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A NEW METHODOLOGY TO CREATE VALID TIME-DEPENDENT CORRELATION MATRICES *VIA* ISOSPECTRAL FLOWS

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Abstract. In many areas of finance and of risk management it is interesting to know how to specify time-dependent correlation matrices. In this work we propose a new methodology to create valid time-dependent instantaneous correlation matrices, which we called correlation flows. In our methodology one needs only an initial correlation matrix to create these correlation flows based on isospectral flows. The tendency of the time-dependent matrices can be controlled by requirements. An application example is presented to illustrate our methodology.

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1. Introduction

In finance and risk management it is very interesting to know how to specify time-dependent instantaneous correlation matrices using real market data. We should naturally recover the real-world correlation matrices. However, the task is not as easy as it might seem, even only for specifying a constant correlation matrix. It is well-known that a valid correlation matrix is a real symmetric matrix with the following constraints (*i.e.*, properties):

(1) all diagonal elements are equal to one and absolute values of all non-diagonal elements are less than one,
(2) non-negative eigenvalues (positive semi-definite).

For example, the estimated correlation using stock data over a period may fail to be semi-definite due to some missing data. In particular, a risk manager wishes to assess the effect on a portfolio of adjusting the correlations between underlying assets, which can be different with those estimated from the historical data. The negative eigenvalues can thus be brought in. In the literature there are numerous methods solving this problem. The basic idea is to find the nearest valid correlation matrix, which should approximate the true correlation matrix "perfectly" as well. The technique proposed in [8] is to increase portions of the correlation matrix. However, as commented in [21], the drawback is that other portions of the matrix can be changed in an uncontrolled fashion. The shrinkage method proposed by Kupiec in [13] has the main drawback that "there is no way of determining

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to what extent the resulting matrix is optimal in any easily quantifiable sense", see [21]. Furthermore, the hyperspherical decomposition method and the unconstrained convex optimization approach are proposed in [21] and [19], respectively. Some Newton-based methods can be found in [4, 16, 18], and many others, see *e.g.*, [3,7,11,12,14,20,29]. Note that all the mentioned methods can address the constraints (1) and (2). Some of those methods can also address more constraints, *e.g.*, some correlations with specified indices (i, j) in the current matrix (estimated based using the historical data) must be kept in the target correlation matrix as well during the correlation stress testing.

In fact, either implied correlation in the context of a model or historical correlation from the market data show us that correlation matrices are time-dependent or might behave even randomly. In [22–24], the authors have shown the necessity of a time-dependent correlation model against a constant correlation. For example, the implied volatility for the Heston model extended by including a time-dependent correlation is much closer to the real market volatility. Therefore, the specification of valid time-dependent instantaneous correlation matrices is still an important application in finance. Note that, when considering stochastic correlation matrices [1, 25, 26, 28], only the expectations of stochastic correlation processes at each time point are actually needed for pricing and hedging, especially when the correlation between assets are traded directly with a correlation swap. Therefore, a suitable time-dependent model for the time-dependent matrix consist of those expectations is desired and will lead to a simpler application.

An isospectral flow is a dynamical system of matrices whose eigenvalues are an invariant of trajectories and has been used in many areas, *e.g.*, molecular dynamics, micromagnetics, linear algebra and so on. In this work we develop a methodology based on isospectral flows to create valid time-dependent instantaneous correlation matrices (correlation flows), *i.e.*, the correlation matrices at each time point satisfy constraints (1) and (2). In [22–24] an instantaneous time-dependent correlation function is proposed. If one uses the proposed correlation function to construct time-dependent correlation matrices, the constraint (1) will be fulfilled automatically for each time point. However, in this way, it cannot be guaranteed that the constraint (2) is enabled for all the time points.

With our new methodology we are able to create correlation flows starting from an initial correlation matrix. Furthermore, in our methodology not only do we control the tendency of the correlation flows but also let the flows provide the assigned correlation values at some time points. One possible application of our methodology is the specification of correlation flows in a time interval only with known correlation values at a few time points. For example, if one knows the correlation matrix between underlying assets in a portfolio at some time points in $[t, T]$, which can be known *e.g.*, by retrieving from the middle office's reporting system. Our methodology can tell us how the valid correlation flows move from t to T, *i.e.*, we can obtain valid correlation matrices for all time points. Furthermore, in the application, we show that the generated time-dependent correlation flow only based on the two time points can already represent correlation matrices consist of the expectations of stochastic correlation processes at each time point.

We show in Section 2 how to create valid time-dependent instantaneous covariance matrices which are called covariance flows based on the isospectral flows. In Section 3, we show the possible practical applications of our methodology. Finally, Section 4 concludes this work.

2. Covariance and correlation flows

Firstly, our purpose is to create covariance flows $(P(t))_{t\in\mathbb{R}_+}$, which must be positive semi-definite for all $t \geq 0$. Applying the singular value decomposition (SVD) one obtains

$$P(t) = \tilde{Q}(t)^\top S(t)\tilde{Q}(t), \tag{2.1}$$

where $\tilde{Q}(t)$ is a unitary matrix consisting of singular vectors and $S(t)$ is a diagonal matrix consisting of singular values of $P(t)$. In fact, without loss of generality, we can assume that $\tilde{Q}(t)$ is a rotation matrix whose determinant is always 1, *i.e.*, $|\tilde{Q}(t)| = 1$. Since if $|\tilde{Q}(t)| = -1$, one can rewrite (2.1) into

$$P(t) = \frac{\tilde{Q}(t)}{|\tilde{Q}(t)|}^{\top} S(t) \frac{\tilde{Q}(t)}{|\tilde{Q}(t)|}, \tag{2.2}$$

where $\frac{\tilde{Q}(t)}{|\tilde{Q}(t)|}$ is a rotation matrix and $(\tilde{Q}(t))_{t\in\mathbb{R}_+}$ is differentiable for $t \geq 0$.

We use $\mathcal{G}(n)$ to denote the Lie group of all non-singular matrices in $\mathbb{R}^{n\times n}$, for this we refer to *e.g.*, [6,10,30]. We then define an isospectral surface

$$\mathcal{M}(X_0) := \{Z^{-1}X_0Z | Z \in \mathcal{G}(n)\} \tag{2.3}$$

with the given $X_0 \in \mathbb{R}^{n\times n}$. Note that matrices in $\mathcal{M}(X_0)$ are similar to X_0 and thus have the same kind of geometric multiplicity as X_0. We consider the subgroup $\mathcal{O}(n)$ of all orthogonal matrices

$$\widetilde{\mathcal{M}}(P_0) := \{Q^{\top}P_0Q | Q \in \mathcal{O}(n)\}. \tag{2.4}$$

Theorem 2.1. *Suppose that $Q(t)$, with $Q(0) = I$ (Identity matrix), represents a differential curve on the manifold $\mathcal{O}(n)$, it holds that*

$$P(t) = Q(t)^{\top}P_0Q(t) \tag{2.5}$$

is the solution (upon differentiation) of the initial value problem

$$\begin{cases} \frac{\mathrm{d}P(t)}{\mathrm{d}t} = [P(t), k(t)], \quad t \geq 0 \\ P(0) = P_0, \end{cases} \tag{2.6}$$

where, for $t \geq 0$, $[P(t), k(t)] = P(t)k(t) - k(t)P(t)$ denotes the Lie bracket and $k(t)$ is defined by

$$k(t) := Q(t)^{\top}\frac{\mathrm{d}Q(t)}{\mathrm{d}t}, \quad t \geq 0. \tag{2.7}$$

The proof can be done by a simple calculation, see also [5]. Note that (2.5) defines a differentiable curve on the surface $\widetilde{\mathcal{M}}(P_0)$ with $P(0) = P_0$. From (2.1), we denote the initial value $P_0 = Q(0)^{\top}S(0)Q(0)$ by $Q_0^{\top}S_0Q_0$.

Remark 2.2. Conversely, if $k(t) \in \mathbb{R}^{n\times n}$ is known, one can find that the solution of (2.6) can be formulated in the form of (2.5), where $Q(t)$ satisfies

$$\begin{cases} \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = Q(t)k(t), \quad t \geq 0 \\ Q(0) = I. \end{cases} \tag{2.8}$$

Therefore, (2.8) is called the dual problem of (2.6), see [5].

Note that different isospectral curves can be defined by (2.6) with different values of $k(t)$, the asymptotic behavior of $P(t)$ on the surface $\mathcal{M}(P_0)$ is related to that of the corresponding $Q(t)$ on the manifold $\mathcal{O}(n)$.

2.1. In the commutative case

Clearly, in the case that the matrices $k(t)$ and $\int_0^t k(s)\,\mathrm{d}s$ commute, *i.e.*, $\left[k(t), \int_0^t k(s)\,\mathrm{d}s\right] = 0$, the unique solution of (2.8) is

$$Q(t) = \mathrm{e}^{\int_0^t k(s)\,\mathrm{d}s}. \tag{2.9}$$

Since $Q(t)$, $t \geq 0$ are orthogonal matrices, actually rotation matrices, $\int_0^t k(s)\,\mathrm{d}s$ must thus be skew-symmetric.

In the following we show how to control the flow $(P(t))_{t\in\mathbb{R}_+}$. Suppose that $Q(t)$ in (2.9) converges to a constant matrix for $t\to\infty$, we can thus *e.g.*, require

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} Q(t)^\top P_0 Q(t) := P^*. \tag{2.10}$$

Similarly, the flow $P(t)$ can be controlled as well by a given $Q(t^*)$, where t^* is a fixed time point. In this work, P^* is supposed to be the target covariance matrix whose singular values must be equal to S_0, since the covariance flows are modelled as isospectral flows. This is to say that the SVD of P^* must hold as

$$P^* = \tilde{Q}^{*\top} S^* \tilde{Q}^* = \tilde{Q}^{*\top} S_0 \tilde{Q}^*, \tag{2.11}$$

The problem becomes how to construct $Q(t)$. By combining (2.10) and (2.11) we need to choose $Q(t)$ such that

$$\lim_{t\to\infty} Q_0 Q(t) = \tilde{Q}^*. \tag{2.12}$$

Together with (2.9) we actually need to find $k(t)$ such that

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} \mathrm{e}^{\int_0^t k(s)\,\mathrm{d}s} := \lim_{t\to\infty} \mathrm{e}^{B(t)} := \mathrm{e}^{B} = Q_0^\top \tilde{Q}^*, \tag{2.13}$$

where $B(0) = \mathbf{0}$ due to $Q(0) = I$. Since Q_0 and $\tilde{Q}^*$ are both rotation matrices, there exists a skew-symmetric matrix B which fulfills $\mathrm{e}^B = Q_0^\top \tilde{Q}^*$.

Given Q_0 and $\tilde{Q}^*$, which are obtained from the initial and target matrices, respectively, we have $B = \log(Q_0^\top \tilde{Q}^*)$. For the covariance flows we then need to find suitable models for $B(t)$, namely $k(t)$. For example, one might find a function $f:\mathbb{R}_+\to\mathbb{R}$ with $f(0)=0$ such that

$$B(t) = Bf(t) \text{ and } k(t) = Bf(t)', \quad t\geq 0 \tag{2.14}$$

where $B(t)$ and $k(t)$ satisfy all the properties mentioned above. The corresponding correlation flows can be obtained by converting the covariance flow.

2.2. In the non-commutative case

Generally, the matrices $k(t)$ and $\int_0^t k(s)\,\mathrm{d}s$ do not commute, *i.e.*, $\left[k(t), \int_0^t k(s)\,\mathrm{d}s\right] \neq 0$. One can solve (2.8) numerically, *e.g.*, using the methods based on the Magnus series Expansion [15].

Theorem 2.3. *The solution of* (2.8) *can be given by* $Q(t) = \mathrm{e}^{\Omega(t)}$ *with* $\Omega(t)$ *defined by*

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = d\,\exp^{-1}_{-\Omega}(k(t)), \quad \Omega(0) = 0, \tag{2.15}$$

where $d\,\exp^{-1}_{-\Omega}(k(t)) = \sum_{j\geq 0} \frac{\mathcal{B}_j}{j!}\mathrm{ad}^j_{-\Omega}(k(t))$ *converges for* $t\in[0,T]$, *when* $\int_0^T \|k(t)\|_2\,\mathrm{d}t < \pi$ *with the matrix norm* $\|\cdot\|_2$, $\mathcal{B}_j$ *are the Bernoulli numbers and* $\mathrm{ad}_\Omega(k(t)) = [\Omega, k(t)]$.

The proof can be found in [9]. However, for our purposes, the analytical solution of (2.8) in a closed form is desired as well in the non-commutative case. For this we use Ascoli-type solution [2], see also [17].

Lemma 2.4. *When* $\left[k(t), \int_0^t k(s)\,\mathrm{d}s\right] \neq 0$, *there exists a constant matrix* C *such that* $\left[\tilde{B}(t), k(t)\right] = 0$, *with* $\tilde{B}(t) = C + \int_0^t k(s)\,\mathrm{d}s$. *The solution of* (2.8) *can thus be given by*

$$Q(t) = \mathrm{e}^{\tilde{B}(t)}\cdot K \tag{2.16}$$

for some constant matrix K.

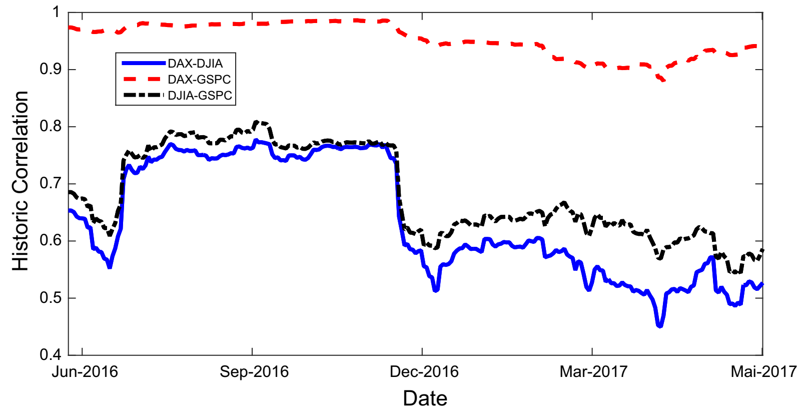

FIGURE 1. The 100-day historical correlations between GSPC, DAX and DJIA, source of data: www.yahoo.com.

For the proof, we refer to [2]. Clearly, we need to set K to be e^{-C} so that $Q(0) = I$. Finally, the solution of (2.6) is given by

$$P(t) = \left(\mathrm{e}^{\tilde{B}(t)}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{\tilde{B}(t)}\mathrm{e}^{-C}\right). \tag{2.17}$$

For example, we choose

$$\int_0^t k(s)\,\mathrm{d}s = \tanh\left(a_1 t + \tanh(a_2 t) + \cdots + \tanh(a_n t)\right) \tag{2.18}$$

and thus

$$k(t) = \left(a_1 + a_2 \operatorname{sech}^2(a_2 t) + \cdots + a_n \operatorname{sech}^2(a_n t)\right) \operatorname{sech}^2\left(a_1 t + \tanh(a_2 t) + \cdots + \tanh(a_n t)\right). \tag{2.19}$$

Similar to the way described in the previous section, we can control the covariance flows to the given target matrices as $t \to \infty$ by specifying the matrices $a_1, a_2, \cdots a_n$ and C as parameters. In practice, one needs to control the covariance and correlation flows to the given target matrices at n time points, this can be done analogously as well and will be considered in the following sections.

3. Practical applications

In this section we show an example of how to use our methodology. Suppose that a risk analyst retrieves from the middle office reporting system the correlation matrix of underlyings at $t = 0$ (initial correlation matrix). Moreover, the analyst is aware of how the relations between underlyings will develop. This means that the analyst can be aware of the correlation matrices at a few time points, *e.g.*, at $t = T/2, T$ (target correlation matrices). Then, the question is how to create the valid time-dependent instantaneous correlation matrices for the time interval $[0, T]$ using historical data.

3.1. Benchmark using historical data

We use the historical prices of S&P 500 index (GSPC), the German stock index (DAX) and the Dow Jones Industrial Average (DJIA) from Jan 04, 2016 to May 26, 2017. We compute moving correlations with the windows size of 100 days and obtain moving correlations from May 27, 2016 to May 26, 2017, which are plotted in Figure 1. Naturally, the moving correlations as sample paths are not appropriate to be the benchmark for

TABLE 1. Estimated stochastic correlation process parameters with the historical data in Figure 1.

	ρ_0	κ	μ	σ
DAX-DJIA	0.653	1.322	0.620	0.345
DAX-GSPC	0.974	0.829	1.47	0.443
DJIA-GSPC	0.686	1.615	0.743	0.329

our correlation flows. For a sensible benchmark we firstly employ the stochastic correlation process proposed in [27], see also [25, 26, 28], for modelling correlation

$$\frac{\mathrm{d}\rho_t}{1-\rho_t^2} = \left(\kappa(\mu - \operatorname{artanh}(\rho_t)) - \rho_t\sigma^2\right)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{3.1}$$

where $t \geq 0, \rho_0 \in (-1,1), \kappa, \sigma > 0$ and $\mu \in \mathbb{R}$. Note that (3.1) is derived from $\rho_t = \tanh(X_t)$, where X_t is the Ornstein–Uhlenbeck process $\mathrm{d}X_t = \kappa(\mu - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$. Therefore, the boundaries $+1$ and -1 are unattainable. We then apply the approach proposed in [27] to estimate the parameters in (3.1) using the historical data in Figure 1. The results are reported in Table 1, whereas we have taken the first historical correlation values as the initial values. We denote the density function of ρ_t by $f_\rho(\tilde{\rho}_t)$. The conditional probability density of $\rho_{s+\Delta t}$ is given by [27]

$$f_\rho(\tilde{\rho}_{s+\Delta t}|\tilde{\rho}_s,\kappa,\mu,\sigma) = \sqrt{\frac{a}{b}}\cdot\frac{1}{1-\tilde{\rho}_{s+\Delta t}^2}\cdot \mathrm{e}^{\frac{-\kappa(\operatorname{artanh}(\tilde{\rho}_{s+\Delta t}) - \operatorname{artanh}(\tilde{\rho}_s)\mathrm{e}^{-\kappa\Delta t} - \mu c)^2}{\sigma^2 b}}, \quad s < t \tag{3.2}$$

with

$$a = \frac{\kappa}{\pi\sigma^2}, \quad b = (1 - \mathrm{e}^{-2\kappa\Delta t}) \quad \text{and} \quad c = (1 - \mathrm{e}^{-\kappa\Delta t}). \tag{3.3}$$

Therefore, the mean values at each time points can be computed by

$$E[\rho_t] = \int_{-1}^{1} \tilde{\rho} f_\rho(\tilde{\rho}_{s+\Delta t}|\tilde{\rho}_s,\kappa,\mu,\sigma)\,\mathrm{d}\tilde{\rho}, \tag{3.4}$$

which are used as our benchmark. Note that (3.4) converges, since (3.1) is stationary and mean-reverting. Using the estimated parameter values in Table 1 we plot the computed expected correlations in Figure 2. Note that the 3-dimensional expected correlation matrices theoretically cannot be guaranteed to be positive semi-definite.

3.2. Preparation for the construction

For the initial matrix we use again the first historical correlation matrix

$$\mathcal{R}(0) := \mathcal{R}_0 = \begin{pmatrix} 1 & 0.6533 & 0.9738 \\ 0.6533 & 1 & 0.6855 \\ 0.9738 & 0.6855 & 1 \end{pmatrix}, \tag{3.5}$$

which is positive semi-definite. We let the expected correlation matrices at $t = 0.5$ and $t = T = 1$ to be the target matrices

$$\mathcal{R}(0.5) := \mathcal{R}^m = \begin{pmatrix} 1 & 0.5895 & 0.9516 \\ 0.5895 & 1 & 0.6419 \\ 0.9516 & 0.6419 & 1 \end{pmatrix} \tag{3.6}$$

and

$$\mathcal{R}(1) := \mathcal{R}^* = \begin{pmatrix} 1 & 0.5547 & 0.9280 \\ 0.5547 & 1 & 0.6217 \\ 0.9280 & 0.6217 & 1 \end{pmatrix}, \tag{3.7}$$

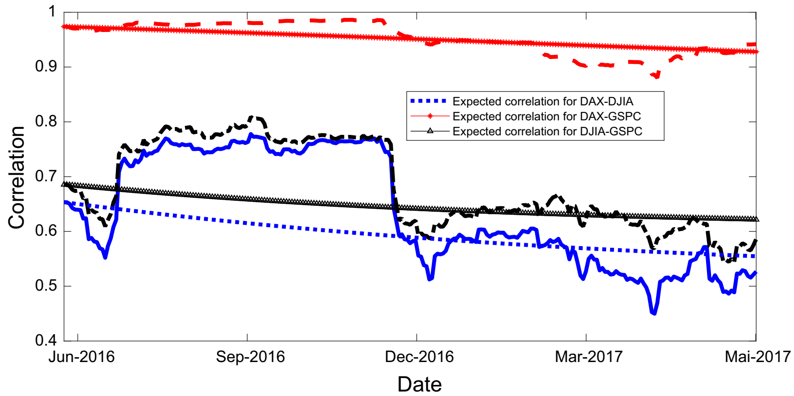

FIGURE 2. The expected correlations between GSPC, DAX and DJIA computed by (3.4) with parameter values in Table 1.

which are both positive semi-definite. In the following, based on those given matrices and the historical data in Figure 1 we create correlation flows applying the proposed methodology in the previous sections. And we will compare the correlation flows to the benchmark, namely expected correlation matrices in Figure 2.

We estimate the covariance matrix of the whole historical data

$$\overline{\Sigma} = \begin{pmatrix} 0.0502 \times 10^{-3} & 0.0501 \times 10^{-3} & 0.0574 \times 10^{-3} \\ 0.0501 \times 10^{-3} & 0.0536 \times 10^{-3} & 0.0625 \times 10^{-3} \\ 0.0574 \times 10^{-3} & 0.0625 \times 10^{-3} & 0.1534 \times 10^{-3} \end{pmatrix} \tag{3.8}$$

whose SVD reads

$$\overline{\Sigma} = \overline{Q}^{\top} \overline{S} \overline{Q}, \tag{3.9}$$

where

$$\overline{Q} = \begin{pmatrix} -0.4084 & -0.4338 & -0.8031 \\ -0.5825 & -0.5535 & 0.5952 \\ -0.7028 & 0.7109 & -0.0266 \end{pmatrix} \tag{3.10}$$

and

$$\overline{S} = \begin{pmatrix} 0.2165 \times 10^{-03} & 0 & 0 \\ 0 & 0.0391 \times 10^{-03} & 0 \\ 0 & 0 & 0.0017 \times 10^{-03} \end{pmatrix}. \tag{3.11}$$

Note that the matrix $\overline{Q}$ is an orthogonal matrix, $\overline{S}$ is a diagonal matrix where the elements are singular values and sorted in descending order. Since the covariance flows are modelled as the isospectral flows, *i.e.*, they must have the same singular values all the time. Thus, it may be meaningful to keep the singular values of all the covariance matrices equal to those in $\overline{S}$, which has been computed based on the whole historical data. This criteria allows us to compute the initial and the target covariance matrices according to the given correlation matrices, because we need to find such covariance matrices which have singular values $\overline{S}$ and can also be converted to the correlation matrices (3.5)–(3.7). More precisely, we need to find the corresponding standard deviations which are needed to compute the initial and target covariance matrices by converting the correlation matrices (3.5)–(3.7), whereas all the computed covariance matrices must have the same singular values as those in $\overline{S}$.

One can use *e.g.*, an optimization procedure. We denote the searched covariance matrices by P_0, P^m and P^* which can be determined by minimizing the corresponding errors

$$\epsilon_i = \|\overline{S} - \hat{S}\|_2^2 = \sum_{ij} \left(\overline{s}_{ij} - s_{0,ij}\right)^2, \quad i = 1, 2, 3, \quad \hat{S} \in \{S_0, S^m, S^*\} \tag{3.12}$$

by varying the parameters $\sigma_0 = (\sigma_{0,1}\ \sigma_{0,2}\ \sigma_{0,3})^\top, \sigma^m = (\sigma_1^m\ \sigma_2^m\ \sigma_3^m)^\top$ and $\sigma^* = (\sigma_1^*\ \sigma_2^*\ \sigma_3^*)^\top$, which are elements on diagonal of P_0, P^m and P^*, respectively. Similar to (3.11), S_0, S^m and S^* are the diagonal matrices whose main diagonal entries are singular values of P_0, P^m and P^*, respectively. $s_{0,ij}, s_{ij}^m$ and s_{ij}^* are used to denote the elements in such matrices. Note that all the matrices $\overline{S}, S_0, S^m$ and S^* are diagonal matrices, they are thus just simple optimization problems. Furthermore, since the singular values in $\overline{S}$ are very small, for the optimizations we need to scale these values by multiplying *e.g.*, by 1000. Note that the factor 1000 used to scale in this example works very well. However, depending on the historical data one may need another scale values for the optimizations. In our experiments, the covariance matrices are found as

$$P_0 = \begin{pmatrix} 0.0478\times10^{-3} & 0.0459\times10^{-3} & 0.0695\times10^{-3} \\ 0.0459\times10^{-3} & 0.1031\times10^{-3} & 0.0717\times10^{-3} \\ 0.0695\times10^{-3} & 0.0717\times10^{-3} & 0.1062\times10^{-3} \end{pmatrix}, \tag{3.13}$$

$$P^m = \begin{pmatrix} 0.0205\times10^{-3} & 0.0243\times10^{-3} & 0.0534\times10^{-3} \\ 0.0243\times10^{-3} & 0.0828\times10^{-3} & 0.0725\times10^{-3} \\ 0.0534\times10^{-3} & 0.0725\times10^{-3} & 0.1539\times10^{-3} \end{pmatrix}, \tag{3.14}$$

$$P^* = \begin{pmatrix} 0.0132\times10^{-3} & 0.0177\times10^{-3} & 0.0435\times10^{-3} \\ 0.0177\times10^{-3} & 0.0770\times10^{-3} & 0.0705\times10^{-3} \\ 0.0435\times10^{-3} & 0.0705\times10^{-3} & 0.1670\times10^{-3} \end{pmatrix} \tag{3.15}$$

which correspond to the correlations in (3.5)–(3.7) and have same singular values to those in $\overline{S}$. Furthermore, we also report the estimated standard deviations

$$\hat{\sigma}_0 = \begin{pmatrix} 0.0069 \\ 0.0102 \\ 0.0103 \end{pmatrix},\ \hat{\sigma}^m = \begin{pmatrix} 0.0045 \\ 0.0091 \\ 0.0124 \end{pmatrix} \text{ and } \hat{\sigma}^* = \begin{pmatrix} 0.0036 \\ 0.0088 \\ 0.0129 \end{pmatrix}. \tag{3.16}$$

3.3. Construction of covariance and correlation flows

We start with the SVDs of (3.13)–(3.15),

$$P_0 = Q_0^\top S_0 Q_0,\ P^m = Q^{m\top} S^m Q^m \text{ and } P^* = Q^{*\top} S^* Q^*, \tag{3.17}$$

note that $S_0 = S^m = S^* = \overline{S}$. Based on our model (2.6) and (2.8) in the non-commutative case, the covariance flows are given by $P(t) = Q^\top(t) P_0 Q(t) = \left(\mathrm{e}^{C+\int_0^t k(s)\,\mathrm{d}s}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\int_0^t k(s)\,\mathrm{d}s}\mathrm{e}^{-C}\right)$.

Similar to (2.12) and (2.13), we need to find suitable models for $k(t)$ such that the covariance flows $P(t)$, $t\in[0,1]$ pass through P^m at $t=0.5$ and P^* at $t=1$. For this we use (2.19), namely

$$k(t) = \left(a + b\,\mathrm{sech}^2(bt)\right)\mathrm{sech}^2\left(at + \tanh(bt)\right). \tag{3.18}$$

and thus

$$\tilde{B}(t) = C + \tanh\left(at + \tanh(bt)\right). \tag{3.19}$$

Note that (3.19) can be constructed using another bounded function as well, *i.e.*, the choice is not unique. Then, the unknown constant matrices a, b and C in (3.19) can be obtained by solving

$$\begin{cases} P(0.5) = Q^\top(0.5) P_0 Q(0.5) = P^m, \\ P(1) = Q^\top(1) P_0 Q(1) = P^*, \end{cases} \tag{3.20}$$

i.e.,

$$\begin{cases} \left(\mathrm{e}^{C+\tanh(0.5a+\tanh(0.5b))}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\tanh(0.5a+\tanh(0.5b))}\mathrm{e}^{-C}\right) = P^m, \\ \left(\mathrm{e}^{C+\tanh(a+\tanh(b))}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\tanh(a+\tanh(b))}\mathrm{e}^{-C}\right) = P^*, \end{cases} \tag{3.21}$$

where P_0, P^m and P^* have been already given in (3.13)–(3.15). Our numerical results read

$$a = \begin{pmatrix} 0.2290 & -0.1756 & -0.3114 \\ 0.3222 & -0.2390 & 0.0051 \\ 0.4569 & 0.4342 & 0.1211 \end{pmatrix}, \tag{3.22}$$

$$b = \begin{pmatrix} -0.2165 & 0.2053 & 1.2017 \\ -0.3834 & 0.2298 & 0.0052 \\ -0.9240 & -0.9742 & -0.1243 \end{pmatrix} \tag{3.23}$$

and

$$C = \begin{pmatrix} -0.6217 & 1.2092 & 0.4554 \\ 0.8537 & 0.0985 & -0.6926 \\ -0.1354 & 0.4918 & 0.4374 \end{pmatrix} \tag{3.24}$$

which can be used to compute $\tilde{B}(t)$, namely $Q(t)$. For example, we obtain

$$\tilde{B}(0.5) := \tilde{B}^m = \begin{pmatrix} -0.5476 & 0.8407 & 0.6844 \\ 0.6780 & 0.0931 & -0.5962 \\ -0.3262 & 0.2515 & 0.4102 \end{pmatrix} \tag{3.25}$$

$$\tilde{B}(1) := \tilde{B}^* = \begin{pmatrix} -0.5412 & 0.8443 & 0.7523 \\ 0.6697 & 0.0851 & -0.5930 \\ -0.3853 & 0.1737 & 0.4094 \end{pmatrix} \tag{3.26}$$

and

$$Q(0.5) := Q^m = \begin{pmatrix} 0.9668 & -0.0739 & 0.2447 \\ 0.0467 & 0.9922 & 0.1153 \\ -0.2513 & -0.1000 & 0.9627 \end{pmatrix} \tag{3.27}$$

$$Q(1) := Q^* = \begin{pmatrix} 0.9423 & -0.0994 & 0.3197 \\ 0.0514 & 0.9866 & 0.1550 \\ -0.3308 & -0.1297 & 0.9347 \end{pmatrix} \tag{3.28}$$

which are both rotations. Finally, the covariance flows can be generated by

$$P(t) = Q(t)^\top P_0 Q(t) = \left(\mathrm{e}^{C+\tanh(at+\tanh(bt))}\mathrm{e}^{-C}\right)^\top P_0 \left(\mathrm{e}^{C+\tanh(at+\tanh(bt))}\mathrm{e}^{-C}\right), \tag{3.29}$$

where the matrices a, b and C are given in (3.22), (3.23) and (3.25), respectively. By converting (3.29) the corresponding correlation flows can be immediately obtained, and they are valid, *i.e.*, all correlation matrices satisfy the constraints (1) and (2) at each time point. In Figure 3 we compare the correlation flows generated by (3.29) to the benchmark. We observe, although we only have used two covariance matrices P^m and P^* to construct $B(t)$, namely $Q(t)$ which can control the tendency of matrix flows, the generated correlation flows approach the benchmark quite well. However, we can actually imagine that there should be infinitely many correlation flows moving from the initial matrix to the target matrices. Therefore, this phenomenon shows that, given the initial and target matrices our methodology have generated meaningful correlation flows. To confirm our observations we do exactly the procedure as above but for another historical data, which are GSPC, DAX and the exchanges rates between US Dollar and Euro. Instead of 100-day moving correlations we consider in this experiment 50-day moving correlations. Furthermore, 1 year historical correlations are analyzed, we thus use the historical prices from March 16, 2016 to May 26, 2017. We plot all the results in Figure 4, from which the same conclusion as those of the previous example can be drawn.

Note that, in the experiments above we have only considered two target matrices. For a better approximation to the benchmark one can choose more target matrices. However, one needs to solve a larger equation system than (3.20).

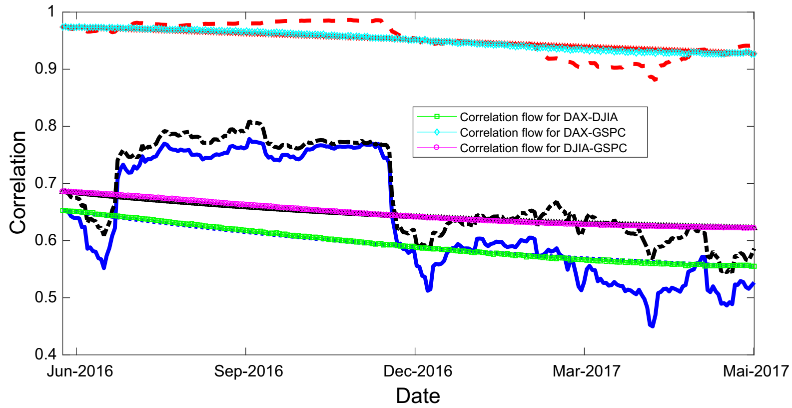

FIGURE 3. The generated correlation flows between GSPC, DAX and DJIA with (3.29).

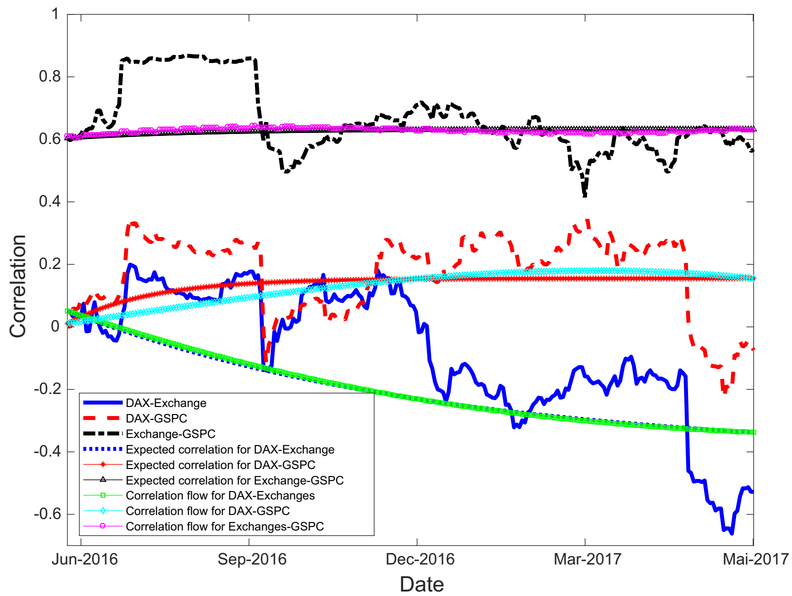

FIGURE 4. The 50-day historical correlations between GSPC, DAX and the exchange rates (US Dollar/Euro) from 27 May, 2016 to 26 May, 2017, the computed expected correlations and the generated correlation flows, source of data: www.yahoo.com.

4. Conclusion

We have proposed a new methodology to create valid time-dependent covariance and correlation matrices (covariance and correlation flows) based on isospectral flows. Given an initial correlation matrix, the tendency of the correlation flows can be controlled by the rotation matrices in the model. For example, one can require that correlation flows give correlation matrices which are equal to prespecified target matrices at some time points or as $t \to \infty$.

As an application, we model correlation as a stochastic process and calibrate the correlation process with the historical data. Then we compute the expected values of correlation processes at each time instant, namely obtain time-dependent expected correlation matrices based on the historical data, which are taken as benchmark. From the benchmark we choose the initial and two target matrices, from which we determine the rotation matrices which are used to generate the flows. By comparing the generated correlation flows to the benchmark, we find that the correlation flows are meaningful in the sense of expected correlation values at each time instant. Many more applications are expected to show the ability of our model, which is regarded as future work.

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