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PERIODIC SOLUTIONS OF A HYSTERESIS MODEL FOR BREATHING

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Abstract. We propose to model the lungs as a viscoelastic deformable porous medium with a hysteretic pressure–volume relationship described by the Preisach operator. Breathing is represented as an isothermal time-periodic process with gas exchange between the interior and exterior of the body. The main result consists in proving the existence of a periodic solution under an arbitrary periodic forcing in suitable function spaces.

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1. Introduction

As pointed out in [18], the first measurements which showed a hysteretic pressure–volume characteristic in mammalian lungs were obtained in [6] in 1913. There exist different hypotheses about the nature of the forces which originate the hysteresis behaviour by opposing the lung distension, but up to now, there is no theory which could explain both small and large volume excursions, as reported in [24]. A mechanical system combining linear viscoelasticity with the rate-independent Prandtl model of elastoplasticity was used by Hildebrandt in [12] to describe the breathing process of cats. A morphological study of pressure–volume hysteresis has been conducted dealing with rat lungs [11]. About twenty years later, the deformation of dogs' lungs, treated as an elastic body, was studied in [17]. In 2000, Lai-Fook and Hyatt proposed a model [16] describing the effect of ages on human lungs, according to the pressure–volume curve.

Several other contributions appeared also in the last years, (see for instance [19] for a survey), with particular emphasis on the role of hysteresis in mammalian lungs [9]. We point out that most of the research is concerned with mammalian lungs [4,5,13], in particular with human lungs [20] for medical purposes; some other studies are concerned with rats [8], some others with pigs [22]. In this paper we refer to the analysis carried out by Flynn in [10], where the Preisach operator is shown to be an appropriate model for the pressure–volume hysteresis relationship in mammalian lungs.

Keywords and phrases. Hysteresis operators – periodic solutions – mammalian lungs – convergence scheme.

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Studying the relationship between pressure and volume in mammalian lungs can be useful in order to ventilate patients with lung disease [3]. In particular, pressure–volume curves could help in setting the ventilator correctly and so to prevent the possibility of damaging the patient's lungs.

One of the strategies used in order to deal with the setting of the ventilator is to understand how the healthy lungs respond to mechanical ventilation. In [20] a ventilation model simulating hysteretic pressure–volume relationships in the lungs of a healthy subject is shown. Moreover, pressure–volume loops were used to develop a new technique for treating patients with Acute Lung Injury (ALI). This procedure gave the same levels of oxygen but with lower airway pressures and less over-inflation than other techniques [2]. Furthermore, [2] also shows that this kind of strategy reduced trauma and increased survival, with respect to methods which do not take into account pressure–volume curves.

Our work focuses on representing the breathing as an isothermal time-periodic process described by a PDE system with hysteresis. Our model system is derived from the principles of conservation of mass and momentum under the small deformation hypothesis, so that we deal with the linearized strain tensor. The resulting system has a similar structure as the model for unsaturated deformable porous media proposed in [1], but with different boundary conditions. Instead of prescribing boundary displacement as in [1], we prescribe here mechanical reaction between lungs and their surroundings. The difficulty here is that we look for periodic solutions under periodic forcing, so that we do not control the initial energy, and the only information available comes from the dissipation mechanisms. Unlike in [7], we cannot expect here any convexity in the hysteretic pressure–volume relation, and viscosity of the solid matrix material has to be assumed instead as the main source of dissipation.

The mathematical problem we solve here consists in proving that our PDE system with a degenerating pressure–mass content term under the time derivative admits a periodic solution for every periodic boundary forcing with a given regularity.

The structure of the paper is as follows. In Section 2 we present our model and state the main result of this work, which is Theorem 2.3. In Section 3 we summarize the main properties of the Preisach operator which are needed in the analysis of the problem. Theorem 2.3 is proved in Sections 4 and 5 in several steps. The degeneracy of the nonlinearities makes it difficult to estimate the time derivative of the pressure. We therefore regularize in Section 4 the nonlinearity by means of a cut-off parameter, and construct the corresponding periodic solutions by Galerkin approximations and a fixed point argument. Subsequently, in Section 5, we use a Moser-type technique adapted to the time-periodic case to obtain L^∞-estimates of the pressure, which allow us to remove the cut-off parameter and prove that the solution of the cut-off system satisfies the original degenerate system, too.

2. Statement of the problem

The modeling idea is to represent the lungs as a nonhomogeneous deformable viscoelastic porous body with gas exchange with the exterior and its motion is controlled by external forces acting on its boundary. Let u denote the displacement vector in the solid, let σ be the stress tensor, let q be the gas mass flux and let s be the gas mass content in the pores. Similarly as in [1], we assume that the system is governed by the momentum balance equation

$$\rho u_{tt} = \operatorname{div} \sigma, \tag{2.1}$$

where ρ is the solid mass density, and by the gas mass balance

$$s_t + \operatorname{div} q = 0, \tag{2.2}$$

where q is the mass flux. We consider constitutive relations

$$\sigma = \mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u - p\delta, \tag{2.3}$$

where $\mathbf{B}(x)$ (viscosity), $\mathbf{A}(x)$ (elasticity) are symmetric positive definite constant tensors of order 4 depending on $x \in \Omega$, the symbol ∇_s denotes the symmetric gradient, p is the air pressure, and δ is the Kronecker tensor.

The pressure–volume relation is assumed in the form

$$f(x,p) + G[p] = \frac{1}{\rho_a} s - \operatorname{div} u, \tag{2.4}$$

where $\rho_a > 0$ is the referential air mass density at standard pressure, $f : \Omega \times \mathbb{R} \to (0, \infty)$ is a given function increasing in p, and G is a Preisach operator defined in Section 3.

We can interpret (2.4) as follows. Under the small deformation hypothesis, the term $\operatorname{div} u$ represents the void volume difference with respect to the reference state, hence, at constant pressure, if $\operatorname{div} u$ increases, then s/ρ_a increases at the same rate. Similarly, at constant void volume, the mass content s is an increasing function (with different inflation and deflation curves) of the pressure. Finally, at constant gas mass content, the pressure increases if the void volume decreases. Notice also that the mass content s cannot be negative, so that, e. g., at constant volume, the pressure term $f(x,p) + G[p]$ must be bounded from below. This is why we have to admit the degeneracy of the partial derivative $f_p(x,p) := \frac{\partial f}{\partial p}(x,p)$ in Hypothesis 2.1 (i) below.

For the mass flux we assume the Darcy law

$$q = -\rho_a \mu(x) \nabla p \tag{2.5}$$

with a space-dependent permeability coefficient $\mu(x) > 0$. The full PDE system thus reads

$$\rho u_{tt} = \operatorname{div}(\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u) - \nabla p\,, \tag{2.6}$$

$$(f(x,p) + G[p])_t = -\operatorname{div} u_t + \operatorname{div} \mu(x)\nabla p\,, \tag{2.7}$$

for x in a bounded connected Lipschitzian domain $\Omega \subset \mathbb{R}^3$ and for $t \geq 0$. On the boundary $\partial\Omega$ we prescribe boundary conditions

$$-\sigma \cdot n\big|_{\partial\Omega} = \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g) + pn\,, \quad \frac{1}{\rho_a} q \cdot n\big|_{\partial\Omega} = \alpha(x)(p - \bar{p}) - u_t \cdot n\,, \tag{2.8}$$

where n is the unit outward normal vector, $g = g(x,t)$ is a given external force acting on the body Ω, $\bar{p} = \bar{p}(x,t)$ is the given outer air pressure, $\mathbf{C}(x), \mathbf{D}(x)$ are symmetric positive definite 3×3 matrices depending on $x \in \partial\Omega$, $\beta(x) \geq 0$ is the relative elasticity modulus of the boundary at the point $x \in \partial\Omega$, and $\alpha(x) \geq 0$ is the boundary permeability at the point $x \in \partial\Omega$.

The physical meaning of the first boundary condition in (2.8) is that on the part of the boundary where β is positive, the body Ω interacts with the exterior, which is viscoelastic with stiffness $\mathbf{C}(x)$, viscosity $\mathbf{D}(x)$, and active traction component g. There is no mechanical interaction with the exterior on the part of boundary where β vanishes. Similarly, the second boundary condition in (2.8) reflects the assumption that gas exchange proportional to the inner and outer pressure difference takes place on the part of the boundary where α is positive.

We denote

$$X_3 = W^{1,2}(\Omega; \mathbb{R}^3)\,, \quad X = W^{1,2}(\Omega)\,, \tag{2.9}$$

and reformulate Problems (2.6) and (2.7) in variational form for all test functions $\phi \in X_3$ and $\psi \in X$ as follows:

$$\begin{aligned} &\int_\Omega \big(\rho u_{tt}\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u) : \nabla_s \phi + \nabla p\ \phi\big)\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi\,\mathrm{d}s(x) = 0\,, \end{aligned} \tag{2.10}$$

$$\int_\Omega \big((f(x,p) + G[p])_t \psi + (\mu(x)\nabla p - u_t)\nabla\psi\big)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p - \bar{p})\psi\,\mathrm{d}s(x) = 0\,, \tag{2.11}$$

and the identities (2.10) and (2.11) are supposed to hold for a.e. $t > 0$.

We fix a period $T > 0$ and denote by L^q_T the L^q-space of T-periodic functions $v : \mathbb{R} \to \mathbb{R}$ for $q \ge 1$, by $W^{1,q}_T$ the associated Sobolev space, and by C_T the space of continuous real T-periodic functions on $\mathbb{R}$. Similarly, we deal with the spaces $L^q_T(X)$ and $L^q_T(X_3)$ of T-periodic L^q-functions $v : \mathbb{R} \to X$ and $v : \mathbb{R} \to X_3$, respectively, as well as with the spaces $L^q(\Omega; C_T)$ and $L^q(\Omega; W^{1,q}_T)$.

Hypothesis 2.1. *We assume that*

(i) $f : \Omega\times\mathbb{R} \to \mathbb{R}$ *is such that* $f(\cdot, p)$ *is bounded and measurable for all* $p \in \mathbb{R}$, $f(x, \cdot)$ *is continuously differentiable in* $\mathbb{R}$ *for a.e.* $x \in \Omega$, *and there exist constants* $0 < f^\flat < f^\sharp$ *and* $\omega \ge 0$ *with the property*

$$\frac{f^\flat}{1+p^2} \le f_p(x,p) := \frac{\partial f}{\partial p}(x,p) \le f^\sharp(1+p^2)^\omega \quad a.e. \ \ \forall p \in \mathbb{R}\,;$$

(ii) *The permeability coefficient* μ *belongs to* $L^\infty(\Omega)$ *and there exists a constant* $\mu_0 > 0$ *such that* $\mu(x) \ge \mu_0$ *a.e.;*

(iii) *The nonnegative functions* α *and* β *belong to* $L^\infty(\partial\Omega)$ *and we have* $\int_{\partial\Omega} \beta(x)\,\mathrm{d}s(x) > 0$, $\int_{\partial\Omega} \alpha(x)\,\mathrm{d}s(x) > 0$;

(iv) *The functions* g, g_t *belong to* $L^2_T(L^2(\partial\Omega;\mathbb{R}^3))$, $\bar p, \bar p_t$ *belong to* $L^2_T(L^2(\partial\Omega))$, $\bar p \in L^\infty(\partial\Omega\times(0,T))$;

(v) *The symmetric positive definite tensors* $\mathbf{A}, \mathbf{B} \in L^\infty(\Omega; \mathbb{R}^{3\times3}_s \times \mathbb{R}^{3\times3}_s)$, *where* $\mathbb{R}^{3\times3}_s$ *is the space of real symmetric tensors of order* 3×3, *and symmetric definite matrices* $\mathbf{C}, \mathbf{D} \in L^\infty(\partial\Omega;\mathbb{R}^3)$ *are given and there exists a constant* $\bar c$, *independent of* x, *such that*

$$\begin{cases} \mathbf{A}(x)\xi : \xi \ge \bar c(\xi:\xi), \quad \mathbf{B}(x)\xi : \xi \ge \bar c(\xi:\xi) \ \ a.e.\ \forall \xi \in \mathbb{R}^{3\times3}_s \\ \mathbf{C}(x)v\cdot v \ge \bar c|v|^2, \quad \mathbf{D}(x)v\cdot v \ge \bar c|v|^2 \qquad a.e.\ \forall v \in \mathbb{R}^3 \end{cases}.$$

Indeed, the lower bound for $f_p(x,p)$ does not exclude the relevant case that f is bounded from below, as mentioned in the comment on the modeling hypothesis (2.4).

The Preisach operator G is characterized by its *density function* γ, see Definition 3.3 below. We suppose that it has the following properties.

Hypothesis 2.2. *Let* $\gamma \in L^\infty(\Omega\times(0,\infty)\times\mathbb{R})$ *be a given function,* $\gamma(x,r,v) \ge 0$ *a.e., and there exists* $B > 0$ *such that*

$$\gamma(x,r,v) = 0 \ \ for \ \ r+|v| \ge B. \tag{2.12}$$

Moreover, assume that there exists a function $\gamma^* \in L^1(0,\infty)$ *such that for a.e.* $(x,r,v) \in \Omega\times(0,\infty)\times\mathbb{R}$ *we have* $0 \le \gamma(x,r,v) \le \gamma^*(r)$, *and we put*

$$C^*_\gamma = \int_0^\infty \gamma^*(r)\,\mathrm{d}r. \tag{2.13}$$

Theorem 2.3. *Let Hypotheses 2.1 and 2.2 hold. Then equations* (2.10) *and* (2.11) *have a solution* (u,p) *such that* $u, u_t, \nabla p \in L^2_T(L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(T,2T;L^2(\Omega;\mathbb{R}^3))$, $u_{tt} \in L^2_T(L^2(\Omega;\mathbb{R}^3))$, $p_t \in L^2_T(L^2(\Omega))$, $p \in L^\infty(\Omega\times(T,2T))$, $\nabla_s u, \nabla_s u_t \in L^2_T(L^2(\Omega;\mathbb{R}^{3\times3})) \cap L^\infty(T,2T;L^2(\Omega;\mathbb{R}^{3\times3}))$.

The main issue is the possibly degenerate character of the function f and of the operator G, which makes it difficult to estimate the time derivative of p. The proof will therefore be carried out in several steps in Sections 4 and 5 below. The reason why $[T,2T]$ is chosen to be the referential interval for T-periodic functions is related to the properties of the operator G and is explained below in Remark 3.7 (i).

3. Preisach operator

We recall here the basic theory of hysteresis operators, in particular of the Preisach operator, which is needed in the sequel. The detailed proofs of the statements of this Section can be found in [15].

We construct the Preisach operator G in terms of the play operator, which is defined as a solution to the following variational inequality

$$\begin{cases} |p(t)-\xi_r(t)| \le r & \forall t \ge 0, \\ (\xi_r(t))_t(p(t)-\xi_r(t)-z) \ge 0 & \text{a.e. } \forall z \in [-r,r], \\ p(0)-\xi_r(0) = \max\{-r, \min\{p(0), r\}\}. \end{cases} \tag{3.1}$$

It is well known (see [14,23]) that for any given input function $p \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ and each parameter $r > 0$, there exists a unique solution $\xi_r \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ of the variational inequality (3.1). The mapping $\mathfrak{p}_r : W^{1,1}_{\mathrm{loc}}(0,\infty) \to W^{1,1}_{\mathrm{loc}}(0,\infty)$ which with each $p \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ associates the solution $\xi_r \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ of (3.1) is called the *play operator* and the parameter $r > 0$ can be interpreted as a *memory parameter*.

As pointed out in [23], the play operator is *causal* in the sense that the values of $\mathfrak{p}_r[p](t)$ depend only on past values $p(\tau)$ of p for $\tau \in [0,t]$. It is therefore meaningful to consider restrictions of the play operator to inputs defined on a bounded time interval $[0,\hat{T}]$ for any $\hat{T} > 0$.

Proposition 3.1. *For each $r > 0$ and $\hat{T} > 0$, the mapping $\mathfrak{p}_r : W^{1,1}(0,\hat{T}) \to W^{1,1}(0,\hat{T})$ is Lipschitz continuous and admits a Lipschitz continuous extension to $\mathfrak{p}_r : C[0,\hat{T}] \to C[0,\hat{T}]$, in the sense that for every $p_1, p_2 \in C[0,\hat{T}]$ and for every $t \in [0,\hat{T}]$ we have*

$$|\mathfrak{p}_r[p_1](t) - \mathfrak{p}_r[p_2](t)| \le \|p_1 - p_2\|_{[0,t]} := \max_{\tau\in[0,t]} |p_1(\tau) - p_2(\tau)|. \tag{3.2}$$

Besides, for every $p \in W^{1,1}_{\mathrm{loc}}(0,\infty)$, the energy balance equation

$$\mathfrak{p}_r[p]_t p - \frac{1}{2}(\mathfrak{p}_r^2[p])_t = |r\mathfrak{p}_r[p]_t| \tag{3.3}$$

and the identity

$$\mathfrak{p}_r[p]_t p_t = (\mathfrak{p}_r[p]_t)^2 \ge 0 \tag{3.4}$$

hold almost everywhere in $(0,\infty)$.

Proposition 3.2. *Let $p \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ be periodic with period $T > 0$. Then $\mathfrak{p}_r[p](t+T) = \mathfrak{p}_r[p](t)$ for all $t \ge T$, that is, $\mathfrak{p}_r[p]$ is periodic for $t \ge T$ for all $r > 0$.*

In what follows, we consider input functions p which depend on $x \in \Omega$ and $t > 0$. For $r > 0$, $q \ge 1$, and $p \in L^q(\Omega; C_T)$ we interpret the play $\mathfrak{p}_r[p](x,t)$ as

$$\mathfrak{p}_r[p](x,t) = \mathfrak{p}_r[p(x,\cdot)](t). \tag{3.5}$$

Definition 3.3. Let γ be given and satisfy Hypothesis 2.2. For $p \in L^q(\Omega; C_T)$ we define the value of the Preisach operator G by the integral

$$G[p](x,t) = \int_0^\infty \int_0^{\xi_r(x,t)} \gamma(x,r,v)\,\mathrm{d}v\,\mathrm{d}r, \tag{3.6}$$

where $\xi_r(x,t) = \mathfrak{p}_r[p](x,t)$ is the output of the play operator applied to p.

From the properties of the play operator in Proposition 3.1, namely (3.2) and (3.4), we can prove the Lipschitz continuity and the local monotonicity of the Preisach operator.

Proposition 3.4. *Let γ be a function fulfilling Hypothesis 2.2. Then, the Preisach operator is pointwise Lipschitz continuous in $L^q(\Omega; C_T)$ for $q \ge 1$, i.e., for all $p_1, p_2 \in L^q(\Omega; C_T)$, a.e. $x \in \Omega$, and all $t > 0$, it holds*

$$|G[p_1](x,t) - G[p_2](x,t)| \le C^*_\gamma \max_{\tau\in[0,t]} |p_1(x,\tau) - p_2(x,\tau)|, \tag{3.7}$$

where C^*_γ is defined in Hypothesis 2.2.

Proposition 3.5. *Let the Preisach operator G from Definition 3.3 satisfy Hypothesis 2.2. Then G is locally monotone in the sense that for every $p \in L^q(\Omega; W_T^{1,1})$, $G[p]$ belongs to $L^q(\Omega; W_T^{1,1})$, and*

$$G[p]_t(x,t)p_t(x,t) \geq 0 \quad a.e. \tag{3.8}$$

Moreover, as a consequence of the energy identity (3.3), we have $\mathfrak{p}_r[p]_t(p - \mathfrak{p}_r[p]) \geq 0$ a.e. Hence, the inequality $\mathfrak{p}_r[p]_t(h(p) - h(\mathfrak{p}_r[p])) \geq 0$ holds almost everywhere for every nondecreasing function $h : \mathbb{R} \to \mathbb{R}$. We then easily conclude that the following Preisach energy inequality holds:

Proposition 3.6. *Let $h : \mathbb{R} \to \mathbb{R}$ be nondecreasing. Then for every $p \in L^q(\Omega; W_T^{1,1})$ and a.e. $(x,t) \in \Omega \times (0,\infty)$ it holds*

$$G[p]_t\, h(p) - V_h[p]_t \geq 0 \tag{3.9}$$

where $V_h[p](x,t) = \int_0^\infty \int_0^{\xi_r(x,t)} \gamma(x,r,v)h(v)\,\mathrm{d}v\,\mathrm{d}r$ is the h-energy potential.

Remark 3.7. Let G be a Preisach operator as in Definition 3.3.

(i) By Propositions 3.1 and 3.2, the output $G[p]$ is time periodic for $t \geq T$, and can be considered as an element of $L^q(\Omega; C_T)$ with referential interval $[T, 2T]$.
(ii) By Proposition 3.4, the Preisach operator G as a mapping $L^q(\Omega; C_T) \to L^q(\Omega; C_T)$ is Lipschitz continuous.
(iii) The "physical" energy inequality in (3.9) corresponds to the choice $h(p) = p$. For the Moser iterations in Section 5, we also choose $h(p)$ approximating suitable powers of p.

4. An approximation scheme

As a first step towards the proof of Theorem 2.3, we choose a cut-off parameter $R > 0$, define the function

$$f^{(R)}(x,p) = \begin{cases} f(x,-R) + f_p(x,-R)(p+R) & \text{for } \ p \leq -R, \\ f(x,p) & \text{for } \ p \in (-R,R), \\ f(x,R) + f_p(x,R)(p-R) & \text{for } \ p \geq R, \end{cases} \tag{4.1}$$

and replace the systems (2.10) and (2.11) with

$$\begin{aligned} &\int_\Omega \big(\rho u_{tt}\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u) : \nabla_s \phi + \nabla p\ \phi\big)\,\mathrm{d}x \\ &\qquad + \int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi\,\mathrm{d}s(x) = 0\,, \end{aligned} \tag{4.2}$$

$$\int_\Omega \big((f^{(R)}(x,p) + G[p])_t\psi + (\mu(x)\nabla p - u_t)\nabla\psi\big)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p - \bar{p})\psi\,\mathrm{d}s(x) = 0. \tag{4.3}$$

We now choose orthonormal bases $\{\phi_l\}_{l=0}^\infty$ in $L^2(\Omega)$ and $\{e_k\}_{k=-\infty}^\infty$ in L_T^2 as

$$\begin{aligned} -\Delta\phi_l = \lambda_l\phi_l \ \text{ in } \ \Omega,\ \nabla\phi_l \cdot n = 0 \ \text{ on } \ \partial\Omega, \\ e_k(t) \ = \begin{cases} \frac{2}{T}\sin\frac{2\pi k}{T}t & \text{for } k \geq 1, \\ \frac{1}{T} & \text{for } k = 0, \\ \frac{2}{T}\cos\frac{2\pi k}{T}t & \text{for } k \leq -1. \end{cases} \end{aligned} \tag{4.4}$$

Note that for every $k \in \mathbb{Z}$ we have

$$\dot{e}_k(t) = \frac{2\pi k}{T} e_{-k}(t), \tag{4.5}$$

where the dot denotes here and in the sequel the derivative with respect to t.

For a fixed $m \in \mathbb{N}$ we consider Galerkin approximations of (u,p) in the form

$$\begin{aligned} u_j^{(m)}(x,t) &= \sum_{k=-m}^{m} \sum_{l=0}^{m} u_{jkl}\phi_l(x)e_k(t), \\ p^{(m)}(x,t) &= \sum_{k=-m}^{m} \sum_{l=0}^{m} p_{kl}\phi_l(x)e_k(t), \end{aligned} \tag{4.6}$$

where the scalars u_{jkl}, p_{kl} are the new unknowns of the problem

$$\begin{aligned} &\int_T^{2T}\int_\Omega \left(\rho u_{tt}^{(m)}\eta_j\phi_l + \left(\mathbf{B}(x)\nabla_s u_t^{(m)} + \mathbf{A}(x)\nabla_s u^{(m)}\right) : \nabla_s(\eta_j\phi_l) + \nabla p^{(m)}\ \eta_j\phi_l\right) e_{-k}(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{2T}\int_{\partial\Omega} \beta(x)\left(\mathbf{C}(x)u^{(m)} + \mathbf{D}(x)u_t^{(m)} - g\right)\eta_j\phi_l e_{-k}(t)\,\mathrm{d}s(x)\,\mathrm{d}t = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\int_T^{2T}\int_\Omega \left(\left(f^{(R)}(x,p^{(m)}) + G[p^{(m)}]\right)_t \phi_l + \left(\mu(x)\nabla p^{(m)} - u_t^{(m)}\right)\nabla\phi_l\right) e_k(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{2T}\int_{\partial\Omega} \alpha(x)\left(p^{(m)} - \bar{p}\right)\phi_l e_k(t)\,\mathrm{d}s(x)\,\mathrm{d}t = 0, \end{aligned} \tag{4.8}$$

for $j = 1,2,3$, $l = 0,\dots,m$, and $k = -m,\dots,m$, where η_j are the vectors

$$\eta_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \eta_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ \eta_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

This is an algebraic system of $3 \times 2(2m+1) \times 2(m+1)$ equations for $3 \times 2(2m+1) \times 2(m+1)$ unknowns (u_{jkl}, p_{kl}), $j = 1,2,3$, $k = -m,\dots,m$, $l = 0,\dots m$.

We prove that it has a solution by a homotopy argument. We first derive some a priori estimates.

4.1. *A priori* estimates

In the sequel, we will systematically use the inequalities

$$\begin{aligned} \int_\Omega |v|^2\,\mathrm{d}x + \int_\Omega |\nabla v|^2\,\mathrm{d}x &\le C\left(\int_\Omega |\nabla v|^2\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)|v|^2\,\mathrm{d}s(x)\right), \\ \int_\Omega |w|^2\,\mathrm{d}x + \int_\Omega |\nabla w|^2\,\mathrm{d}x &\le C\left(\int_\Omega |\nabla_s w|^2\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|w|^2\,\mathrm{d}s(x)\right), \end{aligned} \tag{4.9}$$

for all $v \in X$ and $w \in X_3$, which follow from the Poincaré and Korn inequalities, see [21].

Multiplying (4.7) by $\frac{2\pi k}{T}u_{jkl}$, (4.8) by p_{kl} and summing up using (4.5) we obtain

$$\begin{aligned} &\int_T^{2T}\int_\Omega \left(\mathbf{B}(x)\nabla_s u_t^{(m)}{:}\nabla_s u_t^{(m)} + G\left[p^{(m)}\right]_t p^{(m)} + \mu(x)\left|\nabla p^{(m)}\right|^2\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{2T}\int_{\partial\Omega}\left(\beta(x)\left(\mathbf{D}(x)u_t^{(m)} - g\right)\cdot u_t^{(m)} + \alpha(x)\left(p^{(m)} - \bar{p}\right)p^{(m)}\right)\mathrm{d}s(x)\,\mathrm{d}t = 0. \end{aligned} \tag{4.10}$$

Note that the integral of the time derivative of a T-periodic function over the period vanishes.

From the Preisach energy inequality (3.9), the Hölder inequality, and Hypothesis 2.1 it follows that

$$\int_T^{2T}\int_\Omega \left(\left|\nabla_s u_t^{(m)}\right|^2 + \left|\nabla p^{(m)}\right|^2\right) \mathrm{d}x\,\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega} \left(\beta(x)\left|u_t^{(m)}\right|^2 + \alpha(x)\left|p^{(m)}\right|^2\right) \mathrm{d}s(x)\,\mathrm{d}t$$
$$\le C\int_T^{2T}\int_{\partial\Omega} \left(\beta(x)|g|^2 + \alpha(x)\bar{p}^2\right) \mathrm{d}s(x)\,\mathrm{d}t \le C, \tag{4.11}$$

where C denotes here and in the sequel any constant independent of m and R.

The next estimate is obtained when testing (4.7) by $u_{j(-k)l}$ and summing up, which yields

$$\int_T^{2T}\int_\Omega \left(\mathbf{A}(x)\nabla_s u^{(m)}:\nabla_s u^{(m)} + u^{(m)}\nabla p^{(m)} - \rho\left|u_t^{(m)}\right|^2\right) \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_T^{2T}\int_{\partial\Omega} \left(\beta(x)\left(\mathbf{C}(x)u^{(m)} - g\right)\cdot u^{(m)}\right) \mathrm{d}s(x)\,\mathrm{d}t = 0. \tag{4.12}$$

It thus follows from (4.9), (4.11), and (4.12) together with Hypothesis 2.1 that

$$\int_T^{2T}\int_\Omega \left|\nabla_s u^{(m)}\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega} \beta(x)\left|u^{(m)}\right|^2 \mathrm{d}s(x)\,\mathrm{d}t \le C. \tag{4.13}$$

We finally test (4.7) by $-(\frac{2\pi k}{T})^2 u_{j(-k)l}$, (4.8) by $-\frac{2\pi k}{T}p_{(-k)l}$ and sum up to obtain

$$\int_T^{2T}\int_\Omega \left(\left(f^{(R)}\left(x,p^{(m)}\right) + G\left[p^{(m)}\right]\right)_t p_t^{(m)} + \rho\left|u_{tt}^{(m)}\right|^2 - \mathbf{A}(x)\nabla_s u_t^{(m)} : \nabla_s u_t^{(m)} + 2\nabla p^{(m)}\ u_{tt}^{(m)}\right) \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_T^{2T}\int_{\partial\Omega} \left(\beta(x)\left(-\mathbf{C}(x)u_t^{(m)} + g_t\right)u_t^{(m)} + \alpha(x)\bar{p}_t\ p^{(m)}\right) \mathrm{d}s(x)\,\mathrm{d}t = 0. \tag{4.14}$$

In view of (4.11), Hypothesis 2.1, and the piecewise monotonicity of the operator G this yields

$$\int_T^{2T}\int_\Omega \left(\frac{f^\flat}{1+R^2}\left|p_t^{(m)}\right|^2 + \left|u_{tt}^{(m)}\right|^2\right) \mathrm{d}x\,\mathrm{d}t \le C. \tag{4.15}$$

We prove the existence of a solution to (4.7) and (4.8) by a topological argument. For $\kappa \in [0,1]$ we define a mapping $H_\kappa : \mathbb{R}^3 \times \mathbb{R}^{2(2m+1)} \times \mathbb{R}^{2(m+1)} \to \mathbb{R}^3 \times \mathbb{R}^{2(2m+1)} \times \mathbb{R}^{2(m+1)}$ by the left-hand side of (4.7) and (4.8), with $f^{(R)}(x,p)$ replaced by $(1-\kappa)\frac{f^\flat}{1+R^2}p + \kappa f^{(R)}(x,p)$, $G[p]$ replaced by $\kappa G[p]$, g replaced by κg, and $\bar{p}$ replaced by $\kappa\bar{p}$.

System (4.7) and (4.8) is of the form

$$H_1(Y) = 0, \tag{4.16}$$

where $Y = (u_{jkl}, p_{kl})$, $j = 1,2,3$, $k = -m,\dots,m$, $l = 0,\dots m$. The constant C in the estimates (4.11), (4.13), and (4.15) is independent of $\kappa \in [0,1]$. Hence, the equation $H_\kappa(Y) = 0$ has no solution outside a ball in $\mathbb{R}^3 \times \mathbb{R}^{2(2m+1)} \times \mathbb{R}^{2(m+1)}$ of a constant radius. The topological degree of H_κ with respect to this ball and the point 0 is therefore constant for all $\kappa \in [0,1]$. Since H_0 is linear, its topological degree is odd. We conclude that the degree of H_1 is nonzero, and therefore the Equation (4.16) admits a solution in $\mathbb{R}^3 \times \mathbb{R}^{2(2m+1)} \times \mathbb{R}^{2(m+1)}$, see ([25], Sect. 13.6).

4.2. Solution to the cut-off system (4.2) and (4.3)

The upper bound C in estimates (4.11), (4.13), and (4.15) is independent of m and R. More specifically, the sequences $\{u^{(m)}\}, \{p^{(m)}\}$ admit the bounds

$$\begin{aligned} &\left|u_t^{(m)}\right|_{L_T^2(L^2(\Omega;\mathbb{R}^3))} + \left|u_{tt}^{(m)}\right|_{L_T^2(L^2(\Omega;\mathbb{R}^3))} + \left|\nabla_s u^{(m)}\right|_{L_T^2(L^2(\Omega;\mathbb{R}^{3\times 3}))} + \left|\nabla_s u_t^{(m)}\right|_{L_T^2(L^2(\Omega;\mathbb{R}^{3\times 3}))} \\ &\quad + |u^{(m)}|_{L_T^2(L_\beta^2(\partial\Omega;\mathbb{R}^3))} + \left|u_t^{(m)}\right|_{L_T^2(L_\beta^2(\partial\Omega;\mathbb{R}^3))} \le C, \end{aligned} \tag{4.17}$$

$$\frac{1}{1+R}\left|p_t^{(m)}\right|_{L_T^2(L^2(\Omega))} + \left|\nabla p^{(m)}\right|_{L_T^2(L^2(\Omega;\mathbb{R}^3))} + \left|p^{(m)}\right|_{L_T^2(L_\alpha^2(\partial\Omega))} \le C, \tag{4.18}$$

where we let appear the seminorms

$$\begin{aligned} |v|_{L_T^2(L_\beta^2(\partial\Omega;\mathbb{R}^3))} &:= \left(\int_T^{2T}\int_{\partial\Omega} \beta(x)|v(x)|^2\,\mathrm{d}s(x)\,\mathrm{d}t\right)^{1/2}, \\ |w|_{L_T^2(L_\alpha^2(\partial\Omega))} &:= \left(\int_T^{2T}\int_{\partial\Omega} \alpha(x)w^2(x)\,\mathrm{d}s(x)\,\mathrm{d}t\right)^{1/2}. \end{aligned}$$

Still keeping R fixed for the moment, we pass to the weak limits as $m \to \infty$ in the highest order terms in (4.7) and (4.8). In the boundary terms and in the nonlinear terms we use the compact embeddings which imply that $\{p^{(m)}\}$ is a compact sequence in $L^2(\Omega; C[0,T])$ and $L_T^2(L^2(\partial\Omega))$ and that $\{u^{(m)}\}, \{u_t^{(m)}\}$ are compact in $L_T^2(L^2(\partial\Omega;\mathbb{R}^3))$. Note that the Lipschitz continuity (3.7) of the Preisach operator allows us to pass to the limit in $G[p^{(m)}]$.

Hence, we can find convergent subsequences in suitable function spaces and check that the limits of $(u^{(m)}, p^{(m)})$ satisfy the system

$$\begin{aligned} &\int_T^{2T}\int_\Omega \big(\rho u_{tt}\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u) : \nabla_s\phi + \nabla p\ \phi\big)\lambda(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{2T}\int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi\lambda(t)\,\mathrm{d}s(x)\,\mathrm{d}t = 0, \end{aligned} \tag{4.19}$$

$$\begin{aligned} &\int_T^{2T}\int_\Omega \big((f^{(R)}(x,p) + G[p])_t\psi + (\mu(x)\nabla p - u_t)\nabla\psi\big)\lambda(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{2T}\int_{\partial\Omega} \alpha(x)(p - \bar{p})\psi\lambda(t)\,\mathrm{d}s(x)\,\mathrm{d}t = 0, \end{aligned} \tag{4.20}$$

for every $\phi \in X_3$, $\psi \in X$, and $\lambda \in L_T^2$, which is equivalent to (4.2) and (4.3), and the estimates

$$\begin{aligned} &\int_T^{2T}\int_\Omega \left(|u_{tt}|^2 + |\nabla_s u_t|^2 + |\nabla_s u|^2 + |\nabla p|^2 + \frac{1}{1+R^2}|p_t|^2\right)\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_T^{2T}\int_{\partial\Omega} \big(\beta(x)(|u_t|^2 + |u|^2) + \alpha(x)|p|^2\big)\,\mathrm{d}s(x)\,\mathrm{d}t \le C \end{aligned} \tag{4.21}$$

hold as a consequence of (4.11), (4.13), and (4.15).

To get uniform estimates in time independent of R that we need in the sequel, we now test (4.2) by $\phi = u_t$. This is indeed an admissible choice, and we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \left(\frac{\rho}{2}|u_t|^2 + \frac{1}{2}\mathbf{A}(x)\nabla_s u : \nabla_s u\right)\mathrm{d}x + \int_{\partial\Omega} \frac{\beta(x)}{2}\mathbf{C}(x)u\cdot u\,\mathrm{d}s(x)\right) \\ + \int_\Omega \mathbf{B}(x)\nabla_s u_t : \nabla_s u_t\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)\mathbf{D}(x)u_t\cdot u_t\,\mathrm{d}s(x) \\ = \int_\Omega -\nabla p\ u_t\,\mathrm{d}x + \int_{\partial\Omega}\beta(x) g u_t\,\mathrm{d}s(x).\end{aligned} \tag{4.22}$$

By virtue of (4.21), (4.9), and Hypothesis 2.1, this is an equation of the form $\dot y(t) = z(t)$ with $y, z \in L^1_T$. The elementary identity

$$y(t) = \frac{1}{T}\int_{t-T}^{t}\big(y(s) + (T+s-t)z(s)\big)\,\mathrm{d}s$$

implies that the L^∞-norm of y is bounded by the L^1-norm of y and z. We thus have in particular

$$\sup_{t\in[T,2T]}\operatorname{ess}\left(\int_\Omega \big(|u_t|^2 + |\nabla_s u|^2\big)\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u|^2\,\mathrm{d}s(x)\right) \le C. \tag{4.23}$$

The next step is to test (4.2) by $\phi = u_{tt}$. This is, however, more delicate, since we have no evidence that $u_{tt}(\cdot,t)$ belongs to X_3 for a.e. t. We have to regularize the solution in time by using the operators $D_\varepsilon : L^2_T \to L^2_T$ defined by the formula

$$v^\varepsilon = D_\varepsilon[v] \iff \varepsilon\dot v^\varepsilon + v^\varepsilon = v \ \text{ a.e.}, \quad v^\varepsilon(T) = v^\varepsilon(2T). \tag{4.24}$$

Put $u^\varepsilon = D_\varepsilon[u]$, $p^\varepsilon = D_\varepsilon[p]$, $g^\varepsilon = D_\varepsilon[g]$. It follows from (4.2) that we have

$$\begin{aligned}\int_\Omega \big(\rho u^\varepsilon_{tt}\phi + (\mathbf{B}(x)\nabla_s u^\varepsilon_t + \mathbf{A}(x)\nabla_s u^\varepsilon) : \nabla_s\phi + \nabla p^\varepsilon\ \phi\big)\,\mathrm{d}x \\ + \int_{\partial\Omega}\beta(x)(\mathbf{C}(x)u^\varepsilon + \mathbf{D}(x)u^\varepsilon_t - g^\varepsilon)\phi\,\mathrm{d}s(x) = 0\end{aligned} \tag{4.25}$$

for a.e. t. We now test (4.25) by $\phi = u^\varepsilon_{tt}$ and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \Big(\frac{1}{2}\mathbf{B}(x)\nabla_s u^\varepsilon_t + \mathbf{A}(x)\nabla_s u^\varepsilon\Big):\nabla_s u^\varepsilon_t\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)\Big(\mathbf{C}(x)u^\varepsilon + \frac{1}{2}\mathbf{D}(x)u^\varepsilon_t - g^\varepsilon\Big)u^\varepsilon_t\,\mathrm{d}s(x)\right) \\ + \int_\Omega \rho|u^\varepsilon_{tt}|^2\,\mathrm{d}x = \int_\Omega\Big(\mathbf{A}(x)\nabla_s u^\varepsilon_t : \nabla_s u^\varepsilon_t - \nabla p^\varepsilon u^\varepsilon_{tt}\Big)\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)\big(\mathbf{C}(x)u^\varepsilon_t - g^\varepsilon_t\big)u^\varepsilon_t\,\mathrm{d}s(x).\end{aligned} \tag{4.26}$$

Using the estimates (4.21), (4.22), and Hypothesis 2.1, we conclude similarly as above that

$$\sup_{t\in[T,2T]}\operatorname{ess}\left(\int_\Omega |\nabla_s u^\varepsilon_t|^2\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u^\varepsilon_t|^2\,\mathrm{d}s(x)\right) \le C. \tag{4.27}$$

with a constant C independent of ε and R. Hence, letting ε tend to 0, we have

$$\sup_{t\in[T,2T]}\operatorname{ess}\left(\int_\Omega |\nabla_s u_t|^2\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u_t|^2\,\mathrm{d}s(x)\right) \le C. \tag{4.28}$$

5. Estimates of p

We first test (4.3) by $\psi = \hat{p}(x,t)\lambda(t)$, where $\hat{p}(x,t) = \frac{1}{\tau}(p(x,t) - p(x,t-\tau))$ for $\tau > 0$ and λ is an arbitrary nonnegative T-periodic Lipschitz continuous function. Passing to the limit as $\tau \to 0$ we infer from, e.g., Lusin's Theorem that

$$\begin{aligned} &-\int_T^{2T} \left(\int_\Omega \big(\frac{1}{2}\mu(x)|\nabla p|^2 - u_t \nabla p\big)\,\mathrm{d}x + \int_{\partial\Omega} \frac{1}{2}\alpha(x)(p-\bar{p})^2\,\mathrm{d}s(x)\right)\lambda'(t)\,\mathrm{d}t \\ &\leq -\int_T^{2T}\int_\Omega u_{tt}\nabla p \lambda(t)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{5.1}$$

where we have also used Proposition 3.5, the monotonicity of $f^{(R)}$, and the fact that $p_t \in L^2_T(L^2(\Omega))$. This is an inequality of the form

$$-\int_T^{2T} A(t)\lambda'(t)\,\mathrm{d}t \leq \int_T^{2T} B(t)\lambda(t)\,\mathrm{d}t \tag{5.2}$$

for two nonnegative functions

$$\begin{aligned} A(t) &= \int_\Omega \big(\frac{1}{2}\mu(x)|\nabla p|^2 - u_t\nabla p\big)\,\mathrm{d}x + \int_{\partial\Omega}\frac{1}{2}\alpha(x)(p-\bar{p})^2\,\mathrm{d}s(x) + \bar{C}\operatorname*{sup\,ess}_{s\in[T,2T]}\int_\Omega |u_t(x,s)|^2\,\mathrm{d}x, \\ B(t) &= \int_\Omega |u_{tt}||\nabla p|\,\mathrm{d}x, \end{aligned}$$

with a constant $\bar{C} > 0$. The constant term $\bar{C}\operatorname{sup\,ess}_{s\in[T,2T]}\int_\Omega |u_t(x,s)|^2\,\mathrm{d}x$ in the definition of $A(t)$ is added in order to keep $A(t)$ positive. Both A and B belong to L^1_T by (4.21), (4.28), and (4.9).

We now choose arbitrary points $2T < r < 3T$ and $T < s < r$ such that $r - s < T$. Then for each $\varepsilon < \frac{r-s}{2}$ we set in (5.2)

$$\lambda(t) = \frac{1}{\varepsilon}(t-s) \quad \text{for } t \in (s, s+\varepsilon), \qquad \lambda(t) = \frac{1}{\varepsilon}(r-t) \quad \text{for } t \in (r-\varepsilon, r),$$

choosing λ constant and continuous otherwise, T-periodically extended to the whole real line. This yields

$$\frac{1}{\varepsilon}\int_{r-\varepsilon}^r A(t)\,\mathrm{d}t \leq \frac{1}{\varepsilon}\int_s^{s+\varepsilon} A(t)\,\mathrm{d}t + \int_T^{2T} B(t)\,\mathrm{d}t.$$

Integrating the above inequality over s from $r - T$ to $r - 2\varepsilon$ we obtain

$$\frac{1}{\varepsilon}\int_{r-\varepsilon}^r A(t)\,\mathrm{d}t \leq \frac{1}{T-2\varepsilon}\int_T^{2T} A(t)\,\mathrm{d}t + \int_T^{2T} B(t)\,\mathrm{d}t. \tag{5.3}$$

The function $A(t)$ admits therefore the pointwise bound given by the right hand side of (5.3) at each of its Lebesgue points $t = r$. This enables us to conclude that

$$\operatorname*{sup\,ess}_{t\in[T,2T]}\left(\int_\Omega \mu(x)|\nabla p|^2\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)p^2\,\mathrm{d}s(x)\right) \leq C. \tag{5.4}$$

It remains to remove the cut-off parameter R. This will be done in the following way. We test the Equation (4.3) by suitable approximations of higher and higher powers of p and show by Moser iterations that we can obtain upper bounds for p in $L^\infty_T(L^q(\Omega))$ for an arbitrarily high exponent $q > 1$, and these bounds are independent of q and R. This will imply that p admits an L^∞-bound independent of R, and choosing R sufficiently large, we easily check that the solution of (4.2) and (4.3) that we have constructed in Section 4 is at the same time a solution of (2.10) and (2.11).

More specifically, we define Lipschitz continuous functions $a_{R,k}(p)$ and their antiderivatives $A_{R,k}(p)$ with indices $k \ge 1$ and with $R > 0$ from (4.1) by the formula

$$a_{R,k}(p) = p(1+\min\{p^2, R^2\})^k, \;\; A_{R,k}(p) = \int_0^p a_{R,k}(s)\,\mathrm{d}s. \tag{5.5}$$

We have

$$a'_{R,k}(p) = \begin{cases} (1+p^2)^{k-1}(1+(2k+1)p^2) & \text{for } |p| < R\,, \\ (1+R^2)^k & \text{for } |p| \ge R, \end{cases} \tag{5.6}$$

and

$$A_{R,k}(p) = \begin{cases} \frac{1}{2k+2}(1+p^2)^{k+1} & \text{for } |p| < R, \\ \frac{1}{2k+2}(1+R^2)^{k+1} + \frac{1}{2}(p^2-R^2)(1+R^2)^k \ge \frac{1}{2k+2}(1+p^2)(1+R^2)^k & \text{for } |p| \ge R. \end{cases} \tag{5.7}$$

With the notation from Hypotheses 2.1 and 2.2, put

$$\begin{aligned} F_k(x, p(x,t)) &= \frac{f^\flat}{2k} + \int_0^{p(x,t)} f_p^{(R)}(x,s) a_{R,k}(s)\,\mathrm{d}s, \\ V_k[p](x,t) &= \int_0^\infty \int_0^{\xi_r(x,t)} a_{R,k}(v)\gamma(x,r,v)\,\mathrm{d}v\,\mathrm{d}r. \end{aligned} \tag{5.8}$$

We now test (4.3) by $a_{R,k}(p)$ (note that by (4.21) this is an admissible choice), and obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(x,p) + V_k[p])\,\mathrm{d}x + \int_\Omega \mu(x)|\nabla p|^2 a'_{R,k}(p)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p-\bar{p})a_{R,k}(p)\,\mathrm{d}s(x) \\ &\quad \le \int_\Omega (u_t\cdot\nabla p)a'_{R,k}(p)\,\mathrm{d}x \le C\int_\Omega |u_t|^2 a'_{R,k}(p)\,\mathrm{d}x + \int_\Omega \frac{\mu(x)}{2}|\nabla p|^2 a'_{R,k}(p)\,\mathrm{d}x. \end{aligned} \tag{5.9}$$

We have used here the Preisach energy inequality (3.9) with $h(p) = a_{R,k}(p)$. The boundary term will be estimated from below as follows. We first notice that the function $a_{R,k}$ is increasing, hence

$$(p-\bar{p})a_{R,k}(p) \ge A_{R,k}(p) - A_{R,k}(\bar{p}).$$

Since the function $\bar{p}$ is bounded, we may take

$$R > \sup\operatorname{ess}\bar{p} \tag{5.10}$$

and finally get the estimate

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(x,p) + V_k[p])\,\mathrm{d}x + c\int_\Omega |\nabla p|^2 a'_{R,k}(p)\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)A_{R,k}(p)\,\mathrm{d}s(x) \\ &\quad \le C^{2k+2} + C\int_\Omega |u_t|^2 a'_{R,k}(p)\,\mathrm{d}x \end{aligned} \tag{5.11}$$

with constants c, C independent of R and k. The last term on the right-hand side of (5.11) will again be estimated by Hölder's inequality as follows:

$$\int_\Omega |u_t|^2 a'_k(p)\,\mathrm{d}x \le \left(\int_\Omega |u_t|^6\,\mathrm{d}x\right)^{1/3}\left(\int_\Omega a'_k(p)^{3/2}\,\mathrm{d}x\right)^{2/3}.$$

Using the embedding $W^{1,2}(\Omega) \to L^6(\Omega)$ and (4.9) we obtain from (4.28) the pointwise bound

$$\left(\int_\Omega |u_t|^6(x,t)\,\mathrm{d}x\right)^{1/3} \le C\left(\int_\Omega |\nabla_s u_t|^2(x,t)\,\mathrm{d}x + \int_{\partial\Omega}\beta(x)|u_t|^2(x,t)\,\mathrm{d}s(x)\right) \le C \tag{5.12}$$

with a constant $C > 0$ independent of t, and (5.11) can be reduced to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(x,p) + V_k[p])\,\mathrm{d}x + c\int_\Omega |\nabla p|^2 a'_{R,k}(p)\,\mathrm{d}x + \int_{\partial\Omega} \alpha(x) A_{R,k}(p)\,\mathrm{d}s(x)\\ \le C^{2k+2} + C\left(\int_\Omega a'_{R,k}(p)^{3/2}\,\mathrm{d}x\right)^{2/3}.\end{aligned} \tag{5.13}$$

Put

$$\hat{w}_k = \begin{cases}(1+p^2)^{\frac{k+1}{2}} & \text{for } |p| < R,\\ (1+p^2)^{\frac12}(1+R^2)^{\frac{k}{2}} & \text{for } |p| \ge R,\end{cases} \qquad w_k = \begin{cases}(1+p^2)^{\frac{k}{2}} & \text{for } |p| < R,\\ (1+R^2)^{\frac{k}{2}} & \text{for } |p| \ge R,\end{cases} \tag{5.14}$$

Then $a'_{R,k}(p) \le (2k+1)|w_k|^2$, and

$$|\hat{w}_k|^2 \le (2k+2)A_{R,k}(p)\,, \ \ |\nabla \hat{w}_k|^2 \le (k+1)|\nabla p|^2 a'_{R,k}(p)\,. \tag{5.15}$$

From (5.13) we thus obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(x,p) + V_k[p])\,\mathrm{d}x + \frac{c}{k+1}\left(\int_\Omega |\nabla \hat{w}_k|^2\,\mathrm{d}x + \int_{\partial\Omega}\alpha(x)|\hat{w}_k|^2\,\mathrm{d}s(x)\right)\\ \le C^{2k+2} + C(k+1)\left(\int_\Omega |w_k|^3\,\mathrm{d}x\right)^{2/3}.\end{aligned} \tag{5.16}$$

Note that c and C denote some non-specified constants, possibly different in different estimates, independent of k and R. For simplicity, put

$$\mathbf{F}_k(t) = \int_\Omega F_k(x, p(x,t))\,\mathrm{d}x\,, \quad \mathbf{V}_k(t) = \int_\Omega V_k[p](x,t)\,\mathrm{d}x, \tag{5.17}$$

and let $|\cdot|_q$ and $\|\cdot\|_{1,q}$ for $1 \le q \le \infty$ denote the norms in $L^q(\Omega)$ and in $W^{1,q}(\Omega)$, respectively. Recall the interpolation inequality

$$|w_k(t)|_3 \le |w_k(t)|_{3/2}^{1/3}\,|w_k(t)|_6^{2/3} \le |w_k(t)|_{3/2}^{1/3}\,|\hat{w}_k(t)|_6^{2/3}. \tag{5.18}$$

Using the embedding of $W^{1,2}(\Omega)$ into $L^6(\Omega)$, the Poincaré inequality (4.9), and the Young inequality we have (note that $k \ge 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_k(t) + \mathbf{V}_k(t)) + \frac{c}{k}\|\hat{w}_k(t)\|_{1,2}^2 \le C^{2k+2} + Ck^5|w_k(t)|_{3/2}^2 \tag{5.19}$$

with some constants c, C independent of k and R. By (5.8), (5.17), and Hypothesis 2.2, we have

$$0 \le \mathbf{V}_k(t) \le CB^{2k+2}, \tag{5.20}$$

and from Hypothesis 2.1 we easily get the lower bound independent of t

$$\mathbf{F}_k(t) \ge \frac{f^\flat}{2k}|w_k(t)|_2^2. \tag{5.21}$$

An upper bound for $F_k(x,p)$ for $|p| \le R$ is still straightforward, namely

$$F_k(x,p) \le \frac{f^\flat}{2k} + \int_0^p f^\sharp s(1+s^2)^{k+\omega}\,\mathrm{d}s \le \frac{f^\sharp}{2k}(1+p^2)^{k+\omega+1}. \tag{5.22}$$

The case $p > R$ or $p < -R$ is more delicate. For $p > R$ we have

$$F_k(x,p) \le \frac{f^\flat}{2k} + \int_0^R f^\sharp s(1+s^2)^{k+\omega}\,\mathrm{d}s + \int_R^p f^\sharp s(1+R^2)^{k+\omega}\,\mathrm{d}s$$
$$\le \frac{f^\sharp}{2k}(1+R^2)^{k+\omega}\left(1+R^2+k(p^2-R^2)\right) \le \frac{f^\sharp}{2}(1+p^2)(1+R^2)^{k+\omega}, \tag{5.23}$$

and the same formula holds for $p < -R$ by symmetry. By virtue of (5.14) and (5.21)–(5.23) we thus have

$$\frac{c}{k}|w_k(t)|_2^2 \le \mathbf{F}_k(t) \le C\int_\Omega (1+p^2)|w_k|^{2(k+\omega)/k}\,\mathrm{d}x \le C|\hat{w}_k(t)|_{q_k}^{q_k} \tag{5.24}$$

with constants c, C independent of R, k, and t, and with

$$q_k = \frac{2(k+\omega+1)}{k+1}\cdot \tag{5.25}$$

We now restrict ourselves to k sufficiently large such that $q_k \le 6$, that is,

$$k \ge K_\omega := \max\left\{1, \frac{\omega}{2} - 1\right\}. \tag{5.26}$$

It follows from (5.24)–(5.26) and from the embedding $W^{1,2}(\Omega)$ into $L^6(\Omega)$ that

$$\frac{1}{k}\|\hat{w}_k(t)\|_{1,2}^2 \ge \frac{c}{k}\mathbf{F}_k^{2/q_k}$$

with some constant $c > 0$. From (5.19) we thus obtain (note that $2/q_k < 1$), using also (5.20), that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_k(t) + \mathbf{V}_k(t)) + \frac{c}{k}(\mathbf{F}_k(t)+\mathbf{V}_k(t))^{2/q_k}$$
$$\le C^{2k+2} + Ck^5|w_k(t)|_{3/2}^2 + \frac{c}{k}(\mathbf{V}_k(t))^{2/q_k}$$
$$\le Ck^5 \max\{C^k, |w_k(t)|_{3/2}\}^2 =: M_{R,k}(t). \tag{5.27}$$

Put $M_{R,k}^* = \sup\operatorname{ess}_{t\in[T,2T]} M_{R,k}(t)$. Then (5.27) is of the form

$$\dot{Y}(t) + bY^\kappa(t) \le M \tag{5.28}$$

with a T-periodic function Y and with constants $b > 0$, $\kappa \in (0,1)$, and $M > 0$. Let H be the Heaviside function $H(s) = 1$ for $s > 0$, $H(s) = 0$ for $s \le 0$. Multiplying (5.28) by $H(Y^\kappa(t) - (M/b))$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Y(t) - (M/b)^{1/\kappa}\right)^+ + (bY^\kappa(t) - M)^+ \le 0.$$

Integrating from T to $2T$ and using the fact that Y is periodic, we thus have

$$\int_T^{2T} (bY^\kappa(t) - M)^+\,\mathrm{d}t \le 0,$$

which is only possible if $Y(t) \le (M/b)^{1/\kappa}$ a.e. Hence, (5.27) yields that

$$\mathbf{F}_k(t) + \mathbf{V}_k(t) \le C(kM_{R,k}^*)^{q_k/2}. \tag{5.29}$$

Referring again to (5.20) and (5.24) (note that $q_k \le 6$ by virtue of (5.25) and (5.26)), we thus conclude that

$$\sup_{t\in[T,2T]}\operatorname{ess} |w_k(t)|_2^2 \le Ck^{19}\max\{C^k, \sup_{t\in[T,2T]}\operatorname{ess} |w_k(t)|_{3/2}\}^{q_k}. \tag{5.30}$$

Putting for $q \geq 1$
$$\|w_k\|_q^* = \sup_{t\in[T,2T]} \operatorname{ess} |w_k(t)|_q,$$
we can reformulate (5.30) as
$$\left(\|w_k\|_2^*\right)^2 \leq Ck^{19}\left(\max\{C^k, \|w_k\|_{3/2}^*\}\right)^{q_k} \tag{5.31}$$
with a constant C independent of k and R.

The Moser iteration technique will be applied to the new variable
$$w := 1 + \min\{p^2, R^2\}. \tag{5.32}$$
By (5.14), inequality (5.31) can be rewritten as
$$\left(\|w\|_k^*\right)^k \leq Qk^{19} \max\left\{L^k, \left(\|w\|_{3k/4}^*\right)^{kq_k/2}\right\} \tag{5.33}$$
with some constants $Q \geq 1$, $L \geq 1$ that we keep fixed from now on. For $k \geq K_\omega$ (*cf.* (5.26)) we thus have
$$\|w\|_k^* \leq (Qk^{19})^{1/k} \max\{L, \|w\|_{3k/4}^*\}^{q_k/2}. \tag{5.34}$$
We are ready now to start the Moser iterations. Let $\{k_j\}$ be the sequence
$$k_j = K_\omega \left(\frac{4}{3}\right)^j, \quad j = 0, 1, \ldots \tag{5.35}$$
with K_ω from (5.26), and let
$$P_j = \max\{L, \|w\|_{k_j}^*\}, \quad \delta_j = \frac{\omega}{k_j + 1}.$$
Then, by virtue of (5.25), the inequality (5.34) is of the form
$$P_j \leq (Qk_j^{19})^{1/k_j} P_{j-1}^{1+\delta_j} \quad \text{for } j = 1, 2, \ldots \tag{5.36}$$
The logarithm applied to (5.36) yields
$$\begin{aligned} \log P_j - (1+\delta_j)\log P_{j-1} &\leq \frac{1}{k_j}\left(\log(QK_\omega^{19}) + 19 \log\left(\frac{4}{3}\right)^j\right) \\ &= \frac{1}{K_\omega}\left(\frac{3}{4}\right)^j (\log(QK_\omega^{19}) + 19j\log(4/3)), \end{aligned} \tag{5.37}$$
or, equivalently, putting $\delta_0 = 0$,
$$\frac{\log P_j}{\prod_{i=0}^j (1+\delta_i)} - \frac{\log P_{j-1}}{\prod_{i=0}^{j-1}(1+\delta_i)} \leq \frac{1}{K_\omega}\left(\frac{3}{4}\right)^j \frac{\log(QK_\omega^{19}) + 19j\log(4/3)}{\prod_{i=0}^j (1+\delta_i)}. \tag{5.38}$$
The sequence on the right-hand side of (5.38) forms a convergent series. Hence, there exists a constant $C^* > 0$ such that
$$\sup_{j\geq 1} \frac{\log P_j}{\prod_{i=0}^j (1+\delta_i)} \leq C^* + \log P_0. \tag{5.39}$$
We have
$$\prod_{i=0}^\infty (1+\delta_i) = \exp\left(\sum_{i=1}^\infty \log(1+\delta_i)\right) \leq \exp\left(\sum_{i=1}^\infty \delta_i\right) \leq C,$$

hence

$$\sup_{j\geq 1}\log P_j \leq C^{**}(1+\log P_0). \tag{5.40}$$

It remains to prove that P_0 is bounded by a constant C_0 independent of R, that is,

$$\sup_{t\in[T,2T]}\operatorname{ess}\int_\Omega w^{K_\omega}(x,t)\,\mathrm{d}x \leq C_0. \tag{5.41}$$

Using again the embedding of $W^{1,2}(\Omega)$ into $L^6(\Omega)$ and Hölder's inequality, we obtain from (5.19) for every $k\geq 1$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_k(t)+\mathbf{V}_k(t)) + \frac{c}{k}\left(\int_\Omega w^{3(k+1)}(x,t)\,\mathrm{d}x\right)^{1/3} &\leq C^{2k+2} + Ck^5\left(\int_\Omega w^{3k/4}(x,t)\,\mathrm{d}x\right)^{4/3}\\ &\leq C^{2k+2} + Ck^5\left(\int_\Omega w^{3(k+1)}(x,t)\,\mathrm{d}x\right)^{k/(3(k+1))}.\end{aligned} \tag{5.42}$$

From Young's inequality with exponents $k+1$ and $1+(1/k)$ we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_k(t)+\mathbf{V}_k(t)) + \frac{c}{k}\left(\int_\Omega w^{3(k+1)}(x,t)\,\mathrm{d}x\right)^{1/3} \leq (Ck)^{6k+5}. \tag{5.43}$$

Integrating from T to $2T$, using the fact that $\mathbf{F}_k$ and $\mathbf{V}_k$ are periodic and that $L^3(\Omega)$ is embedded in $L^1(\Omega)$, we conclude that for every $k\geq 1$ we have

$$\int_T^{2T}\!\!\int_\Omega w^{k+1}\,\mathrm{d}x\,\mathrm{d}t \leq C\int_T^{2T}\left(\int_\Omega w^{3(k+1)}(x,t)\,\mathrm{d}x\right)^{1/3}\mathrm{d}t \leq (Ck)^{6k+6}. \tag{5.44}$$

We now choose $t\in(2T,3T)$ and $s\in(T,2T)$, and integrate (5.43) from s to t to get (note that $\mathbf{V}_k$ is bounded by (5.20))

$$\mathbf{F}_k(t)-\mathbf{F}_k(s) \leq (Ck)^{6k+6}. \tag{5.45}$$

By (5.24) and Hölder's inequality we have

$$\mathbf{F}_k(s) \leq C\int_\Omega (1+p^2)w^{k+\omega}(x,s)\,\mathrm{d}x \leq C|1+p^2(s)|_3\left(\int_\Omega w^{3(k+\omega)/2}(x,s)\,\mathrm{d}x\right)^{2/3}. \tag{5.46}$$

We now integrate (5.46) over $s\in(T,2T)$. By virtue of (5.4), (4.9) and the embedding of $W^{1,2}(\Omega)$ into $L^6(\Omega)$ we have

$$\int_T^{2T}\mathbf{F}_k(s)\,\mathrm{d}s \leq C\int_T^{2T}\left(\int_\Omega w^{3(k+\omega)/2}(x,s)\,\mathrm{d}x\right)^{2/3}\mathrm{d}s. \tag{5.47}$$

Using (5.24), (5.14), (5.45), and (5.47) we thus conclude that

$$\sup_{t\in[2T,3T]}\operatorname{ess}\int_\Omega w^k(x,t)\,\mathrm{d}x \leq Ck\int_T^{2T}\left(\int_\Omega w^{3(k+\omega)/2}(x,s)\,\mathrm{d}x\right)^{2/3}\mathrm{d}s + (Ck)^{6k+7}. \tag{5.48}$$

In particular, it follows from (5.44) and (5.32) that choosing $k=K_\omega$ in (5.48), we can find a constant $C_0>0$ independent of R such that (5.41) holds, and the Moser argument is complete.

As a consequence of (5.40) and (5.41), we see that the norms $\|w\|_k^*$ are bounded by a constant P_∞ independent of k and R. We now easily prove that $w(x,t)\leq P_\infty$ for almost all $(x,t)\in\Omega\times(T,2T)$. Indeed, assume that there exist $a>0$ and a set $A\subset\Omega\times(T,2T)$ such that $w(x,t)\geq P_\infty+a$ for a.e. $(x,t)\in A$. Then

$$(P_\infty+a)^q|A| \leq \iint_A w^q\,\mathrm{d}x\,\mathrm{d}t \leq \int_T^{2T}\!\!\int_\Omega w^q\,\mathrm{d}x\,\mathrm{d}t \leq T\sup_{t\in[T,2T]}\operatorname{ess}\int_\Omega w^q\,\mathrm{d}x \leq TP_\infty^q. \tag{5.49}$$

We conclude from (5.49) that

$$|A| \le T \left(\frac{P_\infty}{P_\infty + a} \right)^q$$

for all $q > 1$. This is only possible if $|A| = 0$. Hence, $w(x,t) \le P_\infty$ a.e. By (5.32), we have $w = 1 + \min\{p^2, R^2\}$. Hence, it suffices now to choose $R > P_\infty$, which entails that the solution of (4.2) and (4.3) that we have constructed in Section 4 satisfies the condition $|p| \le R$ a.e., and therefore is the desired solution of (2.10) and (2.11). This completes the proof of Theorem 2.3.

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References

[1] B. Albers and P. Krejčí, Unsaturated porous media flow with thermomechanical interaction. *Math. Methods Appl. Sci.* **39** (2016) 2220–2238.
[2] M.B.P. Amato, C.S.V. Barbas, D.M. Medeiros, R.B. Magaldi, G.P. Schettino, G. Lorenzi-Filho, R.A. Kairalla, D. Deheinzelin, C. Munoz, R. Oliveira and T.Y. Takagaki, Effect of a protective-ventilation strategy on mortality in the acute respiratory distress syndrome. *New Engl. J. Med.* **338** (1998) 347–354.
[3] R. Amini, K. Creeden and U. Narusawa, A mechanistic model for quasistatic pulmonary pressure–volume curves for inflation. *J. Biomech. Eng.* **127** (2005) 619–629.
[4] H. Bachofen and J. Hildebrant, Area analysis of pressure–volume hysteresis in mammalian lungs. *J. Appl. Physiol.* **30** (1971) 493–497.
[5] V. Brusasco and R. Pellegrino, Hysteresis of airways and lung parenchyma. *Respir. Med.* **89** (1995) 317–322.
[6] M. Cloetta, Untersuchungen über die Elastizität der Lunge und deren Bedeutung für die Zirkulation. *Pflüger's Archiv für die gesamte Physiologie des Menschen und der Tiere* **152** (1913) 339–364.
[7] B. Detmann, P. Krejčíand E. Rocca, Periodic waves in unsaturated porous media with hysteresis. In *Proceedings of ECM 2016*, edited by V. Mehrmann and M. Skutella, EMS Publishing House, Zürich (2018) 219–234.
[8] J. Dios Escolar, M. Asunción Escolar and J. Guzmán, Bronchial hysteresis: morphometric study on the rat lung. *Exp. Lung Res.* **29** (2003) 195–209.
[9] J. Dios Escolar, A. Escolar, Lungs histeresis: A morphological view. *Histol. Histopathol.* **19** (2004) 159–166.
[10] D. Flynn, A survey of hysteresis models of mammalian lungs. *Rend. Sem. Mat. Univ. Politec. Torino* **72** (2014) 17–36.
[11] J. Gil and E.R. Weibel, Morphological study of pressure–volume hysteresis in rat lungs fixed by vascular perfusion. *Respir. Physiol.* **15** (1972) 190–213.
[12] J. Hildebrandt, Pressure–volume data of cat lung interpreted by a plastoelastic, linear viscoelastic model. *J. Appl. Physiol.* **28** (1970) 365–372.
[13] B.P. Kramer and M. Fussenegger, Hysteresis in a synthetic mammalian gene network. *PNAS* **102** (2005) 9517–9522.
[14] M.A. Krasnosel'skii and A.V. Pokrovskii, Systems with Hysteresis. Nauka, Moscow (1983).
[15] P. Krejčí, Hysteresis, Convexity and Dissipation in Hyperbolic Equations. Gakkōtosho, Tokyo (1996).
[16] S.J. Lai-Fook and R.E. Hyatt, Effects of age on elastic moduli of human lungs. *J. Appl. Physiol.* **89** (2000) 163–168.
[17] S. Liu, S.S. Margulies and T.A Wilson, Deformation of the dog lung in the chest wall. *J. Appl. Physiol.* **68** (1990) 1979–1987.
[18] J. Mead, Mechanical properties of lungs. *Physiol. Rev.* **41** (1961) 281–330.
[19] W. Mitzner, Mechanics of the lung in the 20th century. *Compr. Physiol.* **1** (2011) 2009–2027.
[20] M.L. Mogensen, *A physiological mathematical model of the respiratory system*, Ph.D. thesis, Aalborg University, Denmark (2011).
[21] J. Nečas, Les méthodes directes en théorie des équations elliptiques. Academia, Prague (1967).
[22] H.J. Schiller, J. Steinberg, J. Halter, U. McCann, M. DaSilva, L.A. Gatto, D. Carney, G. Nieman, Alveolar inflation during generation of a quasi-static pressure/volume curve in the acutely injured lung. *Crit. Care Med.* **31** (2003) 1126–1133.
[23] A. Visintin, Differential Models of Hysteresis. Springer, Berlin-Heidelberg (1994).
[24] T.V. Wilson, Relations among recoil pressure, surface area and surface tension in the lung. *J. Appl. Physiol. Respir. Environ. Exerc. Physiol.* **50** (1981) 921–926.
[25] E. Zeidler, Nonlinear Functional Analysis and its Applications: I. Fixed Point Theorems. Springer-Verlag, New York (1986).