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LONG-TIME BEHAVIOUR OF THE APPROXIMATE SOLUTION TO QUASI-CONVOLUTION VOLTERRA EQUATIONS

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Abstract. The integral representation of some biological phenomena consists in Volterra equations whose kernels involve a convolution term plus a non convolution one. Some significative applications arise in linearised models of cell migration and collective motion, as described in Di Costanzo *et al.* (*Discrete Contin. Dyn. Syst. Ser. B* **25** (2020) 443–472), Etchegaray *et al.* (Integral Methods in Science and Engineering (2015)), Grec *et al.* (*J. Theor. Biol.* **452** (2018) 35–46) where the asymptotic behaviour of the analytical solution has been extensively investigated. Here we consider this type of problems from a numerical point of view and we study the asymptotic dynamics of numerical approximations by linear multistep methods. Through a suitable reformulation of the equation, we collect all the non convolution parts of the kernel into a *generalized forcing function,* and we transform the problem into a convolution one. This allows us to exploit the theory developed in Lubich (*IMA J. Numer. Anal.* **3** (1983) 439–465) and based on discrete variants of Paley–Wiener theorem. The main effort consists in the numerical treatment of the generalized forcing term, which will be analysed under suitable assumptions. Furthermore, in cases of interest, we connect the results to the behaviour of the analytical solution.

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1. Introduction

In some important biological phenomena Volterra integral and integro-differential equations represent an appropriate mathematical model for the description of the dynamics involved (see *e.g.* [2] and the bibliography therein). The results on asymptotic properties of Volterra equations are mainly contained in the monograph by Gripenberg *et al.* [8]. The wide bibliography refers to authors like Miller, Nohel, who greatly contributed to the development of a theory essentially based on the results of Paley and Wiener [11] for equations with convolution kernels. A recent, very clear and exhaustive overview is also presented in Brunner ([2], Chap. 6). The theory on asymptotic behaviour of non convolution equations is not well developed and it is mainly described in Gripenberg *et al.* ([8], Chaps. 9 and 10) where for integral equations the results are based on some assumptions on sign and monotonicity of the kernels, while for integro-differential equations the results presented are of perturbation type

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(*i.e.* the equations are seen as perturbed convolution equations as well as perturbed time dependent differential equations). Also a Lyapunov approach is considered, which however applies in very special cases.

As regards the numerical literature, it is mainly devoted to convolution equations and it often refers to Linear Multistep Methods (LMM) [3,9,14], whose stability properties are described by Ch. Lubich in [9] (1983) in one of the most famous and influential papers on the subject. However, some recent applications, as cell migration and collective motion (see *e.g.* [5–7]), are characterized by kernels of *quasi-convolution* type, namely involving a convolution contribution plus a non-convolution term. The presence of this spurious term prevents from using the Lubich approach for describing the long-term behaviour of the numerical solution. In this paper we focus on problems of this type and the special form of the kernels will be treated here in order to exploit the convolution part of the equation. Our idea consists in recasting the Volterra *quasi-convolution* equation into a convolution one by collecting all the non-convolution parts into what we call *generalized forcing term*. Our major effort here will be the treatment of this so-called forcing term. As a matter of fact it now contains the unknown itself, which is a serious drawback since the results we want to apply involve some hypothesis on it (bounded or vanishing) which, of course, cannot be known in advance. In particular, we consider the family of LMMs for Volterra equations and we want to obtain conditions under which the numerical solutions provided by such methods, vanish at infinity, whenever the analytical solution does.

Since our main intention is to exploit the stability theory developed in Theorem 6.1 and Corollary 6.2 of [9], based on the discrete version of the Paley–Wiener theorem, we report here these two fundamental results.

Theorem 1.1. *Consider the discrete system of Volterra equations*

$$y_n = f_n + \sum_{j=0}^{n} A_{n-j} y_j, \; n \geq 0, \; \text{in} \;\; \mathbb{R}^d, \tag{1.1}$$

where the kernel $(A_n)_0^\infty$ *belongs to* l^1*. Then we have*

$$y_n \to 0 \text{ (bounded)} \;\; \text{whenever} \;\; f_n \to 0 \text{ (bounded)} \;\; (n \to \infty)$$

if and only if

$$\det\left(I - \sum_{n=0}^{\infty} A_n \zeta^n\right) \neq 0, \;\text{for}\;\; |\zeta| \leq 1. \tag{1.2}$$

The case $(A_n)_0^\infty$ not belonging to l^1 is also treated.

Theorem 1.2. *Consider the discrete system of Volterra equations* (1.1) *where* $(A_n - A_\infty)_0^\infty$ *belongs to* l^1*, with* $A_\infty = \begin{pmatrix} 0 \\ \bar{\bar{A}}_\infty \end{pmatrix}$, *rank*$(A_\infty) = l > 0$ *and* $\bar{\bar{A}}_\infty \in \mathcal{R}^{l \times d}$*. Then*

$$y_n \to 0 \text{ (bounded)} \;\; \text{whenever} \;\; \bar{f}_n,\, \Delta\bar{\bar{f}}_n \to 0 \text{ (bounded)} \;\; (n \to \infty)$$

if and only if

$$(1-\zeta)^l \det\left(I - \sum_{n=0}^{\infty} A_n \zeta^n\right) \neq 0, \;\text{for}\;\; |\zeta| \leq 1. \tag{1.3}$$

Here $f_n = \begin{pmatrix} \bar{f}_n \\ \bar{\bar{f}}_n \end{pmatrix}$, *where* $\bar{\bar{A}}_\infty$ *and* $\bar{\bar{f}}_n$ *have the same number of rows* l.

Here $\Delta\bar{\bar{f}}_n = \bar{\bar{f}}_{n+1} - \bar{\bar{f}}_n$ and, for $|\zeta| = 1$, (1.3) has to be interpreted as the limit for $\tilde{\zeta} \to \zeta$, $\tilde{\zeta} < 1$.

In our analysis we use Theorems 1.1 or 1.2, depending on the case, by simply applying the discrete Paley–Wiener conditions (1.2) or (1.3) on the convolution part of the kernel. Thus, the main effort consists in proving that the forcing function, which now contains the unknown weighted by non-convolution terms, tends to zero (or is bounded). This is the subject of our investigations. The organization of the paper is the following.

In Section 2 we study the asymptotic properties of the numerical solution to Volterra Integro-Differential Equations (VIDEs) of the type

$$y'(t) = f(t) + a(t)y(t) + \int_0^t k(t,s)y(s)\,\mathrm{d}s,\ t \geq 0,\ \ y(0) = y_0 \in \mathbb{R}, \tag{1.4}$$

where the kernel has the *quasi-convolution* form

$$k(t,s) = p(t) + q(t-s). \tag{1.5}$$

In our approach we consider the numerical solution of the following system

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ y(0) \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & k(t,s) - a'(t) \\ 1 & a(t) \end{bmatrix} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \mathrm{d}s, \tag{1.6}$$

which is equivalent to (1.4), whenever $a(t)$ is a differentiable function. We prefer formulation (1.6) to (1.4) because we want to take advantage of the wide literature on the numerical treatment of Volterra integral equations (VIEs), in particular the results contained in [9]. In order to relate to the continuous problem the analysis carried out on the numerical method, we present, in Section 3, an analogous result for equation (1.4). What has been done in Sections 2 and 3 for the specific form (1.6) can be generalized to any Volterra system of *quasi-convolution* type. Then in Section 4 we report this extension to systems of the form

$$y(t) = f(t) + \int_0^t (P(t,s) + Q(t-s))\, y(s)\,\mathrm{d}s,\ \ t \geq 0, \tag{1.7}$$

$y, f : [0, +\infty) \to \mathbb{R}^d$, $P : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^{d\times d}$, with $P(t,s) = 0$ when $s > t$, and $Q : [0, +\infty) \to \mathbb{R}^{d\times d}$, $d \geq 1$. Equation (1.4) includes the model for the motion of the centre of mass of a system of N interacting particles described in [5], and the integral formulation of the cell migration model, which appears for example in [6,7], fits into the form of equation (1.7). Therefore, in Section 5 we describe the application to these specific problems. In particular, we study the asymptotic behaviour of the numerical solutions to these two problems (vanishing in the first case and bounded in the second one) and we compare these results with the corresponding already known ones for the analytical solutions. Finally, numerical experiments are reported in Section 6 and concluding remarks in Section 7.

From our findings we observe that, for any unconditionally stable LMM, as soon as the discrete Paley–Wiener condition, involving the convolution part of the kernel is satisfied, the numerical solution of any problem whose non convolution terms show an (even very slow) exponential decay, vanishes or remains bounded at infinity, depending on the behaviour of the corresponding forcing function. The Paley–Wiener condition is straightforwardly satisfied in some particular cases of linearised models for 1D cell migration, described in [6], where due to the boundedness of the forcing term, the numerical solution remains bounded, thus mimicking the behaviour of analytical one. For the motion of the centre of mass of a system of N interacting particles described in [5], the forcing function is zero and the exponential decay of the non convolution terms is readily observed. The difficulty for this problem arises in checking the Paley–Wiener condition which requires a dedicated investigation, directly on the discrete equation obtained from the application of Backward Euler method. The vanishing behaviour of the numerical solution is then proved when a suitable condition on the time stepsize is satisfied.

From now on, any time we refer to equation (1.4) (or (1.6)) and equation (1.7), we assume continuity for all the functions involved.

2. Investigations on the long-time behaviour of the numerical solution

A classical approach for the numerical solution of Volterra integral equations consists in LMM methods (see *e.g.* [3, 9, 14]). Let $y_0 = y(0)$, and $t_n = t_0 + nh$, for $n = 0, 1, \ldots$, with $t_0 = rh$, be the time discretization with mesh size $h > 0$. A r-step LMM, applied to system (1.6), reads

$$\begin{bmatrix} u_n \\ y_n \end{bmatrix} = \begin{bmatrix} f_n \\ g_n \end{bmatrix} + h \sum_{j=0}^{n} \omega_{n-j} \begin{bmatrix} 0 & k(t_n, t_j) - a'(t_n) \\ 1 & a(t_n) \end{bmatrix} \begin{bmatrix} u_j \\ y_j \end{bmatrix}, \tag{2.1}$$

where

$$f_n = f(t_n) + h \sum_{j=-r}^{-1} w_{nj} (k(t_n, t_j) - a'(t_n)) y_j,$$
$$g_n = y_0 + h \sum_{j=-r}^{-1} w_{nj} (u_j + a(t_n) y_j).$$

Here, $y_{-r}, \ldots, y_{-1}$ are given starting values and $y_n \approx y(t_n)$, $u_n \approx u(t_n)$ for $n \geq 0$. For the weights w_{nj} and ω_n, $n \geq 0$, we assume that (see [3], Sects. 2.6.2 and 2.6.4) for all $j = -r, \ldots, -1$, there exist constants $\bar{\omega}_j$, $\Omega, W > 0$ such that

$$0 \leq \omega_n \leq \Omega, \;\; 0 \leq w_{nj} \leq W \text{ and } \lim_{n \to \infty} w_{nj} = \bar{w}_j. \tag{2.2}$$

Exploiting the *quasi-convolution* nature (1.5) of k, equation (2.1) can be reformulated into the following discrete system

$$\begin{bmatrix} u_n \\ y_n \end{bmatrix} = \begin{bmatrix} \tilde{f}_n \\ \tilde{g}_n \end{bmatrix} + h \sum_{j=0}^{n} \omega_{n-j} K_{n-j} \begin{bmatrix} u_j \\ y_j \end{bmatrix}, \tag{2.3}$$

with $n \geq 0$. Here, let $q_n = q(t_n)$,

$$K_n = \begin{bmatrix} 0 & q_n \\ 1 & a^* \end{bmatrix}. \tag{2.4}$$

The convolution *appearance* of (2.3) has been obtained by introducing an arbitrary constant a^* in the method formulation and forcing the lag functions $\tilde{f}_n$ and $\tilde{g}_n$ to collect all the non convolution terms of the discrete system, thus, let $p_n = p(t_n)$,

$$\begin{aligned} \tilde{f}_n &= f_n + h(p_n - a'(t_n)) \sum_{j=0}^{n} \omega_{n-j} y_j \\ \tilde{g}_n &= g_n + h(a(t_n) - a^*) \sum_{j=0}^{n} \omega_{n-j} y_j. \end{aligned} \tag{2.5}$$

Here we prove a preparatory lemma, which represents the fundamental tool to prove that $(\tilde{f}_n)_n$ vanishes at infinity in Theorem 2.2.

Lemma 2.1. *Consider the discrete system* (2.3) *subjected to conditions* (2.2) *on the weights, furthermore let*

$$(f(t_n))_0^\infty, \; (a(t_n))_0^\infty, \; (a'(t_n))_0^\infty, \; (p_n)_0^\infty, (q_n)_0^\infty \in l^\infty. \tag{2.6}$$

Then, for h sufficiently small, there exists $\delta \equiv \delta(h) > 0$ and $B \equiv B(h) > 0$, such that

$$|y_n| \leq \delta \mathrm{e}^{Bn}. \tag{2.7}$$

Proof. The proof is readily obtained by applying the discrete Gronwall inequality, see *e.g.* Section 1.5.3 of [3], to system (2.3) and taking into account (2.2) and (2.6). □

Although all the sequences appearing in (2.3) depend on h (*e.g.* $y_n := y_n(h), \ldots$), such dependence will be reported only in particularly meaningful cases since, in our analysis on the asymptotic behaviour of the numerical solution, the stepsize h is a fixed positive constant and $n \to +\infty$.

Theorem 2.2. *Consider the discrete system* (2.3) *subjected to conditions* (2.2) *on the weights and assume that for a fixed stepsize* $h > 0$*:*
(i) $\lim_{n\to\infty} f(t_n) = 0$,
(ii) *there exists* a^* *and* $\alpha := \alpha(h) > 0$ *such that* $\lim_{n\to\infty} \mathrm{e}^{\alpha(h)n}(|a(t_n) - a^*| + |p_n| + |a'(t_n)|) = 0$,
(iii) $(q_n)_0^\infty \in l^1$,
(iv) $\psi(z) = (1-z)\left(1 - ha^* \sum_{n=0}^\infty \omega_n z^n - h^2 \sum_{n=0}^\infty \omega_n z^n \sum_{n=0}^\infty \omega_n q_n z^n\right)$,

$$\psi(z) \neq 0, \quad z \in \mathbb{C}, |z| < 1 \text{ and } \lim_{\zeta \to z,\ |\zeta|<1} \psi(\zeta) \neq 0, \quad |z| = 1.$$

Then,

$$\lim_{n\to\infty} y_n = 0. \tag{2.8}$$

Proof. Note that (i)–(iii) ensure that all the hypotheses of Lemma 2.1 are fulfilled, hence (2.7) holds. Then, with B and α given in (2.7) and (ii) respectively, we choose $M := M(h) \in \mathbb{N}$ such that

$$B/\alpha \leq M. \tag{2.9}$$

Let $y_n^m = \mathrm{e}^{-(M-m)\alpha n} y_n$, $u_n^m = \mathrm{e}^{-(M-m)\alpha n} u_n$, $1 \leq m \leq M-1$ and premultiply (2.3) by $\mathrm{e}^{-(M-m)n\alpha}$ to get

$$\begin{bmatrix} u_n^m \\ y_n^m \end{bmatrix} = \begin{bmatrix} f_n^m \\ g_n^m \end{bmatrix} + h \sum_{j=0}^n \omega_{n-j} K_{n-j}^m \begin{bmatrix} u_j^m \\ y_j^m \end{bmatrix}, \tag{2.10}$$

with $f_n^m = \mathrm{e}^{-(M-m)\alpha n} \tilde{f}_n$, $g_n^m = \mathrm{e}^{-(M-m)\alpha n} \tilde{g}_n$, and $K_n^m = \mathrm{e}^{-(M-m)\alpha n} K_n$.

In the scaled system (2.10) the convolution kernel $(K_n^m)_{n=0}^\infty$ belongs to l^1, for each $m = 1, \ldots, M-1$, and thus can be treated by Theorem 1.1, whose condition (1.2) now reads

$$\det\left(I - h \sum_{n=0}^\infty \omega_n K_n^m \zeta^n\right) \neq 0,\ \zeta \in \mathbb{C},\ |\zeta| \leq 1. \tag{2.11}$$

This condition is certainly true because, with $z = \mathrm{e}^{-(M-m)\alpha}\zeta$, the determinant in (2.11) takes the form

$$1 - ha^* \sum_{n=0}^\infty \omega_n z^n - h^2 \sum_{n=0}^\infty \omega_n z^n \sum_{n=0}^\infty \omega_n q_n z^n,$$

which, by hypothesis iv), is nonzero for $|z| \leq 1$.

According to Theorem 1.1, in order to prove that, for each $m = 1, \ldots, M-1$, the solution of (2.10) tends to zero when $n \to \infty$, we look at f_n^m and g_n^m, which are in turn generalized forcing terms, because they involve the solution. Set $m = 1$, in view of (2.2) and (2.7) in Lemma 2.1, we have

$$|f_n^1| \leq \mathrm{e}^{-(M-1)\alpha n} \left| f(t_n) + h \sum_{j=-r}^{-1} w_{nj}(k(t_n, t_j) - a'(t_n)) y_j \right| + h\mathrm{e}^{-(M-1)\alpha n}\left(|p_n| + |a'(t_n)|\right) \Omega\delta \frac{\mathrm{e}^{B(n+1)} - 1}{\mathrm{e}^B - 1}.$$

Since from hypothesis (i) $f(t_n) \to 0$ and hypotheses (ii) and (iii) imply that the sequences $(q_n)_n, (p_n)_n$ and $(a'(t_n))_n$ vanish at infinity, the first addendum on the right-hand side obviously tends to zero whereas, as regards the second one, we observe that because of (2.9), this is bounded by

$$h\Omega\delta \frac{\mathrm{e}^B}{\mathrm{e}^B - 1} \mathrm{e}^{\alpha n}(|p_n| + |a'(t_n)|),$$

which tends to zero as n tends to infinity, since (ii) holds. The same can be proved for g_n^1, thus $\lim_{n\to\infty} f_n^1 = \lim_{n\to\infty} g_n^1 = 0$ and hence, according to Theorem 1.1, $\lim_{n\to\infty} y_n^1 = \lim_{n\to\infty} u_n^1 = 0$. From here

$$\exists C_1 : |y_n| \leq C_1 \mathrm{e}^{(M-1)\alpha n}, \quad n \geq 0. \tag{2.12}$$

Now consider $m = 2$, using (2.12) instead of (2.7), the same reasoning applies to prove that $\lim_{n\to\infty} f_n^2 = \lim_{n\to\infty} g_n^2 = 0$ and consequently $|y_n| \leq C_2 \mathrm{e}^{(M-2)\alpha n}$. Finally, for $m = M - 1$ we have $\lim_{n\to\infty} f_n^{M-1} = \lim_{n\to\infty} g_n^{M-1} = 0$ and

$$|y_n| \leq C_{M-1} \mathrm{e}^{n\alpha}. \tag{2.13}$$

In order to prove that this sequence is infinitesimal, observe that y_n is the solution to (2.3), where the convolution kernel $(K_n)_0^\infty$ in (2.4) is such that $(K_n - K_\infty)_0^\infty$, with $K_\infty = \begin{pmatrix} 0 & 0 \\ 1 & a^* \end{pmatrix}$, belongs to l^1, thus we refer to Theorem 1.2 to conclude our analysis. The bound (2.13) leads to

$$|\tilde{f}_n| \leq \left| f(t_n) + h \sum_{j=-r}^{-1} w_{nj}(k(t_n, t_j) - a'(t_n)) y_j \right| + hC_{M-1} \frac{\Omega}{\mathrm{e}^\alpha - 1} (|p_n| + |a'(t_n)|) \mathrm{e}^{\alpha n}. \tag{2.14}$$

Recalling (i)–(iii) it is clear that the first addendum of the right-hand side (2.14) goes to zero; once again, because of (ii), the same is true also for the second addendum, hence $\lim_{n\to\infty} \tilde{f}_n = 0$. Moreover, from (2.13) and (ii), we get $\lim_{n\to\infty} |(a(t_n) - a^*) \sum_{j=0}^n \omega_{n-j} y_j| = 0$. This, together with the third of (2.2) leads to $\lim_{n\to\infty} |\tilde{g}_{n+1} - \tilde{g}_n| = 0$. All the other assumptions of Theorem 1.2 are verified, this will yeld (2.8). □

Remark 2.3. Observe that condition (iv) for the transformed system (2.3) and (2.4), corresponds to the discrete Paley–Wiener condition (1.3) in Theorem 1.2. Thus, the solution y_n of (2.3) tends to zero iff $\tilde{f}_n$, $\Delta\tilde{g}_n \to 0$, as $n \to \infty$, where $\tilde{f}_n$ and $\tilde{g}_n$ are the generalized forcing terms defined in (2.5). The presence of the unknown into $\tilde{f}_n$ and $\tilde{g}_n$ does not allow us to draw this conclusion when the only piece of information we have on the solution is that it is bounded by an exponential growth function, as stated in Lemma 2.1. In order to obtain the required vanishing behaviour of the forcing functions, we assume that some of the known functions in the equation decay exponentially (hypothesis (ii)) and rescale system (2.3) by an exponential decay function which neutralizes the growth in y_n. From the proof it is clear that assumption (ii), which represents the most demanding part of the theorem, is not so severe as one can think, since α can be arbitrarily small. Furthermore, it is not unrealistic since it applies to significant model problems, as can be also seen in Sections 5 and 6.

Remark 2.4. Condition (iv) is not very explicit and it is not straightforward to check, however it is certainly satisfied when the (ρ, σ) numerical method is A-stable and $q(t)$, the convolution part of the kernel, is negative definite (see [9], Thm. 7.2). Such kind of kernels appear in some applications treated in Section 5.

The following result is a straightforward corollary.

Corollary 2.5. *Consider the discrete system* (2.3) *subjected to conditions* (2.2) *on the weights and assume that, for a fixed stepsize* $h > 0$, *(iii), (iv) of the previous theorem hold together with*

(i*) *there exists* $F > 0$ *such that* $|f(t_n)| \leq F$,
(ii*) *there exist constants* $\Gamma > 0$ *and* a^* *and* $\alpha := \alpha(h) > 0$ *such that* $\mathrm{e}^{\alpha(h)n}(|a(t_n) - a^*| + |p_n| + |a'(t_n)|) \leq \Gamma$.

Then, the numerical solution y_n is bounded.

Proof. The proof follows the lines of the proof of Theorem 2.2. Even if, in this case, (i*) and (ii*) ensure only the boundedness of f_n^m and g_n^m, we reach the conclusion (2.13) once again. This, together with (i*) and (ii*), implies the boundedness of $\tilde{f}_n$, and $\Delta\tilde{g}_n$ and then, in view of Theorem 1.2, the desired result. □

The analysis on the asymptotic behaviour of the numerical solution carried out here has to be related to the behaviour of the analytical one. This will be our focus in the following section.

3. The long-time behaviour of the continuous solution

Consider the VIDE (1.4) where the kernel $k(t,s)$ is given in (1.5) and observe that, under the same hypotheses of Lemma 2.1, by Gronwall inequality (see *e.g.* [3], Sect. 1.5) for system (1.6), the analogue of (2.7) holds, *i.e.*

$$\exists\hat{\delta}>0 \text{ and } \hat{B}>0 \ \text{ such that } \ |y(t)|\le\hat{\delta}\mathrm{e}^{\hat{B}t}. \tag{3.1}$$

The following result, for which we give a proof sketch, represents the continuous counterpart of Theorem 2.2.

Theorem 3.1. *Consider equation* (1.4) *and assume that*

(i) $\lim_{t\to\infty} f(t)=0$, $\big($*there exists* $F>0$ *such that* $|f(t)|\le F\big)$ *and* $a(t),p(t),q(t)\in L^\infty(0,\infty)$;
(ii) *there exists* a^* *and* $\alpha>0$ *such that* $\lim_{t\to\infty}\mathrm{e}^{\alpha t}(|a(t)-a^*|+|p(t)|)=0$;
(iii) $q(t)\in L^1(0,\infty)$;
(iv) $z-a^*-\int_0^\infty q(s)\mathrm{e}^{-zs}\,\mathrm{d}s\ne 0,\ z\in\mathbb{C},\ Re(z)\ge 0.$
Then

$$\lim_{t\to\infty} y(t)=0 \ \ \Big(\text{there exists } \ Y>0 \text{ such that } \ |y(t)|\le Y\Big).$$

Proof. Hypotheses (i)–(iii) ensure (3.1). Rewrite (1.4) as

$$y'(t)=\hat{f}(t)+a^*y(t)+\int_0^t q(t-s)y(s)\,\mathrm{d}s \tag{3.2}$$

with

$$\hat{f}(t)=f(t)+(a(t)-a^*))y(t)+p(t)\int_0^t y(s)\,\mathrm{d}s; \tag{3.3}$$

choose $M\in\mathbb{N}$ such that $\frac{\hat{B}}{\alpha}\le M$ and put $y_m(t)=\mathrm{e}^{-(M-m)\alpha t}y(t)$ for $m=1,\dots,M-1$.

For each $m=1,\dots,M-1$, the VIDE

$$y_m'(t)=f_m(t)+(a^*-(M-m)\alpha))y_m(t)+\int_0^t q(t-s)\mathrm{e}^{-(M-m)\alpha(t-s)}y_m(s)\,\mathrm{d}s \tag{3.4}$$

with $f_m(t)=\mathrm{e}^{-(M-m)\alpha t}\hat{f}(t)$, has the form of equation (9.9) in [9]

$$\mathbf{y}'(t)=\mathbf{f}(t)+\mathbf{A}\mathbf{y}(t)+\int_0^t\mathbf{B}(t-s)\mathbf{y}(s)\,\mathrm{d}s,\ \ \mathbf{y}(0)=\mathbf{y}_0,\ \text{in } \mathbb{R}^d,$$

for which (see [9], Thm. 9.2) if $\mathbf{B}\in L^1(0,\infty)$, $\mathbf{y}(t)\to 0$ whenever $\mathbf{f}(t)\to 0$, if and only if

$$\det\left(\mathbf{w}I-\mathbf{A}-\int_0^\infty\mathbf{B}(t)\mathrm{e}^{-\mathbf{w}t}\,\mathrm{d}t\right)\ne 0,\ \text{for } Re(\mathbf{w})\ge 0.$$

In our notation $d = 1$, $\mathbf{B}(t) = q(t)\mathrm{e}^{-(M-m)\alpha t}$ is summable in $(0, +\infty)$ because of (iii),

$$\mathbf{w} - \mathbf{A} - \int_0^\infty \mathbf{B}(t)\mathrm{e}^{-\mathbf{w}t}\,\mathrm{d}t = \mathbf{w} - (a^* - (M-m)\alpha) - \int_0^\infty q(t)\mathrm{e}^{-(M-m)\alpha t}\mathrm{e}^{-\mathbf{w}t}\,\mathrm{d}t \neq 0 \text{ for } Re(\mathbf{w}) \geq 0,$$

for hypothesis (iv), setting $\mathbf{w} = (M-m)\alpha + z$. Furthermore, by using a similar approach to the one in the proof of Theorem 2.2, it is easy to prove that, for each m, $\mathbf{f}(t) = f_m(t) \to 0$, as $t \to \infty$. Thus $\lim_{t\to\infty} y_1(t) = \lim_{t\to\infty} y_2(t) = \ldots = \lim_{t\to\infty} y_{M-1}(t) = 0$, hence $\exists C_{M-1} : |y(t)| \leq C_{M-1}\mathrm{e}^{\alpha t}$.

Then, if, $\lim_{t\to\infty} f(t) = 0$, $\left(|f(t)| \leq F\right)$, since (ii) holds, $\lim_{t\to\infty} \hat{f}(t) = 0$ $\left(|\hat{f}(t)| \leq \hat{F}\right)$. Thus, hypotheses (iii) and (iv), by means of Theorem 9.2 of [9], guarantee a vanishing (bounded) solution for (3.2). □

4. Vector generalization

The results contained in Sections 2 and 3 can be easily extended to LMMs applied to a system of VIEs of the type (1.7). That is

$$y_n = f_n + h\sum_{j=-r}^{-1} w_{nj}(P_{nj} + Q_{n-j})y_j + h\sum_{j=0}^{n} \omega_{n-j}(P_{nj} + Q_{n-j})y_j. \tag{4.1}$$

Now $y_n,\ f_n \in \mathbb{R}^d$, $P_{ij} = P(t_i, t_j)$, $Q_{n-j} = Q(t_n - t_j)$ and Theorems 2.2 and 3.1 respectively read.

Theorem 4.1. *Consider equation (4.1), subjected to conditions (2.2) on the weights and assume that, for a fixed $h > 0$:*

(i) $\lim_{n\to\infty} f(t_n) = 0$ *(there exists $F > 0$ such that $\|f(t_n)\| \leq F, \quad n \geq 0$);*
(ii) $\exists\, (p_n)_0^\infty, \alpha(h)$ *such that* $\|P_{nj}\| \leq p_n, j = 0, \ldots, n$ *and* $\lim_{n\to\infty} \mathrm{e}^{\alpha n} p_n = 0$, *($\mathrm{e}^{\alpha n} p_n \leq p^*$);*
(iii) $(Q_n)_0^\infty \in l^1$ *entry by entry;*
(iv) $\det(I - h\sum_{n=0}^\infty \omega_n z^n \sum_{n=0}^\infty \omega_n Q_n z^n) \neq 0, \quad z \in \mathbb{C}, |z| \leq 1.$
Then

$$\lim_{n\to\infty} y_n = 0 \qquad (\text{there exists } Y > 0 \text{ such that } \|y_n\| \leq Y).$$

Theorem 4.2. *Consider equation (1.7) and assume that:*

(i) $\lim_{t\to\infty} f(t) = 0$ *(there exists $F > 0$ such that $\|f(t)\| \leq F$);*
(ii) $\exists\, p(t), \alpha$ *such that* $\|P(t,s)\| \leq p(t), 0 \leq s \leq t$ *and* $\lim_{t\to\infty} \mathrm{e}^{\alpha t} p(t) = 0$ *($\mathrm{e}^{\alpha t} p(t) \leq p^*$);*
(iii) $Q(t) \in L^1(0, \infty)$ *entry by entry;*
(iv) $\det\left(I - \int_0^\infty Q(s)\mathrm{e}^{-zs}\,\mathrm{d}s\right) \neq 0, \quad z \in \mathbb{C}, Re(z) \geq 0.$
Then

$$\lim_{t\to\infty} y(t) = 0 \qquad (\text{there exists } y^* > 0 \text{ such that } \|y(t)\| \leq y^*).$$

5. Application to model problems

In order to show the applicability of our results, we report here two case studies taken from the literature, which have, in fact, inspired this paper.

The first is a model of cell migration in one dimension based on crawling processes and described in [6]. When the forces exerted on the cell by some finger-like extensions, named filopodia, at time t, are linear, the equation for the velocity $v(t)$ of the centre of mass is the following Volterra equation

$$v(t) = f(t) - k\psi\int_0^t (b(t) - b(t-s))v(s)\,\mathrm{d}s, \tag{5.1}$$

with $f(t) = kl(\psi_r - \psi_l)b(t)$, with $k > 0$. The constant $l > 0$ is the size of filopodia, $\psi_{r,l} > 0$ are the densities of filopodia sent to the right and left, $\psi = \psi_r + \psi_l$ and

$$b(t) = \int_0^t r(x)\,\mathrm{d}x, \tag{5.2}$$

where $r(t)$ is the lifetime function. In [6] the existence and uniqueness of a solution to (5.1) has been proved, furthermore the asymptotic behaviour of the solution is described.

Equation (5.1) fits into the form (1.7) with

$$Q(t) = -P(t,s) = -k\psi(b_\infty - b(t)) \tag{5.3}$$

where, with $r \in L^1([0,+\infty))$, $b_\infty = \int_0^\infty r(x)\,\mathrm{d}x$ and $d = 1$. Since in realistic situations the forcing term can not converge to zero as t goes to infinity, here we are interested in the boundedness of the numerical solution. According to the theory described in Section 4 both the analytical and numerical solution are bounded if the conditions of Theorems 4.2 and 4.1 are satisfied, respectively. In cases where the lifetime function $r(t)$ decays exponentially, as the model described in Section 17.3.5.2 of [6], and more generally when $b_\infty - b(t)$ is positive definite (see *e.g.* [3], p. 463), conditions of Theorem 4.2 are satisfied; in particular (iv) is true for the Bochner characterization of positive definite functions, thus allowing us to use the results in [9] for the stability of linear multistep methods. Equation (5.1) is also presented in [7] as a linear model to describe cell transmigration through the arterial wall and adhesion phenomena, for which analogous numerical considerations hold.

In [5] a simplified mathematical model of collective motion, in which the coupling between an alignment and chemotaxis mechanism acts on a system of interacting particles, is described as an hybrid system, where particles are considered discrete entities and the chemical signal is supposed to be continuous. The rate of change in time of the concentration in signal is equal to the sum of a diffusion term, a source term depending on the position of each particle, and a degradation term. The linearised form of this model, in case in which the source term is given by a characteristic function on a ball centred on each particle, gives rise to the following integro-differential equation for the velocity of the centre of mass

$$y'(t) = -\int_0^t \left(\int_{t-s}^t c(x)\,\mathrm{d}x\right) y(s)\,\mathrm{d}s, \tag{5.4}$$

with

$$c(x) = \gamma_1 \frac{\mathrm{e}^{-\gamma_2 x}\mathrm{e}^{-\frac{\gamma_3}{x}}}{x^2}, \tag{5.5}$$

and $\gamma_i > 0$, $i = 1, \ldots, 3$, given. From the analytic point of view, it has been proved in [5] that the velocity of the centre of mass tends time-asymptotically to zero.

Equation (5.4) assumes the form (1.4) (and thus (1.6)) with

$$f(t) \equiv 0,\ a(t) \equiv 0,\ p(t) = -q(t) = \int_0^\infty c(x)\,\mathrm{d}x - \int_0^t c(x)\,\mathrm{d}x. \tag{5.6}$$

Here we want to prove that the simulation of (5.4) by means of the Backward Euler (BE) method (see *e.g.* [13]) furnishes a numerical solution which vanishes at infinity, thus mimicking the asymptotic behaviour of the analytical one. In [10] we have proved that, if $\lim_{n\to\infty} y_n = \bar{y}$, then $\bar{y} = 0$. Here, we want to exploit the results in Section 2 in order to get a more general result.

According to (5.6), $p(t_n) = \int_{t_n}^\infty c(x)\,\mathrm{d}x$ and $q(t_n) = -p(t_n)$. Since $p(t)$ (and consequently $q(t)$) is not known in a closed form, it must be approximated in turn. Let $\bar{p}_n$ and $\bar{q}_n$ represent the BE approximations to $p(t_n)$ and $q(t_n)$ respectively, then they have the following expressions

$$\bar{p}_n = h \sum_{j=n+1}^\infty c_j(h) \text{ and } \bar{q}_n = -\bar{p}_n, \tag{5.7}$$

with

$$c_n(h) := c(nh) = \gamma_1 \frac{e^{-\gamma_2 nh} e^{-\frac{\gamma_3}{nh}}}{(nh)^2}\cdot \tag{5.8}$$

The fully BE discretization of (5.4) now reads

$$\begin{aligned} u_n &= h\bar{p}_n \sum_{j=1}^{n} y_j + h\sum_{j=0}^{n} \bar{q}_{n-j} y_j \\ y_n &= y_0 + h\sum_{j=1}^{n} u_j, \end{aligned} \tag{5.9}$$

with $n \geq 1$.

A straightforward convergence analysis shows that the additional discretization of $p(t_n)$ and $q(t_n)$ does not affect the order of convergence, if the functions involved are sufficiently smooth. Here it is sufficient that the function $c(x)$ in (5.5) is continuously differentiable; this can be obtained by its continuous extension in zero. The following lemma plays a key role for proving our result.

Lemma 5.1. *Consider the sequence* $(c_n(h))_n$ *in* (5.8) *where* $\gamma_i > 0, i = 1, \dots 3,$ *are given and* $0 < h < +\infty$. *Define* $(b_n(h))_n$ *with*

$$b_n(h) = \sum_{j=n+1}^{\infty} c_j(h), \quad n \geq 1, \tag{5.10}$$

then,

(a) $\exists\ c^*(h)$, $0 < c^*(h) < +\infty$ *such that*

$$\sum_{n=1}^{\infty} c_n(h) = c^*(h), \tag{5.11}$$

$$\lim_{h\to 0} h^3 c^*(h) = 0. \tag{5.12}$$

(b) $\exists\ b^*(h)$, $0 < b^*(h) < +\infty$ *such that*

$$\sum_{n=1}^{\infty} b_n(h) = b^*(h). \tag{5.13}$$

Proof. (a) Observe that for any fixed $h > 0$ it is $0 < c_n(h) \leq \frac{\gamma_1}{h^2 n^2}$, so that

$$\sum_{n=1}^{\infty} c_n(h) \leq \frac{2\gamma_1}{h^2}, \tag{5.14}$$

hence (5.11) is true and $0 < h^3 c^*(h) \leq 2\gamma_1 h$, which ensures (5.12).

(b) In order to prove (5.13) we want to show that

$$\lim_{n\to\infty} n^2 b_n(h) = 0. \tag{5.15}$$

From the definition of $b_n(h)$ in (5.10) and from (5.11) we have that $(b_n(h))_n$ is a decreasing sequence vanishing at infinity, so the classical Cesaro Theorem (see *e.g.* [4] or [12]) ensures that

$$\lim_{n\to\infty} \frac{b_n(h)}{n^{-2}} = \lim_{n\to\infty} \frac{b_{n+1}(h) - b_n(h)}{(n+1)^{-2} - n^{-2}} = -\lim_{n\to\infty} \frac{c_{n+1}(h)}{(n+1)^{-2} - n^{-2}}$$

and (5.15) follows from

$$\left|\frac{b_{n+1}(h)-b_n(h)}{(n+1)^{-2}-n^{-2}}\right| \le \gamma_1 \frac{\mathrm{e}^{-\gamma_2(n+1)h}n^2}{(2n+1)h^2}.$$

□

Theorem 5.2. *Consider equation (5.9), where $\bar{p}_n$ and $\bar{q}_n$ are defined in (5.7) with $c_n(h)$ given in (5.8). Then $\lim_{n\to\infty} y_n = 0$, for all $h>0$, γ_1 and γ_2 such that $\eta(h,\gamma_1,\gamma_2)<1$, with*

$$\eta(h,\gamma_1,\gamma_2) = -\gamma_1 h \frac{(\mathrm{e}^{\gamma_2 h}-1)\log(1-\mathrm{e}^{-\gamma_2 h})-1}{2(\mathrm{e}^{\gamma_2 h}-1)}. \tag{5.16}$$

Proof. We want to prove that all the hypotheses of Theorem 2.2 are fulfilled. Assumption (i) is trivially satisfied because of the first of (5.6), whereas (iii) is ensured by (5.13). Once again by Cesaro Theorem, we get that for any fixed $h>0$ and $\alpha(h)<\gamma_2 h$,

$$\lim_{n\to\infty}\frac{\bar{p}_n}{\mathrm{e}^{-\alpha(h)n}} = \lim_{n\to\infty}\frac{h\gamma_1\mathrm{e}^{-\frac{\gamma_3}{(n+1)h}}}{(n+1)^2h^2(\mathrm{e}^{\alpha(h)}-1)}\mathrm{e}^{(n+1)(-\gamma_2 h+\alpha(h))} = 0,$$

so, also (ii) is accomplished and it remains to be proved that there results

$$\phi(z,h) = (1-z)\left(1-h^2\sum_{n=0}^{\infty}z^n\sum_{n=0}^{\infty}\bar{q}_n z^n\right) \neq 0 \quad |z|\le 1, \tag{5.17}$$

where the expression of $\bar{q}_n = -\bar{p}_n$ is given in (5.7). For $|z|<1$ we can write

$$\phi(z,h) = 1-z+h^3\sum_{n=0}^{\infty}b_n(h)z^n,$$

with $\bar{q}_n$ and $b_n(h)$ defined in (5.7) and (5.10), respectively. Observe that $\phi(1,h)\neq 0$, $\forall h>0$, and

$$\begin{aligned}
\phi(z,h) &= 1-z+h^3\sum_{n=0}^{+\infty}b_n z^n \\
&= 1-z+h^3\sum_{n=0}^{+\infty}b_n(z^n-1)+h^3\sum_{n=0}^{+\infty}b_n \\
&= 1-z-h^3\sum_{n=0}^{+\infty}b_n\sum_{j=0}^{n-1}z^j(1-z)+h^3\sum_{n=0}^{+\infty}b_n \\
&= (1-z)\left(1-h^3\sum_{n=0}^{+\infty}b_n\sum_{j=0}^{n-1}z^j\right)+h^3\sum_{n=0}^{+\infty}b_n.
\end{aligned}$$

Assume by contradiction that there exists $\bar{z}\neq 1$ with $|\bar{z}|\le 1$, such that $\phi(\bar{z},h)=0$, then

$$h^3 Re\left(\sum_{n=0}^{+\infty}b_n\sum_{j=0}^{n-1}\bar{z}^j\right) = 1+h^3 Re\left(\frac{1}{1-\bar{z}}\right)\sum_{n=0}^{+\infty}b_n.$$

Thus

$$h^3 Re\left(\sum_{n=0}^{+\infty}b_n\sum_{j=0}^{n-1}\bar{z}^j\right) \ge 1. \tag{5.18}$$

Since the real part of a number is less than its modulus, for $|\bar{z}| \leq 1$,

$$h^3 Re\left(\sum_{n=0}^{+\infty} b_n \sum_{j=0}^{n-1} \bar{z}^j\right) \leq h^3 \sum_{n=0}^{+\infty} n b_n.$$

From the definition of b_n in (5.10)

$$h^3 \sum_{n=0}^{+\infty} n b_n = h^3 \sum_{n=0}^{+\infty} n \sum_{j=n+1}^{+\infty} c_j(h) \leq h^3 \sum_{n=0}^{+\infty} (n+1) \sum_{j=n+1}^{+\infty} c_j(h).$$

Setting $i = n+1$ and switching sums in the last expression we get

$$h^3 \sum_{n=0}^{+\infty} (n+1) \sum_{j=n+1}^{+\infty} c_j(h) = h^3 \sum_{j=1}^{+\infty} c_j(h) \sum_{i=1}^{j} i = \frac{h^3}{2} \sum_{j=1}^{+\infty} (j^2+j) c_j(h),$$

thus, taking into account that, for (5.8), $c_n(h) \leq \gamma_1 \frac{\mathrm{e}^{-\gamma_2 nh}}{(nh)^2}$, it is

$$h^3 Re\left(\sum_{n=0}^{+\infty} b_n \sum_{j=0}^{n-1} \bar{z}^j\right) \leq h^3 \sum_{n=0}^{+\infty} n b_n \leq \gamma_1 \frac{h}{2} \sum_{n=1}^{+\infty} (n^2+n) \frac{\mathrm{e}^{-\gamma_2 nh}}{n^2} = \eta(h, \gamma_1, \gamma_2).$$

If $\eta(h, \gamma_1, \gamma_2) < 1$, condition (5.18) is violated and then (5.17) holds. □

6. Numerical simulations

The main aim of this section is to show experimentally the theoretical results obtained on the numerical dynamic of certain *quasi-convolution* Volterra equations, when integrated by linear multistep methods. We solve the two model problems (5.1) and (5.4) numerically by Trapezoidal and BE, respectively. As mentioned in Section 5, both equations can be rearranged into VIEs of convolution type.

The LMM method applied to (5.1) for the velocity of migration of the cell, with $P(t) = P(t,s)$ and $Q(t)$ defined in (5.3), reads

$$v_n = \tilde{f}_n + h \sum_{j=0}^{n} \omega_{n-j} Q(t_n - t_j) y_j, \tag{6.1}$$

with $\tilde{f}_n = f(t_n) + h \sum_{j=-r}^{-1} w_{nj}(P(t_n) + Q(t_n - t_j)) y_j + h P(t_n) \sum_{j=0}^{n} \omega_{n-j} y_j$. We first consider an exponential decay for the lifetime function $r(x) = \mathrm{e}^{-x}$. In this case, the expression of the analytical solution is known and we focus on the numerical approximation. We have that hypotheses (i)–(iii) of Theorem 4.1 are satisfied (hypothesis (i)) is accomplished with f bounded). Furthermore, taking into account (5.2), one has $b_\infty - b(t) = \mathrm{e}^{-t}$, which is nonnegative, nonincreasing and convex on $[0, \infty)$, then it is positive definite (see *e.g.* [3], Thm. 7.3.21). Since Trapezoidal, as well as BE, method is A-stable (see [9], Sect. 3 for A-stability definition), the results in [9] (Lem. 3.3 and Thm. 5.2) ensure that for equation (6.1) condition

$$1 - h \sum_{n=0}^{\infty} \omega_n z^n \sum_{n=0}^{\infty} \omega_n Q_n z^n \neq 0, \quad z \in \mathbb{C}, |z| \leq 1,$$

corresponding to (iv) in Theorem 4.1, is satisfied for any $h > 0$. Thus the numerical approximation v_n is bounded. This behaviour is shown in Figure 1, where the solid line represents the numerical solution. From the

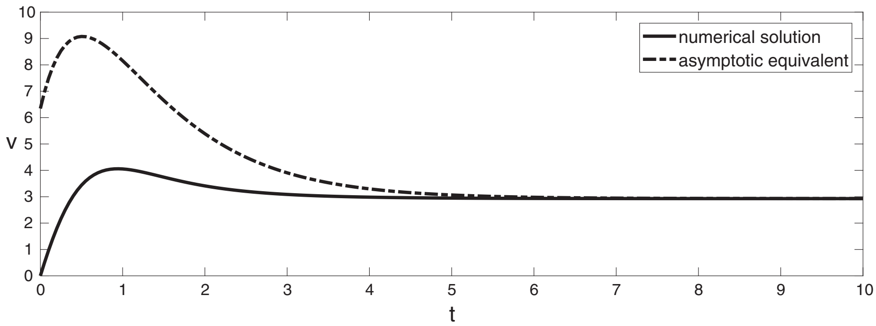

FIGURE 1. Numerical simulation of problem (5.1) ($r(t) = \mathrm{e}^{-t}$, $h = 0.1$) and asymptotic equivalent solution (6.2), with $k = 1$, $l = 20.5$, $\psi_r = 1.5$ and $\psi_l = 1$.

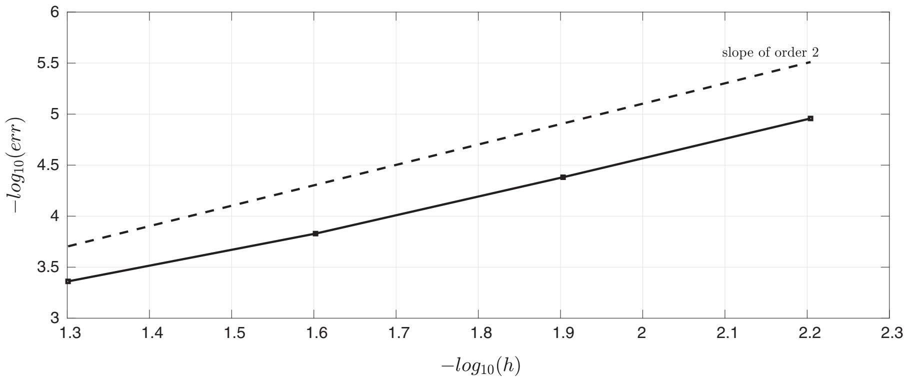

FIGURE 2. Numerical simulation of problem (5.1): error values with respect to stepsize h, in logarithmic scale.

figure it is also clear that the numerical solution behaves asymptotically like the asymptotic equivalent of the analytical solution, described in [6,7]

$$v(t) \rightsquigarrow v_{eq}(t) = kl(\psi_r - \psi_l)\left(1 - \frac{k\psi}{k\psi + 1} - \frac{k\psi}{k\psi - 1}\mathrm{e}^{-2t} + 2\mathrm{e}^{-t}\right), \tag{6.2}$$

as t tends to ∞, and represented by the dash-dotted line. The knowledge of an equivalent solution at infinity allows a check on the order of convergence of the method. So, we compute the errors $|v(t^*) - v_{eq}(t^*)|$ of the numerical solution with respect to v_{eq}, at some time t^* sufficiently large. In Figure 2 these values are reported as functions of the values of the stepsize h, the convergence of order 2 is clear from the slope.

When the lifetime is piecewise constant, $r(x) = 1_{[0,\tau]}(x)$, an explicit solution is known only for $t \leq \tau$. Here, $P(t,s)$ and $Q(t)$ are defined in (5.3), with $b_\infty = \tau$, and we can observe that, once again, all the hypotheses of Theorem 4.1 are satisfied so that the numerical solution is bounded. A plot of the numerical solution together with the exact solution for $t \leq \tau$ and the theoretical bound described in [6] for $t > \tau$, are shown in Figure 3. Finally, the velocity of the centre of mass in the system of particles described in (5.4) has been approximated numerically. According to Theorem 5.2, if $\eta(h, \gamma_1, \gamma_2) < 1$, the BE method produces a sequence $(y_n)_n$ which converges to zero as $n \to \infty$. Figure 4 reproduces this behaviour for $\gamma_1 = 1.5$, $\gamma_2 = 0.9$ and $\gamma_3 = 1$. Here we have integrated the system with $h = 0.1$, according to Theorem 5.2, since $\eta(h, 1.5, 0.9) < 1$ for each $h < h_0 = 0.12$.

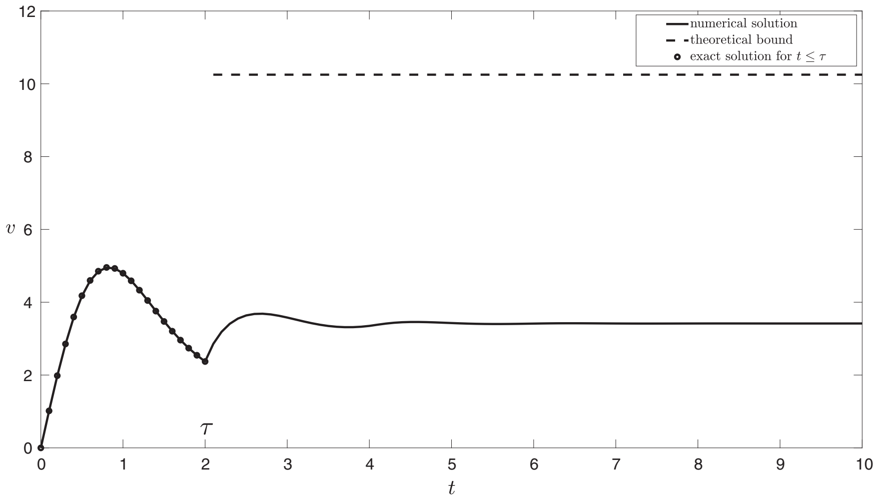

FIGURE 3. Numerical simulation of problem (5.1) ($r(t) = 1_{[0,\tau]}(t)$, $h = 0.1$), exact solution for $t \leq \tau$ and theoretical bound for $t > \tau$, with $\tau = 2$, $k = 1$, $l = 20.5$, $\psi_r = 1.5$ and $\psi_l = 1$.

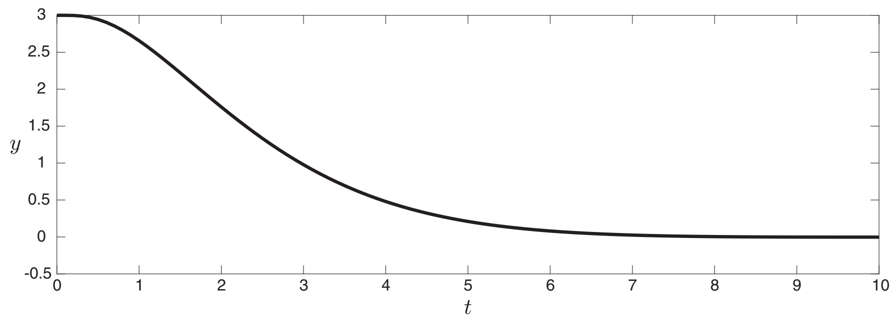

FIGURE 4. Numerical simulation of problem (5.4) ($h = 0.1$), with $\gamma_1 = 1.5$, $\gamma_2 = 0.9$ and $\gamma_3 = 1$.

However, numerical simulations with h values larger than h_0 produce the same asymptotic behaviour and this can be justified with the fact that the condition stated in Theorem 5.2 is just a sufficient one.

7. Conclusions

Motivated by some models of cell migration and collective motions, we have analysed the numerical behaviour of particular Volterra equations of *quasi-convolution* type. In this paper we have presented an alternative formulation for these equations, which moves all the non-convolution terms of the kernels into the forcing function, which now assumes a generalized meaning. The *fake* convolution structure allows us to use the theory in [9], while particular attention is dedicated to the analysis of the generalized forcing term. Here, conditions have been derived on the functions involved in the problem to provide numerical approaches which exhibit the proper end behaviour, thus furnishing a robust tool for the description of the phenomena. For the numerical solution of the two specific problems, representing linearised models for 1D cell migration described in [6] and for the motion of the centre of mass of a system of N interacting particles described in [5], obtained by Trapezoidal and BE methods respectively, the general results developed in the paper apply quite straightforwardly in the first case and with a restriction on the stepsize in the second problem. These properties have been tested trough numerical simulations in complete concordance with the theoretical results. From numerical experiments it turns out that the condition stated in Theorem 5.2 on the stepsize is, in practice, too severe, since simulations with larger stepsizes, although violate this condition, produce again the correct numerical behaviour.

Motivated by real-life problems in which the solutions converge either to zero or to a nonzero value, depending on the parameters of the problem, it is interesting to investigate on the asymptotic behaviour for the corresponding numerical solution. This will be the subject of a future work.

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