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UP-WIND DIFFERENCE APPROXIMATION AND SINGULARITY FORMATION FOR A SLOW EROSION MODEL

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Abstract. We consider a model for a granular flow in the slow erosion limit introduced in [31]. We propose an up-wind numerical scheme for this problem and show that the approximate solutions generated by the scheme converge to the unique entropy solution. Numerical examples are also presented showing the reliability of the scheme. We study also the finite time singularity formation for the model with the singularity tracking method, and we characterize the singularities as *shocks* in the solution.

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1. Introduction

We consider in this paper the model for granular flow proposed in [31] in the *slow erosion* (or deposition) limit. The model can be expressed in the form presented in [1]:

$$u_t + (f(u)E[u])_x = 0, \tag{1.1}$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R} \times (0,+\infty)$. We denote with $(\cdot)_t$ the partial derivative with respect to t and analogously with $(\cdot)_x$ the partial derivative with respect to the x variable. The erosion function $f(u)$ is defined as follows:

$$f(u) = \frac{u}{1+u} \tag{1.2}$$

and $E[u]$ is

$$E[u] = \exp\left(\int_x^{\infty} f(u(\xi,t))\,\mathrm{d}\xi\right). \tag{1.3}$$

The function $u+1$ gives the slope of the standing profile of granular matter, that is influenced by the occurrence of small avalanches. The function $f = f(u)$ is the erosion function and has the meaning of the erosion rate per unit length in space covered by the avalanches. A more detailed derivation of the model can be found in [41]. For more general f we refer to [2, 41].

Keywords and phrases. Entropy solutions, up-wind scheme, Engquist–Osher scheme, spectral analysis, complex singularities, granular flows.

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We augment (1.1) with the initial condition

$$u(x,0)=u^0(x), \qquad x\in\mathbb{R}, \tag{1.4}$$

and we assume that

$$u^0\in L^1(\mathbb{R}), \qquad -1\le u^0\le 0, \qquad f(u^0)\in L^1(\mathbb{R})\cap L^\sigma(\mathbb{R}), \text{ for some } 3\le\sigma\le\infty. \tag{1.5}$$

We use the following notions of solution for (1.1) and (1.4).

Definition 1.1. Let $u:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ be a function. We say that u is a weak solution of (1.1) and (1.4) if for any test function $\phi\in C^\infty(\mathbb{R}^2)$ with compact support we have that

$$\int_0^\infty\int_{\mathbb{R}}\Big(u\partial_t\phi+f(u)E[u]\partial_x\phi\Big)\,\mathrm{d}t\,\mathrm{d}x+\int_{\mathbb{R}}u^0(x)\phi(x,0)\,\mathrm{d}x=0. \tag{1.6}$$

Definition 1.2. Let $u:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ be a function. We say that u is an entropy solution of (1.1) and (1.4) if for any nonnegative test function $\phi\in C^\infty(\mathbb{R}^2)$ with compact support and any convex entropy $\eta\in C^2(\mathbb{R})$ with entropy flux $q\in C^2(\mathbb{R})$, defined by $q'=\eta' f'$, we have that

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}\Big(\eta(u)\partial_t\phi+q(u)E[u]\partial_x\phi+(f(u)\eta'(u)-q(u))f(u)E[u]\phi\Big)\,\mathrm{d}t\,\mathrm{d}x&\\ +\int_{\mathbb{R}}\eta(u^0(x))\phi(x,0)\,\mathrm{d}x\ge 0.&\end{aligned} \tag{1.7}$$

In [1,2] the authors studied the well-posedness of the entropy solutions of (1.1) and (1.4) assuming that

$$u^0\in BV(\mathbb{R}) \tag{1.8}$$

and that (1.5) holds with

$$\sigma=\infty, \tag{1.9}$$

which means

$$-1<-A\le u^0\le 0, \tag{1.10}$$

for some constant A with $A=1-\kappa_0\in(0,1)$. Using a front tracking algorithm, they proved that the Cauchy problem (1.1) and (1.4) admits a unique entropy solution u such that:

$$u\in L^\infty(0,T;L^1(\mathbb{R}))\cap L^\infty(0,T;BV(\mathbb{R})), \qquad T>0; \tag{1.11}$$

$$\text{for any } T>0 \text{ there exists } A_T \text{ s.t. } -1<-A_T\le u\le 0 \ a.e. \text{ in } \mathbb{R}\times(0,T). \tag{1.12}$$

Moreover, they show that the map $u_0\mapsto u$ is Lipschitz continuous, in the sense that if u and v are two entropy solutions of (1.1) satisfying (1.5), (1.8), and (1.10) at time $t=0$, then for any $T>0$ there exists a constant $L_T>0$ such that

$$\left\|u(\cdot,t)-v(\cdot,t)\right\|_{L^1(\mathbb{R})}\le L_T\left\|u(\cdot,0)-v(\cdot,0)\right\|_{L^1(\mathbb{R})}, \qquad a.e.\ 0<t<T. \tag{1.13}$$

In [13] the author proved the existence of entropy solutions for (1.1) and (1.4) under the assumption (1.5) considering the following vanishing viscosity approximation of (1.1) and (1.4) (see [11,12,14])

$$\begin{cases}\partial_t u_\varepsilon+\partial_x\left(f(u_\varepsilon)E[u_\varepsilon]\right)=\varepsilon\partial_{xx}^2u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ u_\varepsilon(x,0)=u_{0,\varepsilon}(x), & x\in\mathbb{R},\end{cases} \tag{1.14}$$

where $\varepsilon > 0$ and $u_{0,\varepsilon}$ is a smooth approximation of u^0. They bypass the lack of BV bounds on u_ε arguing as in [15, 16, 18] and using the compensated compactness result in [44] for conservation laws with discontinuous fluxes.

To the best of our knowledge, neither rigorous numerical results nor information on the process of singularity formation and its character are known for this model. Moreover, in recent years growing interest has been devoted to the numerical analysis for non-local conservation laws, see [9, 24].

The purpose of this paper is twofold. First we propose an Engquist–Osher type numerical scheme, which in this particular case is equivalent to an up-wind difference approximation, and we prove its convergence to the unique entropy solution (1.7) of the Cauchy problem in $L^p_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$, with $1 \leq p < +\infty$, for initial datum in $L^1(\mathbb{R}; [-1 + \kappa_0, 0])$, where κ_0 is a positive constant, using compensated compactness arguments.

The second purpose of this paper is to follow the process of singularity formation for the erosion model (1.1), using the singularity tracking method in the complex plane. This method allows to determine the position and the characterization of the complex singularities of the solution nearest to the real axis. Singularity tracking has been widely used to analyze loss of analyticity in many equations arising in fluid dynamics: Euler flow [4, 10, 25, 36, 38], boundary layer flow [19, 21, 26, 27], Camassa-Holm and Degasperi-Procesi equations [17, 21, 39], KdV equation [28, 33], vortex-sheet flow [3, 6, 29, 34, 40, 42]. We refer also to Caflisch *et al.* [5] for a recent review on the various singularity tracking procedures. By applying the singularity tracking method we determine the time in which the singularities hit the real domain, the positions in which they form, and their character. In a previous paper [41], it was shown that, under suitable regularity assumptions on the erosion function f, different kind of singularities can develop in finite time. We shall see that our initial set-up leads to a finite time singularity having a different character from those reported in [41]. In particular, we show that the solution of the model develops a finite time singularity manifesting with a blow-up of the first derivative; this singularity is similar to the shock singularity of the Burgers solution, see [17, 21, 43].

The paper is organized as follow. We propose a convergent up-wind scheme for (1.1) and (1.4) in Section 2, providing the convergence of the approximate solution to the unique entropy solution of (1.1). Then in Section 3, we study the finite time singularity formation for (1.1) using the singularity tracking method.

2. The up-wind numerical approximation

In this section we introduce the up-wind numerical discretization for equation (1.1), we prove the convergence of the method, and we show some numerical experiments.

We write equation (1.1) as

$$u_t + F(u)_x = 0,$$

with

$$F(u) = f(u)E[u].$$

We denote by Δx the grid spacing and by Δt the uniform time step. Set $x_j = j\Delta x$ and $t^n = n\Delta t$. We use also

$$x_{j\pm 1/2} = x_j \pm \Delta x/2 = (j \pm 1/2)\Delta x.$$

We write

$$u_j^n = u(x_j, t^n), \tag{2.1}$$

$$u^n(x) = u_j^n \ \text{ for } x \in [(j-1/2)\Delta x, (j+1/2)\Delta x), \tag{2.2}$$

$$u_{\Delta x}(x,t) = u^n(x) \ \text{ for } t \in [n\Delta t, (n+1)\Delta t). \tag{2.3}$$

The numerical discretization is given by the following formula (up-wind numerical scheme):

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n \right), \tag{2.4}$$

where, using (2.3), the expression of $E[u]^n_{j+1/2}$ is

$$E[u]^n_{j+1/2} = \mathrm{e}^{\int_{x_{j+1/2}}^{+\infty} \frac{u(\xi,t^n)}{1+u(\xi,t^n)}\,\mathrm{d}\xi} = \exp\left(\sum_{i=j+1}^{+\infty} \frac{u_i^n}{1+u_i^n}\Delta x\right), \tag{2.5}$$

and analogously for $E[u]^n_{j-1/2}$. We observe that in this particular case the up-wind scheme (2.4) is equivalent of the Engquist–Osher discretization. To show this equivalence, consider the Engquist–Osher discretization:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F^{\mathrm{EO}}_{n,j+1/2} - F^{\mathrm{EO}}_{n,j-1/2}\right), \tag{2.6}$$

where the Engquist–Osher flux is defined as (see [22])

$$F^{\mathrm{EO}}_{n,j+1/2} = \left(f^{\mathrm{EO}}(u_j^n, u_{j+1}^n)\right) E[u]^n_{j+1/2}, \tag{2.7}$$
$$F^{\mathrm{EO}}_{n,j-1/2} = \left(f^{\mathrm{EO}}(u_{j-1}^n, u_j^n)\right) E[u]^n_{j-1/2}, \tag{2.8}$$

with

$$f^{\mathrm{EO}}(a,b) = \int_0^a \max(f'(v),0)\,\mathrm{d}v + \int_0^b \min(f'(v),0)\,\mathrm{d}v + f(0), \tag{2.9}$$

and $E[u]^n_j$ given by (2.5). As

$$f'(u) = \frac{1}{(1+u)^2},$$

then

$$f^{\mathrm{EO}}(a,b) = f(a), \tag{2.10}$$

and the numerical scheme (2.6), with (2.7)–(2.9) can be written as (2.4).

We suppose that the initial datum u^0 satisfies

$$-1+\kappa_0 \le u^0 \le 0, \tag{2.11}$$

where κ_0 is a fixed positive constant.

We define the CFL condition as

$$\frac{\Delta t}{\Delta x}\|f^{'}(u)E[u]\|_\infty \le 1, \tag{2.12}$$

which must be valid for the initial datum u_0. In particular, for the initial datum u^0, using (2.11) and the monotonicity of $f^{'}(u)$ and the fact that $E[u] \le 1$, we have

$$\|f^{'}(u^0)E[u^0]\|_\infty \le \frac{1}{{\kappa_0}^2}; \tag{2.13}$$

then we consider a more restrictive, with respect to (2.12), CFL condition:

$$\frac{\Delta t}{\Delta x} \le {\kappa_0}^2. \tag{2.14}$$

The CFL condition (2.14) has the advantage to be dependent only to the initial datum u_0. In the next lemma, we prove the stability of the range $[-1+\kappa_0, 0]$ and the stability in L^∞ norm of the numerical scheme, using (2.11) and (2.14).

We remind that the conservative scheme (2.4) or (2.6), with (2.7)–(2.9) and (2.5), is not monotone (see [20]).

Lemma 2.1. *Suppose that the initial datum $u^0 \in L^1(\mathbb{R}; [-1+\kappa_0, 0])$, where κ_0 is a positive fixed constant, and the CFL condition (2.14) is satisfied, then $u^n \in [-1+\kappa_0, 0]$ for each $n \in \mathbb{N}$ and moreover*

$$\inf_{j\in\mathbb{Z}} u_j^0 \le u_j^n \le \sup_{j\in\mathbb{Z}} u_j^0, \tag{2.15}$$

for each $n \in \mathbb{N}$. In particular,

$$\sup_{j\in\mathbb{Z}} |u_j^{n+1}| \le \sup_{j\in\mathbb{Z}} |u_j^n| \le \|u^0\|_\infty, \tag{2.16}$$

and the CFL condition (2.14) is preserved at each time step.

Proof. We suppose that $u^n \in [-1+\kappa_0, 0]$, we want to prove that u^{n+1} is in $[-1+\kappa_0, 0]$. The preservation of the CFL condition (2.14) at each time step is an easy consequence.

We start from (2.4):

$$\begin{aligned}
u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x}\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right) \\
&= u_j^n - \frac{\Delta t}{\Delta x}\left(f(u_j^n) - f(u_{j-1}^n)\right)E[u]_{j+1/2}^n - \frac{\Delta t}{\Delta x}\left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n\right)f(u_{j-1}^n) \\
&= u_j^n - \frac{\Delta t}{\Delta x}\frac{f(u_j^n) - f(u_{j-1}^n)}{u_j^n - u_{j-1}^n}\left(u_j^n - u_{j-1}^n\right)E[u]_{j+1/2}^n - \frac{\Delta t}{\Delta x}\left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n\right)f(u_{j-1}^n) \\
&= u_j^n - \frac{\Delta t}{\Delta x}\frac{u_j^n - u_{j-1}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}E[u]_{j+1/2}^n - \frac{\Delta t}{\Delta x}\left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n\right)f(u_{j-1}^n) \\
&= u_j^n\left(1 - \frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}\right) \\
&\quad + \frac{\Delta t}{\Delta x}\frac{u_{j-1}^n E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)} - \frac{\Delta t}{\Delta x}\left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n\right)f(u_{j-1}^n).
\end{aligned}$$

We consider the last two terms

$$\begin{aligned}
&\frac{\Delta t}{\Delta x}\frac{u_{j-1}^n E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)} - \frac{\Delta t}{\Delta x}\left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n\right)f(u_{j-1}^n) \\
&\quad = \frac{\Delta t}{\Delta x}f(u_{j-1}^n)\left(\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)} - E[u]_{j+1/2}^n + E[u]_{j-1/2}^n\right) \\
&\quad = \frac{\Delta t}{\Delta x}f(u_{j-1}^n)\left(-f(u_j^n)E[u]_{j+1/2}^n + E[u]_{j-1/2}^n\right) \le 0.
\end{aligned}$$

Then

$$u_j^{n+1} \le u_j^n\left(1 - \frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}\right).$$

As $u_j^n \in [-1+\kappa_0, 0]$ and $0 < E[u]_{j+1/2}^n \le 1$, we have that

$$0 \le \frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)} \le \frac{\Delta t}{{\kappa_0}^2\Delta x},$$

then

$$u_j^{n+1} \le u_j^n\left(1 - \frac{\Delta t}{{\kappa_0}^2\Delta x}\right),$$

and from the CFL condition (2.14) and the fact that $u^n \leq 0$ we obtain that $u^{n+1} \leq 0$.

As $\sup_{j\in\mathbb{Z}} u_j^0 = 0$, because the initial datum u^0 is in $L^1(\mathbb{R};[-1+\kappa_0, 0])$, then the right inequality of (2.15) is proved.

We now prove (2.16) and as a consequence we have that $u^{n+1} \geq -1+\kappa_0$. We start again from (2.4) and using the same procedure, we have:

$$\begin{aligned}
u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x}\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right)\\
&= u_j^n\left(1 - \frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}\right)\\
&\quad + \frac{\Delta t}{\Delta x}\frac{u_{j-1}^n E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)} - \frac{\Delta t}{\Delta x}\left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n\right)f(u_{j-1}^n)\\
&\geq u_j^n\left(1 - \frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}\right) + \frac{\Delta t}{\Delta x}\frac{u_{j-1}^n E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}.
\end{aligned}$$

If $u_j^n \leq u_{j-1}^n$ then, from the previous inequality one obtains that $u_j^{n+1} \geq u_j^n$.

Now suppose that $u_j^n > u_{j-1}^n$, and because by the CFL condition (2.14)

$$\frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)} \leq \frac{\Delta t}{{\kappa_0}^2\Delta x} < 1,$$

we have

$$\begin{aligned}
u_j^{n+1} &\geq u_j^n - \frac{\Delta t}{\Delta x}\frac{E[u]_{j+1/2}^n}{\left(1+u_j^n\right)\left(1+u_{j-1}^n\right)}\left(u_j^n - u_{j-1}^n\right)\\
&> u_j^n - \left(u_j^n - u_{j-1}^n\right) = u_{j-1}^n,
\end{aligned}$$

which completes the proof of inequality (2.15). Because $u^n \leq 0$ and $u^{n+1} \leq 0$, for each $n \in \mathbb{N}$, then

$$|u_j^{n+1}| \leq \max\left(|u_j^n|, |u_{j-1}^n|\right),$$

which gives (2.16) and the proof is complete. □

As a consequence of Lemma 2.1 and the conservation form of the numerical scheme (2.4), we have the following lemma on the stability of the numerical scheme in L^1:

Lemma 2.2. *Suppose that $u^0 \in L^1(\mathbb{R};[-1+\kappa_0, 0])$, where κ_0 is a fixed positive constant, and the CFL condition (2.14) is satisfied, then*

$$\Delta x \sum_j |u_j^n| = \Delta x \sum_j |u_j^0| \leq \|u^0\|_{L^1(\mathbb{R})}, \tag{2.17}$$

for each $n \in \mathbb{N}$.

Proof. Since $u^0 \in L^1(\mathbb{R};[-1+\kappa_0, 0])$ and the CFL condition (2.14) holds, then, by Lemma 2.1, we have that $u^1 \in [-1+\kappa_0, 0]$. We prove (2.17) for $n = 1$: using (2.6) we have

$$\sum_j |u_j^1| - \sum_j |u_j^0| = \frac{\Delta t}{\Delta x}\sum_j\left(F_{0,j+1/2}^{\mathrm{EO}} - F_{0,j-1/2}^{\mathrm{EO}}\right) = 0, \tag{2.18}$$

observing that $|F_{0,j+1/2}^{\mathrm{EO}}| \to 0$ as $|j| \to +\infty$, because $u^0 \in L^1(\mathbb{R};[-1+\kappa_0, 0])$, hence u_j^0 goes to zero for $|j| \to +\infty$.

We proceed by induction on n: $u_j^n \in [-1+\kappa_0, 0]$ by Lemma 2.1, then

$$|u_j^n| - |u_j^{n-1}| = -u_j^n + u_j^{n-1},$$

and from (2.6) we have

$$\sum_j |u_j^n| - \sum_j |u_j^0| = \sum_{1\le k\le n} \sum_j \left(|u_j^k| - |u_j^{k-1}|\right) = \sum_{1\le k\le n} \frac{\Delta t}{\Delta x} \sum_j \left(F^{\mathrm{EO}}_{k-1,j+1/2} - F^{\mathrm{EO}}_{k-1,j-1/2}\right) = 0, \tag{2.19}$$

observing that $|F^{\mathrm{EO}}_{k-1,j+1/2}| \to 0$ as $|j| \to +\infty$, because u_j^{k-1} goes to zero for $|j| \to +\infty$, for $1 \le k \le n$, as (2.17) is valid, *i.e.* $\Delta x \sum_j |u_j^{k-1}| \le \|u^0\|_{L^1(\mathbb{R})}$, for $1 \le k \le n$ by the induction hypothesis and the proof is complete. □

2.1. Convergence analysis

In this subsection we prove the following main theorem:

Theorem 2.3. *Suppose that the initial datum $u^0 \in L^1(\mathbb{R}; [-1+\kappa_0, 0])$, where κ_0 is a fixed positive constant, and suppose that the CFL condition (2.14) is satisfied. If $u_{\Delta x}$ is the numerical solution of the up-wind scheme (2.4), or (2.6) with (2.7)–(2.9), and (2.5), then*

$$u_{\Delta x} \to u \qquad a.e. \quad \text{and in} \quad L^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+), \quad \text{for} \quad \le p < +\infty, \tag{2.20}$$

where $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ is the unique entropy solution (1.7) of the Cauchy problem (1.1)–(1.4).

The convergence proof for the scheme is based on the following crucial lemma, which is an adaptation of the compensated compactness result by Tartar [44].

Let (η, q) be the convex entropy/entropy flux pair for (1.1), *i.e.* $\eta(u)$ is a convex function of u and

$$q'(u) = \eta'(u) f'(u).$$

In particular, we denote by $\eta_0(u) = |u-k|$ and $q_0(u) = \operatorname{sgn}(u-k)(f(u)-f(k))$, the Kružkov entropy/entropy flux, with k constant.

Lemma 2.4 (see [44]). *Let $\{u_\nu\}_{\nu>0}$ be a family of functions defined on $\mathbb{R}\times\mathbb{R}_+$. If $\{u_\nu\}$ is bounded in $L^\infty(\mathbb{R}\times\mathbb{R}_+)$, and*

$$\{(\eta_0(u_\nu))_t + (q_0(u_\nu))_x\}_{\nu>0}$$

lies in a compact set of $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$, for all real constants k, then there exists a sequence $\{\nu_n\}$, $n \in \mathbb{N}$, $\nu_n \to 0$ and a function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ such that

$$u_{\nu_n} \to u \;\; a.e. \text{ and in } L^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+),\ 1 \le p < +\infty.$$

Hence we prove as a first step that the sequence

$$\{\partial_t \eta_0(u_{\Delta x}) + (q_0(u_{\Delta x}))_x\}_{\Delta x}\,, \tag{2.21}$$

with $u_{\Delta x}$ given by (2.3) which satisfies (2.4), or (2.6) with (2.7)–(2.9), and (2.5), is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$ (Lem. 2.7 below). This guarantees, by Lemma 2.4 that there exists a function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ such that (2.20) is verified. Finally, starting from a discrete entropy inequality proved in Lemma 2.8, we prove that u satisfies (1.7), which means that u is the entropy solution and the Theorem 2.3 is proved.

The following to lemmas are needed in order to prove Theorem 2.3. The first lemma gives an estimate of the discrete L^2 bound of the time variation of u_j^n.

We denote by D_x the spatial shift operator:

$$D_x\alpha_j = \alpha_{j+1} - \alpha_j, \tag{2.22}$$

and by D_t the time shift operator

$$D_t\alpha^n = \alpha^{n+1} - \alpha^n. \tag{2.23}$$

We have:

Lemma 2.5. *Suppose that $u^0 \in L^1(\mathbb{R}; [-1+\kappa_0, 0])$, with κ_0 a positive constant, and the CFL condition (2.14) is satisfies, then:*

$$\Delta x \sum_{n,j} \left(D_t u_j^n\right)^2 \leq C, \tag{2.24}$$

where C is a constant which does not depend on Δt and Δx.

Proof. We consider the numerical scheme (2.6) with (2.5)–(2.7):

$$u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\left(F^{\mathrm{EO}}_{n,j+1/2} - F^{\mathrm{EO}}_{n,j-1/2}\right),$$

and multiply both members with $2u_j^{n+1}$. Then we obtain:

$$\begin{aligned}
(u_j^{n+1})^2 - (u_j^n)^2 + (u_j^{n+1} - u_j^n)^2 = 2u_j^{n+1}(u_j^{n+1} - u_j^n) &= -2\frac{\Delta t}{\Delta x}u_j^{n+1}\left(F^{\mathrm{EO}}_{n,j+1/2} - F^{\mathrm{EO}}_{n,j-1/2}\right)\\
&= -2\frac{\Delta t}{\Delta x}u_j^n\left(F^{\mathrm{EO}}_{n,j+1/2} - F^{\mathrm{EO}}_{n,j-1/2}\right)\\
&\quad - 2\frac{\Delta t}{\Delta x}\left(u_j^{n+1} - u_j^n\right)\left(F^{\mathrm{EO}}_{n,j+1/2} - F^{\mathrm{EO}}_{n,j-1/2}\right).
\end{aligned} \tag{2.25}$$

Defining

$$Q^{\mathrm{EO}}_{n,j+1/2} = q^{\mathrm{EO}}(u_j^n)E[u]^n_{j+1/2} - q^{\mathrm{EO}}(u_{j-1}^n)E[u]^n_{j-1/2},$$

where

$$q^{\mathrm{EO}}(a) = \int_0^a sf'(s)\,\mathrm{d}s,$$

and using a simple integration by parts (see [32]), we can write:

$$\begin{aligned}
&u_j^n\left(F^{\mathrm{EO}}_{n,j+1/2} - F^{\mathrm{EO}}_{n,j-1/2}\right) = u_j^n\left(f(u_j^n)E[u]^n_{j+1/2} - f(u_{j-1}^n)E[u]^n_{j-1/2}\right)\\
&\quad = u_j^n f(u_j^n)E[u]^n_{j+1/2} - u_{j-1}^n f(u_{j-1}^n)E[u]^n_{j-1/2} + u_{j-1}^n f(u_{j-1}^n)E[u]^n_{j-1/2} - u_j^n f(u_{j-1}^n)E[u]^n_{j-1/2}\\
&\quad = \left(\int_0^{u_j^n} sf'(s)\,\mathrm{d}s + \int_0^{u_j^n} f(s)\,\mathrm{d}s\right)E[u]^n_{j+1/2} - \left(\int_0^{u_{j-1}^n} sf'(s)\,\mathrm{d}s + \int_0^{u_{j-1}^n} f(s)\,\mathrm{d}s\right)E[u]^n_{j-1/2}\\
&\qquad + u_{j-1}^n f(u_{j-1}^n)E[u]^n_{j-1/2} - u_j^n f(u_{j-1}^n)E[u]^n_{j-1/2}\\
&\quad = q^{\mathrm{EO}}(u_j^n)E[u]^n_{j+1/2} - q^{\mathrm{EO}}(u_{j-1}^n)E[u]^n_{j-1/2} + \left(\int_{u_{j-1}^n}^{u_j^n}\left(f(s) - f(u_{j-1}^n)\right)\,\mathrm{d}s\right)E[u]^n_{j-1/2}\\
&\qquad + \left(\int_0^{u_j^n} f(s)\,\mathrm{d}s\right)\left(E[u]^n_{j+1/2} - E[u]^n_{j-1/2}\right) = \left(Q^{\mathrm{EO}}_{n,j+1/2} - Q^{\mathrm{EO}}_{n,j-1/2}\right) + \Theta_{n,j} + \Xi_{n,j},
\end{aligned}$$

with

$$\Theta_{n,j} = \left(\int_{u_{j-1}^n}^{u_j^n}\left(f(s) - f(u_{j-1}^n)\right)\,\mathrm{d}s\right)E[u]^n_{j-1/2},$$

and

$$\Xi_{n,j} = \left(\int_0^{u_j^n} f(s)\,\mathrm{d}s \right) \left(E[u]_{j+1/2}^n - E[u]_{j-1/2}^n \right).$$

Thus, (2.25) can be written as:

$$\begin{aligned}(u_j^{n+1})^2 - (u_j^n)^2 + (u_j^{n+1} - u_j^n)^2 = &-2\frac{\Delta t}{\Delta x}\left(Q_{n,j+1/2}^{\mathrm{EO}} - Q_{n,j-1/2}^{\mathrm{EO}}\right) - 2\frac{\Delta t}{\Delta x}\left(\Theta_{n,j} + \Xi_{n,j}\right) \\ &- 2\frac{\Delta t}{\Delta x}\left(u_j^{n+1} - u_j^n\right)\left(F_{n,j+1/2}^{\mathrm{EO}} - F_{n,j-1/2}^{\mathrm{EO}}\right).\end{aligned} \tag{2.26}$$

Multiplying (2.26) by Δx, summing over $n = 0, 1, \dots, N-1$ and $j \in \mathbb{Z}$, we obtain:

$$\begin{aligned}\Delta x \sum_j (u_j^N)^2 + \Delta x \sum_{n,j} (u_j^{n+1} - u_j^n)^2 + 2\Delta t \sum_{n,j} \left(\Theta_{n,j} + \Xi_{n,j}\right) \\ \le \Delta x \sum_j (u_j^0)^2 + 2\Delta t \sum_{n,j} \left|u_j^{n+1} - u_j^n\right| \left|F_{n,j+1/2}^{\mathrm{EO}} - F_{n,j-1/2}^{\mathrm{EO}}\right|.\end{aligned} \tag{2.27}$$

Observing that since f is negative and increasing in u and $E[u](x,t)$ is increasing in x, then $\Xi_{n,j} \ge 0$. Following [23], we prove that:

$$\Theta_{n,j} \ge \frac{1}{2\|f'(u)E[u]\|_\infty} \left(F_{n,j+1/2}^{\mathrm{EO}} - F_{n,j-1/2}^{\mathrm{EO}}\right)^2. \tag{2.28}$$

To do that, we suppose that $u_{j-1}^n \le u_j^n$ and we consider the function $h(s) = f(u_{j-1}^n)$ for $s \in \left[u_{j-1}^n, u_j^n - l\right]$ and $h(s) = f(u_{j-1}^n) + (s - u_j^n + l)G$ for $s \in \left[u_j^n - l, u_j^n\right]$, with $lG = f(u_j^n) - f(u_{j-1}^n)$. Therefore we have:

$$\begin{aligned}\Theta_{n,j} &= \left(\int_{u_{j-1}^n}^{u_j^n} \left(f(s) - f(u_{j-1}^n)\right)\,\mathrm{d}s \right) E[u]_{j-1/2}^n \\
&\ge \left(\int_{u_{j-1}^n}^{u_j^n} \left(h(s) - f(u_{j-1}^n)\right)\,\mathrm{d}s \right) E[u]_{j-1/2}^n \\
&= \frac{l}{2}\left(f(u_j^n) - f(u_{j-1}^n)\right) E[u]_{j-1/2}^n \\
&\ge \frac{l}{2}\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right) \\
&= \frac{1}{2G}\left(f(u_j^n) - f(u_{j-1}^n)\right)\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right) \\
&= \frac{1}{2GE[u]_{j-1/2}^n}\left(f(u_j^n) - f(u_{j-1}^n)\right)E[u]_{j-1/2}^n\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right) \\
&\ge \frac{1}{2GE[u]_{j-1/2}^n}\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right)^2 \\
&\ge \frac{1}{2\|f'(u)E[u]\|_\infty}\left(f(u_j^n)E[u]_{j+1/2}^n - f(u_{j-1}^n)E[u]_{j-1/2}^n\right)^2,\end{aligned}$$

and analogously if $u_j^n \le u_{j-1}^n$. We recall that $\|f'(u)E[u]\|_\infty$ is bounded by the CFL condition (2.14) and Lemma 2.1. Then, from (2.27) and using (2.28), we have:

$$\begin{aligned}\Delta x \sum_j (u_j^N)^2 + \Delta x \sum_{n,j} (u_j^{n+1} - u_j^n)^2 + \frac{\Delta t}{\|f'(u)E[u]\|_\infty} \sum_{n,j} \left(F_{n,j+1/2}^{\mathrm{EO}} - F_{n,j-1/2}^{\mathrm{EO}}\right)^2 \\ \le \Delta x \sum_j (u_j^0)^2 + 2\Delta t \sum_{n,j} \left|u_j^{n+1} - u_j^n\right| \left|F_{n,j+1/2}^{\mathrm{EO}} - F_{n,j-1/2}^{\mathrm{EO}}\right|.\end{aligned}$$

Using the expression (2.6) we have

$$\Delta x\left(1+\frac{\Delta x}{\Delta t\|f'(u)E[u]\|_\infty}\right)\sum_{n,j}(u_j^{n+1}-u_j^n)^2\le\|u^0\|^2_{L^2(\mathbb{R})}+2\Delta x\sum_{n,j}\left(u_j^{n+1}-u_j^n\right)^2,$$

which implies

$$\Delta x\left(\frac{\Delta x}{\Delta t\|f'(u)E[u]\|_\infty}-1\right)\sum_{n,j}(u_j^{n+1}-u_j^n)^2\le\|u^0\|^2_{L^2(\mathbb{R})},$$

and by the CFL condition (2.14), we have

$$\Delta x\sum_{n,j}(u_j^{n+1}-u_j^n)^2\le\left(\frac{\Delta x}{\Delta t\|f'(u)E[u]\|_\infty}-1\right)^{-1}\|u^0\|^2_{L^2(\mathbb{R})}\le C. \tag{2.29}$$

In particular (2.24) is proved. □

We prove now the next crucial lemma, which states that (2.21) is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$. To prove this lemma, we use the following important result, given in [37]:

Lemma 2.6 (see [37])**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$, with $d\ge 2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n=\mathcal{L}_{1,n}+\mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ is in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ is in a bounded subset of $\mathcal{M}_{\mathrm{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ is in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$.

Then

Lemma 2.7. *Let (η_0,q_0) be the Kružkov entropy/entropy flux pair, i.e.*

$$\eta_0(u)=|u-k|,\qquad q_0(u)=\operatorname{sgn}(u-k)(f(u)-f(k)),$$

where k is a constant, and suppose that $u^0\in L^1(\mathbb{R};[-1+\kappa_0,0])$, with κ_0 a positive constant, and the CFL conditions (2.14) is satisfied. Then the sequence

$$\{\partial_t\eta_0(u_{\Delta x})+(q_0(u_{\Delta x}))_x\}_{\Delta x},$$

with $u_{\Delta x}$ given by (2.3) which satisfies (2.4), or (2.6) with (2.7)–(2.9), and (2.5), is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Let $\eta_{\Delta x}$ and $q^{\mathrm{EO}}_{\Delta x}$ be a family of smooth convex approximations of η_0 and q_0, such that

$$\begin{aligned}
&\eta_{\Delta x}\in\mathcal{C}^2([-1,0]),\qquad q^{\mathrm{EO}}_{\Delta x}\in\mathcal{C}^2([-1,0]),\\
&\eta''_{\Delta x}\ge 0,\qquad {q^{\mathrm{EO}}_{\Delta x}}'=\eta'_{\Delta x}f',\\
&\eta_{\Delta x}(0)=q^{\mathrm{EO}}_{\Delta x}(0)=0,\qquad |\eta'_{\Delta x}|\le 1,\\
&\|\eta_{\Delta x}-\eta_0\|_{L^\infty(-1,0)}\le\Delta x,\\
&\|\eta'_{\Delta x}-\eta'_0\|_{L^1(-1,0)}\to 0\qquad\text{as }\Delta x\to 0.
\end{aligned}$$

Let ϕ be a function in $\mathcal{C}^1_0(\mathbb{R}\times\mathbb{R}^+)$ and set

$$\begin{aligned}
&\langle\mathcal{L}_{\Delta x},\phi\rangle=\langle\partial_t\eta_0(u_{\Delta x})+(q_0(u_{\Delta x}))_x,\phi\rangle\\
&\langle\tilde{\mathcal{L}}_{\Delta x},\phi\rangle=\langle\partial_t\eta_{\Delta x}(u_{\Delta x})+(q^{\mathrm{EO}}_{\Delta x}(u_{\Delta x}))_x,\phi\rangle,
\end{aligned}$$

and

$$\langle \mathcal{L}_{\Delta x}, \phi\rangle = \langle \tilde{\mathcal{L}}_{\Delta x}, \phi\rangle + \langle \mathcal{L}_{\Delta x} - \tilde{\mathcal{L}}_{\Delta x}, \phi\rangle.$$

It is clear that $\mathcal{L}_{\Delta x} - \tilde{\mathcal{L}}_{\Delta x}$ is compact in $H^{-1}_{\text{loc}}(\mathbb{R}\times\mathbb{R}^+)$ as

$$\begin{aligned}
|\langle \mathcal{L}_{\Delta x} - \tilde{\mathcal{L}}_{\Delta x}, \phi\rangle| &\le \int_{\mathbb{R}\times\mathbb{R}^+} |\eta_{\Delta x}(u_{\Delta x}) - \eta_0(u_{\Delta x})|\,|\phi_t|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}\times\mathbb{R}^+} |\eta'_{\Delta x}(u_{\Delta x}) - \eta'_0(u_{\Delta x})||f'(u_{\Delta x})\,\phi_x|\,\mathrm{d}x\,\mathrm{d}t \\
&\le C\|\phi_t\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\eta_{\Delta x} - \eta_0\|_{L^\infty(-1,0)} \\
&\quad + C\|\phi_x\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\eta'_{\Delta x} - \eta'_0\|_{L^1(-1,0)}\|f'\|_{L^\infty[-1+\kappa_0,0]},
\end{aligned}$$

because $\|f'\|_{L^\infty[-1+\kappa_0,0]}$ is finite and C is a constant that depends on the domain of support of the function ϕ.

Let consider $\phi(x,0)=0$ and we can write $\tilde{\mathcal{L}}_{\Delta x}$ as

$$\begin{aligned}
\langle \tilde{\mathcal{L}}_{\Delta x}, \phi\rangle &= \sum_{n\ge 0,j} \left(\eta_{\Delta x}(u_j^{n+1}) - \eta_{\Delta x}(u_j^n)\right)\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t^{n+1})\,\mathrm{d}x \\
&\quad + \sum_{n\ge 0,j}\left(q_0(u_{j+1}^n) - q_0(u_j^n)\right)\int_{t^n}^{t^{n+1}} \phi(x_{j+1/2},t)\,\mathrm{d}t.
\end{aligned} \tag{2.30}$$

Denote by

$$\phi_j^n = \frac{1}{\Delta t\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

we want to write the above expression of $\tilde{\mathcal{L}}_{\Delta x}$ using ϕ_j^n. The error we make in doing this in the first term of (2.30) is

$$\begin{aligned}
&\left|\sum_{n\ge 0,j}\left(\eta_{\Delta x}(u_j^{n+1}) - \eta_{\Delta x}(u_j^n)\right)\left(\int_{x_{j-1/2}}^{x_{j+1/2}}\phi(x,t^{n+1})\,\mathrm{d}x - \Delta x\phi_j^n\right)\right| \\
&\quad\le \|\eta'_{\Delta x}\|_\infty \sum_{n,j}|u_j^{n+1} - u_j^n|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}|\phi(x,t^{n+1}) - \phi(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \|\eta'_{\Delta x}\|_\infty \sum_{n,j}|u_j^{n+1} - u_j^n|\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_t^{t^{n+1}}|\partial_t\phi(x,s)|\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \sum_{n,j}|u_j^{n+1} - u_j^n|\frac{1}{\Delta t}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t^n}^{t^{n+1}}\sqrt{t^{n+1}-t}\left(\int_{t^n}^{t^{n+1}}|\partial_t\phi(x,s)|^2\,\mathrm{d}s\right)^{1/2}\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \frac{2}{3}\sum_{n,j}|u_j^{n+1} - u_j^n|\sqrt{\Delta t}\int_{x_{j-1/2}}^{x_{j+1/2}}\left(\int_{t^n}^{t^{n+1}}|\partial_t\phi(x,s)|^2\,\mathrm{d}s\right)^{1/2}\mathrm{d}x \\
&\quad\le \frac{2}{3}\sum_{n,j}|u_j^{n+1} - u_j^n|\sqrt{\Delta t\Delta x}\left(\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t^n}^{t^{n+1}}|\partial_t\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\quad\le \frac{4}{3}\|u^0\|_\infty\sqrt{\Delta t\Delta x}\sum_{n,j}\left(\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t^n}^{t^{n+1}}|\partial_t\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{4}{3}\|u^0\|_\infty\sqrt{2\Delta t\Delta x}\left(\sum_{n,j}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{t^n}^{t^{n+1}}|\partial_t\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\leq \frac{2}{3}\|u^0\|_\infty\sqrt{2\Delta t\Delta x}\|\phi\|_{H^1(\mathbb{R}\times\mathbb{R}^+)},
\end{aligned}
$$

where for the last inequalities we have used (2.16). We perform a similar computation for the second term in (2.30):

$$
\begin{aligned}
&\left|\sum_{n\geq 0,j}\left(q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})-q^{\mathrm{EO}}_{\Delta x}(u^n_j)\right)\right)\left(\int_{t^n}^{t^{n+1}}\phi(x_{j+1/2},t)\,\mathrm{d}t-\Delta t\phi^n_j\right)\Bigg|\\
&\quad\leq \sum_{n\geq 0,j}\left|q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})-q^{\mathrm{EO}}_{\Delta x}(u^n_j)\right)\Big|\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}\left|\phi(x_{j+1/2},t)-\phi(x,t)\right|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq C\sum_{n\geq 0,j}\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_{x}^{x_{j+1/2}}|\partial_x\phi(z,t)|\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq C\sum_{n\geq 0,j}\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}\sqrt{x_{j+1/2}-x}\cdot\left(\int_{x_{j-1/2}}^{x_{j+1/2}}|\partial_x\phi(z,t)|^2\,\mathrm{d}z\right)^{1/2}\mathrm{d}x\,\mathrm{d}t\\
&\quad\leq \frac{2}{3}C\sum_{n\geq 0,j}\sqrt{\Delta x}\int_{t^n}^{t^{n+1}}\left(\int_{x_{j-1/2}}^{x_{j+1/2}}|\partial_x\phi(z,t)|^2\,\mathrm{d}z\right)^{1/2}\mathrm{d}t\\
&\quad\leq \frac{2}{3}C\sum_{n\geq 0,j}\sqrt{\Delta x\Delta t}\left(\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}|\partial_x\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\quad\leq \frac{2}{3}C\sum_{n\geq 0,j}\sqrt{\Delta x\Delta t}\left(\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}|\partial_x\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\quad\leq \frac{2}{3}C\sum_{n\geq 0,j}\sqrt{\Delta x\Delta t}\left(\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}|\partial_x\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\quad\leq \frac{2}{3}C\sqrt{2\Delta x\Delta t}\left(\sum_{n\geq 0,j}\int_{t^n}^{t^{n+1}}\int_{x_{j-1/2}}^{x_{j+1/2}}|\partial_x\phi(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\
&\quad\leq \frac{2}{3}C\sqrt{2\Delta x\Delta t}\|\phi\|_{H^1(\mathbb{R}\times\mathbb{R}^+)},
\end{aligned}
$$

using the hypothesis that $q^{\mathrm{EO}}_{\Delta x}\in\mathcal{C}^2([-1,0])$.

Then, by summarizing what we have done, we can write

$$
\begin{aligned}
\langle\tilde{\mathcal{L}}_{\Delta x},\phi\rangle = \Delta x\Delta t\sum_{n,j}&\left(\frac{1}{\Delta t}\left(\eta_{\Delta x}(u^{n+1}_j)-\eta_{\Delta x}(u^n_j)\right)+\frac{1}{\Delta x}\left(q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})-q^{\mathrm{EO}}_{\Delta x}(u^n_j)\right)\right)\phi^n_j\\
&+\text{compact terms in } H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+).
\end{aligned}
\tag{2.31}
$$

We will decompose (2.31) as the sum of three terms:

$$
\langle\tilde{\mathcal{L}}_{\Delta x},\phi\rangle = \langle\mathcal{A},\phi\rangle+\langle\mathcal{B},\phi\rangle+\langle\mathcal{C},\phi\rangle+\text{compact terms in } H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+).
\tag{2.32}
$$

We will prove that $\mathcal{A}$ and $\mathcal{B}$ are in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$, while $|\langle\mathcal{C},\phi\rangle|$ is a compact term in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$, and in this way the proof of the lemma is complete using Lemma 2.6.

We now start to prove (2.32). We consider the first term in the right hand side of (2.31), and using the Taylor expansion of $\eta_{\Delta x}$ and the numerical scheme (2.6), we have:

$$\begin{aligned}
\eta_{\Delta x}(u_j^{n+1})-\eta_{\Delta x}(u_j^n) &= \eta'_{\Delta x}(u_j^{n+1})(u_j^{n+1}-u_j^n)-\frac{1}{2}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2\\
&= -\frac{\Delta t}{\Delta x}\eta'_{\Delta x}(u_j^{n+1})\left(F^{\mathrm{EO}}_{n,j+1/2}-F^{\mathrm{EO}}_{n,j-1/2}\right)-\frac{1}{2}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2\\
&= -\frac{\Delta t}{\Delta x}\eta'_{\Delta x}(u_j^{n})\left(F^{\mathrm{EO}}_{n,j+1/2}-F^{\mathrm{EO}}_{n,j-1/2}\right)\\
&\quad -\frac{\Delta t}{\Delta x}\left(\eta'_{\Delta x}(u_j^{n+1})-\eta'_{\Delta x}(u_j^n)\right)\left(F^{\mathrm{EO}}_{n,j+1/2}-F^{\mathrm{EO}}_{n,j-1/2}\right)\\
&\quad -\frac{1}{2}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2,
\end{aligned}\tag{2.33}$$

where $\xi_j^{n+1/2}$ is an intermediate value. Now, we want to relate the Engquist–Osher flux $F^{\mathrm{EO}}_{n,j+1/2}$ with the numerical entropy flux $Q^{\mathrm{EO}}_{\Delta x}(n,j+1/2)=q^{\mathrm{EO}}_{\Delta x}(u_j^n)E[u]^n_{j+1/2}$, where we recall that

$$q^{\mathrm{EO}}_{\Delta x}(a)=\int_0^a \eta'_{\Delta x}(s)f'(s)\,\mathrm{d}s.$$

To do that, we consider

$$\begin{aligned}
\eta'_{\Delta x}(u_j^n)\left(F^{\mathrm{EO}}_{n,j+1/2}-F^{\mathrm{EO}}_{n,j-1/2}\right) &= \eta'_{\Delta x}(u_j^n)\left(f(u_j^n)E[u]^n_{j+1/2}-f(u_{j-1}^n)E[u]^n_{j-1/2}\right)\\
&= \eta'_{\Delta x}(u_j^n)f(u_j^n)E[u]^n_{j+1/2}-\eta'_{\Delta x}(u_{j-1}^n)f(u_{j-1}^n)E[u]^n_{j-1/2}\\
&\quad +\eta'_{\Delta x}(u_{j-1}^n)f(u_{j-1}^n)E[u]^n_{j-1/2}-\eta'_{\Delta x}(u_j^n)f(u_{j-1}^n)E[u]^n_{j-1/2}\\
&= q^{\mathrm{EO}}_{\Delta x}(u_j^n)E[u]^n_{j+1/2}-q^{\mathrm{EO}}_{\Delta x}(u_{j-1}^n)E[u]^n_{j-1/2}\\
&\quad +\left(\int_{u_{j-1}^n}^{u_j^n}\eta''_{\Delta x}(s)\left(f(s)-f(u_{j-1}^n)\right)\,\mathrm{d}s\right)E[u]^n_{j-1/2}\\
&\quad +\left(\int_0^{u_j^n}\eta''(s)f(s)\,\mathrm{d}s\right)\left(E[u]^n_{j+1/2}-E[u]^n_{j-1/2}\right),
\end{aligned}$$

where the last equality is obtained by a simple integration by parts (see [32]). Introducing the following notation:

$$\begin{aligned}
\Xi_{\Delta x}(n,j) &= \left(\int_0^{u_j^n}\eta''_{\Delta x}(s)f(s)\,\mathrm{d}s\right)\left(E[u]^n_{j+1/2}-E[u]^n_{j-1/2}\right),\\
\Theta_{\Delta x}(n,j) &= \left(\int_{u_{j-1}^n}^{u_j^n}\eta''_{\Delta x}(s)\left(f(s)-f(u_{j-1}^n)\right)\,\mathrm{d}s\right)E[u]^n_{j-1/2},
\end{aligned}$$

we obtain

$$\eta'_{\Delta x}(u_j^n)\left(F^{\mathrm{EO}}_{n,j+1/2}-F^{\mathrm{EO}}_{n,j-1/2}\right)=\left(Q^{\mathrm{EO}}_{\Delta x}(n,j+1/2)-Q^{\mathrm{EO}}(n,j-1/2)\right)+\Theta_{\Delta x}(n,j)+\Xi_{\Delta x}(n,j).\tag{2.34}$$

Observe that since f is increasing in u and $E[u](x,t)$ is increasing in x, then $\Theta_{\Delta x}(n,j)$ and $\Xi_{\Delta x}(n,j)$ are both non negative.

Using (2.34) in (2.33) we can write:

$$\begin{aligned}\eta_{\Delta x}(u_j^{n+1})-\eta_{\Delta x}(u_j^n)= &-\frac{\Delta t}{\Delta x}\left(Q_{\Delta x}^{\mathrm{EO}}(n,j+1/2)-Q_{\Delta x}^{\mathrm{EO}}(n,j-1/2)\right)-\frac{\Delta t}{\Delta x}\left(\Theta_{\Delta x}(n,j)+\Xi_{\Delta x}(n,j)\right)\\ &-\frac{\Delta t}{\Delta x}\left(\eta'_{\Delta x}(u_j^{n+1})-\eta'_{\Delta x}(u_j^n)\right)\left(F_{n,j+1/2}^{\mathrm{EO}}-F_{n,j-1/2}^{\mathrm{EO}}\right)-\frac{1}{2}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2,\end{aligned}\tag{2.35}$$

and finally, using the above expression (2.35) in (2.31), we have proved (2.32), with

$$\langle\mathcal{A},\phi\rangle=\Delta x\Delta t\sum_{n,j}A_j^n\phi_j^n,\qquad\langle\mathcal{B},\phi\rangle=\Delta x\Delta t\sum_{n,j}B_j^n\phi_j^n,\qquad\langle\mathcal{C},\phi\rangle=\Delta x\Delta t\sum_{n,j}C_j^n\phi_j^n,$$

and

$$\begin{aligned}A_j^n&=-\frac{1}{2\Delta}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2-\frac{1}{\Delta x}(\Theta_{\Delta x}(n,j)+\Xi_{\Delta x}(n,j)),\\ B_j^n&=-\frac{1}{\Delta x}\left(\eta'_{\Delta x}(u_j^{n+1})-\eta'_{\Delta x}(u_j^n)\right)\left(F_{n,j+1/2}^{\mathrm{EO}}-F_{n,j-1/2}^{\mathrm{EO}}\right),\\ C_j^n&=-\frac{1}{\Delta x}\left(\left(Q_{\Delta x}^{\mathrm{EO}}(n,j+1/2)-Q_{\Delta x}^{\mathrm{EO}}(n,j-1/2)\right)-\left(q_{\Delta x}^{\mathrm{EO}}(u_{j+1}^n)-q_{\Delta x}^{\mathrm{EO}}(u_j^n)\right)\right).\end{aligned}$$

We now prove that $\mathcal{A}$ is in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$. To do that, we multiply (2.35) by Δx and $|\phi_j^n|$ and we sum over $n=0,\dots,N$ and j, obtaining:

$$\begin{aligned}&\Delta x\sum_j\eta_{\Delta x}(u_j^N)|\phi_j^n|+\frac{1}{2}\Delta x\sum_{n,j}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2|\phi_j^n|+\Delta t\sum_{n,j}\left(\Theta_{\Delta x}(n,j)+\Xi_{\Delta x}(n,j)\right)|\phi_j^n|\\ &\le\Delta x\sum_j\eta_{\Delta x}(u_j^0)|\phi_j^n|+2\|\eta''_{\Delta x}\|_\infty\Delta t\sum_{n,j}\left|u_j^{n+1}-u_j^n\right|\left|F_{n,j+1/2}^{\mathrm{EO}}-F_{n,j-1/2}^{\mathrm{EO}}\right||\phi_j^n|\end{aligned}$$

and finally, using (2.24) and (2.6), we have

$$\begin{aligned}\left|\Delta x\Delta t\sum_{n,j}A_j^n\phi_j^n\right|&\le\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\left(\frac{\Delta x}{2}\sum_{n,j}\eta''_{\Delta x}(\xi_j^{n+1/2})(u_j^{n+1}-u_j^n)^2+\Delta t\sum_{n,j}\left(\Theta_{\Delta x}(n,j)+\Xi_{\Delta x}(n,j)\right)\right)\\ &\le\left(\Delta x\sum_j\eta_{\Delta x}(u_j^0)+2\|\eta''_{\Delta x}\|_\infty\Delta x\sum_{n,j}\left(u_j^{n+1}-u_j^n\right)^2\right)\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\\ &\le C\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}.\end{aligned}$$

Similar is the proof that $\mathcal{B}$ is bounded $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$. In fact, using again the expression of the numerical scheme (2.6) and (2.24), we have

$$\begin{aligned}|\langle\mathcal{B},\phi\rangle|=\left|\Delta x\Delta t\sum_{n,j}B_j^n\phi_j^n\right|&\le\Delta x\Delta t\|\phi\|_\infty\sum_{n,j}\left|B_j^n\right|\\ &\le\Delta t\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\sum_{n,j}\left|\eta'_{\Delta x}(u_j^{n+1})-\eta'_{\Delta x}(u_j^n)\right|\left|F_{n,j+1/2}^{\mathrm{EO}}-F_{n,j-1/2}^{\mathrm{EO}}\right|\\ &\le\Delta t\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\eta''_{\Delta x}\|_\infty\sum_{n,j}\left|u_j^{n+1}-u_j^n\right|\left|F_{n,j+1/2}^{\mathrm{EO}}-F_{n,j-1/2}^{\mathrm{EO}}\right|\\ &\le\Delta x\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\eta''_{\Delta x}\|_\infty\sum_{n,j}\left(u_j^{n+1}-u_j^n\right)^2\\ &\le C\|\phi\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}\|\eta''_{\Delta x}\|_\infty.\end{aligned}$$

Finally, we prove that $|\langle \mathcal{C},\phi\rangle|$ is a compact term in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$:

$$\begin{aligned}
\left|\Delta x\Delta t\sum_{n,j} C_j^n\phi_j^n\right| &= \left|\Delta t\sum_{n,j}\left(\left(Q^{\mathrm{EO}}_{\Delta x}(n,j+1/2)-Q^{\mathrm{EO}}_{\Delta x}(n,j-1/2)\right)-\left(q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})-q^{\mathrm{EO}}_{\Delta x}(u^n_j)\right)\right)\phi^n_j\right|\\
&= \left|\Delta x\Delta t\sum_{n,j}\left(Q^{\mathrm{EO}}_{\Delta x}(n,j+1/2)-q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})\right)\frac{D\phi^n_j}{\Delta x}\right|\\
&\le \Delta x\Delta t\sum_{n,j}\left(\left|\int_0^{u^n_j}\eta'_{\Delta x}(s)f'(s)\,\mathrm{d}s\right|E[u]^n_{j+1/2}+\left|q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})\right|\right)\left|\frac{D\phi^n_j}{\Delta x}\right|\\
&\le \Delta x\Delta t\sum_{n,j}\left(\left|q^{\mathrm{EO}\prime}_{\Delta x}(u^n_j)\right|+\left|q^{\mathrm{EO}}_{\Delta x}(u^n_{j+1})\right|\right)\left|\frac{D\phi^n_j}{\Delta x}\right|\\
&\le C\sqrt{\Delta x\Delta t}\left(\Delta x\Delta t\sum_{n,j}\left(\frac{D\phi^n_j}{\Delta x}\right)^2\right)^{1/2}\le C\sqrt{\Delta x\Delta t}\|\phi\|_{H^1(\mathbb{R}\times\mathbb{R}_+)},
\end{aligned}$$

where we use again the hypothesis that $q^{\mathrm{EO}}_{\Delta x}\in\mathcal{C}^2((-1,0])$. □

We prove the following lemma, where the discrete entropy inequality for the numerical scheme (2.6) is proved:

Lemma 2.8. *Suppose that the initial datum $u^0\in L^1(\mathbb{R};[-1+\kappa_0,0])$, where κ_0 is a fixed positive constant, and suppose that the CFL condition (2.14) is satisfied. If u^{n+1}_j is the numerical solution of the up-wind scheme (2.4), or (2.6) with (2.7)–(2.9), and (2.5), then*

$$\begin{aligned}
\eta(u^{n+1}_j)\le \eta(u^n_j)&-\frac{\Delta t}{\Delta x}\left(q^{\mathrm{EO}}(u^n_j)E[u]^n_{j+1/2}-q^{\mathrm{EO}}(u^n_{j-1})E[u]^n_{j-1/2}\right)\\
&-\frac{\Delta t}{\Delta x}\left(\eta'(u^{n+1}_j)f(u^n_j)-q^{\mathrm{EO}}(u^n_j)\right)\left(E[u]^n_{j+1/2}-E[u]^n_{j-1/2}\right),
\end{aligned}\tag{2.36}$$

for each convex entropy function $\eta\in C^2(\mathbb{R})$, with entropy flux $q^{\mathrm{EO}\prime}=\eta' f'$.

Proof. We write the numerical scheme (2.6) as

$$u^{n+1}_j = z^n_j-\frac{\Delta t}{\Delta x}f(u^n_j)\left(E[u]^n_{j+1/2}-E[u]^n_{j-1/2}\right),\tag{2.37}$$

with

$$z^n_j=u^n_j-\frac{\Delta t}{\Delta x}\left(f(u^n_j)-f(u^n_{j-1})\right)E[u]^n_{j-1/2}.\tag{2.38}$$

We introduce the following notation

$$G(v)=v_j-\frac{\Delta t}{\Delta x}\left(f(v_j)-f(v_{j-1})\right)E[u]^n_{j-1/2},$$

where we use the notation $v=(v_j)_{j\in\mathbb{Z}}$. We prove that $G(v)$ is monotone: let $w=(w_j)_{j\in\mathbb{Z}}$ such that, for each $j\in\mathbb{Z}$, $w_j\le v_j$ and suppose that $v_j\ge -1+\kappa_0$, for each $j\in\mathbb{Z}$, with κ_0 that satisfies the CFL condition (2.14), then $G(w) \le G(v)$, because

$$\begin{aligned} G(v) - G(w) &= (v_j - w_j) - \frac{\Delta t}{\Delta x}\left(f(v_j) - f(v_{j-1})\right) E[u]^n_{j-1/2} + \frac{\Delta t}{\Delta x}\left(f(w_j) - f(w_{j-1})\right) E[u]^n_{j-1/2} \\ &= (v_j - w_j) - \frac{\Delta t}{\Delta x}\left(f(v_j) - f(w_j)\right) E[u]^n_{j-1/2} + \frac{\Delta t}{\Delta x}\left(f(v_{j-1}) - f(w_{j-1})\right) E[u]^n_{j-1/2} \\ &= (v_j - w_j)\left(1 - \frac{\Delta t}{\Delta x}\frac{1}{(1+v_j)(1+w_j)}\right) E[u]^n_{j-1/2} + \frac{\Delta t}{\Delta x}\left(f(v_{j-1}) - f(w_{j-1})\right) E[u]^n_{j-1/2} \\ &\ge (v_j - w_j)\left(1 - \frac{\Delta t}{\Delta x}\frac{1}{(1+v_j)^2}\right) E[u]^n_{j-1/2} + \frac{\Delta t}{\Delta x}\left(f(v_{j-1}) - f(w_{j-1})\right) E[u]^n_{j-1/2} \\ &\ge (v_j - w_j)\left(1 - \frac{\Delta t}{\Delta x}\frac{1}{\kappa_0^2}\right) E[u]^n_{j-1/2} + \frac{\Delta t}{\Delta x}\left(f(v_{j-1}) - f(w_{j-1})\right) E[u]^n_{j-1/2} \ge 0, \end{aligned}$$

where in the last inequality we use that f is non-decreasing. This implies (see [30]) that from (2.38) we have

$$\eta(z_j^n) \le \eta(u_j^n) - \frac{\Delta t}{\Delta x}\left(q^{\mathrm{EO}}(u_j^n) - q^{\mathrm{EO}}(u_{j-1}^n)\right) E[u]^n_{j-1/2}, \tag{2.39}$$

where η is a convex entropy in $C^2(\mathbb{R})$, and q^{EO} is the associated numerical entropy flux

$$q^{\mathrm{EO}}(u) = \int_0^u \eta'(s) f'(s)\,\mathrm{d}s.$$

From (2.37) and the fact that η is convex we have that

$$\eta(z_j^n) \ge \eta(u_j^{n+1}) + \frac{\Delta t}{\Delta x}\eta'(u_j^{n+1}) f(u_j^n)\left(E[u]^n_{j+1/2} - E[u]^n_{j-1/2}\right). \tag{2.40}$$

Collecting (2.39) and (2.40), we prove (2.36). □

We are finally ready to prove Theorem 2.3:

Proof of Theorem 2.3. In Lemma 2.7 was proved that the sequence

$$\{\partial_t \eta_0(u_{\Delta x}) + (q_0(u_{\Delta x}))_x\}_{\Delta x}$$

is compact in $H^{-1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$. Therefore by Lemma 2.4 there exists a subsequence $\{u_{\Delta x}\}$, which we do not relabel, and a function $u \in L^\infty(\mathbb{R}\times\mathbb{R}^+)$ such that

$$u_{\Delta x} \to u \qquad a.e. \text{ and in } L^p_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+), \text{ for any } p \ge 1.$$

We prove now that the limit function u is the unique entropy solution of (1.1). In particular, uniqueness is guarantee from [13]. We prove that the limit function u satisfies (1.7). With the notation of D_t and D_x of (2.23) and (2.22), the discrete entropy inequality (2.36) obtained in the previous lemma can be written as

$$\frac{1}{\Delta t} D_t \eta(u_j^n) + \frac{1}{\Delta x} D_x Q^{\mathrm{EO}}_{n,j-1/2} + \frac{1}{\Delta x}\left(\eta'(u_j^{n+1}) f(u_j^n) - q^{\mathrm{EO}}(u_j^n)\right)\left(E[u]^n_{j+1/2} - E[u]^n_{j-1/2}\right) \le 0, \tag{2.41}$$

where $Q^{\mathrm{EO}}(n, j-1/2) = q^{\mathrm{EO}}(u_{j-1}^n) E[u]^n_{j-1/2}$, with η a convex entropy in $C^2(\mathbb{R})$ and $q^{\mathrm{EO}\prime} = \eta' f'$ the associated entropy flux. Let $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_0^+)$ be a positive test function and let define

$$\phi_j^n = \frac{1}{\Delta x \Delta t}\int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t^n}^{t^{n+1}} \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

We multiply (2.41) by $\Delta x \Delta t \phi_j^n$ and doing a partial summation for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, we find that

$$\Delta x \sum_{j,n} D_t \eta(u_j^n)\phi_j^n = -\Delta x \sum_j \eta(u_j^0)\phi_j^0 - \Delta x \sum_{j,n\geq 1} \eta(u_j^n) D_t \phi_j^{n-1},$$

and

$$\Delta t \sum_{j,n} D_x Q^{\mathrm{EO}}_{n,j-1/2}\phi_j^n = -\Delta t \sum_{n,j} Q^{\mathrm{EO}}_{n,j+1/2} D_x \phi_j^n,$$

then (2.41) can be written as

$$\begin{aligned}\underbrace{\Delta x \sum_j \eta(u_j^0)\phi_j^0}_{I_0} + \underbrace{\Delta x \sum_{j,n\geq 1} \eta(u_j^n) D_t \phi_j^{n-1}}_{I_1} + \underbrace{\Delta t \sum_{n,j} Q^{\mathrm{EO}}_{n,j+1/2} D_x \phi_j^n}_{I_2} \\ \underbrace{-\Delta t \sum_{j,n\geq 1} \left(\eta'(u_j^{n+1}) f(u_j^n) - q^{\mathrm{EO}}(u_j^n)\right) \left(E[u]^n_{j+1/2} - E[u]^n_{j-1/2}\right) \phi_j^n}_{I_3} \geq 0.\end{aligned}$$

Since

$$\begin{aligned} u^n(x) &= u_j^n \text{ for } x \in [(j-1/2)\Delta x, (j+1/2)\Delta x\,), \\ u_{\Delta x}(x,t) &= u^n(x) \text{ for } t \in [n\Delta t, (n+1)\Delta t\,). \end{aligned}$$

the first term can be written as

$$\begin{aligned} I_0 &= \Delta x \sum_j \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(u(x,0)) \int_0^{\Delta t} \frac{\phi(x,s)}{\Delta t} \,\mathrm{d}s \,\mathrm{d}x \\ &= \int_{\mathbb{R}} \eta(u_{\Delta x}(x,0))\phi(x,0)\,\mathrm{d}x + \Delta x \sum_j \eta(u_j^0) \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_0^{\Delta t} \frac{\phi(x,s)-\phi(x,0)}{\Delta t} \,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

The last summation can be estimated by

$$\begin{aligned} &\Delta x \left| \sum_j \eta(u_j^0) \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_0^{\Delta t} \frac{\phi(x,s)-\phi(x,0)}{\Delta t} \,\mathrm{d}s\,\mathrm{d}x \right| \\ &\qquad \leq \Delta x \sum_j |\eta(u_j^0)| \left| \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_0^{\Delta t} \frac{\phi(x,s)-\phi(x,0)}{\Delta t} \,\mathrm{d}s\,\mathrm{d}x \right| \\ &\qquad \leq \|\partial_t \phi\|_\infty \left(\Delta x \sum_j |\eta(u_j^0)| \right) \left| \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{1}{\Delta x \Delta t} \int_0^{\Delta t} s \,\mathrm{d}s\,\mathrm{d}x \right| \leq \frac{\Delta t}{2} \|\partial_t \phi\|_\infty \|\eta(u^0)\|_{L^1(\mathbb{R})}, \end{aligned}$$

which implies that it goes to zero as $\Delta t \to 0$, because $\|\eta(u^0)\|_{L^1}$ is finite by hypothesis and $\|\partial_t \phi\|_\infty$ is finite as ϕ is a test function. Then, using the bounded convergence theorem and the CFL condition, we have:

$$\lim_{\Delta x \to 0} I_0 = \int_{\mathbb{R}} \eta(u^0(x))\phi(x,0)\,\mathrm{d}x. \tag{2.42}$$

We start to analyze the I_1 integral:

$$\begin{aligned}
\Delta x \sum_{j,n\geq 1} \eta(u_j^n) D_t \phi_j^{n-1} &= \Delta x \sum_{j,n\geq 1} \eta(u_j^n) \frac{1}{\Delta x \Delta t} \int_{x_{j-1/2}}^{x^{j+1/2}} \left(\int_{t^n}^{t^{n+1}} \phi(x,t)\,\mathrm{d}t - \int_{t^{n-1}}^{t^n} \phi(x,t)\,\mathrm{d}t \right) \mathrm{d}x \\
&= \sum_{j,n\geq 1} \frac{u_j^n}{\Delta t} \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \left(\phi(x,t) - \phi(x,t-\Delta t)\right) \mathrm{d}t\,\mathrm{d}x \\
&= \sum_{j,n\geq 1} \frac{\eta(u_j^n)}{\Delta t} \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \int_{t-\Delta t}^{t} \partial_t \phi(x,s)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\
&= \sum_{j,n\geq 1} \eta(u_j^n) \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \partial_t \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \underbrace{\sum_{j,n\geq 1} \eta(u_j^n) \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \int_{t-\Delta t}^{t} \frac{\partial_t \phi(x,s) - \partial_t \phi(x,t)}{\Delta t}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x}_{J_1}.
\end{aligned}$$

We prove now that $J_1 \to 0$ as $\Delta t \to 0$:

$$\begin{aligned}
|J_1| &\leq \sum_{j,n\geq 1} |\eta(u_j^n)| \left| \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \int_{t-\Delta t}^{t} \frac{1}{\Delta t} \int_s^t \partial_{tt} \phi(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \right| \\
&\leq \sum_{j,n\geq 1} |\eta(u_j^n)| \left| \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \int_{t-\Delta t}^{t} \frac{\|\partial_{tt}\phi\|_\infty}{\Delta t} (t-s)\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \right| \\
&\leq \sum_{j,n\geq 1} |\eta(u_j^n)| \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} \|\partial_{tt}\phi\|_\infty \frac{\Delta t}{2}\,\mathrm{d}t\,\mathrm{d}x \\
&\leq \|\partial_{tt}\phi\|_\infty \frac{\Delta t}{2} \sum_{j,n\geq 1} \int_{x_{j-1/2}}^{x^{j+1/2}} \int_{t^n}^{t^{n+1}} |\eta(u_j^n)|\,\mathrm{d}t\,\mathrm{d}x \\
&\leq \|\partial_{tt}\phi\|_\infty \frac{\Delta t}{2} \|\eta(u_{\Delta x}(\cdot,t))\|_{L^1(\mathbb{R})} \to 0 \qquad \text{as } \Delta t \to 0,
\end{aligned}$$

and consequently

$$\lim_{\Delta x \to 0} I_1 = \int_0^{+\infty} \int_{\mathbb{R}} \eta(u(x,t)) \partial_t \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x. \tag{2.43}$$

We now analyze the integral I_2

$$\begin{aligned}
I_2 &= \Delta t \sum_{n,j} Q^{\mathrm{EO}}_{n,j+1/2} D_j \phi_j^n \\
&= \sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]^n_{j+1/2} \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left(\int_{x_{j+1/2}}^{x_{j+1}} \phi(x,t)\,\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\,\mathrm{d}x \right) \mathrm{d}t
\end{aligned}$$

$$= \underbrace{\sum_{n,j} \left(q^{\mathrm{EO}}(u_j^n) E[u]_{j+1/2}^n - q^{\mathrm{EO}}(u_j^n) E[u]_j^n \right) \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left(\int_{x_{j+1/2}}^{x_{j+1}} \phi(x,t)\,\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\,\mathrm{d}x \right) \mathrm{d}t}_{J_2}$$

$$+ \underbrace{\sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]_j^n \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left(\int_{x_{j+1/2}}^{x_{j+1}} \phi(x,t)\,\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\,\mathrm{d}x \right) \mathrm{d}t}_{J_3}.$$

Using a change of coordinate we can write J_3 as

$$\begin{aligned}
J_3 &= \sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]_j^n \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\phi(x-\Delta x,t)-\phi(x,t)}{\Delta x}\,\mathrm{d}x\,\mathrm{d}t \\
&= \sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]_j^n \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_0^{-\Delta x} \frac{\partial_s \phi(x+s,t)}{\Delta x}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&= \sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]_j^n \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\int_0^{-\Delta x} \frac{\partial_s \phi(x+s,t) - \partial_x \phi(x,t)}{\Delta x}\,\mathrm{d}s + \partial_x \phi(x,t) \right) \mathrm{d}x\,\mathrm{d}t \\
&= \underbrace{\sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]_j^n \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_0^{-\Delta x} \int_0^s \frac{\partial_{zz} \phi(x+z,t)}{\Delta x}\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t}_{J_4} \\
&\quad + \sum_{n,j} q^{\mathrm{EO}}(u_j^n) E[u]_j^n \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \partial_x \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

then

$$J_3 = J_4 + \int_0^{+\infty} \int_{\mathbb{R}} q^{\mathrm{EO}}(u_{\Delta x}) E[u_{\Delta x}] \partial_x \phi(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

Repeating similar arguments, we have that

$$\begin{aligned}
|J_4| &\leq \sum_{n,j} |q^{\mathrm{EO}}(u_j^n)| |E[u]_j^n| \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\Delta x}{2} \|\partial_{xx}\phi\|_\infty \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{\Delta x}{2} \|\partial_{xx}\phi\|_\infty \|q^{\mathrm{EO}}(u_{\Delta x})\|_{L^1([-M,M]\times[0,T])} \to 0 \qquad \text{as } \Delta x \to 0,
\end{aligned}$$

where M and T are such that $\phi(x,t) = 0$ for $t > T$ and $|x| > M$. Then

$$\lim_{\Delta x \to 0} J_3 = \int_0^{+\infty} \int_{\mathbb{R}} q^{\mathrm{EO}}(u(x,t)) E[u](x,t) \partial_x \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x. \tag{2.44}$$

We now prove that $\lim\limits_{\Delta x \to 0} J_2 = 0$ and this implies with (2.44) that

$$\lim_{\Delta x \to 0} I_2 = \int_0^{+\infty} \int_{\mathbb{R}} q^{\mathrm{EO}}(u(x,t)) E[u](x,t) \partial_x \phi(x,t)\,\mathrm{d}t\,\mathrm{d}x. \tag{2.45}$$

We start from the definition of J_2:

$$
\begin{aligned}
|J_2| &= \left| \sum_{n,j} q^{\mathrm{EO}}(u_j^n) \left(E[u]_{j+1/2}^n - E[u]_j^n \right) \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left(\int_{x_{j+1/2}}^{x_{j+1}} \phi(x,t)\,\mathrm{d}x - \int_{x_{j-1/2}}^{x_{j+1/2}} \phi(x,t)\,\mathrm{d}x \right) \mathrm{d}t \right| \\
&\leq \left| \sum_{n,j} \Delta x q^{\mathrm{EO}}(u_j^n) \frac{E[u]_{j+1/2}^n - E[u]_j^n}{\Delta x} \left(\Delta x \Delta t \frac{D_j \phi_j^n}{\Delta x} \right) \right| \\
&\leq \Delta x \| q^{\mathrm{EO}}(u) \partial_x E[u] \|_\infty \| \partial_x \phi \|_{L^1(\mathbb{R}\times\mathbb{R}_0^+)} \\
&\leq \Delta x \| u^0 \|_\infty \| q^{\mathrm{EO}}(u) f'(u) E[u] \|_\infty \| \partial_x \phi \|_{L^1(\mathbb{R}\times\mathbb{R}_0^+)},
\end{aligned}
$$

where the last inequality is obtained observing that

$$
\partial_x E[u] = -f(u) E[u],
$$

hence

$$
\partial_x E[u] = -f(u) E[u] \leq -u f'(u) E[u],
$$

that implies

$$
\lim_{\Delta x \to 0} J_2 = 0.
$$

We now analyze the integral I_3. As a first step it is easy to consider $\frac{E[u]_{j+1/2}^n - E[u]_{j-1/2}^n}{\Delta x} \approx -f(u_j^n) E[u]_j^n$, plus other terms of order less then Δx. Then we write the integral I_3 as

$$
I_3 = \Delta x \Delta t \sum_{j,n} \left(\eta'(u_j^{n+1}) f(u_j^n) - q^{\mathrm{EO}}(u_j^n) \right) f(u_j^n) E[u]_j^n \phi_j^n. \tag{2.46}
$$

It is easy that

$$
\lim_{\Delta x \to 0} \Delta x \Delta t \sum_{j,n} q^{\mathrm{EO}}(u_j^n) f(u_j^n) E[u]_j^n \phi_j^n = \int_0^{+\infty} \int_{\mathbb{R}} q^{\mathrm{EO}}(u) f(u) E[u] \phi\,\mathrm{d}x. \tag{2.47}
$$

We can consider the other term in (2.46):

$$
\underbrace{\Delta t \Delta x \sum_{n\geq 0} \sum_j \left(\eta'(u_j^{n+1}) - \eta'(u_j^n) \right) f^2(u_j^n) E[u]_j^n \phi_j^n}_{J_5} - \underbrace{\Delta t \Delta x \sum_{n\geq 0} \sum_j \eta'(u_j^n) f^2(u_j^n) E[u]_j^n \phi_j^n}_{J_6}. \tag{2.48}
$$

The term J_6 satisfies

$$
\lim_{\Delta x \to 0} J_6 = \int_0^{+\infty} \int_{\mathbb{R}} \eta'(u) f^2(u) E[u] \phi\,\mathrm{d}x. \tag{2.49}
$$

The term J_5 goes to zero as Δx goes to zero because η' is continuous, then for each $\epsilon < \Delta x$ there exists a δ such that $\left|\eta'(s) - \eta'(u_j^n)\right| \leq \epsilon < \Delta x$, for $\left|s - u_j^n\right| \leq \delta$. Moreover $u_{\Delta x}$ is a Cauchy sequence, then, from (2.3), we have that $\left|u_j^{n+1} - u_j^n\right| \leq \delta$, for $\Delta t \leq \Delta^0$.

Finally, combining (2.42), (2.43) and (2.45) with (2.47) and (2.49) we have that the limit u of $u_{\Delta x}$ satisfies

$$
\int_0^\infty \int_{\mathbb{R}} \left(\eta(u)\partial_t \phi + q^{\mathrm{EO}}(u) E[u] \partial_x \phi + (f(u)\eta'(u) - q^{\mathrm{EO}}(u)) f(u) E[u] \phi \right) \mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} \eta(u^0(x)) \phi(x,0)\,\mathrm{d}x \geq 0.
$$

for every positive test function $\phi \in C_0^\infty(\mathbb{R}\times\mathbb{R}_0^+)$, which means that u satisfies (1.7) and the theorem is proved. $\square$

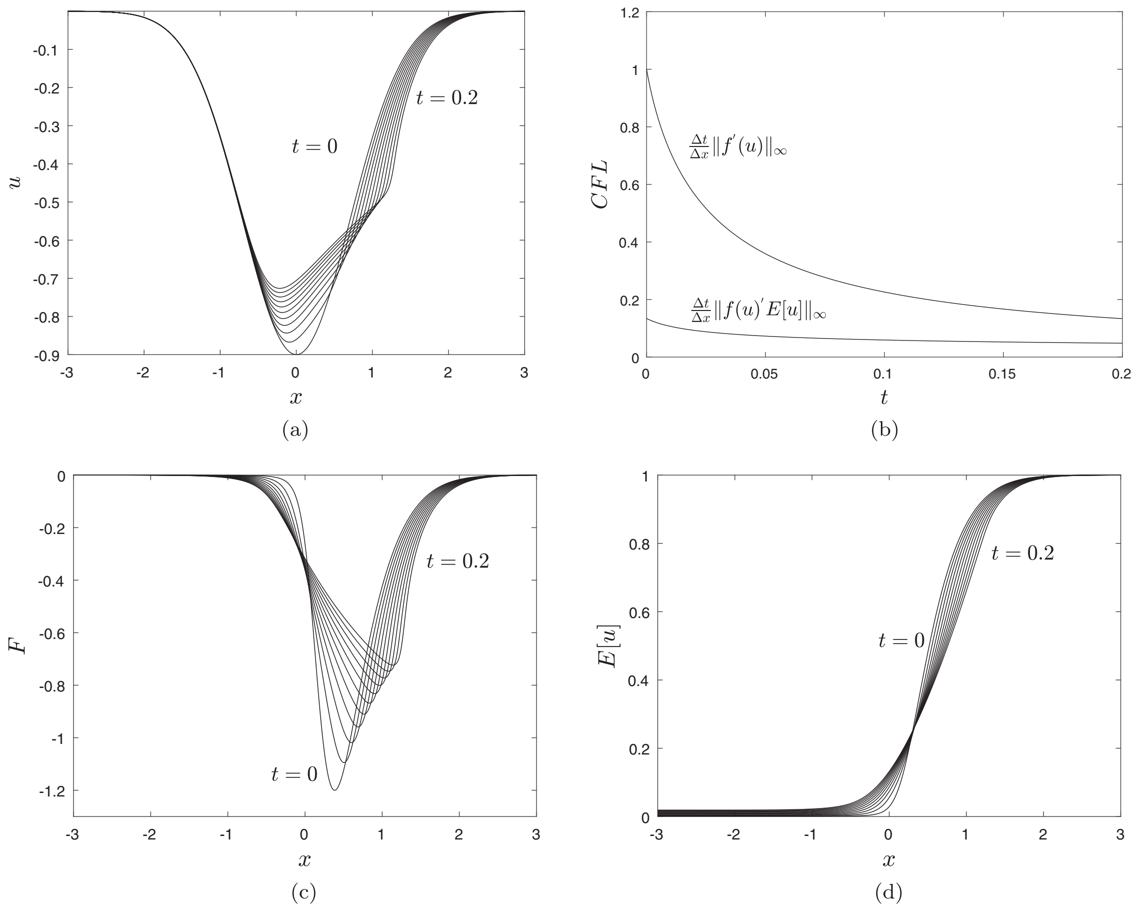

FIGURE 1. (a) Time evolution of the solution of the up-wind scheme (2.4) with initial condition (2.50) with $A = 1 - \kappa_0$ and $\kappa_0 = 0.1$, for $t = 0$ up to $t = 0.2$ (time intervals of 0.04). (b) The evolution in time of the CFL condition (2.14) and the less restrictive CFL condition (2.12), of the solution of the up-wind scheme (2.4) with initial condition (2.50) with $A = 1 - \kappa_0$ and $\kappa_0 = 0.1$. (c) Time evolution of the flux function F with initial condition (2.50) with $A = 1-\kappa_0$ and $\kappa_0 = 0.1$ for $t = 0$ up to $t = 0.2$ (time intervals of 0.04). (d) Time evolution of the $E[u]$ in (2.5) with initial condition (2.50) with $A = 1 - \kappa_0$ and $\kappa_0 = 0.1$ for $t = 0$ up to $t = 0.2$ (time intervals of 0.04).

2.2. Numerical experiments

In this subsection we apply the numerical scheme (2.4), with (2.7)–(2.9) and (2.5). In particular we consider the following initial datum

$$u_0(x) = -A\mathrm{e}^{-x^2}, \tag{2.50}$$

with $A = 1 - \kappa_0$ and we choose the value $\kappa_0 = 0.1$.

The computational domain considered is the finite interval $[-5, 5]$ and we have chosen $\Delta x = 0.01$ and $\Delta t = \kappa_0^2 \Delta x$ according to the CFL condition (2.14). In Figure 1a it is shown the time evolution of the solution of up-wind scheme (2.4) with initial condition (2.50) for $t = 0$ up to $t = 0.2$ with time intervals of 0.04.

In Figures 1c and 1d it is shown the time evolution, respectively, of the flux function F and of $E[u]$ with $\kappa_0 = 0.1$ for $t = 0$ up to $t = 0.2$, with time intervals of 0.04.

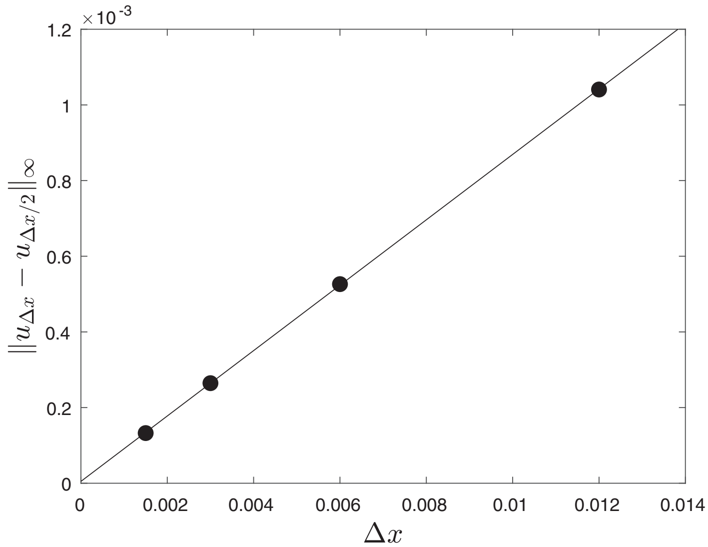

FIGURE 2. The local error with initial condition (2.50) with $A = 1 - \kappa_0$ and $\kappa_0 = 0.1$ at time $t = 0.2$ and CFL condition (2.14) equal 0.95. The convergence rate is of the first order in Δx.

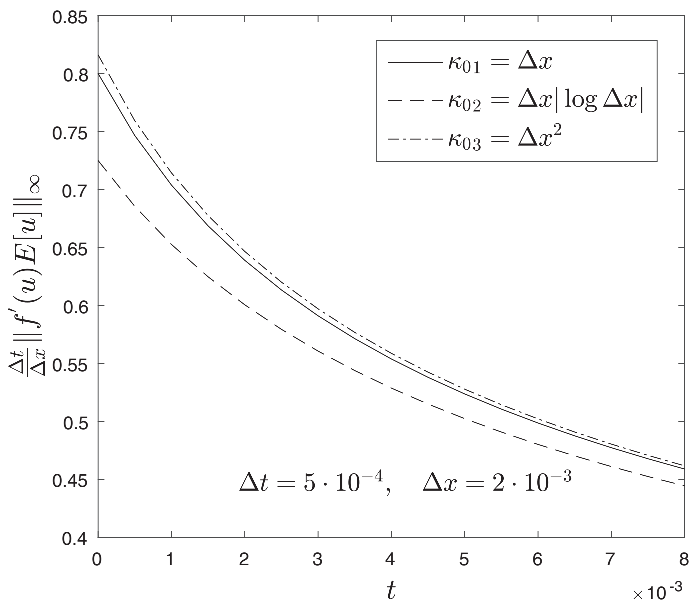

FIGURE 3. Time evolution of the CFL condition (2.12) for numerical solutions of the up-wind scheme (2.4) with initial conditions (2.50) with $\kappa_{01} = \Delta x$, $\kappa_{02} = \Delta x|\log(\Delta x)|$ and $\kappa_{03} = \Delta x^2$.

In Figure 1b it is shown the evolution in time of the CFL condition (2.14) and the less restrictive CFL condition (2.12).

The convergence rate of the numerical method is shown in Figure 2, where it is evident that the convergence rate is of the first order in Δx.

Remark. It is not straightforward to prove that the less restrictive CFL condition (2.12) remains valid in time for the numerical solution $u_{\Delta x}$ for fixed Δx and Δt. However, it is numerically evident in Figure 1b that the CFL value (2.12) decreases in time. We also show, only numerically, that one can consider initial data with minimum close to the critical value -1. In particular, we consider the initial datum defined in (2.50) with $A = 1 - \kappa_0$ and different values of κ_0: $\kappa_{01} = \Delta x$, $\kappa_{02} = \Delta x|\log(\Delta x)|$ and $\kappa_{03} = \Delta x^2$. The evolution in time of the less restrictive CFL condition (2.12) is shown in Figure 3. This is a possible numerical evidence of the convergence of the numerical scheme (2.4), also for initial datum $u^0 \in (-1, 0]$. These are arguments for possible future investigations.

3. Singularity formation

In this section we present the analysis of the singularity formation for the slow erosion model (1.1) using the singularity tracking method.

The singularity tracking is based on the link between the asymptotic properties of the Fourier spectrum and the width of analyticity of a real function. In particular, suppose that $u(z)$ is a real function having a complex singularity in $z^\star = x^\star + i\delta^\star$ and that

$$u(z) \approx (z - z^\star)^{\alpha + i\tau}, \tag{3.1}$$

using a steepest descent argument it is possible to give the asymptotic (in k) behavior of the spectrum u_k of $u(z)$ (see [8]):

$$u_k \approx Ck^{-(1+\alpha)}\mathrm{e}^{-\delta^\star k}\sin(kx^\star + \tau\log(k) + \phi). \tag{3.2}$$

Therefore an estimation of $\delta^\star$ and x^* gives the complex location of the singularity $z^\star$, while determining the rate of algebraic decay $1 + \alpha$ and the rate of oscillatory behavior τ allows to classify the singularity type. C and ϕ are constants related to the strength and phase of the singularity. In particular, if at a specific time t_s results that $\delta^\star(t_s) = 0$ then the width of analyticity of the solution is zero and u or its derivative, depending on the value α, can blow-up.

The evaluation of these parameters involves numerical fitting procedures, and in particular we have used for our analysis the procedure adopted for instance in [3]. Namely, we determine the parameters in specific band of wavenumber k by supposing that (3.2) (actually its logarithmic form) holds point-wise for each k, and equating six modes $u_{k-5}, u_{k-4}, u_{k-3}, u_{k-2}, u_{k-1}, u_k$ to the form in (3.2) we retrieve a nonlinear system for the parameters $C, \alpha, \delta^\star, x^\star, \tau, \phi$ whose solution is obtained numerically with a Newton's method. Although this procedure involves a k-dependent evaluation of the parameters, one actually needs to search for wavenumbers range where the parameters are in practice k-independent, and this generally happens in the first 1000 wavenumbers range. Hence at each time we assume as values for the various parameters in (3.2) those obtained in this range of wavenumbers.

3.1. Fourier spectral numerical scheme

The application of the singularity tracking method is based on the analytic continuation in the complex domain of numerical solutions, and this typically involves high-resolution numerical computation preserving spectral accuracy (see [5]). As the initial datum (2.50) is a rapidly decaying to zero function, we use a large enough numerical domain $[-M, M]$ so that we can impose periodic boundary conditions without spoiling numerical accuracy. This allows to write the Fourier expansion of the numerical solution of (1.1) along with a Fourier-Galerkin method which maintains spectral accuracy (see [7]). In particular, the dynamics of the generic k-th Fourier coefficient $\widehat{u}_k$ of the solution is given by

$$\frac{\mathrm{d}\widehat{u}_k}{\mathrm{d}t} = -i\frac{\pi k}{M}\widehat{f(u)E[u]}_k, \qquad \widehat{u}_k(0) = \widehat{u_0}_k, \tag{3.3}$$

where $\widehat{u_0}_k$ are the Fourier coefficients of the initial data.

TABLE 1. Convergence of the spectral numerical solution of (3.3) to the numerical solution obtained through the numerical scheme (2.4) at $T = 0.2$ for initial datum (2.50) and $A = 1-\kappa_0$ with $\kappa_0 = 0.1$.

Grid	$T = 0.2$	
(K)	L^2	L^∞
128	0.0125	0.024
256	0.0064	0.0129
512	0.0032	0.0067
1024	0.0017	0.0034
2048	0.00086	0.0018
4096	0.00047	0.00092
8192	0.00021	0.00053
16 384	0.00011	0.00028

Notes. The numerical solution of (2.4) is obtained with 16 385 grids points. In order to guarantee the CFL conditions (2.12), the time step is fixed to $\Delta t = 10^{-5}$ in all cases.

The system (3.3) is solved using an explicit Runge–Kutta method of the 4th order as temporal discretization. The product $f(u)E[u]$ is evaluated with a pseudo-spectral multiplication (see [7]), and as $E[u]$ is not periodic in the computational domain $[-M, M]$, we extend both $f(u)$ and $E[u]$ in the domain $[-M, 3M]$ with even symmetry with respect to the axis $x = M$, so that pseudo-spectral multiplication is performed in $[-M, 3M]$. The integral term in $E[u]$ is evaluated with a standard Fourier-spectral integration. The convergence of the spectral numerical solution is shown in Table 1.

3.2. Finite time singularity

We solve (1.1) with the initial condition provided in (2.50), and $0 < A < 1$.

To show how the singularity forms in finite time, we present the solution of (1.1) with initial amplitude $A = 0.95$. The time evolution of u is shown in Figure 4a from $t = 0$ up to $t = 0.25$ (time steps of 0.05). At $t_s = 0.25$ the solution results in the formation of a shock with a blow-up of the first derivative of the solution as shown in Figure 4b. The shock formation is revealed by the singularity tracking method, as at t_s the solution u loses analyticity being $\delta^\star \approx 0$. At t_s we also found that $\alpha \approx 0.36$ and $x^\star \approx 1.423$. The results of the fitting procedure of (3.2) applied to the spectrum of the solution at t_s is shown in Figure 4c in the range of wavenumber 50–500; the values $\alpha, x^\star, \delta^\star$ are almost k-independent in the range 250–450 of wavenumbers. We mention also that the imaginary part τ of the complex characterization is almost negligible (order of 10^{-4}). This implies that, in view of (3.1), the solution has a blow up of the first derivative in $x^\star \approx 1.423$.

The singularity has the typical $\approx 1/3$ character of the nonlinear transport motion, manifesting with a *shock* singularity as in the Burgers solution, see [43], and in particular with a blow-up of the first derivative. More precisely, prior the singularity time t_s, when the exponential decay of the spectrum dominates in (3.2), the characterization α of the singularity is close to the value 1/2, and rapidly approach to the value 1/3 when the complex singularity approaches the real axis. This phenomenon is essentially due to the interaction of this singularity with the symmetric one w.r.t. to the real axis, The variation of the character in time can be observed in Figure 4d where the (almost k-independent) value of α obtained from the fitting procedure is shown at various time up to t_s in the range 200–400 of wavenumbers.

The character $\approx$1/3 of this singularity means that standing profile $z(x) = \int_\infty^x (u + 1)\,\mathrm{d}x$ of the granular matter, experiences a blow-up of its curvature despite its profile remains regular. As the profile curvature affects the acceleration and deceleration of granular matter and therefore erosion and deposition, the physical meaning of this singularity formation is that strong deposition is followed by a strong erosion when singularity forms. This kind of singularity is different from those reported in [41], where under general assumptions on the erosion

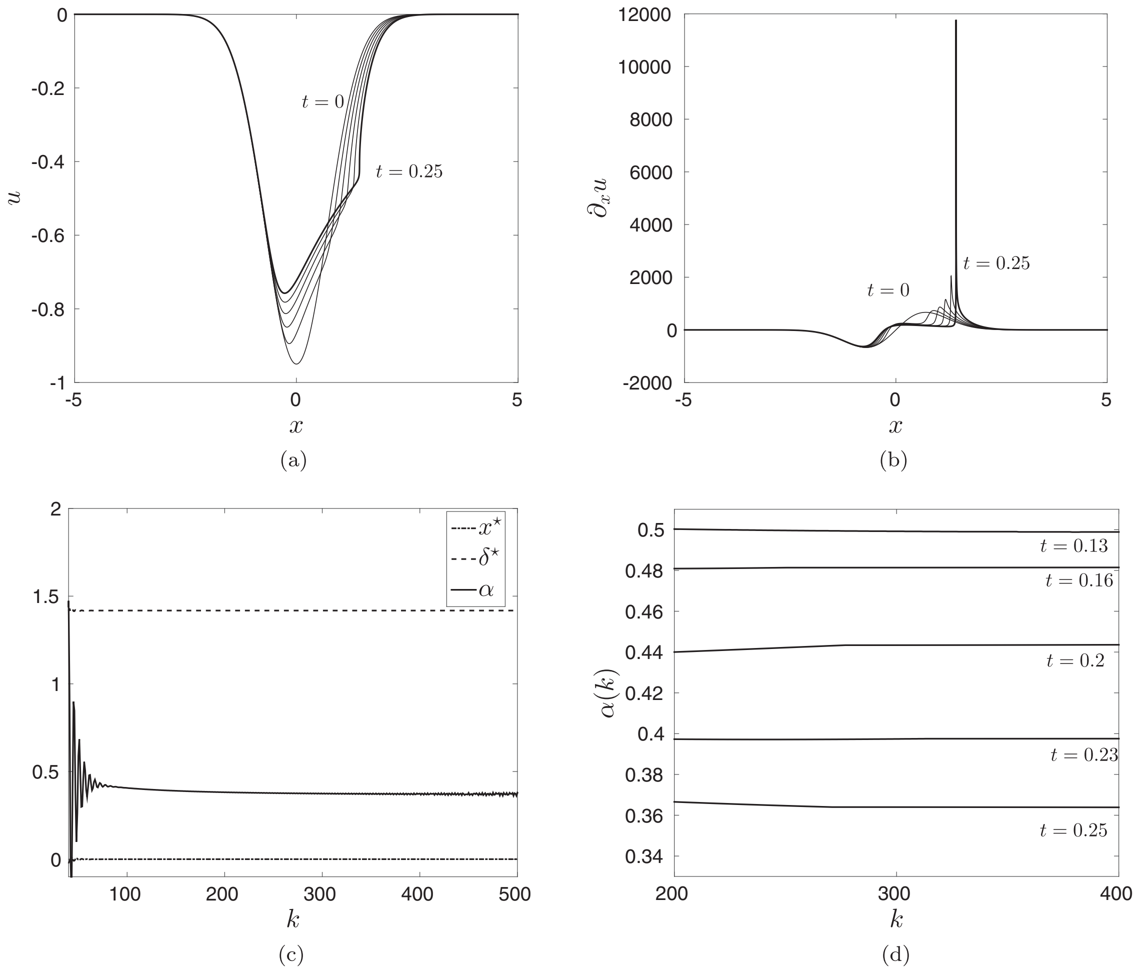

FIGURE 4. (a) Time evolution of the solution of (1.1) with initial condition (2.50) with $A = 0.95 \quad (\kappa_0 = 0.05)$ for $t = 0$ up to $t_s = 0.25$ (time steps of 0.05). At t_s a shock forms in the solution. (b) Time evolution of the derivative $\partial_x u$ for $t = 0$ up to $t_s = 0.25$ (time steps of 0.05). At t_s the derivative blows-up revealing the shock formation. (c) k-dependent values of $x^\star, \delta^\star, \alpha$ in (3.2) evaluated with the fitting procedure at t_s. In the range 200–400 of wavenumbers these values are almost k-independent. (d) k-dependent values of α in (3.2) evaluated with the fitting procedure at various time. In the range 200–400 of wavenumbers these values are almost k-independent.

function f, it was proven that three different kind of singularity, namely the kink, hyperkink and jump, can develop in finite time. Our results shows that these are not the only possible singularities of the model, and other kind of singularities can develop.

We stress here that the shock singularity we have found is related to the specific initial datum we have chosen. It will be interesting to prove analytically under which conditions the initial datum develops a shock singularity, following for instance the analysis performed in [35], or to prove the existence of other kind of singularity by changing the initial datum.

We conclude this section by investigating how the initial amplitude A in (2.50) affect the singularity formation. In Figure 4a it is shown the time t_s of singularity formation for various A, up to $A = 1 - \Delta x$ (where Δx is the

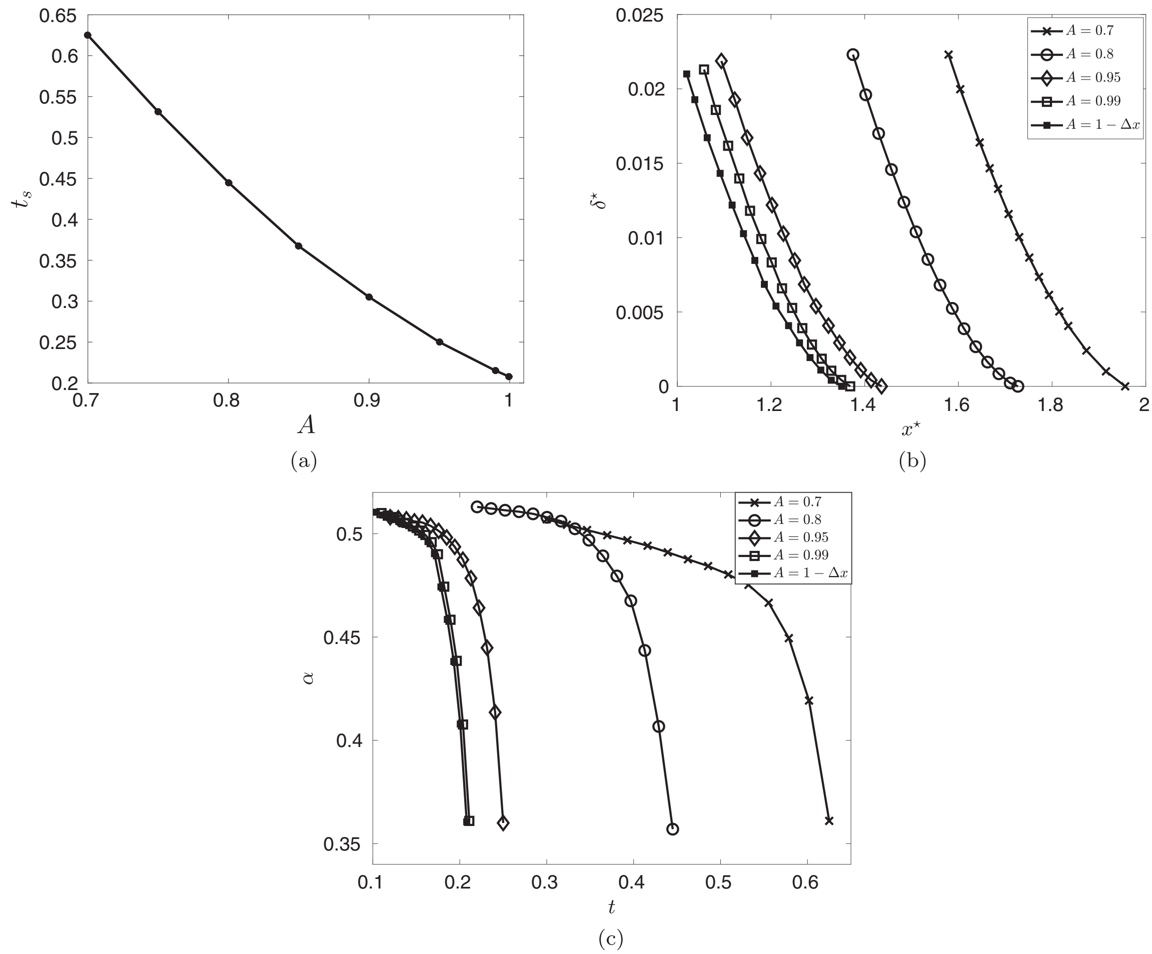

FIGURE 5. (a) Time t_s of singularity formation for various values of the initial amplitude A of the initial condition (2.50). (b) Singularity tracking in the complex plane of the singularity $z^\star$ of the solution from $t_s/2$ up t_s (15 total times) for different initial amplitude A. (c) Time evolution of the characters α in (3.2) from $t_s/2$ up t_s (15 total times) for different initial amplitude A.

spatial mesh size), and one can see that the singularity time is anticipated for increasing A. In Figure 5b the tracking of the singularity $x^\star + i\delta^\star$ in the complex plane is shown for various A from $t_s/2$ up to t_s. The position $x^\star$ where the singularity forms at t_s diminishes with increasing A. Finally in Figure 5c the time evolutions of the characterization α are shown for various A from $t_s/2$ up to t_s, and one can see that, in all cases, the singularity has a ≈1/3 characterization.

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