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A POSTERIORI ERROR ESTIMATES FOR DARCY'S PROBLEM COUPLED WITH THE HEAT EQUATION

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Abstract. This work derives *a posteriori* error estimates, in two and three dimensions, for the heat equation coupled with Darcy's law by a nonlinear viscosity depending on the temperature. We introduce two variational formulations and discretize them by finite element methods. We prove optimal *a posteriori* errors with two types of computable error indicators. The first one is linked to the linearization and the second one to the discretization. Then we prove upper and lower error bounds under regularity assumptions on the solutions. Finally, numerical computations are performed to show the effectiveness of the error indicators.

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1. Introduction

The present work investigates *a posteriori* error estimates of the finite element discretization of a heat equation coupled with Darcy's law by a nonlinear viscosity depending on the temperature in polygonal or polyhedral domains. The *a posteriori* analysis controls the overall discretization error of a problem by providing error indicators that are easy to compute. Once these error indicators are constructed, their efficiency can be proven by bounding each indicator by the local error. *A posteriori* analysis was first introduced by Babuška [3], developed by Verfürth [28], and has been the object of a large number of publications. *A posteriori* error estimations have been studied for several types of partial differential equations. For the stationary Navier–Stokes equations, we can refer for instance to [5,16,17,21,22]. For the stationary Boussinesq model, we refer to [14,15]. Many works have been established for the Darcy flow, see for instance [2,8,10,23]. In [11], Chen and Wang establish optimal *a poteriori* error estimates for the $H(\mathrm{div},\Omega)$ conforming mixed finite element method applied to the coupled Darcy–Stokes system in two dimensions, but excludes the RT_0 element that we shall study in the present work. For the Darcy equations with pressure dependent viscosity, we refer to [18] and the references therein.

In this article, we consider the heat equation coupled with Darcy's law by a nonlinear viscosity depending on the temperature. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded simply-connected domain, with a Lipschitz-continuous

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boundary Γ. This work studies the temperature distribution of a fluid in a porous medium modelled by a convection-diffusion equation coupled with Darcy's law. The system of equations is the following:

$$(P)\begin{cases} \nu(T)\mathbf{u}+\nabla p &= \mathbf{f} \quad \text{in}\,\Omega,\\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in}\,\Omega,\\ -\alpha\Delta T+(\mathbf{u}\cdot\nabla T) &= g \quad \text{in}\,\Omega,\\ \mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on}\,\Gamma,\\ T &= 0 \quad \text{on}\,\Gamma,\end{cases}$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. The unknowns are the velocity $\mathbf{u}$, the pressure p and the temperature T of the fluid. The function $\mathbf{f}$ represents an external density force and g an external heat source. The parameter α is a positive constant that corresponds to the diffusion coefficient. The viscosity ν depends on the temperature; it is a positive-valued function that satisfies the following assumptions:

Assumption 1.1. *We assume that*

- *ν belongs to $W^{1,\infty}(\mathbb{R})$. Therefore ν is a Lipschitz-continuous function with Lipschitz constant λ, i.e.,*

$$\forall s,t\in\mathbb{R},\quad |\nu(s)-\nu(t)|\le\lambda|s-t|. \tag{1.1}$$

- *There exist two positive constants ν_1 and ν_2 such that for any $s\in\mathbb{R}$*

$$\nu_1\le\nu(s)\le\nu_2. \tag{1.2}$$

In [7], the above problem was treated by using finite element methods combined with a Picard iterative algorithm to solve for the nonlinearity. Two numerical schemes were analyzed and an optimal *a priori* error estimate was established, together with convergence of the algorithm. In the present paper, we establish optimal *a posteriori* error estimates including algorithmic effects as well as the influence of the nonlinear function ν. The theory is validated by corresponding numerical experiments.

This article is organized as follows:

- Section 2 is devoted to the continuous problem.
- In Section 3, we introduce the discrete and iterative problems and recall their main properties.
- In Section 4, we introduce the error indicators and prove the upper and lower error *a posteriori* bounds for the first approximation.
- In Section 5, we introduce the error indicators and prove the upper and lower *a posteriori* error bounds for the second approximation.
- The theory is validated by numerical results in Section 6.

2. Variational formulations

In order to introduce the variational formulations, we recall some classical Sobolev spaces and their properties.

Let $\alpha=(\alpha_1,\alpha_2,\ldots\alpha_d)$ be a d-uple of non negative integers, set $|\alpha|=\sum_{i=1}^d\alpha_i$, and define the partial derivative ∂^α by

$$\partial^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}}\cdot$$

Then, for any positive integer m and number $p\ge1$, we recall the classical Sobolev space (Adams [1] or Nečas [24])

$$W^{m,p}(\Omega)=\{v\in L^p(\Omega);\,\forall\,|\alpha|\le m,\ \partial^\alpha v\in L^p(\Omega)\}, \tag{2.1}$$

equipped with the seminorm

$$|v|_{W^{m,p}(\Omega)} = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^p \, d\mathbf{x} \right\}^{\frac{1}{p}} \tag{2.2}$$

and the norm

$$\|v\|_{W^{m,p}(\Omega)} = \left\{ \sum_{0\le k\le m} |v|^p_{W^{k,p}(\Omega)} \right\}^{\frac{1}{p}}. \tag{2.3}$$

When $p = 2$, this space is the Hilbert space $H^m(\Omega)$. In particular, the scalar product of $L^2(\Omega)$ is denoted by $(.,.)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{v}$ be a vector valued function; we set

$$\|\mathbf{v}\|_{L^p(\Omega)} = \left(\int_\Omega |\mathbf{v}|^p \, d\mathbf{x} \right)^{\frac{1}{p}}, \tag{2.4}$$

where $|.|$ denotes the Euclidean vector norm.

For vanishing boundary values, we define

$$H^1_0(\Omega) = \{v \in H^1(\Omega);\ v_{|\Gamma} = 0\}. \tag{2.5}$$

We shall often use the following Sobolev imbeddings: for any real number $p \ge 1$ when $d = 2$, or $1 \le p \le 6$ when $d = 3$, there exist constants S_p and S^0_p such that

$$\forall\, v \in H^1(\Omega),\ \|v\|_{L^p(\Omega)} \le S_p \|v\|_{H^1(\Omega)} \tag{2.6}$$

and

$$\forall\, v \in H^1_0(\Omega),\ \|v\|_{L^p(\Omega)} \le S^0_p |v|_{H^1(\Omega)}. \tag{2.7}$$

When $p = 2$, (2.7) reduces to Poincaré's inequality.

Recall the standard spaces for Darcy's equations

$$L^2_m(\Omega) = \left\{ v \in L^2(\Omega);\ \int_\Omega v \, d\mathbf{x} = 0 \right\}, \tag{2.8}$$

$$H(\mathrm{div}, \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^d;\ \mathrm{div}\, \mathbf{v} \in L^2(\Omega) \right\}, \tag{2.9}$$

$$H_0(\mathrm{div}, \Omega) = \{ \mathbf{v} \in H(\mathrm{div}, \Omega);\ (\mathbf{v} \cdot \mathbf{n})|_\Gamma = 0 \}, \tag{2.10}$$

equipped with the norm

$$\|\mathbf{v}\|^2_{H(\mathrm{div},\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\mathrm{div}\, \mathbf{v}\|^2_{L^2(\Omega)}. \tag{2.11}$$

We also define the kernel of the divergence in $H_0(\mathrm{div}, \Omega)$,

$$V = \{ \mathbf{v} \in H_0(\mathrm{div}, \Omega);\ \mathrm{div}\, \mathbf{v} = 0 \}. \tag{2.12}$$

The spaces $L^2_m(\Omega)$ and $H_0(\mathrm{div}, \Omega)$ (resp. $H^1(\Omega) \cap L^2_m(\Omega)$ and $L^2(\Omega)^d$) satisfy the following inf-sup conditions:

$$\inf_{q \in L^2_m(\Omega)} \sup_{\mathbf{v} \in H_0(\mathrm{div},\Omega)} \frac{\displaystyle\int_\Omega q \, \mathrm{div}\, \mathbf{v} \, d\mathbf{x}}{\|\mathbf{v}\|_{H(\mathrm{div},\Omega)} \|q\|_{L^2(\Omega)}} \ge \beta, \tag{2.13}$$

with a constant $\beta > 0$, and,

$$\inf_{q\in H^1(\Omega)\cap L^2_m(\Omega)} \sup_{\mathbf{v}\in L^2(\Omega)^d} \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\nabla q \,\mathrm{d}\mathbf{x}}{\|\mathbf{v}\|_{L^2(\Omega)^d}|q|_{H^1(\Omega)}} \geq 1. \tag{2.14}$$

For more details we refer to Bernardi *et al.* [7].

We introduce the two following variational problems equivalent to problem (P):

$$(V_1)\begin{cases} \text{Find } (\mathbf{u},p,T)\in H_0(\mathrm{div},\Omega)\times L^2_m(\Omega)\times H^1_0(\Omega) \text{ such that}\\ \forall\,\mathbf{v}\in H_0(\mathrm{div},\Omega),\quad \displaystyle\int_\Omega \nu(T)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_\Omega p\,\mathrm{div}\,\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\\ \forall\,q\in L^2_m(\Omega),\quad \displaystyle\int_\Omega q\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\mathbf{x} = 0,\\ \forall\,S\in H^1_0(\Omega)\cap L^\infty(\Omega),\quad \alpha\displaystyle\int_\Omega \nabla T\cdot\nabla S\,\mathrm{d}\mathbf{x} + \int_\Omega(\mathbf{u}\cdot\nabla T)S\,\mathrm{d}\mathbf{x} = \int_\Omega g\,S\,\mathrm{d}\mathbf{x}, \end{cases}$$

and

$$(V_2)\begin{cases} \text{Find } (\mathbf{u},p,T)\in L^2(\Omega)^d\times (H^1(\Omega)\cap L^2_m(\Omega))\times H^1_0(\Omega) \text{ such that}\\ \forall\,\mathbf{v}\in L^2(\Omega)^d,\quad \displaystyle\int_\Omega \nu(T)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla p\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\\ \forall\,q\in H^1(\Omega)\cap L^2_m(\Omega),\quad \displaystyle\int_\Omega \nabla q\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = 0,\\ \forall\,S\in H^1_0(\Omega)\cap L^\infty(\Omega),\quad \alpha\displaystyle\int_\Omega \nabla T\cdot\nabla S\,\mathrm{d}\mathbf{x} + \int_\Omega(\mathbf{u}\cdot\nabla T)\,S\,\mathrm{d}\mathbf{x} = \int_\Omega g\,S\,\mathrm{d}\mathbf{x}. \end{cases}$$

The variational problem (V_1) is well adapted to locally conservative discrete schemes while (V_2) leads to numerical schemes that are more easily implemented. For the existence and uniqueness of the solutions of problems (V_1) and (V_2), and their equivalence to problem (P), we refer to [7].

3. Discretization

From now on, we assume that Ω is a polygon when $d = 2$ or polyhedron when $d = 3$, so it can be completely meshed. For the space discretization, we consider a regular (see Ciarlet [12]) family of triangulations $(\mathcal{T}_h)_h$ of Ω which is a set of closed non degenerate triangles for $d = 2$ or tetrahedra for $d = 3$, called elements, satisfying,

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- the intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge (or face when $d = 3$);
- the ratio of the diameter h_K of an element K in $\mathcal{T}_h$ to the diameter ρ_K of its inscribed circle when $d = 2$ or ball when $d = 3$ is bounded by a constant independent of h: there exists a positive constant σ independent of h such that,

$$\max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K} \leq \sigma. \tag{3.1}$$

As usual, h denotes the maximal diameter of all elements of $\mathcal{T}_h$. To define the finite element functions, let r be a non negative integer. For each K in $\mathcal{T}_h$, we denote by $\mathbb{P}_r(K)$ the space of restrictions to K of polynomials in d variables and total degree at most r, with a similar notation on the faces or edges of K. For every edge (when $d = 2$) or face (when $d = 3$) e of the mesh $\mathcal{T}_h$, we denote by h_e the diameter of e.

In what follows, $c, c', C, C', c_1, \ldots$ stand for generic constants which may vary from line to line but are always independent of h.

We shall use the following inverse inequalities: for any number $p \geq 2$, for any dimension d, and for any non negative integer r, there exist constants $C_I^0(p)$ and $C_I^1(p)$ such that for any polynomial function v_h of degree r on K,

$$\|v_h\|_{L^p(K)} \leq C_I^0(p) h_K^{\frac{d}{p}-\frac{d}{2}} \|v_h\|_{L^2(K)} \tag{3.2}$$

and

$$|v_h|_{H^1(K)} \leq C_I^1(p) h_K^{\frac{d}{2}-\frac{d}{p}-1} \|v_h\|_{L^p(K)}. \tag{3.3}$$

The constants C_I^0 and C_I^1 depend also on the regularity parameter σ of (3.1), but for the sake of simplicity this is not indicated.

For a given triangulation $\mathcal{T}_h$, we define the following finite dimensional spaces:

$$Z_h = \{S_h \in \mathcal{C}^0(\bar{\Omega});\ \forall K \in \mathcal{T}_h,\ S_h|_K \in \mathbb{P}_1(K)\} \quad \text{and} \quad X_h = Z_h \cap H_0^1(\Omega). \tag{3.4}$$

There exists an approximation operator (when $d = 2$, see Bernardi and Girault [4] or Clément [13]; when $d = 2$ or $d = 3$, see Scott and Zhang [27]) R_h in $\mathcal{L}(W^{1,p}(\Omega); Z_h)$ and in $\mathcal{L}(W^{1,p}(\Omega) \cap H_0^1(\Omega); X_h)$ such that for all K in $\mathcal{T}_h$, $m = 0, 1$, $l = 0, 1$, and all $p \geq 2$,

$$\forall S \in W^{l+1,p}(\Omega),\ |S - R_h(S)|_{W^{m,p}(K)} \leq C(p, m, l)\, h_K^{l+1-m} |S|_{W^{l+1,p}(\omega_K)}, \tag{3.5}$$

where ω_K is the union of elements of $\mathcal{T}_h$ that intersect K, including K itself.

3.1. Discrete schemes

3.1.1. First discrete scheme

The velocity is discretized by the Raviart–Thomas RT_0 [25] elements. More precisely, the discrete spaces $(\mathcal{W}_{h,1}, M_{h,1})$ are defined as follows:

$$\begin{aligned} \mathcal{W}_h &= \{\mathbf{v}_h \in H(\mathrm{div}, \Omega);\ \mathbf{v}_h(\mathbf{x})|_K = a_K \mathbf{x} + \mathbf{b}_K, a_K \in \mathbb{R}, \mathbf{b}_K \in \mathbb{R}^d, \forall K \in \mathcal{T}_h\}, \\ \mathcal{W}_{h,1} &= \mathcal{W}_h \cap H_0(\mathrm{div}, \Omega), \qquad (3.6) \\ M_h &= \{q_h \in L^2(\Omega);\ \forall K \in \mathcal{T}_h,\ q_h|_K \text{ is constant}\} \quad \text{and} \quad M_{h,1} = M_h \cap L_m^2(\Omega). \qquad (3.7) \end{aligned}$$

The kernel of the divergence in $\mathcal{W}_{h,1}$ is denoted by $\mathcal{V}_{h,1}$,

$$\mathcal{V}_{h,1} = \{\mathbf{v}_h \in \mathcal{W}_{h,1};\ \mathrm{div}\, \mathbf{v}_h = 0 \text{ in } \Omega\}. \tag{3.8}$$

The following discrete inf-sup condition holds (see Roberts and Thomas [26]):

$$\forall q_h \in M_{h,1},\ \sup_{\mathbf{v}_h \in \mathcal{W}_{h,1}} \frac{\displaystyle\int_\Omega q_h \,\mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x}}{\|\mathbf{v}_h\|_{H(\mathrm{div},\Omega)}} \geq \beta_1 \|q_h\|_{L^2(\Omega)}, \tag{3.9}$$

with a constant $\beta_1 > 0$ independent of h.

We then consider the straightforward discretization of problem (V_1):

$$(V_{h,1}) \begin{cases} \text{Find } (\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h \text{ such that} \\ \forall \mathbf{v}_h \in \mathcal{W}_{h,1},\ \displaystyle\int_\Omega \nu(T_h)\mathbf{u}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_\Omega p_h \,\mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}, \\ \forall q_h \in M_{h,1},\ \displaystyle\int_\Omega q_h \,\mathrm{div}\, \mathbf{u}_h \,\mathrm{d}\mathbf{x} = 0, \\ \forall S_h \in X_h,\ \alpha \displaystyle\int_\Omega \nabla T_h \cdot \nabla S_h \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u}_h \cdot \nabla T_h) S_h \,\mathrm{d}\mathbf{x} = \int_\Omega g\, S_h \,\mathrm{d}\mathbf{x}. \end{cases}$$

For the existence and uniqueness of the solution of problem $(V_{h,1})$, we refer to [7]. We recall the theorem of *a priori error estimates* [7]:

Theorem 3.1. *Let $d = 3$ and ν satisfy (1.1) and (1.2). We suppose that problem (V_1) has a solution $(\mathbf{u}, p, T) \in H^1(\Omega)^3 \times H^1(\Omega) \times W^{2,3}(\Omega)$, such that*

$$\lambda\,(S_6^0)^2\,\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)} < \alpha\,\nu_1. \tag{3.10}$$

Let the mesh satisfy (3.1). Then the following error inequality between the solutions of (V_1) and $(V_{h,1})$ holds:

$$\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathrm{div},\Omega)} + \|p-p_h\|_{L^2(\Omega)} + |T-T_h|_{H^1(\Omega)} \le C_1\,h\,\big(|\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^1(\Omega)} + |T|_{W^{2,3}(\Omega)}\big). \tag{3.11}$$

3.1.2. Second discrete scheme

Let K be an element of $\mathcal{T}_h$ with vertices $\mathbf{a}_i$, $1 \le i \le d+1$, and corresponding barycentric coordinates λ_i. We denote by $\psi_K \in \mathbb{P}_{d+1}(K)$ the basic bubble function

$$\psi_K(\mathbf{x}) = \lambda_1(\mathbf{x}) \dots \lambda_{d+1}(\mathbf{x}). \tag{3.12}$$

We observe that $\psi_K(\mathbf{x}) = 0$ on ∂K and that $\psi_K(\mathbf{x}) > 0$ in the interior of K.

Let $(\mathcal{W}_{h,2}, M_{h,2})$ be a pair of discrete spaces approximating $L^2(\Omega)^d \times \big(H^1(\Omega) \cap L^2_m(\Omega)\big)$ defined by

$$\mathcal{W}_{h,2} = \Big\{\mathbf{v}_h \in \mathcal{C}^0(\bar\Omega)^d;\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_h|_K \in \mathcal{P}(K)^d\Big\}, \tag{3.13}$$

$$\tilde M_h = \big\{q_h \in \mathcal{C}^0(\bar\Omega);\ \forall K \in \mathcal{T}_h,\ q_h|_K \in \mathbb{P}_1(K)\big\} \quad \text{and} \quad M_{h,2} = \tilde M_h \cap L^2_m(\Omega), \tag{3.14}$$

where

$$\mathcal{P}(K) = \mathbb{P}_1(K) \oplus \mathrm{Vect}\{\psi_K\}.$$

We approximate problem (V_2) by the following discrete scheme:

$$(V_{h,2})\begin{cases} \text{Find } (\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,2} \times M_{h,2} \times X_h \text{ such that} \\ \forall\,\mathbf{v}_h \in \mathcal{W}_{h,2},\ \ \displaystyle\int_\Omega \nu(T_h)\mathbf{u}_h\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla p_h\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x}, \\ \forall\,q_h \in M_{h,2},\ \ \displaystyle\int_\Omega \nabla q_h\cdot\mathbf{u}_h\,\mathrm{d}\mathbf{x} = 0, \\ \forall\,S_h \in X_h,\ \ \alpha\displaystyle\int_\Omega \nabla T_h\cdot\nabla S_h\,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u}_h\cdot\nabla T_h)S_h\,\mathrm{d}\mathbf{x} \\ \qquad\qquad +\displaystyle\frac{1}{2}\int_\Omega (\mathrm{div}\,\mathbf{u}_h)T_h\,S_h\,\mathrm{d}\mathbf{x} = \int_\Omega g\,S_h\,\mathrm{d}\mathbf{x}. \end{cases}$$

For the existence and uniqueness of the solution of problem $(V_{h,2})$, we refer to [7]. In particular, the following inf-sup condition, analogous to (3.9), is valid with another constant $\beta_2 > 0$ independent of h, see [7]:

$$\forall\,q_h \in M_{h,2},\ \sup_{\mathbf{v}_h\in\mathcal{W}_{h,2}} \frac{\displaystyle\int_\Omega \nabla q_h\cdot\mathbf{v}_h\,\mathrm{d}\mathbf{x}}{\|\mathbf{v}_h\|_{L^2(\Omega)^d}} \ge \beta_2\,|q_h|_{H^1(\Omega)}. \tag{3.15}$$

We recall the theorem of *a priori error estimates* [7]:

Theorem 3.2. *We retain the settings and assumptions of Theorem 3.1; in addition, we suppose that $(p, T) \in H^2(\Omega) \times \big(W^{1,\infty}(\Omega) \cap W^{2,3}(\Omega)\big)$ and*

$$\lambda\,S_6^0\,\|\mathbf{u}\|_{L^3(\Omega)^3}\big(S_6^0\,|T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)}\big) < 2\,\alpha\,\nu_1. \tag{3.16}$$

Then the following error inequality between the solutions of problems (V_2) and $(V_{h,2})$ holds:

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)^3} + |p-p_h|_{H^1(\Omega)} + |T-T_h|_{H^1(\Omega)} \le C_2\,h\,\big(|\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^2(\Omega)} + |T|_{W^{2,3}(\Omega)} + |T|_{W^{1,\infty}(\Omega)}\big). \tag{3.17}$$

3.2. Successive approximations

As the problem is nonlinear, we introduce a straightforward successive approximation algorithm which converges to the discrete solution under suitable conditions. We present the following Picard iterative problems:

$$
(V_{h,i,1}) \begin{cases} \text{For given } T_h^i \in X_h, \text{ find } (\mathbf{u}_h^{i+1}, p_h^{i+1}, T_h^{i+1}) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h \text{ such that} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_{h,1}, \ \displaystyle\int_\Omega \nu(T_h^i)\mathbf{u}_h^{i+1} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_\Omega p_h^{i+1} \operatorname{div} \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}, \\ \forall\, q_h \in M_{h,1}, \ \displaystyle\int_\Omega q_h \operatorname{div} \mathbf{u}_h^{i+1} \,\mathrm{d}\mathbf{x} = 0, \\ \forall\, S_h \in X_h, \ \alpha \displaystyle\int_\Omega \nabla T_h^{i+1} \cdot \nabla S_h \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1}) S_h \,\mathrm{d}\mathbf{x} = \int_\Omega g\, S_h \,\mathrm{d}\mathbf{x}, \end{cases}
$$

and

$$
(V_{h,i,2}) \begin{cases} \text{For given } T_h^i \in X_h, \text{ find } (\mathbf{u}_h^{i+1}, p_h^{i+1}, T_h^{i+1}) \in \mathcal{W}_{h,2} \times M_{h,2} \times X_h \text{ such that} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_{h,2}, \ \displaystyle\int_\Omega \nu(T_h^i)\mathbf{u}_h^{i+1} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_\Omega \nabla p_h^{i+1} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}, \\ \forall\, \mathbf{q}_h \in M_{h,2}, \ \displaystyle\int_\Omega \nabla q_h \cdot \mathbf{u}_h^{i+1} \,\mathrm{d}\mathbf{x} = 0, \\ \forall\, S_h \in X_h, \ \alpha \displaystyle\int_\Omega \nabla T_h^{i+1} \cdot \nabla S_h \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1}) S_h \,\mathrm{d}\mathbf{x} \\ \qquad\qquad + \dfrac{1}{2} \displaystyle\int_\Omega (\operatorname{div} \mathbf{u}_h^{i+1}) T_h^{i+1}\, S_h \,\mathrm{d}\mathbf{x} = \int_\Omega g\, S_h \,\mathrm{d}\mathbf{x}. \end{cases}
$$

The following stability bounds are proved for both problems in reference [7]:

$$
\begin{aligned} \|\mathbf{u}_h^i\|_{L^2(\Omega)^d} &\le \frac{1}{\nu_1} \|\mathbf{f}\|_{L^2(\Omega)^d}, \\ |T_h^i|_{H^1(\Omega)} &\le \frac{S_2^0}{\alpha} \|g\|_{L^2(\Omega)}, \\ \|p_h^i\|_{L^2(\Omega)} &\le \frac{1}{\beta_k} \|\mathbf{f}\|_{L^2(\Omega)^d} \left(1 + \frac{\nu_2}{\nu_1}\right), \ k = 1, 2, \end{aligned} \tag{3.18}
$$

where β_k refers to the inf-sup constant of the k-th discretization, $k = 1, 2$. The reader will also find convergence of the algorithm to the solution of the continuous problem when i tends to $+\infty$ and h tends to 0. We will demonstrate in the following theorems the convergence of the iterative solution to the discrete solution when i tends to $+\infty$ for all h sufficiently small. This will be useful when studying the lower bounds of *a posteriori* error estimates in Sections 4.2 and 5.2. To simplify, they are stated in three dimensions, but the two-dimensional analogue is easily derived. In both cases, the regularity assumption (3.1) is not sufficient and is strengthened by prescribing in addition some quasi-uniformity. In three dimensions, to simplify the exposition, we choose the following sufficient condition: There exists a constant $\tau > 0$, independent of h, such that

$$
\forall K \in \mathcal{T}_h, \quad h_K \ge \tau\, h. \tag{3.19}
$$

We refer to Remarks 3.4 and 4.10 below for a discussion on the quasi-uniformity condition.

Theorem 3.3. *Let $d = 3$, let ν satisfy (1.1) and (1.2), and let (3.1) and (3.19) hold. We suppose that problem (V_1) has a solution $(\mathbf{u}, p, T) \in H^1(\Omega)^3 \times H^1(\Omega) \times W^{2,3}(\Omega)$ verifying*

$$
\lambda\, (S_6^0)^2 \|\mathbf{u}\|_{L^3(\Omega)^3} |T|_{W^{1,3}(\Omega)} < \frac{\alpha\, \nu_1}{4} \cdot \tag{3.20}
$$

In addition, we denote by

$$h_0 = C\frac{\min\left(\|\mathbf{u}\|^2_{L^3(\Omega)^3}, |T|^2_{W^{1,3}(\Omega)}\right)}{\left(|\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^1(\Omega)} + |T|_{W^{2,3}(\Omega)}\right)^2}, \tag{3.21}$$

where $C = \frac{\tau}{C_I^0(3)^2 C_1^2}$, $C_I^0(3)$ *is the constant of* (3.2), τ *the constant of* (3.19), *and* C_1 *that of* (3.11). *Then for* $h < h_0$, *the solution* $(\mathbf{u}_h^{i+1}, p_h^{i+1}, T_h^{i+1})$ *of* $(V_{h,i,1})$ *converges, uniformly with respect to* h, *to the discrete solution* $(\mathbf{u}_h, p_h, T_h)$ *of* $(V_{h,1})$, *as* i *tends to infinity.*

Proof. Let $(\mathbf{u}_h, p_h, T_h)$ and $(\mathbf{u}_h^{i+1}, p_h^{i+1}, T_h^{i+1})$ solve respectively $(V_{h,1})$ and $(V_{h,i,1})$.

To estimate the temperature error, we take the difference between the third equations of $(V_{h,1})$ and $(V_{h,i,1})$ tested with $S_h = T_h - T_h^{i+1}$, and insert ∇T_h and ∇T. We obtain by using the antisymmetry of the transport term,

$$\begin{aligned}\alpha|T_h - T_h^{i+1}|^2_{H^1(\Omega)} = &\int_\Omega ((\mathbf{u}_h^{i+1} - \mathbf{u}_h)\cdot\nabla(T_h - T))(T_h - T_h^{i+1})\,\mathrm{d}\mathbf{x}\\ &+ \int_\Omega ((\mathbf{u}_h^{i+1} - \mathbf{u}_h)\cdot\nabla T)(T_h - T_h^{i+1})\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{3.22}$$

Owing to the regularity of T, the bound of the second term in the above right-hand side is straightforward,

$$\left|\int_\Omega ((\mathbf{u}_h^{i+1} - \mathbf{u}_h)\cdot\nabla T)(T_h - T_h^{i+1})\,\mathrm{d}\mathbf{x}\right| \le S_6^0 |T|_{W^{1,3}(\Omega)} |T_h - T_h^{i+1}|_{H^1(\Omega)} \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}.$$

To simplify, let A denote the first term and set

$$C(\mathbf{u}, p, T) = |\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^1(\Omega)} + |T|_{W^{2,3}(\Omega)}. \tag{3.23}$$

By applying (3.2) and Hölder's inequality, we find

$$|A| \le \|T_h - T_h^{i+1}\|_{L^6(\Omega)} |T - T_h|_{H^1(\Omega)} \left(\sum_{K\in\mathcal{T}_h} \left(C_I^0(3) h_K^{-\frac{1}{2}}\right)^3 \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|^3_{L^2(K)^3}\right)^{\frac{1}{3}}.$$

Then Jensen's and Sobolev's inequalities, and the *a priori* error estimates (3.11) yield

$$\begin{aligned}|A| &\le S_6^0 C_I^0(3) |T_h - T_h^{i+1}|_{H^1(\Omega)} |T - T_h|_{H^1(\Omega)} \left(\sum_{K\in\mathcal{T}_h} h_K^{-1} \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|^2_{L^2(K)^3}\right)^{\frac{1}{2}}\\ &\le S_6^0 C_1 C_I^0(3) |T_h - T_h^{i+1}|_{H^1(\Omega)} C(\mathbf{u}, p, T) \left(\sum_{K\in\mathcal{T}_h} (h^2 h_K^{-1}) \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|^2_{L^2(K)^3}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.24}$$

With the quasi-uniform regularity assumption (3.19), this becomes

$$|A| \le \left(\frac{h}{\tau}\right)^{\frac{1}{2}} S_6^0 C_1 C_I^0(3) C(\mathbf{u}, p, T) |T_h - T_h^{i+1}|_{H^1(\Omega)} \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}.$$

Thus by substituting into (3.22) these bounds for the two terms, we find

$$|T_h - T_h^{i+1}|_{H^1(\Omega)} \le \frac{S_6^0}{\alpha}\left[\left(\frac{h}{\tau}\right)^{\frac{1}{2}} C_1 C_I^0(3) C(\mathbf{u}, p, T) + |T|_{W^{1,3}(\Omega)}\right] \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}. \tag{3.25}$$

To bound the velocity error, let $\mathbf{v}_h \in \mathcal{V}_{h,1}$. The difference between the first equations of $(V_{h,1})$ and $(V_{h,i,1})$ leads to the following relation

$$\int_\Omega \nu(T_h)\mathbf{u}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int_\Omega \nu(T_h^i)\mathbf{u}_h^{i+1} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}. \tag{3.26}$$

By inserting $\mathbf{u}_h$ and $\mathbf{u}$, and testing with $\mathbf{v}_h = \mathbf{u}_h - \mathbf{u}_h^{i+1}$ that belongs indeed to $\mathcal{V}_{h,1}$, we easily derive

$$\begin{aligned}\|(\nu(T_h^i))^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}_h^{i+1})\|^2_{L^2(\Omega)^3} = &\int_\Omega (\nu(T_h) - \nu(T_h^i))(\mathbf{u}_h - \mathbf{u}) \cdot (\mathbf{u}_h - \mathbf{u}_h^{i+1}) \,\mathrm{d}\mathbf{x} \\ &+ \int_\Omega (\nu(T_h) - \nu(T_h^i))\mathbf{u} \cdot (\mathbf{u}_h - \mathbf{u}_h^{i+1}) \,\mathrm{d}\mathbf{x}.\end{aligned}$$

Then (1.2), (3.2), (3.19), (3.11), the Lipschitz continuity of ν, and the above argument yield

$$\begin{aligned}\nu_1\|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|^2_{L^2(\Omega)^3} \le\ &\lambda\, S_6^0 |T_h - T_h^i|_{H^1(\Omega)} \\ &\times \left(\|\mathbf{u}\|_{L^3(\Omega)^3}\|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} + C_I^0(3)\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)^3} \left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|^2_{L^2(K)^3} \right)^{\frac{1}{2}} \right) \\ \le\ &\lambda\, S_6^0 |T_h - T_h^i|_{H^1(\Omega)} \left(\|\mathbf{u}\|_{L^3(\Omega)^3} + \left(\tfrac{h}{\tau}\right)^{\frac{1}{2}} C_1 C_I^0(3) C(\mathbf{u}, p, T) \right) \|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}.\end{aligned}$$

By substituting (3.25) at level i into this inequality we obtain

$$\|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \le \frac{\lambda\,(S_6^0)^2}{\alpha\,\nu_1}\Big(C'\, h^{\frac{1}{2}} C(\mathbf{u}, p, T) + \|\mathbf{u}\|_{L^3(\Omega)^3}\Big)\Big(C' h^{\frac{1}{2}} C(\mathbf{u}, p, T) + |T|_{W^{1,3}(\Omega)}\Big)\|\mathbf{u}_h - \mathbf{u}_h^i\|_{L^2(\Omega)^3}, \tag{3.27}$$

where $C' = \frac{C_I^0(3)C_1}{\tau^{\frac{1}{2}}}$. As h is sufficiently small ($h < h_0$), we obtain by using (3.20) the bound

$$\|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \le M\|\mathbf{u}_h - \mathbf{u}_h^i\|_{L^2(\Omega)^3},$$

where $M = \dfrac{4\lambda\,(S_6^0)^2}{\alpha\,\nu_1}(\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)}) < 1$. This implies

$$\|\mathbf{u}_h - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \le M^{i+1}\|\mathbf{u}_h - \mathbf{u}_h^0\|_{L^2(\Omega)^3}, \tag{3.28}$$

which allows us to derive the uniform convergence of $\mathbf{u}_h^{i+1}$ to $\mathbf{u}_h$, and owing to (3.25), that of T_h^{i+1} to T_h.

Finally, the proof of the error estimate for the pressure follows the same lines. By taking the difference between the second equations of $(V_{h,1})$ and $(V_{h,i,1})$, inserting $\nu(T_h^i)$ and $\mathbf{u}$, and testing with $\mathbf{v}_h$ in $\mathcal{W}_{h,1}$, we obtain

$$\begin{aligned}\int_\Omega (p_h^{i+1} - p_h)\mathrm{div}\, \mathbf{v}_h \,\mathrm{d}\mathbf{x} = &\int_\Omega \nu(T_h^i)(\mathbf{u}_h^{i+1} - \mathbf{u}_h) \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_\Omega (\nu(T_h^i) - \nu(T_h))(\mathbf{u}_h - \mathbf{u}) \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} \\ &+ \int_\Omega (\nu(T_h^i) - \nu(T_h))\mathbf{u} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}.\end{aligned} \tag{3.29}$$

Hence

$$\left|\int_\Omega (p_h^{i+1}-p_h)\mathrm{div}\,\mathbf{v}_h\,\mathrm{d}\mathbf{x}\right| \le \nu_2\|\mathbf{u}_h-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}\|\mathbf{v}_h\|_{L^2(\Omega)^3}$$

$$+\lambda\, S_6^0|T_h-T_h^i|_{H^1(\Omega)}\Bigg[\|\mathbf{u}_h-\mathbf{u}\|_{L^2(\Omega)^3}C_I^0(3)\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\mathbf{v}_h\|^2_{L^2(K)^3}\right)^{\frac12}$$

$$+\|\mathbf{u}\|_{L^3(\Omega)^3}\|\mathbf{v}_h\|_{L^2(\Omega)^3}\Bigg] \le \|\mathbf{v}_h\|_{L^2(\Omega)^3}\Bigg(\nu_2\|\mathbf{u}_h-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}$$

$$+\lambda\, S_6^0|T_h-T_h^i|_{H^1(\Omega)}\big(C_1C_I^0(3)\left(\frac{h}{\tau}\right)^{\frac12}C(\mathbf{u},p,T)+\|\mathbf{u}\|_{L^3(\Omega)^3}\big)\Bigg).$$

The uniform convergence of p_h^{i+1} to p_h follows from the inf-sup condition (3.9), the condition $h<h_0$ with h_0 given by (3.21), and the convergences of $\mathbf{u}_h^{i+1}$ and T_h^{i+1}. □

Remark 3.4. Condition (3.19) can be somewhat relaxed. Indeed, it stems from (3.24) and (3.27) that it suffices to prescribe for some small $\delta>0$,

$$\forall K\in\mathcal{T}_h,\quad \tau h_K^{-1}h^2\le h^\delta,$$

i.e.,

$$\forall K\in\mathcal{T}_h,\quad h_K\ge \tau\, h^{2-\delta}, \tag{3.30}$$

a condition less restrictive than (3.19). The situation in two dimensions is more favorable, owing to the wider range of Sobolev's imbeddings. It suffices that for some small $\delta>0$ and some $p>2$, close to two,

$$\forall K\in\mathcal{T}_h,\quad h_K\ge \tau\, h^{(1-\delta)\frac{p}{p-2}}. \tag{3.31}$$

Indeed, the contribution of $\mathbf{u}_h-\mathbf{u}_h^{i+1}$ to the first line of (3.24) can be replaced by

$$\left(\sum_{K\in\mathcal{T}_h} h_K^{2(\frac{2}{p}-1)}\|\mathbf{u}_h-\mathbf{u}_h^{i+1}\|^2_{L^2(K)^2}\right)^{\frac12},$$

for any $p>2$, close to two. Therefore it suffices that for some small $\delta>0$,

$$\forall K\in\mathcal{T}_h,\quad \tau h_K^{\frac{2}{p}-1}h\le h^\delta,$$

i.e.,

$$\forall K\in\mathcal{T}_h,\quad h_K\ge \big(\tau\, h^{(1-\delta)}\big)^{\frac{p}{p-2}}.$$

Since p is a little above two, this brings hardly a restriction (provided of course that $h<1$).

The convergence of the solution of $(V_{h,i,2})$, that holds under similar assumptions, is stated below. The proof is a straightforward adaptation of the proof of Theorem 3.3.

Theorem 3.5. *We retain the settings and assumptions of Theorem 3.3 and we suppose in addition that* $(p,T)\in H^2(\Omega)\times\big(W^{1,\infty}(\Omega)\cap W^{2,3}(\Omega)\big)$ *and*

$$\lambda\, S_6^0\|\boldsymbol{u}\|_{L^3(\Omega)^3}(\|T\|_{L^\infty(\Omega)}+S_6^0|T|_{W^{1,3}(\Omega)})<\frac{\alpha\,\nu_1}{2}.$$

We replace (3.21) *by*

$$h_0=C\frac{\min\big(2\,\|\boldsymbol{u}\|^2_{L^3(\Omega)^3},(S_6^0|T|_{W^{1,3}(\Omega)}+\|T\|_{L^\infty(\Omega)})^2\big)}{2(|\boldsymbol{u}|_{H^1(\Omega)^3}+|p|_{H^2(\Omega)}+|T|_{W^{2,3}(\Omega)})^2}, \tag{3.32}$$

where $C=\frac{\tau}{C_I^0(3)^2C_2^2}$ *and* C_2 *is the constant of* (3.17). *Then for* $h<h_0$, *the solution* $(\mathbf{u}_h^{i+1},p_h^{i+1},T_h^{i+1})$ *of* $(V_{h,i,2})$ *converges, uniformly with respect to* h, *to the discrete solution* $(\mathbf{u}_h,p_h,T_h)$ *of* $(V_{h,2})$, *as* i *tends to infinity.*

4. *A POSTERIORI* ERROR ESTIMATES FOR THE FIRST APPROXIMATION

As usual, for *a posteriori* error estimates, we introduce the following notation. For every element K in $\mathcal{T}_h$, we denote by

- Γ_h^i the set of edges (when $d = 2$) or faces (when $d = 3$) of K that are not contained in $\partial\Omega$;
- Γ_h^b the set of edges (when $d = 2$) or faces (when $d = 3$) of K which are contained in $\partial\Omega$.

For every edge (when $d = 2$) or face (when $d = 3$) e of the mesh $\mathcal{T}_h$, we denote by

- ω_e the union of elements of $\mathcal{T}_h$ adjacent to e;
- $[\cdot]_e$ the jump through e on each edge e of Γ_h^i.

From now on, to simplify, we set $d = 3$. Again, the extension to two dimensions is straightforward and simpler. In this and the next section, the *a posteriori* error estimates are established when the solution is slightly smoother and the data are suitably restricted.

It is well known that by using the Raviart–Thomas finite element, the *a posteriori* error estimates corresponding to the Darcy problem are not optimal [8], since we can not locally bound the indicator with the local error. But when the data $\mathbf{f}$ is sufficiently smooth, optimality can be derived by adding an indicator obtained by taking the **curl** of the first equation of problem (P) [11],

$$\mathbf{curl}(\nu(T)\mathbf{u}) = \mathbf{curl}\,\mathbf{f}. \tag{4.1}$$

To specify the smoothness of $\mathbf{f}$, let us recall the standard space

$$H(\mathbf{curl}, \Omega) = \left\{\mathbf{v} \in L^2(\Omega)^3;\ \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3\right\}, \tag{4.2}$$

equipped with the norm

$$\|\mathbf{v}\|^2_{H(\mathbf{curl},\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(\Omega)^3}, \tag{4.3}$$

and Green's formula, valid in any Lipschitz domain $\mathcal{O}$,

$$\forall\varphi \in H^1(\mathcal{O})^3, \forall\mathbf{v} \in H(\mathbf{curl}, \mathcal{O}), \langle\mathbf{v} \times \mathbf{n}, \varphi\rangle_{\partial\mathcal{O}} = \int_{\mathcal{O}} \mathbf{v}\,\cdot \mathbf{curl}\,\varphi\,\mathrm{d}\mathbf{x} - \int_{\mathcal{O}} \varphi \cdot\, \mathbf{curl}\,\mathbf{v}\ \mathrm{d}\mathbf{x}, \tag{4.4}$$

where $\partial\mathcal{O}$ denotes the boundary of $\mathcal{O}$ and the duality $\langle\cdot,\cdot\rangle_{\partial\mathcal{O}}$ reduces to the surface integral, when $\mathbf{v}$ is smoother, *e.g.* $\mathbf{v} \in H^1(\mathcal{O})^3$. In view of (4.1), the additional regularity $\mathbf{f} \in H(\mathbf{curl}, \Omega)$ would seem sufficient, but considering that Green's formula (4.4) will have to be applied below in each element, thus leading to jumps of tangential components on each face, it is much simpler to assume that $\mathbf{f} \in H^1(\Omega)^3$.

The estimates below rely on the following fundamental result.

Lemma 4.1. *Let Ω be a bounded simply-connected domain of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. To each function $\mathbf{v} \in V$, defined by (2.12), we can associate a unique function $\eta \in H_0^1(\Omega)^3$ such that*

$$\mathbf{v} = \mathbf{curl}\,\eta, \tag{4.5}$$

and there exists a constant C independent of $\mathbf{v}$ and η, such that

$$\|\eta\|_{H^1(\Omega)^3} \leq C\|\mathbf{v}\|_{L^2(\Omega)^3}. \tag{4.6}$$

While the analogue of Lemma 4.1 in two dimensions is a straightforward and well-known result, see Girault and Raviart [19], this is not so in three dimensions, and the reader will find a proof of Lemma 4.1 in the appendix.

4.1. Upper error bound for the first discretization

As usual, the choice of error indicators stems from suitable error equalities. The error equality for the temperature is straightforward. Indeed, a standard argument shows that the solutions of problems (V_1) and $(V_{h,i,1})$ verify for all S in $H^1(\Omega) \cap L^\infty(\Omega)$ and S_h in X_h,

$$\begin{aligned}\alpha \int_\Omega \nabla\left(T - T_h^{i+1}\right) \cdot \nabla S \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla T) S \,\mathrm{d}\mathbf{x} - \int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1}) S \,\mathrm{d}\mathbf{x} \\ = \sum_{K \in \mathcal{T}_h} \Bigg[\int_K (\alpha \Delta T_h^{i+1} - \mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1} + g_h)(S - S_h) \,\mathrm{d}\mathbf{x} \\ + \int_K (g - g_h)(S - S_h) \,\mathrm{d}\mathbf{x} - \frac{\alpha}{2} \sum_{e \in \partial K \cap \Gamma_h^i} \int_e [\nabla T_h^{i+1} \cdot \mathbf{n}]_e (S - S_h) \,\mathrm{d}s \Bigg],\end{aligned} \tag{4.7}$$

where g_h is a piecewise constant approximation of g in each K of $\mathcal{T}_h$.

But the error equality of Darcy's system is less straightforward precisely because of the lack of optimality of the Raviart–Thomas elements. In fact, it is convenient to use two equalities, one where the test functions belong to $H_0(\mathrm{div}, \Omega)$ and one where the test functions have zero divergence. On the one hand, regarding the first equality, the same arguments as with the temperature give for all $\mathbf{v} \in H_0(\mathrm{div}, \Omega)$ and all $\mathbf{v}_h \in \mathcal{W}_{h,1}$,

$$\begin{aligned}\int_\Omega \nu(T) \mathbf{u} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} - \int_\Omega p \,\mathrm{div}\, \mathbf{v} \,\mathrm{d}\mathbf{x} - \int_\Omega \nu(T_h^i) \mathbf{u}_h^{i+1} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \int_\Omega p_h^{i+1} \mathrm{div}\, \mathbf{v} \,\mathrm{d}\mathbf{x} \\ = \sum_{K \in \mathcal{T}_h} \int_K (\mathbf{f} - \mathbf{f}_h) \cdot (\mathbf{v} - \mathbf{v}_h) \,\mathrm{d}\mathbf{x} + \sum_{K \in \mathcal{T}_h} \Bigg[\int_K (\mathbf{f}_h - \nu(T_h^i) \mathbf{u}_h^{i+1} - \nabla p_h^{i+1}) \cdot (\mathbf{v} - \mathbf{v}_h) \,\mathrm{d}\mathbf{x} \\ + \frac{1}{2} \sum_{e \in \partial K \cap \Gamma_h^i} \int_e [p_h^{i+1} \mathbf{n}]_e \cdot (\mathbf{v} - \mathbf{v}_h) \,\mathrm{d}s \Bigg],\end{aligned} \tag{4.8}$$

where $\mathbf{f}_h$ is an approximation of $\mathbf{f}$, which is a polynomial of degree l with $l \geq 1$ in each element K of $\mathcal{T}_h$.

For the second equality, on the other hand, we apply Lemma 4.1 and we have for all $\mathbf{v} \in V$ and all $\eta_h \in X_h$,

$$\begin{aligned}\int_\Omega \nu(T) \mathbf{u} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} - \int_\Omega \nu(T_h^i) \mathbf{u}_h^{i+1} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} \\ = \sum_{K \in \mathcal{T}_h} \Bigg[\int_K \mathbf{curl}(\mathbf{f} - \mathbf{f}_h) \cdot (\eta - \eta_h) \,\mathrm{d}\mathbf{x} + \frac{1}{2} \sum_{e \in \partial K \cap \Gamma_h^i} \int_e [(\mathbf{f} - \mathbf{f}_h) \times \mathbf{n}]_e \cdot (\eta - \eta_h) \,\mathrm{d}s \Bigg] \\ + \sum_{K \in \mathcal{T}_h} \Bigg[\int_K \mathbf{curl}\big(\mathbf{f}_h - \nu(T_h^i) \mathbf{u}_h^{i+1}\big) \cdot (\eta - \eta_h) \,\mathrm{d}\mathbf{x} \\ + \frac{1}{2} \sum_{e \in \partial K \cap \Gamma_h^i} \int_e [(\mathbf{f}_h - \nu(T_h^i) \mathbf{u}_h^{i+1}) \times \mathbf{n}]_e \cdot (\eta - \eta_h) \,\mathrm{d}s \Bigg],\end{aligned} \tag{4.9}$$

where $\mathbf{v} = \mathbf{curl}\, \eta$, $\eta \in H_0^1(\Omega)^3$ associated to $\mathbf{v}$ by Lemma 4.1.

The error equalities (4.7)–(4.9) suggest the following temperature, pressure, and velocity error indicators in each $K \in \mathcal{T}_h$:

$$\eta_{K,i,1}^{(D,1)} = h_K \left\|\alpha \Delta T_h^{i+1} - \mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1} + g_h\right\|_{L^2(K)} + \frac{1}{2} \sum_{e \in \partial K \cap \Gamma_h^i} h_e^{\frac{1}{2}} \left\|\alpha \left[\nabla T_h^{i+1} \cdot \mathbf{n}\right]_e\right\|_{L^2(e)}, \tag{4.10}$$

$$\eta_{K,i,2,1}^{(D,1)} = h_K \left\|-\nabla p_h^{i+1} - \nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h\right\|_{L^2(K)^3} + \frac{1}{2} \sum_{e \in \partial K \cap \Gamma_h^i} h_e^{\frac{1}{2}} \left\|[p_h^{i+1}\mathbf{n}]_e\right\|_{L^2(e)}, \tag{4.11}$$

$$\eta_{K,i,2,2}^{(D,1)} = h_K \left\|\mathbf{curl}(-\nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)\right\|_{L^2(K)^3} + \frac{1}{2} \sum_{e \in \partial K \cap \Gamma_h^i} h_e^{\frac{1}{2}} \left\|[(-\nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h) \times \mathbf{n}]_e\right\|_{L^2(e)^3}. \tag{4.12}$$

In addition, we introduce an indicator for the algorithmic error, in each $K \in \mathcal{T}_h$,

$$\eta_{K,i}^{(L,1)} = |T_h^{i+1} - T_h^i|_{H^1(K)}. \tag{4.13}$$

The next theorem proves an upper bound for the error in terms of these indicators.

Theorem 4.2. *Let ν satisfy (1.1) and (1.2), let the mesh satisfy (3.1), and let $\mathbf{f} \in H^1(\Omega)^3$. We suppose that problem (V_1) has a solution $(\boldsymbol{u}, T) \in L^3(\Omega)^3 \times W^{1,3}(\Omega)$ such that*

$$\lambda\, (S_6^0)^2(\|\mathbf{u}\|_{L^3(\Omega)^3} |T|_{W^{1,3}(\Omega)}) < \alpha\, \nu_1.$$

Then the following error inequalities hold:

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} + |T - T_h^{i+1}|_{H^1(\Omega)} \le C \Bigg[\sum_{K \in \mathcal{T}_h} \Bigg(\left(\eta_{K,i,1}^{(D,1)}\right)^2 + \left(\eta_{K,i,2,2}^{(D,1)}\right)^2 \\ + h_K^2 \|\mathbf{curl}(\mathbf{f} - \mathbf{f}_h)\|_{L^2(K)^3}^2 + \sum_{e \in \partial K \cap \Gamma_h^i} h_e \|[(\mathbf{f} - \mathbf{f}_h) \times \boldsymbol{n}]_e\|_{L^2(e)^3}^2 + h_K^2 \|g - g_h\|_{L^2(K)}^2 \Bigg) \Bigg]^{\frac{1}{2}} \\ + C' \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(L,1)}\right)^2 \right)^{\frac{1}{2}},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\|p - p_h^{i+1}\|_{L^2(\Omega)} \le \frac{C}{\beta} \left(\sum_{K \in \mathcal{T}_h} \left(\left(\eta_{K,i,2,1}^{(D,1)}\right)^2 + h_K^2 \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^3}^2 \right) \right)^{\frac{1}{2}} \\ + \frac{\lambda}{\beta} S_6^0 \|\mathbf{u}\|_{L^3(\Omega)^3} \left(|T - T_h^{i+1}|_{H^1(\Omega)} + \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(L,1)}\right)^2 \right)^{\frac{1}{2}} \right) + \frac{\nu_2}{\beta} S_2^0 \|\boldsymbol{u} - \boldsymbol{u}_h^{i+1}\|_{L^2(\Omega)^3},\end{aligned} \tag{4.15}$$

where β is the constant of the inf-sup condition between $H_0^1(\Omega)^d$ and $L_{\mathrm{m}}^2(\Omega)$, see [19].

Proof. Let us start with the temperature. Consider first the left-hand side of (4.7). By inserting

$$\int_\Omega (\mathbf{u} \cdot \nabla T_h^{i+1}) S \,\mathrm{d}\mathbf{x}$$

into this left-hand side, by testing it with $S = T - T_h^{i+1}$ that is indeed an admissible test function, and by using the antisymmetric property of the transport term we derive,

$$\alpha |T - T_h^{i+1}|_{H^1(\Omega)}^2 = \mathcal{R}(T - T_h^{i+1}) + \int_\Omega \left(\left(\mathbf{u}_h^{i+1} - \mathbf{u}\right) \cdot \nabla T_h^{i+1}\right) (T - T_h^{i+1}) \,\mathrm{d}\mathbf{x}, \tag{4.16}$$

where $\mathcal{R}(T-T_h^{i+1})$ denotes the right-hand side of (4.7) with $S=T-T_h^{i+1}$. The last term of (4.16) becomes, after inserting ∇T,

$$\begin{aligned}\int_\Omega ((\mathbf{u}_h^{i+1}-\mathbf{u})\cdot\nabla T_h^{i+1})(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}\\ = \int_\Omega ((\mathbf{u}_h^{i+1}-\mathbf{u})\cdot\nabla (T_h^{i+1}-T))(T-T_h^{i+1})\,\mathrm{d}\mathbf{x} + \int_\Omega ((\mathbf{u}_h^{i+1}-\mathbf{u})\cdot\nabla T)(T-T_h^{i+1})\,\mathrm{d}\mathbf{x},\end{aligned}$$

and the Hölder inequality and the antisymmetry imply

$$\left|\int_\Omega ((\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\nabla T_h^{i+1})(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}\right| \le S_6^0|T|_{W^{1,3}(\Omega)}\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}|T-T_h^{i+1}|_{H^1(\Omega)}. \tag{4.17}$$

Next, we turn to its right-hand side, $\mathcal{R}(S)$, for general S; it can be bounded as follows:

$$\begin{aligned}\mathcal{R}(S) \le \sum_{K\in\mathcal{T}_h}\Bigg[&\left(\|\alpha\Delta T_h^{i+1}-\mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1}+g_h\|_{L^2(K)}+\|g-g_h\|_{L^2(K)}\right)\|S-S_h\|_{L^2(K)}\\ &+\frac{1}{2}\sum_{e\in\partial K\cap\Gamma_h^i}\|\alpha[\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|_{L^2(e)}\|S-S_h\|_{L^2(e)}\Bigg].\end{aligned}$$

The choice $S_h=R_h(S)$, see (3.5) gives the following bound:

$$\begin{aligned}\mathcal{R}(S) \le C_2\sum_{K\in\mathcal{T}_h}\Bigg[&h_K\left(\|\alpha\Delta T_h^{i+1}-\mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1}+g_h\|_{L^2(K)}+\|g-g_h\|_{L^2(K)}\right)|S|_{H^1(\omega_K)}\\ &+\frac{c_2}{2}\sum_{e\in\partial K\cap\Gamma_h^i}h_e^{\frac{1}{2}}\|\alpha[\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|_{L^2(e)}|S|_{H^1(\omega_K)}\Bigg].\end{aligned} \tag{4.18}$$

Then, by substituting (4.18) with $S=T-T_h^{i+1}$ and (4.17) into (4.16), and by applying the regularity of the mesh, we conclude

$$\alpha|T-T_h^{i+1}|_{1,\Omega} \le C_3\left(\sum_{K\in\mathcal{T}_h}((\eta_{K,i,1}^{(D,1)})^2+h_K^2\|g-g_h\|^2_{L^2(K)})\right)^{\frac{1}{2}} + S_6^0|T|_{W^{1,3}(\Omega)}\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}. \tag{4.19}$$

Now, we examine the velocity. Consider first the left-hand side of (4.9) with $\mathbf{v}\in V$. By inserting $\int_\Omega \nu(T_h^i)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$ and $\int_\Omega \nu(T_h^{i+1})\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$ into this left-hand side, (4.9) becomes

$$\begin{aligned}\int_\Omega \nu(T_h^i)(\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = \mathcal{R}_1(\eta) + \int_\Omega(\nu(T_h^{i+1})-\nu(T))\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\\ + \int_\Omega(\nu(T_h^i)-\nu(T_h^{i+1}))\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\end{aligned} \tag{4.20}$$

where $\mathcal{R}_1(\eta)$ denotes the right-hand side of (4.9) and $\eta\in H_0^1(\Omega)^3$ is the vector potential of $\mathbf{v}$ constructed in Lemma 4.1, see (4.5) and (4.6). We denote by I the sum of the second and third terms of the right-hand side of (4.20). The Lipschitz continuity of ν and Hölder's inequality yield

$$I \le \lambda\, S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}(|T-T_h^{i+1}|_{H^1(\Omega)}+|T_h^{i+1}-T_h^i|_{H^1(\Omega)})\|\mathbf{v}\|_{L^2(\Omega)^3}.$$

Regarding the right-hand side $\mathcal{R}_1(\eta)$ for $\eta \in H_0^1(\Omega)^3$, we choose $\eta_h = R_h(\eta)$, the degree one Scott and Zhang interpolant of η that belongs to X_h, since η vanishes on Γ. By applying the approximation properties (3.5) of R_h to the function η, we obtain

$$\begin{aligned}\mathcal{R}_1(\eta) \leq C_4 \sum_{K\in\mathcal{T}_h} \Bigg[& h_K \left(\|\mathbf{curl}(-\nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)\|_{L^2(K)^3} + \|\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(K)^3}\right) |\eta|_{H^1(\omega_K)^3} \\ & + \frac{1}{2}\sum_{e\in\partial K\cap\Gamma_h^i} h_e^{\frac{1}{2}} \left(\|[(-\nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)\times\mathbf{n}]_e\|_{L^2(e)^3} + \|[(\mathbf{f}-\mathbf{f}_h)\times\mathbf{n}]_e\|_{L^2(e)^3}\right) |\eta|_{H^1(\omega_K)^3}\Bigg]. \end{aligned} \tag{4.21}$$

The choice $\mathbf{v} = \mathbf{u} - \mathbf{u}_h^{i+1}$ in (4.20), the regularity of the mesh (3.1), and the estimate (4.6) relating η and $\mathbf{v}$ give

$$\begin{aligned}\nu_1\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \leq C_5 & \left[\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,2,2}^{(D,1)}\right)^2 + h_K^2\|\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\|^2_{L^2(K)^3} + \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|[(\mathbf{f}-\mathbf{f}_h)\times\mathbf{n}]_e\|^2_{L^2(e)^3}\right)\right]^{\frac{1}{2}} \\ & + \lambda\, S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}|T-T_h^{i+1}|_{H^1(\Omega)} + \lambda\, S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}\left(\sum_{K\in\mathcal{T}_h}(\eta_{K,i}^{(L,1)})^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence (4.19) yields

$$\begin{aligned}\nu_1\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \leq C_6 & \Bigg[\sum_{K\in\mathcal{T}_h}\Bigg(\left(\eta_{K,i,1}^{(D,1)}\right)^2 + \left(\eta_{K,i,2,2}^{(D,1)}\right)^2 + h_K^2\|g-g_h\|^2_{L^2(K)} + h_K^2\|\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\|^2_{L^2(K)^3} \\ & + \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|[(\mathbf{f}-\mathbf{f}_h)\times\mathbf{n}]_e\|^2_{L^2(e)^3}\Bigg)\Bigg]^{\frac{1}{2}} + \frac{\lambda}{\alpha}(S_6^0)^2\|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{W^{1,3}(\Omega)}\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \\ & + \lambda\, S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}\left(\sum_{K\in\mathcal{T}_h}\left(\eta_{K,i}^{(L,1)}\right)^2\right)^{\frac{1}{2}},\end{aligned} \tag{4.22}$$

thus proving (4.14).

Finally, to estimate the pressure error, we consider equation (4.8) with any $\mathbf{v} \in H_0(\mathrm{div},\Omega)$. By inserting $\int_\Omega \nu(T_h^i)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$ and $\int_\Omega \nu(T_h^{i+1})\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}$ into its left-hand side, (4.8) becomes

$$\begin{aligned}\int_\Omega (p_h^{i+1}-p)\mathrm{div}\,\mathbf{v}\,\mathrm{d}\mathbf{x} &= \mathcal{R}_1(\mathbf{v}) + \int_\Omega \left(\nu(T_h^{i+1}) - \nu(T)\right)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} \\ & + \int_\Omega\left(\nu(T_h^i) - \nu(T_h^{i+1})\right)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_\Omega \nu(T_h^i)(\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},\end{aligned} \tag{4.23}$$

where $\mathcal{R}_1(\mathbf{v})$ denotes its right-hand side. Since $p_h^{i+1}-p$ belongs to $L_m^2(\Omega)$, owing to the inf-sup condition between $H_0^1(\Omega)^3$ and $L_m^2(\Omega)$ [19], there exists $\mathbf{v} \in H_0^1(\Omega)^3$, such that

$$\int_\Omega (p_h^{i+1}-p)\mathrm{div}\mathbf{v}\,\mathrm{d}\mathbf{x} = \|p_h^{i+1}-p\|^2_{L^2(\Omega)}, \tag{4.24}$$

and

$$|\mathbf{v}|_{H^1(\Omega)^3} \leq \frac{1}{\beta}\|p_h^{i+1}-p\|_{L^2(\Omega)}. \tag{4.25}$$

We choose this $\mathbf{v}$ and $\mathbf{v}_h = \Pi_h \mathbf{v}$ in (4.23), where Π_h is the interpolation operator from $H^1(\Omega)^3$ in $\mathcal{W}_{h,1}$ (see Brezzi and Fortin [9] or [26]) such that for all $\mathbf{v} \in H^1(\Omega)^3$

$$\forall K \in \mathcal{T}_h, \quad \|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(K)^3} \leq C_5\, h_K |\mathbf{v}|_{H^1(K)^3}, \tag{4.26}$$

and

$$\forall e \in \Gamma_h^i \cup \Gamma_h^b, \quad \|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(e)^3} \leq C_6 h_e^{\frac{1}{2}} |\mathbf{v}|_{H^1(K)^3}, \tag{4.27}$$

where K is an element adjacent to e. With this choice, by applying (4.26) and (4.27), $\mathcal{R}_1(\mathbf{v})$ is bounded as follows:

$$\begin{aligned} \mathcal{R}_1(\mathbf{v}) \leq \sum_{K \in \mathcal{T}_h} \Bigg[& C_7 \left(\| -\nabla p_h^{i+1} - \nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h \|_{L^2(K)^3} + \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^3} \right) h_K |\mathbf{v}|_{H^1(K)^3} \\ & + \frac{C_8}{2} \sum_{e \in \partial K \cap \Gamma_h^i} \|[p_h^{i+1}\mathbf{n}]_e\|_{L^2(e)^3} h_e^{\frac{1}{2}} |\mathbf{v}|_{H^1(K)^3} \Bigg]. \end{aligned} \tag{4.28}$$

Then, by substituting (4.28) into (4.23), and applying (4.24), the regularity of the mesh (3.1), and (4.25), we infer

$$\begin{aligned} \|p - p_h^{i+1}\|_{L^2(\Omega)} \leq \frac{C_9}{\beta} & \left(\sum_{K \in \mathcal{T}_h} \left(\left(\eta_{K,i,2,1}^{(D,1)} \right)^2 + h_K^2 \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^3}^2 \right) \right)^{\frac{1}{2}} \\ & + \frac{\lambda}{\beta} S_6^0 \|\mathbf{u}\|_{L^3(\Omega)^3} \left(|T - T_h^{i+1}|_{H^1(\Omega)} + \left(\sum_{K \in \mathcal{T}_h} (\eta_{K,i}^{(L,1)})^2 \right)^{\frac{1}{2}} \right) + \frac{\nu_2}{\beta} \|\mathbf{u} - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}. \end{aligned} \tag{4.29}$$

This proves (4.15). □

4.2. Lower error bound for the first discretization

As is well known, lower bounds are established locally. This localization is achieved in each element K of $\mathcal{T}_h$ by means of the bubble function ψ_K defined by (3.12). The jump terms on edges e when $d = 2$ or faces e when $d = 3$ are localized by means of similar bubble functions defined on e. For those jumps, we also need a lifting operator $\mathcal{L}_e$ from edges or faces e of Γ_h^i to the two elements K and K' sharing this edge or face. It acts on polynomials that vanish on the boundary of e and produces a globally continuous function with element-wise polynomial values defined individually on K and K' that vanish on $\partial(K \cup K') \setminus e$. The effect of these elements' geometry on $\mathcal{L}_e$ is controlled by first defining $\mathcal{L}_{\hat{e}}$ on the reference edge or face $\hat{e}$ shared by two reference elements and passing to $\mathcal{L}_e$ by a suitable piecewise affine transformation. We refer to Verfürth [28] for this construction and the next results:

Property 4.3. *For any positive integer r, there exist positive constants c and c', independent of h, such that, for all elements K,*

$$\forall v \in \mathbb{P}_r(K), \quad c\|v\|_{L^2(K)} \leq \|v\psi_K^{\frac{1}{2}}\|_{L^2(K)} \leq c'\|v\|_{L^2(K)}. \tag{4.30}$$

Property 4.4. *For any positive integer r, there exist positive constants c, c', and c'', independent of h, such that, for all faces or edges e, according to the dimension, we have*

$$\forall\, v \in \mathbb{P}_r(e), \quad c\|v\|_{L^2(e)} \leq \|v\psi_e^{\frac{1}{2}}\|_{L^2(e)} \leq c'\|v\|_{L^2(e)}, \tag{4.31}$$

and, for all polynomials v in $\mathbb{P}_r(e)$ vanishing on ∂e, if K is an element adjacent to e,

$$\|\mathcal{L}_e v\|_{L^2(K)} + h_e |\mathcal{L}_e v|_{H^1(K)} \leq c'' h_e^{\frac{1}{2}} \|v\|_{L^2(e)}. \tag{4.32}$$

To establish lower bounds for this first discretization, as we need to work with polynomials for the application of inverse inequalities, it is convenient to approximate $\nu(f)$ by a polynomial of degree one. For instance, we choose $\nu_h : f \in H^1(K) \mapsto \nu_h(f) \in \mathbb{P}_1(K)$ defined by

$$\nu_h(f)|_K = \frac{1}{|K|}\int_K \nu(f(\mathbf{y}))\,\mathrm{d}\mathbf{y} + \left[\frac{1}{|K|}\int_K (\nabla\nu(f))(\mathbf{y})\,\mathrm{d}\mathbf{y}\right]\cdot(\mathbf{x}-\mathbf{c}), \tag{4.33}$$

where $\mathbf{c}$ is the center of the element K; thus $|\mathbf{x}-\mathbf{c}| \le h_K$. Clearly, if $\nu(f)|_K$ belongs to $\mathbb{P}_1$, then $\nu_h(f) = \nu(f)$ in K and it is easy to check that the mapping defined by (4.33) is invariant by affine transformations. In other words, if $\hat{K}$ is the reference element and $K = \mathcal{F}_K(\hat{K})$, then

$$\widehat{\nu_h(f)}|_{\hat{K}} = \frac{1}{|\hat{K}|}\int_{\hat{K}} \widehat{\nu(f)}\,d\hat{\mathbf{y}} + \Big[\frac{1}{|\hat{K}|}\int_{\hat{K}} (\hat{\nabla}\widehat{\nu(f)}\,d\hat{\mathbf{y}}\Big]\cdot(\hat{\mathbf{x}}-\hat{\mathbf{c}}),$$

where the hat denotes composition with $\mathcal{F}_K$ and $\hat{\mathbf{c}}$ is the center of $\hat{K}$. As a consequence, for smooth enough ν and f such that $\nu(f)$ belongs to $W^{\ell,p}(K)$ with $\ell = 1, 2$, and a number $p \ge 2$, we have

$$\|\nu_h(f)-\nu(f)\|_{L^p(K)} + h_K\|\nabla(\nu_h(f)-\nu(f))\|_{L^p(K)} \le C h_K^{\ell}|\nu(f)|_{W^{\ell,p}(K)}, \tag{4.34}$$

with a constant C that depends on p but is independent of h and K. The term $\|\nu_h(T)-\nu(T)\|_{W^{1,3}(K)}$ will appear in the subsequent lower bounds and will be treated as an error, considering (4.34).

When $\nu \in W^{2,\infty}(\mathbb{R})$, so that its derivative ν' is bounded by a real number denoted by ν_2' and is a Lipschitz-continuous function with Lipschitz constant λ', the polynomial function ν_h verifies from (1.2) and (3.1), for every $f \in H^1(K)$,

$$\|\nu_h(f)\|_{L^\infty(K)} \le \nu_2 + \nu_2'\,h_K|K|^{-\frac{1}{2}}\|\nabla f\|_{0,K}. \tag{4.35}$$

Furthermore, by observing that for all numbers $p \ge 2$,

$$\|\mathbf{x}-\mathbf{c}\|_{L^p(K)^d} \le C(p)|K|^{\frac{1}{p}}h_K, \tag{4.36}$$

where $C(p)$ depends only on p and the reference element, we readily derive that for every f_1 and f_2 in $H^1(K)$ we have for each number $p \ge 2$,

$$\begin{aligned}\|\nu_h(f_1)-\nu_h(f_2)\|_{L^p(K)} &\le \lambda|K|^{\frac{1}{p}-\frac{1}{6}}\|f_1-f_2\|_{L^6(K)}\\ &\quad + C(p)h_K|K|^{\frac{1}{p}-1}\left(\nu_2'|K|^{\frac{1}{2}}|f_1-f_2|_{H^1(K)} + \lambda'|K|^{\frac{1}{3}}\|f_1-f_2\|_{L^6(K)}|f_2|_{H^1(K)}\right).\end{aligned} \tag{4.37}$$

In the remainder of this section, we assume that $\nu \in W^{2,\infty}(\mathbb{R})$.

Let us start with the temperature error indicators (4.13) and (4.10).

Theorem 4.5. *We retain the settings and assumptions of Theorem 3.3. Then for each h, there exists an integer i_0 (depending on h) such that for all $i \ge i_0$ and for all $K \in \mathcal{T}_h$, the following inequality holds:*

$$\eta_{K,i,1}^{(D,1)} \le C\left(\|\boldsymbol{u}-\boldsymbol{u}_h^{i+1}\|_{L^2(\omega_e)^3} + |T-T_h^{i+1}|_{H^1(\omega_e)} + \sum_{\tilde{K}\subset\omega_e} h_{\tilde{K}}\|g-g_h\|_{L^2(\tilde{K})}\right), \tag{4.38}$$

with a constant C independent of h and K. Moreover, without assumption, we have

$$\eta_{K,i}^{(L,1)} \le |T-T_h^{i+1}|_{H^1(K)} + |T-T_h^{i}|_{H^1(K)}. \tag{4.39}$$

Proof. Clearly (4.39) follows by the triangle inequality.

Next, we bound the volume part of $\eta_{K,i,1}^{(D,1)}$. As usual, this is done by testing the error equation (4.7) with $S_h = 0$ and $S = S_K$, where S_K is the localizing function defined by

$$S_K = \begin{cases} (\alpha \Delta T_h^{i+1} - \mathbf{u}_h^{i+1}.\nabla T_h^{i+1} + g_h)\psi_K & \text{in } \ K, \\ 0 & \text{in } \ \Omega \setminus K. \end{cases}$$

With this choice and after inserting $\displaystyle\int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T) S \, \mathrm{d}\mathbf{x}$, (4.7) reduces to

$$\begin{aligned} \int_K \left(\alpha \Delta T_h^{i+1} - \mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1} + g_h\right)^2 \psi_K \, \mathrm{d}\mathbf{x} &= \alpha \int_K \nabla(T - T_h^{i+1}) \cdot \nabla S_K \, \mathrm{d}\mathbf{x} \\ &+ \int_K ((\mathbf{u} - \mathbf{u}_h^{i+1}) \cdot \nabla T) S_K \, \mathrm{d}\mathbf{x} + \int_K (\mathbf{u}_h^{i+1} \cdot \nabla(T - T_h^{i+1})) S_K \, \mathrm{d}\mathbf{x} \\ &- \int_K (g - g_h) S_K \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{4.40}$$

Now, we bound each term in the right hand side of (4.40). A bound for the last term is obvious; for the first term, we use the inverse inequality (3.3). The second term can be bounded by using Hölder's inequality and the inverse inequality (3.2),

$$\left| \int_K ((\mathbf{u} - \mathbf{u}_h^{i+1}) \cdot \nabla T) S_K \, \mathrm{d}\mathbf{x} \right| \le C_I^0(6) h_K^{-1} |T|_{W^{1,3}(K)} \|\mathbf{u} - \mathbf{u}_h^{i+1}\|_{L^2(K)^3} \|S_K\|_{L^2(K)}.$$

For the third term, we insert first $\mathbf{u}_h$ and next $R_h(\mathbf{u})$ (that is well defined since $\mathbf{u} \in H^1(\Omega)^3$) to apply inverse inequalities,

$$\begin{aligned} \left| \int_K (\mathbf{u}_h^{i+1} \cdot \nabla(T - T_h^{i+1})) S_K \, \mathrm{d}\mathbf{x} \right| &= \left| \int_K ((\mathbf{u}_h^{i+1} - \mathbf{u}_h) \cdot \nabla(T - T_h^{i+1})) S_K \, \mathrm{d}\mathbf{x} \right. \\ &+ \int_K ((\mathbf{u}_h - R_h(\mathbf{u})) \cdot \nabla(T - T_h^{i+1})) S_K \, \mathrm{d}\mathbf{x} \\ &+ \left. \int_K (R_h(\mathbf{u}) \cdot \nabla(T - T_h^{i+1})) S_K \, \mathrm{d}\mathbf{x} \right| \\ &\le C_I^0(6) h_K^{-1} \Big(C_I^0(3) h_K^{-\frac{1}{2}} \big(\|\mathbf{u}_h^{i+1} - \mathbf{u}_h\|_{L^2(K)^3} \\ &+ \|\mathbf{u}_h - R_h(\mathbf{u})\|_{L^2(K)^3} \big) + \|R_h(\mathbf{u})\|_{L^3(K)^3} \Big) \\ &\times |T - T_h^{i+1}|_{H^1(K)} \|S_K\|_{L^2(K)}. \end{aligned} \tag{4.41}$$

According to Theorem 3.3, see (3.28), there exists $h_0 > 0$ such that for any $h < h_0$, we have

$$\|\mathbf{u}_h^{i+1} - \mathbf{u}_h\|_{L^2(K)^3} \le c_1^i \|\mathbf{u}_h^1 - \mathbf{u}_h\|_{L^2(\Omega)^3},$$

where $c_1 < 1$. As $c_1^i \|\mathbf{u}_h^1 - \mathbf{u}_h\|_{L^2(\Omega)^3}$ tends to 0 when i tends to $+\infty$, then there exists i_0 depending on h, such that for any $i \ge i_0$,

$$\|\mathbf{u}_h^{i+1} - \mathbf{u}_h\|_{L^2(K)^3} \le c_2 \, h. \tag{4.42}$$

Using the *a priori* error estimates (3.11) and the approximation properties (3.5) of R_h, we obtain with the constant C_1 of (3.11) and interpolation constants C_2 and C_3 independent of h and K,

$$\begin{aligned} \left| \int_K (\mathbf{u}_h^{i+1} \cdot \nabla(T - T_h^{i+1})) S_K \, \mathrm{d}\mathbf{x} \right| &\le C_I^0(6) h_K^{-1} \Big(C_I^0(3) h_K^{-\frac{1}{2}} h \left(c_2 + C_1 C(\mathbf{u}, p, T) + C_2 |\mathbf{u}|_{H^1(\omega_K)} \right) + C_3 \|\mathbf{u}\|_{H^1(\omega_K)} \Big) \\ &\times |T - T_h^{i+1}|_{H^1(K)} \|S_K\|_{L^2(K)}. \end{aligned}$$

Hence, by collecting these estimates and using (3.3), we infer

$$
\begin{aligned}
h_K \left\|(\alpha\Delta T_h^{i+1} - \mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1} + g_h)\psi_K^{\frac{1}{2}}\right\|_{L^2(K)} \leq\; & h_K\|g-g_h\|_{L^2(K)} + C_I^0(6)|T|_{W^{1,3}(K)}\left\|\mathbf{u}-\mathbf{u}_h^{i+1}\right\|_{L^2(K)^3} \\
& + |T-T_h^{i+1}|_{H^1(K)}\Bigg(\alpha C_I^1(2) + C_I^0(3)C_I^0(6)\left(\frac{h}{\tau}\right)^{\frac{1}{2}} \\
& \times \left(c_2 + C_1 C(\mathbf{u},p,T) + C_2|\mathbf{u}|_{H^1(\omega_K)}\right) \\
& + C_3 C_I^0(6)\|\mathbf{u}\|_{H^1(\omega_K)}\Bigg), \qquad (4.43)
\end{aligned}
$$

which yields the first part of (4.38).

Finally, we estimate the surface part of $\eta_{K,i,1}^{(D,1)}$ by testing (4.7) with $S_h = 0$ and $S = S_e$, where S_e is the localizing function defined by

$$
S_e = \begin{cases} \mathcal{L}_e(\alpha[\nabla T_h^{i+1}\cdot\mathbf{n}]_e\psi_e) & \text{on } \; K\cup K', \\ 0 & \text{on } \; \Omega\setminus(K\cup K'), \end{cases}
$$

and K and K' are the two elements adjacent to e. Then (4.7) reduces to

$$
\begin{aligned}
\alpha\int_e [\nabla T_h^{i+1}\cdot\mathbf{n}]_e^2\psi_e\,\mathrm{d}s \;=\; & \int_{K\cup K'}(\alpha\Delta T_h^{i+1} - \mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1} + g_h)S_e\,\mathrm{d}\mathbf{x} \\
& + \alpha\int_{K\cup K'}\nabla(T_h^{i+1}-T)\cdot\nabla S_e\,\mathrm{d}\mathbf{x} + \int_{K\cup K'}((\mathbf{u}_h^{i+1}-\mathbf{u})\cdot\nabla T)S_e\,\mathrm{d}\mathbf{x} \\
& + \int_{K\cup K'}(\mathbf{u}_h^{i+1}\cdot\nabla(T_h^{i+1}-T))S_e\,\mathrm{d}\mathbf{x} + \int_{K\cup K'}(g-g_h)S_e\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

In view of the continuity properties of $\mathcal{L}_e$ in (4.32), a bound for the above left-hand side is derived by the same arguments; for instance, by combining it with (3.2), we have on the elements K sharing e

$$
\|\mathcal{L}_e(v)\|_{L^6(K)} \leq c'' C_I^0(6) h_K^{-1} h_e^{\frac{1}{2}}\|v\|_{L^2(e)}.
$$

Thus, by applying (4.43), we obtain

$$
h_e^{\frac{1}{2}}\|\alpha[\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|_{L^2(e)} \leq C\left(\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K\cup K')^3} + |T-T_h^{i+1}|_{H^1(K\cup K')} + h_e\|g-g_h\|_{L^2(K\cup K')}\right). \qquad (4.44)
$$

This gives the second part of (4.38). $\square$

Now, we turn to the first velocity error indicator (4.11). Beforehand, we establish the following preliminary results.

Lemma 4.6. *Let the mesh satisfy* (3.1), $\nu \in W^{2,\infty}(\mathbb{R})$ *and* $\mathbf{u}\in L^6(\Omega)^3$*. Then*

$$
\begin{aligned}
\|(\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\|_{L^2(K)^3} \leq\; & \left(2\nu_2 + \nu_2' h_K|K|^{-\frac{1}{2}}\frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}\right)\|\boldsymbol{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3} \\
& + \|\nu_h(T)-\nu(T)\|_{L^3(K)}\|\mathbf{u}\|_{L^6(K)^3} + \lambda\|T-T_h^i\|_{L^6(K)}\|\mathbf{u}\|_{L^3(K)^3} \\
& + \Big[C(3)|K|^{-\frac{2}{3}}h_K\big(\nu_2'|K|^{\frac{1}{2}}|T_h^i-T|_{H^1(K)} \\
& + \lambda'|K|^{\frac{1}{3}}\|T_h^i-T\|_{L^6(K)}|T|_{H^1(K)}\big) + \lambda|K|^{\frac{1}{6}}\|T_h^i-T\|_{L^6(K)}\Big]\|\mathbf{u}\|_{L^6(K)^3}, \quad (4.45)
\end{aligned}
$$

where $C(3)$ *is the constant of* (4.36).

Proof. The left-hand side of (4.45) can be split into

$$\|(\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\|_{L^2(K)^3} \le \|(\nu(T_h^i)-\nu_h(T_h^i))(\mathbf{u}_h^{i+1}-\mathbf{u})\|_{L^2(K)^3} + \|(\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}\|_{L^2(K)^3} = I_1 + I_2.$$

The bound for I_1 follows from (4.35), (1.2), and (3.18),

$$\|(\nu(T_h^i)-\nu_h(T_h^i))(\mathbf{u}_h^{i+1}-\mathbf{u})\|_{L^2(K)^3} \le \big(2\nu_2 + \nu_2' h_K |K|^{-\frac{1}{2}} \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}\big)\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3}. \tag{4.46}$$

To bound I_2, we split it in turn into three parts

$$I_2 \le \|(\nu_h(T_h^i)-\nu_h(T))\mathbf{u}\|_{L^2(K)^3} + \|(\nu_h(T)-\nu(T))\mathbf{u}\|_{L^2(K)^3} + \|(\nu(T)-\nu(T_h^i))\mathbf{u}\|_{L^2(K)^3}.$$

The second and third terms have a straightforward bound

$$\begin{aligned}
&\|(\nu_h(T)-\nu(T))\mathbf{u}\|_{L^2(K)^3} \le \|\nu_h(T)-\nu(T)\|_{L^3(K)}\|\mathbf{u}\|_{L^6(K)^3},\\
&\|(\nu(T)-\nu(T_h^i))\mathbf{u}\|_{L^2(K)^3} \le \lambda\|T-T_h^i\|_{L^6(K)}\|\mathbf{u}\|_{L^3(K)^3}.
\end{aligned}$$

Finally, we deal with the first term by a simple variant of (4.37)

$$\begin{aligned}
\|(\nu_h(T_h^i)-\nu_h(T))\mathbf{u}\|_{L^2(K)^3} \le \Big[&\lambda|K|^{\frac{1}{6}}\|T_h^i-T\|_{L^6(K)} + C(3)|K|^{-\frac{2}{3}}h_K\big(\nu_2'|K|^{\frac{1}{2}}|T_h^i-T|_{H^1(K)}\\
&+\lambda'|K|^{\frac{1}{3}}\|T_h^i-T\|_{L^6(K)}|T|_{H^1(K)}\big)\Big]\|\mathbf{u}\|_{L^6(K)^3},
\end{aligned} \tag{4.47}$$

where $C(3)$ is the constant of (4.36). The result follows by collecting these inequalities. □

Lemma 4.7. *Let the mesh satisfy (3.1), $\nu \in W^{2,\infty}(\mathbb{R})$, $\mathbf{u} \in L^6(\Omega)^3$, and $T \in W^{1,3}(\Omega)$. To simplify, the constants arising from inverse inequalities are not specified. Then*

$$\begin{aligned}
\|\nabla(\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\|_{L^2(K)^3} \le\ & Ch_K\|\nu''\|_{L^\infty(\mathbb{R})}\big(|K|^{-1}|T_h^i-R_h(T)|^2_{H^1(K)} + |K|^{-\frac{2}{3}}|R_h(T)|^2_{W^{1,3}(K)}\big)\\
&\times\|\boldsymbol{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3} + \Big[|\nu_h(T)-\nu(T)|_{W^{1,3}(K)} + |K|^{-\frac{1}{3}}|T_h^i-R_h(T)|_{H^1(K)}\\
&\times\|T_h^i-T\|_{L^6(K)} + |K|^{-\frac{1}{6}}|R_h(T)|_{W^{1,3}(K)}\|T_h^i-T\|_{L^6(K)}\\
&+|K|^{-\frac{1}{6}}|T_h^i-T|_{H^1(K)} + |R_h(T)-T|_{W^{1,3}(K)}\Big]\|\mathbf{u}\|_{L^6(K)^3}.
\end{aligned} \tag{4.48}$$

Proof. As previously,

$$\begin{aligned}
\|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\times\mathbf{u}_h^{i+1}\|_{L^2(K)^3} \le\ & \|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\times(\mathbf{u}_h^{i+1}-\mathbf{u})\|_{L^2(K)^3}\\
&+\|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\times\mathbf{u}\|_{L^2(K)^3} = I_1 + I_2.
\end{aligned}$$

An application of (4.34) gives

$$I_1 \le \|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\|_{L^\infty(K)^3}\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3} \le Ch_K\|\nabla(\nabla\,\nu(T_h^i))\|_{L^\infty(K)^{3\times 3}}\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3}.$$

As T_h^i is a polynomial of degree one in K, we have

$$\|\nabla(\nabla\,\nu(T_h^i))\|_{L^\infty(K)^{3\times 3}} \le \|\nu''\|_{L^\infty(\mathbb{R})}\|\nabla\,T_h^i\|^2_{L^\infty(K)^3}.$$

But of course, since the $W^{1,\infty}$ norm of T_h^i is very unfavorable, we insert the interpolant $R_h(T)$ and write

$$\|\nabla(\nabla\,\nu(T_h^i))\|_{L^\infty(K)^{3\times 3}} \le 2\|\nu''\|_{L^\infty(\mathbb{R})}\|\nabla\big(T_h^i-R_h(T)\big)\|^2_{L^\infty(K)^3} + 2\|\nu''\|_{L^\infty(\mathbb{R})}\|\nabla\,R_h(T)\|^2_{L^\infty(K)^3}.$$

Therefore, applying inverse inequalities (for simplicity, we do not specify the constants in the remainder of the proof)

$$\|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\|_{L^\infty(K)^3} \le C h_K \|\nu''\|_{L^\infty(\mathbf{R})}\big(|K|^{-1}|T_h^i - R_h(T)|^2_{H^1(K)} + |K|^{-\frac{2}{3}}|R_h(T)|^2_{W^{1,3}(K)}\big).$$

Thus

$$I_1 \le C h_K \|\nu''\|_{L^\infty(\mathbf{R})}\big(|K|^{-1}|T_h^i - R_h(T)|^2_{H^1(K)} + |K|^{-\frac{2}{3}}|R_h(T)|^2_{W^{1,3}(K)}\big)\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3}. \tag{4.49}$$

Next, the term I_2 has the bound

$$I_2 \le \|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\|_{L^3(K)^3}\|\mathbf{u}\|_{L^6(K)^3}.$$

As previously, we split the factor involving ν,

$$\begin{aligned}\|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\|_{L^3(K)^3} \le{}& \|\nabla\big(\nu_h(T_h^i)-\nu_h(T)\big)\|_{L^3(K)^3} + \|\nabla\big(\nu_h(T)-\nu(T)\big)\|_{L^3(K)^3}\\ &+\|\nabla\big(\nu(T)-\nu(T_h^i)\big)\|_{L^3(K)^3}.\end{aligned}$$

We only need to bound the first and third terms since the second one is an error, see (4.34). By virtue of (4.33), the first term has the expression

$$\begin{aligned}\big\|\nabla\big(\nu_h(T_h^i)-\nu_h(T)\big)\big\|_{L^3(K)^3} &= \left\|\frac{1}{|K|}\int_K \nabla\big(\nu(T_h^i)-\nu(T)\big)\,\mathrm{d}\mathbf{x}\right\|_{L^3(K)^3}\\ &\le |K|^{-\frac{2}{3}}\left|\int_K \nabla\big(\nu(T_h^i)-\nu(T)\big)\,\mathrm{d}\mathbf{x}\right|.\end{aligned} \tag{4.50}$$

Now, by splitting as follows:

$$\begin{aligned}\nabla\big(\nu(T_h^i)-\nu(T)\big) &= \nu'(T_h^i)\nabla T_h^i - \nu'(T)\nabla T = \big(\nu'(T_h^i)-\nu'(T)\big)\nabla T_h^i + \nu'(T)\nabla(T_h^i - T)\\ &= \big(\nu'(T_h^i)-\nu'(T)\big)\nabla(T_h^i - R_h(T)) + \big(\nu'(T_h^i)-\nu'(T)\big)\nabla R_h(T) + \nu'(T)\nabla(T_h^i - T),\end{aligned}$$

the first term is bounded by

$$\begin{aligned}\big\|\nabla\big(\nu_h(T_h^i)-\nu_h(T)\big)\big\|_{L^3(K)^3} \le{}& |K|^{-\frac{2}{3}}\Big(\lambda'|T_h^i - R_h(T)|_{H^1(K)}\|T_h^i - T\|_{L^2(K)}\\ &+ \lambda'|R_h(T)|_{H^1(K)}\|T_h^i - T\|_{L^2(K)} + \nu_2'|K|^{\frac{1}{2}}|T_h^i - T|_{H^1(K)}\Big)\\ \le{}& |K|^{-\frac{2}{3}}\Big(\lambda'|K|^{\frac{1}{3}}|T_h^i - R_h(T)|_{H^1(K)}\|T_h^i - T\|_{L^6(K)}\\ &+ \lambda'|K|^{\frac{1}{6}}|R_h(T)|_{W^{1,3}(K)}|K|^{\frac{1}{3}}\|T_h^i - T\|_{L^6(K)} + \nu_2'|K|^{\frac{1}{2}}|T_h^i - T|_{H^1(K)}\Big).\end{aligned}$$

There remains the third term. By recalling (4.50), we see that, we have to bound again $\nabla\big(\nu(T)-\nu(T_h^i)\big)$, but now in L^3 instead of L^1, as was done above. The same splitting gives

$$\begin{aligned}\|\nabla\big(\nu(T)-\nu(T_h^i)\big)\|_{L^3(K)^3} \le{}& \|\big(\nu'(T_h^i)-\nu'(T)\big)\nabla(T_h^i - R_h(T))\|_{L^3(K)^3} + \|\big(\nu'(T_h^i)-\nu'(T)\big)\nabla R_h(T)\|_{L^3(K)^3}\\ &+ \|\nu'(T)\nabla(T_h^i - T)\|_{L^3(K)^3}\\ \le{}& \lambda'\|T_h^i - T\|_{L^6(K)}\big(|T_h^i - R_h(T)|_{W^{1,6}(K)} + |R_h(T)|_{W^{1,6}(K)}\big)\\ &+ \nu_2'|T_h^i - R_h(T)|_{W^{1,3}(K)} + \nu_2'|R_h(T) - T|_{W^{1,3}(K)}\\ \le{}& \lambda' C\|T_h^i - T\|_{L^6(K)}\big(|K|^{-\frac{1}{3}}|T_h^i - R_h(T)|_{H^1(K)} + |K|^{-\frac{1}{6}}|R_h(T)|_{W^{1,3}(K)}\big)\\ &+ \nu_2'\big(C|K|^{-\frac{1}{6}}|T_h^i - R_h(T)|_{H^1(K)} + |R_h(T) - T|_{W^{1,3}(K)}\big).\end{aligned}$$

Summing up,

$$
\begin{aligned}
I_2 \leq C \Big[& |\nu_h(T) - \nu(T)|_{W^{1,3}(K)} + |K|^{-\frac{1}{3}} |T_h^i - R_h(T)|_{H^1(K)} \|T_h^i - T\|_{L^6(K)} \\
& + |K|^{-\frac{1}{6}} |R_h(T)|_{W^{1,3}(K)} \|T_h^i - T\|_{L^6(K)} + |K|^{-\frac{1}{6}} \left(|T_h^i - T|_{H^1(K)} + |T_h^i - R_h(T)|_{H^1(K)} \right) \\
& + |R_h(T) - T|_{W^{1,3}(K)} \Big] \|\mathbf{u}\|_{L^6(K)^3}.
\end{aligned} \tag{4.51}
$$

Then (4.48) follows from (4.49) and (4.51). □

Theorem 4.8. *Let the mesh satisfy (3.1). In addition to (1.1) and (1.2), we suppose that* $\mathbf{u} \in L^6(\Omega)^3$ *and* $\nu \in W^{2,\infty}(\mathbb{R})$. *Then there exists a constant* C, *independent of* h *and* K, *such that*

$$
\begin{aligned}
\eta_{K,i,2,1}^{(D,1)} \leq C \Bigg[& \left\| p - p_h^{i+1} \right\|_{L^2(\omega_e)} + \sum_{\tilde{K} \subset \omega_e} h_{\tilde{K}}^{\frac{1}{2}} \left\| \boldsymbol{u} - \boldsymbol{u}_h^{i+1} \right\|_{L^2(\tilde{K})^3} + \sum_{\tilde{K} \subset \omega_e} h_{\tilde{K}} \Bigg(\|\mathbf{f} - \mathbf{f}_h\|_{L^2(\tilde{K})^3} \\
& + \|T_h^i - T\|_{L^6(\tilde{K})} + |T_h^i - T|_{H^1(\tilde{K})} + \|\nu(T) - \nu_h(T)\|_{L^3(\tilde{K})} \Bigg) \Bigg].
\end{aligned} \tag{4.52}
$$

Proof. We test the error equation (4.8) with $\mathbf{v}_h = \mathbf{0}$ and $\mathbf{v} = \mathbf{v}_K$ where each component of the localizing function $\mathbf{v}_K$ is defined by

$$
(\mathbf{v}_K)_j = \begin{cases} (-\nabla p_h^{i+1} - \nu_h(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)_j \psi_K & \text{in } \ K, \\ 0 & \text{in } \ \Omega \setminus K. \end{cases} \tag{4.53}
$$

After inserting $\displaystyle\int_\Omega \nu(T)\mathbf{u}_h^{i+1} \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x}$ and $\displaystyle\int_\Omega \nu_h(T_h^i)\mathbf{u}_h^{i+1} \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x}$, (4.8) becomes,

$$
\begin{aligned}
\int_K |-\nabla p_h^{i+1} - \nu_h(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h|^2 \psi_K \, \mathrm{d}\mathbf{x} = & \int_K (\nu(T) - \nu(T_h^i))\mathbf{u} \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x} - \int_K \nu(T_h^i)(\mathbf{u}_h^{i+1} - \mathbf{u}) \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x} \\
& + \int_K (\nu(T_h^i) - \nu_h(T_h^i))\mathbf{u}_h^{i+1} \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x} + \int_K (p_h^{i+1} - p) \operatorname{div} \mathbf{v}_K \, \mathrm{d}\mathbf{x} \\
& - \int_K (\mathbf{f} - \mathbf{f}_h) \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x}.
\end{aligned} \tag{4.54}
$$

The bound for the last term in the right-hand side of (4.54) is obvious. The third term is bounded by applying (4.45). The next to last term is easily handled by considering that $\mathbf{v}_K$ vanishes on ∂K and by using (3.3),

$$
\left| \int_K (p_h^{i+1} - p) \operatorname{div} \mathbf{v}_K \, \mathrm{d}\mathbf{x} \right| \leq \|p_h^{i+1} - p\|_{L^2(K)} |\mathbf{v}_K|_{H^1(K)} \leq C_I^1(2) h_K^{-1} \|p_h^{i+1} - p\|_{L^2(K)} \|\mathbf{v}_K\|_{L^2(K)^3}.
$$

The second term is bounded by

$$
\left| \int_K \nu(T_h^i)(\mathbf{u} - \mathbf{u}_h^{i+1}) \cdot \mathbf{v}_K \right| \leq \nu_2 \|\mathbf{u} - \mathbf{u}_h^{i+1}\|_{L^2(K)^3} \|\mathbf{v}_K\|_{L^2(K)^3}.
$$

We have for the first term,

$$
\left| \int_K (\nu(T) - \nu(T_h^i))\mathbf{u} \cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x} \right| \leq \lambda \|T - T_h^i\|_{L^6(K)} \|\mathbf{u}\|_{L^3(K)^3} \|\mathbf{v}_K\|_{L^2(K)^3}.
$$

By collecting the above bounds, we deduce

$$\begin{aligned}
h_K \big\| | -\nabla p_h^{i+1} - \nu_h(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h|\psi_K^{\frac12}\big\|_{L^2(K)^3} &\le h_K\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^3} + C_I^1(2)\|p - p_h^{i+1}\|_{L^2(K)} \\
&+ \|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K)^3}\big(3\nu_2 h_K + \nu_2' h_K^2 |K|^{-\frac12}\frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}\big) + h_K\|\mathbf{u}\|_{L^6(K)^3}\|\nu(T)-\nu_h(T)\|_{L^3(K)} \\
&+ |T - T_h^i|_{H^1(K)} C(3)\nu_2' h_K^2 |K|^{-\frac16}\|\mathbf{u}\|_{L^6(K)^3} \\
&+ \|T-T_h^i\|_{L^6(K)}\Big[\lambda h_K\big(2\|\mathbf{u}\|_{L^3(K)^3} + |K|^{\frac16}\|\mathbf{u}\|_{L^6(K)^3}\big) + C(3)\lambda' h_K^2|K|^{-\frac13}\|\mathbf{u}\|_{L^6(K)^3}|T|_{H^1(K)}\Big].
\end{aligned} \tag{4.55}$$

The estimate for the volume part of $\eta_{K,i,2,1}^{(D,1)}$ follows from (4.55), (3.18), (4.30), and another application of Lemma 4.6.

Regarding the surface part of $\eta_{K,i,2,1}^{(D,1)}$, let e belong to Γ_h^i; by testing (4.8) with $\mathbf{v}_h = 0$ and $\mathbf{v} = \mathbf{v}_e$, where each component is defined by

$$(\mathbf{v}_e)_j = \begin{cases} \mathcal{L}_e([p_h^{i+1}\mathbf{n}_j]_e\psi_e) & \text{on } K\cup K', \\ 0 & \text{on } \Omega\setminus(K\cup K'), \end{cases}$$

we deduce

$$\begin{aligned}
\int_e |[p_h^{i+1}\mathbf{n}]_e|^2\psi_e\,\mathrm{d}s \;=\; &\int_{K\cup K'}(\nabla p_h^{i+1} + \nu(T_h^i)\mathbf{u}_h^{i+1} - \mathbf{f}_h)\cdot\mathbf{v}_e\,\mathrm{d}\mathbf{x} + \int_{K\cup K'}(\nu(T)\mathbf{u} - \nu(T_h^i)\mathbf{u}_h^{i+1})\cdot\mathbf{v}_e\,\mathrm{d}\mathbf{x} \\
&+ \int_{K\cup K'}(p_h^{i+1}-p)\operatorname{div}\mathbf{v}_e\,\mathrm{d}\mathbf{x} - \int_{K\cup K'}(\mathbf{f}-\mathbf{f}_h)\cdot\mathbf{v}_e\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

A straightforward application of (4.32) gives

$$\begin{aligned}
\left|\int_{K\cup K'}(\nu(T)\mathbf{u} - \nu(T_h^i)\mathbf{u}_h^{i+1})\cdot\mathbf{v}_e\,\mathrm{d}\mathbf{x}\right| &= \left|\int_{K\cup K'}\big(\nu(T_h^i)(\mathbf{u}_h^{i+1}-\mathbf{u}) + (\nu(T_h^i)-\nu(T))\mathbf{u}\big)\cdot\mathbf{v}_e\,\mathrm{d}\mathbf{x}\right| \\
&\le c''\,h_e^{\frac12}\Big(\nu_2\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K\cup K')^3} \\
&\quad + \lambda\|T_h^i - T\|_{L^6(K\cup K')}\|\mathbf{u}\|_{L^3(K\cup K')^3}\Big)\|\mathbf{v}_e\|_{L^2(e)^3}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_e |[p_h^{i+1}\mathbf{n}]_e|^2\psi_e\,\mathrm{d}s \;\le c''\,h_e^{\frac12}\Big(&\|\nabla p_h^{i+1} + \nu(T_h^i)\mathbf{u}_h^{i+1} - \mathbf{f}_h\|_{L^2(K\cup K')^3} + \|\mathbf{f}-\mathbf{f}_h\|_{L^2(K\cup K')^3} \\
&+ \nu_2\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K\cup K')^3} + \lambda\|T_h^i - T\|_{L^6(K\cup K')}\|\mathbf{u}\|_{L^3(K\cup K')^3} \\
&+ h_e^{-1}\|p_h^{i+1}-p\|_{L^2(K\cup K')}\Big)\|\mathbf{v}_e\|_{L^2(e)^3}.
\end{aligned} \tag{4.56}$$

The surface part of $\eta_{K,i,2,1}^{(D,1)}$ follows readily from (4.56), (4.31), (4.32), and the volume part of $\eta_{K,i,2,1}^{(D,1)}$. $\square$

Next, we estimate the second velocity error indicator.

Theorem 4.9. *In addition to the assumptions of Theorem 3.1, let the mesh satisfy (3.19), $\nu \in W^{2,\infty}(\mathbb{R})$ and $\mathbf{f} \in H^1(\Omega)^3$. Then for each h, there exists an integer i_0 (depending on h) such that for all $i \ge i_0$ and for all*

$K \in \mathcal{T}_h$, the following inequalities hold:

$$\begin{aligned}\eta^{(D,1)}_{K,i,2,2} \leq C' \Big(&\|\boldsymbol{u} - \boldsymbol{u}_h^{i+1}\|_{L^2(\omega_e)^3} + \sum_{\tilde{K} \subset \omega_e} \big[h_{\tilde{K}} \big(\|\mathbf{curl}(\mathbf{f} - \mathbf{f}_h)\|_{L^2(\tilde{K})^3} + |\nu(T) - \nu_h(T)|_{W^{1,3}(\tilde{K})} \big) \\ &+ \|T_h^i - T\|_{L^6(\tilde{K})} + h_{\tilde{K}}^{\frac{1}{2}} |T_h^i - T|_{H^1(\tilde{K})} + \|\nu(T) - \nu_h(T)\|_{L^3(\tilde{K})} \big] \\ &+ \sum_{e \in \partial K \cap \Gamma_h^i} h_e^{\frac{1}{2}} \|[(\mathbf{f} - \mathbf{f}_h) \times \boldsymbol{n}]_e\|_{L^2(e)^3} \Big).\end{aligned} \tag{4.57}$$

Proof. To bound the volume part of $\eta^{(D,1)}_{K,i,2,2}$, (4.9) is tested with $\eta_h = \mathbf{0}$ and $\eta = \eta_K$, with each component defined by

$$(\eta_K)_j = \begin{cases} \big(\mathbf{curl}(-\nu_h(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h\big)_j \psi_K & \text{in} \quad K, \\ 0 & \text{in} \quad \Omega \setminus K. \end{cases}$$

This gives

$$\int_K |\mathbf{curl}(-\nu_h(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)|^2 \psi_K \,\mathrm{d}\mathbf{x} = \int_K (\nu(T)\mathbf{u} - \nu_h(T_h^i)\mathbf{u}_h^{i+1}) \cdot \mathbf{curl}\,\eta_K \,\mathrm{d}\mathbf{x} - \int_K \mathbf{curl}(\mathbf{f} - \mathbf{f}_h) \cdot \eta_K \,\mathrm{d}\mathbf{x}.$$

There does not seem to be much gain in applying Green's formula to the first integral in the above right-hand side. Hence, applying the inverse inequality (3.3), we have

$$\begin{aligned}\int_K |\mathbf{curl}(-\nu_h(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)|^2 \psi_K \,\mathrm{d}\mathbf{x} \leq\ & C_I^1(2) h_K^{-1} \|\nu(T)\mathbf{u} - \nu_h(T_h^i)\mathbf{u}_h^{i+1}\|_{L^2(K)^3} \|\eta_K\|_{L^2(K)^3} \\ &+ \|\mathbf{curl}(\mathbf{f} - \mathbf{f}_h)\|_{L^2(K)^3} \|\eta_K\|_{L^2(K)^3},\end{aligned}$$

which means that a bound for the first factor cannot contain a negative power of h_K. As usual, we expand

$$\begin{aligned}\|\nu(T)\mathbf{u} - \nu_h(T_h^i)\mathbf{u}_h^{i+1}\|_{L^2(K)^3} &\leq \|(\nu_h(T_h^i) - \nu(T_h^i))\mathbf{u}_h^{i+1}\|_{L^2(K)^3} + \|\nu(T_h^i)(\mathbf{u}_h^{i+1} - \mathbf{u})\|_{L^2(K)^3} \\ &\quad + \|(\nu(T_h^i) - \nu(T))\mathbf{u}\|_{L^2(K)^3} \\ &\leq \|(\nu_h(T_h^i) - \nu(T_h^i))\mathbf{u}_h^{i+1}\|_{L^2(K)^3} + \nu_2 \|\mathbf{u}_h^{i+1} \\ &\quad - \mathbf{u}\|_{L^2(K)^3} + \lambda \|\mathbf{u}\|_{L^3(K)^3} \|T_h^i - T\|_{L^6(K)},\end{aligned} \tag{4.58}$$

and we treat the first term on the right hand side. The bound in (4.45) is not sharp enough here because the factor multiplying $\|\mathbf{u}_h^{i+1} - \mathbf{u}\|_{L^2(K)^3}$ does have a negative power of h_K. Thus, we must sharpen (4.35). To this end, we insert ∇T as follows:

$$\begin{aligned}\|\nu_h(T_h^i) - \nu(T_h^i)\|_{L^\infty(K)} &\leq \nu_2 + \|\nu_h(T_h^i)\|_{L^\infty(K)} \\ &\leq 2\nu_2 + \nu_2' h_K |K|^{-\frac{1}{2}} (|T_h^i - T|_{H^1(K)} + |T|_{H^1(K)}) \\ &\leq 2\nu_2 + \nu_2' \big(h_K |K|^{-\frac{1}{2}} |T_h^i - T|_{H^1(K)} + h_K |K|^{-\frac{1}{3}} |T|_{W^{1,3}(K)} \big).\end{aligned}$$

With this inequality, (4.46) is replaced by

$$\begin{aligned}\|(\nu_h(T_h^i) - \nu(T_h^i))(\mathbf{u}_h^{i+1} - \mathbf{u})\|_{L^2(K)^3} \leq\ & \Big(2\nu_2 + \nu_2' \big(h_K |K|^{-\frac{1}{2}} |T_h^i - T|_{H^1(K)} + h_K |K|^{-\frac{1}{3}} |T|_{W^{1,3}(K)} \big) \Big) \\ &\times \|\mathbf{u}_h^{i+1} - \mathbf{u}\|_{L^2(K)^3}.\end{aligned} \tag{4.59}$$

By replacing the first line of (4.45) with (4.59) and substituting it into (4.58), we derive

$$\begin{aligned}\|\nu(T)\mathbf{u}-\nu_h(T_h^i)\mathbf{u}_h^{i+1}\|_{L^2(K)^3} \le& \Big[3\nu_2+\nu_2'\big(h_K|K|^{-\frac{1}{2}}|T-T_h^i|_{H^1(K)}+h_K|K|^{-\frac{1}{3}}|T|_{W^{1,3}(K)}\big)\Big]\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3}\\ &+\|\nu_h(T)-\nu(T)\|_{L^3(K)}\|\mathbf{u}\|_{L^6(K)^3}+2\lambda\|T-T_h^i\|_{L^6(K)}\|\mathbf{u}\|_{L^3(K)^3}\\ &+\Big[C(3)h_K|K|^{-\frac{2}{3}}\big(\nu_2'|K|^{\frac{1}{2}}|T-T_h^i|_{H^1(K)}+\lambda'|K|^{\frac{1}{3}}\|T-T_h^i\|_{L^6(K)}|T|_{H^1(K)}\big)\\ &+\lambda|K|^{\frac{1}{6}}\|T-T_h^i\|_{L^6(K)}\Big]\|\mathbf{u}\|_{L^6(K)^3}. \end{aligned}\tag{4.60}$$

By proceeding as in Theorem 4.5 and using the convergence of the iterates of Theorem 3.3 (see (3.28) and (3.25)), we derive the volume part of $\eta_{K,i,2,2}^{(D,1)}$.

Regarding the surface part of $\eta_{K,i,2,2}^{(D,1)}$, let $e\in\Gamma_h^i$; by testing (4.9) with $\eta_h=\mathbf{0}$ and $\eta=\eta_e$, $\mathbf{v}=\mathbf{curl}\,\eta_e$, with each component defined by

$$(\eta_e)_j=\begin{cases}\mathcal{L}_e\big(([(-\nu_h(T_h^i)\mathbf{u}_h^{i+1}+\mathbf{f}_h)\times\mathbf{n}]_e)_j\psi_e\big) & \text{on}\quad K\cup K',\\ 0 & \text{on}\quad \Omega\setminus(K\cup K'),\end{cases}$$

we deduce

$$\begin{aligned}\int_e\big|[(\mathbf{f}_h-\nu_h(T_h^i)\mathbf{u}_h^{i+1})\times\mathbf{n}]_e\big|^2\psi_e\,\mathrm{d}s=&-\int_{K\cup K'}\mathbf{curl}\big(\mathbf{f}_h-\nu(T_h^i)\mathbf{u}_h^{i+1}\big)\cdot\eta_e\,\mathrm{d}\mathbf{x}-\int_e[(\mathbf{f}-\mathbf{f}_h)\times\mathbf{n}]_e\cdot\eta_e\,\mathrm{d}s\\ &-\int_{K\cup K'}\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\cdot\eta_e\,\mathrm{d}\mathbf{x}+\int_{K\cup K'}(\nu(T)\mathbf{u}-\nu(T_h^i)\mathbf{u}_h^{i+1})\cdot\mathbf{curl}\,\eta_e\,\mathrm{d}\mathbf{x}\\ &-\int_e[(\nu_h(T_h^i)-\nu(T_h^i))\mathbf{u}_h^{i+1}\times\mathbf{n}]_e\cdot\eta_e\,\mathrm{d}s.\end{aligned}\tag{4.61}$$

On account of the last term in the above right-hand side, the estimate of the surface part is more complex than that of the volume part. Indeed, by (4.4) and the fact that η_e vanishes on $\partial(K\cup K')$, this term reads

$$\begin{aligned}\int_e[(\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\times\mathbf{n}]_e\cdot\eta_e\,\mathrm{d}s=&\int_{K\cup K'}(\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\cdot\mathbf{curl}\,\eta_e\\ &-\int_{K\cup K'}\mathbf{curl}\big((\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\big)\cdot\eta_e\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{4.62}$$

The structure of the RT_0 element implies that $\mathbf{curl}\,\mathbf{u}_h^{i+1}=\mathbf{0}$ in each cell, and so

$$\big|\mathbf{curl}\big((\nu(T_h^i)-\nu_h(T_h^i))\mathbf{u}_h^{i+1}\big)\big|=\big|\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\times\mathbf{u}_h^{i+1}\big|.$$

Before applying (4.48), we substitute (4.62) into (4.61),

$$\begin{aligned}\int_e\big|[(\mathbf{f}_h-\nu_h(T_h^i)\mathbf{u}_h^{i+1})\times\mathbf{n}]_e\big|^2\psi_e\,\mathrm{d}s=&-\int_{K\cup K'}\mathbf{curl}\big(\mathbf{f}_h-\nu(T_h^i)\mathbf{u}_h^{i+1}\big)\cdot\eta_e\,\mathrm{d}\mathbf{x}-\int_e[(\mathbf{f}-\mathbf{f}_h)\times\mathbf{n}]_e\cdot\eta_e\,\mathrm{d}s\\ &-\int_{K\cup K'}\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\cdot\eta_e\,\mathrm{d}\mathbf{x}+\int_{K\cup K'}(\nu(T)\mathbf{u}-\nu_h(T_h^i)\mathbf{u}_h^{i+1})\cdot\mathbf{curl}\,\eta_e\,\mathrm{d}\mathbf{x}\\ &-\int_{K\cup K'}\big(\nabla\big(\nu(T_h^i)-\nu_h(T_h^i)\big)\times\mathbf{u}_h^{i+1}\big)\cdot\eta_e\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{4.63}$$

Let us denote by T_1 and T_2 the last two terms in the right-hand side of (4.63). Then

$$\begin{aligned}\int_e\big|[(\mathbf{f}_h-\nu_h(T_h^i)\mathbf{u}_h^{i+1})\times\mathbf{n}]_e\big|^2\psi_e\,\mathrm{d}s\le c''h_e^{\frac{1}{2}}\Big(&\|\mathbf{curl}\big(\mathbf{f}_h-\nu(T_h^i)\mathbf{u}_h^{i+1}\big)\|_{L^2(K\cup K')^3}+\|[(\mathbf{f}-\mathbf{f}_h)\times\mathbf{n}]\|_{L^2(e)^3}\\ &+\|\mathbf{curl}(\mathbf{f}-\mathbf{f}_h)\|_{L^2(K\cup K')^3}\Big)\|\eta_e\|_{L^2(e)^3}+T_1+T_2.\end{aligned}\tag{4.64}$$

The term T_1 is bounded by (4.60) via

$$|T_1| \le C_I^1(2) \sum_{\tilde{K}\in\omega_e} h_{\tilde{K}}^{-1} \|\nu(T)\mathbf{u} - \nu_h(T_h^i)\mathbf{u}_h^{i+1}\|_{L^2(\tilde{K})^3} \|\eta_e\|_{L^2(\tilde{K})^3},$$

and the term T_2 by (4.48) via

$$|T_2| \le \sum_{\tilde{K}\in\omega_e} \|\nabla\big(\nu(T_h^i) - \nu_h(T_h^i)\big)\mathbf{u}_h^{i+1}\|_{L^2(\tilde{K})^3} \|\eta_e\|_{L^2(\tilde{K})^3}.$$

A comparison between (4.60) and (4.51) shows that the most unfavorable term here, which is the first term in (4.49), has the same order as that in (4.60). As above, this is resolved by iterating sufficiently the algorithm, and we readily derive the desired bound on the surface part of the indicator $\eta_{K,i,2,2}^{(D,1)}$ by substituting (4.60), (4.49), and (4.51) into (4.64), and by applying to the first term of (4.61) the bound for the volume part found above. □

Remark 4.10. Condition (3.30) when $d = 3$ and (3.31) when $d = 2$, used in the efficiency bounds, are undesirable but also inevitable. They are caused by the discrepancy between the norms used in measuring the *a priori* error estimates (including stability bounds) see for instance (3.11), and the norms used in measuring continuity of the operators. Indeed, if the velocity is in L^2 and the temperature in H^1, the nonlinear convection term is only in L^1. This is why the test functions are taken in $L^\infty \cap H^1$. This difficulty is inherent to the model and is independent of the choice of discretization or indicators. In addition, it is aggravated by the two following factors:

(1) The analysis is performed on the sequence produced by a computing algorithm. If such algorithm had not been taken into account, the parameter δ would not be necessary and (3.30) could be replaced by

$$\forall K \in \mathcal{T}_h, \quad h_K \ge \tau\, h^2.$$

(2) The analysis is done in three dimensions. In two dimensions, the relevant condition is (3.31) (with $\delta = 0$, when the algorithm is not considered), that is almost negligible.

The same observations apply also to the second approximation analyzed in the section below.

In any case, we suppose that (3.30) (or (3.31) when $d = 2$) is verified for all levels of the refinement iteration.

5. *A POSTERIORI* ERROR ESTIMATES FOR THE SECOND APPROXIMATION

5.1. Upper error bound for the second discretization

In order to establish upper bounds for the second variational formulation, we introduce, on every edge e of the mesh, the function

$$\phi_{h,1}^e = \begin{cases} \dfrac{1}{2}\,[\mathbf{u}_h^{i+1}\cdot\mathbf{n}]_e & \text{if } \ e \in \Gamma_h^i, \\ \mathbf{u}_h^{i+1}\cdot\mathbf{n} & \text{if } \ e \in \Gamma_h^b. \end{cases} \tag{5.1}$$

A standard calculation shows that the solutions of problems (V_2) and $(V_{h,i,2})$ verify for all $(\mathbf{v}, q, S) \in L^2(\Omega)^3 \times (H^1(\Omega) \cap L^2_m(\Omega)) \times (H^1(\Omega) \cap L^\infty(\Omega))$ and $(\mathbf{v}_h, q_h, S_h) \in \mathcal{W}_{h,2} \times M_{h,2} \times X_h$:

$$\begin{aligned} \alpha \int_\Omega \nabla(T - T_h^{i+1})\cdot\nabla S\,\mathrm{d}\mathbf{x} &+ \int_\Omega (\mathbf{u}\cdot\nabla T)S\,\mathrm{d}\mathbf{x} - \int_\Omega (\mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1})S\,\mathrm{d}\mathbf{x} - \frac{1}{2}\int_\Omega \mathrm{div}\,\mathbf{u}_h^{i+1} T_h^{i+1} S\,\mathrm{d}\mathbf{x} \\ = \sum_{K\in\mathcal{T}_h} \Bigg[&\int_K (\alpha\Delta T_h^{i+1} - \mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1} - \frac{1}{2}\mathrm{div}\,\mathbf{u}_h^{i+1} T_h^{i+1} + g_h)(S - S_h)\,\mathrm{d}\mathbf{x} \\ &+ \int_K (g - g_h)(S - S_h)\,\mathrm{d}\mathbf{x} - \frac{\alpha}{2}\sum_{e\in\partial K\cap\Gamma_h^i} \int_e [\nabla T_h^{i+1}\cdot\mathbf{n}]_e (S - S_h)\,\mathrm{d}s \Bigg], \end{aligned} \tag{5.2}$$

$$\begin{aligned}\int_\Omega \nu(T)\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} &+ \int_\Omega \nabla(p-p_h^{i+1})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_\Omega \nu(T_h^i)\mathbf{u}_h^{i+1}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\\ &= \sum_{K\in\mathcal{T}_h}\left[\int_K(-\nabla p_h^{i+1}-\nu(T_h^i)\mathbf{u}_h^{i+1}+\mathbf{f}_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x} + \int_K(\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x}\right],\end{aligned}\tag{5.3}$$

and

$$\int_\Omega \nabla q\cdot(\mathbf{u}-\mathbf{u}_h^{i+1})\,\mathrm{d}\mathbf{x} = \sum_{K\in\mathcal{T}_h}\left[\int_K(q-q_h)\mathrm{div}\,\mathbf{u}_h^{i+1}\,\mathrm{d}\mathbf{x} - \sum_{e\in\partial K}\int_e \phi_{h,1}^e(q-q_h)\,\mathrm{d}s\right],\tag{5.4}$$

where g_h and $\mathbf{f}_h$ are an approximation of g and $\mathbf{f}$ which are constant on each element K of $\mathcal{T}_h$.

From these error equations we deduce the following error indicators for each $K\in\mathcal{T}_h$,

$$\begin{aligned}\eta_{K,i}^{(L,2)} &= |T_h^{i+1}-T_h^i|_{H^1(K)},\\ \eta_{K,i,1}^{(D,2)} &= h_K\|\alpha\Delta T_h^{i+1}-\mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1}-\frac{1}{2}\mathrm{div}\,\mathbf{u}_h^{i+1}T_h^{i+1}+g_h\|_{L^2(K)}\\ &\quad+\frac{1}{2}\sum_{e\in\partial K\cap\Gamma_h^i} h_e^{\frac{1}{2}}\|\alpha[\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|_{L^2(e)},\end{aligned}\tag{5.5}$$

$$\eta_{K,i,2}^{(D,2)} = \|-\nabla p_h^{i+1}-\nu(T_h^i)\mathbf{u}_h^{i+1}+\mathbf{f}_h\|_{L^2(K)^3} + h_K\|\mathrm{div}\,\mathbf{u}_h^{i+1}\|_{L^2(K)} + \sum_{e\in\partial K} h_e^{\frac{1}{2}}\|\phi_{h,1}^e\|_{L^2(e)}.\tag{5.6}$$

Theorem 5.1. *Let $d=3$, let the mesh satisfy (3.1), and ν satisfy (1.1) and (1.2). We suppose that problem (V_2) has a solution $(\boldsymbol{u},T)\in L^3(\Omega)^3\times(W^{1,3}(\Omega)\cap L^\infty(\Omega))$ such that*

$$\lambda\, S_6^0\|\boldsymbol{u}\|_{L^3(\Omega)^3}(S_6^0|T|_{W^{1,3}(\Omega)}+\|T\|_{L^\infty(\Omega)}) < 2\,\alpha\,\nu_1.\tag{5.7}$$

Then the following error inequalities hold:

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h^{i+1}\|_{L^2(\Omega)^3}+|p-p_h^{i+1}|_{H^1(\Omega)}+|T-T_h^{i+1}|_{H^1(\Omega)} \le C\Bigg[\sum_{K\in\mathcal{T}_h}\bigg(\Big(\eta_{K,i,1}^{(D,2)}\Big)^2+\Big(\eta_{K,i,2}^{(D,2)}\Big)^2+\|\boldsymbol{f}-\boldsymbol{f}_h\|_{L^2(K)^3}^2\\ +h_K^2\|g-g_h\|_{L^2(K)}^2\bigg)\Bigg]^{\frac{1}{2}} + C'\left(\sum_{K\in\mathcal{T}_h}\Big(\eta_{K,i}^{(L,2)}\Big)^2\right)^{\frac{1}{2}}.\end{aligned}\tag{5.8}$$

Proof. Let us start with the temperature equation (5.2) tested with $S=T-T_h^{i+1}$. Its left-hand side can be written as

$$\begin{aligned}\alpha|T-T_h^{i+1}|_{H^1(\Omega)}^2+\int_\Omega((\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\nabla T)(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}+\int_\Omega(\mathbf{u}_h^{i+1}\cdot\nabla(T-T_h^{i+1}))(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}\\ -\frac{1}{2}\int_\Omega(\mathrm{div}\mathbf{u}_h^{i+1})T_h^{i+1}(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}.\end{aligned}$$

By applying Green's formula and the zero divergence of $\mathbf{u}$, the sum of last two nonlinear terms has the expression

$$\begin{aligned}\int_\Omega(\mathbf{u}_h^{i+1}\cdot\nabla(T-T_h^{i+1}))(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}-\frac{1}{2}\int_\Omega(\operatorname{div}\mathbf{u}_h^{i+1})T_h^{i+1}(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}\\=-\frac{1}{2}\int_\Omega(\operatorname{div}\mathbf{u}_h^{i+1})\big(T-T_h^{i+1}+T_h^{i+1}\big)(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}=\frac{1}{2}\int_\Omega\operatorname{div}(\mathbf{u}-\mathbf{u}_h^{i+1})T(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}\\=-\frac{1}{2}\int_\Omega(\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\big(\nabla T(T-T_h^{i+1})+T\nabla(T-T_h^{i+1})\big)\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Thus the sum of the three nonlinear terms is

$$\begin{aligned}\int_\Omega((\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\nabla T)(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}+\int_\Omega(\mathbf{u}_h^{i+1}\cdot\nabla(T-T_h^{i+1}))(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}-\frac{1}{2}\int_\Omega(\operatorname{div}\mathbf{u}_h^{i+1})T_h^{i+1}(T-T_h^{i+1})\,\mathrm{d}\mathbf{x}\\=\frac{1}{2}\int_\Omega(\mathbf{u}-\mathbf{u}_h^{i+1})\cdot\big(\nabla T(T-T_h^{i+1})-T\nabla(T-T_h^{i+1})\big)\,\mathrm{d}\mathbf{x}\\\le\frac{1}{2}\big(S_6^0|T|_{W^{1,3}(\Omega)}+\|T\|_{L^\infty(\Omega)}\big)\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}|T-T_h^{i+1}|_{H^1(\Omega)}.\end{aligned}\tag{5.9}$$

Now, the right-hand side of (5.2) is bounded straightforwardly by

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\Big[\big(\|\alpha\Delta T_h^{i+1}-\mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1}-\frac{1}{2}\operatorname{div}\mathbf{u}_h^{i+1}T_h^{i+1}+g_h\|_{L^2(K)}+\|g-g_h\|_{L^2(K)}\big)\|S-S_h\|_{L^2(K)}\\+\frac{1}{2}\sum_{e\in\partial K\cap\Gamma_h^i}\|\alpha[\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|_{L^2(e)}\|S-S_h\|_{L^2(e)}\Big].\end{aligned}$$

Then the choice $S_h=R_h(S)$, (5.9), the approximation properties of R_h, and the regularity of $\mathcal{T}_h$ yield

$$\begin{aligned}\alpha|T-T_h^{i+1}|_{H^1(\Omega)}\le C_1\left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,1}^{(D,2)}\right)^2+h_K^2\|g-g_h\|^2_{L^2(K)}\right)\right)^{\frac{1}{2}}\\+\frac{1}{2}\left(S_6^0|T|_{W^{1,3}(\Omega)}+\|T\|_{L^\infty(\Omega)}\right)\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3}.\end{aligned}\tag{5.10}$$

Next, we turn to the velocity and pressure errors. The velocity error equation (5.3) can be written as

$$\begin{aligned}\int_\Omega\Big(\big(\nu(T)-\nu(T_h^i)\big)\mathbf{u}+\nu(T_h^i)(\mathbf{u}-\mathbf{u}_h^{i+1})\Big)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}+\int_\Omega\nabla(p-p_h^{i+1})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}\\=\sum_{K\in\mathcal{T}_h}\left[\int_K(-\nabla p_h^{i+1}-\nu(T_h^i)\mathbf{u}_h^{i+1}+\mathbf{f}_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x}+\int_K(\mathbf{f}-\mathbf{f}_h)\cdot(\mathbf{v}-\mathbf{v}_h)\,\mathrm{d}\mathbf{x}\right].\end{aligned}\tag{5.11}$$

As usual, a bound for the velocity is derived by eliminating the pressure from (5.11). This is obtained from the divergence error equation (5.4). Indeed, it follows from the inf-sup condition (2.14) that there exists a velocity $\mathbf{v}_r$ in $L^2(\Omega)^3$ that solves

$$\forall q\in H^1(\Omega)\cap L^2_m(\Omega),\quad\int_\Omega\nabla q\cdot\mathbf{v}_r\,\mathrm{d}\mathbf{x}=\sum_{K\in\mathcal{T}_h}\left[\int_K(q-R_h(q))\operatorname{div}\mathbf{u}_h^{i+1}\,\mathrm{d}\mathbf{x}-\sum_{e\in\partial K}\int_e\phi_{h,1}^e(q-R_h(q)\,\mathrm{d}s\right],\tag{5.12}$$

and satisfies

$$\|\mathbf{v}_r\|_{L^2(\Omega)^3}\le\sup_{q\in H^1(\Omega)\cap L^2_m(\Omega)}\frac{1}{|q|_{H^1(\Omega)}}\left|\sum_{K\in\mathcal{T}_h}\left[\int_K(q-R_h(q))\operatorname{div}\mathbf{u}_h^{i+1}\,\mathrm{d}\mathbf{x}-\sum_{e\in\partial K}\int_e\phi_{h,1}^e(q-R_h(q)\,\mathrm{d}s\right]\right|.$$

Thus, from the approximation properties of R_h, and the regularity of $\mathcal{T}_h$, we infer

$$\|\mathbf{v}_r\|_{L^2(\Omega)^3} \le C_2 \left(\sum_{K\in\mathcal{T}_h} \left[h_K^2 \|\mathrm{div}\mathbf{u}_h^{i+1}\|^2_{L^2(K)} + \sum_{e\in\partial K} h_e \|\phi^e_{h,1}\|^2_{L^2(e)} \right] \right)^{\frac{1}{2}}. \tag{5.13}$$

Now, to simplify we set $\mathbf{z}_0 = \mathbf{u} - \mathbf{u}_h^{i+1} - \mathbf{v}_r$ and we test (5.11) with $\mathbf{v} = \mathbf{z}_0$ and $\mathbf{v}_h = \mathbf{0}$. By construction, (5.12) and (5.4) with $q = p - p_h^{i+1}$ and $q_h = R_h(q)$, imply that

$$\int_\Omega \nabla(p - p_h^{i+1}) \cdot \mathbf{z}_0 \,\mathrm{d}\mathbf{x} = 0.$$

Hence (5.11) reduces to

$$\begin{aligned}\int_\Omega \big(\nu(T) - \nu(T_h^i)\big)\mathbf{u}\cdot\mathbf{z}_0 \,\mathrm{d}\mathbf{x} + \int_\Omega \nu(T_h^i)\mathbf{z}_0\cdot\mathbf{z}_0\,\mathrm{d}\mathbf{x} + \int_\Omega \nu(T_h^i)\mathbf{v}_r\cdot\mathbf{z}_0\,\mathrm{d}\mathbf{x} \\ = \sum_{K\in\mathcal{T}_h}\left[\int_K (-\nabla p_h^{i+1} - \nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)\cdot\mathbf{z}_0\,\mathrm{d}\mathbf{x} + \int_K (\mathbf{f}-\mathbf{f}_h)\cdot\mathbf{z}_0\,\mathrm{d}\mathbf{x}\right].\end{aligned}$$

This yields the bound

$$\nu_1\|\mathbf{z}_0\|_{L^2(\Omega)^3} \le \big(\lambda S_6^0 |T - T_h^i|_{H^1(\Omega)}\|\mathbf{u}\|_{L^3(\Omega)^3} + \nu_2\|\mathbf{v}_r\|_{L^2(\Omega)^3}\big) + \left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,2}^{(D,2)}\right)^2 + \|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(K)^3}\right)\right)^{\frac{1}{2}}.$$

With (5.13), and after inserting T_h^{i+1}, this implies

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} \le{}& C_2\big(1+\frac{\nu_2}{\nu_1}\big)\Big(\sum_{K\in\mathcal{T}_h}\big[h_K^2\|\mathrm{div}\mathbf{u}_h^{i+1}\|^2_{L^2(K)} + \sum_{e\in\partial K} h_e\|\phi^e_{h,1}\|^2_{L^2(e)}\big]\Big)^{\frac{1}{2}} \\ &+ \frac{\lambda}{\nu_1}S_6^0|T - T_h^{i+1}|_{H^1(\Omega)}\|\mathbf{u}\|_{L^3(\Omega)^3} + \frac{\lambda}{\nu_1}S_6^0|T_h^{i+1} - T_h^i|_{H^1(\Omega)}\|\mathbf{u}\|_{L^3(\Omega)^3} \\ &+ \frac{1}{\nu_1}\Big(\sum_{K\in\mathcal{T}_h}\big((\eta_{K,i,2}^{(D,2)})^2 + \|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(K)^3}\big)\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.14}$$

When substituted into (5.10), this estimate for the velocity error gives

$$\begin{aligned}&\big(2\,\alpha\,\nu_1 - \lambda\, S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}(S_6^0|T|_{W^{1,3}(\Omega)} + \|T\|_{L^\infty(\Omega)})\big)|T - T_h^{i+1}|_{H^1(\Omega)} \\ &\quad\le C_3\left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,1}^{(D,2)}\right)^2 + h_K^2\|g - g_h\|^2_{L^2(K)}\right)\right)^{\frac{1}{2}} + C_4\left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,2}^{(D,2)}\right)^2 + \|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(K)^3}\right)\right)^{\frac{1}{2}} \\ &\quad\quad + C_5\left(\sum_{K\in\mathcal{T}_h}\left[h_K^2\|\mathrm{div}\mathbf{u}_h^{i+1}\|^2_{L^2(K)} + \sum_{e\in\partial K} h_e\|\phi^e_{h,1}\|^2_{L^2(e)}\right]\right)^{\frac{1}{2}} + C_6|T_h^{i+1} - T_h^i|_{H^1(\Omega)}.\end{aligned} \tag{5.15}$$

In view of (5.7), the temperature estimate in (5.8) follows from (5.15), and in turn, the velocity estimate follows by substituting (5.15) into (5.14).

Finally, we obtain the pressure error by testing (5.11) with $\mathbf{v} = \nabla(p - p_h^{i+1})$ and $\mathbf{v}_h = \mathbf{0}$,

$$|p - p_h^{i+1}|_{H^1(\Omega)} \le \lambda\, S_6^0|T - T_h^i|_{H^1(\Omega)}\|\mathbf{u}\|_{L^3(\Omega)^3} + \nu_2\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(\Omega)^3} + \left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,2}^{(D,2)}\right)^2 + \|\mathbf{f}-\mathbf{f}_h\|^2_{L^2(K)^3}\right)\right)^{\frac{1}{2}}, \tag{5.16}$$

and by substituting the temperature and velocity error bounds. □

5.2. Lower error bound for the second discretization

Let us start with the temperature errors.

Theorem 5.2. *We retain the settings and assumptions of Theorem 3.3. Then for each h, there exists an integer i_0 (depending on h) such that for all $i \geq i_0$ and for all $K \in \mathcal{T}_h$, the following inequality holds:*

$$\eta_{K,i,1}^{(D,2)} \leq C \left(\|\boldsymbol{u} - \boldsymbol{u}_h^{i+1}\|_{L^2(\omega_e)^3} + |T - T_h^{i+1}|_{H^1(\omega_e)} + \sum_{\tilde{K} \subset \omega_e} h_{\tilde{K}} \|g - g_h\|_{L^2(\tilde{K})} \right), \tag{5.17}$$

with a constant C independent of h and K. Moreover, without assumption, we have

$$\eta_{K,i}^{(L,2)} \leq |T - T_h^{i+1}|_{H^1(K)} + |T - T_h^i|_{H^1(K)}. \tag{5.18}$$

Proof. To derive an upper bound for the interior part of $\eta_{K,i,1}^{(D,2)}$, we test the error equation (5.2) with $S = S_K$ and $S_h = 0$, where in each element K, S_K is the localizing function

$$S_K = \left(\alpha \Delta T_h^{i+1} - \mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1} - \frac{1}{2} \operatorname{div} \mathbf{u}_h^{i+1} T_h^{i+1} + g_h \right) \psi_K,$$

extended by zero outside K. By arguing as in the proof of Lemma 4.5, we derive the analogue of (4.40)

$$\begin{aligned} \int_K \left(\alpha \Delta T_h^{i+1} - \mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1} - \frac{1}{2} \operatorname{div} \mathbf{u}_h^{i+1} T_h^{i+1} + g_h \right)^2 \psi_K \, \mathrm{d}\mathbf{x} &= \alpha \int_K \nabla \left(T - T_h^{i+1} \right) \cdot \nabla S_K \, \mathrm{d}\mathbf{x} \\ + \int_K \left(\left(\mathbf{u} - \mathbf{u}_h^{i+1} \right) \cdot \nabla T \right) S_K \, \mathrm{d}\mathbf{x} + \int_K \left(\mathbf{u}_h^{i+1} \cdot \nabla \left(T - T_h^{i+1} \right) \right) S_K \, \mathrm{d}\mathbf{x} \\ - \int_K (g - g_h) S_K \, \mathrm{d}\mathbf{x} - \frac{1}{2} \int_K \operatorname{div} \mathbf{u}_h^{i+1} T_h^{i+1} S_K \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{5.19}$$

By comparing with (4.40), we see that all terms except the last one have been bounded in the proof of Theorem 4.5. This term can be written as

$$\begin{aligned} &-\frac{1}{2} \int_K \operatorname{div}(\mathbf{u}_h^{i+1} - \mathbf{u}) T_h^{i+1} S_K \, \mathrm{d}\mathbf{x} \\ &\quad = -\frac{1}{2} \int_K \operatorname{div}(\mathbf{u}_h^{i+1} - \mathbf{u})(T_h^{i+1} - R_h(T)) S_K \, \mathrm{d}\mathbf{x} - \frac{1}{2} \int_K \operatorname{div}(\mathbf{u}_h^{i+1} - \mathbf{u}) R_h(T) S_K \, \mathrm{d}\mathbf{x} \\ &\quad = \frac{1}{2} \int_K \left((\mathbf{u}_h^{i+1} - \mathbf{u}) \cdot \nabla (T_h^{i+1} - R_h(T)) \right) S_K \, \mathrm{d}\mathbf{x} + \frac{1}{2} \int_K \left((\mathbf{u}_h^{i+1} - \mathbf{u}) \cdot \nabla S_K \right) (T_h^{i+1} - R_h(T)) \, \mathrm{d}\mathbf{x} \\ &\qquad + \frac{1}{2} \int_K (\mathbf{u}_h^{i+1} - \mathbf{u}) \cdot \left((\nabla R_h(T)) S_K + (\nabla S_K) R_h(T) \right) \mathrm{d}\mathbf{x}. \end{aligned}$$

Thus

$$\begin{aligned} \left| \frac{1}{2} \int_K \operatorname{div}(\mathbf{u}_h^{i+1} - \mathbf{u}) T_h^{i+1} S_K \, \mathrm{d}\mathbf{x} \right| \leq \frac{1}{2} \|\mathbf{u}_h^{i+1} - \mathbf{u}\|_{L^2(K)^3} \Big[&C_I^0(3) |K|^{-\frac{1}{6}} |K|^{-\frac{1}{3}} |T_h^{i+1} - R_h(T)|_{H^1(K)} \\ &+ C_I^1(2) h_K^{-1} |K|^{-\frac{1}{6}} \|T_h^{i+1} - R_h(T)\|_{L^6(K)} + |K|^{-\frac{1}{3}} |R_h(T)|_{W^{1,3}(K)} \\ &+ C_I^1(2) h_K^{-1} \|R_h(T)\|_{L^\infty(K)} \Big] \|S_K\|_{L^2(K)}. \end{aligned}$$

By arguing as in the proof of Theorem 4.5, we deduce the interior bound in (5.17). The upper bound for the surface part of $\eta_{K,i,1}^{(D,2)}$ is treated in the same way. Finally, formula (5.18) is obvious.

□

To establish the remaining lower bounds for the second variational formulation, since there is no need to take the curl of the velocity equation, it suffices to approximate ν by a piecewise constant function, say $\nu_{0,h}$, defined for every $f \in L^1(K)$ by

$$\nu_{0,h}(f)|_K = \frac{1}{|K|} \int_K \nu(f(\mathbf{y}))\, \mathrm{d}\mathbf{y}. \tag{5.20}$$

Clearly, $\nu_{0,h}$ verifies the following properties:

– In view of (1.2), for any function $f \in L^1(K)$,

$$\nu_1 \le \nu_{0,h}(f)|_K \le \nu_2. \tag{5.21}$$

– In view of (1.1), $\nu_{0,h}$ is Lipschitz-continuous with Lipschitz constant λ, *i.e.*,

$$\forall f_1, f_2 \in L^p(K), \quad \|\nu_{0,h}(f_1) - \nu_{0,h}(f_2)\|_{L^p(K)} \le \lambda \|f_1 - f_2\|_{L^p(K)}. \tag{5.22}$$

Moreover, since ν belongs to $W^{1,\infty}(\mathbb{R})$, the analogue of the first part of (4.34) holds for all numbers $p \ge 2$ and functions f in $W^{1,p}(K)$,

$$\|\nu_{0,h}(f) - \nu(f)\|_{L^p(K)} \le C\lambda h_K |f|_{W^{1,p}(K)}, \tag{5.23}$$

with a constant C that depends on p but is independent of h and K. The term $\|\nu_{0,h}(f) - \nu(f)\|_{L^6(K)}$ will be treated as an error.

Theorem 5.3. *Let $d = 3$ and ν satisfy (1.1) and (1.2). We suppose that the velocity solution $\boldsymbol{u}$ of problem (V_2) belongs to $L^3(\Omega)^3$. Then*

$$\begin{aligned}\eta^{(D,2)}_{K,i,2} \le C\Big(&\|T - T_h^i\|_{L^6(K)} + \|\boldsymbol{u} - \boldsymbol{u}_h^{i+1}\|_{L^2(K)^3} + |p - p_h^{i+1}|_{H^1(K)} + \|\boldsymbol{f} - \boldsymbol{f}_h\|_{L^2(K)^3} \\ &+ \|\nu(T) - \nu_{0,h}(T)\|_{L^6(K)}\Big).\end{aligned} \tag{5.24}$$

Proof. To bound the interior part of $\eta^{(D,2)}_{K,i,2}$, we test (5.3) with $\mathbf{v}_h = \mathbf{0}$ and $\mathbf{v} = \mathbf{v}_K$, where each component of the localizing function $\mathbf{v}_K$ is defined in each element K by

$$(\mathbf{v}_K)_j = \big(-\nabla p_h^{i+1} - \nu_{0,h}(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h\big)_j \psi_K,$$

extended by 0 outside K. As in the proof of Theorem 4.8 we obtain the analogue of (4.54)

$$\begin{aligned}\int_K |-\nabla p_h^{i+1} - \nu_{0,h}(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h|^2 \psi_K \, \mathrm{d}\mathbf{x} \;=\; &\int_K (\nu(T) - \nu(T_h^i))\mathbf{u}\cdot\mathbf{v}_K \, \mathrm{d}\mathbf{x} - \int_K \nu(T_h^i)(\mathbf{u}_h^{i+1} - \mathbf{u})\cdot \mathbf{v}_K \, \mathrm{d}\mathbf{x} \\ &+ \int_K (\nu(T_h^i) - \nu_{0,h}(T_h^i))\mathbf{u}_h^{i+1}\cdot\mathbf{v}_K \, \mathrm{d}\mathbf{x} \\ &+ \int_K \nabla(p - p_h^{i+1})\cdot\mathbf{v}_K \, \mathrm{d}\mathbf{x} - \int_K (\mathbf{f} - \mathbf{f}_h)\cdot\mathbf{v}_K \, \mathrm{d}\mathbf{x}.\end{aligned} \tag{5.25}$$

Owing to the Lipschitz continuity of ν and (5.21), the right-hand side of (5.25) has the straightforward bound

$$\begin{aligned}\int_K |-\nabla p_h^{i+1} - \nu_{0,h}(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h|^2 \psi_K \, \mathrm{d}\mathbf{x} \;\le\; &\Big[\lambda \|T - T_h^i\|_{L^6(K)} \|\mathbf{u}\|_{L^3(K)^3} + \nu_2 \|\mathbf{u} - \mathbf{u}_h^{i+1}\|_{L^2(K)^3} \\ &+ \|\mathbf{u}_h^{i+1}(\nu(T_h^i) - \nu_{0,h}(T_h^i))\|_{L^2(K)^3} + |p - p_h^{i+1}|_{H^1(K)} + \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^3}\Big] \|\mathbf{v}_K\|_{L^2(K)^3}.\end{aligned} \tag{5.26}$$

The bound for the third term in the right-hand side is derived as in Lemma 4.6, but is made simpler by the simpler structure of $\nu_{0,h}$. We have

$$\begin{aligned}\|\mathbf{u}_h^{i+1}(\nu(T_h^i)-\nu_{0,h}(T_h^i))\|_{L^2(K)^3} &\le \|(\mathbf{u}_h^{i+1}-\mathbf{u})(\nu(T_h^i)-\nu_{0,h}(T_h^i))\|_{L^2(K)^3}+\|\mathbf{u}\|_{L^3(K)^3}\big(\|\nu(T_h^i)-\nu(T)\|_{L^6(K)}\\ &\quad+\|\nu(T)-\nu_{0,h}(T)\|_{L^6(K)}+\|\nu_{0,h}(T)-\nu_{0,h}(T_h^i)\|_{L^6(K)}\big)\\ &\le 2\nu_2\|\mathbf{u}_h^{i+1}-\mathbf{u}\|_{L^2(K)^3}+\|\mathbf{u}\|_{L^3(K)^3}\big(2\lambda\|T-T_h^i\|_{L^6(K)}+\|\nu(T)\\ &\quad-\nu_{0,h}(T)\|_{L^6(K)}\big).\end{aligned} \tag{5.27}$$

By substituting (5.27) into (5.26), we infer

$$\begin{aligned}\int_K |-\nabla p_h^{i+1}-\nu_{0,h}(T_h^i)\mathbf{u}_h^{i+1}+\mathbf{f}_h|^2\psi_K\,\mathrm{d}\mathbf{x} &\le \Big[3\lambda\|T-T_h^i\|_{L^6(K)}\|\mathbf{u}\|_{L^3(K)^3}+3\nu_2\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K)^3}\\ &+|p-p_h^{i+1}|_{H^1(K)}+\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^3}+\|\mathbf{u}\|_{L^3(K)^3}\|\nu(T)-\nu_{0,h}(T)\|_{L^6(K)}\Big]\|\mathbf{v}_K\|_{L^2(K)^3},\end{aligned}$$

and the equivalence of norms yields (to simplify, we do not specify the constants)

$$\begin{aligned}\|-\nabla p_h^{i+1}-\nu_{0,h}(T_h^i)\mathbf{u}_h^{i+1}+\mathbf{f}_h\|_{L^2(K)^3} &\le C\Big(\|T-T_h^i\|_{L^6(K)}\|\mathbf{u}\|_{L^3(K)^3}+\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K)^3}\\ &+|p-p_h^{i+1}|_{H^1(K)}+\|\mathbf{f}-\mathbf{f}_h\|_{L^2(K)^3}+\|\mathbf{u}\|_{L^3(K)^3}\|\nu(T)-\nu_{0,h}(T)\|_{L^6(K)}\Big).\end{aligned} \tag{5.28}$$

In view of (5.27), this yields a similar bound for the first part of the indicator.

Regarding the divergence part of $\eta_{K,i,2}^{(D,2)}$, (5.4) is tested with $q_h=0$ and $q=q_K$, where

$$q_K=(\operatorname{div}\mathbf{u}_h^{i+1})\psi_K.$$

Then

$$\int_K((\operatorname{div}\mathbf{u}_h^{i+1})^2\psi_K\,\mathrm{d}\mathbf{x}=\int_K\nabla q_K\cdot(\mathbf{u}-\mathbf{u}_h^{i+1})\,\mathrm{d}\mathbf{x}\le C_I^1(2)h_K^{-1}\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K)^3}\|q_K\|_{L^2(K)},$$

so that

$$h_K\int_K((\operatorname{div}\mathbf{u}_h^{i+1})^2\psi_K\,\mathrm{d}\mathbf{x}\le C_I^1(2)\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K)^3}\|q_K\|_{L^2(K)}. \tag{5.29}$$

The bound for the volume part of $\eta_{K,i,2}^{(D,2)}$ follows from (5.28) and (5.29).

For the surface parts, (5.4) is tested with $q_h=0$ and $q=q_e$, where

$$q_e=\mathcal{L}(\phi_{h,1}^e\psi_e),\quad \text{in } K\cup K' \text{ or in } K,$$

according that e is an interior face or a boundary face (see (5.1)). Then

$$\begin{aligned}\int_e(\phi_{h,1}^e)^2\psi_e &= \int_{K\cup K'}q_e\operatorname{div}\mathbf{u}_h^{i+1}\,\mathrm{d}\mathbf{x}-\int_{K\cup K'}\nabla q_e\cdot(\mathbf{u}-\mathbf{u}_h^{i+1})\,\mathrm{d}\mathbf{x}\\ &\le \big(\|\operatorname{div}\mathbf{u}_h^{i+1}\|_{L^2(K\cup K')}+C_I^1(2)h_K^{-1}\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K\cup K')}\big)\|q_e\|_{L^2(K\cup K')}.\end{aligned}$$

Hence

$$h_e^{\frac12}\|\phi_{h,1}^e\|_{L^2(e)}\le C\big(h_K\|\operatorname{div}\mathbf{u}_h^{i+1}\|_{L^2(K\cup K')}+\|\mathbf{u}-\mathbf{u}_h^{i+1}\|_{L^2(K\cup K')}\big),$$

thus implying (5.24). $\square$

Remark 5.4. The approximation error of ν is measured here in the L^6 norm because $\mathbf{u}$ is assumed to be in $L^3(\Omega)^3$. If instead, we were to take $\mathbf{u}$ in $L^6(\Omega)^3$, then the approximation error of ν would be measured in L^3.

6. Numerical results

The theory developed here is validated by numerical simulations using Freefem++ (see [20]). The domain Ω is the unit square $\Omega =]0,3[^2$ and all computations start on a uniform initial triangular mesh obtained by dividing Ω into N^2 equal squares, each one subdivided into 2 triangles, so that the initial triangulation consists of $2N^2$ triangles.

The theory is tested by applying the numerical schemes $(V_{h,i,1})$ and $(V_{h,i,2})$ to the exact solution $(\mathbf{u}, p, T) = (\mathbf{curl}\,\psi, p, T)$ where ψ, p, and T are given by

$$\psi(x,y) = e^{-\gamma((x-1)^2+(y-1)^2)}, \tag{6.1}$$

$$p(x,y) = \cos\left(\frac{\pi}{3}x\right)\cos\left(\frac{\pi}{3}y\right), \tag{6.2}$$

and

$$T(x,y) = x^2(x-3)^2y^2(y-3)^2e^{-\gamma((x-1)^2+(y-1)^2)}, \tag{6.3}$$

with the choice $\alpha = 10$, $\gamma = 50$, $N = 30$, and different functions ν:

$$\begin{aligned} &\nu_1(T) = T + 1,\\ &\nu_2(T) = e^{-T} + 1/10,\\ &\nu_3(T) = \sin(T) + 2. \end{aligned} \tag{6.4}$$

For $D = 1, 2$, it is convenient to compute the following expression $\eta_i^{(D)}$ for the indicators, equivalent to the l^2 norm,

$$\eta_i^{(D)} = \left(\sum_{K\in\mathcal{T}_h}\left(\left(\eta_{K,i,1}^{(D)}\right)^2 + \left(\eta_{K,i,2}^{(D)}\right)^2\right)^{\frac{1}{2}}\right),$$

where,

– for the first variational formulation $(V_{h,i,1})$:

$$\left(\eta_{K,i,1}^{(D)}\right)^2 = h_K^2\|\alpha\Delta T_h^{i+1} - \mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1} + g_h\|^2_{L^2(K)} + \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|[\alpha\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|^2_{L^2(e)},$$

and

$$\begin{aligned}\left(\eta_{K,i,2}^{(D)}\right)^2 = {}& h_K^2\|-\nabla p_h^{i+1} - \nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h\|^2_{L^2(K)^2} + h_K^2\|\mathbf{curl}(-\nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)\|^2_{L^2(K)^2}\\ &+ \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|[p_h^{i+1}\mathbf{n}]_e\|^2_{L^2(e)^2} + \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|[(-\nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h)\times\mathbf{n}_e]_e\|^2_{L^2(e)^2};\end{aligned}$$

– for the second variational formulation $(V_{h,i,2})$:

$$\left(\eta_{K,i,1}^{(D)}\right)^2 = h_K^2\|\alpha\Delta T_h^{i+1} - \mathbf{u}_h^{i+1}\cdot\nabla T_h^{i+1} - \frac{1}{2}\operatorname{div}\mathbf{u}_h^{i+1}T_h^{i+1} + g_h\|^2_{L^2(K)} + \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|[\alpha\nabla T_h^{i+1}\cdot\mathbf{n}]_e\|^2_{L^2(e)},$$

and

$$\left(\eta_{K,i,2}^{(D)}\right)^2 = \|-\nabla p_h^{i+1} - \nu(T_h^i)\mathbf{u}_h^{i+1} + \mathbf{f}_h\|^2_{L^2(K)^2} + h_K^2\|\operatorname{div}\mathbf{u}_h^{i+1}\|^2_{L^2(K)} + \sum_{e\in\partial K\cap\Gamma_h^i} h_e\|\phi_{h,1}^e\|^2_{L^2(e)}.$$

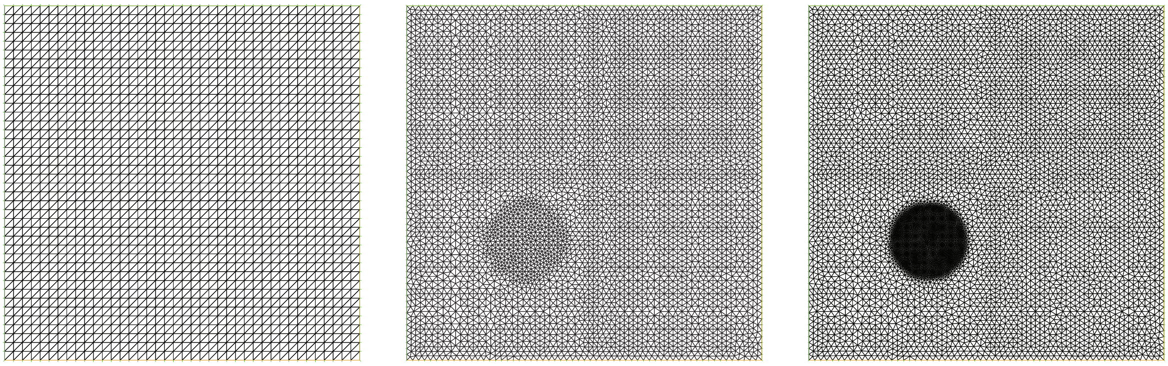


FIGURE 1. Evolution of the mesh for the first discrete scheme with $\nu_1(T) = T + 1$.

Likewise, we compute the algorithmic error indicator $\eta_i^{(L)}$ by,

$$\eta_i^{(L)} = \left(\sum_{K \in \mathcal{T}_h} \left(\eta_{K,i}^{(L)} \right)^2 \right)^{\frac{1}{2}},$$

where

$$\eta_{K,i}^{(L)} = |T_h^{i+1} - T_h^i|_{H^1(K)}.$$

These indicators are used for mesh adaptation by the adapted mesh algorithm introduced in [6]; the mesh is adapted so as to satisfy the following criteria

$$\eta_i^{(L)} \leq \varepsilon \quad \text{and} \quad \eta_i^{(D)} \leq \upsilon, \tag{6.5}$$

with $\varepsilon \leq 10^{-7}$ and $\upsilon \leq 10^{-8}$. For the adaptive mesh (refinement and coarsening), we use routines in FreeFem++.

In Figure 1, we present the evolution of the mesh during the iterations for the first discrete scheme $(V_{h,i,1})$ with $\nu_1(T) = T + 1$. We notice that the mesh is concentrated in the region where the solution needs to be well described. Similar figures are obtained for the second numerical scheme $(V_{h,i,2})$ and for different values of the viscosity (ν_2 and ν_3).

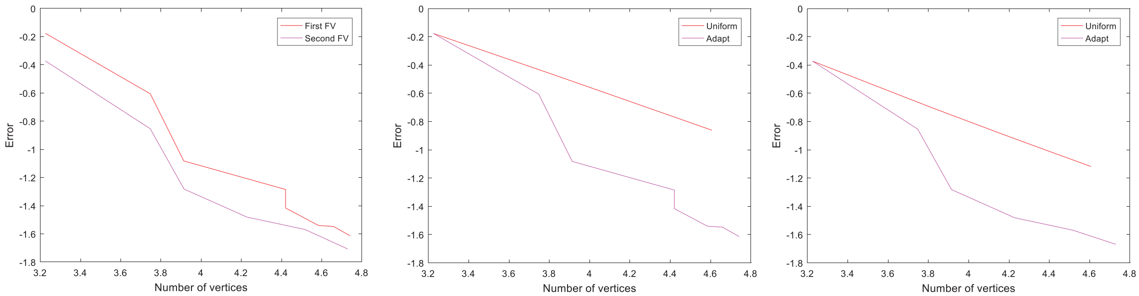
Next, we plot and study the error curves between the exact and numerical solutions corresponding to the first and second schemes for different values of the viscosity ν.

Figure 2A (respectively 3A and 4A) plots the comparison of the global error curves versus the number of vertices in logarithmic scales for the first discrete scheme and for ν_1 (respectively ν_2 and ν_3); global in the sense that they depict the sum of the velocity, pressure and temperature errors. For each case of ν, the left figure shows the comparison of the adapt mesh method between the two proposed numerical schemes. We remark that the errors corresponding to $(V_{h,i,2})$ are smaller than those corresponding to $(V_{h,i,1})$; this is expected because the finite elements used for the second numerical scheme $(V_{h,i,2})$ contain much more of degrees of freedom than $(V_{h,i,1})$.

In Figures 2B and 2C (respectively 3B and 3C, 4B and 4C), we present comparisons of the global error versus the number of vertices in logarithmic scale for the adapt and uniform methods, for the three cases of the viscosity and the two discrete schemes. We notice that the errors of the adaptive mesh method are much smaller than those obtained with the uniform method, hence the efficiency of this method.

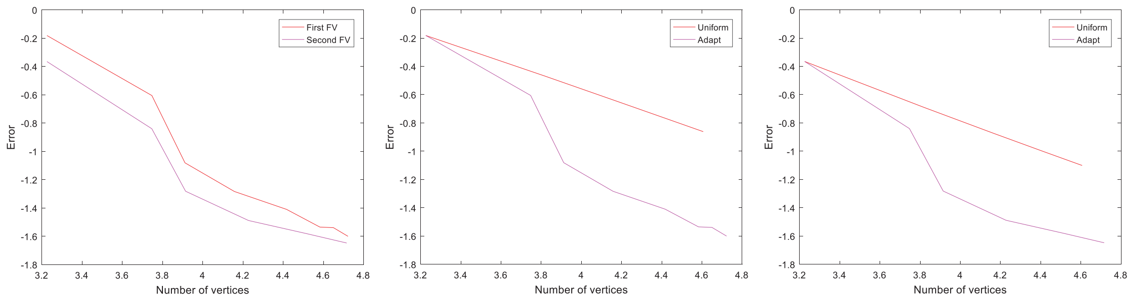
In Table 1, we present the effectivity index defined as

$$\mathrm{EI} = \left(\frac{(\eta_i^{(L)})^2 + (\eta_i^{(D)})^2}{\|\mathbf{u}_h^i - \mathbf{u}\|^2_{L^2(\Omega)^2} + \|p_h^i - p\|^2_{L^2(\Omega)} + |T_h^i - T|^2_{H^1(\Omega)}} \right)^{1/2}$$



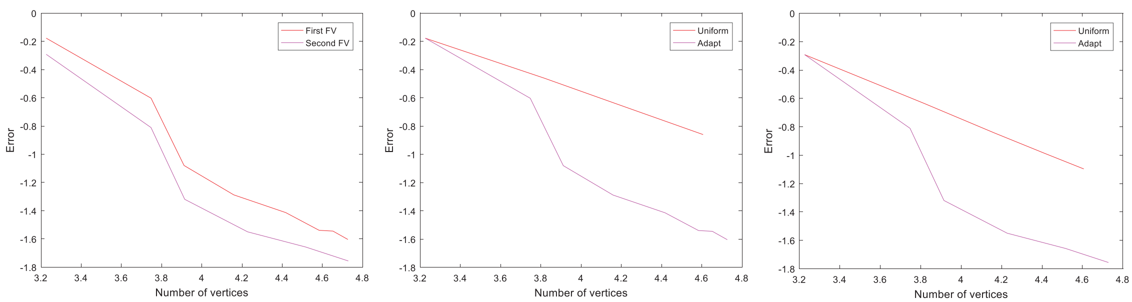
(A) Comparison of the error curve. (B) First discrete scheme. (C) Second discrete scheme.

FIGURE 2. Comparison of the errors for $\nu_1(T) = T + 1$.



(A) Comparison of the error curve. (B) First discrete scheme. (C) Second discrete scheme.

FIGURE 3. Comparison of the errors for $\nu_2(T) = e^{-T} + 1/10$.



(A) Comparison of the error curve. (B) First discrete scheme. (C) Second discrete scheme.

FIGURE 4. Comparison of the errors for $\nu_3(T) = \sin(T) + 2$.

with respect to the number of vertices during the iterations (refinement levels) for the first discrete scheme and for ν_1. We remark that it decreases from 46.09 (refinement level 1) to 27.96 (refinement level 6).

Furthermore, we compare the CPU times between the uniform and the adaptive methods for the first and second discrete schemes $(V_{h,i,1})$ and $(V_{h,i,2})$. Table 2 shows results corresponding to the viscosity ν_1 for $(V_{h,i,1})$. For example, a uniform mesh of 40401 vertices produces an error of 0.137 with a CPU time of 418.5 s while an adaptive mesh of 14280 vertices gives an error of 0.051 with a CPU time of 48.7 s.

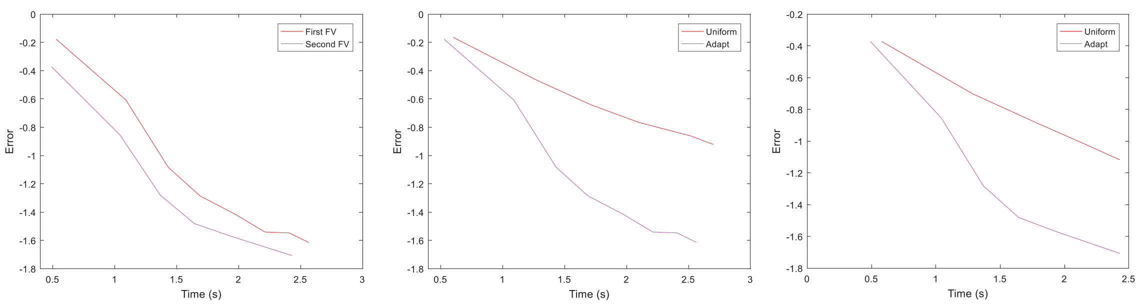
To complete this comparison, we plot in Figure 5A (respectively 6A and 7A) the comparison of the CPU time of computation between the adapted first and second schemes for ν_1 (respectively ν_2 and ν_3). The second scheme

TABLE 1. EI with respect to the iterations for the first discrete scheme and for ν_1.

Refinement level	$i = 1$ 1681 vertices	$i = 2$ 5587 vertices	$i = 3$ 8198 vertices	$i = 4$ 14 280 vertices	$i = 5$ 26 314 vertices	$i = 6$ 38 381 vertices
Effectivity index	46.09	45.52	44.07	28.46	28.23	27.96

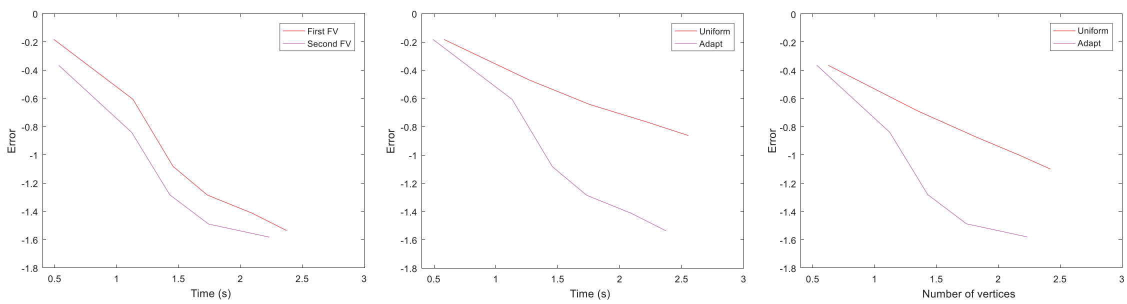
TABLE 2. Comparison between the error and CPU time between the uniform mesh and adaptive mesh for the first discrete scheme with $\nu_1(T) = T + 1$.

Type					
Uniform mesh			Adaptive mesh		
Nbr vertices	Error	Time (s)	Nbr vertices	Error	Time (s)
6561	0.341	18.85	5587	0.247	12.3
14 641	0.228	81.7	14 280	0.051	48.7
25 921	0.171	204.1	26 314	0.038	94.4
40 401	0.137	418.5	38 381	0.028	162.7



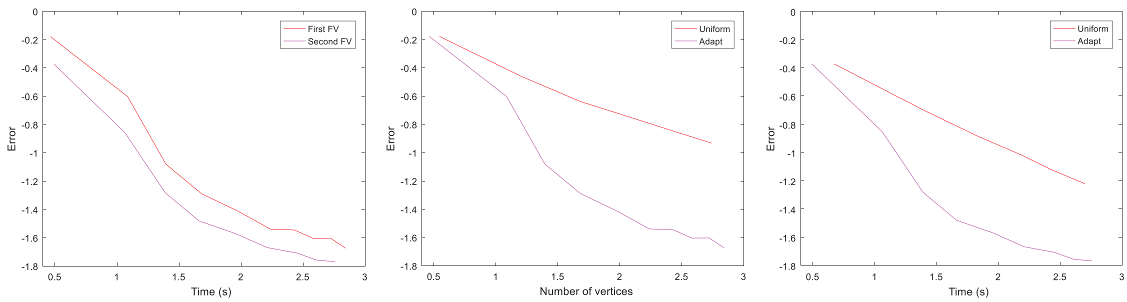
(A) Comparison of the error curve. (B) First discrete scheme. (C) Second discrete scheme.

FIGURE 5. Comparison of the CPU time for $\nu_1(T) = T + 1$.



(A) Comparison of the error curve. (B) First discrete scheme. (C) Second discrete scheme.

FIGURE 6. Comparison of the CPU time for $\nu_2(T) = e^{-T} + 1/10$.



(A) Comparison of the error curve. (B) First discrete scheme. (C) Second discrete scheme.

FIGURE 7. Comparison of the CPU time for $\nu_3(T) = \sin(T) + 2$.

is more precise than the first one for the same reason indicated above. The other figures show comparisons of the CPU time of computation versus the global error, for the uniform and adaptive methods and for the three examples of viscosity. We deduce that for a given global error, the uniform method is much more expensive than the adapted method for the two numerical schemes and for the three choices of viscosity.

7. Conclusion

In this article, we discretize a steady Darcy system coupled with a heat equation. We use two variational formulations and for each one introduce error indicators and establish optimal *a posteriori* error estimates. We perform several numerical simulations where these indicators are used for mesh adaptation, confirming the efficiency of these adaptive methods.

Appendix A.

This section is devoted to the proof of Lemma 4.1. Let $\mathbf{v}$ be given in V. The idea is first to extend $\mathbf{v}$ by zero to $\mathbb{R}^3$ and construct η by Fourier transforms in the whole space. Since by construction $\mathbf{curl}\,\eta$ vanishes in the exterior of Ω and since Ω is simply-connected, η is the gradient of a smoother function in this exterior. The fact that it is a gradient will permit to suitably correct η so as to satisfy the desired boundary condition.

Thus, let $\tilde{\mathbf{v}}$ denote the extended function. As $\mathbf{v} \cdot \mathbf{n}$ vanishes on $\partial\Omega$, $\tilde{\mathbf{v}}$ belongs to $H(\mathrm{div}, \mathbb{R}^3)$ and $\mathrm{div}\,\mathbf{v} = 0$ in $\mathbb{R}^3$. Since the support of $\mathbf{v}$ is bounded, it can be shown, cf. [19], that this construction produces a unique function η in $H^1(\mathcal{O})^3$ on any bounded subset $\mathcal{O}$ of $\mathbb{R}^3$, and

$$\tilde{\mathbf{v}} = \mathbf{curl}\,\eta.$$

The function η has the desired regularity and it remains to correct it in order to satisfy the boundary condition. Since this must not affect the curl of the resulting function, the correcting function must be a gradient. To this end, let us pick a large enough bounded open ball B containing $\bar{\Omega}$ such that the distance between the boundary of $\bar{\Omega}$ and B is strictly positive. Since $\mathrm{div}\,\tilde{\mathbf{v}} = 0$ in $\mathbb{R}^3$ and $\tilde{\mathbf{v}} = \mathbf{0}$ in $B \setminus \Omega$, we have the following bound:

$$\|\eta\|_{H^1(B)^3} \leq C\|\tilde{\mathbf{v}}\|_{H(\mathrm{div},\mathbf{R}^3)} = C\|\mathbf{v}\|_{L^2(\Omega)^3}. \tag{A.1}$$

As $\tilde{\mathbf{v}} = \mathbf{0}$ in $B \setminus \Omega$, we have $\mathbf{curl}\,\eta = \mathbf{0}$ in $B \setminus \Omega$ and as $B \setminus \Omega$ is also simply-connected, there exists a function q in $B \setminus \Omega$ such that

$$\eta = \nabla q \quad \text{in } B \setminus \Omega, \tag{A.2}$$

and $q \in H^2(B \setminus \Omega)$ since $\eta \in H^1(B \setminus \Omega)^3$. The function q is determined by (A.2) up to a constant. Let us choose this constant so that

$$\int_{B\setminus\Omega} q \, \mathrm{d}\mathbf{x} = 0.$$

With this choice

$$\|q\|_{L^2(B\setminus\Omega)} \le C\|\nabla q\|_{L^2(B\setminus\Omega)^3}, \tag{A.3}$$

and by (A.1)–(A.3),

$$\|q\|_{H^2(B\setminus\Omega)} \le C\|\eta\|_{H^1(B\setminus\Omega)^3} \le C\|\mathbf{v}\|_{L^2(\Omega)^3}. \tag{A.4}$$

The function q can be extended continuously in Ω to a function $\tilde{q} \in H^2(B)$ such that

$$\|\tilde{q}\|_{H^2(B)} \le C\|q\|_{H^2(B\setminus\Omega)} \le C\|\mathbf{v}\|_{L^2(\Omega)^3}. \tag{A.5}$$

Finally, the function $\nabla\tilde{q}$ is the required correction of η. Indeed, we define

$$\tilde{\eta} = \eta - \nabla\tilde{q}, \quad \text{in } B. \tag{A.6}$$

As $\tilde{q}$ belongs to $H^2(B)$, the trace of $\nabla\tilde{q}$ is continuous across $\partial\Omega$. Therefore by (A.2), the trace of $\nabla\tilde{q}$ on $\partial\Omega$ coincides with that of η, *i.e.*,

$$\eta|_{\partial\Omega} = \nabla\tilde{q}|_{\partial\Omega}.$$

Hence

$$\tilde{\eta}|_{\partial\Omega} = \mathbf{0}.$$

Summing up, $\tilde{\eta}$ belongs to $H^1_0(\Omega)^3$,

$$\mathbf{curl}\,\tilde{\eta} = \mathbf{curl}\,\eta = \mathbf{v} \quad \text{in } \Omega,$$

and

$$\|\tilde{\eta}\|_{H^1(\Omega)^3} \le C\|\mathbf{v}\|_{L^2(\Omega)^3}.$$

References

[1] R.A. Adams and J.J. Fournier, Sobolev spaces. In: Vol. 140 of Pure and Applied Mathematics. Academic Press, New York, London (2003).

[2] A. Alonso, Error estimators for a mixed method. *Numer. Math.* **74** (1996) 385–395.

[3] I. Babuška and W.C. Rheinboldt, Error estimates for adaptive finite element computations. *SIAM J. Numer. Anal.* **15** (1978) 736–754.

[4] C. Bernardi and V. Girault, A local regularisation operation for triangular and quadrilateral finite elements. *SIAM J. Numer. Anal.* **35** (1998) 1893–1916.

[5] C. Bernardi, F. Hecht and R. Verfürth, A finite element discretization of the three-dimensional Navier–Stokes equations with mixed boundary conditions. *ESAIM: M2AN* **43** (2009) 185–1201.

[6] C. Bernardi, J. Dakroub, G. Mansour and T. Sayah, A posteriori analysis of iterative algorithms for Navier–Stokes problem. *ESAIM: M2AN* **50** (2016) 1035–1055.

[7] C. Bernardi, S. Dib, V. Girault, F. Hecht, F. Murat and T. Sayah, Finite element methods for Darcy's problem coupled with the heat equation. *Numer. Math.* **139** (2018) 315–348.

[8] D. Braess and R. Verfürth, A posteriori error estimators for the Raviart–Thomas element. *SIAM J. Numer. Anal.* **33** (1996) 2431–2444.

[9] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer-Verlag, New York, NY **15** (1991).

[10] C. Carstensen, A posteriori error estimate for the mixed finite element method. *Math. Comput.* **66** (1997) 465–476.

[11] W. Chen and Y. Wang, A posteriori estimate for the $h(div, \omega)$ conforming mixed finite element for the coupled Darcy–Stokes system. *J. Comput. Appl. Math.* **255** (2014) 502–516.

[12] P.G. Ciarlet, Basic error estimates for elliptic problems. In: Vol. II of Handbook of Numerical Analysis. North-Holland, Amsterdam (1991) 17–351.

[13] P. Clément, Approximation by finite element functions using local regularization. *RAIRO Anal. Numér.* **9** (1975) 77–84.

[14] E. Colmenares, G.N. Gatica and R.Y. Oyarzua, A posteriori error analysis of an augmented mixed-primal formulation for the stationary boussinesq model. *Calcolo* **54** (2017) 1055–1095.

[15] E. Colmenares, G.N. Gatica and R.Y. Oyarzua, A posteriori error analysis of an augmented fully-mixed formulation for the stationary boussinesq model. *Comput. Math. App.* **77** (2019) 693–714.

[16] A. El Akkad, A. El Khalfi and N. Guessous, An a posteriori estimate for mixed finite element approximations of the Navier–Stokes equations. *J. Korean Math. Soc.* **48** (2011) 529–550.

[17] V. Ervin, W. Layton and J. Maubach, A posteriori error estimators for a two-level finite element method for the Navier–Stokes equations. I.C.M.A. Tech. Report, Univ. of Pittsburgh (1995).

[18] G.N. Gatica, R. Ruiz-Baier and G. Tierra, A mixed finite element method for Darcy's equations with pressure dependent porosity. *Math. Comput.* **85** (2016) 1–33.

[19] V. Girault and P.-A. Raviart, Finite element methods for Navier–Stokes equations. In: Vol. 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin (1986). Theory and algorithms.

[20] F. Hecht, New development in FreeFem++. *J. Numer. Math.* **20** (2012) 251–266.

[21] H. Jin and S. Prudhomme, A posteriori error estimation in finite element analysis. *Comput. Methods Appl. Mech. Eng.* **159** (1998) 19–48.

[22] V. John, Residual a posteriori error estimates for two-level finite element methods for the Navier–Stokes equations. *Appl. Numer. Math.* **37** (2001) 503–518.

[23] C. Lovadina and R. Stenberg, Energy norm a posteriori error estimates for mixed finite element methods. *Math. Comput.* **75** (2006) 1659–1674.

[24] J. Nečas, Les méthodes directes en théorie des équations elliptiques. Masson, Paris (1967).

[25] P.-A. Raviart and J.-M. Thomas, A mixed finite element method for second order elliptic problems (rome, 1975). In: Mathematical Aspects of Finite Element Methods. Springer, Berlin (1977) 292–315.

[26] J.E. Roberts and J.-M. Thomas, Mixed and hybrid methods. In: Vol. II of Handbook of Numerical Analysis, Finite Element Methods (Part I). North-Holland, Amsterdam (1991) 523–637.

[27] L.R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comput.* **54** (1990) 483–493.

[28] R. Verfürth, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Mathematics. Wiley and Teubner, New York, NY (1996).