of jump on the boundary edges. Next we bound J_1 to J_4, respectively.
Estimate of J_1. Recall that $f'(u) = 3u^2 - 1$ and $\phi^n = u(t_n) - P_h u(t_n)$ yield

$$
f'(u(t_n))\nabla P_h u(t_n) - \nabla f(P_h u(t_n)) = 3\phi^n(u(t_n) + P_h u(t_n))\nabla P_h u(t_n).
$$

By using summation by parts in Lemma 4.1 and choosing $u_h^0 = P_h u_0$, we have

$$
\begin{aligned}
J_1 &= -\frac{k}{\epsilon}\sum_{n=1}^{\ell}(3\phi^n(u(t_n) + P_h u(t_n))\nabla P_h u(t_n), d_t\nabla\theta^n)_h \qquad (4.29)\\
&= \frac{k}{\epsilon}\sum_{n=1}^{\ell}(3d_t[\phi^n(u(t_n) + P_h u(t_n))\nabla P_h u(t_n)], \nabla\theta^{n-1})_h - \frac{1}{\epsilon}(3\phi^{\ell}(u(t_\ell) + P_h u(t_\ell))\nabla P_h u(t_\ell), \nabla\theta^{\ell})_h.
\end{aligned}
$$

Let $\tilde{\phi}(t) = u(t) - P_h u(t)$. Then the first term on the right hand side of (4.29) can be bounded by

$$
\begin{aligned}
&\frac{k}{\epsilon}\sum_{n=1}^{\ell}(3d_t[\phi^n(u(t_n) + P_h u(t_n))\nabla P_h u(t_n)], \nabla\theta^{n-1})_h \qquad (4.30)\\
&\quad \le \frac{1}{k}\sum_{n=1}^{\ell}\| \int_{t_{n-1}}^{t_n}[\tilde{\phi}(u + P_h u)\nabla P_h u]_t(s)\,\mathrm{d}s\|_{0,2,h}^2 + C\epsilon^{-2}k\sum_{n=1}^{\ell}|\theta^{n-1}|_{1,2,h}^2\\
&\quad \le C\operatorname*{ess\,sup}_{t\in[0,T]}\|\nabla P_h u(t)\|_{0,2,h}^2\int_0^T\|\tilde{\phi}_t(s)\|_{L^\infty}^2\,\mathrm{d}s + C\operatorname*{ess\,sup}_{t\in[0,T]}\|\tilde{\phi}(t)\|_{L^\infty}^2\int_0^T\|\nabla(P_h u)_t(s)\|_{0,2,h}^2\,\mathrm{d}s\\
&\qquad + C\operatorname*{ess\,sup}_{t\in[0,T]}\left(\|\tilde{\phi}(t)\|_{L^\infty}^2\|\nabla P_h u(t)\|_{0,2,h}^2\right)\int_0^T\|u_t(s) + (P_h u)_t(s)\|_{L^\infty}^2\,\mathrm{d}s + C\epsilon^{-2}k\sum_{n=1}^{\ell}|\theta^{n-1}|_{1,2,h}^2\\
&\quad \le C\epsilon^{-2\sigma_1-1}(\rho_3(\epsilon) + \epsilon^{-4}\rho_5(\epsilon)|\ln h|)h^2 + C\epsilon^{-4}\rho_0(\epsilon)\rho_4(\epsilon)h^2\\
&\qquad + C\epsilon^{-2\sigma_1-6-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}}\rho_4(\epsilon)h^2\\
&\qquad + C\epsilon^{-6}\tilde{\rho}_0(\epsilon)|\ln h|h^2 + C\epsilon^{-6}\tilde{\rho}_1(\epsilon)k^2,
\end{aligned}
$$

where the fundamental theorem of calculus is used in the first inequality, equations (2.11), (3.26), and the Cauchy–Schwarz inequality are used in the second inequality, equations (2.3), (2.7), (2.9), (3.20), (3.21), (3.24), (3.25), the embedding theorem, and the piecewise $L^2(H^1)$ estimate in Theorem 4.5 are used in the third inequality.

Recall that $|\theta^\ell|_{2,2,h}^2 \le C a_h(\theta^\ell,\theta^\ell)$. The second term on the right hand of (4.29) can be bounded by

$$
\begin{aligned}
&-\frac{1}{\epsilon}(3\phi^\ell(u(t_\ell)+P_hu(t_\ell))\nabla P_hu(t_\ell),\nabla\theta^\ell)_h \qquad (4.31)\\
&\qquad\le C\epsilon^{-2}\|\phi^\ell\|_{L^\infty}^2|P_hu(t_\ell)|_{1,2,h}^2+C\epsilon^{-1}\|\theta^\ell\|_{L^2}^2+\frac{\epsilon}{8}a_h(\theta^\ell,\theta^\ell)\\
&\qquad\le C\epsilon^{-2\sigma_1-7}\rho_4(\epsilon)h^2+C\epsilon^{-1}\tilde\rho_2(\epsilon)|\ln h|^2h^2+C\epsilon^{-1}\tilde\rho_3(\epsilon)|\ln h|k^2+\frac{\epsilon}{8}a_h(\theta^\ell,\theta^\ell),
\end{aligned}
$$

where equations (2.11), (3.26), and interpolation in [14, Lemma 2.4] (namely, $|\theta^\ell|_{1,2,h}^2\le C_\beta\|\theta^\ell\|_{L^2}^2+\beta|\theta^\ell|_{2,2,h}^2$ for any $\beta>0$; see also the equation between (3.13) and (3.14)) are used in the first inequality, equations (2.3), (3.20), (3.24), the embedding theorem, and the $L^\infty(L^2)$ estimate in Theorem 4.6 are used in the second inequality.

Combining (4.30) and (4.31), and simplifying the coefficients according to the definition of $\rho_i(\epsilon)$ and $\tilde\rho_i(\epsilon)$, we obtain the bound for J_1:

$$
\begin{aligned}
J_1 &\le C(\epsilon^{-2\sigma_1-1}\rho_3(\epsilon)+\epsilon^{-4}\rho_0(\epsilon)\rho_4(\epsilon)+\epsilon^{-2\sigma_1-5}\rho_5(\epsilon)+\epsilon^{-1}\tilde\rho_2(\epsilon))|\ln h|^2h^2 \qquad (4.32)\\
&\quad+C\epsilon^{-1}\tilde\rho_3(\epsilon)|\ln h|k^2+\frac{\epsilon}{8}a_h(\theta^l,\theta^l).
\end{aligned}
$$

Estimate of J_2. Define $f(P_hu(t_n))-f(u_h^n):=M^n\theta^n$, where M^n is given as

$$M^n:=(P_hu(t_n))^2+P_hu(t_n)u_h^n+(u_h^n)^2-1.$$

Using the summation by parts in Lemma 4.1 and choosing $u_h^0=P_hu_0$, we have

$$
\begin{aligned}
J_2 &= -\frac{k}{\epsilon}\sum_{n=1}^{\ell}(d_t(M^n\theta^n),\Delta\theta^{n-1})_h+\frac{1}{\epsilon}(M^\ell\theta^\ell,\Delta\theta^\ell)_h \qquad (4.33)\\
&\le \frac{Ck}{\epsilon}\sum_{n=1}^{\ell}\|d_t(M^n\theta^n)\|_{L^2}|\theta^{n-1}|_{2,2,h}+\frac{C}{\epsilon}\|M^\ell\theta^\ell\|_{L^2}|\theta^\ell|_{2,2,h}.
\end{aligned}
$$

Since $d_tu_h^n=d_t(P_hu(t_n))-d_t\theta^n$, a direct calculation shows that

$$
\begin{aligned}
d_t(M^n\theta^n) &= \theta^nd_tM^n+M^{n-1}d_t\theta^n \qquad (4.34)\\
&= M^{n-1}d_t\theta^n+\theta^n(P_hu(t_n)+P_hu(t_{n-1}))d_t(P_hu(t_n))\\
&\quad+\theta^nu_h^nd_t(P_hu(t_n))+\theta^nP_hu(t_{n-1})d_t(P_hu(t_n))-\theta^nP_hu(t_{n-1})d_t\theta^n\\
&\quad+\theta^n(u_h^n+u_h^{n-1})d_t(P_hu(t_n))-\theta^n(u_h^n+u_h^{n-1})d_t\theta^n\\
&= (M^{n-1}-\theta^nP_hu(t_{n-1})-\theta^n(u_h^n+u_h^{n-1}))d_t\theta^n\\
&\quad+(P_hu(t_n)+2P_hu(t_{n-1})+2u_h^n+u_h^{n-1})\theta^nd_t(P_hu(t_n)).
\end{aligned}
$$

The first term on the right hand side of (4.33) can be bounded by

$$
\begin{aligned}
&\frac{Ck}{\epsilon}\sum_{n=1}^{\ell}\|d_t(M^n\theta^n)\|_{L^2}|\theta^{n-1}|_{2,2,h} \qquad (4.35)\\
&\quad\le C\epsilon^{-2\gamma_1-1}k\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}|\theta^{n-1}|_{2,2,h}+C\epsilon^{-\gamma_1-1}k\sum_{n=1}^{\ell}\|\theta^nd_t(P_hu(t_n))\|_{L^2}|\theta^{n-1}|_{2,2,h}\\
&\quad\le \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2+C\epsilon^{-4\gamma_1-2}k\sum_{n=1}^{\ell}|\theta^{n-1}|_{2,2,h}^2+C\epsilon^{2\gamma_1}k\sum_{n=1}^{\ell}\|\theta^nd_t(P_hu(t_n))\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 + C\epsilon^{-4\gamma_1-3}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2+\tilde{\rho}_3(\epsilon)|\ln h|k^2)\\
&\quad + C\epsilon^{2\gamma_1-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma2+4,2\sigma_4\}-1}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2+\tilde{\rho}_3(\epsilon)|\ln h|k^2),
\end{aligned}
$$

where (3.26), (4.2) and (4.34) are used in the first inequality, the Young's inequality is used in the second inequality, and the $L^2(H^2)$ error estimate (4.8) (Here, $|\theta^{n-1}|_{2,2,h}^2 \leq Ca_h(\theta^{n-1},\theta^{n-1})$ is used) and the following bounds are used in the third inequality:

$$
\begin{aligned}
k\sum_{n=1}^{\ell}\|\theta^n d_t(P_hu(t_n))\|_{L^2}^2 &\leq \sup_{1\leq n\leq \ell}\|\theta^n\|_{L^2}^2\sum_{n=1}^{\ell}\frac{1}{k}\Big\|\int_{t_{n-1}}^{t_n}(P_hu)_t(s)\,\mathrm{d}s\Big\|_{L^\infty}^2\\
&\leq C\sup_{1\leq n\leq \ell}\|\theta^n\|_{L^2}^2\int_0^T\|(P_hu)_t(s)\|_{L^\infty}^2\,\mathrm{d}s\\
&\leq C\epsilon^{-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma2+4,2\sigma_4\}-1}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2+\tilde{\rho}_3(\epsilon)|\ln h|k^2).
\end{aligned}
$$

Here, the fundamental theorem of calculus is used in the first inequality, the Cauchy–Schwarz inequality is used in the second inequality, and equations (2.7), (3.25) and the $L^\infty(L^2)$ error estimate (4.8) are used in the third inequality.

The second term on the right hand side of (4.33) can be bounded by

$$
\begin{aligned}
\frac{C}{\epsilon}\|M^\ell\theta^\ell\|_{L^2}|\theta^\ell|_{2,2,h} &\leq C^{-4\gamma_1-3}\|\theta^\ell\|_{L^2}^2+\frac{\epsilon}{8}a_h(\theta^\ell,\theta^\ell) \qquad (4.36)\\
&\leq C\epsilon^{-4\gamma_1-3}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2+\tilde{\rho}_3(\epsilon)|\ln h|k^2)+\frac{\epsilon}{8}a_h(\theta^\ell,\theta^\ell),
\end{aligned}
$$

where (3.26) and (4.2) are used in the first inequality, and the $L^\infty(L^2)$ error estimate (4.8) is used in the second inequality. Combining (4.35) and (4.36), we obtain the bound for J_2:

$$
\begin{aligned}
J_2 &\leq \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2+\frac{\epsilon}{8}a_h(\theta^\ell,\theta^\ell)+C\epsilon^{-4\gamma_1-3}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2+\tilde{\rho}_3(\epsilon)|\ln h|k^2) \qquad (4.37)\\
&\quad + C\epsilon^{2\gamma_1-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-1}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2+\tilde{\rho}_3(\epsilon)|\ln h|k^2).
\end{aligned}
$$

Estimate of J_3. Notice that $\theta^n \in S_E^h$ and

$$
\int_E [\![\nabla\theta^n]\!]\,\mathrm{d}s = 0 \qquad \forall E\in\mathcal{E}_h.
$$

Using the summation by parts in Lemma 4.1, Lemma 2.1 in [14], the inverse inequality $h|d_t(M^n\theta^n)|_{1,2,h} \leq C\|d_t(M^n\theta^n)\|_{L^2}$, and choosing $u_h^0 = P_hu_0$, we have

$$
\begin{aligned}
J_3 &= \frac{k}{\epsilon}\sum_{n=1}^{\ell}\sum_{E\in\mathcal{E}_h}(d_t\{M^n\theta^n\},[\![\nabla\theta^{n-1}]\!])_E - \frac{1}{\epsilon}\sum_{E\in\mathcal{E}_h}(\{M^\ell\theta^\ell\},[\![\nabla\theta^\ell]\!])_E\\
&\leq \frac{Ck}{\epsilon}\sum_{n=1}^{\ell}\|d_t(M^n\theta^n)\|_{L^2}|\theta^{n-1}|_{2,2,h}+\frac{C}{\epsilon}\|M^\ell\theta^\ell\|_{L^2}|\theta^\ell|_{2,2,h}.
\end{aligned}
$$

Hence, J_3 has the same bound as J_2.

Estimate of J_4. Since $P_h u(t_n)$ and u_h^n are continuous at vertices of $\mathcal{T}_h$, we have

$$J_4 = -\frac{k}{\epsilon}\sum_{n=1}^{\ell}\sum_{E\in\mathcal{E}_h}([\![M^n\theta^n]\!],\{\nabla d_t\theta^n\})_E \leq \frac{Ck}{\epsilon}\sum_{n=1}^{\ell} h|M^n\theta^n|_{2,2,h}|d_t\theta^n|_{1,2,h} \tag{4.38}$$
$$\leq \frac{Ck}{\epsilon}\sum_{n=1}^{\ell}|M^n\theta^n|_{2,2,h}\|d_t\theta^n\|_{L^2} \leq \frac{Ck}{\epsilon^2}\sum_{n=1}^{\ell}|M^n\theta^n|_{2,2,h}^2 + \frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2,$$

where Lemma 2.6 in [14] is used in the first inequality (the $h|M^n\theta^n|_{3,2,h}$ term is absorbed into $|M^n\theta^n|_{2,2,h}$ term by the inverse inequality), the inverse inequality is used in the second inequality, and the Young's inequality is used in the third inequality.

The first term on the right-hand side of (4.38) can be bounded by

$$\frac{Ck}{\epsilon^2}\sum_{n=1}^{\ell}|M^n\theta^n|_{2,2,h}^2 \leq \frac{Ck}{\epsilon^2}\sum_{n=1}^{\ell}\left(\|M^n\|_{L^\infty}^2|\theta^n|_{2,2,h}^2 + |M^n|_{1,4,h}^2|\theta^n|_{1,4,h}^2 + |M^n|_{2,2,h}^2\|\theta^n\|_{L^\infty}^2\right) \tag{4.39}$$
$$\leq \frac{C}{\epsilon^2}\sup_{1\leq n\leq\ell}\|M^n\|_{2,2,h}^2 k\sum_{n=1}^{\ell}\|\theta^n\|_{2,2,h}^2$$
$$\leq C(\epsilon^{-4\gamma_2-2} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}-2})\epsilon^{-1}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2 + \tilde{\rho}_3(\epsilon)|\ln h|k^2),$$

where the embedding theorem is used in the second inequality, and the piecewise $L^2(H^2)$ estimate given in Theorem 4.6 and the following result is used in the third inequality:

$$\|M^n\|_{2,2,h} \leq C(\|(P_hu(t_n))^2\|_{2,2,h} + \|u_h^nP_hu(t_n)\|_{2,2,h} + \|(u_h^n)^2\|_{2,2,h}+1) \tag{4.40}$$
$$\leq C(\|P_hu(t_n)\|_{2,2,h} + \|P_hu(t_n)\|_{1,4,h}^2 + \|u_h^n\|_{L^\infty}\|u_h^n\|_{2,2,h} + \|u_h^n\|_{1,4,h}^2$$
$$+ \|u_h^n\|_{2,2,h} + \|u_h^n\|_{L^\infty}\|P_hu(t_n)\|_{2,2,h} + \|u_h^n\|_{1,4,h}\|P_hu(t_n)\|_{1,4,h}+1)$$
$$\leq C(\epsilon^{-2\gamma_2} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}}).$$

Here (3.26) is used in the second inequality, and equations (2.5), (3.24), (4.6) and the embedding theorem are used in the third inequality.

Piecewise $L^\infty(H^2)$ *error estimate.* Taking (4.26), (4.27), (4.32), (4.37)–(4.40) into (4.25), we obtain

$$\frac{k}{8}\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 + \frac{\epsilon}{8}a_h(\theta^\ell,\theta^\ell) + \frac{\epsilon k^2}{2}\sum_{n=1}^{\ell}a_h(d_t\theta^n,d_t\theta^n) \tag{4.41}$$
$$\leq C(\epsilon^4\rho_3(\epsilon) + \epsilon^{-6}\rho_4(\epsilon))h^2 + C\rho_5(\epsilon)|\ln h|^2h^2 + C\rho_3(\epsilon)k^2$$
$$+ C(\epsilon^{-2\sigma_1-1}\rho_3(\epsilon) + \epsilon^{-4}\rho_0(\epsilon)\rho_4(\epsilon) + \epsilon^{-2\sigma_1-5}\rho_5(\epsilon) + \epsilon^{-1}\tilde{\rho}_2(\epsilon))|\ln h|^2h^2$$
$$+ C\epsilon^{-1}\tilde{\rho}_3(\epsilon)|\ln h|k^2 + C\epsilon^{-4\gamma_1-3}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2 + \tilde{\rho}_3(\epsilon)|\ln h|k^2)$$
$$+ C\epsilon^{2\gamma_1-\max\{2\sigma_1+\frac{13}{2},2\sigma_3+\frac{7}{2},2\sigma_2+4,2\sigma_4\}-1}(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2 + \tilde{\rho}_3(\epsilon)|\ln h|k^2)$$
$$+ C(\epsilon^{-4\gamma_2-3} + \epsilon^{-\max\{2\sigma_1+5,2\sigma_3+2\}-3})(\tilde{\rho}_2(\epsilon)|\ln h|^2h^2 + \tilde{\rho}_3(\epsilon)|\ln h|k^2).$$

Then the theorem can be proved by simplifying the coefficients according to the definitions of $\rho_i(\epsilon)$ and $\tilde{\rho}_i(\epsilon)$. $\square$

Remark 4.8. If the summation by parts for time and integration by parts for space techniques are not employed simultaneously, we can only obtain a coarse estimate

$$\|\theta^\ell\|_{2,2,h}^2 + k\sum_{n=1}^{\ell}\|d_t\theta^n\|_{L^2}^2 + \epsilon k^2\sum_{n=1}^{\ell}a_h(d_t\theta^n,d_t\theta^n) \leq Ck^{-\frac{1}{2}}(\epsilon^{-\gamma_4}|\ln h|^2h^2 + \epsilon^{-\gamma_5}|\ln h|k),$$

where γ_4, γ_5 denote some positive constants. To see this, consider the first term $-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(f'(u(t_n))\nabla P_h u(t_n) - \nabla f(P_h u(t_n)), d_t\nabla\theta^n)_h$ and the second item $-\frac{k}{\epsilon}\sum_{n=1}^{\ell}(\nabla(f(P_h u(t_n)) - f(u_h^n)), d_t\nabla\theta^n)_h$ on the right-hand side of (4.28), the scaling of k will be lost at least $k^{-\frac{1}{2}}$.

Finally, using (4.5), Theorem 4.7 and the Sobolev embedding theorem, we can prove the desired $L^\infty(L^\infty)$ error estimate.

Theorem 4.9. *Assume u is the solution of (1.1)–(1.4) and u_h^n is the numerical solution of scheme (3.9) and (3.10). The initial condition is chosen to be $u_h^0 = P_h u_0$. Then, under the mesh constraints (3.23) and (4.3), we have the $L^\infty(L^\infty)$ error estimate*

$$\|u(t_n) - u_h^n\|_{L^\infty} \le C|\ln h|^{\frac{1}{2}}(\tilde\rho_6^{\frac{1}{2}}(\epsilon)|\ln h|^{\frac{1}{2}}h + \tilde\rho_7^{\frac{1}{2}}(\epsilon)k) \quad \forall 1 \le n \le \ell, \tag{4.42}$$

where $\tilde\rho_6(\epsilon) = \epsilon^{-1}\tilde\rho_4(\epsilon)$ and $\tilde\rho_7(\epsilon) = \epsilon^{-1}\tilde\rho_5(\epsilon)$.

Remark 4.10. The mesh constraints (3.23) and (4.3) can be achieved by $h = O(\epsilon^{p+2})$ and $k = O(\epsilon^p)$ for a sufficiently large constant p. More precisely, we need

$$\epsilon^{p+2} \le C\min\left\{\epsilon^2\rho_4^{-\frac{1}{2}}(\epsilon), \rho_3^{-\frac{1}{2}}(\epsilon), \epsilon^2|\ln\epsilon|^{-\frac{1}{2}}\rho_5^{-\frac{1}{2}}(\epsilon), \epsilon^{3\sigma_1+13}, \epsilon^{4\gamma_1+4}, \epsilon^{\gamma_3+3}\right\}. \tag{4.43}$$

Hence, the term $|\ln h|h$ (resp. $|\ln h|^{\frac{1}{2}}k$) decreases asymptotically as h (resp. k) when ϵ goes to zero.

5. Convergence of the numerical interface

In this section, we prove that the numerical interfaces defined as the zero level sets of the Morley element interpolation of the solutions U^n converge to the moving interface of the Hele-Shaw problem under the assumption that the Hele-Shaw problem has a unique global (in time) classical solution. We first cite the following convergence result established in [2, 19].

Theorem 5.1. *Let Ω be a given smooth domain and Γ_{00} be a smooth closed hypersurface in Ω. Suppose that the Hele-Shaw problem (1.5)–(1.9) starting from Γ_{00} has a unique smooth solution $\big(w, \Gamma := \bigcup_{0\le t\le T}(\Gamma_t \times \{t\})\big)$ in the time interval $[0,T]$ such that $\Gamma_t \subseteq \Omega$ for all $t \in [0,T]$. Then there exists a family of smooth functions $\{u^\epsilon(x)\}_{0<\epsilon\le 1}$ which are uniformly bounded in $\epsilon \in (0,1]$ and $(x,t) \in \overline\Omega_T$, such that if u^ϵ solves the Cahn–Hilliard problem (1.1)–(1.4),*

(i) $\lim\limits_{\epsilon\to 0} u^\epsilon(x,t) = \begin{cases} 1 & \text{if } (x,t) \in \mathcal{O} \\ -1 & \text{if } (x,t) \in \mathcal{I} \end{cases}$ *uniformly on compact subsets, where $\mathcal{I}$ and $\mathcal{O}$ stand for the "inside" and "outside" of Γ;*

(ii) $\lim\limits_{\epsilon\to 0}\big(\epsilon^{-1}f(u^\epsilon) - \epsilon\Delta u^\epsilon\big)(x,t) = w(x,t)$ *uniformly on $\overline\Omega_T$.*

We are now ready to state the first main theorem in this section.

Theorem 5.2. *Let $\{\Gamma_t\}_{t\ge 0}$ denote the zero level set of the Hele-Shaw problem and $U_{\epsilon,h,k}(x,t)$ denote the piecewise linear interpolation in time of the numerical solution u_h^n, namely,*

$$U_{\epsilon,h,k}(x,t) := \frac{t - t_{n-1}}{k}u_h^n(x) + \frac{t_n - t}{k}u_h^{n-1}(x), \tag{5.1}$$

for $t_{n-1} \le t \le t_n$ and $1 \le n \le M$. Then, under the mesh and starting value constraints in Theorem 4.9, by taking $h = O(\epsilon^{p+2})$ and $k = O(\epsilon^p)$ for a sufficiently large constant p, we have

(i) $U_{\epsilon,h,k}(x,t) \xrightarrow{\epsilon\searrow 0} 1$ *uniformly on compact subset of $\mathcal{O}$,*

(ii) $U_{\epsilon,h,k}(x,t) \xrightarrow{\epsilon \searrow 0} -1$ *uniformly on compact subset of* $\mathcal{I}$.

Proof. For any compact set $A \subset \mathcal{O}$ and for any $(x,t) \in A$, we have

$$\begin{aligned} |U_{\epsilon,h,k}(x,t) - 1| &\leq |U_{\epsilon,h,k}(x,t) - u^\epsilon(x,t)| + |u^\epsilon(x,t) - 1| \\ &\leq \|U_{\epsilon,h,k}(x,t) - u^\epsilon(x,t)\|_{L^\infty(\Omega_T)} + |u^\epsilon(x,t) - 1|. \end{aligned} \tag{5.2}$$

Theorem 4.9 infers that

$$\|U_{\epsilon,h,k}(x,t) - u^\epsilon(x,t)\|_{L^\infty(\Omega_T)} \leq C\epsilon^p \big[(p+2)\tilde{\rho}_6^{\frac{1}{2}}(\epsilon)|\ln \epsilon|\epsilon^2 + (p+2)^{\frac{1}{2}}\tilde{\rho}_7^{\frac{1}{2}}(\epsilon)|\ln \epsilon|^{\frac{1}{2}}\big]. \tag{5.3}$$

Since both $\tilde{\rho}_6(\epsilon)$ and $\tilde{\rho}_7(\epsilon)$ depend on $\frac{1}{\epsilon}$ polynomially, the first term on the right-hand side of (5.2) tends to 0 when $\epsilon \searrow 0$ for a sufficiently large p that also satisfies (4.43). The second term converges uniformly to 0 on the compact set A, which is ensured by (i) of Theorem 5.1. Hence, the assertion (i) holds.

To show (ii), we only need to replace $\mathcal{O}$ by $\mathcal{I}$ and 1 by -1 in the proof above. $\square$

The second main theorem addresses the convergence of the numerical interfaces.

Theorem 5.3. *Let* $\Gamma_t^{\epsilon,h,k} := \{x \in \Omega; U_{\epsilon,h,k}(x,t) = 0\}$ *be the zero level set of* $U_{\epsilon,h,k}(x,t)$*. Then, under the assumptions of Theorem 5.2, we have*

$$\sup_{x \in \Gamma_t^{\epsilon,h,k}} \operatorname{dist}(x, \Gamma_t) \xrightarrow{\epsilon \searrow 0} 0 \quad \textit{uniformly on } [0,T].$$

Proof. For any $\eta \in (0,1)$, define the tabular neighborhood $\mathcal{N}_\eta$ of width 2η of Γ_t

$$\mathcal{N}_\eta := \{(x,t) \in \Omega_T; \operatorname{dist}(x, \Gamma_t) < \eta\}. \tag{5.4}$$

Let A and B denote the complements of the neighborhood $\mathcal{N}_\eta$ in $\mathcal{O}$ and $\mathcal{I}$, respectively,

$$A = \mathcal{O} \setminus \mathcal{N}_\eta \qquad \text{and} \qquad B = \mathcal{I} \setminus \mathcal{N}_\eta.$$

Note that A is a compact subset outside Γ_t and B is a compact subset inside Γ_t. By Theorem 5.2, there exists $\epsilon_1 > 0$, which only depends on η, such that for any $\epsilon \in (0, \epsilon_1)$

$$|U_{\epsilon,h,k}(x,t) - 1| \leq \eta \quad \forall (x,t) \in A, \tag{5.5}$$

$$|U_{\epsilon,h,k}(x,t) + 1| \leq \eta \quad \forall (x,t) \in B. \tag{5.6}$$

Now for any $t \in [0,T]$ and $x \in \Gamma_t^{\epsilon,h,k}$, from $U_{\epsilon,h,k}(x,t) = 0$ we have

$$|U_{\epsilon,h,k}(x,t) - 1| = 1, \tag{5.7}$$

$$|U_{\epsilon,h,k}(x,t) + 1| = 1. \tag{5.8}$$

Equations (5.5) and (5.7) imply that (x,t) is not in A, and (5.6) and (5.8) imply that (x,t) is not in B. Then (x,t) must lie in the tubular neighborhood $\mathcal{N}_\eta$. Therefore, for any $\epsilon \in (0, \epsilon_1)$,

$$\sup_{x \in \Gamma_t^{\epsilon,h,k}} \operatorname{dist}(x, \Gamma_t) \leq \eta \qquad \text{uniformly on } [0,T]. \tag{5.9}$$

The proof is complete. $\square$

6. Numerical experiments

In this section, we present two two-dimensional numerical tests to gauge the performance of the proposed fully discrete Morley finite element method for the Cahn–Hilliard equation. The square domain $\Omega = [-1,1]^2$ is used in both tests.

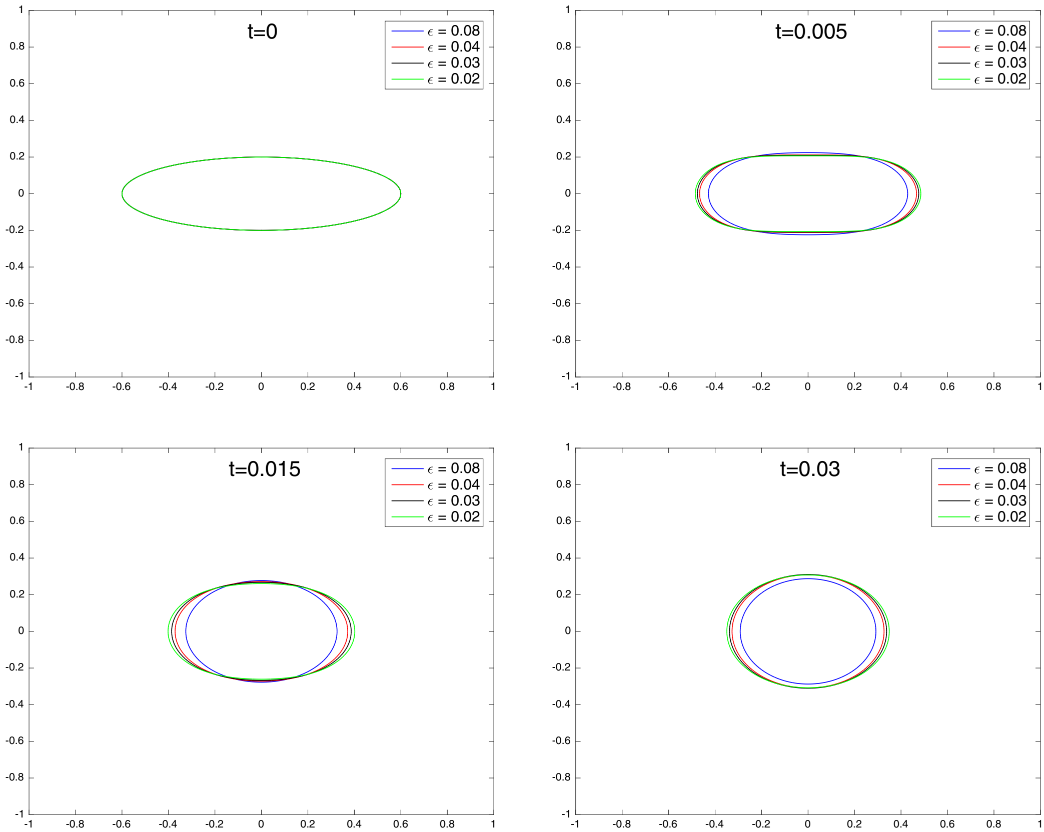

FIGURE 2. Test 1: Snapshots of the zero-level sets of $U_{\epsilon,h,k}$ when $t = 0, 0.005, 0.015, 0.03$ and $\epsilon = 0.08, 0.04, 0.03, 0.02$.

TABLE 1. Test 2: Spatial errors and convergence rates when $\epsilon = 0.08$, $k = 1 \times 10^{-5}$, $T = 0.002$.

	L^2 error	order	H^1 error	order	H^2 error	order
$h = 0.2\sqrt{2}$	0.079659	–	1.761563	–	34.097686	–
$h = 0.1\sqrt{2}$	0.023142	1.7833	0.642870	1.4543	21.604986	0.6583
$h = 0.05\sqrt{2}$	0.007598	1.6067	0.183600	1.8080	11.783724	0.8746
$h = 0.025\sqrt{2}$	0.002151	1.8201	0.048042	1.9342	6.045416	0.9629
$h = 0.0125\sqrt{2}$	0.000557	1.9501	0.012167	1.9813	3.042138	0.9908

Test 1. Consider the Cahn–Hilliard problem with an elliptical initial interface determined by $\Gamma_0 : \frac{x^2}{0.36} + \frac{y^2}{0.04} = 1$. The initial condition is chosen to have the form $u_0(x, y) = \tanh(\frac{d_0(x,y)}{\sqrt{2}\epsilon})$, where $d_0(x, y)$ denotes the signed distance from (x, y) to the initial interface Γ_0 and $\tanh(t) = (e^t - e^{-t})/(e^t + e^{-t})$.

Figure 2 displays four snapshots at four fixed time points of the numerical interfaces with four different ϵ's. Here the time step size is $k = 1 \times 10^{-4}$ and the space size is $h = 0.01$. They clearly indicate that at each time point the numerical interfaces converge to the sharp interface Γ_t of the Hele-Shaw flow as ϵ tends to zero. Note that this initial condition may not satisfy the General Assumption (GA) due to the singularity of the signed distance function. We will adopt a smooth initial condition in the later test.

TABLE 2. Test 2: Spatial errors and convergence rates when $\epsilon = 0.08$, $k = 1 \times 10^{-5}$, $T = 0.001$.

	L^2 error	order	H^1 error	order	H^2 error	order
$h = 0.2\sqrt{2}$	0.137170	–	2.469582	–	43.008910	–
$h = 0.1\sqrt{2}$	0.032310	2.0859	0.710340	1.7977	23.320078	0.8831
$h = 0.05\sqrt{2}$	0.008830	1.8715	0.183932	1.9493	11.774451	0.9859
$h = 0.025\sqrt{2}$	0.002349	1.9103	0.046810	1.9743	5.927408	0.9902
$h = 0.0125\sqrt{2}$	0.000597	1.9746	0.011764	1.9924	2.970322	0.9968

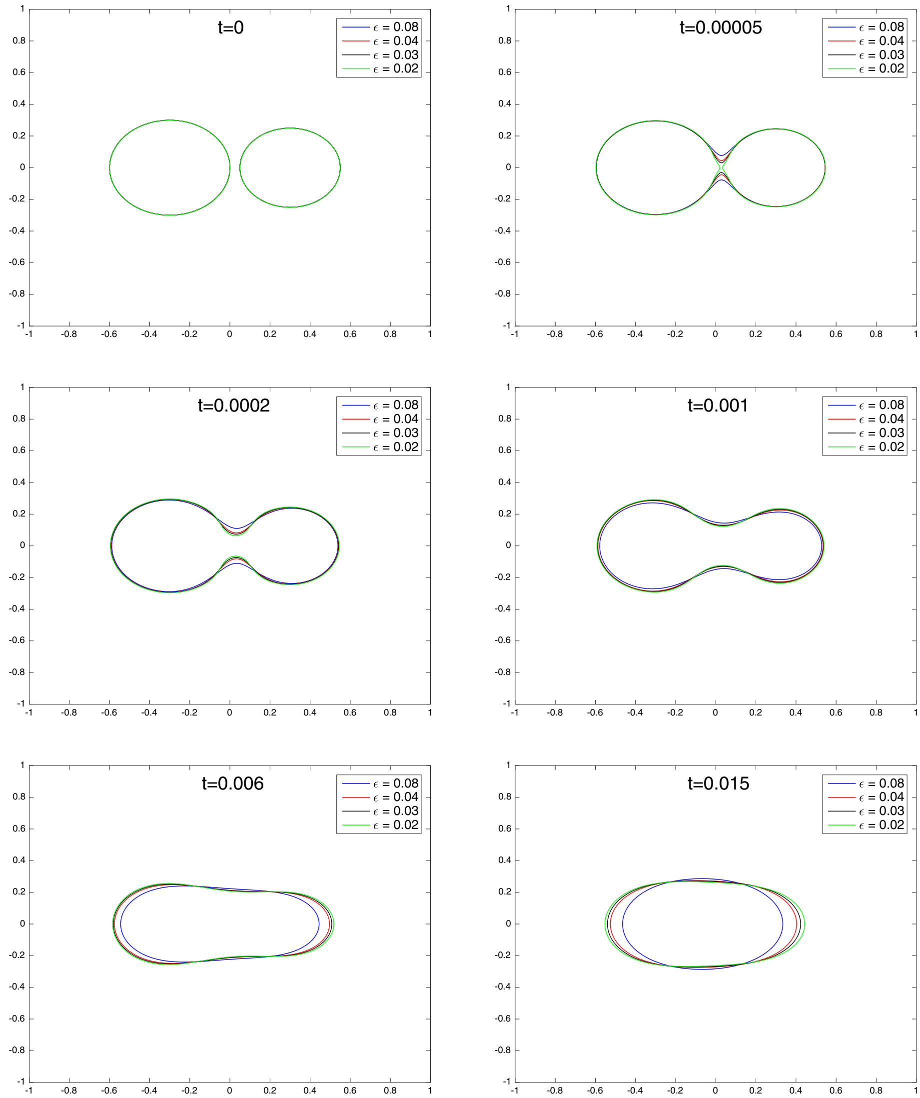

FIGURE 3. Test 2: Snapshots of the zero-level sets of $U_{\epsilon,h,k}$ when $t = 0, 0.00005, 0.0002, 0.001, 0.006, 0.015$ and $\epsilon = 0.08, 0.04, 0.03, 0.02$.

Test 2. Consider the following initial condition, which is also adopted in [20],

$$u_0(x,y) = \tanh\Big(((x-0.3)^2 + y^2 - 0.25^2)/\epsilon\Big)\tanh\Big(((x+0.3)^2 + y^2 - 0.3^2)/\epsilon\Big).$$

Tables 1 and 2 show the errors of spatial L^2, H^1 and H^2 semi-norms and the rates of convergence at $T = 0.0002$ and $T = 0.001$. Here $\epsilon = 0.08$ is used to generate the table, and $k = 1 \times 10^{-5}$ is chosen such that the error in time is relatively small compared to the error in space. The error in H^2 norm is in agreement with the convergence theorem, but the errors in L^2 and H^1 norms are one order higher than our theoretical results. We note that in [14], the second order convergence in both L^2 and H^1 norms were proved, whereas only $\frac{1}{\epsilon}$-exponential dependence could be derived.

Figure 3 displays six snapshots at six fixed time points of the numerical interfaces with four different ϵ's. Again, they clearly indicate that at each time point the numerical interfaces converge to the sharp interface Γ_t of the Hele-Shaw flow as ϵ tends to zero.

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References

[1] R.A. Adams and J.J. Fournier, Sobolev Spaces, vol. 140. Elsevier (2003).

[2] N.D. Alikakos, P.W. Bates and X. Chen, Convergence of the Cahn–Hilliard equation to the Hele-Shaw model. *Arch. Ration. Mech. Anal.* **128** (1994) 165–205.

[3] S.M. Allen and J.W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.* **27** (1979) 1085–1095.

[4] A.C. Aristotelous, O. Karakashian and S.M. Wise, A mixed discontinuous Galerkin, convex splitting scheme for a modified Cahn–Hilliard equation and an efficient nonlinear multigrid solver. *Discrete Continuous Dyn. Syst.-Ser. B* **18** (2013) 2211–22238.

[5] D.N. Arnold, F. Brezzi, B. Cockburn and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* **39** (2002) 1749–1779.

[6] S. Bartels, R. Müller and C. Ortner, Robust a priori and a posteriori error analysis for the approximation of Allen–Cahn and Ginzburg–Landau equations past topological changes. *SIAM J. Numer. Anal.* **49** (2011) 110–134.

[7] S.C. Brenner, Two-level additive Schwarz preconditioners for nonconforming finite element methods. *Math. Comput.* **65** (1996) 897–921.

[8] S.C. Brenner, Convergence of nonconforming multigrid methods without full elliptic regularity. *Math. Comput.* **68** (1999) 25–53.

[9] S.C. Brenner, Forty years of the Crouzeix-Raviart element. *Numer. Methods Partial Differ. Equ.* **31** (2015) 367–396.

[10] S.C. Brenner, L.-Y. Sung, H. Zhang and Y. Zhang, A Morley finite element method for the displacement obstacle problem of clamped Kirchhoff plates. *J. Comput. Appl. Math.* **254** (2013) 31–42.

[11] J.W. Cahn and J.E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy. *J. Chem. Phys.* **28** (1958) 258–267.

[12] X. Chen, Spectrum for the Allen–Chan, Chan–Hillard, and phase-field equations for generic interface. *Commun. Partial Differ. Equ.* **19** (1994) 1371–1395.

[13] Q. Du and R.A. Nicolaides, Numerical analysis of a continuum model of phase transition. *SIAM J. Numer. Anal.* **28** (1991) 1310–1322.

[14] C.M. Elliott and D.A. French, A nonconforming finite-element method for the two-dimensional Cahn–Hilliard equation. *SIAM J. Numer. Anal.* **26** (1989) 884–903.

[15] L.C. Evans, H.M. Soner and P.E. Souganidis, Phase transitions and generalized motion by mean curvature. *Commun. Pure Appl. Math.* **45** (1992) 1097–1123.

[16] X. Feng and Y. Li, Analysis of symmetric interior penalty discontinuous Galerkin methods for the Allen–Cahn equation and the mean curvature flow. *IMA J. Numer. Anal.* **35** (2014) 1622–1651.

[17] X. Feng and A. Prohl, Numerical analysis of the Allen–Cahn equation and approximation for mean curvature flows. *Numer. Math.* **94** (2003) 33–65.

[18] X. Feng and A. Prohl, Error analysis of a mixed finite element method for the Cahn–Hilliard equation. *Numer. Math.* **99** (2004) 47–84.

[19] X. Feng and A. Prohl, Numerical analysis of the Cahn–Hilliard equation and approximation for the Hele-Shaw problem. *Interfaces Free Boundaries* **7** (2005) 1–28.

[20] X. Feng and H.-J. Wu, A posteriori error estimates for finite element approximations of the Cahn–Hilliard equation and the Hele-Shaw flow. *J. Comput. Math.* **26** (2008) 767–796.

[21] X. Feng, Y. Li and A. Prohl, Finite element approximations of the stochastic mean curvature flow of planar curves of graphs. *Stochastic Partial Differ. Equ.: Anal. Comput.* **2** (2014) 54–83.

[22] X. Feng, Y. Li and Y. Xing, Analysis of mixed interior penalty discontinuous Galerkin methods for the Cahn–Hilliard equation and the Hele-Shaw flow. *SIAM J. Numer. Anal.* **54** (2016) 825–847.

[23] X. Feng, Y. Li and Y. Zhang, Finite element methods for the stochastic Allen–Cahn equation with gradient-type multiplicative noise. *SIAM J. Numer. Anal.* **55** (2017) 194–216.

[24] T. Ilmanen, Convergence of the Allen–Cahn equation to Brakke's motion by mean curvature. *J. Differ. Geom.* **38** (1993) 417–461.

[25] M. Kovács, S. Larsson and A. Mesforush, Finite element approximation of the Cahn–Hilliard–Cook equation. *SIAM J. Numer. Anal.* **49** (2011) 2407–2429.

[26] Y. Li, *Numerical methods for deterministic and stochastic phase field models of phase transition and related geometric flows*. Ph.D. thesis, The University of Tennessee (2015).

[27] Y. Li, Error analysis of a fully discrete Morley finite element approximation for the Cahn–Hilliard equation. *J. Sci. Comput.* **78** (2019) 1862–1892.

[28] A. Miranville, The Cahn–Hilliard Equation: Recent Advances and Applications. SIAM (2019).

[29] B.E. Stoth, Convergence of the Cahn–Hilliard equation to the Mullins–Sekerka problem in spherical symmetry. *J. Differ. Equ.* **125** (1996) 154–183.

[30] J. Xu, Y. Li, S. Wu and A. Bousquet, On the stability and accuracy of partially and fully implicit schemes for phase field modeling. *Comput. Methods Appl. Mech. Eng.* **345** (2019) 826–853.

[31] S. Wu and J. Xu, Multiphase Allen–Cahn and Cahn–Hilliard models and their discretizations with the effect of pairwise surface tensions. *J. Comput. Phys.* **343** (2017) 10–32.