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CONVERGENCE ANALYSIS OF A LDG METHOD FOR TEMPERED FRACTIONAL CONVECTION–DIFFUSION EQUATIONS

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Abstract. This paper proposes a local discontinuous Galerkin method for tempered fractional convection–diffusion equations. The tempered fractional convection–diffusion is converted to a system of the first order of differential/integral equation, then, the local discontinuous Galerkin method is employed to solve the system. The stability and order of convergence of the method are proven. The order of convergence $O(h^{k+1})$ depends on the choice of numerical fluxes. The provided numerical examples confirm the analysis of the numerical scheme.

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1. Introduction

Tempered fractional calculus is the generalization of fractional calculus [43]. One of the mathematical modeling of particle kinetics is the continuum-time random walk (CTRW) which incorporates waiting times and/or non-Gaussian jump distributions with divergent second moments to account for the anomalous jumps called Lévy flights [9, 37, 38, 50]. The continuous limit for such models resultes in a fractional diffusion equation [30, 39, 41]. Exponentially tempering the Lévy measure ensures that the moments of Lévy distributions are finite. The corresponding fluid (continuous) limit for such models leads to the tempered fractional diffusion equation [50, 53]. Cartea and del-Castillo-Negrete [9] presented the tempered fractional diffusion equation that governs the probability densities of the tempered Lévy flight. Probability densities of the tempered stable motion solve a tempered fractional diffusion equation that describes the particle plume shape [2]. The tempered fractional derivative is firstly introduced in [9] and used to model of various problems in finance [25, 33, 34, 36, 44], biology [4, 29, 32], and ground water hydrology [3, 5, 6, 18, 22, 45] and poroelasticity [28]. The tempered diffusion model has been arisen in many applications such as geophysics [37, 51, 52] and finance [7, 8].

The numerical solutions of three kinds of fractional Black-Merton-Scholes equations with tempered fractional derivatives have been compared by Marom and Momoniat [34]. A general finite difference scheme to solve a Black-Merton-Scholes model with tempered fractional derivatives has been introduced by Cartea and del-Castillo-Negrete [9]. Baeumera and Meerschaert [2] developed a second-order Crank–Nicolson scheme, using a variant of the Grünwald finite difference formula for tempered fractional derivatives combined with a Richardson extrapolation. Dehghan and Abbaszadeh [19] proposed a finite difference/finite element technique for space

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fractional tempered diffusion-wave equations. Yu *et al.* [49] gave the third order finite difference schemes for the tempered fractional diffusion equation. Guo *et al.* [26] employed the weighted and shifted Lubich difference operator for tempered fractional diffusion equations. Deng *et al.* [23] applied fast predictor-corrector approach for the tempered fractional differential equations. Hanert and Piret [27] presented Chebyshev pseudospectral method to solve the space-time tempered fractional diffusion equation. Chen and Deng [12,13] solved space-time tempered fractional diffusion-wave equation and time tempered Feynman–Kac equation, respectively. Zhao *et al.* [53] proposed spectral Galerkin and Petrov–Galerkin methods for tempered fractional advection and diffusion problems. Zayernouri *et al.* [50] introduced a Petrov–Galerkin spectral method for approximating tempered fractional ODEs. Chen *et al.* [14] solved the tempered fractional differential equations by using efficient spectral methods and generalized Laguerre functions.

We focus on the following tempered fractional convection–diffusion equations

$$u_t(x,t) + c\,u_x(x,t) - d\,{}_a\mathcal{D}_x^{\alpha,\lambda}u(x,t) \;=\; f(x,t), \text{ on } \;\Omega\times[0,T], \tag{1.1}$$

where $1<\alpha<2$ and $\Omega=[a,b]$ with the Dirichlet boundary conditions,

$$u(a,t)=0, \quad u(b,t)=g(t), \quad t\in[0,T],$$

and initial condition,

$$u(x,0)=l(x), \quad x\in\Omega,$$

where coefficients c and d are positive real numbers. ${}_a\mathcal{D}_x^{\alpha,\lambda}u(x)$ denotes the Reimann–Liouville tempered fractional derivatives of function u. An introduction of fractional derivative is introduced in the next section. This paper presents and analyzes local discontinuous Galerkin method (LDG method) to solve tempered fractional convection–diffusion equations (1.1). DG method is a finite element method which is very efficient for numerical simulations because of their physical and numerical properties. The DG methods are employed for the numerical solution of PDEs based on piecewise polynomial functions. The choice of fluxes helps us to achieve higher order convergence of the method. LDG method is to convert a higher order differential equation to a first order system of differential equations and solving the first order system of differential equations by DG method. Deng and Hesthaven [20, 21] solved fractional diffusion and fractional ODE equation by LDG method. The analysis of LDG method for fractional convection–diffusion equations with a fractional Laplacian operator is considered by Xu and Hesthaven [48]. HDG method has been proposed to solve 2D fractional convection–diffusion equations and fractional diffusion problems by Wang *et al.* [47] and Cockburn and Mustapha [16], respectively. Mustapha *et al.* [40] introduced discontinuous Petrov Galerkin method to solve the time fractional differential equation. A superconvergence of the DG method for the fractional diffusion and wave equations is presented by McLean and Mustapha [35]. Ahmadinia *et al.* [1] employed LDG method for time-space fractional convection–diffusion equations. Wang and Deng [46] considered the adaptive DG methods, namely interior penalty method, for the two–dimensional space tempered fractional differential equation with Riesz tempered fractional derivatives. The paper is structured as follows. Section 2 presents some basic notations and definitions. Section 3 presents the LDG method for tempered fractional convection–diffusion equation and introduces numerical fluxes which are crucial for the stability, and the convergence of the LDG method. Proof of the stability of the method and estimates the error are given in Section 4. The last section presents some numerical results to illustrate the accuracy and applicability of the method.

2. Preliminaries

This section presents some definitions and properties of tempered fractional calculus.

Definition 2.1 (Riemann–Liouville tempered fractional integral [9, 53])**.** For any $\alpha>0$, $\lambda\geq 0$, the left and right tempered Riemann–Liouville fractional integrals of function $u(x)$ defined by

$${}_a\mathcal{I}_x^{\alpha,\lambda}u(x) := \frac{1}{\Gamma(\alpha)}\int_a^x e^{-\lambda(x-\xi)}(x-\xi)^{\alpha-1}u(\xi)\mathrm{d}\xi,$$

and

$$ {}_x\mathcal{I}_b^{\alpha,\lambda}u(x) := \frac{1}{\Gamma(\alpha)}\int_x^b e^{-\lambda(\xi-x)}(\xi-x)^{\alpha-1}u(\xi)\mathrm{d}\xi, $$

where Γ represents the Euler gamma function.

Definition 2.2 (Riemann–Liouville tempered fractional derivative [11,31,53])**.** For any $n-1 \le \alpha < n$ ($n \in \mathbb{N}^+$) and fixed parameter $\lambda \ge 0$, then the left Riemann–Liouville tempered fractional derivative is defined as follows

$$ \begin{aligned} {}_a\mathcal{D}_x^{\alpha,\lambda}u(x) &:= \frac{e^{-\lambda x}}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_a^x \frac{e^{\lambda\xi}u(\xi)}{(x-\xi)^{\alpha-n+1}}\mathrm{d}\xi \\ &= (D+\lambda)^n\, {}_a\mathcal{I}_x^{n-\alpha,\lambda}u(x), \end{aligned} \tag{2.1} $$

where $(D+\lambda)^n = \left(\frac{\partial}{\partial x}+\lambda\right)^n = \left(\frac{\partial}{\partial x}+\lambda\right)\dots\left(\frac{\partial}{\partial x}+\lambda\right)$, and the following notation denotes the right Riemann–Liouville tempered fractional derivative

$$ {}_x\mathcal{D}_b^{\alpha,\lambda}u(x) := \frac{e^{\lambda x}}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^b \frac{e^{-\lambda\xi}}{(\xi-x)^{\alpha-n+1}}u(\xi)\mathrm{d}\xi $$

Lemma 2.3 ([11])**.** *Let u be a continuous function on $[a,\infty)$ and $\alpha,\beta > 0$. Then for all $x \ge a$,*

$$ {}_a\mathcal{I}_x^{\alpha,\lambda}[{}_a\mathcal{I}_x^{\beta,\lambda}u(x)] = {}_a\mathcal{I}_x^{\alpha+\beta,\lambda}u(x) = {}_a\mathcal{I}_x^{\beta,\lambda}[{}_a\mathcal{I}_x^{\alpha,\lambda}u(x)]. $$

Lemma 2.4 ([11])**.** *Let u be a continuously differentiable function on $[a,\infty)$, and $\alpha > 0$. Then for all $x \ge a$,*

$$ (D+\lambda)[{}_a\mathcal{I}_x^{\alpha,\lambda}u(x)] = \frac{u(a)e^{-\lambda(x-a)}}{\Gamma(\alpha)(x-a)^{1-\alpha}} + {}_a\mathcal{I}_x^{\alpha,\lambda}[(D+\lambda)u(x)]. \tag{2.2} $$

Lemma 2.5 (Adjoint property [53])**.** *Let $\alpha > 0$, $\lambda \ge 0$. If u and $v \in L^2(a,b)$, then*

$$ ({}_a\mathcal{I}_x^{\alpha,\lambda}u, v) = (u, {}_x\mathcal{I}_b^{\alpha,\lambda}v). $$

Notation (u,v) denotes inner product of functions u and v and norm $||.||$ denotes the L^2-norm throughout the paper.

Lemma 2.6 ([53])**.** *Let $0 < \alpha < \frac{1}{2}$, and $\lambda \ge 0$. If $u(x) \in L_p(a,b)$, $p = \frac{2}{1+2\alpha}$ then*

$$ ({}_a\mathcal{I}_x^{\alpha,\lambda}u, {}_x\mathcal{I}_b^{\alpha,\lambda}u) \ge \cos(\pi\alpha)||{}_a\mathcal{I}_x^{\alpha,\lambda}u||^2 = \cos(\pi\alpha)||{}_x\mathcal{I}_b^{\alpha,\lambda}u||^2, $$

Theorem 2.7. *For $0 \le \alpha_1 < \alpha_2$, and $u \in L^2(\Omega)$, it follows that*

$$ ||{}_a\mathcal{I}_x^{\alpha_2,\lambda}u|| \le \frac{(b-a)^{\alpha_2-\alpha_1}}{\Gamma(\alpha_2-\alpha_1+1)}||{}_a\mathcal{I}_x^{\alpha_1,\lambda}u||, $$

and

$$ ||{}_a\mathcal{I}_x^{\alpha_2,\lambda}u|| \le \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)}||u||. $$

Proof. Proof is similar to [20,24]. □

Lemma 2.8 (Continuous Gronwall inequality [42])**.** *Let f, g and h be piecewise contiguous nonnegative functions defined on (a,b). Assume that g is nondecreasing function and a positive constant C is independent of t such that*

$$ f(t) + h(t) \le g(t) + C\int_a^t f(s)\mathrm{d}s, \quad t \in (a,b). $$

Then

$$ f(t) + h(t) \le \exp(C(t-a))g(t), \quad t \in (a,b). $$

3. The LDG method for the tempered fractional convection–diffusion

This section presents LDG method for the tempered fractional convection–diffusion (1.1). Therefore, tempered fractional convection–diffusion (1.1) is converted to the first order system of differential equations and a fractional integral equation. Let $u(a,t)=0$ and $1<\alpha<2$. Definition 2.2, equality (2.2) and Lemma 2.3 imply that

$$\begin{aligned}
{}_a\mathcal{D}_x^{\alpha,\lambda}u(x,t) &= (D+\lambda)^2\left[{}_a\mathcal{I}_x^{2-\alpha,\lambda}u(x,t)\right]\\
&= (D+\lambda)\left[{}_a\mathcal{I}_x^{2-\alpha,\lambda}(D+\lambda)u(x,t)\right]\\
&= \frac{\partial}{\partial x}{}_a\mathcal{I}_x^{2-\alpha,\lambda}\frac{\partial u(x,t)}{\partial x}+\lambda_a\mathcal{I}_x^{2-\alpha,\lambda}\frac{\partial u(x,t)}{\partial x}+\lambda_a\mathcal{I}_x^{2-\alpha,\lambda}(D+\lambda)u(x,t)\\
&= \frac{\partial}{\partial x}{}_a\mathcal{I}_x^{2-\alpha,\lambda}\frac{\partial u(x,t)}{\partial x}+2\lambda_a\mathcal{I}_x^{2-\alpha,\lambda}\frac{\partial u(x,t)}{\partial x}+\lambda^2{}_a\mathcal{I}_x^{2-\alpha,\lambda}u(x,t).
\end{aligned}$$

Therefore,

$${}_a\mathcal{D}_x^{\alpha,\lambda}u(x,t) = \frac{\partial}{\partial x}{}_a\mathcal{I}_x^{2-\alpha,\lambda}\frac{\partial u(x,t)}{\partial x}+2\lambda_a\mathcal{I}_x^{2-\alpha,\lambda}\frac{\partial u(x,t)}{\partial x}+\lambda^2{}_a\mathcal{I}_x^{2-\alpha,\lambda}u(x,t).$$

Define

$$q(x,t) := {}_a\mathcal{I}_x^{2-\alpha,\lambda}p(x,t), \quad p(x,t) := \sqrt{d}\,\frac{\partial u(x,t)}{\partial x},$$

then the tempered fractional convection–diffusion equations (1.1) can be written as follows,

$$\begin{cases}
u_t(x,t)+c\,u_x(x,t)-\sqrt{d}\,q_x(x,t)-2\sqrt{d}\lambda_a\mathcal{I}_x^{2-\alpha,\lambda}p(x,t) & \\
-d\lambda^2{}_a\mathcal{I}_x^{2-\alpha,\lambda}u(x,t)=f(x,t), & \text{on } \Omega\times[0,T],\\
q(x,t)-{}_a\mathcal{I}_x^{2-\alpha,\lambda}p(x,t)=0, & \text{on } \Omega\times[0,T],\\
p(x,t)-\sqrt{d}\,\dfrac{\partial u(x,t)}{\partial x}=0, & \text{on } \Omega\times[0,T],\\
u(a,t)=0, \quad u(b,t)=g(t), & t\in[0,T],\\
u(x,0)=l(x), & x\in\Omega.
\end{cases} \tag{3.1}$$

To describe the LDG method we need the following notations. Consider mesh $J: a=x_0<x_1<\ldots<x_N=b$ on interval $[a,b]$, and set

$$I_i := [x_{i-1},x_i], \quad h_i := x_i-x_{i-1},\ i=1,\ldots,N, \quad h:=\max_{i=1}^{N} h_i.$$

$[v]_n$ denotes the jump across an interior node x_n which is defined by $[v]_n := v_n^- - v_n^+$ where

$$v_n^+ := \lim_{x\to x_n^+} v(x), \quad v_n^- := \lim_{x\to x_n^-} v(x).$$

$L^2(\Omega,J)$ and $H^1(\Omega,J)$ are defined as

$$L^2(\Omega,J) := \{v:\Omega\longrightarrow\mathbb{R} \mid v|_{I_i}\in L^2(I_i), i=1,2,\ldots,N\},$$

and

$$H^1(\Omega,J) := \{v:\Omega\longrightarrow\mathbb{R} \mid v|_{I_i}\in H^1(I_i), i=1,2,\ldots,N\}.$$

$\mathbb{P}^k(I_j)$ is the space of polynomials P (restricted on I_j) whose degree is at most k and space V^k defined by

$$V^k := \{v : (a,b) \to \mathbb{R} \,|\, v \in \mathbb{P}^k(I_j), j = 1, 2, \ldots, N\}.$$

Assume that (u, p, q) as the exact solution of (3.1) belongs to

$$H^1(0,T; H^1(\Omega, J)) \times L^2(0,T; L^2(\Omega, J)) \times L^2(0,T; H^1(\Omega, J)),$$

then the following system is obtained by inner product of test function $(v, w, z) \in H^1(0,T; H^1(\Omega, J)) \times L^2(0,T; L^2(\Omega, J)) \times L^2(0,T; H^1(\Omega, J))$ and integration by parts technique.

$$\begin{cases} (u_t(x,t), v)_{I_j} - c(u(x,t), v_x)_{I_j} + c\, u(x,t)v\Big|_{x_{j-1}^+}^{x_j^-} + \sqrt{d}\,(q(x,t), v_x)_{I_j} \\ -\sqrt{d}\, q(x,t)v\Big|_{x_{j-1}^+}^{x_j^-} - 2\sqrt{d}\lambda({}_a\mathcal{I}_x^{2-\alpha,\lambda} p(x,t), v)_{I_j} - d\lambda^2({}_a\mathcal{I}_x^{2-\alpha,\lambda} u(x,t), v)_{I_j} \\ = (f(x,t), v), \\ (q(x,t), w) - ({}_a\mathcal{I}_x^{2-\alpha,\lambda} p(x,t), w) = 0, \\ (p(x,t), z) + \sqrt{d}(u(x,t), z_x) - \sqrt{d}u(x,t)z\Big|_{x_{j-1}^+}^{x_j^-} = 0. \end{cases} \tag{3.2}$$

Now, we are ready to describe the numerical method. In the LDG method, the choice of numerical fluxes is very important to hold the stability, and the convergence of the scheme. The introduced numerical fluxes in [10] help us to choose the following numerical fluxes,

$$\hat{Q}(x_j,t) := Q(x_j^+,t), \quad \tilde{U}(x_j,t) := U(x_j^-,t), \quad U^*(x_j,t) := U(x_j^-,t), \tag{3.3}$$

for $j = 1, \ldots, N-1$ and at the boundary we define

$$\hat{Q}(a,t) := Q(a^+,t), \quad \hat{Q}(b,t) := Q(b^-,t),$$
$$U^*(a,t) := 0, \quad \tilde{U}(a,t) := 0,$$

and

$$U^*(b,t) := g(t), \quad c\,\tilde{U}(b,t) := c\,U(b^-,t) - \delta(g(t) - U(b^-,t)),$$

where

$$\delta := \frac{d}{h}\max\{1,k\}. \tag{3.4}$$

Let (U, Q, P) be the approximation of $(u(.,t), q(.,t), p(.,t))$. The approximate solution of LDG method $(U, Q, P) \in H^1(0,T; V^k) \times L^2(0,T; V^k) \times L^2(0,T; V^k)$ is obtained by the following system

$$\begin{cases} (U_t(.,t), v)_{I_j} - c(U(.,t), v_x)_{I_j} + c\,\tilde{U}(x,t)v\Big|_{x_{j-1}^+}^{x_j^-} + \sqrt{d}\,(Q(.,t), v_x)_{I_j} - \sqrt{d}\,\hat{Q}(x,t)v\Big|_{x_{j-1}^+}^{x_j^-} \\ -2\sqrt{d}\lambda({}_a\mathcal{I}_x^{2-\alpha,\lambda} P(.,t), v)_{I_j} - d\lambda^2({}_a\mathcal{I}_x^{2-\alpha,\lambda} U(.,t), v)_{I_j} = (f(.,t), v), \\ (Q(.,t), w) - ({}_a\mathcal{I}_x^{2-\alpha,\lambda} p(.,t), w) = 0, \\ (P(.,t), z) + \sqrt{d}(U(.,t), z_x) - \sqrt{d}U^*(x,t)z\Big|_{x_{j-1}^+}^{x_j^-} = 0, \end{cases}$$

where $(v, w, z) \in H^1(0,T; V^k) \times L^2(0,T; V^k) \times L^2(0,T; V^k)$ for $j = 1, \ldots, N$.

4. Stability and convergence analysis

This section presents the stability and the error estimate for the LDG method for tempered fractional convection–diffusion equation (1.1). Consider the linear form L defined as

$$L(v,w,z) := \int_0^T (f(.,t),v) + (\delta v(b^-,t) + \sqrt{d} z(b^-,t))g(t)\mathrm{d}t,$$

and the following discrete bilinear form

$$\begin{aligned}\mathcal{B}(U,P,Q;v,w,z) :=& \int_0^T (U_t(.,t),v(.,t))\mathrm{d}t - c\int_0^T (U(.,t),v_x(.,t))\mathrm{d}t + c\sum_{j=1}^{N-1}\int_0^T \tilde{U}(x_j,t)[v]_j\mathrm{d}t \\ &+ \sqrt{d}\int_0^T (Q(.,t),v_x(.,t))\mathrm{d}t - 2\sqrt{d}\lambda\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}P(.,t),v(.,t))\mathrm{d}t \\ &- d\lambda^2\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}U(.,t),v(.,t))\mathrm{d}t + \int_0^T (Q(.,t),w(.,t))\mathrm{d}t \\ &- \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}P(.,t),w(.,t))\mathrm{d}t + \int_0^T (P(.,t),z(.,t))\mathrm{d}t \\ &+ \sqrt{d}\int_0^T (U(.,t),z_x(.,t))\mathrm{d}t - \sqrt{d}\sum_{j=1}^{N-1}\int_0^T (\hat{Q}(x_j,t)[v]_j + U^*(x_j,t)[z]_j)\mathrm{d}t \\ &+ \sqrt{d}\int_0^T \left(Q(x_0^+,t)v(x_0^+,t) - Q(x_N^-,t)v(x_N^-,t) + (c+\delta)U(x_N^-,t)v(x_N^-,t)\right)\mathrm{d}t. \end{aligned} \tag{4.1}$$

The numerical scheme is to obtain $(U,P,Q) \in H^1(0,T;V^k)\times L^2(0,T;V^k)\times L^2(0,T;V^k)$ such that the following weak form holds

$$\mathcal{B}(U,P,Q;v,w,z) = L(v,w,z), \tag{4.2}$$

for all test functions $(v,w,z)\in H^1(0,T;V^k)\times L^2(0,T;V^k)\times L^2(0,T;V^k)$. Since the scheme is consistent with (3.2), the exact solution (u,p,q) of (3.2) in the space $H^1(0,T;H^1(\Omega,J))\times L^2(0,T;L^2(\Omega,J))\times L^2(0,T;H^1(\Omega,J))$ satisfies

$$\mathcal{B}(u,p,q;v,w,z) = L(v,w,z), \tag{4.3}$$

for all $(v,w,z)\in H^1(0,T;V^k)\times L^2(0,T;V^k)\times L^2(0,T;V^k)$.

Lemma 4.1. *For all $(v,w,z)\in H^1(\Omega,J)\times L^2(\Omega,J)\times H^1(\Omega,J)$, the following equality holds*

$$\begin{aligned}\mathcal{B}(v,w,z;v,-w,z) \ge& \frac{1}{2}||v(.,T)||^2 - \frac{1}{2}||v(.,0)||^2 + \frac{c}{2}\int_0^T\left(\sum_{j=1}^{N-1}([v]_j)^2 + v(x_0^+,t)^2\right)\mathrm{d}t \\ &+ \left(\frac{c}{2}+\delta\right)\int_0^T v(x_N^-,t)^2\mathrm{d}t - 2\sqrt{d}\lambda\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}w(.,t),v(.,t))\mathrm{d}t \\ &- d\lambda^2\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}v(.,t),v(.,t))\mathrm{d}t + \cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}w(.,t)\|\mathrm{d}t, \end{aligned} \tag{4.4}$$

where constants c, d and λ are the coefficients in equation (1.1).

Proof. Bilinear form (4.1) yields

$$\mathcal{B}(v,w,z;v,-w,z)=\frac{1}{2}\int_0^T\frac{\partial}{\partial t}||v(.,t)||^2\mathrm{d}t+A_1+A_2, \tag{4.5}$$

where

$$\begin{aligned}A_1:=&-c\int_0^T(v(.,t),v_x(.,t))\mathrm{d}t+c\sum_{j=1}^{N-1}\int_0^T\tilde{v}(x_j,t)[v]_j\mathrm{d}t-\int_0^T(z(.,t),w(.,t))\mathrm{d}t\\&+\int_0^T(w(.,t),z(.,t))\mathrm{d}t+\sqrt{d}\int_0^T(z(.,t),v_x(.,t))\mathrm{d}t+\sqrt{d}\int_0^T(v(.,t),z_x(.,t))\mathrm{d}t\\&-\sqrt{d}\sum_{j=1}^{N-1}\int_0^T(\hat{z}(x_j,t)[v]_j+v^*(x_j,t)[z]_j)\mathrm{d}t+\sqrt{d}\int_0^T z(x_0^+,t)v(x_0^+,t)\mathrm{d}t\\&-\sqrt{d}\int_0^T z(x_N^-,t)v(x_N^-,t)\mathrm{d}t+(c+\delta)\int_0^T v^2(x_N^-,t)\mathrm{d}t,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}A_2:=&-2\sqrt{d}\lambda\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}w(.,t),v(.,t))\mathrm{d}t-d\lambda^2\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}v(.,t),v(.,t))\mathrm{d}t\\&+\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}w(.,t),w(.,t))\mathrm{d}t.\end{aligned} \tag{4.7}$$

The following inequalities have been obtained by integration by parts technique

$$(z,v_x)+(v,z_x)=\sum_{i=1}^{N-1}[vz]_i-z(x_0^+,t)v(x_0^+,t)+z(x_N^-,t)v(x_N^-,t), \tag{4.8}$$

and

$$-c(v,v_x)=-\frac{c}{2}\sum_{j=1}^{N-1}[v^2]_j+\frac{c}{2}v^2(x_0^+,t)-\frac{c}{2}v^2(x_N^-,t). \tag{4.9}$$

The following inequality holds by introduced numerical fluxes in (3.3)

$$\sum_{j=1}^{N-1}(\hat{z}(x_j,t)[v]_j+v^*(x_j,t)[z]_j)=\sum_{j=1}^{N-1}\left(z(x_j^+,t)[v]_j+v(x_j^-,t)[z]_j\right)=\sum_{j=1}^{N-1}[vz]_j. \tag{4.10}$$

Substituting equalities (4.8)–(4.10) into (4.6) and definition (3.3) imply that

$$\begin{aligned}A_1=&\frac{c}{2}\int_0^T\left(-\sum_{j=1}^{N-1}[v^2]_j+v(x_0^+,t)^2-v(x_N^-,t)^2\right)\mathrm{d}t+c\sum_{j=1}^{N-1}\int_0^T\tilde{v}(x_j,t)[v]_j\mathrm{d}t\\&+\sqrt{d}\int_0^T\left(\sum_{i=1}^{N-1}[vz]_i-z(x_0^+,t)v(x_0^+,t)+z(x_N^-,t)v(x_N^-,t)\right)\mathrm{d}t\\&-\sqrt{d}\sum_{j=1}^{N-1}\int_0^T(\hat{z}(x_j,t)[v]_j+v^*(x_j,t)[z]_j)\mathrm{d}t+\sqrt{d}\int_0^T z(x_0^+,t)v(x_0^+,t)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&-\sqrt{d}\int_0^T z(x_N^-,t)v(x_N^-,t)\mathrm{d}t+(c+\delta)\int_0^T v^2(x_N^-,t)\mathrm{d}t\\
&=\frac{c}{2}\int_0^T\left(\sum_{j=1}^{N-1}([v]_j)^2+v(x_0^+,t)^2\right)\mathrm{d}t+\left(\frac{c}{2}+\delta\right)\int_0^T v(x_N^-,t)^2\mathrm{d}t,
\end{aligned}
\tag{4.11}
$$

and Lemmas 2.3, 2.5 and 2.6 yield

$$
\begin{aligned}
A_2\geq\ &-2\sqrt{d}\lambda\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}w(.,t),v(.,t))\mathrm{d}t-d\lambda^2\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}v(.,t),v(.,t))\mathrm{d}t\\
&+\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T\|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}w(.,t)\|^2\mathrm{d}t.
\end{aligned}
\tag{4.12}
$$

Equalities (4.5), (4.11) and inequality (4.12) complete proof of the lemma. □

4.1. The stability of the method

Let $(\overline{U},\overline{P},\overline{Q})$ and (U,P,Q) be the solution (4.2) such that (U,P,Q) satisfies initial condition but $(\overline{U},\overline{P},\overline{Q})$ satisfies perturbed initial condition. Consider $e_U:=\overline{U}-U$, $e_P:=\overline{P}-P$ and $e_Q:=\overline{Q}-Q$ as the errors. Now, we ready to prove the stability of the numerical method by the following theorem.

Theorem 4.2. *LDG method scheme (4.2) is L^2-stable, and*

$$
\begin{aligned}
||e_U(.,T)||^2\leq C||e_U(.,0)||^2-c\int_0^T\left(\sum_{j=1}^{N-1}([e_U]_j)^2+e_U(x_0^+,t)^2\right)\mathrm{d}t\\
-(c+\delta)\int_0^T e_U(x_N^-,t)^2\mathrm{d}t-2\left(\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)-\varepsilon C\right)\int_0^T\|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P(.,t)\|^2\mathrm{d}t,
\end{aligned}
$$

where constant c is the coefficient in equation (1.1).

Proof. It is clear that $\mathcal{B}(e_U,e_P,e_Q;v,w,z)=0$ for all test function

$$(v,w,z)\in H^1(0,T;V^k)\times L^2(0,T;V^k)\times L^2(0,T;V^k).$$

The following inequality holds by taking $v:=e_U$, $w:=-e_P$, $z:=e_Q$ and Lemma 4.1

$$
\begin{aligned}
0&=\mathcal{B}(e_U,e_P,e_Q;e_U,-e_P,e_Q)\\
&\geq\frac{1}{2}||e_U(.,T)||^2-\frac{1}{2}||e_U(.,0)||^2+\frac{c}{2}\int_0^T\left(\sum_{j=1}^{N-1}([e_U]_j)^2+e_U(x_0^+,t)^2\right)\mathrm{d}t\\
&+\left(\frac{c}{2}+\delta\right)\int_0^T e_U(x_N^-,t)^2\mathrm{d}t-2\sqrt{d}\lambda\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}e_P,e_U)\mathrm{d}t\\
&-d\lambda^2\int_0^T({}_a\mathcal{I}_x^{2-\alpha,\lambda}e_U,e_U)\mathrm{d}t+\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T\|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P\|^2\mathrm{d}t.
\end{aligned}
$$

Therefore,

$$\frac{1}{2}||e_U(.,T)||^2 + \frac{c}{2}\int_0^T \left(\sum_{j=1}^{N-1}([e_U]_j)^2 + e_U(x_0^+,t)^2\right)\mathrm{d}t$$
$$+\left(\frac{c}{2}+\delta\right)\int_0^T e_U(x_N^-,t)^2\mathrm{d}t + \cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P(.,t)\|^2\mathrm{d}t$$
$$\leq \frac{1}{2}||e_U(.,0)||^2 + 2\sqrt{d}\lambda\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}e_P(.,t), e_U(.,t))\mathrm{d}t$$
$$+ d\lambda^2\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}e_U(.,t), e_U(.,t))\mathrm{d}t, \tag{4.13}$$

inequality (4.13) and Young's inequality imply that

$$\frac{1}{2}||e_U(.,T)||^2 + \frac{c}{2}\int_0^T \left(\sum_{j=1}^{N-1}([e_U]_j)^2 + e_U(x_0^+,t)^2\right)\mathrm{d}t$$
$$+\left(\frac{c}{2}+\delta\right)\int_0^T e_U(x_N^-,t)^2\mathrm{d}t + \cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P(.,t)\|\mathrm{d}t$$
$$\leq \frac{1}{2}||e_U(.,0)||^2 + \varepsilon\int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}e_P(.,t)||^2\mathrm{d}t + \frac{d\lambda^2}{\varepsilon}\int_0^T ||e_U(.,t)||^2\mathrm{d}t$$
$$+\frac{d\lambda^2}{2}\int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}e_U(.,t)||^2\mathrm{d}t + \frac{d\lambda^2}{2}\int_0^T ||e_U(.,t)||^2\mathrm{d}t$$
$$\leq \frac{1}{2}||e_U(.,0)||^2 + C\varepsilon\int_0^T ||{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P(.,t)||^2\mathrm{d}t$$
$$+\frac{d}{2}\lambda^2(C+1+2/\varepsilon)\int_0^T ||e_U(.,t)||^2\mathrm{d}t. \tag{4.14}$$

The last inequality is obtained by Theorem 2.7. Choose a sufficiently small $\varepsilon > 0$ such that $\cos((\frac{\alpha}{2}-1)\pi) - \varepsilon C > 0$, hence,

$$||e_U(.,T)||^2 + c\int_0^T \left(\sum_{j=1}^{N-1}([e_U]_j)^2 + e_U(x_0^+,t)^2\right)\mathrm{d}t + (c+\delta)\int_0^T e_U(x_N^-,t)^2\mathrm{d}t$$
$$+2\left(\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right) - \varepsilon C\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P(.,t)\|^2\mathrm{d}t$$
$$\leq ||e_U(.,0)||^2 + \frac{d}{2}\lambda^2(C+1+2/\varepsilon)\int_0^T ||e_U(.,t)||^2\mathrm{d}t. \tag{4.15}$$

Set $\tilde{C} := 0.5d\lambda^2(C+1+2/\varepsilon)$. The continuous Gronwall inequality (Lem. 2.8) and inequality (4.15) yield

$$||e_U(.,T)||^2 \leq \exp(\tilde{C}T)||e_U(.,0)||^2 - c\int_0^T \left(\sum_{j=1}^{N-1}([e_U]_j)^2 + e_U(x_0^+,t)^2\right)\mathrm{d}t$$
$$-(c+\delta)\int_0^T e_U(x_N^-,t)^2\mathrm{d}t - 2\left(\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right) - \varepsilon C\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}e_P(.,t)\|^2\mathrm{d}t.$$

□

4.2. Convergence analysis

To analyze the convergence of the LDG method we need to introduce three projections $\Pi^{\mp}$ and R from H^1 to V^k as follows:

$$\int_{I_j} \Pi^+ u(x)v(x)\mathrm{d}x = \int_{I_j} u(x)v(x)\mathrm{d}x, \ v \in \mathbb{P}^{k-1}(I_j), \ \Pi^+ u(x_{j-1}) = u(x_{j-1}^+), \tag{4.16}$$

$$\int_{I_j} \Pi^- u(x)v(x)\mathrm{d}x = \int_{I_j} u(x)v(x)\mathrm{d}x, \ v \in \mathbb{P}^{k-1}(I_j), \ \Pi^- u(x_j) = u(x_j^-), \tag{4.17}$$

for $u \in H^1$ and $I_j = (x_{j-1}, x_j)$, $j = 1, \ldots, N$. R is L^2-projection by the standard definition

$$\int_{I_j} Ru(x)v(x)\mathrm{d}x = \int_{I_j} u(x)v(x)\mathrm{d}x, \quad v \in \mathbb{P}^k(I_j). \tag{4.18}$$

Cockburn *et al.* [17] presented some results for $\Pi^{\mp}$ and R as follows

$$\|Sw - w\| + h\|Sw - w\|_\infty + \sqrt{h}\|Sw - w\|_{\tau_h} \leq C_2 h^{k+1}, \tag{4.19}$$

where $S = \Pi^{\mp}$ or $S = R$, the positive constant C_2 is independent of w. The norm with subscript τ_h means that the discrete infinity norm on the all points $\{x_j\}_{j=0}^N$.

Theorem 4.3. *Let $u(x,t)$ be the exact solution of (1.1) and $U(x,t)$ is the numerical solution of scheme (4.2), then the following error estimate holds*

$$\|u(.,T) - U(.,T)\| \leq \tilde{C} h^{k+1}$$

when u is a sufficiently regular function of x and $\tilde{C}$ is positive constant independent of h.

Proof. Let (u,p,q) be the exact solution of (3.1) which satisfies (4.3) and (U,P,Q) be the numerical solution of LDG scheme (4.2). Define the errors by the following notation,

$$e_u := u(x,t) - U(x,t), \quad e_p := p(x,t) - P(x,t), \quad e_q := q(x,t) - Q(x,t).$$

The errors can be split in the following form,

$$\begin{aligned} e_u &= (u - \Pi^- u) - (U - \Pi^- u) := \theta_u - \eta_u, \\ e_q &= (q - \Pi^+ q) - (Q - \Pi^+ q) := \theta_q - \eta_q, \\ e_p &= (p - Rp) - (P - Rp) := \theta_p - \eta_p. \end{aligned}$$

Relations (4.2) and (4.3) imply that

$$\mathcal{B}(e_u, e_p, e_q; v, w, z) = 0,$$

for all $v, w, z \in V^k$, then

$$\mathcal{B}(\eta_u, \eta_p, \eta_q; v, w, z) = \mathcal{B}(\theta_u, \theta_p, \theta_q; v, w, z). \tag{4.20}$$

Choose the test functions as

$$v := \eta_u, \quad w := -\eta_p, \quad z := \eta_q,$$

hence, the definition bilinear form $\mathcal{B}$ in (4.1) yields

$$
\begin{aligned}
\mathcal{B}(\theta_u,\theta_p,\theta_q;\eta_u,-\eta_p,\eta_q) = & \int_0^T \left(\frac{\partial \theta_u(.,t)}{\partial t},\eta_u(.,t)\right)\mathrm{d}t - c\int_0^T \left(\theta_u(.,t),\frac{\partial \eta_u(.,t)}{\partial x}\right)\mathrm{d}t \\
& + c\sum_{j=1}^{N-1}\int_0^T \tilde{\theta}_u(x_j,t)[\eta_u]_j\mathrm{d}t + \sqrt{d}\int_0^T \left(\theta_q(.,t),\frac{\partial \eta_u(.,t)}{\partial x}\right)\mathrm{d}t \\
& - 2\sqrt{d}\lambda\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t),\eta_u(.,t))\mathrm{d}t - d\lambda^2\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_u(.,t),\eta_u(.,t))\mathrm{d}t \\
& - \int_0^T (\theta_q(.,t),\eta_p(.,t))\mathrm{d}t + \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t),\eta_p(.,t))\mathrm{d}t + \int_0^T (\theta_p(.,t),\eta_q(.,t))\mathrm{d}t \\
& + \sqrt{d}\int_0^T \left(\theta_u(.,t),\frac{\partial \eta_q(.,t)}{\partial x}\right)\mathrm{d}t - \sqrt{d}\sum_{j=1}^{N-1}\int_0^T (\hat{\theta}_q(x_j,t)[\eta_u]_j + \theta_u^*(x_j,t)[\eta_q]_j)\mathrm{d}t \\
& + \sqrt{d}\int_0^T (\theta_q(x_0^+,t)\eta_u(x_0^+,t) - \theta_q(x_N^-,t)\eta_u(x_N^-,t))\mathrm{d}t \\
& + (c+\delta)\int_0^T \theta_u(x_N^-,t)\eta_u(x_N^-,t)\mathrm{d}t. \qquad (4.21)
\end{aligned}
$$

The orthogonality property of the projections $\Pi^{\mp}$ and R in (4.16)–(4.18) and the fact that η_u,η_p and $\eta_q \in V^k$ imply that

$$
\begin{aligned}
& \left(\theta_u(.,t),\frac{\partial \eta_u(.,t)}{\partial x}\right) = 0, \quad \left(\theta_q(.,t),\frac{\partial \eta_u(.,t)}{\partial x}\right) = 0, \\
& \left(\theta_u(.,t),\frac{\partial \eta_q(.,t)}{\partial x}\right) = 0, \quad (\theta_p(.,t),\eta_q(.,t)) = 0, \\
& \hat{\theta}_q(x_j,t) = 0, \quad \theta_u^*(x_j,t) = 0, \quad \tilde{\theta}_u(x_j,t) = 0, \quad j = 1,\dots,N-1, \\
& \theta_u(x_N^-,t) = 0, \quad \theta_q(x_0^+,t) = 0,
\end{aligned}
$$

then, relation (4.21) is reduced to

$$
\begin{aligned}
\mathcal{B}(\theta_u,\theta_p,\theta_q;\eta_u,-\eta_p,\eta_q) = & \int_0^T \left(\frac{\partial \theta_u(.,t)}{\partial t},\eta_u(.,t)\right)\mathrm{d}t - 2\sqrt{d}\lambda\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t),\eta_u(.,t))\mathrm{d}t \\
& - d\lambda^2\int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_u(.,t),\eta_u(.,t))\mathrm{d}t - \int_0^T (\theta_q(.,t),\eta_p(.,t))\mathrm{d}t \\
& + \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t),\eta_p(.,t))\mathrm{d}t - \sqrt{d}\int_0^T \theta_q(x_N^-,t)\eta_u(x_N^-,t)\mathrm{d}t. \qquad (4.22)
\end{aligned}
$$

Lemma 4.1 implies that

$$
\begin{aligned}
\mathcal{B}(\eta_u,\eta_p,\eta_q;\eta_u,-\eta_p,\eta_q) \geq & \frac{1}{2}||\eta_u(.,T)||^2 - \frac{1}{2}||\eta_u(.,0)||^2 \\
& + \int_0^T \left(\frac{c}{2}\sum_{j=1}^{N-1}([\eta_u]_j)^2 + \frac{c}{2}\eta_u(x_0^+,t)^2\right)\mathrm{d}t + \left(\frac{c}{2}+\delta\right)\int_0^T \eta_u(x_N^-,t)^2\mathrm{d}t \\
& + \cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}\eta_p(.,t)\|^2\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- 2\sqrt{d}\lambda \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t), \eta_u(.,t))\mathrm{d}t \\
&- d\lambda^2 \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_u(.,t), \eta_u(.,t))\mathrm{d}t. \qquad (4.23)
\end{aligned}
$$

The following inequality is obtained by substitution (4.22) and (4.23) into (4.20)

$$
\begin{aligned}
&\frac{1}{2}||\eta_u(.,T)||^2 + \cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}\eta_p(.,t)\|^2\mathrm{d}t \\
&\quad + \frac{c}{2}\int_0^T \left(\sum_{j=1}^{N-1}([\eta_u]_j)^2 + \eta_u(x_0^+,t)^2\right)\mathrm{d}t + \left(\frac{c}{2}+\delta\right)\int_0^T \eta_u(x_N^-,t)^2\mathrm{d}t \\
&\quad \le \frac{1}{2}||\eta_u(.,0)||^2 + \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3, \qquad (4.24)
\end{aligned}
$$

where

$$
\mathcal{A}_1 := \int_0^T \left(\frac{\partial\theta_u}{\partial t}, \eta_u\right),
$$

$$
\begin{aligned}
\mathcal{A}_2 :=\ & - 2\sqrt{d}\lambda \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t), \eta_u(.,t))\mathrm{d}t - d\lambda^2 \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_u(.,t), \eta_u(.,t))\mathrm{d}t \\
& + d\lambda^2 \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_u(.,t), \eta_u(.,t))\mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_3 :=\ & 2\sqrt{d}\lambda \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t), \eta_u(.,t))\mathrm{d}t + \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t), \eta_p(.,t))\mathrm{d}t \\
& - \int_0^T (\theta_q(.,t), \eta_p(.,t))\mathrm{d}t - \sqrt{d}\int_0^T \theta_q^n(x_N^-,t)\eta_u(x_N^-,t)\mathrm{d}t.
\end{aligned}
$$

Young's inequality and standard approximation theory [15] imply that

$$
\begin{aligned}
\mathcal{A}_1 &\le \frac{1}{2}\int_0^T \left\|\frac{\partial\theta_u(.,t)}{\partial t}\right\|^2\mathrm{d}t + \frac{1}{2}\int_0^T ||\eta_u(.,t)||^2\mathrm{d}t \\
&\le Ch^{2k+2} + \frac{1}{2}\int_0^T ||\eta_u(.,t)||^2\mathrm{d}t, \qquad (4.25)
\end{aligned}
$$

where C is constant. Cauchy–Schwartz inequality, Young's inequality and Lemma 2.7 imply that

$$
\begin{aligned}
\mathcal{A}_2 &\le 2\sqrt{d}\lambda \int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_p(.,t)||\,||\eta_u(.,t)||\mathrm{d}t + d\lambda^2 \int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\theta_u(.,t)||||\eta_u(.,t)||\mathrm{d}t \\
&\quad + d\lambda^2 \int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_u(.,t)||\,||\eta_u(.,t)||\mathrm{d}t \\
&\le 2\sqrt{d}C\lambda \int_0^T ||\theta_p(.,t)||\,||\eta_u(.,t)||\mathrm{d}t + Cd\lambda^2 \int_0^T ||\theta_u(.,t)||\,||\eta_u(.,t)||\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&+ Cd\lambda^2 \int_0^T ||\eta_u(.,t)||^2 \mathrm{d}t \\
&\le Cd\lambda^2 \int_0^T ||\eta_u(.,t)||^2 \mathrm{d}t + C\int_0^T ||\theta_p(.,t)||^2 \mathrm{d}t \\
&+ \frac{3Cd\lambda^2}{2} \int_0^T ||\eta_u(.,t)||^2 \mathrm{d}t + \frac{Cd\lambda^2}{2} \int_0^T ||\theta_u(.,t)||^2 \mathrm{d}t \\
&\le \frac{Cd\lambda^2}{2} \int_0^T ||\theta_u(.,t)||^2 \mathrm{d}t + C\int_0^T ||\theta_p(.,t)||^2 \mathrm{d}t + \frac{5}{2} Cd\lambda^2 \int_0^T ||\eta_u(.,t)||^2 \mathrm{d}t,
\end{aligned}
$$

where $C = \frac{(b-a)^{2-\alpha}}{\Gamma(3-\alpha)}$. Therefore, relation (4.19) yields

$$
\mathcal{A}_2 \le C(1+0.5d\lambda^2)h^{2k+2} + \frac{5}{2}Cd\lambda^2 \int_0^T ||\eta_u(.,t)||^2 \mathrm{d}t. \tag{4.26}
$$

Lemma 2.5, Young's inequality ($ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2, \quad \epsilon > 0$) and Cauchy–Schwartz inequality imply that

$$
\begin{aligned}
\mathcal{A}_3 &= 2\sqrt{d}\lambda \int_0^T ({}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t), \eta_u(.,t))\mathrm{d}t + \int_0^T (\theta_p(.,t), {}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t))\mathrm{d}t \\
&\quad - \int_0^T (\theta_q(.,t), \eta_p(.,t))\mathrm{d}t - \sqrt{d}\theta_q(x_N^-,t)\eta_u(x_N^-,t)\mathrm{d}t \\
&\le 2\sqrt{d}\lambda \int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t)||\, ||\eta_u(.,t)||\mathrm{d}t + \int_0^T ||\theta_p(.,t)||\, ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t)||\mathrm{d}t \\
&\quad + \int_0^T ||\theta_q(.,t)||\, ||\eta_p(.,t)||\mathrm{d}t + \sqrt{d}\int_0^T |\theta_q(x_N^-,t)|\, |\eta_u(x_N^-,t)|\mathrm{d}t \\
&\le \varepsilon \int_0^T ||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t)||^2\mathrm{d}t + \frac{2d\lambda^2}{\varepsilon}\int_0^T ||\eta_u(.,t)||^2\mathrm{d}t + \frac{1}{2\varepsilon}\int_0^T ||\theta_p(.,t)||^2\mathrm{d}t \\
&\quad + \frac{1}{2}\left(\frac{1}{\epsilon}\int_0^T ||\theta_p(.,t)||^2\mathrm{d}t + \epsilon\int_0^T ||\eta_p(.,t)||^2\mathrm{d}t + \frac{d}{\delta}\int_0^T |\theta_q(x_N^-,t)|^2\mathrm{d}t + \delta\int_0^T |\eta_u(x_N^-,t)|^2\mathrm{d}t\right).
\end{aligned}
$$

Note that $||{}_a\mathcal{I}_x^{2-\alpha,\lambda}v(.,t)||$ for $v \in V^k$ is a norm on vector space V^k. Since vector space V^k is a finite dimensional vector space, the norm-equivalence on the finite dimensional vector spaces yields

$$
||\eta_p(.,t)|| \le M||{}_a\mathcal{I}_x^{1-\frac{\alpha}{2},\lambda}\eta_p(.,t)||, \tag{4.27}
$$

for $\eta_p \in V^k$. Since $1-\frac{\alpha}{2} < 2-\alpha$ (for $1<\alpha<2$), Theorem 2.7 implies that

$$
||{}_a\mathcal{I}_x^{2-\alpha,\lambda}\eta_p(.,t)|| \le \frac{(b-a)^{1-\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}||{}_a\mathcal{I}_x^{1-\frac{\alpha}{2},\lambda}\eta_p(.,t)||, \tag{4.28}
$$

inequalities (4.19), (4.27), (4.28) and definition of δ in (3.4) yield

$$
\begin{aligned}
\mathcal{A}_3 \le C(h^{2k+2}+h^{2k+2}) + C_1\varepsilon\int_0^T ||{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}\eta_p(.,t)||^2\mathrm{d}t \\
+ \frac{2d\lambda^2}{\varepsilon}\int_0^T ||\eta_u(.,t)||^2\mathrm{d}t + \frac{\delta}{2}\int_0^T |\eta_u(x_N^-,t)|^2\mathrm{d}t,
\end{aligned} \tag{4.29}
$$

TABLE 1. Example 5.1, $m = 4.2$.

	N	$k=1$ L^2-error	Order	L^∞-error	Order	$k=2$ L^2-error	Order	L^∞-error	Order
$\lambda = 1.8$	5	5.7060e-2	–	1.3187e-1	–	1.8828e-2	–	2.4995e-2	–
	10	1.4623e-2	1.964	3.8128e-2	1.790	2.3969e-3	2.973	3.0417e-3	3.038
	15	6.6014e-3	1.961	1.7252e-2	1.955	7.1519e-4	2.982	9.7357e-4	2.809
$\alpha = 1.5$	20	3.7278e-3	1.986	9.7679e-3	1.977	3.0261e-4	2.989	4.2737e-4	2.861
	25	2.3959e-1	1.981	6.3144e-3	1.955	1.5522e-4	2.991	2.1857e-4	3.005
$\lambda = 1.8$	5	5.7060e-2	–	1.3187e-1	–	1.8828e-2	–	2.4995e-2	–
	10	1.4623e-2	1.964	3.8128e-2	1.790	2.3969e-3	2.973	3.0417e-3	3.038
	15	6.6014e-3	1.961	1.7252e-2	1.955	7.1519e-4	2.982	9.7357e-4	2.809
$\alpha = 1.1$	20	3.7278e-3	1.986	9.7679e-3	1.977	3.0261e-4	2.989	4.2737e-4	2.861
	25	2.3959e-3	1.981	6.3144e-3	1.955	1.5522e-4	2.991	2.1857e-4	3.005
$\lambda = 1.8$	5	6.2509e-2	–	1.5924e-1	–	2.0429e-2	–	2.3901e-2	–
	10	1.6113e-2	1.955	4.1485e-2	1.940	2.5951e-3	2.976	3.0145e-3	2.987
	15	7.2720e-3	1.962	1.8758e-2	1.957	7.7438e-4	2.982	8.9905e-4	2.983
$\alpha = 1.9$	20	4.1038e-3	1.988	1.0589e-2	1.987	3.2768e-4	2.989	3.8417e-4	2.955
	25	2.6362e-3	1.983	6.8034e-3	1.982	1.6809e-4	2.991	1.9741e-4	2.983
$\lambda = 0$	5	1.5333e-1	–	5.7391e-1	–	3.4877e-2	–	5.0465e-2	–
	10	4.0653e-2	1.915	1.6867e-1	1.766	4.4030e-3	2.985	6.7468e-3	2.903
	15	1.8563e-2	1.933	7.8723e-2	1.879	1.3163e-3	2.977	2.0343e-3	2.956
$\alpha = 1.9$	20	1.0577e-2	1.955	4.5566e-2	1.900	5.5791e-4	2.983	8.6220e-4	2.983
	25	6.8490e-3	1.947	3.0025e-2	1.869	2.8654e-4	2.986	4.4221e-4	2.992
$\lambda = 1.6$	5	5.8919e-2	–	1.4258e-1	–	1.9489e-2	–	2.4648e-2	–
	10	1.5132e-2	1.961	3.8094e-2	1.904	2.4754e-3	2.976	2.9590e-3	3.058
	15	6.8288e-3	1.962	1.7599e-2	1.904	7.3846e-4	2.983	8.6701e-4	3.027
$\alpha = 1.6$	20	3.8557e-3	1.987	9.9949e-3	1.966	3.1245e-4	2.989	3.7425e-4	2.920
	25	2.4781e-3	1.981	6.4366e-3	1.972	1.6027e-4	2.991	1.8987e-4	3.041
$\lambda = 0.4$	5	1.0290e-1	–	3.4418e-1	–	2.7184e-2	–	3.6512e-2	–
	10	2.6615e-2	1.950	8.9267e-2	1.947	3.4351e-3	2.984	4.6278e-3	2.980
	15	1.2063e-2	1.951	4.0080e-2	1.974	1.0258e-3	2.980	1.3966e-3	2.954
$\alpha = 1.6$	20	6.8240e-3	1.980	2.2824e-2	1.957	4.3446e-4	2.986	5.8882e-4	3.002
	25	4.3909e-3	1.976	1.4739e-2	1.959	2.2301e-4	2.988	3.0346e-4	2.970
$\lambda = 1.2$	5	6.5638e-2	–	1.6871e-1		2.1100e-2	–	2.3662e-2	–
	10	1.6772e-2	1.968	4.2498e-2	1.989	2.6716e-3	2.981	3.1520e-3	2.908
	15	7.5526e-3	1.967	1.9443e-2	1.928	7.9713e-4	2.982	9.4916e-4	2.960
$\alpha = 1.3$	20	4.2613e-3	1.989	1.0945e-2	1.997	3.3736e-4	2.988	4.0231e-4	2.983
	25	2.7379e-3	1.982	7.0234e-3	1.988	1.7308e-4	2.990	2.0642e-4	2.990

The following inequality is obtained by substitution (4.25), (4.26) and (4.29) into (4.24)

$$
\begin{aligned}
&||\eta_u(.,T)||^2 + 2\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}\eta_p(.,t)\|^2 \mathrm{d}t \\
&\quad + c\int_0^T \left(\sum_{j=1}^{N-1}([\eta_u]_j)^2 + \eta_u(x_0^+,t)^2\right)\mathrm{d}t + 2\left(\frac{c}{2}+\delta\right)\int_0^T \eta_u(x_N^-,t)^2 \mathrm{d}t \\
&\quad \le ||\eta_u(.,0)||^2 + Ch^{2k+2} + \left(1 + 5Cd\lambda^2 + \frac{4d\lambda^2}{\varepsilon}\right)\int_0^T ||\eta_u(.,t)||^2 \mathrm{d}t \\
&\quad + 2C_1\varepsilon\int_0^T ||{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}\eta_p(.,t)||^2 \mathrm{d}t + \delta\int_0^T |\eta_u(x_N^-,t)|^2 \mathrm{d}t,
\end{aligned}
$$

TABLE 2. Example 5.1, $m = 6$.

		$k=1$				$k=2$			
	N	L^2-error	Order	L^∞-error	Order	L^2-error	Order	L^∞-error	Order
$\lambda = 1.2$,	5	1.5556e-1	–	5.7439e-1	–	3.5289e-2	–	5.2695e-2	–
	10	4.0131e-2	1.954	1.5494e-1	1.890	4.4490e-3	2.987	7.0064e-3	2.910
	15	1.8065e-2	1.968	6.9459e-2	1.978	1.3292e-3	2.979	2.0778e-3	2.997
$\alpha = 1.2$,	20	1.0174e-2	1.995	3.8822e-2	2.022	5.6313e-4	2.985	8.9218e-4	2.938
	25	6.5298e-3	1.987	2.4759e-2	2.015	2.8911e-4	2.987	4.5781e-4	2.990
$\lambda = 2$	5	8.5199e-2	–	2.3733e-1	–	2.4719e-2	–	3.4142e-2	–
	10	2.1852e-2	1.963	6.6508e-2	1.835	3.1297e-3	2.981	4.1502e-3	3.040
	15	9.8430e-3	1.966	2.9956e-2	1.967	9.3442e-4	2.981	1.2557e-3	2.948
$\alpha = 1.2$	20	5.5538e-3	1.989	1.6788e-2	2.012	3.9556e-4	2.988	5.3514e-4	2.964
	25	3.5690e-3	1.981	1.0831e-2	1.964	2.0295e-4	2.990	2.7573e-4	2.971
$\lambda = 1.3$	5	1.4431e-1	–	5.1838e-1	–	3.3891e-2	–	4.9972e-2	–
	10	3.7749e-2	1.934	1.3417e-1	1.950	4.2843e-3	2.983	6.5301e-3	2.935
	15	1.7126e-2	1.949	6.1960e-2	1.905	1.2796e-3	2.980	1.9517e-3	2.978
$\alpha = 1.8$	20	9.7042e-3	1.974	3.5154e-2	1.970	5.4192e-4	2.986	8.3308e-4	2.959
	25	6.2484e-3	1.972	2.2502e-2	1.999	2.7814e-4	3.989	4.2592e-4	3.006
$\lambda = 3$	5	5.9218e-2	–	1.5930e-1	–	1.8989e-2	–	3.0058e-2	-
	10	1.5650e-2	1.919	4.7201e-2	1.754	2.3758e-3	2.998	3.5575e-3	3.078
	15	7.0786e-3	1.956	2.2004e-2	1.882	7.0820e-4	2.985	1.0799e-3	2.940
$\alpha = 1.8$	20	4.0004e-3	1.983	1.2512e-2	1.962	2.9942e-4	2.992	4.5072e-4	3.037
	25	2.5725e-3	1.978	8.0556e-3	1.973	1.5344e-4	2.995	2.2988e-4	3.017
$\lambda = 0.6$	5	2.7029e-1	–	1.062	–	5.3001e-2	–	9.5102e-2	–
	10	7.1989e-2	1.908	3.3357e-1	1.671	6.6525e-3	2.994	1.2258e-2	2.955
	15	3.3020e-2	1.922	1.5939e-1	1.821	1.9880e-3	2.978	3.6434e-3	2.992
$\alpha = 1.7$	20	1.8888e-2	1.941	9.3468e-2	1.855	8.4275e-4	2.983	1.5373e-3	2.999
	25	1.2222e-2	1.950	6.1440e-2	1.880	4.3290e-4	2.985	8.000e-4	2.927
$\lambda = 0.6$	5	2.7225e-1	–	1.066	–	5.2915e-2	–	9.5224e-2	-
	10	7.1827e-2	1.922	3.3888e-1	1.653	6.6207e-3	2.998	1.2243e-2	2.959
	15	3.2814e-2	1.932	1.6385e-1	1.792	1.9764e-3	2.981	3.6531e-3	2.982
$\alpha = 1.4$	20	1.8563e-2	1.980	9.5235e-2	1.886	8.3725e-4	2.985	1.5412e-3	2.999
	25	1.1938e-2	1.978	6.2214e-2	1.908	4.2982e-4	2.988	7.9451e-4	2.969
$\lambda = 1$	5	1.8691e-1	–	7.1176e-1	–	3.9919e-2	–	5.9499e-2	–
	10	4.8653e-2	1.941	2.0172e-1	1.819	5.0144e-3	2.993	8.2344e-3	2.853
	15	2.2158e-2	1.939	9.3467e-2	1.897	1.4973e-3	2.980	2.4757e-3	2.964
$\alpha = 1.5$	20	1.2538e-2	1.979	5.3192e-2	1.959	6.3428e-4	2.985	1.0466e-3	2.993
	25	8.0674e-3	1.975	3.4312e-2	1.964	3.2563e-4	2.987	5.3565e-4	3.001
$\lambda = 2.5$	10	2.5646e-02	–	6.5256e-02	–	4.6142e-03	-	5.9270e-03	–
	20	6.5595e-03	1.9671	1.6603e-02	1.9747	5.8563e-04	2.9780	7.3137e-04	3.0186
	40	1.6557e-03	1.9957	4.2239e-03	1.9983	7.3712e-05	2.9927	9.2613e-05	2.9773
$\alpha = 1.5$	60	7.1215e-04	2.0807	1.8215e-03	2.0744	2.1851e-05	2.9988	2.6851e-05	3.0535
	80	4.0121e-04	1.9945	9.9821e-04	2.0906	9.2482e-06	2.9887	1.1382e-05	2.9833

therefore,

$$
\begin{aligned}
&||\eta_u(.,T)||^2 + 2\left(\cos\left(\left(\frac{\alpha}{2}-1\right)\pi\right) - C_1\varepsilon\right)\int_0^T \|{}_a\mathcal{I}_x^{1-\alpha/2,\lambda}\eta_p(.,t)\|^2\mathrm{d}t \\
&\quad + c\int_0^T\left(\sum_{j=1}^{N-1}([\eta_u]_j)^2 + \eta_u(x_0^+,t)^2\right)\mathrm{d}t + (c+\delta)\int_0^T \eta_u(x_N^-,t)^2\mathrm{d}t \\
&\quad \le C(h^{2k+2}) + 2\left(1 + 5Cd\lambda^2 + \frac{4d\lambda^2}{\varepsilon}\right)\int_0^T ||\eta_u(.,t)||^2\mathrm{d}t. \qquad (4.30)
\end{aligned}
$$

The following inequality is obtained by choosing sufficiently small $\varepsilon > 0$ in relation (4.30) and continuous Gronwall inequality (Lem. 2.8)

$$||\eta_u(.,T)||^2 \leq \exp(\tilde{C}T)(h^{2k+2}), \tag{4.31}$$

where $\tilde{C} := 2\left(1 + 5Cd\lambda^2 + \dfrac{4d\lambda^2}{\varepsilon}\right)$. Inequalities (4.19) and (4.31) complete the proof as follows

$$\|u(.,T) - U(.,T)\| \leq \|\theta_u(.,T)\| + \|\eta_u(.,T)\| \leq Ch^{k+1}. \tag{4.32}$$

□

5. Numerical examples and conclusion

Numerical examples are presented to confirm the theory of the proposed scheme. The tables show that the convergence rate of the method and the reduction of errors by decreasing h and increasing k (h is the mesh size on space and k is the degree of polynomial). We employ RK4 (fourth order explicit Runge-Kutta method) to solve the method-of-line fractional PDE. Then we apply the LDG method on the classical ODE system. We set $\triangle t = 0.1h^{k+1}$.

Example 5.1. Consider tempered fractional convection–diffusion equation (1.1), where $c = 1$ and $d = \frac{\Gamma(6-\alpha)}{6}$ with exact solution $u(x,t) = t^3((1+x)^m - 2(1+x)^{m-1})\exp(-\lambda x)$ on $[-1,1]\times[0,1]$. Using the above conditions, it is easy to obtain $f(x,t)$. Tables 1 and 2 show L^2-error, L^∞-error and order of the method by different values of λ and α for $m = 6$ and $m = 4.2$. Figure 1 illustrates L^2-error, L^∞-error and the rate of convergence the method.

Example 5.2. Consider tempered fractional convection–diffusion equation (1.1), where $c = 3$ and $d = \frac{\Gamma(m-\alpha)}{m}$ with exact solution $u(x,t) = \exp(-\lambda x - t)(x)^m(x-1)$ on $[0,1]\times[0,1]$. Tables 3 and 4 illustrate L^2-error, L^∞-error and order of the method by different values of λ and α, for $m = 5$ and $m = 7.3$ (Fig. 2).

A discontinuous Galerkin method is proposed for tempered fractional convection–diffusion equations. The stability and convergence analysis of the method have been presented. The suitable choice of numerical fluxes helped us to achieve order of convergence $O(h^{k+1})$. The theory of the proposed method is confirmed by the presented numerical examples for different values of α and λ.

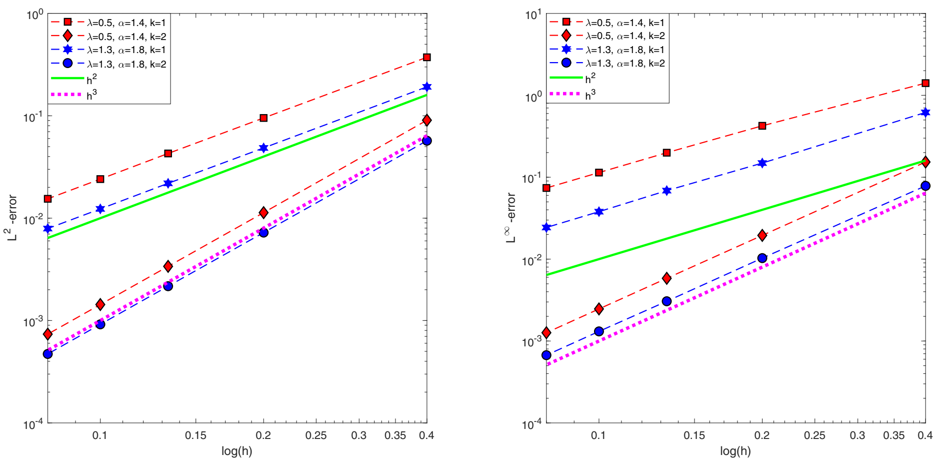

Figure 1. Convergence rate for Example 5.1 for $\mathbb{P}^k, k = 1, 2$ and $m = 5$.

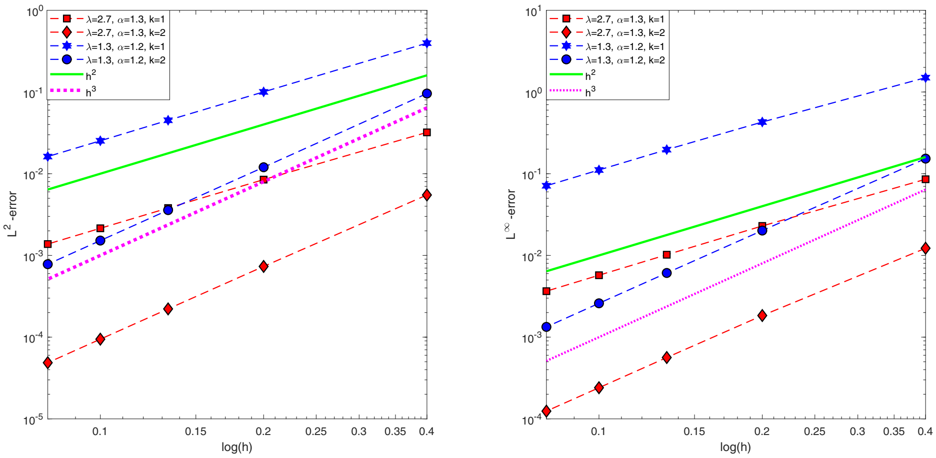

FIGURE 2. Convergence rate for Example 5.2 for $\mathbb{P}^k, k = 1, 2$ and $m = 7.2$.

TABLE 3. Example 5.2, $m = 5$.

		$k=1$				$k=2$			
	N	L^2-error	Order	L^∞-error	Order	L^2-error	Order	L^∞-error	Order
$\lambda = 2$	5	3.3186e-2	–	5.8542e-2	–	1.0017e-2	–	1.5275e-2	–
	10	7.4684e-3	2.1517	1.4551e-2	2.0083	1.3686e-3	2.8716	2.0126e-3	2.9240
$\alpha = 1.4$	15	3.2945e-3	2.0185	5.8235e-3	2.2586	4.1504e-4	2.9428	5.8638e-4	3.0415
	20	1.8698e-3	1.9690	3.4369e-3	1.8330	1.7570e-4	2.9881	2.4105e-4	3.0901
	25	1.2070e-3	1.9614	2.2402e-3	1.9182	8.9798e-5	3.0079	1.2515e-4	2.9376
$\lambda = 0.6$	5	4.5546e-2	–	1.8238e-1	–	5.7544e-3	–	1.6627e-2	–
	10	1.1648e-2	1.9673	5.6051e-2	1.7021	7.3584e-4	2.9672	1.9069e-3	3.1242
$\alpha = 1.4$	15	5.2317e-3	1.9740	2.6835e-2	1.8165	2.1521e-4	3.0321	5.5157e-4	3.0594
	20	2.9547e-3	1.9860	1.5620e-2	1.8810	8.9934e-5	3.0329	2.3149e-4	3.0180
	25	1.8925e-3	1.9963	1.0165e-2	1.9255	4.5761e-5	3.0279	1.1735e-4	3.0445
$\lambda = 0.9$	5	3.4837e-2	–	1.2626e-1	–	3.4878e-3	–	9.5755e-3	–
	10	8.7037e-3	2.0009	3.4933e-2	1.8537	4.6149e-4	2.9179	1.2982e-3	2.8829
$\alpha = 1.8$	15	3.9119e-3	1.9724	1.6071e-2	1.9149	1.3906e-4	2.9585	3.9271e-4	2.9488
	20	2.2150e-3	1.9770	9.1308e-3	1.9651	5.9342e-5	2.9601	1.6739e-4	2.9643
	25	1.4223e-3	1.9853	5.8494e-3	1.9957	3.0694e-5	2.9544	8.6304e-5	2.9686
$\lambda = 1.2$	5	2.4549e-2	–	8.4976e-2	–	6.3092e-3	–	1.1891e-2	–
	10	6.8210e-3	1.8476	2.8193e-2	1.5917	8.2205e-4	2.9402	1.9654e-3	2.5970
$\alpha = 1.1$	15	3.1044e-3	1.9415	1.3655e-2	1.7879	2.4435e-4	2.9921	6.3386e-4	2.7909
	20	1.7626e-3	1.9676	8.0110e-3	1.8539	1.0297e-4	3.0040	2.7849e-4	2.8589
	25	1.1317e-3	1.9853	5.2503e-3	1.8935	5.2617e-5	3.0087	1.4600e-4	2.8939
$\lambda = 1.2$	5	2.5946e-2	–	8.6520e-2	–	3.9978e-3	–	7.3907e-3	–
	10	6.5948e-3	1.9761	2.4310e-2	1.8315	5.3124e-4	2.9118	1.0704e-3	2.7876
$\alpha = 1.6$	15	2.9709e-3	1.9667	1.1161e-2	1.9200	1.5938e-4	2.9693	3.2208e-4	2.9619
	20	1.6836e-3	1.9742	6.2897e-3	1.9934	6.7671e-5	2.9777	1.3805e-4	2.9448
	25	1.0815e-3	1.9833	3.9878e-3	2.0421	3.4812e-5	2.9788	7.1330e-5	2.9591
$\lambda = 0.2$	5	7.0122e-2	–	2.8276e-1	–	7.7594e-3	–	3.2959e-2	–
	10	1.7799e-2	1.9781	8.8483e-2	1.6761	9.9746e-4	2.9596	3.9458e-3	3.0623
$\alpha = 1.6$	15	8.1000e-3	1.9417	4.3015e-2	1.7789	2.9367e-4	3.0157	1.0979e-3	3.1551
	20	4.6106e-3	1.9588	2.5230e-2	1.8544	1.2358e-4	3.0087	4.5431e-4	3.0671
	25	2.9692e-3	1.9721	1.6509e-2	1.9008	6.3254e-5	3.0015	2.3080e-4	3.0349

Table 4. Example 5.2, $m = 7.3$.

		$k=1$				$k=2$			
	N	L^2- error	Order	L^∞- error	order	L^2- error	order	L^∞- error	order
$\lambda = 2.3$	5	4.7231e-2	–	1.5554e-1	–	5.8369e-3	–	1.6629e-2	–
	10	1.1437e-2	2.0461	3.9667e-2	1.9713	7.9887e-4	2.8692	2.2899e-3	2.8604
$\lambda = 1.5$	15	15.0997e-3	1.9919	1.8071e-2	1.9390	2.4402e-4	2.9249	7.0693e-4	2.8987
	20	2.9017e-3	1.9601	1.0596e-2	1.8556	1.0441e-4	2.9509	3.0284e-4	2.9467
	25	1.8676e-3	1.9747	6.9375e-3	1.8981	5.3891e-5	2.9638	1.5611e-4	2.9695
$\lambda = 3.5$	5	2.3057e-2	–	6.7479e-2	–	5.2590e-3	–	1.0536e-2	–
	10	5.6374e-3	2.0321	1.7844e-2	1.9190	7.2847e-4	2.8518	1.5967e-3	2.7222
$\alpha = 1.9$	15	2.5565e-3	1.9503	7.5717e-3	2.1143	2.2129e-4	2.9386	5.3059e-4	2.7172
	20	1.4664e-3	1.9322	4.1398e-3	2.0987	9.4468e-5	2.9588	2.2870e-4	2.9254
	25	9.5089e-4	1.9411	2.6410e-3	2.0144	4.8726e-5	2.9669	1.1805e-4	2.96347
$\lambda = 1.9$	5	7.2169e-2	–	2.7200e-1	–	6.6739e-3	–	2.4221e-2	–
	10	1.7467e-2	2.0467	7.3300e-2	1.8917	8.9239e-4	2.9028	3.1679e-3	2.9347
$\alpha = 2.3$	15	7.7384e-3	2.0079	3.3305e-2	1.9455	2.7068e-4	2.9422	9.7385e-4	2.9092
	20	4.3624e-3	1.9924	1.8939e-2	1.9622	1.1574e-4	2.9532	4.1371e-4	2.9759
	25	2.7956e-3	1.9941	1.2167e-2	1.9830	5.9861e-5	2.9547	2.1382e-4	2.9578
$\lambda = 1.3$	10	3.2188e-2	–	1.6286e-1	–	1.8451e-3	–	7.7056e-3	–
$\alpha = 1.3$	15	1.4422e-2	1.9800	7.8409e-2	1.8027	5.5088e-4	2.9811	2.2252e-3	3.0634
	20	8.1481e-3	1.9848	4.6081e-2	1.8477	2.3238e-4	3.0004	9.2894e-4	3.0365
	25	5.2174e-3	1.9978	3.0241e-2	1.8876	1.1880e-4	3.0067	4.7487e-4	3.0071
$\lambda = 0.1$	10	1.2475e-1	–	7.1039e-1	–	1.0381e-2	–	6.5433e-2	–
$\alpha = 1.2$	15	5.5762e-2	1.9860	3.6306e-1	1.6555	3.1462e-3	2.9442	2.0973e-2	2.8061
	20	3.1309e-2	2.0063	2.1913e-1	1.7551	1.3388e-3	2.9699	9.1745e-3	2.8741
	25	1.9952e-2	2.0193	1.4613e-1	1.8157	6.8834e-4	2.9814	4.7949e-3	2.9079
$\lambda = 0.1$	5	4.7766e-1	–	1.8946	–	7.1437e-2	–	3.7866e-1	–
	10	1.2197e-1	1.9694	6.8707e-1	1.4634	9.8365e-3	2.8605	6.2617e-2	2.5963
	15	5.4142e-2	2.0030	3.4499e-1	1.6991	2.9692e-3	2.9541	2.0142e-2	2.7974
$\alpha = 1.5$	20	3.0892e-2	1.9505	2.1007e-1	1.7243	1.2547e-3	2.9943	8.8037e-3	2.8769
	25	1.9876e-2	1.9761	1.4069e-1	1.7965	6.4053e-4	3.0131	4.5921e-3	2.9166

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