ESAIM: M2AN 54 (2020) 929–956
https://doi.org/10.1051/m2an/2019082

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

EXPONENTIAL CONVERGENCE OF CARTESIAN PML METHOD FOR MAXWELL'S EQUATIONS IN A TWO-LAYER MEDIUM

Xiaoqi Duan[1,2], Xue Jiang[3] and Weiying Zheng[4,1,*]

Abstract. The perfectly matched layer (PML) method is extensively studied for scattering problems in homogeneous background media. However, rigorous studies on the PML method in layered media are very rare in the literature, particularly, for three-dimensional electromagnetic scattering problems. Cartesian PML method is favorable in numerical solutions since it is apt to deal with anisotropic scatterers and to construct finite element meshes. Its theories are more difficult than circular PML method due to anisotropic wave-absorbing materials. This paper presents a systematic study on the Cartesian PML method for three-dimensional electromagnetic scattering problem in a two-layer medium. We prove the well-posedness of the PML truncated problem and that the PML solution converges exponentially to the exact solution as either the material parameter or the thickness of PML increases. To the best of the authors' knowledge, this is the first theoretical work on Cartesian PML method for Maxwell's equations in layered media.

Mathematics Subject Classification. 35Q60 and 65N30.

Received April 30, 2019. Accepted November 22, 2019.

1. Introduction

In this paper, we study the Cartesian perfectly matched layer (PML) method for solving the electromagnetic scattering problem in a two-layer medium:

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - k^2\boldsymbol{E} = 0 \qquad \text{in} \quad D_c, \tag{1.1a}$$

$$\boldsymbol{n}_D \times \boldsymbol{E} = \boldsymbol{g} \qquad \text{on} \quad \Gamma_D, \tag{1.1b}$$

$$[\![\boldsymbol{n} \times \mathbf{curl}\,\boldsymbol{E}]\!] = [\![\boldsymbol{n} \times \boldsymbol{E}]\!] = 0 \qquad \text{on} \quad \Sigma, \tag{1.1c}$$

$$\lim_{\rho\to\infty}\int_{\partial B(\rho)} \left|\mathbf{curl}\,\boldsymbol{E} \times \boldsymbol{n} - \mathbf{i}k\boldsymbol{E}\right|^2 = 0, \tag{1.1d}$$

Keywords and phrases. Cartesian perfectly matched layer, exponential convergence, electromagnetic scattering problem, Maxwell's equation, two-layer medium.

[1] School of Mathematical Science, University of Chinese Academy of Sciences, 100190 Beijing, P.R. China.
[2] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, P.R. China.
[3] College of Applied Sciences, Beijing University of Technology, Beijing, P.R. China.
[4] LSEC, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, P.R. China.
*Corresponding author: `zwy@lsec.cc.ac.cn`

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{g}$ determined by the incoming wave, $D \subset \mathbb{R}^3$ a bounded domain with Lipschitz-continuous boundary Γ_D, $D_c = \mathbb{R}^3_\pm \setminus \bar{D}$ the complement of D, $B(\rho)$ the open ball of radius ρ and centering at the origin, and $\boldsymbol{n}_D$ the unit outer normal to D, to $\mathbb{R}^3_-$, or to $B(\rho)$ on their respective boundaries. Here $[\![u]\!]$ denotes the jump of function u across $\Sigma := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_3 = 0\}$, and $\boldsymbol{n} = (0, 0, 1)^\top$ is the unit normal to Σ. We assume that the wave number k is positive and piecewise constant

$$k(\boldsymbol{x}) = \begin{cases} k_+, & \text{if } \boldsymbol{x} \in \mathbb{R}^3_+, \\ k_-, & \text{if } \boldsymbol{x} \in \mathbb{R}^3_-, \end{cases} \tag{1.2}$$

where $\mathbb{R}^3_\pm = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ \pm x_3 > 0\}$. Without loss of generality, we assume $k_- > k_+ > 0$ in this paper. For convenience, we scale the system such that the diameter of the scatterer satisfies $\mathrm{diam}(D) \geq 1$.

The idea of PML is to design a layer of artificial material which attenuates the outgoing waves scattered by the obstacle. One can truncate the exterior domain into a bounded one and impose homogeneous boundary conditions on the truncation boundary. In this way, an approximate problem is proposed on the bounded domain and the approximate solution could converge to the scattering solution exponentially in the region surrounded by PML. Since the pioneering work of Bérénger on PML method [4], it is extensively studied in both the engineering and the mathematical communities. In 2003, Chen and Wu proposed the adaptive PML finite element method for grating problems [12]. Based on a posteriori error estimate, the adaptive method can determine the thickness of PML automatically and reduce finite element errors quasi-optimally. The adaptive PML finite element method was extended to accoustic scattering problems [11,13], to electromagnetic scattering problems [10, 17], to multiple scattering problems [24, 29], and to grating problems [2,3]. The works show that adapt finite element method combined with PML provides an efficient way for solving scattering problems in homogeneous background media.

Generally, the convergence theory of PML method relies on two important aspects:

– exponential decay of the original solution under the complex coordinate stretching, which motivates us to truncate the exterior domain and use zero boundary condition,
– the inf-sup condition for the weak formulation of the truncated problem, which yields the stability and error estimate of the PML solution.

Using dual arguments, Bao and Wu proved the exponential convergence of PML method for Maxwell's equations in homogeneous media [1]. In [5, 6], Bramble and Pasciak introduced the technique of reflection extension to prove the inf-sup condition for the truncated PML problem. They studied cartesian PML methods for both acoustic and electromagnetic scattering problems. For elastic scattering problems, we refer to the recent papers [7,18] on the PML method. The readers are also referred to earlier papers for theoretical studies on PML methods [20, 23, 25, 26]. In this paper, we study the Cartesian PML method for electromagnetic scattering problem in a two-layer medium. Inspired by Bramble and Pasciak, we prove the inf-sup condition for the truncated problem by combining the reflection extension and the exponential decay of the solution.

Wave propagations in inhomogeneous media are difficult for both theoretical analysis and numerical computation [19, 22]. One typical application of this study is to layered media. For homogeneous media, the Green's function usually contains a factor $e^{\mathbf{i}k\rho}$ where ρ denotes the distance function. This term leads directly to the exponential decay of the Green's function when we replace ρ with the complex stretched distance $\tilde{\rho}$ and let $\mathrm{Im}\,\tilde{\rho} \to +\infty$. While for two-layer media, the Green's function comprises emitted waves from the point source, reflected waves by the interface, and transmitted waves through the interface. It is usually written in the form of Fourier integrals from which one can not derive the attenuation factor directly. In [15], Chen and Zheng proved an equivalent form of the Green's function for two-dimensional (2D) Helmholtz equaiton in a two-layer medium by the Cagniard-de Hoop transform [16, 19]. The new form contains a factor $e^{\mathbf{i}k\rho t}$, $t \geq 1$ which decays exponentially when replacing ρ with $\tilde{\rho}$ and letting $\mathrm{Im}\,\tilde{\rho} \to +\infty$. They established the stability and exponential convergence of the PML method. More recently, they extended the results to circular PML method for three-dimensional (3D) Maxwell's equation in a two-layer medium.

In practical computations, uniaxial PML method or Cartesian PML method is preferable to circular PML method when dealing with anisotropic scatterers or when subdividing the domain into finite element meshes. However, its convergence theory is still absent in the literature for Maxwell's equation in two-layer media. The objective of this paper is to fulfill this task. The contributions of this work are listed as follows.

(1) We prove that the original solution is attenuated exponentially in PML. The key ingredient of the proof lies in the analytic continuation of the Cagniard-de Hoop transform from real coordinates to complex coordinates. Since the medium parameter of Cartesian PML in one axis direction degenerates in the others, the extension of theories from circular PML to Cartesian PML method is nontrivial.
(2) We prove the stability and exponential convergence of the PML solution. We develop new techniques to prove the well-posedness of Maxwell's equation under anisotropic coordinate stretching. The proofs are simpler than those in [16] for circular PML.

The layout of this paper is organized as follows. In Section 2, we introduce some Sobolev spaces and trace operators and present the dyadic Green's function in the two-layer medium. The well-posedness of problem (1.1) is also cited from the literature. In Section 3, we propose the Cartesian PML and present the main result of the paper. In Section 4, we prove the exponential attenuation of the stretched Green's function by the analytic extension of the Cagniard-de Hoop transform. Based on this and the analytic extension of the Stratton-Chu formula, we prove the exponential attenuation of the exact solution in PML. In Section 5, we prove the well-posedness of the exterior problem of modified Maxwell equation under the PML transform. In Section 6, we prove that the PML problem has a unique solution which converges exponentially to the exact solution.

2. Well-posedness of the scattering problem

The purpose of this section is to introduce an integral representation of the scattering solution. It is an important tool for studying the propagating behavior of scattering waves in the layered medium. Throughout the paper, we shall always use the convention that, for any $\xi \in \mathbb{C}$, $\xi^{1/2}$ is the branch of the square root $\sqrt{\xi}$ such that $\operatorname{Re}(\xi^{1/2}) \geq 0$. This corresponds to the left half real axis as the branch cut in the complex plane. This yields

$$\xi^{1/2} = \sqrt{\frac{|\xi| + \operatorname{Re}\xi}{2}} + \mathbf{i}\,\operatorname{sign}(\operatorname{Im}\xi)\sqrt{\frac{|\xi| - \operatorname{Re}\xi}{2}}. \tag{2.1}$$

2.1. Sobolev spaces and trace operators

First we introduce some Sobolev spaces and trace operators which will be used in this paper. For a domain $\Omega \subset \mathbb{R}^3$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$, let $L^2(\Omega)$ be the space of square-integrable functions, $H^1(\Omega) \subset L^2(\Omega)$ the subspace whose functions have square-integrable gradients, and $\boldsymbol{H}(\mathbf{curl}, \Omega) \subset \boldsymbol{L}^2(\Omega)$ the subspace whose functions have square-integrable curls. Throughout the paper we denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^3$.

From [8], we have the surjective mappings

$$\begin{aligned}
&\gamma: \ H^1(\Omega) \to H^{1/2}(\Gamma), && \gamma\varphi = \varphi && \text{on} \quad \Gamma,\\
&\gamma_t: \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma), && \gamma_t \boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{u} && \text{on} \quad \Gamma,\\
&\gamma_T: \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\mathrm{Curl}, \Gamma), && \gamma_T \boldsymbol{u} = \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) && \text{on} \quad \Gamma,
\end{aligned}$$

where Div, Curl stand for the surface divergence operator and the surface scalar curl operator respectively. For any $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, it holds that

$$\mathrm{Div}(\gamma_t \boldsymbol{u}) = -\boldsymbol{n} \cdot \mathbf{curl}\, \boldsymbol{u}, \quad \mathrm{Curl}(\gamma_T \boldsymbol{u}) = \boldsymbol{n} \cdot \mathbf{curl}\, \boldsymbol{u} \quad \text{on} \quad \Gamma.$$

It is known that $\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma)$ and $\boldsymbol{H}^{-1/2}(\mathrm{Curl}, \Gamma)$ are dual spaces.

For any $S \subset \Gamma$, the subspaces with zero trace and zero tangential trace on S are denoted respectively by

$$\begin{aligned} H^1_S(\Omega) &:= \{v \in H^1(\Omega) : \gamma v = 0 \ \text{ on } S\}, \\ \boldsymbol{H}_S(\mathbf{curl}, \Omega) &:= \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \gamma_t \boldsymbol{v} = 0 \ \text{ on } S\}. \end{aligned}$$

In particular, the conventional notations will be used

$$H^1_0(\Omega) := H^1_\Gamma(\Omega), \qquad \boldsymbol{H}_0(\mathbf{curl}, \Omega) := \boldsymbol{H}_\Gamma(\mathbf{curl}, \Omega).$$

2.2. Dyadic Green's function

Let $\mathbb{G}(k; \boldsymbol{x}, \boldsymbol{y})$ be the dyadic Green's function for the electromagnetic scattering problem. It satisfies the time-harmonic Maxwell's equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbb{G}(k; \boldsymbol{x}, \cdot) - k_\pm^2 \mathbb{G}(k; \boldsymbol{x}, \cdot) = \delta_{\boldsymbol{x}} \mathbb{I} \qquad \text{in} \quad \mathbb{R}^3_\pm, \tag{2.2a}$$

$$[\![\gamma_t (\mathbf{curl}\,\mathbb{G})(k; \boldsymbol{x}, \cdot)]\!] = [\![(\gamma_t \mathbb{G})(k; \boldsymbol{x}, \cdot)]\!] = 0 \qquad \text{on} \quad \Sigma, \tag{2.2b}$$

$$\lim_{\rho \to \infty} \int_{\partial B(\rho)} |\gamma_t (\mathbf{curl}\,\mathbb{G})(k; \boldsymbol{x}, \cdot) - \mathbf{i} k \mathbb{G}(k; \boldsymbol{x}, \cdot)|^2 = 0, \tag{2.2c}$$

where $\delta_{\boldsymbol{x}}(\boldsymbol{y}) = \delta(|x_1 - y_1|)\delta(|x_2 - y_2|)\delta(|x_3 - y_3|)$ stands for the Dirac source at $\boldsymbol{x} \in \mathbb{R}^3$ and $\mathbb{I}$ is the 3×3 identity matrix. From [16] (also see [21, 27]), we can write $\mathbb{G}$ with the Hertz tensor $\mathbb{H}$ as follows

$$\mathbb{G}(k; \boldsymbol{x}, \boldsymbol{y}) = \mathbb{H}(k; \boldsymbol{x}, \boldsymbol{y}) + k_\pm^{-2} \nabla_{\boldsymbol{y}} \operatorname{div}_{\boldsymbol{y}} \mathbb{H}(k; \boldsymbol{x}, \boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \mathbb{R}^3_\pm. \tag{2.3}$$

Here $\mathbb{H}$ satisfies the Helmholtz equation in matrix form

$$\Delta \mathbb{H}(k; \boldsymbol{x}, \cdot) + k_\pm^2 \mathbb{H}(k; \boldsymbol{x}, \cdot) = -\delta_{\boldsymbol{x}} \mathbb{I} \qquad \text{in} \quad \mathbb{R}^3_\pm, \tag{2.4a}$$

$$[\![\mathbb{H}(k; \boldsymbol{x}, \cdot)]\!] = [\![\gamma_t (\mathbf{curl}\,\mathbb{H})(k; \boldsymbol{x}, \cdot)]\!] = 0 \qquad \text{on} \quad \Sigma, \tag{2.4b}$$

$$\left[\!\!\left[k^{-2} \operatorname{div} \mathbb{H}(k; \boldsymbol{x}, \cdot) \right]\!\!\right] = 0 \qquad \text{on} \quad \Sigma, \tag{2.4c}$$

$$\lim_{\rho \to \infty} \int_{\partial B(\rho)} \left| \frac{\partial \mathbb{H}(k; \boldsymbol{x}, \cdot)}{\partial \boldsymbol{n}} - \mathbf{i} k_\pm \mathbb{H}(k; \boldsymbol{x}, \cdot) \right|^2 = 0. \tag{2.4d}$$

Let $\Phi(\omega; \boldsymbol{x}, \boldsymbol{y}) := \frac{e^{\mathbf{i}\omega|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$ be the fundamental solution of the scalar Helmholtz equation

$$\Delta \Phi(\omega; \boldsymbol{x}, \cdot) + \omega^2 \Phi(\omega; \boldsymbol{x}, \cdot) = -\delta_{\boldsymbol{x}} \qquad \text{in} \quad \mathbb{R}^3$$

with constant wave number $\omega > 0$. It is well-known that $\Phi(\omega; \boldsymbol{x}, \boldsymbol{y})$ stands for waves emitted from the point source at $\boldsymbol{x}$. Due to the presence of the interface Σ, the point source has a reflection $\Phi(\omega; \boldsymbol{x}', \boldsymbol{y})$ where $\boldsymbol{x}' = (x_1, x_2, -x_3)^\top$. Define the double source tensor by

$$\mathbb{S}(k; \boldsymbol{x}, \boldsymbol{y}) = \mathbb{I} \times \begin{cases} \Phi(k_+; \boldsymbol{x}, \boldsymbol{y}) - \Phi(k_+; \boldsymbol{x}', \boldsymbol{y}) & \text{if} \quad x_3 > 0, \ y_3 > 0, \\ \Phi(k_-; \boldsymbol{x}, \boldsymbol{y}) - \Phi(k_-; \boldsymbol{x}', \boldsymbol{y}) & \text{if} \quad x_3 < 0, \ y_3 < 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.5}$$

The residual $\mathbb{P} := \mathbb{S} - \mathbb{H}$ is called the perturbation tensor. The analytic forms of $\mathbb{P}$ are presented in Appendix A. Both $\Phi(k_+; \boldsymbol{x}, \cdot)$ and $\Phi(k_+; \boldsymbol{x}', \cdot)$ satisfy the Helmholtz equations in homogeneous material. The treatment for $\mathbb{S}$ is easy and standard. While the form of $\mathbb{P}$ is complicated and need elaborate analysis.

Now we cite Lemma 2.1 of [16] to estimate the singularities of $\mathbb{P}$ and its partial derivatives. From (2.5) and the definition of Φ, $\mathbb{S}$ satisfies the same estimates as $\mathbb{P}$.

Lemma 2.1. *There exists a constant $C > 0$ depending only on $k_\pm$ such that, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm$, $n = 0, 1$, and $j = 1, 2, 3$,*

$$\left|\frac{\partial^n}{\partial x_j^n}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| + \left|\frac{\partial^n}{\partial y_j^n}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\left(1 + |\boldsymbol{x}-\boldsymbol{y}|^{-n-1}\right) \quad \text{if } x_3y_3 < 0, \tag{2.6}$$

$$\left|\frac{\partial^n}{\partial x_j^n}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| + \left|\frac{\partial^n}{\partial y_j^n}\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})\right| \le C\left(1 + |\boldsymbol{x}-\boldsymbol{y}'|^{-n-1}\right) \quad \text{if } x_3y_3 > 0. \tag{2.7}$$

2.3. Integral representation of $\boldsymbol{E}$

For completeness of the paper, we cite Theorem 2.2 of [16] to give the well-posedness of the scattering problem (1.1).

Theorem 2.2. *For any $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$, (1.1) has a unique solution. For any bounded domain $\Omega \subset D_c$, there is a constant C depending only on k, Ω such that*

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\,\|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}. \tag{2.8}$$

For any $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$, we define the Maxwell single-layer and double-layer potentials as follows

$$\boldsymbol{\Psi}_{\mathrm{SL}}(\boldsymbol{\mu})(\boldsymbol{x}) = \int_{\Gamma_D} \gamma_T \mathbb{G}^\top(k;\boldsymbol{x},\boldsymbol{y})\boldsymbol{\mu}(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}}, \tag{2.9}$$

$$\boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{\lambda})(\boldsymbol{x}) = \int_{\Gamma_D} \gamma_T (\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G})^\top(k;\boldsymbol{x},\boldsymbol{y})\,\boldsymbol{\lambda}(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}}. \tag{2.10}$$

By Section 12.4.3 of [27], $\boldsymbol{E}$ admits the integral representation

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{\Psi}_{\mathrm{SL}}(\boldsymbol{\mu})(\boldsymbol{x}) + \boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{g})(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in D_c, \tag{2.11}$$

where $\boldsymbol{g} = \gamma_t \boldsymbol{E}$ and $\boldsymbol{\mu} = \gamma_t(\mathbf{curl}\,\boldsymbol{E})$ are the Dirichlet and Neumann traces of $\boldsymbol{E}$ on Γ_D respectively.

3. The PML problem

This section presents the main result of the paper, namely, the stability and exponential convergence of the Cartesian PML method. Unlike circular PML which stretches the vector norm $|\boldsymbol{x}|$, the Cartesian PML stretches three coordinates x_1, x_2, x_3 independently.

3.1. Cartesian perfectly matched layer

For any positive triple index $\boldsymbol{l} = (l_1, l_2, l_3)$, let $B_{\boldsymbol{l}} := \{(x_1, x_2, x_3) : |x_i| < l_i,\ i = 1, 2, 3\}$ be the open cuboid which contains $\bar{D}$ and define $\Gamma_{\boldsymbol{l}} := \partial B_{\boldsymbol{l}}$ and $\Omega_{\boldsymbol{l}} := B_{\boldsymbol{l}} \backslash \bar{D}$.

Let $\boldsymbol{L}$ be a given positive triple index such that $B_{\boldsymbol{L}}$ contains D and the scattering field in $B_{\boldsymbol{L}}$ is interested. The Cartesian PML is realized by means of complex coordinate stretching

$$\tilde{x}_j = x_j + (1+\mathbf{i})\sigma_0 \int_0^{x_j} \sigma(t/L_j)\,\mathrm{d}t, \qquad j = 1, 2, 3, \tag{3.1}$$

where $\sigma_0 > 0$ is the constant medium parameter and $\sigma \in C^1(\mathbb{R})$ is defined by

$$\sigma(t) = \begin{cases} 0 & \text{if } \ |t| \le 1, \\ 8(7-4|t|)(|t|-1)^2 & \text{if } \ 1 < |t| < 1.5, \\ 2 & \text{if } \ |t| \ge 1.5. \end{cases} \tag{3.2}$$

It is easy to see that the complex stretching $\boldsymbol{F}(\boldsymbol{x}) := \tilde{\boldsymbol{x}}$ is C^2-smooth. Hereafter, both $\boldsymbol{F}(\boldsymbol{x})$ and $\tilde{\boldsymbol{x}}$ will denote the same vector.

Remark 3.1. The definition of $\sigma(t)$ can be more general. Here we just choose one case that σ is even and defined by a third-order polynomial in $[1, 1.5]$. Moreover, the requirement that $\sigma = \text{Constant}$ in $(-\infty, -1.5] \cup [1.5, +\infty)$ is in favor of defining reflection extension in Section 6.

The Jacobi matrix and the Jacobi determinant of $\boldsymbol{F}(\boldsymbol{x})$ are given by

$$\mathbb{B} := D\boldsymbol{F} = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3), \qquad J = \det(\mathbb{B}) = \alpha_1\alpha_2\alpha_3, \tag{3.3}$$

where

$$\alpha_j = 1 + (1 + \mathbf{i})\beta_j, \qquad \beta_j(x_j) = \sigma_0\sigma(x_j/L_j), \qquad j = 1, 2, 3.$$

Clearly $\mathbb{B}$ and J are C^1-smooth. Moreover, $\operatorname{Re}\alpha_j > \operatorname{Im}\alpha_j$, which is useful in proving the stability of the solution of the stretched Maxwell equation in $\mathbb{R}^3$. We extend the distance functions analytically from real coordinates to complex coordinates

$$\begin{aligned} r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) &= \left[(\tilde{x}_1 - \tilde{y}_1)^2 + (\tilde{x}_2 - \tilde{y}_2)^2\right]^{1/2}, \\ \rho(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) &= \left[(\tilde{x}_1 - \tilde{y}_1)^2 + (\tilde{x}_2 - \tilde{y}_2)^2 + (\tilde{x}_3 - \tilde{y}_3)^2\right]^{1/2}. \end{aligned}$$

3.2. Analytic extension of $\mathbb{G}$

The analytic extension of $\mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is defined by replacing $\boldsymbol{x}$, $\boldsymbol{y}$, $|\boldsymbol{x} - \boldsymbol{y}|$ with $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, and $\rho(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ in (2.5) respectively. The analytic extension of $\mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is defined by replacing

$$\boldsymbol{x}, \quad \boldsymbol{y}, \quad \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad |x_3| + |y_3|, \quad |\boldsymbol{x} - \boldsymbol{y}|$$

in (A.12)–(A.14) respectively with

$$\tilde{\boldsymbol{x}}, \quad \tilde{\boldsymbol{y}}, \quad r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \quad (\tilde{x}_3^2)^{1/2} + (\tilde{y}_3^2)^{1/2}, \quad \rho(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}).$$

Therefore, we obtain the analytic extension of the Hertz tensor

$$\mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{S}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) - \mathbb{P}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}). \tag{3.4}$$

Similar to (2.3), the analytic extension of the dyadic Green's function is defined by

$$\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) + k_\pm^{-2}\nabla_{\tilde{\boldsymbol{y}}} \operatorname{div}_{\tilde{\boldsymbol{y}}} \mathbb{H}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_\pm, \tag{3.5}$$

where $\nabla_{\tilde{\boldsymbol{y}}}$ denotes the gradient operator with respect to $\tilde{\boldsymbol{y}}$ and $\operatorname{div}_{\tilde{\boldsymbol{y}}}$ the divergence operator with respect to $\tilde{\boldsymbol{y}}$.

3.3. Analytic extension of the scattering solution

Now we define the analytic extension of the scattering solution $\boldsymbol{E}$. In view of (2.9) and (2.10), we define the analytic extensions of the single-layer and double-layer potentials as follows, for any $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \boldsymbol{H}^{-1/2}(\operatorname{Div}, \Gamma_D)$,

$$\begin{aligned} \boldsymbol{\Psi}_{\mathrm{SL}}(\boldsymbol{\mu})(\tilde{\boldsymbol{x}}) &= \int_{\Gamma_D} \gamma_T \mathbb{G}^\top(k; \tilde{\boldsymbol{x}}, \boldsymbol{y})\boldsymbol{\mu}(\boldsymbol{y})\, \mathrm{d}S_{\boldsymbol{y}}, & (3.6) \\ \boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{\lambda})(\tilde{\boldsymbol{x}}) &= \int_{\Gamma_D} \gamma_T (\mathbf{curl}_{\boldsymbol{y}}\, \mathbb{G})^\top(k; \tilde{\boldsymbol{x}}, \boldsymbol{y})\, \boldsymbol{\lambda}(\boldsymbol{y})\, \mathrm{d}S_{\boldsymbol{y}}. & (3.7) \end{aligned}$$

The analytic extension of $\boldsymbol{E}$ is defined by

$$\boldsymbol{E}(\tilde{\boldsymbol{x}}) = \boldsymbol{\Psi}_{\mathrm{SL}}(\gamma_t(\mathbf{curl}\,\boldsymbol{E}))(\tilde{\boldsymbol{x}}) + \boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{g})(\tilde{\boldsymbol{x}}) \qquad \forall\, \boldsymbol{x} \in D_c \cap \mathbb{R}^3_\pm. \tag{3.8}$$

3.4. The modified Maxwell equation

Under the complex stretching, we define the stretched gradient, curl, divergence, and Laplace operators respectively as follows

$$\begin{aligned} \tilde{\nabla} v &:= \mathbb{B}^{-\top}\nabla v, & \tilde{\nabla}\times \boldsymbol{u} &:= J^{-1}\mathbb{B}\,\mathbf{curl}(\mathbb{B}^\top \boldsymbol{u}),\\ \tilde{\nabla}\cdot \boldsymbol{u} &:= J^{-1}\operatorname{div}(J\mathbb{B}^{-1}\boldsymbol{u}), & \tilde{\Delta} v &:= J^{-1}\operatorname{div}(\mathbb{A}^{-1}\nabla v), \end{aligned} \tag{3.9}$$

where the coefficient matrix $\mathbb{A}$ is defined by

$$\mathbb{A} := J^{-1}\mathbb{B}^\top\mathbb{B} = J^{-1}\mathrm{diag}\left(\alpha_1^2, \alpha_2^2, \alpha_3^2\right). \tag{3.10}$$

For two functions $\xi(\tilde{\boldsymbol{x}})$ and $\boldsymbol{\eta}(\tilde{\boldsymbol{x}})$, it is easy to see that $v := \xi\circ \boldsymbol{F}$, $\boldsymbol{u} := \boldsymbol{\eta}\circ\boldsymbol{F}$ satisfy

$$\tilde{\nabla} v = \nabla_{\tilde{\boldsymbol{x}}}\xi, \quad \tilde{\Delta} v = \tilde{\nabla}\cdot(\tilde{\nabla}\xi), \quad \tilde{\nabla}\cdot\boldsymbol{u} = \nabla_{\tilde{\boldsymbol{x}}}\cdot\boldsymbol{\eta}, \quad \tilde{\nabla}\times\boldsymbol{u} = \nabla_{\tilde{\boldsymbol{x}}}\times\boldsymbol{\eta}. \tag{3.11}$$

Lemma 3.2. *For any fixed $\boldsymbol{x}\in\mathbb{R}^3_\pm$, $\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ satisfies the stretched equations*

$$\tilde{\nabla}\times\tilde{\nabla}\times\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) - k_\pm^2\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = J^{-1}\delta_{\boldsymbol{x}}\mathbb{I} \quad \text{in} \quad \mathbb{R}^3_\pm, \tag{3.12}$$

$$[\![\boldsymbol{n}\times\mathbb{B}^\top\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))]\!] = \left[\!\!\left[\boldsymbol{n}\times\mathbb{B}^\top\tilde{\nabla}\times\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))\right]\!\!\right] = 0 \quad \text{on} \quad \Sigma. \tag{3.13}$$

Proof. Equation (3.12) follows directly from (2.2a) and (3.9). It is left to show (3.13).

For any fixed $\boldsymbol{x}\in\mathbb{R}^3_\pm$, write $\mathbb{J}_0(\boldsymbol{x},\cdot) := [\![\mathbb{G}(k;\boldsymbol{x},\cdot)]\!]$ and $\mathbb{J}_1(\boldsymbol{x},\cdot) := [\![(\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G})(k;\boldsymbol{x},\cdot)]\!]$ on Σ. From (2.2b), we have $\boldsymbol{n}\times\mathbb{J}_0(\boldsymbol{x},\cdot) = \boldsymbol{n}\times\mathbb{J}_1(\boldsymbol{x},\cdot) = 0$ on Σ. Since $\tilde{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ is C^2-smooth, we have

$$\boldsymbol{n}\times\mathbb{J}_0(\tilde{\boldsymbol{x}},\cdot) = \boldsymbol{n}\times\mathbb{J}_1(\tilde{\boldsymbol{x}},\cdot) = 0 \quad \text{on} \quad \Sigma.$$

Similarly the smoothness of $\boldsymbol{F}(\cdot)$ implies

$$\boldsymbol{n}\times\mathbb{J}_0(\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = \boldsymbol{n}\times\mathbb{J}_1(\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = 0 \quad \text{on} \quad \Sigma.$$

Let J_{ij} be the entries of $\mathbb{J}_0$. The equality $\boldsymbol{n}\times\mathbb{J}_0 = 0$ is equivalent to

$$\begin{pmatrix} \mathrm{J}_{11} & \mathrm{J}_{12} & \mathrm{J}_{13}\\ \mathrm{J}_{21} & \mathrm{J}_{22} & \mathrm{J}_{23}\\ 0 & 0 & 0 \end{pmatrix}(\tilde{\boldsymbol{x}},\boldsymbol{F}(\boldsymbol{y})) = 0 \quad \forall\,\boldsymbol{y}\in\Sigma.$$

Since $\mathbb{B}$ is diagonal, it follows that

$$\boldsymbol{n}\times\mathbb{B}^\top\mathbb{J}_0(\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = \begin{pmatrix} \mathrm{B}_{11}\mathrm{J}_{11} & \mathrm{B}_{11}\mathrm{J}_{12} & \mathrm{B}_{11}\mathrm{J}_{13}\\ \mathrm{B}_{22}\mathrm{J}_{21} & \mathrm{B}_{22}\mathrm{J}_{22} & \mathrm{B}_{22}\mathrm{J}_{23}\\ 0 & 0 & 0 \end{pmatrix}(\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = 0 \quad \text{on} \quad \Sigma, \tag{3.14}$$

where $\mathrm{B}_{11}, \mathrm{B}_{22}$ are diagonal entries of $\mathbb{B}$. Similarly we obtain

$$\boldsymbol{n}\times\mathbb{B}^\top\mathbb{J}_1(\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = 0 \quad \text{on} \quad \Sigma. \tag{3.15}$$

Note that $(\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G})(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ is obtained by taking the curl of $\mathbb{G}(k;\cdot,\cdot)$ with respect to the second argument and evaluating $\mathbf{curl}\,\mathbb{G}$ at $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$. It equals to taking the curl of $\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ with respect to $\tilde{\boldsymbol{y}}$. Therefore, we have

$$\mathbb{J}_1(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}) = [\![(\mathbf{curl}_{\boldsymbol{y}}\,\mathbb{G})(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})]\!] = [\![\mathbf{curl}_{\tilde{\boldsymbol{y}}}\,\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})]\!] \quad \forall\,\boldsymbol{y}\in\Sigma.$$

Since $\mathbb{B}$ is C^1-smooth, using (3.11), we find that

$$\mathbb{B}^\top\mathbb{J}_1(\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = \mathbb{B}^\top\left[\!\!\left[\tilde{\nabla}\times\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))\right]\!\!\right] = \left[\!\!\left[\mathbb{B}^\top\tilde{\nabla}\times\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))\right]\!\!\right] \quad \text{on} \quad \Sigma.$$

The second equality of (3.13) follows directly from (3.15). Similarly, from (3.14), we also have $[\![\boldsymbol{n}\times\mathbb{B}^\top\mathbb{G}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))]\!] = 0$ on Σ. The proof is completed. $\square$

Remember that the Hertz tensor $\mathbb{H}$ satisfies (2.4). By arguments similar to the proof of Lemma 3.2, we can also show that the stretched Hertz tensor $\mathbb{H}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ satisfies

$$\begin{aligned}
&\tilde{\Delta}\mathbb{H}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) + k_\pm^2\mathbb{H}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot)) = -J^{-1}\delta_{\boldsymbol{x}}\mathbb{I} && \text{in} \quad \mathbb{R}^3_\pm,\\
&[\![\mathbb{H}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))]\!] = \left[\!\!\left[\boldsymbol{n}\times\mathbb{B}^\top\tilde{\nabla}\times\mathbb{H}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))\right]\!\!\right] = 0 && \text{on} \quad \Sigma,\\
&\left[\!\!\left[k^{-2}\tilde{\nabla}\cdot\mathbb{H}(k;\tilde{\boldsymbol{x}},\boldsymbol{F}(\cdot))\right]\!\!\right] = 0 && \text{on} \quad \Sigma.
\end{aligned}$$

The details are omitted here.

In view of (1.1) and by arguments similar to the proof of Lemma 3.2, we find that the stretched solution $\boldsymbol{E}(\tilde{\boldsymbol{x}}) = (\boldsymbol{E}\circ\boldsymbol{F})(\boldsymbol{x})$ satisfies

$$\begin{aligned}
&\tilde{\nabla}\times\tilde{\nabla}\times(\boldsymbol{E}\circ\boldsymbol{F}) - k_\pm^2(\boldsymbol{E}\circ\boldsymbol{F}) = 0 && \text{in} \quad D_c\cap\mathbb{R}^3_\pm,\\
&\left[\!\!\left[\boldsymbol{n}\times\mathbb{B}^\top(\boldsymbol{E}\circ\boldsymbol{F})\right]\!\!\right] = \left[\!\!\left[\boldsymbol{n}\times\mathbb{B}^\top\tilde{\nabla}\times(\boldsymbol{E}\circ\boldsymbol{F})\right]\!\!\right] = 0 && \text{on} \quad \Sigma,\\
&\gamma_t(\boldsymbol{E}\circ\boldsymbol{F}) = \boldsymbol{g} && \text{on} \quad \Gamma_D.
\end{aligned}$$

Define $\tilde{\boldsymbol{E}} := \mathbb{B}^\top\boldsymbol{E}\circ\boldsymbol{F}$. Since the Jacobian determinant J is C^1-smooth, we have

$$\left[\!\!\left[\boldsymbol{n}\times\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}\right]\!\!\right] = J^{-1}\left[\!\!\left[\boldsymbol{n}\times\mathbb{B}^\top\tilde{\nabla}\times(\boldsymbol{E}\circ\boldsymbol{F})\right]\!\!\right] = 0 \quad \text{on} \quad \Sigma.$$

Since $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$ for any $\boldsymbol{x}\in\Gamma_D$, the relations in (3.11) show that $\tilde{\boldsymbol{E}}$ satisfies

$$\boxed{\begin{aligned}
&\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}}) - k^2\mathbb{A}^{-1}\tilde{\boldsymbol{E}} = 0 && \text{in} \quad D_c\cap\mathbb{R}^3_\pm, && (3.16a)\\
&\left[\!\!\left[\gamma_t(\mathbb{A}\,\mathbf{curl}\,\tilde{\boldsymbol{E}})\right]\!\!\right] = \left[\!\!\left[\gamma_t\tilde{\boldsymbol{E}}\right]\!\!\right] = 0 && \text{on} \quad \Sigma, && (3.16b)\\
&\gamma_t\tilde{\boldsymbol{E}} = \boldsymbol{g} && \text{on} \quad \Gamma_D. && (3.16c)
\end{aligned}}$$

3.5. The main result

The modified problem (3.16) is still proposed on unbounded domain. The essence of the PML method is to solve the modified Maxwell equation on a bounded truncation domain.

Let $\boldsymbol{l} = (l_1, l_2, l_3)$ satisfy $l_j \geq 3L_j$, $j = 1,2,3$, and define $\Omega_{\boldsymbol{l}} = B_{\boldsymbol{l}}\cap D_c$. The approximate problem of (3.16) is proposed on $\Omega_{\boldsymbol{l}}$ as follows:

$$\boxed{\begin{aligned}
&\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{E}}) - k^2\mathbb{A}^{-1}\hat{\boldsymbol{E}} = 0 && \text{in} \quad \Omega_{\boldsymbol{l}}\cap\mathbb{R}^3_\pm, && (3.17a)\\
&\left[\!\!\left[\gamma_t(\mathbb{A}\,\mathbf{curl}\,\hat{\boldsymbol{E}})\right]\!\!\right] = \left[\!\!\left[\gamma_t\hat{\boldsymbol{E}}\right]\!\!\right] = 0 && \text{on} \quad \Sigma\cap\Omega_{\boldsymbol{l}}, && (3.17b)\\
&\gamma_t\hat{\boldsymbol{E}} = \boldsymbol{g} && \text{on} \quad \Gamma_D, && (3.17c)\\
&\gamma_t\hat{\boldsymbol{E}} = 0 && \text{on} \quad \Gamma_{\boldsymbol{l}} := \partial B_{\boldsymbol{l}}. && (3.17d)
\end{aligned}}$$

Now we present the main result of this paper, which states that, $\hat{\boldsymbol{E}}$ converges exponentially to the scattering solution $\boldsymbol{E}$ in $\Omega_{\boldsymbol{L}}$ as $|\boldsymbol{l}|\to+\infty$. The proof of Theorem 3.3 will be given in Section 6.2. For convenience, we define

$$L_{\max} = \max(L_1, L_2, L_3), \qquad l_{\max} = \max(l_1, l_2, l_3), \qquad l_{\min} = \min(l_1, l_2, l_3).$$

Theorem 3.3. *Suppose $\sigma_0 \geq 5$ and $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$, and $l_j \geq 3L_j$, $j = 1, 2, 3$. There exists a constant $C_1 > 0$ independent of $\boldsymbol{l}$ and σ_0 such that, when $l_{\min} \geq C_1 \sigma_0^{11}$ and $l_{\max} \geq 7k_- k_+^{-1} L_{\max}$,*

- *the PML problem (3.17) has a unique solution $\hat{\boldsymbol{E}} \in \boldsymbol{H}_{\Gamma_{\boldsymbol{l}}}(\mathbf{curl}, \Omega_{\boldsymbol{l}})$, and*
- *there exists a constant $C > 0$ independent of σ_0 and $\boldsymbol{l}$ such that*

$$\left\| \hat{\boldsymbol{E}} - \boldsymbol{E} \right\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{L}})} \leq C \sigma_0^2 l_{\max}^4 e^{-0.7 k_+ \sigma_0 d_{\mathrm{PML}}} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)},$$

where $d_{\mathrm{PML}} := l_{\min} - L_{\max}$ is the thickness of the PML, and $\boldsymbol{E}$ is the solution of (1.1).

Remark 3.4. The requirements for $l_j \geq 3L_j$ and $\sigma = \mathrm{Const.}$ in $(0.5 l_j, l_j)$ will be used in defining the reflection extension in Section 6.1.

4. Exponential attenuation of $\boldsymbol{E}(\tilde{\boldsymbol{x}})$

The purpose of this section is to prove that $\boldsymbol{E}(\tilde{\boldsymbol{x}})$ decays exponentially as $|\boldsymbol{x}| \to +\infty$. First we prove some important estimates on $\rho(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ and $\mathbb{G}(k; \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$. Write

$$\begin{aligned} &\rho = \rho(\boldsymbol{x}, \boldsymbol{y}), \quad r = r(\boldsymbol{x}, \boldsymbol{y}), \quad z = x_3 - y_3, \\ &\tilde{\rho} = \rho(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \quad \tilde{r} = r(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}), \quad \tilde{z} = \tilde{x}_3 - \tilde{y}_3, \end{aligned} \tag{4.1}$$

for convenience. We also use the notation

$$d(\boldsymbol{x}, \boldsymbol{y}) := \max_{j=1,2,3} |x_j - y_j|. \tag{4.2}$$

4.1. Useful estimates on $\boldsymbol{\rho(\tilde{x}, \tilde{y})}$

Since $\sigma \geq 0$, it is easy to see

$$(x_j - y_j) \operatorname{Im}(\tilde{x}_j - \tilde{y}_j) = \sigma_0 (x_j - y_j) \int_{y_j}^{x_j} \sigma(t/L_j) \, \mathrm{d}t \geq 0, \qquad j = 1, 2, 3.$$

Use the convention in (2.1), we know that

$$0 \leq \operatorname{Im} \tilde{r} \leq \operatorname{Re} \tilde{r}, \qquad 0 \leq \operatorname{Im} \tilde{z} \leq \operatorname{Re} \tilde{z}, \qquad 0 \leq \operatorname{Im} \tilde{\rho} \leq \operatorname{Re} \tilde{\rho}. \tag{4.3}$$

Since $\operatorname{Re}(\tilde{x}_j - \tilde{y}_j) = x_j - y_j + \operatorname{Im}(\tilde{x}_j - \tilde{y}_j)$, we have

$$\operatorname{Re} \tilde{r}^2 \leq \operatorname{Re} \tilde{\rho}^2, \qquad \operatorname{Im} \tilde{r}^2 \leq \operatorname{Im} \tilde{\rho}^2, \qquad |\tilde{r}|^2 = \left|\tilde{r}^2\right| \leq \left|\tilde{\rho}^2\right| = |\tilde{\rho}|^2. \tag{4.4}$$

Lemma 4.1. *Assume $\sigma_0 \geq 5$ and $|x_j| > |y_j|$ for $j = 1, 2, 3$. Then*

- *if $x_j - y_j \geq 4L_j$, $\operatorname{Im}(\tilde{x}_j - \tilde{y}_j) \geq 0.68 \sigma_0 (x_j - y_j)$,*
- *if $x_j - y_j \geq 6L_j$, $\operatorname{Im}(\tilde{x}_j - \tilde{y}_j) \geq 1.1 \sigma_0 (x_j - y_j)$, for $j = 1, 2, 3$.*

Proof. We only prove the case of $j = 1$ and $x_1 - y_1 \geq 4L_1$. The proof for $x_j - y_j \geq 6L_j$ is similar and omitted here. In this case, we have $x_1 \geq 2L_1$ and $\operatorname{Im} \tilde{x}_1 = \sigma_0 (2x_1 - 2.5L_1)$. Now we prove the inequality for different ranges of y_1.

- For $y_1 \geq 1.5L_1$, we have $\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) = 2\sigma_0 (x_1 - y_1)$.

– For $y_1 \in [L_1, 1.5L_1]$, we have

$$\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) \geq 2\sigma_0(x_1 - 1.5L_1) \geq 1.5\sigma_0(x_1 - y_1).$$

– For $y_1 \in [-1.35L_1, L_1]$, we have

$$\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) \geq \operatorname{Im}\tilde{x}_1 = \sigma_0[2(x_1 - y_1) + 2y_1 - 2.5L_1] \geq 0.7\sigma_0(x_1 - y_1).$$

– For $y_1 \in [-1.5L_1, -1.35L_1]$, we have

$$\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) \geq \sigma_0(2x_1 - 2.5L_1) + 0.22\sigma_0 \geq 0.68\sigma_0(x_1 - y_1).$$

– For $y_1 < -1.5L_1$, we have

$$\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) = 2\sigma_0(x_1 - y_1 - 2.5L_1) \geq \sigma_0(x_1 - y_1).$$

We conclude that $\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) \geq 0.68\sigma_0(x_1 - y_1)$ for all cases. □

Lemma 4.2. *Suppose $\sigma_0 \geq 5$ and that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ satisfy $d(\boldsymbol{x}, \boldsymbol{y}) \geq 6L_{\max}$. Then*

$$\operatorname{Re}\tilde{\rho} \leq 2\operatorname{Im}\tilde{\rho}.$$

Proof. The result is equivalent to $\operatorname{Re}^2\tilde{\rho} \leq 0.8\,|\tilde{\rho}|^2$. From (2.1), we have

$$\operatorname{Re}^2\tilde{\rho} \leq \frac{1}{2}\sum_{j=1}^{3}\left[\left|(\tilde{x}_j - \tilde{y}_j)^2\right| + \operatorname{Re}(\tilde{x}_j - \tilde{y}_j)^2\right] = \sum_{j=1}^{3}\operatorname{Re}^2(\tilde{x}_j - \tilde{y}_j), \tag{4.5}$$

$$\left|\tilde{\rho}^2\right| \geq \operatorname{Im}\tilde{\rho}^2 = 2\sum_{j=1}^{3}\operatorname{Re}(\tilde{x}_j - \tilde{y}_j)\operatorname{Im}(\tilde{x}_j - \tilde{y}_j). \tag{4.6}$$

It suffices to show

$$\sum_{j=1}^{3}\operatorname{Re}^2(\tilde{x}_j - \tilde{y}_j) \leq 1.6\sum_{j=1}^{3}\operatorname{Re}(\tilde{x}_j - \tilde{y}_j)\operatorname{Im}(\tilde{x}_j - \tilde{y}_j). \tag{4.7}$$

We assume $x_j \geq |y_j|$, $j = 1, 2, 3$ and $x_1 - y_1 = \max\limits_{j=1,2,3}|x_j - y_j| \geq 6L_{\max}$ without loss of generality. By Lemma 4.1 and the assumption that $\sigma_0 \geq 5$, we have

$$\operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) \geq 1.1\sigma_0(x_1 - y_1), \qquad \operatorname{Im}(\tilde{x}_1 - \tilde{y}_1) \geq \frac{11}{13}\operatorname{Re}(\tilde{x}_1 - \tilde{y}_1). \tag{4.8}$$

Now we consider different ranges of $x_j - y_j$ for $j = 2, 3$.

– If $x_j - y_j \geq 4L_j$, by Lemma 4.1, we have $\operatorname{Im}(\tilde{x}_j - \tilde{y}_j) \geq 0.68\sigma_0(x_j - y_j)$ and

$$\operatorname{Im}(\tilde{x}_j - \tilde{y}_j) \geq \frac{0.68\sigma_0}{1 + 0.68\sigma_0}\operatorname{Re}(\tilde{x}_j - \tilde{y}_j) \geq \frac{17}{22}\operatorname{Re}(\tilde{x}_j - \tilde{y}_j). \tag{4.9}$$

– If $x_j - y_j \leq 0.6\operatorname{Im}(\tilde{x}_j - \tilde{y}_j)$, we have $\operatorname{Re}(\tilde{x}_j - \tilde{y}_j) \leq 1.6\operatorname{Im}(\tilde{x}_j - \tilde{y}_j)$ and

$$\operatorname{Re}^2(\tilde{x}_j - \tilde{y}_j) \leq 1.6\operatorname{Re}(\tilde{x}_j - \tilde{y}_j)\operatorname{Im}(\tilde{x}_j - \tilde{y}_j). \tag{4.10}$$

– If $0.6\,\mathrm{Im}(\tilde{x}_j - \tilde{y}_j) \le x_j - y_j \le 4L_j$, from (4.8) we have

$$\begin{aligned}\mathrm{Re}^2(\tilde{x}_j - \tilde{y}_j) &\le \frac{64}{9}(x_j - y_j)^2 \le \frac{1024}{9}L_j^2 \le \frac{256}{81}(x_1 - y_1)^2 \\ &\le 0.09\,\mathrm{Re}(\tilde{x}_1 - \tilde{y}_1)\cdot \mathrm{Im}(\tilde{x}_1 - \tilde{y}_1).\end{aligned} \tag{4.11}$$

Combining (4.9)–(4.11) leads to

$$\begin{aligned}\sum_{j=1}^{3}\mathrm{Re}^2(\tilde{x}_j - \tilde{y}_j) &\le 1.6\sum_{j=2}^{3}\mathrm{Re}(\tilde{x}_j - \tilde{y}_j)\,\mathrm{Im}(\tilde{x}_j - \tilde{y}_j) + \frac{16}{11}\,\mathrm{Re}(\tilde{x}_1 - \tilde{y}_1)\,\mathrm{Im}(\tilde{x}_1 - \tilde{y}_1) \\ &\le 1.6\sum_{j=1}^{3}\mathrm{Re}(\tilde{x}_j - \tilde{y}_j)\,\mathrm{Im}(\tilde{x}_j - \tilde{y}_j).\end{aligned}$$

The proof is completed. □

Lemma 4.3. *Suppose $\sigma_0 \ge 5$ and that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ satisfy $d(\boldsymbol{x}, \boldsymbol{y}) \ge 6L_{\max}$. Then*

$$\mathrm{Im}\,\tilde{\rho} \ge 0.77\sigma_0 d(\boldsymbol{x}, \boldsymbol{y}).$$

Proof. We assume $|x_1| > |y_1|$ and $x_1 - y_1 = d(\boldsymbol{x}, \boldsymbol{y})$ without loss of generality. From Lemma 4.2 we know that

$$\mathrm{Im}^2\,\tilde{\rho} \ge \frac{1}{2}\,\mathrm{Re}\,\tilde{\rho}\cdot \mathrm{Im}\,\tilde{\rho} = \frac{1}{4}\,\mathrm{Im}\,\tilde{\rho}^2 \ge \frac{1}{4}\,\mathrm{Im}(\tilde{x}_1 - \tilde{y}_1)^2 \ge \frac{1}{2}\,\mathrm{Im}^2(\tilde{x}_1 - \tilde{y}_1).$$

It follows from Lemma 4.1 that

$$\mathrm{Im}\,\tilde{\rho} \ge \frac{\sqrt{2}}{2}\,\mathrm{Im}(\tilde{x}_1 - \tilde{y}_1) \ge 0.77\sigma_0(x_1 - y_1) = 0.77\sigma_0 d(\boldsymbol{x}, \boldsymbol{y}).$$

The proof is completed. □

4.2. Analytic extension of the Cagniard-de Hoop transform

From Appendix A.2, the perturbation tensor is represented by the Cagniard-de Hoop transform. So we have to extend the Cagniard-de Hoop transform to complex coordinates.

Lemma 4.4. *Suppose $\boldsymbol{x} \in \mathbb{R}^3_+$ and $\boldsymbol{y} \in \mathbb{R}^3_-$. For any $q \ge 0$, define $\kappa_1 = \kappa_1(0, q)$ and $\xi_\pm = \frac{\kappa_1}{\tilde{\rho}}\left(\tilde{r}t \pm \mathbf{i}\tilde{z}\sqrt{t^2 - 1}\right)$. Then*

$$(\kappa_1^2 - \xi_\pm^2)^{1/2} = \frac{\kappa_1}{\tilde{\rho}}\left(\tilde{z}t \mp \mathbf{i}\tilde{r}\sqrt{t^2 - 1}\right).$$

Proof. We only prove the lemma for ξ_+. The proof for ξ_- is similar. Write

$$\Lambda_+ = \frac{\kappa_1}{\tilde{\rho}}\left(\tilde{z}t - \mathbf{i}\tilde{r}t_1\right), \qquad t_1 = \sqrt{t^2 - 1}.$$

It is easy to see $\Lambda_+^2 = \kappa_1^2 - \xi_+^2$. Using the convention in (2.1), it suffices to show $\mathrm{Re}\,\Lambda_+ \ge 0$.

From (4.3), we can write $\tilde{\rho} = \rho_1 + \mathbf{i}\rho_2$, $\tilde{r} = r_1 + \mathbf{i}r_2$, $\tilde{z} = z_1 + \mathbf{i}z_2$ with $\rho_1 \ge \rho_2 \ge 0$, $r_1 \ge r_2 \ge 0$, and $z_1 \ge z_2 \ge 0$. Then

$$\frac{|\tilde{\rho}|^2}{\kappa_1}\,\mathrm{Re}\,\Lambda_+ = \rho_1(z_1 t + r_2 t_1) + \rho_2(z_2 t - r_1 t_1) \ge t_1(M - N), \tag{4.12}$$

where $M = \rho_1 z_1 + \rho_1 r_2 + \rho_2 z_2$ and $N = \rho_2 r_1$. By direct calculations, we find that

$$\begin{aligned}\frac{1}{2}\left(M^2 - N^2\right) &\geq \frac{1}{2}\left(\rho_1^2 z_1^2 + \rho_2^2 z_2^2 + \rho_1^2 r_2^2 - \rho_2^2 r_1^2\right)\\ &= \left|\tilde{\rho}^2\right|\left|\tilde{z}^2\right| + \operatorname{Re}\tilde{\rho}^2 \operatorname{Re}\tilde{z}^2 + \operatorname{Re}\tilde{z}^2\left|\tilde{r}^2\right| + \left(\left|\tilde{r}^2\right| - \left|\tilde{\rho}^2\right|\right)\operatorname{Re}\tilde{r}^2\\ &\geq \operatorname{Re}\tilde{\rho}^2 \operatorname{Re}\tilde{z}^2 + \operatorname{Re}\tilde{z}^2\left|\tilde{r}^2\right| + \left(\left|\tilde{z}^2\right| + \left|\tilde{r}^2\right| - \left|\tilde{\rho}^2\right|\right)\operatorname{Re}\tilde{r}^2 \geq 0.\end{aligned}$$

This implies $M > N$. Then (4.12) shows $\operatorname{Re}\Lambda_+ \geq 0$. □

Lemma 4.5. *Suppose $\sigma_0 \geq 5$ and that $\boldsymbol{x} \in \mathbb{R}^3_+$, $\boldsymbol{y} \in \mathbb{R}^3_-$ satisfy $d(\boldsymbol{x}, \boldsymbol{y}) \geq 6L_{\max}$. Define $\kappa_j = \kappa_j(0, q)$ and $\mu_j = \left(\kappa_j^2 - \xi^2\right)^{1/2}$ for any $q \geq 0$ and $j = 1, 2$, where*

$$\xi = \frac{\kappa_1}{\tilde{\rho}}\left(\tilde{r}t \pm \mathbf{i}\tilde{z}\sqrt{t^2 - 1}\right).$$

For $\phi = \tilde{z}$, or $\phi = \tilde{x}_3$ with $x_3 \geq 1.5L_3$, or $\phi = -\tilde{y}_3$ with $|y_3| \geq 1.5L_3$,

$$\operatorname{Im}\left[(\mu_1 - \mu_2)\phi\right] \leq 0. \tag{4.13}$$

Proof. From (4.3), we have $\operatorname{Re}\tilde{r}^2 \geq 0$, $\operatorname{Re}\tilde{z}^2 \geq 0$, and $\operatorname{Re}\tilde{\rho}^2 \geq 0$. Write

$$z = x_3 - y_3, \quad \tilde{\rho} = \rho_1 + \mathbf{i}\rho_2, \quad \tilde{r} = r_1 + \mathbf{i}r_2, \quad \tilde{z} = z_1 + \mathbf{i}z_2, \quad \phi = \phi_1 + \mathbf{i}\phi_2,$$

with $\rho_1 \geq \rho_2 \geq 0$, $r_1 \geq r_2 \geq 0$, $z_1 \geq z_2 \geq 0$, and $\phi_1 \geq \phi_2 \geq 0$. Moreover, let $\mu_j = A_j + \mathbf{i}B_j$ with $A_j, B_j \in \mathbb{R}$, $j = 1, 2$. Using Lemma 4.4 and (2.1), we have

$$A_1 = \frac{\kappa_1}{|\tilde{\rho}|^2}(\rho_1 z_1 t + \rho_1 r_2 t_1 + \rho_2 z_2 t - \rho_2 r_1 t_1), \tag{4.14}$$

$$B_1 = \frac{\kappa_1}{|\tilde{\rho}|^2}(\rho_1 z_2 t - \rho_1 r_1 t_1 - \rho_2 z_1 t - \rho_2 r_2 t_1), \tag{4.15}$$

$$A_2 = \frac{\sqrt{2}}{2}\left[\sqrt{\left(k_-^2 - k_+^2 + A_1^2 - B_1^2\right)^2 + 4A_1^2 B_1^2} + \left(k_-^2 - k_+^2 + A_1^2 - B_1^2\right)\right]^{1/2}, \tag{4.16}$$

$$|B_2| = \frac{\sqrt{2}}{2}\left[\sqrt{\left(k_-^2 - k_+^2 + A_1^2 - B_1^2\right)^2 + 4A_1^2 B_1^2} - \left(k_-^2 - k_+^2 + A_1^2 - B_1^2\right)\right]^{1/2}. \tag{4.17}$$

Since $\mu_2^2 - \mu_1^2 = k_-^2 - k_+^2$, we have

$$A_2^2 - B_2^2 = k_-^2 - k_+^2 + A_1^2 - B_1^2, \qquad A_1 B_1 = A_2 B_2. \tag{4.18}$$

Direct calculations show that

$$A_2 \geq A_1 \geq 0, \qquad |B_2| \leq |B_1|, \qquad \operatorname{sign}(B_1) = \operatorname{sign}(B_2). \tag{4.19}$$

In view of Lemma 4.2, we have $\rho_1 \leq 2\rho_2$ and deduce that

$$\begin{aligned}\operatorname{Im}\left[(\mu_1 - \mu_2)\phi\right] &= \phi_1(B_1 - B_2) + \phi_2(A_1 - A_2) = \frac{A_1 - A_2}{A_2}(\phi_2 A_2 - \phi_1 B_1)\\ &\leq \frac{A_1 - A_2}{A_2}\frac{\kappa_1 t}{|\tilde{\rho}|^2}\left[\rho_2(\phi_1 z_1 + \phi_2 z_2) + \rho_1(\phi_2 z_1 - \phi_1 z_2)\right]\\ &\leq \frac{A_1 - A_2}{A_2}\frac{\kappa_1 t \rho_1}{|\tilde{\rho}|^2}\left[\phi_2 z_2 + \phi_2 z_1 - \phi_1 z_2\right].\end{aligned} \tag{4.20}$$

If $\phi = \tilde{z}$, (4.20) shows $\operatorname{Im}\left[(\mu_1 - \mu_2)\phi\right] \leq 0$ directly. If $\phi = \tilde{x}_3$ and $x_3 \geq 1.5L_3$, then

$$\phi_2 z_2 + \phi_2 z_1 - \phi_1 z_2 \geq (x_3 - \operatorname{Im}\tilde{x}_3)\operatorname{Im}\tilde{y}_3 = \left[(1 - 2\sigma_0)x_3 + 2.5\sigma_0 L_3\right]\operatorname{Im}\tilde{y}_3 \geq 0.$$

So we have $\operatorname{Im}\left[(\mu_1 - \mu_2)\phi\right] \leq 0$. The proof for $\phi = -\tilde{y}_3$ is similar and omitted. □

4.3. Partial derivatives of $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ with $\varepsilon=0$

In view of (A.15) and (A.16) in the appendix, the partial derivatives of $\mathbb{P}$ can be represented with the Cagniard-de Hoop transform. The derivatives of $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ with respect to $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ can be obtained by replacing $\boldsymbol{x},\boldsymbol{y}$ with $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ in the corresponding derivatives of $\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})$ with respect to $\boldsymbol{x},\boldsymbol{y}$, namely,

$$\frac{\partial^{m+n}}{\partial\tilde{x}_i^m\partial\tilde{y}_j^n}\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=\left(\frac{\partial^{m+n}\mathbb{P}}{\partial x_i^m\partial y_j^n}\right)(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}),\quad 1\le i,j\le 3,\quad m,n\ge 0.$$

For example, replacing $\boldsymbol{x},\boldsymbol{y}$ with $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ in (A.15) and (A.16), we get the explicit expressions of

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_1^l\partial\tilde{x}_2^m\partial\tilde{x}_3^n},\qquad\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{y}_1^l\partial\tilde{y}_2^m\partial\tilde{y}_3^n}.$$

On one hand, (A.12) and (A.13) state that the Cagniard-de Hoop representation of $\mathbb{P}$ is defined upon an arbitrary parameter $0<\varepsilon\ll 1$. On the other hand, (A.4)–(A.6) imply that $\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})$ is actually independent of the choice of ε. In this subsection, we are going to prove that, under the complex stretching, the Cagniard-de Hoop representation of $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ can be given by taking $\varepsilon=0$.

Lemma 4.6. *For $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$, the partial derivatives of $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ are given by setting $\varepsilon=0$.*

Proof. Without loss of generality, we only prove the lemma for $\dfrac{\partial^m}{\partial\tilde{x}_1^m}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ and for $\boldsymbol{x}\in\mathbb{R}^3_+$, $\boldsymbol{y}\in\mathbb{R}^3_-$. The results can be extended straightforwardly to other cases of $\boldsymbol{x},\boldsymbol{y}$, to other entries of $\mathbb{P}$, and to other derivatives of $\mathbb{P}$. The details are omitted here.

In the Cagniard-de Hoop representation, we write $\mathbb{P}^{(\varepsilon)}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}):=\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ to specify its dependency on ε. Moreover, let $\mathbb{P}^{(0)}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ be given by setting $\varepsilon=0$ and replacing $\boldsymbol{x},\boldsymbol{y}$ with $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ in (A.7). It suffices to show

$$\frac{\partial^m}{\partial\tilde{x}_1^m}\mathrm{P}_{33}^{(0)}(k,0;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=\lim_{\varepsilon\to 0^+}\frac{\partial^m}{\partial\tilde{x}_1^m}\mathrm{P}_{33}^{(\varepsilon)}(k,\varepsilon;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}),\qquad m\ge 0.\tag{4.21}$$

For convenience, let $\tilde{r}$, $\tilde{z}$, $\tilde{\rho}$, r, z, ρ be defined in (4.1). The analytic extension of the Cagniard-de Hoop transform is defined by, for any $t\ge 1$,

$$\xi_\pm(t)=\frac{\kappa_1(\varepsilon,q)}{\tilde{\rho}}\left(\tilde{r}t\pm\mathbf{i}\tilde{z}\sqrt{t^2-1}\right),\qquad\Lambda_\pm(t)=\frac{\kappa_1(\varepsilon,q)}{\tilde{\rho}}\left(\tilde{z}t\mp\mathbf{i}\tilde{r}\sqrt{t^2-1}\right).$$

Let C be a constant independent of $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, ε, q, and t. From Lemma 4.3 and (A.9), it is easy to see that

$$|\xi_\pm(t)|+|\Lambda_\pm(t)|\le C\frac{|\tilde{r}|+|\tilde{z}|}{|\tilde{\rho}|}(k_++q)t\le C(k_++q)t.\tag{4.22}$$

Since $\operatorname{Im}\mu_1^2=\operatorname{Im}\mu_2^2$, the convention in (2.1) implies

$$\operatorname{Re}\mu_1\ge 0,\qquad\operatorname{Re}\mu_2\ge 0,\qquad\operatorname{sign}(\operatorname{Im}\mu_1)=\operatorname{sign}(\operatorname{Im}\mu_2).$$

This shows $|\mu_1-\mu_2|\le|\mu_1+\mu_2|$. Since $\mu_1^2-\mu_2^2=k_+^2-k_-^2$, we have

$$\left|k_-^2\mu_1+k_+^2\mu_2\right|\ge k_+^2\left|\mu_1+\mu_2\right|\ge k_+^2\left|\mu_1^2-\mu_2^2\right|^{1/2}\ge k_+^2(k_--k_+).\tag{4.23}$$

Replacing $\boldsymbol{x},\boldsymbol{y}$ with $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ in (A.7), we find that, for any $z\ge 2\varepsilon r$,

$$\left|\frac{\partial^m}{\partial\tilde{x}_1^m}\mathrm{P}_{33}^{(\varepsilon)}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right|\le Ce^{k_-|y_3|}\int_0^\infty\int_1^\infty[(k_++q)t]^{m+1}\frac{\left|e^{\mathbf{i}\kappa_1(\varepsilon,q)\tilde{\rho}t}\right|}{\sqrt{t^2-1}}\,\mathrm{d}t\,\mathrm{d}q.\tag{4.24}$$

For $\varepsilon < 1/2$, from (2.1) and (A.9), we find that

$$\operatorname{Im}[\kappa_1(\varepsilon,q)\tilde{\rho}] \geq \operatorname{Re}\kappa_1(\varepsilon,q)\cdot \operatorname{Im}\tilde{\rho} \geq 0.4(k_+ + q)\operatorname{Im}\tilde{\rho}.$$

Inserting the estimates into (4.24) leads to

$$\left|\frac{\partial^m}{\partial \tilde{x}_1^m}\mathrm{P}_{33}^{(\varepsilon)}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right| \leq C e^{k_-|y_3|}\int_0^\infty\int_1^\infty \frac{[(k_+ + q)t]^{m+1}}{\sqrt{t^2-1}} e^{-0.4\operatorname{Im}\tilde{\rho}\,(k_+ + q)t}\,\mathrm{d}t\,\mathrm{d}q.$$

Remember from (4.3) that $\operatorname{Im}\tilde{\rho} > 0$. The integral on the righthand side is convergent and independent of ε. So (4.21) follows from the dominated convergence theorem. □

Remark 4.7. In (A.12)–(A.14), the Cagniard-de Hoop representation of $\mathbb{P}(k;\boldsymbol{x},\boldsymbol{y})$ is only defined for $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$ satisfying

$$|x_3| + |y_3| \geq 2\varepsilon\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}.$$

By Lemma 4.6, we can take $\varepsilon = 0$ in the expression of $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$. Therefore, $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ is well-defined for all $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$.

4.4. Exponential attenuation of $\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$

From (3.5) and (3.4), we must first prove the exponential attenuation of $\mathbb{S}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ and $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ as $|\boldsymbol{x}-\boldsymbol{y}|\to+\infty$. For convenience, through this subsection, we use

$$\zeta_j = \tilde{x}_j,\quad \zeta_{j+3} = \tilde{y}_j,\quad j=1,2,3.$$

Lemma 4.8. *Suppose $\sigma_0\geq 5$ and $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$. There is a constant C depending only on $k_\pm$ and $L_{\max}$ such that, for any $1\leq i,j\leq 6$ and $m,n\geq 0$,*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m\partial\zeta_j^n}\mathbb{S}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right| \leq C\times\begin{cases} e^{-0.77k_+\sigma_0 d(\boldsymbol{x},\boldsymbol{y})}, & \text{if}\quad d(\boldsymbol{x},\boldsymbol{y})\geq 6L_{\max},\\ |\boldsymbol{x}-\boldsymbol{y}|^{-m-n-1}, & \text{otherwise.}\end{cases}$$

Proof. Since $\Phi(k_\pm;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}) = [4\pi\rho(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})]^{-1}e^{ik_\pm\rho(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}$, the lemma follows directly from (2.5) and Lemma 4.3. □

Lemma 4.9. *Suppose $\sigma_0\geq 5$ and $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$. There is a constant C depending only on $k_\pm$ and $L_{\max}$ such that, for any $1\leq i,j\leq 6$ and $m,n\geq 0$,*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m\partial\zeta_j^n}\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right| \leq C\times\begin{cases} e^{k_-\sigma_0 L_3}e^{-0.77k_+\sigma_0 d(\boldsymbol{x},\boldsymbol{y})}, & \text{if}\quad d(\boldsymbol{x},\boldsymbol{y})\geq 6L_{\max},\\ |\boldsymbol{x}-\boldsymbol{y}|^{-m-n-1}, & \text{otherwise.}\end{cases}$$

Proof. Without loss of generality, we assume $\boldsymbol{x}\in\mathbb{R}^3_+$, $\boldsymbol{y}\in\mathbb{R}^3_-$ and only consider the derivatives of $\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ with respect to $\tilde{\boldsymbol{x}}$. The results can be extended straightforwardly to other entries of $\mathbb{P}$, to other derivatives, and to other cases of $\boldsymbol{x},\boldsymbol{y}$. The estimates for $d(\boldsymbol{x},\boldsymbol{y}) < 6L_{\max}$ follow directly from Lemma 2.1. We only consider the case of $d(\boldsymbol{x},\boldsymbol{y})\geq 6L_{\max}$. Moreover, by Remark 4.7, it suffices to take $\varepsilon=0$ in defining $\mathbb{P}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$, namely,

$$\kappa_1 = \left(k_+^2 + q^2\right)^{1/2},\qquad \kappa_2 = \left(k_-^2+q^2\right)^{1/2},\qquad \forall\, q\geq 0.$$

Using (A.15) and the notations in (4.1), we know that

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})}{\partial\tilde{x}_1^l\partial\tilde{x}_2^m\partial\tilde{x}_3^n} = \mathbf{i}^{l+m+n-1}\frac{k_-^2}{2\pi^2}\left[F_+(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}) + F_-(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right], \tag{4.25}$$

where

$$F_\pm(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}) = \int_0^\infty \int_1^\infty \lambda_1^l(\xi_\pm)\lambda_2^m(\xi_\pm)\Lambda_\pm^{n+1} \frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]\tilde{y}_3}}{k_-^2\mu_1(\xi_\pm)+k_+^2\mu_2(\xi_\pm)} \frac{e^{\mathbf{i}\kappa_1\tilde{\rho}t}}{\sqrt{t^2-1}}\,\mathrm{d}t\,\mathrm{d}q.$$

Let ϕ be the polar angle such that $x_1-y_1=r\cos\phi$ and $x_2-y_2=r\sin\phi$. We have

$$\lambda_1(\xi)=\xi\cos\phi-\mathbf{i}q\sin\phi, \quad \lambda_2(\xi)=\xi\sin\phi+\mathbf{i}q\cos\phi.$$

Moreover, $\mu_j(\xi)=\left(\kappa_j^2-\xi^2\right)^{1/2}$ for $j=1,2$ and $\xi_\pm$, $\Lambda_\pm$ are given by

$$\xi_\pm(t)=\frac{\kappa_1}{\tilde{\rho}}\left(\tilde{r}t\pm\mathbf{i}\tilde{z}\sqrt{t^2-1}\right), \quad \Lambda_\pm(t)=\frac{\kappa_1}{\tilde{\rho}}\left(\tilde{z}t\mp\mathbf{i}\tilde{r}\sqrt{t^2-1}\right).$$

From Lemma 4.4, we know that

$$\mu_1(\xi_\pm)=\Lambda_\pm, \qquad \mu_2(\xi_\pm)=\left(k_-^2-k_+^2+\Lambda_\pm^2\right)^{1/2}.$$

It suffices to estimate $F_\pm(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$. Similar to (4.22), there is a generic constant C independent of $\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}}$ such that

$$|\lambda_1(\xi_\pm)|+|\lambda_2(\xi_\pm)|+|\xi_\pm|+|\Lambda_\pm|\le C(k_++q)t,$$

Note that $|\mu_1-\mu_2|^2\le\left|\mu_1^2-\mu_2^2\right|=k_-^2-k_+^2$. If $|y_3|\le 1.5L_3$, we have

$$|\tilde{y}_3|\le\sqrt{(1.5L_3+0.5\sigma_0L_3)^2+0.25\sigma_0^2L_3^2}\le\sigma_0L_3.$$

This implies

$$\left|\frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]\tilde{y}_3}}{k_-^2\mu_1(\xi_\pm)+k_+^2\mu_2(\xi_\pm)}\right|\le Ce^{k_-\sigma_0L_3}.$$

If $|y_3|>1.5L_3$, from Lemma 4.5 we have

$$\left|\frac{e^{\mathbf{i}[\mu_1(\xi_\pm)-\mu_2(\xi_\pm)]\tilde{y}_3}}{k_-^2\mu_1(\xi_\pm)+k_+^2\mu_2(\xi_\pm)}\right|\le C.$$

Substituting the estimates into $F_\pm$ and using Lemma 4.3, we easily get

$$|F_\pm(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})|\le Ce^{k_-\sigma_0L_3}e^{-0.77k_+\sigma_0d(\boldsymbol{x},\boldsymbol{y})}.$$

The proof is finished by using (4.25). □

Lemma 4.10. *Suppose $\sigma_0\ge 5$ and that $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3_\pm$ satisfy $d(\boldsymbol{x},\boldsymbol{y})\ge 6L_{\max}$. There is a constant C depending only on $k_\pm$, $L_{\max}$ such that, for any $1\le i,j\le 6$ and $m,n\ge 0$,*

$$\left|\frac{\partial^{m+n}}{\partial\zeta_i^m\partial\zeta_j^n}\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})\right|\le Ce^{k_-\sigma_0L_3}e^{-0.77k_+\sigma_0d(\boldsymbol{x},\boldsymbol{y})}.$$

Proof. Since $\mathbb{G}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=\mathbb{H}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})+k_\pm^{-2}\nabla_{\tilde{\boldsymbol{y}}}\operatorname{div}_{\tilde{\boldsymbol{y}}}\mathbb{H}(k;\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})$ and $\mathbb{H}=\mathbb{S}-\mathbb{P}$, the lemma is a direct consequence of Lemma 4.8 and Lemma 4.9. □

4.5. Exponential attenuation of $\boldsymbol{E}(\tilde{\boldsymbol{x}})$

Now we prove the exponential attenuation of the stretched solution. It is the key step to prove the well-posedness of the PML truncation problem and the exponential convergence of the PML solution.

Theorem 4.11. *Let $\boldsymbol{E}$ be the solution of (1.1) and let $\boldsymbol{E}(\tilde{\boldsymbol{x}})$ be the analytic continuation of $\boldsymbol{E}(\boldsymbol{x})$ defined in (3.8). Assume $\sigma_0 \geq 5$, $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$ and let $\boldsymbol{x} \in \mathbb{R}^3_\pm$ satisfy $d(\boldsymbol{x}, 0) \geq 7k_- k_+^{-1} L_{\max}$. There is a constant C depending only on k, $\boldsymbol{L}$ such that*

$$|\boldsymbol{E}(\tilde{\boldsymbol{x}})| + |\mathbf{curl}_{\tilde{\boldsymbol{x}}}\, \boldsymbol{E}(\tilde{\boldsymbol{x}})| \leq C e^{-0.45 k_+ \sigma_0 d(\boldsymbol{x},0)} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.$$

Proof. For any $\boldsymbol{y} \in \Gamma_D$, since $d(\boldsymbol{y}, 0) \leq L_{\max}$, the assumptions of the theorem imply $d(\boldsymbol{x}, \boldsymbol{y}) \geq 6 L_{\max} k_-/k_+$. By Lemma 4.10, we have

$$\begin{aligned}\|\mathbf{curl}\, \mathbb{G}(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} + \|\mathbb{G}(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} &\leq C e^{k_- \sigma_0 L_3} e^{-0.77 k_+ \sigma_0 [d(\boldsymbol{x},0) - L_{\max}]} \\ &\leq C e^{-0.45 k_+ \sigma_0 d(\boldsymbol{x},0)}.\end{aligned}$$

Since γ_t and γ_T are bounded, using (1.1a) and Theorem 2.2, we have

$$\begin{aligned}|\boldsymbol{\Psi}_{\mathrm{SL}}(\gamma_t(\mathbf{curl}\, \boldsymbol{E}))(\tilde{\boldsymbol{x}})| &\leq \|\gamma_t(\mathbf{curl}\, \boldsymbol{E})\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \|\gamma_T \mathbb{G}(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_D)} \\ &\leq \|\mathbf{curl}\, \boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} \|\mathbb{G}(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} \\ &\leq C \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} \|\mathbb{G}(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} \\ &\leq C e^{-0.45 k_+ \sigma_0 d(\boldsymbol{x},0)} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)},\end{aligned}$$

where we have used $\tilde{\boldsymbol{y}} = \boldsymbol{y}$ in $\Omega_{\boldsymbol{L}}$. Similarly, the double layer potential can be estimated as follows

$$\begin{aligned}|\boldsymbol{\Psi}_{\mathrm{DL}}(\boldsymbol{g})(\tilde{\boldsymbol{x}})| &\leq \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \|\gamma_T(\mathbf{curl}\, \mathbb{G})(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}^{-1/2}(\mathrm{Curl},\Gamma_D)} \\ &\leq \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)} \|\mathbf{curl}\, \mathbb{G}(\tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{L}})} \\ &\leq C e^{-0.45 k_+ \sigma_0 d(\boldsymbol{x},0)} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}.\end{aligned}$$

From (3.8), we find that $|\boldsymbol{E}(\tilde{\boldsymbol{x}})| \leq C e^{-0.45 k_+ \sigma_0 d(\boldsymbol{x},0)} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div},\Gamma_D)}$. The estimate for $\mathbf{curl}_{\tilde{\boldsymbol{x}}}\, \boldsymbol{E}(\tilde{\boldsymbol{x}})$ is similar and omitted here. □

5. Well-posedness of the modified Maxwell equation

In this section, we prove the well-posedness of the exterior problem (3.16). It plays an important role in proving the well-posedness of the PML problem. In Section 3.4, we already know that (3.16) has at least one solution $\tilde{\boldsymbol{E}} := \mathbb{B}^\top \boldsymbol{E} \circ \boldsymbol{F}$ where $\boldsymbol{E}$ is the solution of problem (1.1). We shall prove that $\tilde{\boldsymbol{E}}$ is the only solution of (3.16) in $\boldsymbol{H}(\mathbf{curl}, D_c)$.

5.1. The stretched Maxwell equation in $\mathbb{R}^3$

Given $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3)$, we first study the stretched Maxwell equation in the layered medium

$$\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u} - k_\pm^2 \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \mathbb{R}^3_\pm, \tag{5.1a}$$

$$\left[\!\left[\gamma_t(\mathbb{B}^\top \tilde{\nabla} \times \boldsymbol{u})\right]\!\right] = \left[\!\left[\gamma_t(\mathbb{B}^\top \boldsymbol{u})\right]\!\right] = 0 \quad \text{on} \quad \Sigma. \tag{5.1b}$$

Lemma 5.1. *Problem* (5.1) *has one solution given by*

$$\boldsymbol{u}(\boldsymbol{x}) := \int_{\mathbb{R}^3} \mathbb{G}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\boldsymbol{x}))\boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3_\pm. \tag{5.2}$$

Proof. From Lemma 3.2, we know that $\mathbb{G}$ satisfies, for any fixed $\boldsymbol{y} \in \mathbb{R}^3_\pm$,

$$\begin{aligned} &\tilde{\nabla} \times \tilde{\nabla} \times \mathbb{G}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\cdot)) - k_\pm^2 \mathbb{G}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\cdot)) = J^{-1}\delta_{\boldsymbol{y}}\mathbb{I} && \text{in} \quad \mathbb{R}^3_\pm, \\ &\left[\!\left[\boldsymbol{n} \times \mathbb{B}^\top \mathbb{G}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\cdot))\right]\!\right] = \left[\!\left[\boldsymbol{n} \times \mathbb{B}^\top \tilde{\nabla} \times \mathbb{G}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\cdot))\right]\!\right] = 0 && \text{on} \quad \Sigma. \end{aligned}$$

Applying $(\tilde{\nabla} \times \tilde{\nabla}\times) - k_\pm^2$ to both sides of (5.4) yields (5.1a). Furthermore, we have

$$\left[\!\left[\gamma_t(\mathbb{B}^\top \boldsymbol{u})\right]\!\right] = \left[\!\left[\boldsymbol{n} \times \mathbb{B}^\top \boldsymbol{u}\right]\!\right] = \int_{\mathbb{R}^3} \left[\!\left[\boldsymbol{n} \times \mathbb{B}^\top \mathbb{G}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\cdot))\right]\!\right] \boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0 \quad \text{on} \quad \Sigma.$$

Similarly, we have $\left[\!\left[\gamma_t(\mathbb{B}^\top \tilde{\nabla} \times \boldsymbol{u})\right]\!\right] = 0$ on Σ. ☐

Lemma 5.2. *Let $\boldsymbol{u}$ be defined in* (5.2)*. There is a constant C independent of σ_0 such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\mathbb{R}^3)} + \left\|\tilde{\nabla} \times \boldsymbol{u}\right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\sigma_0^6 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{5.3}$$

Proof. Using (3.5) and (3.11), $\boldsymbol{u}$ admits a splitting

$$\boldsymbol{u} = \boldsymbol{w} + \tilde{\nabla}\phi, \quad \phi := k_\pm^{-2}\tilde{\nabla} \cdot \boldsymbol{w} \quad \text{in} \quad \mathbb{R}^3_\pm,$$

where

$$\boldsymbol{w}(\boldsymbol{x}) = \int_{\mathbb{R}^3} \mathbb{H}(k; \tilde{\boldsymbol{y}}, \boldsymbol{F}(\boldsymbol{x}))\boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{5.4}$$

From (2.4b), $\mathbb{H}$ is continuous across Σ. So $\boldsymbol{w}$ is also continuous across Σ.

From (3.4) and Lemmas 4.8–4.9, we know that

$$|\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| + |\nabla_{\tilde{\boldsymbol{x}}}\mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})| \le \begin{cases} Ce^{-0.6k_+\sigma_0 d(\boldsymbol{x},\boldsymbol{y})}, & \text{if} \quad d(\boldsymbol{x}, \boldsymbol{y}) \ge 6L_{\max}k_-/k_+, \\ C\left[1 + d(\boldsymbol{x}, \boldsymbol{y})^{-2}\right], & \text{otherwise}. \end{cases}$$

Writing $\zeta(\boldsymbol{x}, \boldsymbol{y}) = \left[1 + d(\boldsymbol{x}, \boldsymbol{y})^{-2}\right] e^{-0.6k_+\sigma_0 d(\boldsymbol{x},\boldsymbol{y})}$, we find that

$$\begin{aligned} \|\boldsymbol{w}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)} &\le C\sigma_0^6 \int_{\mathbb{R}^3} \left[\int_{\mathbb{R}^3} \zeta(\boldsymbol{x}, \boldsymbol{y}) \left|\boldsymbol{f}(\boldsymbol{y})\right|^2 \,\mathrm{d}\boldsymbol{y}\right] \left[\int_{\mathbb{R}^3} \zeta(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right] \mathrm{d}\boldsymbol{x} \\ &\le C\sigma_0^6 \int_{\mathbb{R}^3} |\boldsymbol{f}(\boldsymbol{y})|^2 \int_{\mathbb{R}^3} \zeta(\boldsymbol{x}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \le C\sigma_0^6 \|\boldsymbol{f}\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}. \end{aligned} \tag{5.5}$$

Since $\tilde{\nabla} \times \boldsymbol{w}(\boldsymbol{x}) = \int_{\mathbb{R}^3} \mathbf{curl}_{\tilde{\boldsymbol{x}}}\, \mathbb{H}(k; \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{x}})\boldsymbol{f}(\boldsymbol{y})J(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$, similarly we have

$$\left\|\tilde{\nabla} \times \boldsymbol{u}\right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = \left\|\tilde{\nabla} \times \boldsymbol{w}\right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\sigma_0^3 \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}.$$

To estimate $\tilde{\nabla}\phi$, we multiply both sides of (5.1a) with $J\tilde{\nabla}\bar{\phi}$ and integrate the equation over $\mathbb{R}^3$. An application of integration by part shows that

$$\int_{\mathbb{R}^3} k^2 J\tilde{\nabla}\phi \cdot \tilde{\nabla}\bar{\phi} = -\int_{\mathbb{R}^3} J(\boldsymbol{f} + k^2\boldsymbol{w}) \cdot \tilde{\nabla}\bar{\phi}.$$

From (3.3), the equation can be written equivalently into

$$\int_{\mathbb{R}^3} k^2 \sum_{j=1}^{3} J \frac{\bar{\alpha}_j}{\alpha_j} \left| \frac{\partial \phi}{\partial \tilde{x}_j} \right|^2 = - \int_{\mathbb{R}^3} J \overline{\mathbb{B}} \mathbb{B}^{-1} (\boldsymbol{f} + k^2 \boldsymbol{w}) \cdot \overline{\tilde{\nabla} \phi} \, .$$

The real parts of the coefficients satisfy

$$\operatorname{Re}(J \bar{\alpha}_j / \alpha_j) \geq 1 + \beta_1 + \beta_2 + \beta_3 + 2\beta_1\beta_2\beta_3, \qquad j = 1, 2, 3. \tag{5.6}$$

Therefore, with a constant $C > 0$ independent of σ_0, we have

$$\left\| \tilde{\nabla} \phi \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \leq C \sigma_0^3 \left\| \boldsymbol{f} + k^2 \boldsymbol{w} \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \leq C \sigma_0^6 \left\| \boldsymbol{f} \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} .$$

Together with (5.5), this finishes the proof. □

5.2. The stretched Maxwell equation in D_c

Next we study the stretched Maxwell equation in the exterior domain D_c:

$$\begin{aligned}
&\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u} - k_\pm^2 \boldsymbol{u} = \boldsymbol{f} && \text{in} \quad \mathbb{R}^3_\pm \cap D_c, && (5.7\text{a})\\
&\left[\!\!\left[\gamma_t (\mathbb{B}^\top \tilde{\nabla} \times \boldsymbol{u}) \right]\!\!\right] = \left[\!\!\left[\gamma_t (\mathbb{B}^\top \boldsymbol{u}) \right]\!\!\right] = 0 && \text{on} \quad \Sigma, && (5.7\text{b})\\
&\gamma_t \boldsymbol{u} = 0 && \text{on} \quad \Gamma_D. && (5.7\text{c})
\end{aligned}$$

Lemma 5.3. *For any $\boldsymbol{f} \in \boldsymbol{L}^2(D_c)$, there exist a function $\boldsymbol{u}$ which satisfies (5.7) and a constant $C > 0$ independent of σ_0 such that*

$$\left\| \boldsymbol{u} \right\|_{\boldsymbol{L}^2(D_c)} + \left\| \tilde{\nabla} \times \boldsymbol{u} \right\|_{\boldsymbol{L}^2(D_c)} \leq C \sigma_0^6 \left\| \boldsymbol{f} \right\|_{\boldsymbol{L}^2(D_c)} . \tag{5.8}$$

Proof. First we extend $\boldsymbol{f}$ by zero to the interior of D and denote the extension still by $\boldsymbol{f}$. By Lemma 5.1, there exists a $\boldsymbol{u}_0 \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ satisfying

$$\begin{aligned}
&\tilde{\nabla} \times \tilde{\nabla} \times \boldsymbol{u}_0 - k_\pm^2 \boldsymbol{u}_0 = \boldsymbol{f} && \text{in} \quad \mathbb{R}^3_\pm,\\
&\left[\!\!\left[\gamma_t (\mathbb{B}^\top \tilde{\nabla} \times \boldsymbol{u}_0) \right]\!\!\right] = \left[\!\!\left[\gamma_t (\mathbb{B}^\top \boldsymbol{u}_0) \right]\!\!\right] = 0 && \text{on} \quad \Sigma,\\
&\left\| \boldsymbol{u}_0 \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} + \left\| \tilde{\nabla} \times \boldsymbol{u}_0 \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \leq C \sigma_0^6 \left\| \boldsymbol{f} \right\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = C \sigma_0^6 \left\| \boldsymbol{f} \right\|_{\boldsymbol{L}^2(D_c)} .
\end{aligned}$$

From Theorem 2.2, the scattering problem

$$\begin{aligned}
&\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u}_1 - k_\pm^2 \boldsymbol{u}_1 = 0 && \text{in} \quad \mathbb{R}^3_\pm \cap D_c,\\
&[\![\gamma_t (\mathbf{curl}\, \boldsymbol{u}_1)]\!] = [\![\gamma_t \boldsymbol{u}_1]\!] = 0 && \text{on} \quad \Sigma,\\
&\gamma_t \boldsymbol{u}_1 = \gamma_t \boldsymbol{u}_0 && \text{on} \quad \Gamma_D,\\
&\lim_{\rho \to \infty} \int_{\partial B(\rho)} \left| \mathbf{curl}\, \boldsymbol{u}_1 \times \boldsymbol{n} - \mathrm{i} k \boldsymbol{u}_1 \right|^2 = 0,
\end{aligned}$$

has a unique solution which satisfies

$$\left\| \boldsymbol{u}_1 \right\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_L)} \leq C \left\| \gamma_t \boldsymbol{u}_0 \right\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)} . \tag{5.9}$$

Similar to (3.8), the analytic continuation of $\boldsymbol{u}_1$ is given by

$$\boldsymbol{u}_1(\tilde{\boldsymbol{x}}) = \boldsymbol{\Psi}_{\mathrm{SL}}(\gamma_t(\mathbf{curl}\, \boldsymbol{u}_1))(\tilde{\boldsymbol{x}}) + \boldsymbol{\Psi}_{\mathrm{DL}}(\gamma_t(\boldsymbol{u}_1))(\tilde{\boldsymbol{x}}) \qquad \forall\, \boldsymbol{x} \in D_c. \tag{5.10}$$

We divide D_c into $D_c = \Omega_{\boldsymbol{L}} \cup \bar{D}_1 \cup D_2$ where

$$D_1 = \left\{\boldsymbol{x} \in D_c \backslash \bar{\Omega}_{\boldsymbol{L}} : d(\boldsymbol{x}, 0) < 7k_- k_+^{-1} L_{\max}\right\}, \quad D_2 = \left\{\boldsymbol{x} : d(\boldsymbol{x}, 0) > 7k_- k_+^{-1} L_{\max}\right\}.$$

First we consider the case of $\boldsymbol{x} \in D_1$. From (3.4) to (3.5), it is easy to see that

$$\|\mathbb{G}(k; \tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{L}^\infty(\Gamma_D)} \le C\left[\|\mathbb{S}(k; \tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{W}^{2,\infty}(\Gamma_D)} + \|\mathbb{P}(k; \tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{W}^{2,\infty}(\Gamma_D)}\right].$$

Since $\boldsymbol{F}(\boldsymbol{y}) = \boldsymbol{y}$ for any $\boldsymbol{y} \in \Gamma_D$, using Lemmas 4.8 and 4.9, there exists a constant $C > 0$ depending only on $k_\pm$ and $\Omega_{\boldsymbol{L}}$ such that

$$\|\mathbb{G}(k; \tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{L}^\infty(\Gamma_D)} \le C\left[\operatorname{dist}(\Gamma_{\boldsymbol{L}}, \Gamma_D)^{-1} + \operatorname{dist}(\Gamma_{\boldsymbol{L}}, \Gamma_D)^{-3}\right] \le C,$$

where $\operatorname{dist}(\Gamma_{\boldsymbol{L}}, \Gamma_D)$ denotes the distance between $\Gamma_{\boldsymbol{L}}$ and Γ_D. Similarly, we have

$$\|(\mathbf{curl}\,\mathbb{G})(k; \tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{L}^\infty(\Gamma_D)} + \|(\mathbf{curl}\,\mathbf{curl}\,\mathbb{G})(k; \tilde{\boldsymbol{x}}, \cdot)\|_{\boldsymbol{L}^\infty(\Gamma_D)} \le C.$$

By arguments similar to the proof of Theorem 4.11, we have

$$|(\boldsymbol{u}_1 \circ \boldsymbol{F})(\boldsymbol{x})| = |\boldsymbol{u}_1(\tilde{\boldsymbol{x}})| \le C\,\|\gamma_t \boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)},$$
$$\left|\tilde{\nabla} \times (\boldsymbol{u}_1 \circ \boldsymbol{F})(\boldsymbol{x})\right| = |\mathbf{curl}_{\tilde{\boldsymbol{x}}}\, \boldsymbol{u}_1(\tilde{\boldsymbol{x}})| \le C\,\|\gamma_t \boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}.$$

This leads to

$$\|\boldsymbol{u}_1 \circ \boldsymbol{F}\|_{\boldsymbol{L}^2(D_1)} + \left\|\tilde{\nabla} \times (\boldsymbol{u}_1 \circ \boldsymbol{F})\right\|_{\boldsymbol{L}^2(D_1)} \le C\,\|\gamma_t \boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}. \tag{5.11}$$

For any $\boldsymbol{x} \in D_2$, Theorem 4.11 indicates that

$$|(\boldsymbol{u}_1 \circ \boldsymbol{F})(\boldsymbol{x})| + \left|\tilde{\nabla} \times (\boldsymbol{u}_1 \circ \boldsymbol{F})(\boldsymbol{x})\right| \le C e^{-0.45 k_+ \sigma_0 d(\boldsymbol{x}, 0)}\,\|\gamma_t \boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}.$$

This shows

$$\|\boldsymbol{u}_1 \circ \boldsymbol{F}\|_{\boldsymbol{L}^2(D_2)} + \left\|\tilde{\nabla} \times (\boldsymbol{u}_1 \circ \boldsymbol{F})\right\|_{\boldsymbol{L}^2(D_2)} \le C\,\|\gamma_t \boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}. \tag{5.12}$$

The proof is finished by combining (5.9), (5.11), (5.12) and using the fact that $\|\gamma_t \boldsymbol{u}_0\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)} \le C\,\|\boldsymbol{u}_0\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{L}})} \le C\sigma_0^6\,\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D_c)}$. ☐

5.3. The inf-sup condition in D_c

To study the weak solution of problem (3.16), we define the bilinear form

$$A_\Omega(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \left(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} - k^2 \mathbb{A}^{-1} \boldsymbol{u} \cdot \boldsymbol{v}\right) \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega). \tag{5.13}$$

For $\Omega = D_c$, we use the abbreviation $A(\boldsymbol{u}, \boldsymbol{v}) = A_{D_c}(\boldsymbol{u}, \boldsymbol{v})$.

Lemma 5.4. *There exists a constant $C_{\sup} > 0$ depending only on $k, \boldsymbol{L}$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)} \le C_{\sup} \sigma_0^{10} \sup_{\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)} \frac{|A(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, D_c)}} \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c).$$

Proof. Let $l_{\boldsymbol{u}}$: $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c) \to \mathbb{C}$ be the linear functional defined by

$$l_{\boldsymbol{u}}(\boldsymbol{v}) = \int_{D_c} (\mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} + \boldsymbol{u} \cdot \bar{\boldsymbol{v}}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c).$$

It is clear that $\|l_{\boldsymbol{u}}\|_{\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c)'} = \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}$. Let A_+: $\boldsymbol{H}(\mathbf{curl}, D_c) \times \boldsymbol{H}(\mathbf{curl}, D_c) \to \mathbb{C}$ be the bilinear form defined by

$$A_+(\boldsymbol{w}, \boldsymbol{v}) = \int_{D_c} \mathbb{A} \left(\mathbf{curl}\, \boldsymbol{w} \cdot \mathbf{curl}\, \boldsymbol{v} + k^2 \boldsymbol{w} \cdot \boldsymbol{v}\right).$$

Consider the weak problem: Find $\boldsymbol{u}_+ \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ such that

$$A_+(\boldsymbol{u}_+, \boldsymbol{v}) = l_{\boldsymbol{u}}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c). \tag{5.14}$$

The real parts of the diagonal entries of $\mathbb{A}$ are given by

$$\mathrm{Re}\, \frac{\alpha_j^2}{J} \geq \frac{|\alpha_j|^2}{|J|^2}(1 + 2\beta_1\beta_2\beta_3 + \beta_j) > 0, \quad j = 1, 2, 3.$$

There is a constant $C > 0$ depending only on k such that

$$\mathrm{Re}\, [A_+(\boldsymbol{v}, \bar{\boldsymbol{v}})] \geq C\sigma_0^{-2} \|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathbf{curl},D_c)}.$$

So A_+ is coercive on $\boldsymbol{H}(\mathbf{curl}, D_c)$. By the Lax–Milgram lemma, (5.14) has a unique solution. Moreover, there exists a constant $C > 0$ depending only on k such that

$$\|\boldsymbol{u}_+\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \leq C\sigma_0^2 \|l_{\boldsymbol{u}}\|_{\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c)'} = C\sigma_0^2 \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}. \tag{5.15}$$

Write $\boldsymbol{f} := k^2(\mathbb{A} + \mathbb{A}^{-1})\boldsymbol{u}_+ \in \boldsymbol{L}^2(D_c)$. Recall from (3.9) that, for any $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, D_c)$ satisfying $\mathbb{A}\,\mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, D_c)$,

$$\mathbf{curl}\, (\mathbb{A}\, \mathbf{curl}\, \boldsymbol{v}) = J\mathbb{B}^{-1}\tilde{\nabla} \times \tilde{\nabla} \times \left(\mathbb{B}^{-\top}\boldsymbol{v}\right). \tag{5.16}$$

Using Lemma 5.3, there exists a function $\boldsymbol{u}_1 \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ such that

$$\mathbf{curl}(\mathbb{A}\, \mathbf{curl}\, \boldsymbol{u}_1) - k_\pm^2 \mathbb{A}^{-1}\boldsymbol{u}_1 = \boldsymbol{f} \quad \text{in} \quad \mathbb{R}^3_\pm \cap D_c, \tag{5.17a}$$

$$[\![\gamma_t(\mathbb{A}\, \mathbf{curl}\, \boldsymbol{u}_1)]\!] = [\![\gamma_t \boldsymbol{u}_1]\!] = 0 \quad \text{on} \quad \Sigma, \tag{5.17b}$$

$$\|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \leq C\sigma_0^7 \left\|J^{-1}\mathbb{B}\boldsymbol{f}\right\|_{\boldsymbol{L}^2(D_c)} \leq C\sigma_0^8 \|\boldsymbol{u}_+\|_{\boldsymbol{L}^2(D_c)}. \tag{5.17c}$$

Multiplying both sides of (5.17a) with $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(D_c)$ and integrating by part, we have

$$A(\boldsymbol{u}_1, \boldsymbol{v}) = \int_{D_c} \boldsymbol{f} \cdot \boldsymbol{v} = l_{\boldsymbol{u}}(\boldsymbol{v}) - A(\boldsymbol{u}_+, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{C}_0^\infty(D_c).$$

The denseness of $\boldsymbol{C}_0^\infty(D_c)$ in $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ implies

$$A(\boldsymbol{u}_1, \boldsymbol{v}) = l_{\boldsymbol{u}}(\boldsymbol{v}) - A(\boldsymbol{u}_+, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c).$$

Clearly $\boldsymbol{w} := \boldsymbol{u}_1 + \boldsymbol{u}_+ \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$ and satisfies

$$A(\boldsymbol{w}, \boldsymbol{v}) = l_{\boldsymbol{u}}(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c), \tag{5.18a}$$

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \leq C\sigma_0^8 \|\boldsymbol{u}_+\|_{\boldsymbol{L}^2(D_c)} \leq C\sigma_0^{10} \|\boldsymbol{u}\|_{\boldsymbol{L}^2(D_c)}. \tag{5.18b}$$

Since $A(\cdot, \cdot)$ is symmetric, using (5.18), we obtain

$$\sup_{\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c)} \frac{|A(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}} \geq \frac{|A(\boldsymbol{u}, \boldsymbol{w})|}{\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}} = \frac{|l_{\boldsymbol{u}}(\boldsymbol{u})|}{\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}} \geq \frac{\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}}{C\sigma_0^{10}}.$$

This completes the proof. □

5.4. The well-posedness of problem (3.16)

Now we are ready to prove the well-posedness of problem (3.16). Multiplying both sides of (3.16a) with any $\boldsymbol{v} \in \boldsymbol{C}_0^\infty(D_c)$ and using integration by part, we get

$$A(\tilde{\boldsymbol{E}}, \boldsymbol{v}) = 0.$$

Since $\boldsymbol{C}_0^\infty(D_c)$ is dense in $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$, the equality holds for all $\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c)$. Therefore, a weak formulation of (3.16) reads: Find $\tilde{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl}, D_c)$ such that $\gamma_t \tilde{\boldsymbol{E}} = \boldsymbol{g}$ on Γ_D and

$$A(\tilde{\boldsymbol{E}}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, D_c). \tag{5.19}$$

Theorem 5.5. *For any $\boldsymbol{g} \in \boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)$, the exterior problem (3.16) or the weak problem (5.19) has a unique solution in $\boldsymbol{H}(\mathbf{curl}, D_c)$ which is just $\tilde{\boldsymbol{E}} := \mathbb{B}^\top(\boldsymbol{E} \circ \boldsymbol{F})$ where $\boldsymbol{E}$ is the solution of problem (1.1).*

Proof. Clearly Lemma 5.4 shows that (5.19) has a unique solution. Any solution of (3.16) also satisfies (5.19). So the solution of (3.16) is unique in $\boldsymbol{H}(\mathbf{curl}, D_c)$. Since $\tilde{\boldsymbol{E}} := \mathbb{B}^\top(\boldsymbol{E} \circ \boldsymbol{F}) \in \boldsymbol{H}(\mathbf{curl}, D_c)$ satisfies (3.16), it is just the unique solution of (3.16), or equivalently, the unique solution of (5.19). ☐

6. The well-posedness and exponential convergence of the PML problem

The purpose of this section is to prove the main result of this paper, that is, Theorem 3.3. Multiplying both sides of (3.17a) with $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}})$ and using integration by part, we obtain an equivalent weak formulation of the PML problem (3.17): Find $\hat{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})$ such that $\gamma_t \hat{\boldsymbol{E}} = \boldsymbol{g}$ on Γ_D, $\gamma_t \hat{\boldsymbol{E}} = 0$ on $\Gamma_{\boldsymbol{l}}$, and

$$A_{\Omega_{\boldsymbol{l}}}(\hat{\boldsymbol{E}}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}}). \tag{6.1}$$

The well-posedness of (6.1) depends greatly on the inf-sup condition for $A_{\Omega_{\boldsymbol{l}}}$.

6.1. The inf-sup condition for $A_{\Omega_{\boldsymbol{l}}}$

To prove the inf-sup condition, we adopt the technique of reflection extension introduced by Bramble and Pasciak [5].

Lemma 6.1. *For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}})$, there exists an extension $\boldsymbol{u}_e$ of $\boldsymbol{u}$ to $\Omega_{1.5\boldsymbol{l}}$ such that $\boldsymbol{u}_e \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{1.5\boldsymbol{l}})$ and*

$$\|\boldsymbol{u}_e\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{1.5\boldsymbol{l}})} \leq 2\sqrt{2}\, \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})}\,. \tag{6.2}$$

Proof. Let $\boldsymbol{e}_j$ be the unit vector in the x_j-direction and define the reflection operators

$$\mathcal{R}_j \boldsymbol{v} = 2(\boldsymbol{v} \cdot \boldsymbol{e}_j)\boldsymbol{e}_j - \boldsymbol{v}, \qquad j = 1, 2, 3. \tag{6.3}$$

The mirror mappings with respect to $x_j = \pm l_j$ are defined by

$$\mathcal{M}_j^{\pm} \boldsymbol{x} = 2(\pm l_j - x_j)\boldsymbol{e}_j + \boldsymbol{x}, \qquad j = 1, 2, 3. \tag{6.4}$$

It is easy to see that

$$\mathcal{R}_j \boldsymbol{v} \times \boldsymbol{e}_j = -\boldsymbol{v} \times \boldsymbol{e}_j. \tag{6.5}$$

Moreover, we have the commuting diagram

$$\mathbf{curl}_{\mathcal{M}_j^{\pm}(\boldsymbol{x})} \left[(\mathcal{R}_j \boldsymbol{v}) \circ (\mathcal{M}_j^{\pm})^{-1} \right] = -\left[\mathcal{R}_j (\mathbf{curl}_{\boldsymbol{x}}\, \boldsymbol{v}) \right] \circ (\mathcal{M}_j^{\pm})^{-1}. \tag{6.6}$$

Now we extend $\boldsymbol{u}$ sequentially to sub-domains of $\Omega_{1.5\boldsymbol{l}}$. For $\boldsymbol{x} \in \Omega_{(1.5l_1,l_2,l_3)}$, define

$$\boldsymbol{u}_e(\boldsymbol{x}) = \begin{cases} (\mathcal{R}_1\boldsymbol{u})\left((\mathcal{M}_1^+)^{-1}\boldsymbol{x}\right) & \text{if} \quad x_1 \in (l_1, 1.5l_1), \\ (\mathcal{R}_1\boldsymbol{u})\left((\mathcal{M}_1^-)^{-1}\boldsymbol{x}\right) & \text{if} \quad x_1 \in (-1.5l_1, -l_1), \\ \boldsymbol{u}(\boldsymbol{x}) & \text{otherwise.} \end{cases}$$

For $\boldsymbol{x}$ satisfying $-1.5l_1 < x_1 < 1.5l_1$ and $-l_3 < x_3 < l_3$, define

$$\boldsymbol{u}_e(\boldsymbol{x}) = \begin{cases} (\mathcal{R}_2\boldsymbol{u}_e)\left((\mathcal{M}_2^+)^{-1}\boldsymbol{x}\right) & \text{if} \quad x_2 \in (l_2, 1.5l_2), \\ (\mathcal{R}_2\boldsymbol{u}_e)\left((\mathcal{M}_2^-)^{-1}\boldsymbol{x}\right) & \text{if} \quad x_2 \in (-1.5l_2, -l_2). \end{cases}$$

For $\boldsymbol{x}$ satisfying $-1.5l_1 < x_1 < 1.5l_1$ and $-1.5l_2 < x_2 < 1.5l_2$, define

$$\boldsymbol{u}_e(\boldsymbol{x}) = \begin{cases} (\mathcal{R}_3\boldsymbol{u}_e)\left((\mathcal{M}_3^+)^{-1}\boldsymbol{x}\right) & \text{if} \quad x_3 \in (l_3, 1.5l_3), \\ (\mathcal{R}_3\boldsymbol{u}_e)\left((\mathcal{M}_3^-)^{-1}\boldsymbol{x}\right) & \text{if} \quad x_3 \in (-1.5l_3, -l_3). \end{cases}$$

Since $\gamma_t\boldsymbol{u} = 0$ on $\partial\Omega_{\boldsymbol{l}}$ and the mirror mappings are continuous, the equality (6.5) implies

$$[\gamma_t\boldsymbol{u}_e] = 0 \qquad \text{on} \quad \partial\Omega_{\boldsymbol{l}}.$$

Together with (6.6), we have $\boldsymbol{u}_e \in \boldsymbol{H}(\mathbf{curl}, \Omega_{1.5\boldsymbol{l}})$ and get (6.2). $\square$

Let $\mathcal{L}_\Omega$: $\boldsymbol{H}(\mathbf{curl}, \Omega) \to (\boldsymbol{H}_0(\mathbf{curl}, \Omega))'$ be the linear operator defined by

$$\langle \mathcal{L}_\Omega\boldsymbol{u}, \boldsymbol{v}\rangle := A_\Omega(\boldsymbol{u}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega). \tag{6.7}$$

It is easy to see that

$$\|\mathcal{L}_\Omega\boldsymbol{u}\|_{[\boldsymbol{H}_0(\mathbf{curl},\Omega)]'} = \sup_{\boldsymbol{v}\in\boldsymbol{H}_0(\mathbf{curl},\Omega)} \frac{|A_\Omega(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}}. \tag{6.8}$$

Lemma 6.2. *For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}})$, let $\boldsymbol{u}_e \in \boldsymbol{H}(\mathbf{curl}, \Omega_{1.5\boldsymbol{l}})$ be the extension of $\boldsymbol{u}$ given in Lemma 6.1. Then*

$$\|\mathcal{L}_{\Omega_{1.5\boldsymbol{l}}}\boldsymbol{u}_e\|_{[\boldsymbol{H}_0(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})]'} \le 2\sqrt{2}\,\|\mathcal{L}_{\Omega_{\boldsymbol{l}}}\boldsymbol{u}\|_{[\boldsymbol{H}_0(\mathbf{curl},\Omega_{\boldsymbol{l}})]'}\,. \tag{6.9}$$

Proof. Let $\mathcal{R}_j$, $\mathcal{M}_j^\pm$ be defined in (6.3) and (6.4). Given a function $\boldsymbol{\phi} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{1.5\boldsymbol{l}})$, we define $\boldsymbol{\phi}_3$ as follows, for any $\boldsymbol{x} \in B_{1.5\boldsymbol{l}}$,

$$\boldsymbol{\phi}_3(\boldsymbol{x}) = \begin{cases} (\mathcal{R}_3\boldsymbol{\phi})\left(\mathcal{M}_3^+\boldsymbol{x}\right) & \text{if} \quad x_3 \in (0.5l_3, l_3), \\ (\mathcal{R}_3\boldsymbol{\phi})\left(\mathcal{M}_3^-\boldsymbol{x}\right) & \text{if} \quad x_3 \in (-l_3, -0.5l_3), \\ 0 & \text{otherwise.} \end{cases}$$

Similar to (6.5) and (6.6), $\boldsymbol{\phi}_3 + \boldsymbol{\phi} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{(1.5l_1,1.5l_2,l_3)})$. For any $\boldsymbol{x} \in B_{1.5\boldsymbol{l}}$, write $\boldsymbol{y} = \mathcal{M}_3^+\boldsymbol{x}$ if $x_3 \in (0.5l_3, l_3)$ or $\boldsymbol{y} = \mathcal{M}_3^-\boldsymbol{x}$ if $x_3 \in (-l_3, -0.5l_3)$. Then

$$(\mathbf{curl}\,\boldsymbol{\phi}_3)(\boldsymbol{x}) = -\mathcal{R}_3(\mathbf{curl}_{\boldsymbol{y}}\,\boldsymbol{\phi})(\boldsymbol{y}).$$

Since $\boldsymbol{u}_e$ is the reflection extension of $\boldsymbol{u}$ from $\Omega_{\boldsymbol{l}}$ to $\Omega_{1.5\boldsymbol{l}}$, the definition of $\boldsymbol{\phi}_3$ implies that, for any $\boldsymbol{x} \in B_{\boldsymbol{l}}\backslash B_{(l_1,l_2,0.5l_3)}$,

$$\boldsymbol{u}_e(\boldsymbol{x}) \cdot \boldsymbol{\phi}_3(\boldsymbol{x}) = \boldsymbol{u}_e(\boldsymbol{y}) \cdot \boldsymbol{\phi}(\boldsymbol{y}), \tag{6.10a}$$

$$\mathbf{curl}\,\boldsymbol{u}_e(\boldsymbol{x}) \cdot \mathbf{curl}\,\boldsymbol{\phi}_3(\boldsymbol{x}) = \mathbf{curl}\,\boldsymbol{u}_e(\boldsymbol{y}) \cdot \mathbf{curl}\,\boldsymbol{\phi}(\boldsymbol{y}). \tag{6.10b}$$

Next we define, for any $\boldsymbol{x} \in B_{(1.5l_1,1.5l_2,l_3)}$,

$$\boldsymbol{\phi}_2(\boldsymbol{x}) = \begin{cases} (\mathcal{R}_2(\boldsymbol{\phi}+\boldsymbol{\phi}_3))\left(\mathcal{M}_2^+\boldsymbol{x}\right) & \text{if} \quad x_2 \in (0.5l_2, l_2), \\ (\mathcal{R}_2(\boldsymbol{\phi}+\boldsymbol{\phi}_3))\left(\mathcal{M}_2^-\boldsymbol{x}\right) & \text{if} \quad x_2 \in (-l_2, -0.5l_2), \\ 0 & \text{otherwise}. \end{cases}$$

This yields $\boldsymbol{\phi}+\boldsymbol{\phi}_3+\boldsymbol{\phi}_2 \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{(1.5l_1,l_2,l_3)})$ and for any $\boldsymbol{x} \in B_{\boldsymbol{l}} \backslash B_{(l_1,0.5l_2,l_3)}$,

$$\boldsymbol{u}_e(\boldsymbol{x}) \cdot \boldsymbol{\phi}_2(\boldsymbol{x}) = \boldsymbol{u}_e(\boldsymbol{y}) \cdot [\boldsymbol{\phi}(\boldsymbol{y}) + \boldsymbol{\phi}_3(\boldsymbol{y})] \,, \tag{6.11a}$$

$$\mathbf{curl}\, \boldsymbol{u}_e(\boldsymbol{x}) \cdot \mathbf{curl}\, \boldsymbol{\phi}_2(\boldsymbol{x}) = \mathbf{curl}\, \boldsymbol{u}_e(\boldsymbol{y}) \cdot \mathbf{curl}\, [\boldsymbol{\phi}(\boldsymbol{y}) + \boldsymbol{\phi}_3(\boldsymbol{y})] \,, \tag{6.11b}$$

where $\boldsymbol{y} = \mathcal{M}_2^+\boldsymbol{x}$ if $x_2 > 0$ and $\boldsymbol{y} = \mathcal{M}_2^-\boldsymbol{x}$ if $x_2 < 0$.

Similarly we define, for any $\boldsymbol{x} \in B_{(1.5l_1,l_2,l_3)}$,

$$\boldsymbol{\phi}_1(\boldsymbol{x}) = \begin{cases} (\mathcal{R}_1(\boldsymbol{\phi}+\boldsymbol{\phi}_2+\boldsymbol{\phi}_3))\left(\mathcal{M}_1^+\boldsymbol{x}\right) & \text{if} \quad x_1 \in (0.5l_1, l_1), \\ (\mathcal{R}_1(\boldsymbol{\phi}+\boldsymbol{\phi}_2+\boldsymbol{\phi}_3))\left(\mathcal{M}_1^-\boldsymbol{x}\right) & \text{if} \quad x_1 \in (-l_1, -0.5l_1), \\ 0 & \text{otherwise}. \end{cases}$$

This yields $\boldsymbol{\phi}^e := \boldsymbol{\phi} + \sum_{i=1}^3 \boldsymbol{\phi}_i \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}})$ and for any $\boldsymbol{x} \in B_{\boldsymbol{l}} \backslash B_{(0.5l_1,l_2,l_3)}$,

$$\boldsymbol{u}_e(\boldsymbol{x}) \cdot \boldsymbol{\phi}_1(\boldsymbol{x}) = \boldsymbol{u}_e(\boldsymbol{y}) \cdot [\boldsymbol{\phi}(\boldsymbol{y}) + \boldsymbol{\phi}_2(\boldsymbol{y}) + \boldsymbol{\phi}_3(\boldsymbol{y})] \,, \tag{6.12a}$$

$$\mathbf{curl}\, \boldsymbol{u}_e(\boldsymbol{x}) \cdot \mathbf{curl}\, \boldsymbol{\phi}_1(\boldsymbol{x}) = \mathbf{curl}\, \boldsymbol{u}_e(\boldsymbol{y}) \cdot \mathbf{curl}\, [\boldsymbol{\phi}(\boldsymbol{y}) + \boldsymbol{\phi}_2(\boldsymbol{y}) + \boldsymbol{\phi}_3(\boldsymbol{y})] \,, \tag{6.12b}$$

where $\boldsymbol{y} = \mathcal{M}_1^+\boldsymbol{x}$ if $x_1 > 0$ and $\boldsymbol{y} = \mathcal{M}_1^-\boldsymbol{x}$ if $x_1 < 0$. By the definitions of $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$ and direct calculations, we easily get

$$\|\boldsymbol{\phi}^e\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})} \le 2\sqrt{2}\, \|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})} \,.$$

Recall from Section 3.5 that $0.5l_j \ge 1.5L_j$ for $j = 1, 2, 3$. So $\mathbb{A}$ is a constant matrix outside of $B_{0.5\boldsymbol{l}}$. Using (6.10)–(6.12) and the definition of $\mathcal{L}_{\Omega_{1.5\boldsymbol{l}}}$, we have

$$\begin{aligned} \langle \mathcal{L}_{\Omega_{1.5\boldsymbol{l}}} \boldsymbol{u}_e, \boldsymbol{\phi} \rangle &= A_{\Omega_{1.5\boldsymbol{l}}}(\boldsymbol{u}_e, \boldsymbol{\phi}) = A_{\Omega_{(1.5l_1,1.5l_2,l_3)}}(\boldsymbol{u}_e, \boldsymbol{\phi}+\boldsymbol{\phi}_3) \\ &= A_{\Omega_{(1.5l_1,l_2,l_3)}}(\boldsymbol{u}_e, \boldsymbol{\phi}+\boldsymbol{\phi}_2+\boldsymbol{\phi}_3) = A_{\Omega_{\boldsymbol{l}}}(\boldsymbol{u}, \boldsymbol{\phi}+\boldsymbol{\phi}^e) = \langle \mathcal{L}_{\Omega_{\boldsymbol{l}}} \boldsymbol{u}, \boldsymbol{\phi}^e \rangle. \end{aligned}$$

We conclude that

$$\begin{aligned} |\langle \mathcal{L}_{\Omega_{1.5\boldsymbol{l}}} \boldsymbol{u}_e, \boldsymbol{\phi} \rangle| &\le \|\mathcal{L}_{\Omega_{\boldsymbol{l}}} \boldsymbol{u}\|_{\boldsymbol{H}_0(\mathbf{curl},\Omega_{\boldsymbol{l}})'} \|\boldsymbol{\phi}^e\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})} \\ &\le 2\sqrt{2}\, \|\mathcal{L}_{\Omega_{\boldsymbol{l}}} \boldsymbol{u}\|_{\boldsymbol{H}_0(\mathbf{curl},\Omega_{\boldsymbol{l}})'} \|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})} \,. \end{aligned}$$

The proof is completed by using (6.8) and the arbitrariness of $\boldsymbol{\phi}$. ☐

Theorem 6.3. *There exists a constant $C_1 > 0$ independent of $\boldsymbol{l}$ and σ_0 such that, when $l_{\min} \ge C_1 \sigma_0^{11}$,*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})} \le C\sigma_0^{10} \sup_{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega_{\boldsymbol{l}})} \frac{|A_{\Omega_{\boldsymbol{l}}}(\boldsymbol{u}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})}} \quad \forall \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}}), \tag{6.13}$$

where C is a constant independent of $\boldsymbol{l}$ and σ_0.

Proof. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^3)$ be cutoff functions satisfying

$$\begin{aligned} &\chi_1 \geq 0, \quad \operatorname{supp}(\chi_1) \subset B_{1.2\boldsymbol{l}}, \quad \chi_1 \equiv 1 \quad \text{in} \quad B_{\boldsymbol{l}}, \\ &\chi_2 \geq 0, \quad \operatorname{supp}(\chi_2) \subset B_{1.5\boldsymbol{l}}, \quad \chi_2 \equiv 1 \quad \text{in} \quad B_{1.2\boldsymbol{l}}. \end{aligned}$$

Let $\boldsymbol{u}_e \in \boldsymbol{H}(\mathbf{curl}, \Omega_{1.5\boldsymbol{l}})$ be the extension of $\boldsymbol{u}$ given in Lemma 6.1. Define

$$\boldsymbol{u}_1 = \chi_1 \boldsymbol{u}_e \quad \text{in} \quad \Omega_{1.5\boldsymbol{L}}, \qquad \boldsymbol{u}_1 \equiv 0 \quad \text{in} \quad D_c \backslash \bar{\Omega}_{1.5\boldsymbol{L}}.$$

By Lemma 5.4, there exists a constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})} \leq \|\boldsymbol{u}_1\|_{\boldsymbol{H}(\mathbf{curl},D_c)} \leq C\sigma_0^{10} \sup_{\boldsymbol{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},D_c)} \frac{|A(\boldsymbol{u}_1, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},D_c)}}. \tag{6.14}$$

Write $\boldsymbol{w} = \chi_1 \chi_2 \boldsymbol{v}$. Since $\|\chi_1\|_{W^{1,\infty}(\mathbb{R}^3)} + \|\chi_2\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C(1 + l_{\min}^{-1}) \leq C$, we have

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})} \leq C \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})}.$$

By Lemmas 6.1 and 6.2 and direct calculations, we find that

$$\begin{aligned} |A(\boldsymbol{u}_1, \boldsymbol{v})| &= |A_{\Omega_{1.5\boldsymbol{l}}}(\boldsymbol{u}_1, \chi_2 \boldsymbol{v})| \\ &= \left| A_{\Omega_{1.5\boldsymbol{l}}}(\boldsymbol{u}_e, \boldsymbol{w}) + \int_{\Omega_{1.5\boldsymbol{l}}} [\mathbb{A}(\nabla \chi_1 \times \boldsymbol{u}_e) + (\mathbb{A}\,\mathbf{curl}\,\boldsymbol{u}_e) \times \nabla \chi_1] \cdot \mathbf{curl}(\chi_2 \boldsymbol{v}) \right| \\ &\leq |\langle \mathcal{L}_{\Omega_{1.5\boldsymbol{l}}} \boldsymbol{u}_e, \boldsymbol{w} \rangle| + C\sigma_0 l_{\min}^{-1} \|\boldsymbol{u}_e\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})} \\ &\leq C \left[\|\mathcal{L}_{\Omega_{\boldsymbol{l}}} \boldsymbol{u}\|_{\boldsymbol{H}_0(\mathbf{curl},\Omega_{\boldsymbol{l}})'} + \sigma_0 l_{\min}^{-1} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})} \right] \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{1.5\boldsymbol{l}})}. \end{aligned}$$

Substituting the estimate into (6.14) shows

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})} \leq C_0 \sigma_0^{10} \|\mathcal{L}_{\Omega_{\boldsymbol{l}}} \boldsymbol{u}\|_{\boldsymbol{H}_0(\mathbf{curl},\Omega_{\boldsymbol{l}})'} + C_1 \sigma_0^{11} l_{\min}^{-1} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_{\boldsymbol{l}})},$$

where C_0, C_1 are constants independent of σ_0 and $l_{\min}$. The proof is completed by assuming $l_{\min} \geq 2C_1\sigma_0^{11}$ and using (6.8). □

6.2. The proof of Theorem 3.3

With the inf-sup condition for $A_{\Omega_{\boldsymbol{l}}}$, we are ready to prove Theorem 3.3.

Proof. Using the inf-sup condition (6.13) and the Lax–Milgram lemma, we know that problem (6.1) has a unique solution. Since (6.1) is equivalent to (3.17), we infer that (3.17) has a unique solution $\hat{\boldsymbol{E}} \in \boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})$. It is left to show the error estimate.

From (3.16) and (3.17), the error function $\boldsymbol{e} := \tilde{\boldsymbol{E}} - \hat{\boldsymbol{E}}$ satisfies

$$\mathbf{curl}(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{e}) - k^2 \mathbb{A}^{-1} \boldsymbol{e} = 0 \qquad \text{in} \quad \Omega_{\boldsymbol{l}} \cap \mathbb{R}^3_\pm, \tag{6.15a}$$

$$[\![\gamma_t(\mathbb{A}\,\mathbf{curl}\,\boldsymbol{e})]\!] = [\![\gamma_t \boldsymbol{e}]\!] = 0 \qquad \text{on} \quad \Sigma \cap \Omega_{\boldsymbol{l}}, \tag{6.15b}$$

$$\gamma_t \boldsymbol{e} = 0 \quad \text{on} \quad \Gamma_D, \qquad \gamma_t \boldsymbol{e} = \gamma_t \tilde{\boldsymbol{E}} \qquad \text{on} \quad \Gamma_{\boldsymbol{l}}. \tag{6.15c}$$

Its weak form reads as follows: Find $\boldsymbol{e} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega_{\boldsymbol{l}})$ such that $\gamma_t \boldsymbol{e} = \gamma_t \tilde{\boldsymbol{E}}$ on $\Gamma_{\boldsymbol{l}}$ and

$$A_{\Omega_{\boldsymbol{l}}}(\boldsymbol{e}, \boldsymbol{v}) = 0 \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}}). \tag{6.16}$$

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be the cutoff function satisfying

$$\chi \geq 0, \qquad \operatorname{supp}(\chi) \subset B_{\boldsymbol{l}}, \qquad \chi \equiv 1 \quad \text{in} \quad B_{0.9\boldsymbol{l}}.$$

Clearly $\|\chi\|_{W^{1,\infty}(\mathbb{R}^3)} \leq C$ and $\hat{\boldsymbol{e}} := \boldsymbol{e} - (1-\chi)\tilde{\boldsymbol{E}} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_l)$. For any $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}})$, from Theorem 4.11 we have

$$\begin{aligned} |A_{\Omega_{\boldsymbol{l}}}(\hat{\boldsymbol{e}}, \boldsymbol{v})| = \left| A_{\Omega_{\boldsymbol{l}}}((1-\chi)\tilde{\boldsymbol{E}}, \boldsymbol{v}) \right| &\leq C\sigma_0^2 \left\| \tilde{\boldsymbol{E}} \right\|_{\boldsymbol{H}(\mathbf{curl}, B_{\boldsymbol{l}} \setminus \bar{B}_{0.9\boldsymbol{l}})} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, B_{\boldsymbol{l}} \setminus \bar{B}_{0.9\boldsymbol{l}})} \\ &\leq C l_{\max}^3 \sigma_0^3 e^{-0.7k_+ \sigma_0 d_{\mathrm{PML}}} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})}. \end{aligned}$$

Using the inf-sup condition (6.13), we have

$$\begin{aligned} \|\hat{\boldsymbol{e}}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})} &\leq C\sigma_0^{10} \sup_{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega_{\boldsymbol{l}})} \frac{|A_{\Omega_{\boldsymbol{l}}}(\hat{\boldsymbol{e}}, \boldsymbol{v})|}{\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})}} \\ &\leq C l_{\max}^3 \sigma_0^{13} e^{-0.7k_+ \sigma_0 d_{\mathrm{PML}}} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}. \end{aligned}$$

Since $\left\| (1-\chi)\tilde{\boldsymbol{E}} \right\|_{\boldsymbol{H}(\mathbf{curl}, B_{\boldsymbol{l}})} \leq C\sigma_0 e^{-0.7k_+ \sigma_0 d_{\mathrm{PML}}} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}$ and $l_{\max} \geq C_1 \sigma_0^{11}$, we have

$$\begin{aligned} \|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})} &\leq \|\hat{\boldsymbol{e}}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_{\boldsymbol{l}})} + \left\| (1-\chi)\tilde{\boldsymbol{E}} \right\|_{\boldsymbol{H}(\mathbf{curl}, B_{\boldsymbol{l}})} \\ &\leq C l_{\max}^4 \sigma_0^2 e^{-0.7k_+ \sigma_0 d_{\mathrm{PML}}} \|\boldsymbol{g}\|_{\boldsymbol{H}^{-1/2}(\mathrm{Div}, \Gamma_D)}. \end{aligned}$$

The proof is completed. □

Appendix A. Explicit expressions of the perturbation tensor

In this appendix, we present two explicit expressions of $\mathbb{P}$ which will be used in the PML analysis. One is the Fourier form and the other is the Cagniard-de Hoop form. From [16, 21], $\mathbb{P}$ has the lower diagonal form

$$\mathbb{P} = \begin{pmatrix} \mathrm{P}_{11} & 0 & 0 \\ 0 & \mathrm{P}_{22} & 0 \\ \mathrm{P}_{13} & \mathrm{P}_{23} & \mathrm{P}_{33} \end{pmatrix}. \tag{A.1}$$

The rest of this appendix is quoted from [16]. It is presented here for completeness.

A.1. The Fourier form of $\mathbb{P}$

Let $\mu_\pm$ be the square roots defined by the limiting absorption principle

$$\mu_\pm(\lambda_1, \lambda_2) = \lim_{s \to 0^+} \left[(k_\pm + \mathbf{i}s)^2 - \lambda_1^2 - \lambda_2^2 \right]^{1/2}, \qquad (\lambda_1, \lambda_2) \in \mathbb{R}^2. \tag{A.2}$$

The convention in (2.1) shows $\operatorname{Im} \mu_\pm \geq 0$. For any function $f(\lambda_1, \lambda_2)$, we define

$$J(f; \boldsymbol{x}, \boldsymbol{y}) := \frac{\mathbf{i}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\lambda_1, \lambda_2) e^{\mathbf{i}[(x_1-y_1)\lambda_1 + (x_2-y_2)\lambda_2 + (|x_3|+|y_3|)\mu_+]} \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2.$$

By (2.1), the above integral is convergent absolutely if

$$|f(\lambda_1, \lambda_2)| \leq C \left(1 + \lambda_1^2 + \lambda_2^2 \right)^m, \qquad m \in \mathbb{R}.$$

For convenience, we introduce the notations

$$h_1 = \frac{1}{\mu_+ + \mu_-}, \qquad h_2 = \frac{1}{k_-^2\mu_+ + k_+^2\mu_-}, \qquad h_3 = \frac{k_-^2 - k_+^2}{k_-^2\mu_+ + k_+^2\mu_-} h_1. \tag{A.3}$$

The entries of $\mathbb{P}$ are defined as follows: for $j = 1, 2$,

$$\mathrm{P}_{jj}(k; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} J(h_1; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(h_1 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x} \in \mathbb{R}_+^3, \ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J(h_1 e^{\mathbf{i}(\mu_+ - \mu_-)x_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x} \in \mathbb{R}_-^3, \ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(h_1 e^{\mathbf{i}(\mu_+ - \mu_-)(x_3 + y_3)}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^3, \end{cases} \tag{A.4}$$

$$\mathrm{P}_{j3}(k; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} J(\lambda_j h_3; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x} \in \mathbb{R}_+^3, \ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+ - \mu_-)x_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x} \in \mathbb{R}_-^3, \ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(\lambda_j h_3 e^{\mathbf{i}(\mu_+ - \mu_-)(x_3 + y_3)}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^3, \end{cases} \tag{A.5}$$

$$\mathrm{P}_{33}(k; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} J(k_-^2 h_2; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(k_-^2 h_2 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x} \in \mathbb{R}_+^3, \ \boldsymbol{y} \in \mathbb{R}_-^3, \\ J(k_+^2 h_2 e^{\mathbf{i}(\mu_+ - \mu_-)x_3}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x} \in \mathbb{R}_-^3, \ \boldsymbol{y} \in \mathbb{R}_+^3, \\ J(k_+^2 h_2 e^{\mathbf{i}(\mu_+ - \mu_-)(x_3 + y_3)}; \boldsymbol{x}, \boldsymbol{y}), & \text{if } \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^3. \end{cases} \tag{A.6}$$

The derivatives of $\mathbb{P}$ are computed from (A.4) to (A.6), for example, for $\boldsymbol{x} \in \mathbb{R}_+^3$ and $\boldsymbol{y} \in \mathbb{R}_-^3$,

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k; \boldsymbol{x}, \boldsymbol{y})}{\partial x_1^l \partial x_2^m \partial x_3^n} = \mathbf{i}^{l+m+n} k_-^2 J(\lambda_1^l \lambda_2^m \mu_+^n h_2 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}), \tag{A.7}$$

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k; \boldsymbol{x}, \boldsymbol{y})}{\partial y_1^l \partial y_2^m \partial y_3^n} = (-\mathbf{i})^{l+m+n} k_-^2 J(\lambda_1^l \lambda_2^m \mu_-^n h_2 e^{\mathbf{i}(\mu_+ - \mu_-)y_3}; \boldsymbol{x}, \boldsymbol{y}). \tag{A.8}$$

A.2. The Cagniard-de Hoop form of $\mathbb{P}$

For any fixed $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, we write

$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \quad z = |x_3| + |y_3|, \quad \rho = \sqrt{r^2 + z^2}.$$

For any $0 < \varepsilon \ll 1$ and $q \geq 0$, we define

$$\kappa_1 = \kappa_1(\varepsilon, q) := \left[k_+^2 + (1 + \mathbf{i}\varepsilon)^2 q^2\right]^{1/2}, \tag{A.9a}$$

$$\kappa_2 = \kappa_2(\varepsilon, q) := \left[k_-^2 + (1 + \mathbf{i}\varepsilon)^2 q^2\right]^{1/2}. \tag{A.9b}$$

The Cagniard-de Hoop transform maps $[1, +\infty)$ to the right branch of the hyperbola in the complex ξ-plane

$$\left\{\xi_\pm(t) = \frac{\kappa_1}{\rho}\left(rt \pm \mathbf{i}z\sqrt{t^2 - 1}\right) : \ t \in [1, +\infty)\right\}. \tag{A.10}$$

Define $\mu_j(\xi) = \left(\kappa_j^2 - \xi^2\right)^{1/2}$ for $j = 1, 2$. For any $z \geq 2\varepsilon r$, it is easy to verify that

$$\mu_1(\xi_\pm(t)) = \Lambda_\pm(t) := \frac{\kappa_1}{\rho}\left(zt \mp \mathbf{i}r\sqrt{t^2 - 1}\right). \tag{A.11}$$

Let h_1, h_2, h_3 be defined in (A.3) by replacing μ_+, μ_- with μ_1, μ_2 respectively. For any $z \ge 2\varepsilon r$, the diagonal entries of $\mathbb{P}$ can be rewritten as follows, for $j = 1, 2$,

$$\mathrm{P}_{jj}(k;\boldsymbol{x},\boldsymbol{y}) = \begin{cases} J_{\mathrm{cdh}}(h_1;\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_+, \\ J_{\mathrm{cdh}}(h_1 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3_+,\ \boldsymbol{y} \in \mathbb{R}^3_-, \\ J_{\mathrm{cdh}}(h_1 e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3_-,\ \boldsymbol{y} \in \mathbb{R}^3_+, \\ J_{\mathrm{cdh}}(h_1 e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_-, \end{cases} \tag{A.12}$$

$$\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y}) = \begin{cases} k_-^2 J_{\mathrm{cdh}}(h_2;\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_+, \\ k_-^2 J_{\mathrm{cdh}}(h_2 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3_+,\ \boldsymbol{y} \in \mathbb{R}^3_-, \\ k_+^2 J_{\mathrm{cdh}}(h_2 e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3_-,\ \boldsymbol{y} \in \mathbb{R}^3_+, \\ k_+^2 J_{\mathrm{cdh}}(h_2 e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_-, \end{cases} \tag{A.13}$$

where J_{cdh} is defined by, for a function $f(\xi)$,

$$J_{\mathrm{cdh}}(f;\boldsymbol{x},\boldsymbol{y}) = \frac{\varepsilon - \mathbf{i}}{2\pi^2} \int_0^\infty \int_1^\infty \left[\Lambda_+(t) f(\xi_+(t)) + \Lambda_-(t) f(\xi_-(t))\right] \frac{e^{\mathbf{i}\kappa_1 \rho t}}{\sqrt{t^2-1}}\, \mathrm{d}t\, \mathrm{d}q.$$

Write $x_1 - y_1 = r\cos\phi$, $x_2 - y_2 = r\sin\phi$. Then $\mathrm{P}_{23} = \tan\phi\, \mathrm{P}_{13}$ and

$$\frac{\mathrm{P}_{13}(k;\boldsymbol{x},\boldsymbol{y})}{\cos\phi} = \begin{cases} J_{\mathrm{cdh}}(\xi h_3;\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_+, \\ J_{\mathrm{cdh}}(\xi h_3 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3_+,\ \boldsymbol{y} \in \mathbb{R}^3_-, \\ J_{\mathrm{cdh}}(\xi h_3 e^{\mathbf{i}(\mu_1-\mu_2)x_3};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3_-,\ \boldsymbol{y} \in \mathbb{R}^3_+, \\ J_{\mathrm{cdh}}(\xi h_3 e^{\mathbf{i}(\mu_1-\mu_2)(x_3+y_3)};\boldsymbol{x},\boldsymbol{y}), & \text{if} \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3_-. \end{cases} \tag{A.14}$$

The partial derivatives of $\mathbb{P}$ can also be written with the Cagniard-de Hoop transform. For example, for $\boldsymbol{x} \in \mathbb{R}^3_+$ and $\boldsymbol{y} \in \mathbb{R}^3_-$, (A.7) and (A.8) can be rewritten into

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l \partial x_2^m \partial x_3^n} = \mathbf{i}^{l+m+n} k_-^2 J_{\mathrm{cdh}}(\lambda_1^l \lambda_2^m \mu_1^n h_2 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), \tag{A.15}$$

$$\frac{\partial^{l+m+n}\mathrm{P}_{33}(k;\boldsymbol{x},\boldsymbol{y})}{\partial x_1^l \partial x_2^m \partial x_3^n} = (-\mathbf{i})^{l+m+n} k_-^2 J_{\mathrm{cdh}}(\lambda_1^l \lambda_2^m \mu_2^n h_2 e^{\mathbf{i}(\mu_1-\mu_2)y_3};\boldsymbol{x},\boldsymbol{y}), \tag{A.16}$$

where $\lambda_1 = \xi\cos\phi + (\varepsilon - \mathbf{i})q\sin\phi$ and $\lambda_2 = \xi\sin\phi - (\varepsilon - \mathbf{i})q\cos\phi$.

Acknowledgements. The second author is supported in part by China NSF grant 11771057 and 11671052. The third author was supported in part by the National Science Fund for Distinguished Young Scholars 11725106 and by China NSF grant 11831016.

References

[1] G. Bao and H. Wu, Convergence analysis of the perfectly matched layer problems for time-harmonic Maxwell's equations. *SIAM J. Numer. Anal.* **43** (2005) 2121–2143.

[2] G. Bao, Z. Chen and H. Wu, An adaptive finite element method for diffraction gratings. *J. Opt. Soc. Am. A* **22** (2005) 1106–1114.

[3] G. Bao, P. Li and H. Wu, An adaptive finite element method with perfectly matched absorbing layers for wave scattering by periodic structures. *Math. Comput.* **79** (2010) 1–34.

[4] J.P. Bérénger, A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.* **114** (1994) 185–200.

[5] J.H. Bramble and J.E. Pasciak, Analysis of a Cartesian PML approximation to the three dimensional electromagnetic wave scattering problem. *Int. J. Numer. Anal. Modeling* **9** (2012) 543–561.

[6] J.H. Bramble and J.E. Pasciak, Analysis of a Cartesian PML approximation to acoustic scattering problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. *J. Comput. Appl. Math.* **247** (2013) 209–230.

[7] J.H. Bramble, J.E. Pasciak and D. Trenev, Analysis of a finite PML approximation to the three dimensional elastic wave scattering problem. *Math. Comput.* **79** (2010) 2079–2101.

[8] A. Buffa, M. Costabel and D. Sheen, On traces for $\boldsymbol{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.* **276** (2002) 845–867.

[9] Z. Chen, Convergence of the time-domain perfectly matched layer method for acoustic problems. *Int. J. Numer. Anal. Model.* **6** (2009) 124–146.

[10] J. Chen and Z. Chen, An adaptive perfectly matched layer technique for 3-D time-harmonic electromagnetic scattering problems. *Math. Comput.* **77** (2008) 673–698.

[11] Z. Chen and X. Liu, An adaptive perfectly matched layer technique for time-harmonic scattering problems. *SIAM J. Numer. Anal.* **43** (2005) 645–671.

[12] Z. Chen and H. Wu, An adaptive finite element method with perfectly matched absorbing layers for the wave scattering by periodic structures. *SIAM J. Numer. Anal.* **41** (2003) 799–826.

[13] Z. Chen and X. Wu, An adaptive uniaxial perfectly matched layer method for time-harmonic scattering problems. *Numer. Math.: Theory Methods App.* **1** (2008) 113–137.

[14] Z. Chen and X. Wu, Long-time stability and convergence of the uniaxial perfectly matched layer method for time-domain acoustic scattering problems. *SIAM J. Numer. Anal.* **50** (2012) 2632–2655.

[15] Z. Chen and W. Zheng, Convergence of the uniaxial perfectly matched layer method for time-harmonic scattering problems in two-layer media. *SIAM J. Numer. Anal.* **48** (2010) 2158–2185.

[16] Z. Chen and W. Zheng, PML method for electromagnetic scattering problem in a two-layer medium. *SIAM J. Numer. Anal.* **55** (2017) 2050–2084.

[17] Z. Chen, T. Cui and L. Zhang, An adaptive uniaxial perfectly matched layer method for time harmonic Maxwell scattering problems. *Numer. Math.* **125** (2013) 639–677.

[18] Z. Chen, X. Xiang and X. Zhang, Convergence of the PML method for elastic wave scattering problems. *Math. Comput.* **85** (2016) 2687–2714.

[19] W.-C. Chew, Waves and Fields in Inhomogenous Media. Van Nodtrand Reimhold, New York (1990).

[20] F. Collino and P. Monk, The perfectly matched layer in curvilinear coordinates. *SIAM J. Sci. Comput.* **19** (1998) 2061–2090.

[21] P.M. Cutzach and C. Hazard, Existence and uniqueness and analyticity properties for electromagnetic scattering in a two-layered medium. *Math. Meth. Appl. Sci.* **21** (1998) 433–461.

[22] T. Hagstrom, Radiation boundary conditions for the numerical simulation of waves. *Acta Numer.* **8** (1999) 47–106.

[23] T. Hohage, F. Schmidt and L. Zschiedrich, Solving time-harmonic scattering problems based on the pole condition. II: convergence of the PML method. *SIAM J. Math. Anal.* **35** (2003) 547–560.

[24] X. Jiang and W. Zheng, Adaptive uniaxial perfectly matched layer method for multiple scattering problems. *Comput. Methods Appl. Mech. Eng.* **201** (2012) 42–52.

[25] M. Lassas and E. Somersalo, On the existence and convergence of the solution of PML equations. *Computing* **60** (1998) 229–241.

[26] M. Lassas and E. Somersalo, Analysis of the PML equations in general convex geometry. *Proc. R. Soc. Edinburg* (2001) 1183–1207.

[27] P. Monk, Finite Element Methods for Maxwell's Equations. Oxford University Press (2003).

[28] J.C. Nédélec, Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems. Springer-Verlag, New York (2001).

[29] X. Wu and W. Zheng, An adaptive perfectly matched layer method for multiple cavity scattering problems. *Commun. Comput. Phys.* **19** (2016) 534–558.