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NONLOCAL GRADIENT OPERATORS WITH A NONSPHERICAL INTERACTION NEIGHBORHOOD AND THEIR APPLICATIONS

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Abstract. Nonlocal gradient operators are prototypical nonlocal differential operators that are very important in the studies of nonlocal models. One of the simplest variational settings for such studies is the nonlocal Dirichlet energies wherein the energy densities are quadratic in the nonlocal gradients. There have been earlier studies to illuminate the link between the coercivity of the Dirichlet energies and the interaction strengths of radially symmetric kernels that constitute nonlocal gradient operators in the form of integral operators. In this work we adopt a different perspective and focus on nonlocal gradient operators with a non-spherical interaction neighborhood. We show that the truncation of the spherical interaction neighborhood to a half sphere helps making nonlocal gradient operators well-defined and the associated nonlocal Dirichlet energies coercive. These become possible, unlike the case with full spherical neighborhoods, without any extra assumption on the strengths of the kernels near the origin. We then present some applications of the nonlocal gradient operators with non-spherical interaction neighborhoods. These include nonlocal linear models in mechanics such as nonlocal isotropic linear elasticity and nonlocal Stokes equations, and a nonlocal extension of the Helmholtz decomposition.

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1. Introduction

Nonlocal continuum models have become increasingly popular in many scientific fields [1, 2, 4–9, 11, 12, 23, 25, 27, 28, 33, 44, 47, 53–57]. In particular nonlocal models given in terms of integral equations have received much attention as alternatives, tools and bridges to classical local differential equation models and discrete models; see Du [13] for more references and further discussions. Nonlocal models can be particularly effective in modeling singular physical phenomena. Peridyanmics, for example, was proposed by Silling [47] as a continuum theory to study materials failure to which classical continuum theories of elasticity are not well suited. Central to the nonlocal models are nonlocal operators that are integral relaxations of the counterpart local differential operators, and this work is concerned with a class of representatives of such nonlocal operators, namely nonlocal gradient operators.

Nonlocal gradient operators have been studied in various contexts ranging from rigorous theoretical analysis to computational studies for applications [13]. For example, the development of nonlocal vector calculus [18]

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has initiated a systematic foundation to the study of nonlocal gradient operators for tensor and vector fields in connection to applications to nonlocal mechanics and diffusion. On the analytical front, inspired by the work of Bourgain *et al.* [4] on nonlocal characterizations of Sobolev spaces, Mengesha and Spector [39] investigated localization of nonlocal gradients of scalar-valued functions, which was further extended to the cases of vector-valued functions [38]. A related notion of fractional gradient and divergence has been studied by Mazowiecka and Schikorra [35] to provide a fractional generalization of the local div-curl lemmas in geometric analysis. More towards applications of nonlocal gradients include works on a posteriori nonlocal stress analysis [19], a nonlocal chemotactic model [27], nonlocal image analysis [25], nonlocal mechanics [32], and many others. We are focused in this work on nonlocal gradient operators similar to those studied in [38,39] that are characterized with a finite range of nonlocal interaction. What motivates our work are in two folds: the most relevant one is the stability of nonlocal systems associated with nonlocal gradient operators as discussed in the context of correspondence theory of peridynamics [14] and as utilized in the setting of fluid dynamics [15]. Du and Tian [14, 15] have established the stability provided that nonlocal interaction kernels are radially symmetric and have suitably strong singularity of fractional type at the center of the nonlocal interaction neighborhood. We extend their analysis by proving that the use of nonspherical interaction neighborhood (or non-radially defined nonlocal interaction kernel) allows the assumptions on singular interaction at the center to be removed when establishing the coercivity of the nonlocal Dirichlet energies. The idea of breaking the radial symmetry in the nonlocal interaction contributes to the second motivation of our work in connection to the study of such nonlocal operators for nonlocal convections [52] and nonlocal in time dynamic processes [20]. The former is important to preserve the upwinding feature of the physical transport process while the latter reflects the time-irreversibility.

Existing studies in the literature have demonstrated successful applications of nonlocal operators with non-radially symmetric kernels to nonlocal modeling. In [20], the well-posedness and localization of nonlocal in time parabolic equation is established for a wide class of kernels when the support of the kernels for nonlocal-in-time derivative operators are truncated to yield one-sided backward in time nonlocal derivative operators. In study of compact embeddings of nonlocal function spaces of L^p vector fields [21], it is shown that the monotonicity assumption of the kernels for the nonlocal diffusion operators can be relaxed if they are non-radially symmetric and remain non-negative on a conic region. In [24], the unique solvability is examined for nonlocal diffusion operators with kernels that may not have any symmetry at all. One of our main contributions here is to establish the coercivity result on the correspondence nonlocal Dirichlet energies provided that, for nonlocal gradient operators used, one starts with radially symmetric kernels and breaks the symmetry by truncating the support of the kernels *via* multiplication by characteristic functions associated with half-spheres.

The organization of the paper is as follows. In Section 2 we propose nonlocal nonsymmetric gradient operators and verify their consistency in the case of linear functions. We then introduce associated adjoint operators, namely nonlocal divergence operators, as well as corresponding nonlocal diffusion operators. We present the spectra of the nonlocal operators introduced thus far by imposing the periodic boundary conditions. This allows us to furnish, without technicalities of nonlocal boundary conditions, a straightforward proof of the main result of the paper that our nonlocal diffusion operators are positive definite *uniformly* in two nonlocality parameters: one for the length scale and the other for the geometric specifications of nonlocal interaction neighborhoods. We conclude the section with a discussion on the issue of many choices of nonlocal interaction neighborhoods in our multi-dimensional formulations of the nonlocal operators. In Section 3 we demonstrate the applications of our nonlocal gradient operators within the purview of linearized local continuum mechanics models. Specifically we revisit the unique solvability of the nonlocal Stokes equation [15] and provide a nonlocal version of the classical Helmholtz decompositions. We also study a nonlocal model for isotropic linear elasticity. In Section 4 we return to our nonlocal diffusion operators and draw connections to other existing formulations of the operators. In Section 5 we give concluding remarks.

2. Nonlocal nonsymmetric gradient operators

A general form of the nonlocal gradient operator $\mathcal{G}_\delta$ studied in earlier works [13,18,19,38] is given by

$$\mathcal{G}_\delta u(x) = 2\int_{\mathbb{R}^d} \underline{\omega}_\delta(y-x)(y-x)\otimes\frac{u(y)-u(x)}{|y-x|}\,\mathrm{d}y \tag{2.1}$$

for a suitably defined function $u = u(x) : \mathbb{R}^d \to \mathbb{R}^1$ or $\mathbb{R}^d$, and a nonlocal interaction kernel $\underline{\omega}_\delta = \underline{\omega}_\delta(z)$. A common case for the latter is given by a radially symmetric kernel. In this work, we define a nonlocal gradient operator $\mathcal{G}_\delta^{\vec{n}}$ acting on u as

$$\mathcal{G}_\delta^{\vec{n}} u(x) = 2\int_{\mathbb{R}^d} \chi_{\vec{n}}(y-x)w_\delta(|y-x|)(y-x)\otimes\frac{u(y)-u(x)}{|y-x|}\,\mathrm{d}y \tag{2.2}$$

where $\chi_{\vec{n}}(z)$ denotes the characteristic function of the half-space $\mathcal{H}_{\vec{n}} = \{\vec{z}\in\mathbb{R}^d : \vec{z}\cdot\vec{n}\geq 0\}$ parameterized by the unit vector $\vec{n}$. Here, the scalar-valued nonnegative function w_δ is a radially symmetric nonlocal kernel that measures the strength of the nonlocal interaction. We consider in particular a scaled kernel of the form $w_\delta(|x|) = \frac{1}{\delta^{d+1}}w(\frac{|x|}{\delta})$ where w is nonnegative and compactly supported on the unit ball with a bounded first moment

$$\int_{\mathbb{R}^d} w(|x|)|x|\,\mathrm{d}x = \int_{|x|\leq 1} w(|x|)|x|\,\mathrm{d}x = d. \tag{2.3}$$

Corresponding to (2.1), we have $\underline{\omega}_\delta(z) = \chi_{\vec{n}}(z)w_\delta(|z|)$, which is no longer radially symmetric. The support of w_δ is given by the ball of radius δ, which is called a nonlocal horizon or smoothing length depending on different contexts [15,47]. Effectively, the nonlocal interaction neighborhood (the domain of integration) in (2.2) is given by the support of $\underline{\omega}_\delta$ inside a half sphere defined by $\vec{n}$ with radius δ. The operators $\mathcal{G}_\delta^{\vec{n}}$ are d-dimensional versions of the following one dimensional one-sided nonlocal gradient operators [15,19,20]

$$\mathcal{G}_\delta^{\pm} u(x) = \pm 2\int_0^\delta \omega_\delta(s)(u(x\pm s)-u(x))\,\mathrm{d}s. \tag{2.4}$$

Nonlocal derivative operators with a nonradial interaction kernel or a nonspherical neighborhood have also been used to study spatially inhomogeneous nonlocal convection in multidimensional cases [52]. However, systematic studies remain limited. We thus present some further analysis in this section.

2.1. Consistency for linear functions

Let us first note that the operators $\mathcal{G}_\delta^{\vec{n}}$ coincide with their local counterparts for linear functions.

Lemma 2.1. *For every unit vector $\vec{n}$ in $\mathbb{R}^d$ and every affine function $u : \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$ where $\tilde{d} = 1$ or d,*

$$\mathcal{G}_\delta^{\vec{n}}(u)(x) = \nabla u(x) \quad \forall x\in\mathbb{R}^d.$$

Proof. We assume without loss of generality $\tilde{d} = d$ and $x = 0$. Let us write $u(x) = Ax + b$ for some $d\times d$ real-valued matrix A and $b\in\mathbb{R}^d$. If R is a rotation matrix that aligns $\vec{e}_1 = (1,0,\dots,0)\in\mathbb{R}^d$ with $\vec{n}$, then we have

$$\begin{aligned}\mathcal{G}_\delta^{\vec{n}}(u)(0) &= 2\int_{\mathcal{H}_{\vec{n}}} w_\delta(|y|)y\otimes\frac{Ay}{|y|}\,\mathrm{d}y = 2\int_{\mathcal{H}_{\vec{e}_1}} w_\delta(|z|)Rz\otimes\frac{ARz}{|z|}\,\mathrm{d}z \\ &= R\left(2\int_{\mathcal{H}_{\vec{e}_1}} w_\delta(|z|)z\otimes\frac{z}{|z|}\,\mathrm{d}z\right)R^TA^T = RI_dR^TA^T = A^T = \nabla u(0)\end{aligned}$$

where T, I_d denote the transpose operator and the $d \times d$ identity matrix, respectively, and the moment condition (2.3) is used in the third to the last equality. $\square$

2.2. Adjoint operator

In analogy with the local setting, the definition of the operator $\mathcal{G}_\delta^{\vec{n}}$ leads to the consideration of its adjoint nonlocal divergence operator $\mathcal{D}_\delta^{\vec{n}}$. That is, we define the operator $\mathcal{D}^{\vec{n}}$ by

$$\int\limits_{\mathbb{R}^d} u(x) \cdot \mathcal{G}_\delta^{\vec{n}} v(x)\,\mathrm{d}x = -\int\limits_{\mathbb{R}^d} \mathcal{D}_\delta^{\vec{n}} u(x) v(x)\,\mathrm{d}x \tag{2.5}$$

for suitable functions u and v such that both sides of the equality make sense. In explicit terms, the operator $\mathcal{D}_\delta^{\vec{n}}$ takes the form

$$\mathcal{D}_\delta^{\vec{n}} u(x) = 2\int\limits_{\mathbb{R}^d} \chi_{\vec{n}}(s) \frac{w^\delta(|s|)}{|s|} s \cdot (u(x) - u(x-s))\,\mathrm{d}s.$$

Note that in an analogy to the local diffusion operator $\Delta = \operatorname{div}\operatorname{grad}$, we may also define the associated nonlocal diffusion operator $\mathcal{L}_\delta^{\vec{n}} = \mathcal{D}_\delta^{\vec{n}} \circ \mathcal{G}_\delta^{\vec{n}}$ where $\circ$ denotes the composition. More discussions on $\mathcal{L}_\delta^{\vec{n}}$ are given later.

2.3. Representation in Fourier space

For simplicity and definiteness, we assume from here on the two dimensional setting $d = 2$ unless otherwise noted. Moreover we focus only on the action of the nonlocal operators on functions that are periodic on the domain $\Omega = (-\pi, \pi)^2$. We can then exploit the periodicity to examine the Fourier symbols of the nonlocal operators introduced thus far. In particular, for any periodic function $u(x)$ on Ω, we let $\widehat{u} = \widehat{u}(\xi)$ denote its Fourier coefficients, hence the Fourier expansion

$$u(x) = \sum_{\xi \in \mathbb{Z}^2} \widehat{u}(\xi) e^{i\xi\cdot x}.$$

The loss of symmetry in the integration domain in the definitions of the nonlocal operators, in contrast to the symmetric case studied in [15], is manifested in terms of the real parts of the Fourier symbols.

Lemma 2.2. *For locally integrable, periodic u, v, w where $u, w : \Omega \to \mathbb{R}^1$ or $\mathbb{R}^2$ and $v : \Omega \to \mathbb{R}^2$ or $\mathbb{R}^{2\times 2}$, the Fourier symbols of the operators $\mathcal{G}_\delta^{\vec{n}}, \mathcal{D}_\delta^{\vec{n}}, \mathcal{L}_\delta^{\vec{n}}$ are given by*

$$\begin{aligned}
\widehat{\mathcal{G}_\delta^{\vec{n}} u}(\xi) &= \lambda_\delta^{\vec{n}}(\xi)(\widehat{u}(\xi))^T \\
\widehat{\mathcal{D}_\delta^{\vec{n}} v}(\xi) &= -\overline{\lambda_\delta^{\vec{n}}(\xi)}^T \widehat{v}(\xi) = \lambda_\delta^{-\vec{n}}(\xi)^T \widehat{v}(\xi) \\
\widehat{\mathcal{L}_\delta^{\vec{n}} w}(\xi) &= -|\lambda_\delta^{\vec{n}}(\xi)|^2 \widehat{w}(\xi)
\end{aligned}$$

where $\xi \in \mathbb{Z}^2$ and

$$\begin{aligned}
\Re(\lambda_\delta^{\vec{n}}(\xi)) &= 2\int\limits_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}(\cos(\xi\cdot s) - 1)\,\mathrm{d}s \\
\Im(\lambda_\delta^{\vec{n}}(\xi)) &= 2\int\limits_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}\sin(\xi\cdot s)\,\mathrm{d}s.
\end{aligned}$$

The above results are immediate from the definitions of the operators. It is natural to compare $\lambda_\delta^{\vec{n}}(\xi)$ to the Fourier symbol of the local gradient operator given by $i\xi$ to relate the imaginary part of $\lambda_\delta^{\vec{n}}(\xi)$ to its local counterpart. The former is shown to be a scalar multiple of the latter independently of $\vec{n}$ due to some symmetry of the integrand on half-spheres.

Lemma 2.3. *For each $\vec{n}$ and $\xi \in \mathbb{Z}^2\backslash\{0\}$, the Fourier symbol $\lambda_\delta^{\vec{n}}(\xi)$ in Lemma 2.2 can be expressed as*

$$\lambda_\delta^{\vec{n}}(\xi) = i\Lambda_\delta(|\xi|)\frac{\xi}{|\xi|} + \Re(\lambda_\delta^{\vec{n}}(\xi)) \tag{2.6}$$

where

$$\Lambda_\delta(|\xi|) = 4\int\limits_{s_1\geq 0, s_2\geq 0} \frac{w_\delta(|s|)s_1}{|s|}\sin(|\xi|s_1)\,\mathrm{d}s. \tag{2.7}$$

On the other hand, $\Re(\lambda_\delta^{\vec{n}}(\xi))$ is a scalar multiple of $\frac{\xi}{|\xi|}$ if and only if $\vec{n}$ is a scalar multiple of ξ.

Proof. We observe that

$$\Im(\lambda_\delta^{\vec{n}}(\xi)) = 2\int\limits_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|}\sin(\xi\cdot s)\,\mathrm{d}s = \int\limits_{B_\delta(0)} \frac{w_\delta(|s|)s}{|s|}\sin(\xi\cdot s)\,\mathrm{d}s$$

where the equality is due to the odd symmetry of $s\sin(\xi\cdot s)$. Then the first claim follows from Lemma 3 in [15]. Next we consider the real part of $\lambda_\delta^{\vec{n}}(\xi)$. If we let $\xi^\perp$ denote a vector orthogonal to ξ, then it follows that

$$|\xi^\perp\cdot\Re(\lambda_\delta^{\vec{n}}(\xi)| = 2\int\limits_{\mathcal{I}_{\vec{n},\xi}} \frac{w_\delta(|s|)|\xi^\perp\cdot s|}{|s|}(1-\cos(\xi\cdot s))\,\mathrm{d}s$$

where $\mathcal{I}_{\vec{n},\xi} = \{s\in\mathcal{H}_{\vec{n}} : s - 2(s\cdot\xi)\xi \in \mathcal{H}_{\vec{n}}\}$, thus the second claim holds since $|\mathcal{J}_{\vec{n},\xi}| = 0$ precisely when $\vec{n}$ is a scalar multiple of ξ. $\square$

2.4. Spectral estimates

We now present a key theorem concerning the spectral property of the nonlocal gradient operator $\mathcal{G}_\delta^{\vec{n}}$. The theorem implies, in particular the strong coercivity, uniformly in δ and $\vec{n}$, of the Dirichlet energies given by $\|\mathcal{G}_\delta^{\vec{n}}(\cdot)\|_2^2$ with respect to the norm $\|\cdot\|_2 + \|\mathcal{G}_\delta^{\vec{n}}(\cdot)\|_2$.

Theorem 2.4. *There exists a positive constant C independent of $\vec{n}$, ξ and δ (as $\delta\to 0$) such that*

$$C \leq |\lambda_\delta^{\vec{n}}(\xi)| \leq 2\sqrt{2}|\xi|, \quad \forall\,\xi\in\mathbb{Z}^2/\{0\}.$$

Proof. Let $k = \delta|\xi|$, we show the bound on $|\lambda_\delta^{\vec{n}}(\xi)|$ using two separate estimates on the imaginary and real parts of $\lambda_\delta^{\vec{n}}(\xi)$ respectively depending on $k<1$ or $k\geq 1$.

For $k<1$, we estimate the imaginary part. Using Lemma 2.3, we have

$$\begin{aligned}\Lambda_\delta(\xi) = 4\int\limits_{s_1\geq 0, s_2\geq 0} \frac{w_\delta(|s|)s_1}{|s|}\sin(|\xi|s_1)\,\mathrm{d}s &= 4\int\limits_{r=0}^{\delta}\int\limits_{\theta=0}^{\frac{\pi}{2}} w_\delta(r)r\cos(\theta)\sin(|\xi|r\cos(\theta))\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{4}{\delta}\int\limits_{r=0}^{1}\int\limits_{\theta=0}^{\frac{\pi}{2}} w(r)r\cos(\theta)\sin(|\xi|\delta r\cos(\theta))\,\mathrm{d}r\,\mathrm{d}\theta.\end{aligned}$$

By the inequality $\sin(x) \geq x - x^3/6$ for $0 \leq x \leq 1$, we obtain

$$\begin{aligned}\Lambda_\delta(\xi) &\geq \frac{4k}{\delta}\int\limits_{r=0}^{1}\int\limits_{\theta=0}^{\frac{\pi}{2}} w(r)r^2\cos^2(\theta)\,\mathrm{d}r\,\mathrm{d}\theta - \frac{4k^3}{6\delta}\int\limits_{r=0}^{1}\int\limits_{\theta=0}^{\frac{\pi}{2}} w(r)r^4\cos^4(\theta)\,\mathrm{d}r\,\mathrm{d}\theta \\ &\geq \frac{Ck}{\delta} = C|\xi|,\end{aligned}$$

for a constant $C > 0$, independent of ξ, δ and $\vec{n}$.

Next, for $1 \geq k$, we consider $\Re(\lambda_\delta^{\vec{n}}(\xi))$. Note that

$$\begin{aligned}|\Re(\lambda_\delta^{\vec{n}}(\xi))| \geq |\vec{n}\cdot\Re(\lambda_\delta^{\vec{n}}(\xi))| &= 2\int\limits_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)}{|s|}(n_1s_1 + n_2s_2)(1 - \cos(\xi\cdot s))\,\mathrm{d}s \\ &\geq 2\cos\left(\frac{\pi}{4}\right)\int\limits_{H_{\vec{n},\frac{\pi}{4}}} w_\delta(|s|)(1 - \cos(\xi\cdot s))\,\mathrm{d}s\end{aligned}$$

where $\mathcal{H}_{\vec{n},\frac{\pi}{4}}$ denotes the set of those points in $\mathcal{H}_{\vec{n}}$ that have angles with $\vec{n}$ between $-\pi/4$ and $\pi/4$. In terms of polar coordinates, we can then write

$$\begin{aligned}|\Re(\lambda_\delta^{\vec{n}}(\xi))| &\geq C\int\limits_{r=0}^{\delta}\int\limits_{\theta_1}^{\theta_1+\frac{\pi}{2}} w_\delta(r)(1 - \cos(|\xi|r\cos(\psi_\theta)))r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{C}{\delta}\underbrace{\int\limits_{r=0}^{1}\int\limits_{\theta_1}^{\theta_1+\frac{\pi}{2}} w(r)(1 - \cos(kr|\cos(\psi_\theta)|))r\,\mathrm{d}r\,\mathrm{d}\theta}_{J_\delta(\xi)}\end{aligned}$$

for some θ_1 depending on $\vec{n}$. Here $0 \leq \psi_\theta \leq \pi$ denotes the angle between the vector ξ and the vector with the polar coordinates (r, θ).

We now introduce a possible cut-off of w at the origin to get an absolutely integrable kernel ϕ, that is, ϕ is a radial function such that

$$0 \leq \phi(|x|) \leq w(|x|) \text{ and } 0 < I := \int\limits_{\mathbb{R}^2} \phi(|x|)\,\mathrm{d}x < \infty.$$

We then discuss the different cases separately. First, let us consider the case with $1 \leq k \leq \lambda$ where λ is to be specified.

If we let

$$A_\xi = \left(\theta_1, \theta_1 + \frac{\pi}{2}\right) - \left\{\theta \in (0, 2\pi) : \left|\psi_\theta - \frac{\pi}{2}\right| \leq \frac{\pi}{8}\right\},$$

then

$$\frac{\cos\left(\frac{3\pi}{8}\right)}{2\lambda} \leq kr|\cos(\psi_\theta)| \leq 1 \quad \text{for } (r,\theta) \in \left(\frac{1}{2\lambda}, \frac{1}{\lambda}\right) \times A_\xi$$

so that

$$J_\delta(\xi) \geq \frac{1}{\delta} \int_{r=\frac{1}{2\lambda}}^{\frac{1}{\lambda}} \int_{A_\xi} w(r) r (1 - \cos(kr|\cos(\psi_\theta)|)) \, \mathrm{d}r \, \mathrm{d}\theta$$

$$\geq \frac{1}{\delta} \left(1 - \cos\left(\frac{\cos\left(\frac{3\pi}{8}\right)}{2\lambda}\right)\right) \int_{r=\frac{1}{2\lambda}}^{\frac{1}{\lambda}} \int_{A_\xi} w(r) r \, \mathrm{d}r \, \mathrm{d}\theta \geq \frac{C}{\delta}$$

where the last inequality is due to the non-degeneracy of $|A_\xi| \geq \frac{\pi}{4}$ uniformly in ξ.

Next, we consider the case $\lambda < k$. Then with the same A_ξ defined in the case (a)

$$J_\delta(\xi) \geq \frac{1}{\delta} \int_{A_\xi} \int_{r=0}^{1} w(r) r \left(1 - \cos(kr|\cos(\psi_\theta)|)\right) \, \mathrm{d}r \, \mathrm{d}\theta$$

$$\geq \frac{1}{\delta} \int_{A_\xi} \left(\frac{I}{2\pi} - \int_{r=0}^{1} \phi(r) r \cos(kr|\cos(\psi_\theta)|) \, \mathrm{d}r \right) \mathrm{d}\theta.$$

By the Riemann–Lebesgue lemma, there exists a constant $c > 0$ such that for $j > c$

$$\left| \int_{r=0}^{1} \phi(r) r \cos(jr) \, \mathrm{d}r \right| < \frac{I}{4\pi}\cdot$$

Then, since $\cos\left(\frac{3\pi}{8}\right) \leq |\cos(\psi_\theta)|$ for $\theta \in A_\xi$ we set $\lambda = c/\cos\left(\frac{3\pi}{8}\right)$ to obtain that for some $\tilde{C}$,

$$J_\delta(\xi) \geq \frac{\tilde{C}}{\delta}\cdot$$

In summary, we have $|\lambda_\delta^{\vec{n}}(\xi)| \geq \min\{C_1, \frac{C_2}{\delta}\}$ for positive constants C_1 and C_2.

In order to prove the uniform upper bound on $|\lambda_\delta^{\vec{n}}(\xi)|$ we observe

$$|\Re(\lambda_\delta(\xi))| = \left| 2 \int_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|) s}{|s|} (\cos(\xi \cdot s) - 1) \, \mathrm{d}s \right| \leq 2 \int_{\mathcal{H}_{\vec{n}}} w_\delta(|s|) |\xi \cdot s| \, \mathrm{d}s \leq 2|\xi|$$

and

$$|\Im(\lambda_\delta(\xi))| = \left| \int_{B_\delta(0)} \frac{w_\delta(|s|) s}{|s|} \sin(\xi \cdot s) \, \mathrm{d}s \right| \leq \int_{B_\delta(0)} w_\delta(|s|) |\xi| |s| \, \mathrm{d}s = 2|\xi|$$

following the derivation in the proof of Lemma 2.3. □

2.5. Orientation dependence

Before we turn to applications of the nonlocal operators, we discuss an important issue that is naturally concerned with the introduction of the parameter $\vec{n}$ in our formulation of the operators. We recall that the $\vec{n}$ itself belongs to the non-trivial ambient space $S^{d-1} = \{\vec{n} \in \mathbb{R}^d : \|\vec{n}\|_2 = 1\}$ for $d \geq 2$ when the one dimensional half-spaces $(0, \infty)$ and $(-\infty, 0)$ are generalized to the higher dimensional analogues $\mathcal{H}_{\vec{n}}$. A specific choice of

$\vec{n}$ leads to orientation dependence. It is a legitimate question to ask if such dependence is necessary (while maintaining a coercive Dirichret integral). One can also ask how to pick $\vec{n}$, in case that it is needed, in practice. Since the coercivity result of Theorem 2.4 is true for all $\vec{n}$, one possible approach is to eliminate the dependence on $\vec{n}$ by defining a new energy functional in terms of the average over $\vec{n} \in S^{d-1}$. Indeed, if we remain in the two dimensional domain $\Omega = (-\pi, \pi)^2$ as in Section 2.4 and take for concreteness a periodic scalar valued $u : \Omega \to \mathbb{R}$, we may consider the averaged nonlocal Dirichlet integral

$$\mathcal{E}_\delta^{\mathrm{avg}}(u) = \frac{1}{2\pi} \int_{S^1} \int_{\Omega} |\mathcal{G}_\delta^{\vec{n}} u(x)|^2 \, \mathrm{d}x \, \mathrm{d}S.$$

The coercivity of $\mathcal{E}_\delta^{\mathrm{avg}}$ is immediate from that of each Dirichlet integral associated with $\mathcal{G}_\delta^{\vec{n}}$. One can view $\mathcal{E}_\delta^{\mathrm{avg}}$ as a stabilized symmetric nonlocal Dirichlet integral

$$\mathcal{E}_\delta^{\mathrm{sym}}(u) = \int_{\Omega} \left| \frac{1}{2} (\mathcal{G}_\delta^{\vec{n}} + \mathcal{G}_\delta^{\vec{n}}) u(x) \right|^2 \mathrm{d}x$$

since

$$\mathcal{E}_\delta^{\mathrm{avg}}(u) = \mathcal{E}_\delta^{\mathrm{sym}}(u) + \sum_{\xi \in \mathbb{Z}^2/\{0\}} \frac{2}{\pi} \int_{B_\delta(0)} \int_{B_\delta(0)} \frac{w_\delta(|a|) w_\delta(|b|) a \cdot b}{|a||b|} \arcsin\left(\frac{a \cdot b}{|a||b|}\right) (\cos(\xi \cdot a) - 1)(\cos(\xi \cdot b) - 1) \, \mathrm{d}a \, \mathrm{d}b.$$

Unfortunately, we are unable to express $\mathcal{E}_\delta^{\mathrm{avg}}(u)$ as a Dirichlet integral of a nonlocal gradient operator. However, based on the above calculation and using a crude estimate $\pi x^2 \geq x \arcsin(x)$ we may alternatively consider a simpler looking, yet still coercive functional

$$\mathcal{E}_\delta^{\star,\vec{k}}(u) = \int_{\Omega} |\mathcal{G}_\delta^{\vec{k}} u(x)|^2 \, \mathrm{d}x$$

where for any vector $\vec{k} \in \mathbb{R}^d$, $\mathcal{G}_\delta^{\star,\vec{k}} u(x)$ is a modified nonlocal gradient operator given by

$$\mathcal{G}_\delta^{\star,\vec{k}} u(x) = \int_{\mathbb{R}^d} w_\delta(|y-x|) \frac{y-x}{|y-x|} (u(y) - u(x)) \, \mathrm{d}y + \left(\int_{\mathbb{R}^d} w_\delta(|y-x|)(u(y) - u(x)) \, \mathrm{d}y \right) \vec{k}. \tag{2.8}$$

We remark that the adjoint operator of $\mathcal{G}_\delta^{\star,\vec{k}}$ can be defined in a similar fashion as in Section 2.2. This may prove to be useful in nonlocal modeling. Note that $\mathcal{G}_\delta^{\star,\vec{k}}$ can be seen as a special form of more general nonlocal gradient operators studied in [38]. It is also related to the more standard nonlocal gradient operator with a spherical interaction neighborhood corresponding to the form of $\mathcal{G}_\delta^{\star,\vec{k}}$ with $\vec{k} = \vec{0}$ (the zero vector). According to Du and Tian [14, 15], the coercivity of the Dirichret integral corresponding to this case, *i.e.*, $\vec{k} = \vec{0}$, depends on the choices of the kernel w_δ. For nonzero $\vec{k}$, we can have the coercivity of the Dirichret integral for $\mathcal{G}_\delta^{\star,\vec{k}}$ due to the second term in (2.8). This orientation dependent term is $O(\delta)$, due to the moment condition in (2.3), similar in spirit to how stability is attained in our recent work on the deterministic particle methods [31].

3. Applications of the nonlocal gradient operators

We now illustrate how the coercivity of the nonlocal Dirichlet energies can be utilized in several applications. These include the application of our nonlocal operators as building blocks for an alternative formulation of the nonlocal Stokes equation studied in [15]. Another application is to establish a nonlocal version of the Helmholtz decomposition. In addition, the operators are used to construct well-defined models of nonlocal isotropic linear elasticity that converge to the classical counterpart for any Poisson's ratio.

We use the nonsymmetric operator $\mathcal{G}_\delta^{\vec{n}}$ with a unit vector $\vec{n}$ defined in (2.2) for illustration, though similar discussions can be made for $\mathcal{G}_\delta^{\star,\vec{k}}$ with a constant nonzero vector $\vec{k} \neq \vec{0}$ defined in (2.8). To avoid further technical complications, we assume from here on the kernels adopted in our nonlocal operators are positive almost everywhere. Moreover, as in Section 2 we present our results in the two dimensional setting on the domain $\Omega = (-\pi, \pi)^2$ unless specified otherwise.

3.1. Nonlocal Stokes equation

We first consider the steady nonlocal stokes equation

$$\begin{aligned} -\mathcal{L}_\delta^{\vec{n}} \mathbf{u}_\delta^{\vec{n}} + \mathcal{G}_\delta^{\vec{n}} p_\delta^{\vec{n}} &= \mathbf{f} \text{ in } \Omega \\ -\mathcal{D}_\delta^{\vec{n}} \mathbf{u}_\delta^{\vec{n}} &= 0 \text{ in } \Omega \end{aligned} \tag{3.1}$$

where $\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}}, \mathbf{f}$ are periodic functions on Ω and assumed to have zero means. One of the motivation for studying nonlocal Stokes model is to better understand methods like the smoothed particle hydronamics (SPH) [26,34], see [15] for more references and discussions.

The nonlocal Stokes equation is obtained by applications of the nonlocal operators $\mathcal{L}_\delta^{\vec{n}}, \mathcal{G}_\delta^{\vec{n}}, \mathcal{D}_\delta^{\vec{n}}$ to their local counterparts in the classical Stokes equation

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \text{ in } \Omega \\ -\nabla \cdot \mathbf{u} &= 0, \text{ in } \Omega. \end{aligned} \tag{3.2}$$

In the Fourier space the system (3.1) can be written as

$$A_\delta^{\vec{n}}(\xi) \begin{bmatrix} \widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi) \\ \widehat{p_\delta^{\vec{n}}}(\xi) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}}(\xi) \\ 0 \end{bmatrix} \tag{3.3}$$

where

$$A_\delta^{\vec{n}} = \begin{bmatrix} |\lambda_\delta^{\vec{n}}(\xi)|^2 I_2 & \lambda_\delta^{\vec{n}}(\xi) \\ \overline{\lambda_\delta^{\vec{n}}(\xi)}^T & 0 \end{bmatrix}.$$

The non-degeneracy of $\lambda_\delta^{\vec{n}}$ assured by Theorem 2.4 yields the well-posedness of (3.3).

Lemma 3.1. *The system (3.3) has a unique solution given by*

$$\begin{bmatrix} \widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi) \\ \widehat{p_\delta^{\vec{n}}}(\xi) \end{bmatrix} = (A_\delta^{\vec{n}})^{-1} \begin{bmatrix} \widehat{\mathbf{f}}(\xi) \\ 0 \end{bmatrix} \tag{3.4}$$

where

$$(A_\delta^{\vec{n}})^{-1} = \begin{bmatrix} \frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\left(I - \frac{\lambda_\delta^{\vec{n}}(\xi)\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\right) & \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \\ \frac{\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2} & -1 \end{bmatrix}.$$

Moreover, there exists a constant $C > 0$ independent of f and δ, as $\delta \to 0$, such that

$$\|\mathbf{u}_\delta^{\vec{n}}\|_{[S_\delta^{\vec{n}}(\Omega)]^2} + \|p_\delta^{\vec{n}}\|_{L^2(\Omega)} \leq C \|\mathbf{f}\|_{[(S_\delta^{\vec{n}}(\Omega))^\star]^2} \tag{3.5}$$

where $S_\delta^{\vec{n}}(\Omega) = \{h \in L^2(\Omega) : h$ is periodic and $\mathcal{G}_\delta^{\vec{n}} h \in [L^2(\Omega)]^2\}$ is the energy space equipped with the energy norm $\|h\|_{S_\delta^{\vec{n}}(\Omega)} = \left(\|\mathcal{G}_\delta^{\vec{n}} h\|_{[L^2(\Omega)]^2}\right)^{1/2}$ for $h \in S_\delta^{\vec{n}}(\Omega)$, and $(S_\delta^{\vec{n}}(\Omega))^\star$ is its dual space with respect to the standard L^2 duality pairing.

Let us examine the limit of the nonlocal solutions $(u_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ as $\delta \to 0$. As can be expected by Lemma 2.3, it is worthy noting that in the case studied in this work, the nonlocal velocity is only approximately local divergence free, which is in contrast with the nonlocal Stokes equation in [15] that gives equivalent local and nonlocal divergence free vector fields.

Proposition 3.2. *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ *denote the solutions of (3.1) and (3.2) respectively. Then there exists a constant* $C > 0$ *independent of* f *and* δ*, as* $\delta \to 0$*, such that*

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_{[L^2(\Omega)]^2} + \|p_\delta^{\vec{n}} - p\|_{L^2(\Omega)} + \|\nabla \cdot \mathbf{u}_\delta^{\vec{n}}\|_{L^2(\Omega)} \le C\delta \|\mathbf{f}\|_{[L^2(\Omega)]^2}. \tag{3.6}$$

Proof. We can see from (3.4)

$$\begin{aligned}
|\widehat{\mathbf{u}}(\xi) - \widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi)| &\le \left(\left| \frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{1}{|\xi|^2} \right| + \left| \frac{\lambda_\delta^{\vec{n}}(\xi)\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^4} - \frac{i\xi(-i\xi)^T}{|\xi|^4} \right| \right) |\widehat{\mathbf{f}}(\xi)| \\
&\le 2\left(\frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|} + \frac{1}{|\xi|} \right) \left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| |\widehat{\mathbf{f}}(\xi)| \\
&\le C \left(\left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| \right) |\widehat{\mathbf{f}}(\xi)|
\end{aligned}$$

where the last inequality is due to Theorem 2.4. Then since

$$|\widehat{p}(\xi) - \widehat{p_\delta^{\vec{n}}}(\xi)| \le \left| \frac{-\overline{\lambda_\delta^{\vec{n}}(\xi)}}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| |\widehat{\mathbf{f}}(\xi)| = \left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| |\widehat{\mathbf{f}}(\xi)|,$$

it is sufficient to prove

$$\left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| \le C\delta \qquad (\dagger)$$

for some C independent of δ and ξ. To this end let us explicitly write $\lambda_\delta^{\vec{n}}(\xi) = a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)$ to obtain

$$\begin{aligned}
\left| \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| &= \left| \frac{a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)}{|a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right| \\
&\le \underbrace{\left| \frac{ib_\delta(\xi)}{|ib_\delta(\xi)|^2} - \frac{i\xi}{|\xi|^2} \right|}_{I_1} + \underbrace{\left| \frac{ib_\delta(\xi)}{|a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)|^2} - \frac{ib_\delta(\xi)}{|ib_\delta(\xi)|^2} \right|}_{I_2} + \underbrace{\left| \frac{a_\delta^{\vec{n}}(\xi)}{|a_\delta^{\vec{n}}(\xi) + ib_\delta(\xi)|^2} \right|}_{I_3}
\end{aligned}$$

and consider cases depending on the values of $k = \delta|\xi|$ as in the proof of Theorem 2.4.

(1) $k < 1$. We first consider I_1. Using Lemma 2.3 we have

$$I_1 \le \left| \frac{1}{\Lambda_\delta(\xi)} - \frac{1}{|\xi|} \right| = \delta \left| \frac{1}{4\displaystyle\int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}} w(r) r \cos(\theta) \sin(kr\cos(\theta))\, \mathrm{d}r\, \mathrm{d}\theta} - \frac{1}{k} \right|.$$

Since

$$x - \frac{x^3}{6} \le \sin(x) \le x \text{ for } 0 \le x \le 1,$$

we obtain

$$2\pi k - \frac{\pi k^3}{4} \le 4 \int\limits_{r=0}^{1} \int\limits_{\theta=0}^{\frac{\pi}{2}} w(r) r \cos(\theta) \sin(kr\cos(\theta))\, \mathrm{d}r\, \mathrm{d}\theta \le 2\pi k,$$

which implies

$$\frac{1}{\delta}\left|\frac{1}{\Lambda_\delta(\xi)} - \frac{1}{|\xi|}\right| \le \frac{1}{2\pi}\left(\frac{1}{k - \frac{k^3}{8}} - \frac{1}{k}\right) = \frac{1}{2\pi}\frac{k}{8-k^2} \le \frac{k}{2\pi}.$$

Hence $I_1 \le \frac{\delta k}{2\pi} \le \frac{\delta}{2\pi}$.

Next, for I_2, since $|\cos(x) - 1| \le \frac{x^2}{2}$, it is clear that

$$|a_\delta^{\vec{n}}(\xi)| = \left| 2 \int\limits_{\mathcal{H}_{\vec{n}}} \frac{w_\delta(|s|)s}{|s|} (\cos(\xi \cdot s) - 1)\, \mathrm{d}s \right| \le C\delta|\xi|^2.$$

where we have used the moment condition (2.3).

On the other hand we can see from the proof of Theorem 2.4 that

$$|b_\delta(\xi)| \ge C|\xi|.$$

Hence it follows that

$$I_2 \le \frac{|a_\delta^{\vec{n}}(\xi)|^2}{(|a_\delta^{\vec{n}}(\xi)|^2 + |b_\delta(\xi)|^2)|b_\delta(\xi)|} \le \frac{C\delta^2|\xi|^4}{|\xi|^3} \le C\delta.$$

As for I_3, similar calculation as in the case of I_2 shows $I_3 \le C\delta$.

(2) $k \ge 1$. We observe from the proof of Theorem 2.4 that $|\lambda_\delta^{\vec{n}}(\xi)| \ge \frac{C}{\delta}$, hence

$$\frac{1}{\delta}\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{\xi}{|\xi|^2}\right| \le \frac{1}{\delta|\lambda_\delta^{\vec{n}}(\xi)|} + \frac{1}{\delta|\xi|} \le C.$$

Lastly the local divergence of u_δ can be estimated as

$$\begin{aligned}|\widehat{\nabla \cdot \mathbf{u}_\delta^{\vec{n}}}(\xi)| &\le \frac{|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|}\left(\frac{|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|}\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right| + \left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\right|\right)|\widehat{\mathbf{f}}(\xi)| \\ &\le \frac{|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|}\left(\frac{2|\xi|}{|\lambda_\delta^{\vec{n}}(\xi)|} + 1\right)\left|\frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} - \frac{i\xi}{|\xi|^2}\right||\widehat{\mathbf{f}}(\xi)| \le C\delta|\widehat{\mathbf{f}}(\xi)|\end{aligned}$$

where the last inequality is due to the estimate (†). □

The study on the nonlocal Stokes equation can be extended to time-dependent case. Let us consider

$$\begin{aligned}(\mathbf{u}_\delta^{\vec{n}})_t - \mathcal{L}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} + \mathcal{G}_\delta^{\vec{n}}p_\delta^{\vec{n}} &= \mathbf{f} \text{ in } (0,T)\times\Omega \\ -\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= 0 \text{ in } (0,T)\times\Omega \\ \mathbf{u}_\delta^{\vec{n}}|_{t=0} &= \mathbf{u}_0 \text{ in } \Omega\end{aligned} \tag{3.7}$$

along with its counterpart local equation

$$\begin{aligned}\mathbf{u}_t - \Delta\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } (0,T)\times\Omega \\ -\nabla\cdot\mathbf{u} &= 0 \text{ in } (0,T)\times\Omega \\ \mathbf{u}|_{t=0} &= \mathbf{u_0} \text{ in } \Omega\end{aligned} \tag{3.8}$$

where all the local and nonlocal field variables as well as the data are assumed to be periodic on Ω with zero means. We then have the analogous results as in the steady case.

Proposition 3.3. *Assume* $\mathbf{f} \in L^2(0,T : [(S_\delta^{\vec{n}}(\Omega))^\star]^2)$ *and* $\mathbf{u}_0 \in [L^2(\Omega)]^2$ *with* $-\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_0 = 0$ *in* Ω *. Then the nonlocal Stokes equation (3.7) has a unique solution* $(\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ *where* $\mathbf{u}_\delta^{\vec{n}} \in L^2(0,T : [S_\delta^{\vec{n}}(\Omega)]^2) \cap C(0,T : [L^2(\Omega)]^2)$*,* $(\mathbf{u}_\delta^{\vec{n}})_t \in L^2(0,T : [(S_\delta^{\vec{n}}(\Omega))^\star]^2)$ *and* $p_\delta^{\vec{n}} \in L^2(0,T : L^2(\Omega))$*.*

Proof. Let us write $P_\delta^{\vec{n}}$ to denote the nonlocal Leray operator which in the Fourier space is given by

$$\widehat{P_\delta^{\vec{n}}}(\xi) := I - \frac{\lambda_\delta^{\vec{n}}(\xi)\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}.$$

One can check that $P_\delta^{\vec{n}}\mathbf{u} = \mathbf{u}$ for $\mathcal{D}_\delta^{\vec{n}}\mathbf{u} = 0$, $P_\delta^{\vec{n}}$ commutes with $\mathcal{L}_\delta^{\vec{n}}$, and $P_\delta^{\vec{n}} \circ \mathcal{G}_\delta^{\vec{n}} = 0$, hence the nonlocal system (3.7) is equivalent to

$$\begin{aligned}(\mathbf{u}_\delta^{\vec{n}})_t - \mathcal{L}_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= P_\delta^{\vec{n}}\mathbf{f} \text{ in } (0,T)\times\Omega\\ \mathcal{G}_\delta^{\vec{n}}p_\delta^{\vec{n}} &= \mathbf{f} - P_\delta^{\vec{n}}\mathbf{f} \text{ in } (0,T)\times\Omega\\ \mathbf{u}_\delta^{\vec{n}}|_{t=0} &= \mathbf{u}_0 \text{ in } \Omega\end{aligned}$$

of which the unique solutions are given by Duhamel's principle

$$\widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi,t) = \widehat{\mathbf{u}_0}(\xi)\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2 t) + \int_0^t \exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2(t-s))\widehat{P_\delta^{\vec{n}}}(\xi)\widehat{\mathbf{f}}(\xi,s)\,\mathrm{d}s$$

and

$$\widehat{p_\delta^{\vec{n}}}(\xi,t) = \frac{\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}(I - \widehat{P_\delta^{\vec{n}}}(\xi))\widehat{\mathbf{f}}(\xi,t).$$

We may then apply the standard energy arguments to show $\mathbf{u}_\delta^{\vec{n}}, (\mathbf{u}_\delta^{\vec{n}})_t$ and $p_\delta^{\vec{n}}$ belong to the appropriate spaces. In order to show the continuity of $\mathbf{u}_\delta$, we first deduce from Theorem 2.4 that $S_\delta^{\vec{n}}(\Omega) \subset L^2(\Omega) \subset (S_\delta^{\vec{n}}(\Omega))^\star$ is a Hilbert triple and then apply the classical interpolation result [50]. ☐

To conclude our discussion of the nonlocal Stokes equation we prove that the nonlocal solutions of (3.7) converge to the corresponding local ones as the nonlocal parameter δ vanishes. Given a locally divergence free initial velocity u_0, however, we need to exercise care in prescribing the initial velocity for the nonlocal Stokes equations since Lemma 2.3 shows that $\mathbf{u}_0$ is in general not nonlocally divergence free.

Proposition 3.4. *Suppose* $\mathbf{f} \in L^2(0,T : [L^2(\Omega)]^2)$ *and* $u_0 \in L^2(\Omega)$ *with* $-\nabla\cdot\mathbf{u}_0 = 0$ *in* Ω*. Assume* $\mathbf{u}_{0,\delta}^{\vec{n}} \to \mathbf{u}_0$ *in* $[L^2(\Omega)]^2$ *as* $\delta \to 0$ *with* $-\mathcal{D}_\delta^{\vec{n}}\mathbf{u}_{0,\delta}^{\vec{n}} = 0$*. Then the unique solution* $(\mathbf{u}_\delta^{\vec{n}}, p_\delta^{\vec{n}})$ *of (3.7) where* $\mathbf{u}_0$ *is replaced by* $\mathbf{u}_{0,\delta}^{\vec{n}}$ *converges to the unique solution* $(\mathbf{u}, p)$ *of (3.8) in* $L^2(0,T : [L^2(\Omega)]^2)$ *as* $\delta \to 0$*.*

Proof. Let us define the local Leray projector P_0 in the Fourier space

$$\widehat{P_0}(\xi) = I - \frac{\xi\xi^T}{|\xi|^2}.$$

The local solutions u and p are then given by

$$\widehat{\mathbf{u}}(\xi,t) = \widehat{\mathbf{u}_0}(\xi)\exp(-|\xi|^2 t) + \int_0^t \exp(-|\xi|^2(t-s))\widehat{P_0}(\xi)\widehat{\mathbf{f}}(\xi,s)\,\mathrm{d}s$$

and

$$\widehat{p}(\xi,t) = \frac{-i\xi^T}{|\xi|^2}(I - \widehat{P_0}(\xi))\widehat{\mathbf{f}}(\xi,t).$$

We first consider

$$\widehat{\mathbf{u}_\delta^{\vec{n}}}(\xi,t)-\widehat{\mathbf{u}}(\xi,t)=\widehat{\mathbf{u}_{0,\delta}^{\vec{n}}}(\xi)\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2t)-\widehat{\mathbf{u}_0}(\xi)\exp(-|\xi|^2t)$$
$$+\int_0^t\Big(\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2(t-s))\widehat{P}_\delta^{\vec{n}}(\xi)-\exp(-|\xi|^2(t-s))P_0(\xi)\Big)\widehat{\mathbf{f}}(\xi,s)\,\mathrm{d}s$$

where $\widehat{P_\delta^{\vec{n}}}(\xi)$ is the nonlocal Leray operator used in the proof of Proposition 3.3, hence

$$\int_0^T\|\mathbf{u}_\delta^{\vec{n}}-\mathbf{u}\|^2_{[L^2(\Omega)]^2}\mathrm{d}t\le C_T\int_0^T\Bigg(\sum_{\xi\in\mathbb{Z}^2,\xi\neq0}|\widehat{\mathbf{u}_0}(\xi)|^2\left|\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2t)-\exp(-|\xi|^2t)\right|^2+|\widehat{\mathbf{u}_{0,\delta}^{\vec{n}}}(\xi)-\widehat{\mathbf{u}_0}(\xi)|^2$$
$$\times|\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2t)|^2+\int_0^t\Big|\Big(\exp(-|\lambda_\delta^{\vec{n}}(\xi)|^2(t-s))\widehat{P}_\delta^{\vec{n}}(\xi)$$
$$-\ \exp(-|\xi|^2(t-s))\widehat{P}_0(\xi)\Big)\widehat{\mathbf{f}}(\xi,s)\Big|^2\,\mathrm{d}s\Bigg)\,\mathrm{d}t$$

where C_T is a constant depending only on T. We observe that since $\mathbf{u}_{0,\delta}^{\vec{n}}\to\mathbf{u}_0$ in $[L^2(\Omega)]^2$ as $\delta\to0$ the integrand in the parentheses can be bounded, uniformly in δ (as $\delta\to0$) and t, by some constant C depending only on $\mathbf{u}_0$ and $\mathbf{f}$. One can easily verify

$$|\lambda_\delta^{\vec{n}}(\xi)-i\xi|\to0,\ \text{hence}\ \left|\widehat{P}_\delta^{\vec{n}}(\xi)-\widehat{P}_0(\xi)\right|\le\left|\widehat{P}_\delta^{\vec{n}}(\xi)-\widehat{P}_0(\xi)\right|_F\to0\ \text{as}\ \delta\to0\ \text{for each}\ \xi\in\mathbb{Z}^2,\xi\neq0$$

so that it follows the dominated convergence theorem yields $\mathbf{u}_\delta^{\vec{n}}\to\mathbf{u}$ in $L^2(0,T:[L^2(\Omega)]^2)$ as $\delta\to0$.

We apply the similar argument to the expression

$$\int_0^T\|p_\delta^{\vec{n}}-p\|^2_{L^2(\Omega)}\mathrm{d}t=\int_0^T\sum_{\xi\in\mathbb{Z}^2,\xi\neq0}\left(\frac{\overline{\lambda_\delta^{\vec{n}}(\xi)}^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}(I-\widehat{P_\delta}(\xi))\widehat{\mathbf{f}}(\xi,t)+\frac{i\xi^T}{|\xi|^2}(I-\widehat{P_0}(\xi))\widehat{\mathbf{f}}(\xi,t)\right)^2\mathrm{d}t$$

to conclude $p_\delta^{\vec{n}}\to p$ in $L^2(0,T:L^2(\Omega))$ as $\delta\to0$.

Lastly we remark that nonlocally divergence free initial velocity $\mathbf{u}_{0,\delta}^{\vec{n}}$ as in the assumption of the theorem can be explicitly constructed by taking $\widehat{\mathbf{u}_{0,\delta}^{\vec{n}}}(\xi)=\widehat{P_\delta^{\vec{n}}}(\xi)\widehat{\mathbf{u_0}}(\xi)$. □

We note that one can also get the order of convergence as in the time-independent case. The details are omitted.

3.2. Nonlocal Helmholtz decomposition

The nonlocal Leray operators introduced in the proof of Proposition 3.3 clearly imply a nonlocal version of the classical Helmholtz decomposition theorem which warrants a more detailed discussion.

We begin with the following two dimensional result.

Theorem 3.5. *If $\mathbf{u}:\Omega\to\mathbb{R}^2$ is square integrable and periodic with zero mean, then there exist unique periodic, zero mean scalar potentials $p_\delta^{\vec{n}},q_\delta^{\vec{n}}\in S_\delta^{\vec{n}}(\Omega)$ such that*

$$\mathbf{u}(x)=\mathcal{G}_\delta^{\vec{n}}p_\delta^{\vec{n}}(x)+\begin{bmatrix}0&-1\\1&0\end{bmatrix}\mathcal{G}_\delta^{-\vec{n}}q_\delta^{\vec{n}}(x)$$

with $\mathcal{D}_\delta^{\vec{n}}\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\mathcal{G}_\delta^{-\vec{n}} q_\delta^{\vec{n}}\right)(x) = 0$. *In addition we have the estimate*

$$\|p_\delta^{\vec{n}}\|_{S_\delta^{\vec{n}}(\Omega)} + \|q_\delta^{\vec{n}}\|_{S_\delta^{\vec{n}}(\Omega)} \le C\|\mathbf{u}\|_{[L^2(\Omega)]^2}$$

for some constant C *independent of* $\vec{n}$ *and also of* δ *as* $\delta \to 0$. *Here* $S_\delta^{\vec{n}}(\Omega)$ *is the energy space as in Lemma 3.1.*

Proof. In the Fourier space the unique solutions are given by

$$\widehat{p_\delta^{\vec{n}}}(\xi) = -\frac{\lambda_\delta^{-\vec{n}}(\xi)^T\widehat{\mathbf{u}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \quad \text{and} \quad \widehat{q_\delta^{\vec{n}}}(\xi) = -\frac{\lambda_\delta^{\vec{n}}(\xi)^T}{|\lambda_\delta^{-\vec{n}}(\xi)|^2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\left(I + \frac{\lambda_\delta^{\vec{n}}(\xi)\lambda_\delta^{-\vec{n}}(\xi)^T}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\right)\widehat{\mathbf{u}}(\xi)$$

and the rest of the claim is due to $\lambda_\delta^{-\vec{n}}(\xi) = -\overline{\lambda_\delta^{\vec{n}}(\xi)}$ and Theorem 2.4. ☐

In three dimensions we introduce nonlocal curl operators

$$\mathcal{C}_\delta^{\vec{n}} v(x) = 2\int_{\mathbb{R}^3} \chi_{\vec{n}}(y-x)\underline{\omega}_\delta(y-x)(y-x)\times\frac{v(y)-v(x)}{|y-x|}\,\mathrm{d}y,$$

based on which we deduce the following.

Theorem 3.6. *If* $\mathbf{u} : \widetilde{\Omega} = (-\pi,\pi)^3 \to \mathbb{R}^3$ *is square integrable and periodic with zero mean, then there exist unique periodic, zero mean scalar and vector potentials* $p_\delta^{\vec{n}} \in S_\delta^{\vec{n}}(\widetilde{\Omega})$ *and* $\mathbf{v}_\delta^{\vec{n}} \in [S_\delta^{\vec{n}}(\widetilde{\Omega})]^3$*, respectively, such that*

$$\mathbf{u}(x) = \mathcal{G}_\delta^{\vec{n}} p_\delta^{\vec{n}}(x) + \mathcal{C}_\delta^{-\vec{n}}\mathbf{v}_\delta^{\vec{n}}(x)$$

with the nonlocal Gauge condition $\mathcal{D}_\delta^{-\vec{n}}\mathbf{v}_\delta^{\vec{n}}(x) = 0$. *Moreover* $(\mathcal{C}_\delta^{\vec{n}} \circ \mathcal{G}_\delta^{\vec{n}})p_\delta^{\vec{n}}(x)$ *and* $(\mathcal{D}_\delta^{\vec{n}} \circ \mathcal{C}_\delta^{-\vec{n}})\mathbf{v}_\delta^{\vec{n}}(x)$ *vanish along with the analogous estimate as in Theorem 3.5 with* $\mathbf{v}_\delta^{\vec{n}}$ *in place of* $p_\delta^{\vec{n}}$.

Proof. One can verify that a nonlocal vector identity

$$(\mathcal{C}_\delta^{-\vec{n}} \circ \mathcal{C}_\delta^{\vec{n}})\mathbf{f}(x) = (\mathcal{G}_\delta^{\vec{n}} \circ \mathcal{D}_\delta^{\vec{n}})\mathbf{f}(x) - \mathcal{L}_\delta^{\vec{n}}\mathbf{f}(x)$$

holds for any periodic $\mathbf{f} : (-\pi,\pi)^3 \to \mathbb{R}^3$. Solving for $\mathbf{f}$ in $-\mathcal{L}_\delta^{\vec{n}}\mathbf{f} = \mathbf{u}$ then yields the Fourier representations of the unique solutions

$$\widehat{p}_\delta^{\vec{n}}(\xi) = -\frac{\lambda_\delta^{-\vec{n}}(\xi)^T\widehat{\mathbf{u}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2} \quad \text{and} \quad \widehat{\mathbf{v}}_\delta^{\vec{n}}(\xi) = \frac{\lambda_\delta^{\vec{n}}(\xi)}{|\lambda_\delta^{\vec{n}}(\xi)|^2}\times\widehat{\mathbf{u}}(\xi).$$

We refer to the proof of Theorem 3.5 for the rest of the claim. ☐

Let us point out that the proof of Theorem 3.6 reveals the well-posedness of a nonlocal version of the classical first order div-curl elliptic system as stated below.

Theorem 3.7. *Given periodic, zero mean data* $f \in L^2((-\pi,\pi)^3)$ *and* $\mathbf{g} \in [L^2((-\pi,\pi)^3)]^3$ *with* $\mathcal{D}_\delta^{-\vec{n}}\mathbf{g} = 0$, *there exist a unique periodic, zero mean vector field* $\mathbf{u} \in [S_\delta^{\vec{n}}((-\pi,\pi)^3)]^3$ *satisfying the following nonlocal div-curl system*

$$\begin{aligned} \mathcal{D}_\delta^{\vec{n}}\mathbf{u} &= f \text{ in } (-\pi,\pi)^3\\ \mathcal{C}_\delta^{\vec{n}}\mathbf{u} &= \mathbf{g} \text{ in } (-\pi,\pi)^3 \end{aligned} \tag{3.9}$$

and the estimate

$$\|\mathbf{u}\|_{[S_\delta^{\vec{n}}((-\pi,\pi)^3)]^3} \le C\left(\|f\|_{L^2((-\pi,\pi)^3)} + \|\mathbf{g}\|_{[L^2((-\pi,\pi)^3)]^3}\right) \tag{3.10}$$

for some positive constant C *independent of* $\mathbf{u}$, $\vec{n}$ *and* δ *(as* $\delta \to 0$*).*

The estimate (3.10) is a nonlocal version of the second Friedrichs inequality [30,46], which can be easily shown with the help of Theorem 2.4. We omit the details.

We note that classical Helmholtz decomposition have wide applications of mechanics and electromagnetics. For nonlocal and fractional versions, one can also check [49]. The further study of nonlocal div-curl systems, including nonlocal versions of the div-curl lemma [10, 40], may also be of interests and will be left for future works.

3.3. Nonlocal correspondence models of isotropic linear elasticity

We study the elastic potential energy given by

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) = \frac{1}{2}\lambda\|\mathcal{D}_\delta^{\vec{n}}\mathbf{u}(\mathbf{x})\|_2^2 + \mu\|e_\delta^{\vec{n}}(\mathbf{u}(\mathbf{x}))\|_2^2 \tag{3.11}$$

where λ, μ are Lamé coefficients and $e_\delta^{\vec{n}}(\mathbf{u})$ is the nonlocal strain tensor

$$e_\delta^{\vec{n}}(\mathbf{u}) = \frac{\mathcal{G}_\delta^{\vec{n}}\mathbf{u} + (\mathcal{G}_\delta^{\vec{n}}\mathbf{u})^T}{2}$$

for a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$. This can be viewed as a so-called nonlocal correspondence model where the local stress tensors in the local energy density are replaced by the nonlocal counterparts [14,48]. We assume $\mu > 0$ and $\lambda + 2\mu > 0$ for which it is well known that the corresponding local elastic energy is variationally stable over the energy space

$$V_0 = \{\mathbf{u} \in \mathbf{H}^1(\Omega) | \mathbf{u} \text{ is periodic with zero mean}\}, \qquad \|\cdot\|_{V_0} = \|\cdot\|_{\mathbf{H}^1(\Omega)},$$

where $\mathbf{H}^1(\Omega)$ denotes the standard $\mathbf{R}^2$-valued Sobolev space. The energy $\mathcal{E}_\delta^{\vec{n}}$ can be seen as a linear elastic potential energy of isotropic materials in the peridynamics correspondence theory [47,48]. As in the one dimensional case with radially symmetric interaction kernels [14], let us first define the nonlocal energy space $V_\delta^{\vec{n}}$ to be the closure of C^∞, zero mean, periodic $\mathbb{R}^2$-valued functions on Ω with respect to the norm

$$\|\mathbf{u}\|_{V_\delta^{\vec{n}}} = (\|\mathbf{u}\|_2^2 + \mathcal{E}_\delta^{\vec{n}}(\mathbf{u}))^{1/2}.$$

A precise statement of the variational stability of $\mathcal{E}_\delta^{\vec{n}}$ is then given by

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) \geq C\|\mathbf{u}\|_{V_\delta^{\vec{n}}}^2 \qquad \forall \mathbf{u} \in V_\delta^{\vec{n}} \tag{3.12}$$

for some constant $C > 0$ independent of $\vec{n}$ and δ, as $\delta \to 0$. In order to establish the stability let us introduce the nonlocal Naviér operator $M_\delta^{\vec{n}}$ defined as

$$M_\delta^{\vec{n}}(\mathbf{u}) = -\mu\mathcal{L}_\delta^{\vec{n}}\mathbf{u} - (\lambda + \mu)\mathcal{G}_\delta^{\vec{n}}(\mathcal{D}_\delta^{\vec{n}}\mathbf{u}) \tag{3.13}$$

so that for $\mathbf{u} = (u_1, u_2) \in V_\delta^{\vec{n}}$

$$(M_\delta^{\vec{n}}(\mathbf{u}), \mathbf{u})_2 = \lambda\int_\Omega |\mathcal{D}_\delta^{\vec{n}}\mathbf{u}(\mathbf{x})|^2 d\mathbf{x} + \mu\int_\Omega \left(|\mathcal{D}_\delta^{\vec{n}}\mathbf{u}(\mathbf{x})|^2 + \sum_{i=1,2}|\mathcal{G}_\delta^{\vec{n}}u_i(\mathbf{x})|^2\right) \mathrm{d}\mathbf{x} = 2\mathcal{E}_\delta^{\vec{n}}(\mathbf{u})$$

due to (2.5). We observe from Lemma 2.2 $\mathcal{D}_\delta^{\vec{n}}\mathbf{u} = \mathrm{Tr}(e_\delta^{-\vec{n}}(\mathbf{u}))$ is not in general equal to $\mathrm{Tr}(e_\delta^{\vec{n}}(\mathbf{u}))$ where Tr denotes the trace operator, which is different from the local case.

Similar to the studies on the classical Korn's inequality and the nonlocal versions in [29,36,41], we have a nonlocal version for the nonlocal Naviér system as follows.

Lemma 3.8 (Nonlocal Korn's inequality)**.** *There exists a constant $C > 0$ independent of $\vec{n}$ and δ such that*

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) \geq C\|\mathcal{G}_\delta^{\vec{n}}\mathbf{u}(\mathbf{x})\|_2^2 \qquad \forall \mathbf{u} \in V_\delta^{\vec{n}}.$$

Proof. We can see

$$\begin{aligned}\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) = \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2,\xi\neq(0,0)} M_\delta^{\vec{n}}(\xi)\widehat{\mathbf{u}}(\xi)\cdot\widehat{\mathbf{u}}(\xi) &= \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2,\xi\neq(0,0)} \mu|\lambda_\delta^{\vec{n}}(\xi)|^2|\widehat{\mathbf{u}}(\xi)|^2 + (\mu+\lambda)|\lambda_\delta^{\vec{n}}(\xi)\cdot\widehat{\mathbf{u}}(\xi)|^2 \\ &\geq \frac{1}{2}\min(\mu,\lambda+2\mu)\sum_{\xi\in\mathbb{Z}^2,\xi\neq(0,0)} |\lambda_\delta^{\vec{n}}(\xi)|^2|\widehat{\mathbf{u}}(\xi)|^2\end{aligned}$$

which proves the claim. □

We point out that by setting $\lambda = 0$, one can recover the periodic versions of nonlocal Korn's inequality studied in [36, 41].

Now using Theorem 2.4, which serves like a nonlocal Poincare inequality [13, 38], we readily have the following coercivity result and the well-posedness.

Lemma 3.9. *There exists a constant $C > 0$ independent of $\vec{n}$ and δ as $\delta \to 0$ such that*

$$\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) \geq C\|\mathbf{u}\|_2^2 \qquad \forall \mathbf{u} \in V_\delta^{\vec{n}}.$$

Moreover, the problem

$$M_\delta^{\vec{n}}(\mathbf{u}) = \mathbf{f} \tag{3.14}$$

is well-posed over $V_\delta^{\vec{n}}$ where $\mathbf{f} \in \mathbf{L^2}(\Omega)$ is periodic with zero mean.

We note that the same remains valid for $\mathbf{f}$ belonging to the dual space $(V_\delta^{\vec{n}})^*$. Indeed, given the equivalence of $\mathcal{E}_\delta^{\vec{n}}(\cdot)$ with $\|\cdot\|_{V_\delta^{\vec{n}}}$ due to Lemma 3.9, an explicit characterization of the dual space with the L^2 duality pairing can also be obtained as done in [57]. Not only is the nonlocal solution $\mathbf{u}_\delta^{\vec{n}} \in V_\delta^{\vec{n}}$ to (3.14) important in its own right as a unique minimizer of $\mathcal{E}_\delta^{\vec{n}}(\mathbf{u}) - (\mathbf{f}, \mathbf{u})_2$ but it can also be shown to recover the corresponding local solution to the local Naviér equation, when the latter is well-posed. To this end let us first present the following embedding result.

Lemma 3.10. *There exists a constant C independent of δ and $\vec{n}$ such that*

$$\|\mathbf{u}\|_{V_\delta^{\vec{n}}} \leq C\|\mathbf{u}\|_{V_0} \quad \forall \mathbf{u} \in V_0.$$

Proof. As can be observed in the proof of Lemma 3.8, we have

$$M_\delta^{\vec{n}}(\xi)\widehat{\mathbf{u}}(\xi)\cdot\widehat{\mathbf{u}}(\xi) \leq \max(\mu,\lambda+2\mu)|\lambda_\delta^{\vec{n}}(\xi)|^2|\widehat{\mathbf{u}}(\xi)|^2$$

to which applying Theorem 2.4 proves the claim. □

Not only is it in $\|\cdot\|_2$ but also in the nonlocal norm $\|\cdot\|_{V_\delta^{\vec{n}}}$ that we have the convergence of the nonlocal solution $\mathbf{u}_\delta$ to its local counterpart.

Proposition 3.11. *Assume $\mathbf{f} \in \mathbf{L^2}(\Omega)$ is periodic with zero mean. Let $\mathbf{u}_\delta^{\vec{n}}$ denote the solution of the nonlocal Naviér equation (3.14). Then there exists a constant C independent of δ and $\vec{n}$ as $\delta \to 0$ such that*

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_2 \leq C\delta\|\mathbf{f}\|_2$$

where $\mathbf{u}$ is the solution to the local Naviér equation

$$M_0(\mathbf{u}) := -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} = \mathbf{f} \qquad \text{in} \quad \Omega.$$

If we further assume that $\mathbf{f}$ belongs to the fractional Sobolev space $\mathbf{H}^{\frac{1}{2}}(\Omega)$, then we have

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_{V_\delta^{\vec{n}}} \leq C\delta\|\mathbf{f}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)}.$$

Proof. Let us first write

$$\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_2^2 \le \sum_{\xi \in \mathbb{Z}^2, \xi \ne (0,0)} \left| (\widehat{M}_\delta^{\vec{n}}(\xi))^{-1} - (\widehat{M}_0(\xi))^{-1} \right|^2 \left| \widehat{\mathbf{f}}(\xi) \right|^2 .$$

Based on the explicit expressions for the matrices $M_\delta^{\vec{n}}(\xi)$ and $M_0(\xi)$, we can apply Theorem 2.4 and the following estimate shown in the proof of Proposition 3.2

$$\left| \frac{1}{|\lambda_\delta^{\vec{n}}(\xi)|} - \frac{1}{|\xi|} \right| \le C\delta \qquad (\star)$$

to verify that

$$|(\widehat{M}_\delta^{\vec{n}}(\xi))^{-1} - (\widehat{M}_0(\xi))^{-1}|_F \le C\delta$$

where $|\cdot|_F$ denotes the Frobenius norm, from which the convergence in L^2 follows. We next consider

$$\begin{aligned}\|\mathbf{u}_\delta^{\vec{n}} - \mathbf{u}\|_{V_\delta^{\vec{n}}}^2 &= \sum_{\xi \in \mathbb{Z}^2, \xi \ne (0,0)} (I - \widehat{M}_\delta^{\vec{n}}(\xi)(\widehat{M}_0(\xi))^{-1})\widehat{\mathbf{f}}(\xi) \cdot ((\widehat{M}_\delta^{\vec{n}}(\xi))^{-1} - (\widehat{M}_0(\xi))^{-1})\widehat{\mathbf{f}}(\xi) \\ &\le C\delta \sum_{\xi \in \mathbb{Z}^2, \xi \ne (0,0)} \left| I - \widehat{M}_\delta^{\vec{n}}(\xi)(\widehat{M}_0(\xi))^{-1}) \right| \left| \widehat{\mathbf{f}}(\xi) \right|^2 .\end{aligned}$$

If we denote $\xi = (\xi_1, \xi_2)$ and $\lambda_\delta^{\vec{n}}(\xi) = ([\lambda_\delta^{\vec{n}}(\xi)]_1, [\lambda_\delta^{\vec{n}}(\xi)]_2)$, Theorem 2.4 and the estimate $(\star)$ from above allow us to obtain

$$\begin{aligned}&\left| I - \widehat{M}_\delta^{\vec{n}}(\xi)(\widehat{M}_0(\xi))^{-1} \right| \\ &\le \left| I - \frac{1}{\mu(\lambda+2\mu)|\xi|^4} \begin{bmatrix} (\lambda+2\mu)|[\lambda_\delta^{\vec{n}}(\xi)]_1|^2 + \mu|[\lambda_\delta^{\vec{n}}(\xi)]_2|^2 & (\lambda+\mu)[\lambda_\delta^{\vec{n}}(\xi)]_1\overline{[\lambda_\delta^{\vec{n}}(\xi)]_2} \\ (\lambda+\mu)\overline{[\lambda_\delta^{\vec{n}}(\xi)]_1}[\lambda_\delta^{\vec{n}}(\xi)]_2 & \mu|[\lambda_\delta^{\vec{n}}(\xi)]_1|^2 + (\lambda+2\mu)|[\lambda_\delta^{\vec{n}}(\xi)]_2|^2 \end{bmatrix} \right. \\ &\qquad \left. \cdot \begin{bmatrix} \mu\xi_1^2 + (\lambda+2\mu)\xi_2^2 & -(\lambda+\mu)\xi_1\xi_2 \\ -(\lambda+\mu)\xi_1\xi_2 & \mu\xi_2^2 + (\lambda+2\mu)\xi_1^2 \end{bmatrix} \right|_F \\ &\le C\delta|\xi|,\end{aligned}$$

which proves the convergence in $V_\delta^{\vec{n}}$. ◻

As in the previous section on the nonlocal Stokes equation, we now consider the time dependent nonlocal Naviér equation

$$\begin{aligned}(\mathbf{u}_\delta^{\vec{n}})_{tt} + M_\delta^{\vec{n}}\mathbf{u}_\delta^{\vec{n}} &= \mathbf{f} \text{ in } (0,T)\times\Omega \\ \mathbf{u}_\delta^{\vec{n}}|_{t=0} &= \mathbf{g} \text{ in } \Omega \\ (\mathbf{u}_\delta^{\vec{n}})_t|_{t=0} &= \mathbf{h} \text{ in } \Omega \end{aligned} \tag{3.15}$$

in juxtaposition with the local Naviér equation

$$\begin{aligned}\mathbf{u}_{tt} + M_0\mathbf{u} &= \mathbf{f} \text{ in } (0,T)\times\Omega \\ \mathbf{u}|_{t=0} &= \mathbf{g} \text{ in } \Omega \\ \mathbf{u}_t|_{t=0} &= \mathbf{h} \text{ in } \Omega \end{aligned} \tag{3.16}$$

where all the local and nonlocal field variables as well as the data are assumed to be periodic on Ω with zero means. Since the Hermitian matrix $P_\delta^{\vec{n}}$ is positive definite, we can apply the similar argument as in [16] to establish the following well-posedness result for which we omit the details.

Proposition 3.12. *Suppose* $\mathbf{g} \in V_\delta^{\vec{n}}, \mathbf{h} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^2(0,T : L^2(\Omega))$*. Then there exists a unique solution* $\mathbf{u}_\delta^{\vec{n}}$ *to (3.15) such that*

$$\mathbf{u}_\delta^{\vec{n}} \in C(0,T : V_\delta^{\vec{n}}), \quad (\mathbf{u}_\delta^{\vec{n}})_t \in L^2(0,T : L^2(\Omega)).$$

For completeness we consider convergence of the time dependent nonlocal solution $\mathbf{u}_\delta^{\vec{n}}$ to the corresponding local solution as $\delta \to 0$. As in the steady case this can be readily established using the explicit Fourier representations of both nonlocal and local solutions.

Proposition 3.13. *Suppose* $\mathbf{g} \in V_0(\Omega), \mathbf{h} \in L^2(\Omega)$ *and* $\mathbf{f} \in L^2(0,T : L^2(\Omega))$*. Let* $\mathbf{u}_\delta^{\vec{n}}$ *and* $\mathbf{u}$ *denote the solutions of nonlocal and local Naviér equations, respectively, with the same initial displacement field* $\mathbf{g}$ *and velocity field* $\mathbf{h}$*. Then we have* $\mathbf{u}_\delta^{\vec{n}} \to \mathbf{u}$ *in* $L^2(0,T : V_\delta^{\vec{n}}(\Omega)) \bigcap H^1(0,T : L^2(\Omega))$*.*

Proof. We use the explicit Fourier representation of the solutions as given in the proof of Theorem 2.25 in [16] to write

$$\widehat{\mathbf{u}_\delta^{\vec{n}}}(t,\xi) - \widehat{\mathbf{u}}(t,\xi) = \underbrace{\left(\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)}_{\widehat{\mathbf{u}}_{c1}} + \underbrace{\left(\frac{\sin\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right)}{\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)}} - \frac{\sin\left(\sqrt{\widehat{M_0}(\xi)t}\right)}{\sqrt{\widehat{M_0}(\xi)}}\right)\widehat{\mathbf{h}}(\xi)}_{\widehat{\mathbf{u}}_{c2}}$$

$$+ \underbrace{\int_0^t \left(\frac{\sin\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)s}\right)}{\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)}} - \frac{\sin\left(\sqrt{\widehat{M_0}(\xi)s}\right)}{\sqrt{\widehat{M_0}(\xi)}}\right)\widehat{\mathbf{f}}(t-s,\xi)\,\mathrm{d}s}_{\widehat{\mathbf{u}}_{c3}}.$$

We cannot argue as in [16] since the operators $M_\delta^{\vec{n}}$ and M_0 do not commute, hence not simultaneously diagonalizable. Instead we argue directly as follows. We have

$$\begin{aligned}
&\widehat{M_\delta^{\vec{n}}}(\xi)\left(\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\cdot\left(\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\\
&\le C\widehat{M_0}(\xi)\left(\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\cdot\left(\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right)\widehat{\mathbf{g}}(\xi)\\
&\le C|\widehat{M_0}(\xi)||\widehat{\mathbf{g}}(\xi)|^2
\end{aligned}$$

where the second inequality is due to Lemma 3.10, hence $\|\mathbf{u}_{c1}\|^2_{V_\delta^{\vec{n}}} \le C\|\mathbf{g}\|^2_{V_0}$ uniformly in $t \in [0,T]$, which in turn implies

$$\int_0^T \|\mathbf{u}_{c1}\|^2_{V_\delta^{\vec{n}}}\mathrm{d}t \le CT\|\mathbf{g}\|^2_{V_0}.$$

Let us now note that for each fixed $\xi \in \mathbb{Z}^2, \xi \neq (0,0)$ and $t \in (0,T)$ we get

$$\left|\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right| \le \left|\cos\left(\sqrt{\widehat{M_\delta^{\vec{n}}}(\xi)t}\right) - \cos\left(\sqrt{\widehat{M_0}(\xi)t}\right)\right|_F \to 0 \quad \text{as } \delta \to 0$$

since it can be easily checked that $|\lambda_\delta^{\vec{n}}(\xi) - i\xi| \to 0$, hence $\left|\widehat{M_\delta^{\vec{n}}}(\xi) - \widehat{M_0}(\xi)\right|_F \to 0$, as $\delta \to 0$. The dominated convergence theorem then implies

$$\int_0^T \|\mathbf{u}_{c1}\|^2_{V_\delta^{\vec{n}}}\mathrm{d}t \to 0 \quad \text{as } \delta \to 0.$$

Similarly the rest of the claims can be proved and we omit the details. $\square$

4. Another look at the nonlocal gradient operators

We are motivated to revisit the nonlocal operators upon juxtaposing our application results in Section 3 with the popular practice in nonlocal modelling of taking into account a full spherical neighborhood. In more specific terms we consider the queston of whether there is a connection between the nonlocal diffusion operator based on the non-symmetric formulations of nonlocal divergence and gradient and other existing formulations of nonlocal diffusion (Laplacian) operator. We begin by recalling from [14] that symmetric gradient operator is closely related to a bond-based, in the language of peridynamics, nonlocal diffusion operator. We show that the similar conclusion can be drawn in our formulation with some choices of kernels.

To illustrate this point it is sufficient to consider the one dimensional setting wherein the nonlocal Dirichlet integrals are

$$\mathcal{E}_\delta^{\pm}(u) = \int_{-\pi}^{\pi} |\mathcal{G}_\delta^{\pm}(u)(x)|^2 \,\mathrm{d}x$$

for which we have the following.

Lemma 4.1. *For a smoothly defined periodic $u : (-\pi, \pi) \to \mathbb{R}$, if $w_\delta(|x|)$ is integrable then*

$$\mathcal{E}_\delta^{+}(u) = \mathcal{E}_\delta^{-}(u) = 2\int_{-\pi}^{\pi}\int_{0}^{\delta} \rho_\delta(a)\left|\frac{u(x+a)-u(x)}{a}\right|^2 \mathrm{d}a\,\mathrm{d}x$$

where $\rho_\delta(a) = \rho_\delta(|a|)$ is a radial (even) function given by

$$\rho_\delta(a) = 2a^2\int_0^\delta w_\delta(b)\left(w_\delta(a) - w_\delta\left(a+b\right)\right)\mathrm{d}b = -a^2(\mathcal{G}_\delta^{+} w_\delta)(a), \quad \forall a \in (0,\delta) \tag{4.1}$$

supported on $(-\delta,\delta)$ and satisfies $\|\rho_\delta\|_{L^1} = 1$. Moreover if ω is non-increasing in $(0,1)$, then ρ_δ is non-negative.

Proof. Lemma 2.3 shows $|\lambda_\delta^{\vec{n}}(\xi)| = |\lambda_\delta^{-\vec{n}}(\xi)|$, thus it is immediate to see $\mathcal{E}_\delta^{+}(u) = \mathcal{E}_\delta^{-}(u)$. We consider only $\mathcal{E}_\delta^{+}(u)$ and follow the argument in [14] to write $D_\delta = [-\delta,0]^2 \bigcup [0,\delta]^2$ so that

$$\mathcal{E}_\delta^{+}(u) = \underbrace{2\int_{-\pi}^{\pi}\int_{D_\delta} w_\delta(|s|)w_\delta(|t|)(u(x+t)-u(x))^2 \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x}_{I_1} + \dots$$
$$\underbrace{-\int_{-\pi}^{\pi}\int_{D_\delta} w_\delta(|s|)w_\delta(|t|)((u(x+s)-u(x+t))^2 \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x}_{I_2}.$$

Let us first observe

$$I_1 = \int_{-\pi}^{\pi}\int_{-\delta}^{\delta} k_\delta(|a|)\left|\frac{u(x+a)-u(x)}{a}\right|^2 \mathrm{d}a\,\mathrm{d}x$$

where

$$k_\delta(|a|) = 2w_\delta(|a|)|a|^2\int_0^\delta w_\delta(|b|)\,\mathrm{d}b.$$

Next we consider I_2 and rewrite it as

$$I_2 = -\int_{-\pi}^{\pi}\int_{D_\delta} w_\delta(|s|)w_\delta(|t|)((u(x+s-t)-u(x))^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}y$$
$$= -\int_{-\pi}^{\pi}\int_{\hat{D}_\delta} \frac{a^2}{2} w_\delta\left(\left|\frac{a+b}{2}\right|\right) w_\delta\left(\left|\frac{b-a}{2}\right|\right)\frac{((u(x+a)-u(x))^2}{a^2}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}x$$

where we have used the periodicity of u in the first equality and the change of variables $a = s-t$ and $b = s+t$ in the second with the corresponding change of integration domain from D_δ to $\hat{D}_\delta$. That is we have

$$I_2 = \int_{-\pi}^{\pi}\int_{-\delta}^{\delta} h_\delta(|a|)\frac{(u(y+a)-u(y))^2}{a^2}\,\mathrm{d}a\,\mathrm{d}x$$

where

$$h_\delta(|a|) = -\frac{a^2}{2}\left(\int_{|a|}^{-|a|+2\delta} + \int_{|a|-2\delta}^{-|a|}\right) w_\delta\left(\left|\frac{a+b}{2}\right|\right) w_\delta\left(\left|\frac{b-a}{2}\right|\right)\,\mathrm{d}b = -a^2\int_{|a|}^{-|a|+2\delta} w_\delta\left(\left|\frac{a+b}{2}\right|\right) w_\delta\left(\left|\frac{b-a}{2}\right|\right)\,\mathrm{d}b.$$

We note the almost everywhere finiteness of $h_\delta(|a|)$ since

$$\int_{-a}^{a} h_\delta(|a|)\,\mathrm{d}a = 1 - \left(\int_{-\delta}^{\delta} s^2 w_\delta(|s|)\,\mathrm{d}s\right)\left(\int_{-\delta}^{\delta} w_\delta(|t|)\mathrm{d}t\right)$$

which can be easily verified. Then the desired claim follows from setting $\rho_\delta = k_\delta + h_\delta$ and observing that for $0 < a < \delta$,

$$h_\delta(|a|) = -2a^2\int_0^{\delta-a} w_\delta(|z+a|)w_\delta(|z|)\,\mathrm{d}z = -2a^2\int_0^{\delta} w_\delta(|z+a|)w_\delta(|z|)\,\mathrm{d}z$$

where the last equality holds since w_δ is supported on $(-\delta,\delta)$, and analogously for $-\delta < a < 0$.

Finally it is clear from (4.1) that ρ_δ is non-negative for non-increasing ω_δ. $\square$

We note that Lemma 4.1 remains valid for any periodic function u with well-defined $\mathcal{E}_\delta^\pm(u)$. Naturally, one may be interested in extending the result of Lemma 4.1 to more general kernels w. Let us consider first a special example

$$w(|x|) = \begin{cases} \frac{C_\beta}{|x|^\beta}, & |x| \le 1, \\ 0, & \text{otherwise} \end{cases}$$

for $1 \le \beta < 2$, and $C_\beta > 0$ is chosen to satisfy the moment condition (2.3). Then the result of Lemma 4.1 also holds. Indeed, let us fix $0 < a < \delta$ without loss of generality. We then have

$$\frac{\rho_\delta(a)}{2a^2C_\beta^2} = \int_0^{\delta-a}\frac{1}{z^\beta}\left(\frac{1}{a^\beta}-\frac{1}{(a+z)^\beta}\right)\mathrm{d}z + \int_{\delta-a}^{\delta}\frac{1}{z^\beta}\frac{1}{a^\beta}\,\mathrm{d}z \le \int_0^{\delta-a}\frac{1}{z^\beta}\left(\frac{\beta\delta^{\beta-1}z}{a^\beta(a+z)^\beta}\right)\mathrm{d}z + \int_{\delta-a}^{\delta}\frac{1}{z^\beta}\frac{1}{a^\beta}\,\mathrm{d}z < \infty.$$

Moreover if we define

$$\rho_\delta^\epsilon(a) = \chi_{(\epsilon,\infty)}(|a|)2a^2\int_\epsilon^\delta w_\delta(b)\left(w_\delta(a) - w_\delta\left(a+\frac{a}{|a|}b\right)\right)\mathrm{d}b$$

which are nonnegative, monotonically increasing in ϵ, pointwise convergent approximations of ρ_δ as $\epsilon \to 0$, then a direct calculation shows

$$\|\rho_\delta\|_{L^1(\mathbb{R})} = \lim_{\epsilon\to 0} \int_{-\delta}^{\delta} \rho_\delta^\epsilon(a)\,\mathrm{d}a = 1$$

where the first equality is due to the Monotone Convergence Theorem.

We can extend the definition of ρ_δ to a more broader class of non-integrable kernels w_δ that include the above fractional ones as a special case. To this end let us assume that w_δ is non-increasing and consider

$$w_\delta^\epsilon(|x|) = \begin{cases} w_\delta(|x|), & |x| > \epsilon \\ \inf_{|y|\le\epsilon} w_\delta(|y|), & \text{otherwise} \end{cases}. \tag{4.2}$$

Note that the modified nonnegative and radial kernel w_δ^ϵ is integrable and also nonincreasing in $(0,\delta)$. Then

$$\rho_\delta^\epsilon(a) = 2a^2 \int_0^\delta w_\delta^\epsilon(b)\left(w_\delta^\epsilon(a) - w_\delta^\epsilon\left(a + \frac{a}{|a|}b\right)\right)\mathrm{d}b \tag{4.3}$$

is non-negative, monotonically increasing in ϵ and satisfies $\lim_{\epsilon\to 0}\|\tilde{\rho}_\delta^\epsilon\|_{L^1(\mathbb{R})} = 1$. Hence we may define

$$\rho_\delta(a) = \lim_{\epsilon\to 0} \rho_\delta^\epsilon(a) \tag{4.4}$$

which satisfies $\|\rho_\delta\|_{L^1(\mathbb{R})} = 1$ due to the Monotone Convergence Theorem. We thus get the following more general result.

Lemma 4.2. *For a smoothly defined periodic function $u : (-\pi,\pi) \to \mathbb{R}$, if $w_\delta(|x|)$ is non-increasing with bounded first moment then*

$$\mathcal{E}_\delta^+(u) = \mathcal{E}_\delta^-(u) = 2\int_{-\pi}^{\pi}\int_0^\delta \rho_\delta(a)\left|\frac{u(x+a)-u(x)}{a}\right|^2 \mathrm{d}a\,\mathrm{d}x$$

where $\rho_\delta(a) = \rho_\delta(|a|)$ is a radial (even) function defined by (4.2)–(4.4).

Again, the Lemma 4.2 remains valid for any periodic function u with well-defined $\mathcal{E}_\delta^\pm(u)$. We remark that the non-negativity of ρ_δ for non-increasing w_δ is in a clear contrast with the case of symmetric nonlocal gradient operators wherein the corresponding kernel is always sign changing. On the other hand it is easy to see that ρ_δ is not always non-negative and may change sign as in the case of

$$\rho_\delta(x) = \pi x^2 \sin(\pi|x|) + \frac{\pi^2 x^2}{4}\left((|x|-1)\cos(\pi x) - \frac{\sin(\pi|x|)}{\pi}\right)$$

which amounts to $w_\delta(x) = \frac{\pi}{2}\sin(\pi|x|)$ with $\delta = 1$.

As a corollary of Lemmas 4.1 and 4.2, we have the equivalence of the corresponding nonlocal diffusion operator based on the nonsymmetric gradient and divergence

$$\mathcal{L}_\delta^+ u(x) = (\mathcal{D}_\delta^+ \circ \mathcal{G}_\delta^+)u(x) = 4\int_0^\delta\int_0^\delta w_\delta(|s|)w_\delta(|t|)(u(x+s) - u(x) - u(x+s-t) + u(x-t))\,\mathrm{d}s\,\mathrm{d}t$$

and the conventional bond based nonlocal diffusion operator

$$\mathcal{L}_\delta u(x) = 2\int_{-\delta}^{\delta} k_\delta(|a|)(u(x+a) - u(x))\,\mathrm{d}a$$

upon setting $k_\delta(|a|) = \frac{\rho_\delta(|a|)}{a^2}$. At the same time, under the same assumptions on u and w_δ as in Lemma 4.1, we can further relate the operator $\mathcal{L}_\delta^+$ to another nonlocal diffusion operator in a similar form, namely a doubly nonlocal Laplace operator $\mathcal{L}_{\delta,\epsilon}^{\text{double}}$ proposed in [45]

$$\mathcal{L}_{\delta,\epsilon}^{\text{double}} u(x) = \int_{-\delta}^{\delta}\int_{-\epsilon}^{\epsilon} \gamma_\delta(y)\eta_\epsilon(r)(u(x+y+r) - u(x) - u(x+r) + u(x+y))\,\mathrm{d}y\,\mathrm{d}r.$$

Here γ_δ and η_ϵ are assumed to be radial, non-negative and compactly supported on $(-\delta,\delta)$ and $(-\epsilon,\epsilon)$, respectively. Further assumptions on the moments of the kernels are made, namely the normalized second moment of γ_δ and integrability of η_ϵ with unit mass. For clarity of comparison, we first observe

$$\mathcal{L}_{\delta,\epsilon}^{\text{double}} u(x) = (\mathcal{L}_\delta \circ \mathcal{A}_\epsilon)u(x) = (\mathcal{A}_\epsilon \circ \mathcal{L}_\delta)u(x)$$

if we let $k_\delta = \gamma_\delta$ in $\mathcal{L}_\delta$ and define the averaging operator $\mathcal{A}_\epsilon$ by

$$\mathcal{A}_\epsilon u(x) = \frac{1}{2}\int_{-\epsilon}^{\epsilon} \eta_\epsilon(|z|)(u(x+z) + u(x))\,\mathrm{d}z = \frac{1}{2}u(x) + \frac{1}{2}\int_{-\epsilon}^{\epsilon}\eta_\epsilon(|z|)u(x+z)\,\mathrm{d}z.$$

Evidently the operator $\mathcal{A}_\epsilon$ provides a simple averaging, and does not fundamentally alter the spectral properties of $\mathcal{L}_{\delta,\epsilon}^{\text{double}}$. Thus, while involving an extra kernel, it does not change the modeling capability overall.

5. Conclusion

We have studied nonlocal gradient operators $\mathcal{G}_\delta^{\vec{n}}$ wherein the support of a positive kernel is prescribed to be any half-sphere parameterized by a unit vector $\vec{n}$. This can be seen as extensions of the one-sided nonlocal derivative operators for scalar functions of a single variable. It is interesting to observe that our nonlocal gradient operators with nonspherical interaction neighborhood can be effectively applied to model inherently symmetric phenomena as illustrated in our study of nonlocal Naviér equation of isotropic linear elasticity and nonlocal Stokes models of incompressible viscous flows. We also remark that by removing any singular growth assumption on the kernels, our nonsymmetric gradient operators are well suited to numerical quadrature based discretizations.

This work demonstrates that the symmetry of the nonlocal interaction neighborhood is not essential for nonlocal modeling and the related mathematical theory. While the analysis is focused on the half-sphere case, one may study further extensions that may involve only sectors of the sphere such as those used [52] for nonlocal convection and in the studies of [3, 21, 24] on nonlocal variational problems. The analytical results in this work are largely based on the Fourier analysis which is limited to problems defined over periodic cells. On one hand it will be interesting to consider the analogue on more general domains with more general boundary conditions or nonlocal constraints, which could be facilitated by further developments of nonlocal vector calculus leading to the results like nonlocal vector identities that are tailored to our nonlocal operators, particularly so for Neumann and Robin conditions [13,17,18,38,39]. Without the Fourier analysis mathematical investigation of the associated nonlocal energy spaces will need to be carried out using different techniques and more sophisticated analytical tools, such as those in similar spirits to the works of [4, 37–39]. On the other hand our analysis on a periodic setting does have a direct impact on the scientific communities working on numerical methods that are based on nonlocal integro differential operators. For example, one such method is the SPH mentioned earlier, which has indeed been directly applied for numerical simulations on periodic domains [42, 58]. Among a myriad of choices of kernels our current work selects for researchers a particular family of kernels that are theoretically verified to be effective. Our study also imparts to the researchers the message that the underlying nonlocal continuum formulations are bona fide mathematical objects deserving to be scrutinized in view of the local setting wherein successful developments of numerical methods have been built upon rigorous theoretical understandings on the continuum level. Meanwhile we point out that the well-posedness of our nonlocal Stokes and Naviér equations in the periodic setting is naturally linked to the consideration of their Fourier spectral discretizations and related numerical issues such as the asymptotic compatibility [51] as in [15]. Further numerical analysis of other discretizations are also important for applications and will be left to future works.

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