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CONVERGENCE OF A FINITE VOLUME SCHEME FOR THE COMPRESSIBLE NAVIER–STOKES SYSTEM

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Abstract. We study convergence of a finite volume scheme for the compressible (barotropic) Navier–Stokes system. First we prove the energy stability and consistency of the scheme and show that the numerical solutions generate a dissipative measure-valued solution of the system. Then by the dissipative measure-valued-strong uniqueness principle, we conclude the convergence of the numerical solution to the strong solution as long as the latter exists. Numerical experiments for standard benchmark tests support our theoretical results.

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1. Introduction

We study the flow of a viscous fluid governed by the compressible Navier–Stokes system:

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \\ &\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mu \Delta_x \mathbf{u} + (\mu + \lambda)\nabla_x \operatorname{div}_x \mathbf{u} \end{aligned} \tag{1.1}$$

in the time–space domain $(0, T) \times \Omega$. Here $\varrho = \varrho(t, x)$, and $\mathbf{u} = \mathbf{u}(t, x)$ are the fluid density and velocity, constants $\mu > 0$, $\lambda \geq -\mu$ are the viscosity coefficients. The pressure p is assumed to satisfy the *isentropic* state equation

$$p(\varrho) = a\varrho^\gamma, \quad a > 0, \quad \gamma > 1. \tag{1.2}$$

For the sake of simplicity we impose the periodic boundary conditions, meaning that the domain Ω can be identified with the flat torus $\Omega = ([0, 1]|_{0,1})^d$, $d = 1, 2, 3$. The problem is (formally) closed by prescribing the initial conditions

$$\varrho(0) = \varrho_0 \in L^\gamma(\Omega),\ \varrho_0 > 0,\ \mathbf{u}(0) = \mathbf{u}_0 \in L^2(\Omega; R^d). \tag{1.3}$$

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In the literature we can find a variety of numerical schemes for viscous compressible flows, such as the Marker–And–Cell schemes [18–21, 23], the finite element schemes [1, 26, 35], the finite volume schemes [7, 10, 22, 28] and the discontinuous Galerkin schemes [6, 9, 24]. In this paper we want to concentrate on the finite volume methods that are standardly used for physical or engineering applications, see, *e.g.* [17, 27, 28, 31, 34, 35] and the references therein. In the cell-centered finite volumes the unknown quantities (numerical solution) are located at the centers of mass of the mesh cells (finite volume cells). This is very typical for the compressible inviscid flows governed by the Euler equations. By means of the Gauss theorem the inviscid fluxes at cell interfaces are approximated by suitable numerical flux functions. The latter are based on the flux-vector splitting or upwinding strategy as we will explain below.

For the compressible Navier–Stokes equations in addition the viscous fluxes need to be approximated, which means that the gradients of the numerical solution are to be represented at the cell interfaces. Having piecewise discontinuous approximate functions this requires an additional reconstruction step, which is usually realized by introducing the so-called dual grid around the cell interfaces of a primary grid. We refer a reader to Kozel *et al.* [17, 27, 31], where the viscous terms are approximated by the second order central differences using a dual finite volume grid of octahedrons constructed over each face of the primary hexagonal finite volume grid. In Meister and Sonar [28] and Feistauer *et al.* [8] the barycentric subdivision is used to define dual finite volumes, in [33, 34] a special reconstruction satisfying maximum principle is developed for the viscous fluxes. A nice overview of various finite volume methods with the gradient approximations at cell interfaces can be found in [3].

Although these methods are frequently used in practical simulations, their convergence for multi-dimensional viscous compressible flows remains open in general. For a mixed finite element-discontinuous Galerkin method, the convergence to a weak solution has been shown by Karper in his pioneering work [26] under the assumption that the adiabatic coefficient $\gamma > 3$. Note that the convergence in this case holds up to a subsequence as the weak solutions are not known to be unique. Moreover, any generalization of the proof of Karper [26] for other numerical schemes, in particular for cell-centered finite volume methods discussed in the present paper, is highly non-trivial. In [25] Jovanović obtained the error estimate for the isentropic Navier–Stokes equations for entropy dissipative finite volume–finite difference methods under some rather restrictive assumptions on the global smooth solution. In [13] Feireisl and Lukáčová proposed a new way of the convergence proof *via* the dissipative measure-valued (DMV) solutions. They show the convergence of the scheme proposed in [26] for the isentropic Navier–Stokes equations for *physically relevant range of the adiabatic coefficient* $\gamma \in (1, 2)$.

We should also mention the recent results on the analysis of the Marker-And-Cell schemes, *cf.* [19, 20, 23], which are based on the staggered grid approximation of the velocity and the primary grid approximation of the density. In Gallouët *et al.* [19] the convergence to a weak solution of stationary Navier–Stokes equations for $\gamma > 3$ has been proved. In Hošek and She [23] the consistency and the energy stability of the Marker–And–Cell scheme has been shown for instationary Navier–Stokes equations. The error estimates for $\gamma > 3/2$ have been presented in [20] using the relative entropy method.

The main aim of this paper is to demonstrate that the strategy proposed in [13] can be adapted to investigate the convergence of finite volume methods. More precisely, we consider the first order cell-centered finite volume method, where the inviscid fluxes are approximated by the upwinding and the viscous fluxes by the central differences. See also our recent works [14, 15] where analogous finite volume schemes have been applied to show the convergence for the complete Euler system. We adapt this approach to a time-implicit finite volume method for the barotropic Navier–Stokes system and show the stability as well as the convergence of numerical solutions to the (unique) strong solution of (1.1) provided the latter exists. The adiabatic coefficient stays in a physically reasonable range $\gamma \in (1, 2)$. To the best of our knowledge, there is no convergence proof of a finite volume method for the multi-dimensional Navier–Stokes system (1.1) available in literature assuming only the existence of the strong solution.

The rest of the paper is organized as follows. In Section 2 we introduce the mesh, basic notations, the numerical method, and some preliminary (in)equalities. Next, in Section 3 we show the energy stability of the scheme and derive all necessary *a priori* bounds. Then we establish the consistency formulation of the scheme in

Section 4. Further, we address the convergence of approximate solutions in Section 5. Finally, we present some numerical experiments in Section 6.

2. Numerical scheme

We introduce the basic notations, mesh, space and time discretizations, and, finally, we define the numerical scheme along with some useful (in)equalities.

2.1. Space discretization

Mesh. A discretization of Ω is given by $\mathcal{M} = (\mathcal{T}, \mathcal{E})$, where:

– The primary grid $\mathcal{T}$ is the set of all compact regular quadrilateral elements K such that

$$\Omega = \bigcup_{K\in\mathcal{T}} K.$$

Let h_i be the mesh size in the ith Cartesian direction, and $h = \max_{i=1,\ldots,d} h_i$ be the mesh size. The mesh is regular in the sense that there exists a positive η_h such that $\eta_h = \max_{i=1,\ldots,d}\left\{\frac{h}{h_i}\right\}$.

– We denote by $\mathcal{E}$ the set of all faces, and by $\mathcal{E}_i$ the set of all faces that are orthogonal to the standard basis vector $\mathbf{e}_i,\ i = 1,\ldots,d$, of the Cartesian coordinate system. By $\mathcal{E}(K)$ we denote the set of faces of an element K, and define $\mathcal{E}_i(K) = \mathcal{E}(K)\cap\mathcal{E}_i$. We further denote by $\mathbf{n}$ the outer normal vector of a generic face $\sigma\in\mathcal{E}$. By $\mathbf{x}_K$ and $\mathbf{x}_\sigma$ we denote the position of the mass centers of an element $K\in\mathcal{T}$ and a face $\sigma\in\mathcal{E}$, respectively.

– The intersection $K\cap L$, for $K,\ L\in\mathcal{T},\ K\neq L$, is either a vertex, or an edge, or a face $\sigma\in\mathcal{E}$. For any $\sigma\in\mathcal{E}$ we write $\sigma = K|L$ if $\sigma = \mathcal{E}(K)\cap\mathcal{E}(L)$, and further write $\sigma = \overrightarrow{K|L}$ if $\mathbf{x}_L = \mathbf{x}_K + h_i\mathbf{e}_i$ or $\mathbf{x}_L = \mathbf{x}_K + (h_i - 1)\mathbf{e}_i$ for any $\sigma\in\mathcal{E}_i$. Similarly, we write $K = \overrightarrow{[\sigma\sigma']}$ for $\sigma,\sigma'\in\mathcal{E}_i(K)$ if $\mathbf{x}_{\sigma'} = \mathbf{x}_\sigma + h_i\mathbf{e}_i$. For any $\sigma = K|L\in\mathcal{E}_i,\ i\in 1,\ldots,d$, we also denote by $d_\sigma = h_i$ the periodic distance between the points $\mathbf{x}_K$ and $\mathbf{x}_L$.

– By $|K|$ and $|\sigma|$ we denote the (d- and $(d-1)$-dimensional) Lebesgue measure of an element K, and a face σ, respectively. Obviously, $|K| = h_i|\sigma|$ for any $\sigma\in\mathcal{E}_i(K)$. In what follows, we shall suppose

$$|K|\approx h^d,\ |\sigma|\approx h^{d-1} \text{ for any } K\in\mathcal{T},\ \sigma\in\mathcal{E}.$$

Function space. In order to define a finite volume scheme we introduce the space of piecewise constant functions Q_h defined on the primary grid $\mathcal{T}$. We also introduce a standard projection operator

$$\Pi_{\mathcal{T}} : L^1(\Omega)\to Q_h. \quad \Pi_{\mathcal{T}}\phi = \sum_{K\in\mathcal{T}} 1_K \frac{1}{|K|}\int_K \phi\,\mathrm{d}x.$$

For a piecewise (elementwise) continuous function v we define

$$v^{\mathrm{out}}(x) = \lim_{\delta\to 0+} v(x+\delta\mathbf{n}),\ v^{\mathrm{in}}(x) = \lim_{\delta\to 0+} v(x-\delta\mathbf{n}),\ \overline{v}(x) = \frac{v^{\mathrm{in}}(x)+v^{\mathrm{out}}(x)}{2},\ [\![v]\!] = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x),$$

whenever $x\in\sigma\in\mathcal{E}$. Hereafter we mean by $\mathbf{v}\in Q_h$ that $\mathbf{v}\in Q_h(\Omega;R^d)$, *i.e.*, $v_i\in Q_h$, for all $i = 1,\ldots,d$.

Diffusive upwind flux. Given the velocity field $\mathbf{v}\in Q_h$, the upwind flux for any function $r\in Q_h$ is defined at each face $\sigma\in\mathcal{E}$ by

$$Up[r,\mathbf{v}] = r^{\mathrm{up}}\overline{\mathbf{v}}\cdot\mathbf{n} = r^{\mathrm{in}}[\overline{\mathbf{v}}\cdot\mathbf{n}]^+ + r^{\mathrm{out}}[\overline{\mathbf{v}}\cdot\mathbf{n}]^- = \overline{r}\ \overline{\mathbf{v}}\cdot\mathbf{n} - \frac{1}{2}|\overline{\mathbf{v}}\cdot\mathbf{n}|\,[\![r]\!]\,,$$

where

$$[f]^{\pm} = \frac{f \pm |f|}{2} \quad \text{and} \quad r^{\mathrm{up}} = \begin{cases} r^{\mathrm{in}} & \text{if } \overline{\mathbf{v}} \cdot \mathbf{n} \geq 0, \\ r^{\mathrm{out}} & \text{if } \overline{\mathbf{v}} \cdot \mathbf{n} < 0. \end{cases}$$

Furthermore, we consider a diffusive numerical flux function of the following form

$$F_h(r, \mathbf{v}) = Up[r, \mathbf{v}] - h^{\varepsilon} [\![r]\!] \,,\ \varepsilon > 0. \tag{2.1}$$

When r is a vector function, *e.g.* $r = \varrho \mathbf{u}$ in the momentum equation, we write the above numerical flux in bold font as $\mathbf{F}_h(\varrho \mathbf{u}, \mathbf{v}) \equiv \big(F_h(\varrho u_1, \mathbf{v}), \ldots, F_h(\varrho u_d, \mathbf{v})\big)^T$ and $\mathbf{Up}(\varrho \mathbf{u}, \mathbf{v}) \equiv \big(Up(\varrho u_1, \mathbf{v}), \ldots, Up(\varrho u_d, \mathbf{v})\big)^T$.

Discrete divergence. We define the discrete divergence operator as

$$\mathrm{div}_h \mathbf{u}_h(\mathbf{x}) := \sum_{K \in \mathcal{T}} (\mathrm{div}_h \mathbf{u}_h)_K 1_K, \quad (\mathrm{div}_h \mathbf{u}_h)_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \overline{\mathbf{u}_h} \cdot \mathbf{n}, \quad \text{for all } \ \mathbf{u}_h \in Q_h. \tag{2.2}$$

2.2. Time discretization

For a given time step $\Delta t \approx h > 0$, we denote the approximation of a function v_h at time $t^k = k \Delta t$ by v_h^k for $k = 1, \ldots, N_T (= T/\Delta t)$. The time derivative is discretized by the backward Euler method,

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}, \quad \text{for } k = 1, 2, \ldots, N_T.$$

Furthermore, we introduce the piecewise constant extension of discrete values,

$$\begin{aligned} \varrho_h(t, \cdot) &= \varrho_h^0 \text{ for } t < \Delta t, & \varrho_h(t, \cdot) &= \varrho_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \ k = 1, 2, \ldots, N_T, \\ \mathbf{u}_h(t, \cdot) &= \mathbf{u}_h^0 \text{ for } t < \Delta t, & \mathbf{u}_h(t, \cdot) &= \mathbf{u}_h^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \ k = 1, 2, \ldots, N_T, \end{aligned}$$

and $p_h = p(\varrho_h)$, for which the discrete time derivative then reads

$$D_t v_h = \frac{v_h(t, \cdot) - v_h(t - \Delta t, \cdot)}{\Delta t}.$$

We shall write $A \lesssim B$ if $A \leq cB$ for a generic positive constant c independent of h.

2.3. Numerical scheme

Using the above notation we introduce the implicit finite volume scheme to approximate system (1.1).

Definition 2.1 (Numerical scheme)**.** Given the initial values $(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$, find $(\varrho_h, \mathbf{u}_h) \in Q_h \times Q_h$ satisfying for $k = 1, \ldots, N_T$ the following equations

$$\int_{\Omega} D_t \varrho_h^k \phi_h \,\mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_{\sigma} F_h(\varrho_h^k, \mathbf{u}_h^k) [\![\phi_h]\!] \,\mathrm{d}S(x) = 0 \quad \text{for all } \phi_h \in Q_h, \tag{2.3a}$$

$$\begin{aligned} &\int_{\Omega} D_t(\varrho_h^k \mathbf{u}_h^k) \cdot \boldsymbol{\phi}_h \,\mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_{\sigma} \mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S(x) - \sum_{\sigma \in \mathcal{E}} \int_{\sigma} \overline{p_h^k} \mathbf{n} \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S(x) \\ &\quad = -\mu \sum_{\sigma \in \mathcal{E}} \int_{\sigma} \frac{1}{d_\sigma} [\![\mathbf{u}_h^k]\!] \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S(x) - (\mu + \lambda) \int_{\Omega} \mathrm{div}_h \mathbf{u}_h^k \,\mathrm{div}_h \boldsymbol{\phi}_h \,\mathrm{d}x \quad \text{for all } \boldsymbol{\phi}_h \in Q_h. \end{aligned} \tag{2.3b}$$

The weak formulation (2.3) of the scheme can be rewritten in the standard per cell finite volume formulation for all $K \in \mathcal{T}$,

$$\begin{aligned} &D_t \varrho_K^k + \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0, \\ &D_t(\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p_h^k}\mathbf{n} - \mu \frac{[\![\mathbf{u}_h^k]\!]}{d_\sigma} - (\mu+\lambda)\overline{\mathrm{div}_h \mathbf{u}_h^k}\mathbf{n} \right) = 0. \end{aligned} \tag{2.4}$$

Remark 2.2. Let us explain the role of h^ε-terms in (2.4), which are hidden in the numerical flux F_h, see (2.1). Clearly, they are additional diffusion terms

$$\sum_{\sigma\in\mathcal{E}(K)} \frac{|\sigma|}{|K|} h^\varepsilon [\![r_h]\!] = h^{\varepsilon+1}(\Delta_h r_h)_K.$$

In this paper we require $0 < \varepsilon < \min\{1, 2(\gamma-1)\}$ as a compromise between minimality of the additional numerical diffusion and necessary consistency estimates, see Section 4.

The approximate solutions resulting from scheme (2.3) enjoy the following properties:

1. **Conservation of mass.**
   Taking $\phi_h \equiv 1$ in the equation of continuity (2.3a) yields the total mass conservation
   $$\int_\Omega \varrho_h(t,\cdot)\,\mathrm{d}x = \int_\Omega \varrho_h^0\,\mathrm{d}x > 0,\ t \geq 0.$$
2. **Existence of numerical solution.**
   The discrete problem (2.3) admits a solution $(\varrho_h^k, \mathbf{u}_h^k)$ for any $k = 1, \ldots, N_T$. We refer a reader to Theorem 3.5 from [23] for the proof, as it can be done exactly in the same way.
3. **Positivity of numerical density.**
   Any solution $(\varrho_h^k, \mathbf{u}_h^k)$ to (2.3) satisfies $\varrho_h^k > 0$ provided $\varrho_h^{k-1} > 0$, $k = 1, \ldots, N_T$, see Lemma 3.2 from [23] for the proof.

2.4. Preliminaries

To investigate theoretical properties of our finite volume method it is convenient to define a dual grid. We emphasize that the dual grid is not needed for the implementation of the scheme.

Dual grid. A dual element D_σ is associated to a generic face $\sigma = K|L \in \mathcal{E}$, where $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, and $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) is built by half of K (resp. L), see Figure 1 for an example of such cell. We denote the set of all dual cells as $\mathcal{D}$. Furthermore, we define $\mathcal{D}_i = \{D_\sigma\}_{\sigma\in\mathcal{E}_i}, i = 1, \ldots, d$. Note that for each i it holds that $\Omega = \bigcup_{\sigma\in\mathcal{E}_i} D_\sigma$.

Let $W_h^{(i)}$, $i = 1, \ldots, d$, be the space of piecewise constant functions defined on the dual grid $\mathcal{D}_i$. By $\mathbf{q} = (q_1, \ldots, q_d) \in W_h := \left(W_h^{(1)}, \ldots, W_h^{(d)}\right)$ we mean that $q_i \in W_h^{(i)}$, for all $i = 1, \ldots, d$. We define the standard projection of $\phi \in L^1(\Omega)$ into the discrete functional spaces W_h,

$$\Pi_\mathcal{D}: \quad L^1(\Omega) \to W_h, \quad \Pi_\mathcal{D} = (\Pi_\mathcal{D}^{(1)}, \ldots, \Pi_\mathcal{D}^{(d)}), \quad \Pi_\mathcal{D}^{(i)}\phi = \sum_{\sigma\in\mathcal{E}_i} \frac{1_{D_\sigma}}{|D_\sigma|} \int_{D_\sigma} \phi\,\mathrm{d}x.$$

Discrete differential operators. We need some discrete operators that are not directly used to discretize the Navier–Stokes system, but are essential to establish the consistency formulation in Section 4. For any $r_h \in Q_h$ and $\mathbf{q}_h = (q_{1,h}, \ldots, q_{d,h}) \in W_h$, we define the difference operators based on the dual grid

$$\eth_\mathcal{E}^{(i)} r_h(\mathbf{x}) := \sum_{\sigma\in\mathcal{E}_i} 1_{D_\sigma} \left(\eth_\mathcal{E}^{(i)} r_h\right)_{D_\sigma}, \quad \left(\eth_\mathcal{E}^{(i)} r_h\right)_{D_\sigma} := \frac{r_L - r_K}{d_\sigma}, \text{ for all } \sigma = \overrightarrow{K|L} \in \mathcal{E}_i,$$

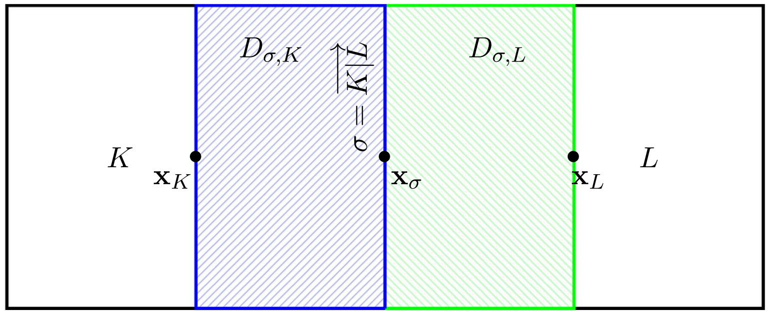


FIGURE 1. Dual grid.

and the primary grid

$$\eth_{\mathcal{T}}^{(i)} q_{i,h}(\mathbf{x}) := \sum_{K\in\mathcal{T}} \left(\eth_{\mathcal{T}}^{(i)} q_{i,h}\right)_K 1_K, \quad i = 1,\dots,d,$$

where

$$\left(\eth_{\mathcal{T}}^{(i)} q_{i,h}\right)_K := \frac{q_{i,h}|_{\sigma'} - q_{i,h}|_{\sigma}}{h}, \quad \text{for all} \quad \sigma,\sigma' \in \mathcal{E}_i \text{ and } K = \overrightarrow{[\sigma\sigma']}.$$

Using the above notations we define the gradient operators for $r_h \in Q_h$ and $\mathbf{q}_h \in W_h$ by

$$\nabla_{\mathcal{E}} r_h(\mathbf{x}) := (\eth_{\mathcal{E}}^{(1)} r_h, \dots, \eth_{\mathcal{E}}^{(d)} r_h)(\mathbf{x}) \quad \text{and} \quad \nabla_{\mathcal{T}} \mathbf{q}_h := (\eth_{\mathcal{T}}^{(1)} q_{1,h}, \dots, \eth_{\mathcal{T}}^{(d)} q_{d,h})(\mathbf{x}),$$

respectively. Note that the divergence operator div_h defined in (2.2) can be rewritten for all $\mathbf{u}_h \in Q_h$

$$\mathrm{div}_h \mathbf{u}_h = \sum_{i=1}^{d} \eth_{\mathcal{T}}^{(i)} \overline{u_{i,h}}, \tag{2.5}$$

which for a regular rectangular grid is equivalent to

$$\mathrm{div}_h \mathbf{u}_h = \sum_{i=1}^{d} \eth_{\mathcal{T}}^{(i)} \left(\Pi_{\mathcal{D}}^{(i)} \mathbf{u}_h\right).$$

Moreover, we define the discrete Laplace operator for $r_h \in Q_h$ on the primary grid

$$\Delta_h r_h(\mathbf{x}) = \sum_{i=1}^{d} \Delta_h^{(i)} r_h(\mathbf{x}) = \sum_{K\in\mathcal{T}} (\Delta_h r_h)_K 1_K, \quad \Delta_h^{(i)} r_h(\mathbf{x}) = \sum_{K\in\mathcal{T}} (\Delta_h^{(i)} r_h)_K 1_K,$$

where $i = 1,\dots,d$, and

$$(\Delta_h^{(i)} r_h)_K := \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}_i(K)} |\sigma| \frac{[\![r_h]\!]}{d_\sigma}, \quad (\Delta_h r_h)_K := \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \frac{[\![r_h]\!]}{d_\sigma}, \quad \text{for all} \quad K \in \mathcal{T}.$$

In addition, it is worth mentioning that

$$\Delta_h^{(i)} r_h = \eth_{\mathcal{T}}^{(i)} (\eth_{\mathcal{E}}^{(i)} r_h), \; i = 1,\dots,d.$$

Integration by parts. Let us start with recalling the algebraic identity

$$\overline{u_h v_h} - \overline{u_h}\;\overline{v_h} = \frac{1}{4} [\![u_h]\!] \, [\![v_h]\!] , \tag{2.6}$$

together with the product rule

$$[\![u_h v_h]\!] = \overline{u_h}\,[\![v_h]\!] + [\![u_h]\!]\,\overline{v_h}, \tag{2.7}$$

which are valid for any $u_h, v_h \in Q_h$. A direct application of the product rule (2.7) further implies

$$[\![r_h \mathbf{v}_h]\!] \cdot [\![\mathbf{v}_h]\!] - \frac{1}{2}[\![r_h]\!]\,[\![|\mathbf{v}_h|^2]\!] = \overline{r_h}|\,[\![\mathbf{v}_h]\!]\,|^2 \text{ for } r_h \in Q_h,\ \mathbf{v}_h \in Q_h, \tag{2.8}$$

and the following lemma.

Lemma 2.3. *For any $r_h \in Q_h$ and $\mathbf{v}_h \in Q_h$ it holds*

$$\sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\overline{r_h}\,[\![\mathbf{v}_h]\!] + \overline{\mathbf{v}_h}\,[\![r_h]\!]\right)\cdot \mathbf{n}\,\mathrm{d}S(x) = 0. \tag{2.9}$$

Proof. For the functions r_h, $\mathbf{v}_h$ that are constant on each element $K \in \mathcal{T}$, it holds that

$$\sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\overline{r_h}\,[\![\mathbf{v}_h]\!] + \overline{\mathbf{v}_h}\,[\![r_h]\!]\right)\cdot \mathbf{n}\,\mathrm{d}S(x) = \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![r_h\mathbf{v}_h]\!]\cdot \mathbf{n}\,\mathrm{d}S(x) = -\sum_{K\in\mathcal{T}} r_K \mathbf{v}_K \cdot \sum_{\sigma\in\mathcal{E}(K)} \int_\sigma \mathbf{n}\,\mathrm{d}S(x) = 0.$$

□

Consequently, for any $r_h, \phi_h \in Q_h$ and $\mathbf{q}_h \in W_h$, it is easy to observe the following discrete integration by parts formulae

$$\int_\Omega \Delta_h r_h \phi_h \,\mathrm{d}x = -\int_\Omega \nabla_{\mathcal{E}} r_h \cdot \nabla_{\mathcal{E}} \phi_h \,\mathrm{d}x = \int_\Omega r_h \Delta_h \phi_h \,\mathrm{d}x, \tag{2.10a}$$

$$\int_\Omega q_{i,h} \eth_{\mathcal{E}}^{(i)} r_h \,\mathrm{d}x = -\int_\Omega r_h \eth_{\mathcal{T}}^{(i)} q_{i,h} \,\mathrm{d}x, \quad \text{for all} \quad i = 1,\ldots,d. \tag{2.10b}$$

Useful estimates. Next, we list some basic inequalities used in the numerical analysis. We assume the reader is fairly familiar with this matter, for which we refer to the monograph [10], and the article paper [20]. If $\phi \in C^1(\Omega)$ we have

$$\left| [\![\Pi_{\mathcal{T}}\phi]\!] \right|_\sigma \lesssim h\|\phi\|_{C^1}, \quad \text{for any } x \in \sigma \in \mathcal{E}, \text{ and } \|\phi - \Pi_{\mathcal{T}}\phi\|_{L^p(\Omega)} \lesssim h\|\phi\|_{C^1}. \tag{2.11}$$

Furthermore, if $\phi \in C^2(\Omega)$ we have for all $1 < p \leq \infty$

$$\|\nabla_x \phi - \nabla_{\mathcal{E}} \Pi_{\mathcal{T}}\phi\|_{L^p} \lesssim h, \quad \|\nabla_{\mathcal{E}} \Pi_{\mathcal{T}}\phi\|_{L^p} \lesssim \|\phi\|_{C^1} + h, \tag{2.12}$$

$$\left\|\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}}\big(\Pi_{\mathcal{T}}\phi\big)\right\|_{L^p} \lesssim h, \quad \|\mathrm{div}_x \phi - \mathrm{div}_h(\Pi_{\mathcal{T}}\phi)\|_{L^p} \lesssim h. \tag{2.13}$$

If in addition, $\phi \in C^3(\Omega)$ we get

$$\|\Delta_h \Pi_{\mathcal{T}}\phi - \Delta_x \phi\|_{L^p} \lesssim h\|\phi\|_{C^3}, \quad \|\Delta_h \Pi_{\mathcal{T}}\phi\|_{L^p} \lesssim \|\phi\|_{C^2} + h\|\phi\|_{C^3}, \text{ for all } 1 < p \leq \infty. \tag{2.14}$$

The inverse estimates [4] for $r_h \in Q_h$ read

$$\|r_h\|_{L^p(\Omega)} \lesssim h^{d(\frac{1}{p}-\frac{1}{q})}\|r_h\|_{L^q(\Omega)} \text{ for any } 1 \leq q \leq p \leq \infty. \tag{2.15}$$

Finally, we need a discrete analogous of the Sobolev-type inequality that can be proved exactly as ([16], Thm. 11.23).

Lemma 2.4 (Sobolev inequality). *Let the function $r \geq 0$ be such that*

$$0 < \int_\Omega r \,\mathrm{d}x = c_M, \text{and} \int_\Omega r^\gamma \,\mathrm{d}x \leq c_E \text{ for } \gamma > 1,$$

where c_M and c_E are some positive constants. Then the following Poincaré-Sobolev type inequality holds true

$$\|v_h\|_{L^6(\Omega)} \leq c\,\|\nabla_{\mathcal{E}} v_h\|^2_{L^2(\Omega)} + c\left(\int_\Omega r|v_h| \,\mathrm{d}x\right)^2 \lesssim c\,\|\nabla_{\mathcal{E}} v_h\|^2_{L^2(\Omega)} + c_M + c\int_\Omega r|v_h|^2 \,\mathrm{d}x \tag{2.16}$$

for any $v_h \in Q_h$, where the constant c depends on c_M and c_E but not on the mesh parameter.

The following lemma shall be useful for analysing the error between the continuous convective term and its numerical analogue.

Lemma 2.5. *For any $r_h, \mathbf{v}_h \in Q_h$, and $\phi \in C^1(\Omega)$, it holds*

$$\begin{aligned}\int_\Omega r_h \mathbf{v}_h \cdot \nabla_x \phi \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h[r_h, \mathbf{v}_h]\ [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) &= \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\frac{1}{2}|\overline{\mathbf{v}_h} \cdot \mathbf{n}| + h^\varepsilon + \frac{1}{4}[\![\mathbf{v}_h]\!] \cdot \mathbf{n}\right) [\![r_h]\!]\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) \\ &\quad + \int_\Omega r_h \mathbf{v}_h \cdot \left(\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}} \left(\Pi_{\mathcal{T}} \phi\right)\right) \,\mathrm{d}x.\end{aligned}$$

Proof. Using the basic equalities (2.6)–(2.9), we have

$$\begin{aligned}\int_\Omega r_h \mathbf{v}_h \cdot \nabla_x \phi \,\mathrm{d}x &= \sum_{K\in\mathcal{T}} \int_K r_h \mathbf{v}_h \cdot \nabla_x \phi \ \mathrm{d}x \\
&= \sum_{K\in\mathcal{T}} \int_K r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}}(\Pi_{\mathcal{T}} \phi)) \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} r_h \mathbf{v}_h \cdot \mathbf{n} \overline{\Pi_{\mathcal{T}} \phi} \,\mathrm{d}S(x) \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}}(\Pi_{\mathcal{T}} \phi)) \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![r_h \mathbf{v}_h]\!] \cdot \mathbf{n} \overline{\Pi_{\mathcal{T}} \phi} \,\mathrm{d}S(x) \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}}(\Pi_{\mathcal{T}} \phi)) \,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{r_h \mathbf{v}_h} \cdot \mathbf{n}\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}}(\Pi_{\mathcal{T}} \phi)) \,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma (\overline{r_h \mathbf{v}_h} - \overline{r_h}\ \overline{\mathbf{v}_h}) \cdot \mathbf{n}\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) \\
&\quad + \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{r_h}\ \overline{\mathbf{v}_h} \cdot \mathbf{n}\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) \pm \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\frac{1}{2}|\overline{\mathbf{v}_h} \cdot \mathbf{n}| + h^\varepsilon\right) [\![r_h]\!]\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) \\
&= \int_\Omega r_h \mathbf{v}_h \cdot (\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}}(\Pi_{\mathcal{T}} \phi)) \,\mathrm{d}x + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{1}{4} [\![r_h]\!]\, [\![\mathbf{v}_h]\!] \cdot \mathbf{n}\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) \\
&\quad + \sum_{\sigma\in\mathcal{E}} \int_\sigma F_h[r_h, \mathbf{v}_h]\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x) + \sum_{\sigma\in\mathcal{E}} \int_\sigma \left(\frac{1}{2}|\overline{\mathbf{v}_h} \cdot \mathbf{n}| + h^\varepsilon\right) [\![r_h]\!]\, [\![\Pi_{\mathcal{T}} \phi]\!] \,\mathrm{d}S(x).\end{aligned}$$

□

3. Stability

In this section we show the stability of the scheme and derive the energy estimates that will be necessary for the consistency formulation in Section 4. For simplicity, we will hereafter denote the norms $\|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{L^p(0,T;L^q(\Omega))}$ by $\|\cdot\|_{L^q}$ and $\|\cdot\|_{L^pL^q}$, respectively. We also denote $\mathrm{co}\{A, B\} = [\min\{A, B\}, \max\{A, B\}]$.

To begin, we recall the discrete internal energy balance, which is a result of the renormalization of the continuity equation, see, *e.g.* Section 4.1 from [12], or Lemma 3.1 from [23]. Indeed, multiplying (2.3a) by $\mathcal{H}'(\varrho_h^k)$ of the internal energy $\mathcal{H}(\varrho) = \frac{p(\varrho)}{\gamma-1}$ gives rise to the result of the following lemma.

Lemma 3.1 (Discrete internal energy balance)**.** *Let $(\varrho_h, \mathbf{u}_h) \in Q_h \times Q_h$ satisfy the discrete continuity equation (2.3a). Then there exists $\xi \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$ and $\zeta \in \mathrm{co}\{\varrho_K^k, \varrho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$ such that*

$$\begin{aligned} &\int_\Omega D_t \mathcal{H}(\varrho_h^k)\,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\mathbf{u}_h^k} \cdot \mathbf{n} \left[\!\left[p(\varrho_h^k) \right]\!\right] \,\mathrm{d}S(x) \\ &\qquad = -\frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\xi) |D_t \varrho_h^k|^2 \,\mathrm{d}x - \frac{1}{2} \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) \left[\!\left[\varrho_h^k \right]\!\right]^2 (h^\varepsilon + |\overline{\mathbf{u}_h} \cdot \mathbf{n}|) \,\mathrm{d}S(x). \end{aligned} \tag{3.1}$$

Next, we recall the renormalization of the transport equation, see Lemma A.1 from [12].

Lemma 3.2 (Discrete renormalized transport equation)**.** *Suppose that $b_h^k \in Q_h$, $\chi \in C^2(R)$. Then there exists $\xi \in \mathrm{co}\{b_h^{k-1}, b_h^k\}$, $\zeta \in \mathrm{co}\{b_h^k, (b_h^k)^{\mathrm{out}}\}$ for any $\phi_h \in Q_h$ such that*

$$\begin{aligned} &\int_\Omega D_t(\varrho_h^k b_h^k) \chi'(b_h^k) \phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma Up[\varrho_h^k b_h^k, \mathbf{u}_h^k] \left[\!\left[\chi'(b_h^k)\phi_h \right]\!\right] \,\mathrm{d}S(x) \\ &\quad = \int_\Omega D_t \left(\varrho_h^k \chi(b_h^k) \right) \phi_h \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma Up[\varrho_h^k \chi(b_h^k), \mathbf{u}_h] \left[\!\left[\phi_h \right]\!\right] \,\mathrm{d}S(x) + \frac{\Delta t}{2} \int_\Omega \chi''(\xi) \varrho_h^{k-1} |D_t b_h^k|^2 \phi_h \,\mathrm{d}x \\ &\qquad + \sum_{\sigma\in\mathcal{E}} \int_\sigma h^\varepsilon \left[\!\left[\varrho_h^k \right]\!\right] \left[\!\left[\left(\chi(b_h^k) - \chi'(b_h^k) b_h^k \right) \phi_h \right]\!\right] \,\mathrm{d}S(x) \\ &\qquad - \frac{1}{2} \sum_{K\in\mathcal{T}} \sum_{\sigma\subset\partial K} \int_\sigma \phi_h \chi''(\zeta) \left[\!\left[b_h^k \right]\!\right]^2 (\varrho_h^k)^{\mathrm{out}} \left[\overline{\mathbf{u}_h^k} \cdot \mathbf{n} \right]^- \,\mathrm{d}S(x). \end{aligned} \tag{3.2}$$

3.1. Total energy balance

Now, we are ready to derive the discrete counterpart of the total energy balance.

Theorem 3.3 (Discrete energy balance)**.** *Let $(\varrho_h, \mathbf{u}_h)$ be a numerical solution obtained from scheme (2.3). Then, for any $k = 1, \ldots, N_T$, there exists $\xi \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$ and $\zeta \in \mathrm{co}\{\varrho_K^k, \varrho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$ such that*

$$\begin{aligned} &D_t \int_\Omega \left(\frac{1}{2} \varrho_h^k |\mathbf{u}_h^k|^2 + \mathcal{H}(\varrho_h^k) \right) \,\mathrm{d}x + h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h^k} \left[\!\left[\mathbf{u}_h^k \right]\!\right]^2 \,\mathrm{d}S(x) + \mu \left\| \nabla_{\mathcal{E}} \mathbf{u}_h^k \right\|_{L^2}^2 + (\mu+\lambda) \int_\Omega |\mathrm{div}_h \mathbf{u}_h^k|^2 \,\mathrm{d}x \\ &\quad = -\frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\xi) |D_t \varrho_h^k|^2 \,\mathrm{d}x - \frac{1}{2} \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) \left[\!\left[\varrho_h^k \right]\!\right]^2 \left(h^\varepsilon + |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \right) \,\mathrm{d}S(x) \\ &\qquad - \frac{\Delta t}{2} \int_\Omega \varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \,\mathrm{d}x - \frac{1}{2} \sum_{\sigma\in\mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \left[\!\left[\mathbf{u}_h^k \right]\!\right]^2 \,\mathrm{d}S(x). \end{aligned} \tag{3.3}$$

Proof. First, taking $\boldsymbol{\phi}_h = \mathbf{u}_h^k$ in (2.3b) we get

$$\begin{aligned} &\int_\Omega D_t(\varrho_h^k \mathbf{u}_h^k) \cdot \mathbf{u}_h^k \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) \cdot \left[\!\left[\mathbf{u}_h^k \right]\!\right] \,\mathrm{d}S(x) - \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{p_h^k} \mathbf{n} \cdot \left[\!\left[\mathbf{u}_h^k \right]\!\right] \,\mathrm{d}S(x) \\ &\quad = -\mu \left\| \nabla_{\mathcal{E}} \mathbf{u}_h^k \right\|_{L^2} - (\mu+\lambda) \int_\Omega |\mathrm{div}_h \mathbf{u}_h^k|^2 \,\mathrm{d}x. \end{aligned}$$

Next, we use relation (3.2) for $b_h = \mathbf{u}_h^k$, $\chi(|\mathbf{u}_h^k|) = \frac{1}{2}|\mathbf{u}_h^k|^2$, and $\phi_h = 1$ to compute

$$
\begin{aligned}
&\int_\Omega D_t(\varrho_h^k \mathbf{u}_h^k)\cdot \mathbf{u}_h^k \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{Up}[\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k]\cdot [\![\mathbf{u}_h^k]\!] \,\mathrm{d}S(x) \\
&\quad = \int_\Omega D_t\left(\frac{1}{2}\varrho_h^k |\mathbf{u}_h^k|^2\right)\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathbf{Up}\left[\frac{1}{2}\varrho_h^k|\mathbf{u}_h^k|^2, \mathbf{u}_h^k\right] \underbrace{[\![1]\!]}_{=0} \,\mathrm{d}S(x) + \frac{\Delta t}{2}\int_\Omega \varrho_h^{k-1}|D_t \mathbf{u}_h^k|^2 \,\mathrm{d}x \\
&\qquad - \sum_{\sigma\in\mathcal{E}} \int_\sigma h^\varepsilon [\![\varrho_h^k]\!] \left[\!\!\left[\frac{1}{2}|\mathbf{u}_h^k|^2\right]\!\!\right] \mathrm{d}S(x) - \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\subset\partial K} \int_\sigma (\varrho_h^k)^{\mathrm{out}} \left[\overline{\mathbf{u}_h^k}\cdot\mathbf{n}\right]^- [\![\mathbf{u}_h^k]\!]^2 \,\mathrm{d}S(x) \\
&\qquad = \int_\Omega D_t\left(\frac{1}{2}\varrho_h^k |\mathbf{u}_h^k|^2\right)\mathrm{d}x + \frac{\Delta t}{2}\int_\Omega \varrho_h^{k-1}|D_t \mathbf{u}_h^k|^2 \,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}} \int_\sigma h^\varepsilon [\![\varrho_h^k]\!] \left[\!\!\left[\frac{1}{2}|\mathbf{u}_h^k|^2\right]\!\!\right] \mathrm{d}S(x) \\
&\qquad\quad + \frac{1}{2}\sum_{\sigma\in\mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{out}} \left[\overline{\mathbf{u}_h^k}\cdot\mathbf{n}\right]^- [\![\mathbf{u}_h^k]\!]^2 \,\mathrm{d}S(x).
\end{aligned}
$$

Further, summing up the previous two observations we infer that

$$
\begin{aligned}
&D_t \int_\Omega \frac{1}{2}\varrho_h |\mathbf{u}_h^k|^2 \,\mathrm{d}x + \mu \left\|\nabla_\mathcal{E} \mathbf{u}_h^k\right\|_{L^2} + (\mu+\lambda)\int_\Omega |\mathrm{div}_h \mathbf{u}_h^k|^2 \,\mathrm{d}x \\
&\quad = \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{p_h^k}\mathbf{n}\cdot [\![\mathbf{u}_h^k]\!] \,\mathrm{d}S(x) - \sum_{\sigma\in\mathcal{E}} \int_\sigma h^\varepsilon [\![\varrho_h^k \mathbf{u}_h^k]\!] [\![\mathbf{u}_h^k]\!] \,\mathrm{d}S(x) + \sum_{\sigma\in\mathcal{E}} \int_\sigma h^\varepsilon [\![\varrho_h^k]\!] \left[\!\!\left[\frac{1}{2}|\mathbf{u}_h^k|^2\right]\!\!\right] \mathrm{d}S(x) \\
&\qquad - \frac{\Delta t}{2}\int_\Omega \varrho_h^{k-1}|D_t \mathbf{u}_h^k|^2 \,\mathrm{d}x - \frac{1}{2}\sum_{\sigma\in\mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k}\cdot\mathbf{n}| [\![\mathbf{u}_h^k]\!]^2 \,\mathrm{d}S(x).
\end{aligned}
\tag{3.4}
$$

Finally, combining (3.4) with (3.1) and using the equalities (2.8)–(2.9) we get

$$
\begin{aligned}
&D_t \int_\Omega \left(\frac{1}{2}\varrho_h^k |\mathbf{u}_h^k|^2 + \mathcal{H}(\varrho_h^k)\right)\mathrm{d}x + h^\varepsilon \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h^k}\, [\![\mathbf{u}_h^k]\!]^2 \,\mathrm{d}S(x) + \mu \left\|\nabla_\mathcal{E} \mathbf{u}_h^k\right\|_{L^2}^2 + (\mu+\lambda)\int_\Omega |\mathrm{div}_h \mathbf{u}_h^k|^2 \,\mathrm{d}x \\
&\quad = -\frac{\Delta t}{2}\int_\Omega \mathcal{H}''(\xi)|D_t \varrho_h^k|^2 \,\mathrm{d}x - \frac{1}{2}\sum_{\sigma\in\mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) [\![\varrho_h^k]\!]^2 \left(h^\varepsilon + |\overline{\mathbf{u}_h^k}\cdot\mathbf{n}|\right) \mathrm{d}S(x) \\
&\qquad - \frac{\Delta t}{2}\int_\Omega \varrho_h^{k-1}|D_t \mathbf{u}_h^k|^2 \,\mathrm{d}x - \frac{1}{2}\sum_{\sigma\in\mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k}\cdot\mathbf{n}| [\![\mathbf{u}_h^k]\!]^2 \,\mathrm{d}S(x),
\end{aligned}
$$

which completes the proof. □

3.2. Uniform bounds

Having established all necessary ingredients, we are ready to discuss the available *a priori* bounds for solutions of scheme (2.3). From the total energy balance (3.3) and the Sobolev inequality (2.16), we directly get the estimates comprised in the following corollary.

Corollary 3.4. *Let $(\varrho_h, \mathbf{u}_h)$ satisfy scheme (2.3) for $\gamma > 1$. Then the following estimates hold*

$$\left\|\varrho_h \mathbf{u}_h^2\right\|_{L^\infty L^1} \lesssim 1, \tag{3.5a}$$
$$\left\|\varrho_h\right\|_{L^\infty L^\gamma} \lesssim 1, \tag{3.5b}$$
$$\left\|\varrho_h \mathbf{u}_h\right\|_{L^\infty L^{\frac{2\gamma}{\gamma+1}}} \lesssim 1, \tag{3.5c}$$
$$\left\|\nabla_\mathcal{E} \mathbf{u}_h\right\|_{L^2 L^2} \lesssim 1, \tag{3.5d}$$
$$\left\|\mathrm{div}_h \mathbf{u}_h\right\|_{L^2 L^2} \lesssim 1, \tag{3.5e}$$

$$\|\mathbf{u}_h\|_{L^2L^6} \lesssim 1, \tag{3.5f}$$

$$h^\varepsilon \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h} \left[\!\left[\mathbf{u}_h\right]\!\right]^2 \,\mathrm{d}S(x)\,\mathrm{d}t \lesssim 1, \tag{3.5g}$$

$$\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \mathcal{H}''(\zeta) \left[\!\left[\varrho_h\right]\!\right]^2 (h^\varepsilon + |\overline{\mathbf{u}_h}\cdot\mathbf{n}|)\,\mathrm{d}S(x)\,\mathrm{d}t \lesssim 1, \tag{3.5h}$$

where $\zeta \in \mathrm{co}\{\varrho_K, \varrho_L\}$ *for any* $\sigma = K|L \in \mathcal{E}$.

To show the consistency of the numerical scheme we shall need further bounds on the numerical solution, which can be derived provided the adiabatic coefficient in (1.2) lies in the physically realistic range $\gamma \in (1,2)$.

Lemma 3.5. *Let* $(\varrho_h, \mathbf{u}_h)$ *satisfy scheme* (2.3), $h \in (0,1)$ *and* $\gamma \in (1,2)$. *Then there hold*

$$\|\varrho_h\|_{L^2L^2} \lesssim h^{-\frac{\varepsilon+2}{2\gamma}}, \tag{3.6a}$$

$$\|\varrho_h\mathbf{u}_h\|_{L^2L^2} \lesssim h^{-\frac{\varepsilon+2}{2\gamma}}. \tag{3.6b}$$

Proof. We start the proof by recalling the Sobolev inequality for the broken norm

$$\|f_h\|_{L^6}^2 \lesssim \|f_h\|_{L^2}^2 + \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{\left[\!\left[f_h\right]\!\right]^2}{d_\sigma}\,\mathrm{d}S(x) = \|f_h\|_{L^2}^2 + \|\nabla_{\mathcal{E}} f_h\|_{L^2}^2\,,\ \forall f_h \in Q_h,$$

and the algebraic inequality

$$a\gamma(\varrho_L^{\gamma/2} - \varrho_K^{\gamma/2})^2 \le \frac{\partial^2\mathcal{H}(z)}{\partial\varrho^2}(\varrho_L - \varrho_K)^2,\ \forall\, z \in \mathrm{co}\{\varrho_L, \varrho_K\}, \varrho_L, \varrho_K > 0 \text{ if } \gamma \in (1,2).$$

Then we indicate from the estimate of the density jumps (3.5h) that

$$\left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2L^2}^2 = \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \frac{1}{d_\sigma} \left[\!\left[\varrho_h^{\gamma/2}\right]\!\right]^2 \,\mathrm{d}S(x) \lesssim h^{-(\varepsilon+1)}.$$

Applying the above inequalities, the inverse estimate and the estimate (3.5b) we derive

$$\begin{aligned}
\|\varrho_h\|_{L^1L^\infty} &= \int_0^T \left\|\varrho_h^{\gamma/2}\right\|_{L^\infty}^{2/\gamma} \mathrm{d}t \le \int_0^T \left(h^{-1/2}\left\|\varrho_h^{\gamma/2}\right\|_{L^6}\right)^{2/\gamma} \mathrm{d}t \\
&\le h^{-1/\gamma} \int_0^T \left(\left\|\varrho_h^{\gamma/2}\right\|_{L^2}^2 + \left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2}^2\right)^{1/\gamma} \mathrm{d}t \le h^{-1/\gamma}\left(\|\varrho_h\|_{L^1L^\gamma} + \left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^{\gamma/2}L^2}^{2/\gamma}\right) \\
&\le h^{-1/\gamma}\left(\|\varrho_h\|_{L^1L^\gamma} + \left\|\nabla_{\mathcal{E}}\varrho_h^{\gamma/2}\right\|_{L^2L^2}^{2/\gamma}\right) \le h^{-\frac{\varepsilon+2}{\gamma}}.
\end{aligned}$$

Further application of the above inequality together with the Gagliardo-Nirenberg interpolation inequality, Hölder's inequality, and the density estimate (3.5b) immediately yield (3.6a), *i.e.*,

$$\|\varrho_h\|_{L^2L^2} = \left(\int_0^T \|\varrho_h\|_{L^2}^2 \,\mathrm{d}t\right)^{1/2} \le \left(\int_0^T \|\varrho_h\|_{L^1}\|\varrho_h\|_{L^\infty}\,\mathrm{d}t\right)^{1/2} \le \|\varrho_h\|_{L^\infty L^1}^{1/2}\|\varrho_h\|_{L^1L^\infty}^{1/2} \lesssim h^{-\frac{\varepsilon+2}{2\gamma}}.$$

Finally, the estimate (3.6b) can be shown in the following way

$$\|\varrho_h\mathbf{u}_h\|_{L^2L^2} \lesssim \|\sqrt{\varrho_h}\|_{L^2L^\infty}\|\sqrt{\varrho_h}\mathbf{u}_h\|_{L^\infty L^2} = \|\varrho_h\|_{L^1L^\infty}^{1/2}\left\|\varrho_h\mathbf{u}_h^2\right\|_{L^\infty L^1}^{1/2} \lesssim h^{-\frac{\varepsilon+2}{2\gamma}}.$$

$\square$

4. Consistency

Next step towards the convergence of the approximate solutions is the consistency of the numerical scheme. In particular, we require the numerical solution to satisfy the weak formulation of the continuous problem up to a residual term vanishing for $h \to 0$.

Theorem 4.1. *Let $(\varrho_h, \mathbf{u}_h)$ be a solution of the approximate problem (2.3) on the time interval $[0,T]$ with $\Delta t \approx h$, $1 < \gamma < 2$ and $0 < \varepsilon < \min\{1, 2(\gamma-1)\}$. Then*

$$-\int_\Omega \varrho_h^0 \phi(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega [\varrho_h \partial_t \phi + \varrho_h \mathbf{u}_h \cdot \nabla_x \phi]\,\mathrm{d}x\,\mathrm{d}t + \int_0^T e_{1,h}(t,\phi)\,\mathrm{d}t, \tag{4.1}$$

for any $\phi \in C_c^3([0,T) \times \Omega)$;

$$\begin{aligned} -\int_\Omega \varrho_h^0 \mathbf{u}_h^0 \boldsymbol{\phi}(0,\cdot)\,\mathrm{d}x &= \int_0^T \int_\Omega [\varrho_h \mathbf{u}_h \cdot \partial_t \boldsymbol{\phi} + \varrho_h \mathbf{u}_h \otimes \mathbf{u}_h : \nabla_x \boldsymbol{\phi} + p_h \mathrm{div}_x \boldsymbol{\phi}]\,\mathrm{d}x\,\mathrm{d}t, \\ -\mu \int_0^T \int_\Omega \nabla_\mathcal{E} \mathbf{u}_h : \nabla_x \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t &- (\mu+\lambda) \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h\, \mathrm{div}_x \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T e_{2,h}(t,\boldsymbol{\phi})\,\mathrm{d}t, \end{aligned} \tag{4.2}$$

for any $\boldsymbol{\phi} \in C_c^3([0,T] \times \Omega; R^d)$;

$$\|e_{j,h}(\cdot,\phi)\|_{L^1(0,T)} \lesssim h^\beta \left(\|\phi\|_{C^2} + h\|\phi\|_{C^3}\right),\ j = 1,2,\ \text{ for some }\ \beta > 0.$$

Proof. Let $\phi \in C_c^3([0,T) \times \Omega)$ and $\boldsymbol{\phi} \in C_c^3([0,T) \times \Omega; R^d)$. We test the equations (2.3a) and (2.3b) with $\Pi_\mathcal{T}\phi$ and $\Pi_\mathcal{T}\boldsymbol{\phi}$, respectively, and deal with each term separately.

Step 1 – Time derivative terms:

$$\begin{aligned} \int_0^T \int_\Omega D_t r_h \Pi_\mathcal{T} \phi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \frac{r_h(t) - r_h(t-\Delta t)}{\Delta t} \phi(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{\Delta t}\int_0^T \int_\Omega r_h(t)\phi(t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\Delta t}\int_{-\Delta t}^{T-\Delta t} \int_\Omega r_h(t)\phi(t+\Delta t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega r_h(t) D_t\phi(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\Delta t}\int_{T-\Delta t}^{T} \int_\Omega r_h(t)\phi(t+\Delta t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \frac{1}{\Delta t}\int_{-\Delta t}^{0} \int_\Omega r_h(t)\phi(t+\Delta t)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_\Omega r_h(t) D_t\phi(t)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega r_h^0 \phi(0)\,\mathrm{d}x \\ &= -\int_0^T \int_\Omega r_h(t) \partial_t\phi(t)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega r_h^0 \phi(0)\,\mathrm{d}x + \Delta t\,\|\phi\|_{C^2}\,\|r_h\|_{L^1L^1}, \end{aligned}$$

where r_h stands for ϱ_h or $\varrho_h u_{i,h}$, $i = 1,\ldots,d$. Recalling the estimates (3.5b) and (3.5c) we know that

$$\|\varrho_h\|_{L^1L^1} \lesssim \|\varrho_h\|_{L^\infty L^\gamma} \lesssim 1 \quad \text{and} \quad \|\varrho_h \mathbf{u}_h\|_{L^1L^1} \lesssim \|\varrho_h \mathbf{u}_h\|_{L^\infty L^{\frac{2\gamma}{\gamma+1}}} \lesssim 1.$$

Thus, we have

$$\int_0^T \int_\Omega D_t \varrho_h \Pi_\mathcal{T} \phi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varrho_h(t) \partial_t \phi(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_h^0 \phi(0)\,\mathrm{d}x \lesssim \Delta t, \tag{4.3a}$$

$$\int_0^T \int_\Omega D_t(\varrho_h \mathbf{u}_h) \Pi_\mathcal{T} \boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varrho_h(t)\mathbf{u}_h(t) \partial_t \boldsymbol{\phi}(t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_h^0 \mathbf{u}_h^0 \boldsymbol{\phi}(0)\,\mathrm{d}x \lesssim \Delta t, \tag{4.3b}$$

for the continuity and the momentum equations, respectively.

Step 2 – Convective terms:
To deal with the convective terms, it is convenient to recall Lemma 2.5:

$$\int_0^T \int_\Omega r_h \mathbf{u}_h \cdot \nabla_x \phi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma F[r_h, \mathbf{u}_h] \llbracket \Pi_{\mathcal{T}} \phi \rrbracket \,\mathrm{d}S(x)\,\mathrm{d}t = \sum_{j=1}^4 E_j(r_h),$$

where

$$\begin{aligned}
E_1(r_h) &= \frac{1}{2}\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}| \llbracket r_h \rrbracket \llbracket \Pi_{\mathcal{T}} \phi \rrbracket \,\mathrm{d}S(x)\,\mathrm{d}t,\\
E_2(r_h) &= \frac{1}{4}\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \llbracket \mathbf{u}_h \rrbracket\cdot\mathbf{n} \llbracket r_h \rrbracket \llbracket \Pi_{\mathcal{T}} \phi \rrbracket \,\mathrm{d}S(x)\,\mathrm{d}t,\\
E_3(r_h) &= \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma h^\varepsilon \llbracket r_h \rrbracket \llbracket \Pi_{\mathcal{T}} \phi \rrbracket \,\mathrm{d}S(x)\,\mathrm{d}t,\\
E_4(r_h) &= \int_0^T \int_\Omega r_h \mathbf{u}_h \cdot \Big(\nabla_x \phi - \nabla_{\mathcal{T}} \Pi_{\mathcal{D}} \big(\Pi_{\mathcal{T}} \phi\big)\Big) \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

are the error terms to be estimated. Again, r_h is either ϱ_h or $\varrho_h u_{i,h}$, $i = 1, \dots, d$.

- Firstly, for the error term E_1 we can write

$$\begin{aligned}
E_1(r_h) &= \frac{1}{2}\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{\mathbf{u}_h}\cdot\mathbf{n}| \llbracket r_h \rrbracket \llbracket \Pi_{\mathcal{T}} \phi \rrbracket \,\mathrm{d}S(x)\,\mathrm{d}t = \frac{1}{2}\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\overline{u_{i,h}}| \llbracket r_h \rrbracket \llbracket \Pi_{\mathcal{T}} \phi \rrbracket \,\mathrm{d}S(x)\,\mathrm{d}t\\
&= \frac{1}{2}\int_0^T \sum_{i=1}^d \sum_{\sigma\in\mathcal{E}_i} \int_{D_\sigma} h_i |\overline{u_{i,h}}| \eth_{\mathcal{E}}^{(i)} r_h \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \phi \,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\int_0^T \sum_{i=1}^d \sum_{K\in\mathcal{T}} \int_K h_i r_h \eth_{\mathcal{T}}^{(i)} \left(|\overline{u_{i,h}}| \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \phi\right) \,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\int_0^T \sum_{i=1}^d \sum_{K\in\mathcal{T}} \int_K r_K h_i \left(\Pi_{\mathcal{T}} |\overline{u_{i,h}}|\, \eth_{\mathcal{T}}^{(i)} (\eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \phi) + \left(\eth_{\mathcal{T}}^{(i)} |\overline{u_{i,h}}|\right) \Pi_{\mathcal{T}} (\eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}} \phi)\right) \mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where we have used the integration by parts formula (2.10b), the product rule

$$r_2 q_2 - r_1 q_1 = \frac{r_1 + r_2}{2}(q_2 - q_1) + \frac{q_1 + q_2}{2}(r_2 - r_1).$$

Further, employing the inequality $\left(\frac{a+b}{2}\right)^2 \le \frac{a^2+b^2}{2}$ twice, we claim $\|\Pi_{\mathcal{T}} |\overline{u_{i,h}}|\|_{L^2} \lesssim \|u_{i,h}\|_{L^2}$. Similarly, we claim $\left\|\eth_{\mathcal{T}}^{(i)} \overline{u_{i,h}}\right\|_{L^2} \lesssim \left\|\eth_{\mathcal{E}}^{(i)} u_{i,h}\right\|_{L^2}$ as $\left(\eth_{\mathcal{T}}^{(i)} \overline{u_{i,h}}\right)_K = \Pi_{\mathcal{T}} \left(\eth_{\mathcal{E}}^{(i)} u_{i,h}\right)_K$. Then applying Hölder's inequality, interpolation error estimates (2.12), (2.14), the velocity estimates (3.5d), (3.5f), the fact $|\partial_x u_i| \ge \partial_x |u_i|$, and noticing $\Delta_h^{(i)} r := \eth_{\mathcal{T}}^{(i)} \eth_{\mathcal{E}}^{(i)} r$, we derive

$$
\begin{aligned}
E_1(r_h) = & -\frac{1}{2}\int_0^T \sum_{i=1}^d \sum_{K\in\mathcal{T}} \int_K r_K h_i \Big(\Pi_{\mathcal{T}} |\overline{u_{i,h}}|\, \eth_{\mathcal{T}}^{(i)} (\eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}}\phi) + \Big(\eth_{\mathcal{T}}^{(i)} |\overline{u_{i,h}}|\Big) \Pi_{\mathcal{T}} (\eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}}\phi) \Big) \,\mathrm{d}x\,\mathrm{d}t \\
& \lesssim \sum_{i=1}^d h_i \left(\int_0^T \sum_K \int_K r_K^2 \right)^{1/2} \Bigg[\left(\int_0^T \sum_{K\in\mathcal{T}} \int_K (\Pi_{\mathcal{T}} |\overline{u_{i,h}}|)_K^2 \right)^{1/2} \left\| \Delta_h^{(i)} \Pi_{\mathcal{T}}\phi \right\|_{L^\infty L^\infty} \\
& \qquad + \left(\int_0^T \sum_{K\in\mathcal{T}} \int_K \left(\eth_{\mathcal{T}}^{(i)} \overline{u_{i,h}} \right)^2 \right)^{1/2} \left\| \Pi_{\mathcal{T}} (\eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}}\phi) \right\|_{L^\infty L^\infty} \Bigg] \\
& \lesssim h \sum_{i=1}^d \|r_h\|_{L^2L^2} \left(\left\| \Delta_h^{(i)} \Pi_{\mathcal{T}}\phi \right\|_{L^\infty L^\infty} \|u_{i,h}\|_{L^2L^2} + \left\| \eth_{\mathcal{E}}^{(i)} \Pi_{\mathcal{T}}\phi \right\|_{L^\infty L^\infty} \left\| \eth_{\mathcal{E}}^{(i)} u_{i,h} \right\|_{L^2L^2} \right) \\
& \lesssim h \|r_h\|_{L^2L^2} \left(\|\Delta_h \Pi_{\mathcal{T}}\phi\|_{L^\infty L^\infty} \|\mathbf{u}_h\|_{L^2L^2} + \|\nabla_{\mathcal{E}} \Pi_{\mathcal{T}}\phi\|_{L^\infty L^\infty} \|\nabla_{\mathcal{E}} \mathbf{u}_h\|_{L^2L^2} \right) \\
& \lesssim h \|r_h\|_{L^2L^2}.
\end{aligned}
$$

Consequently, applying the density estimate (3.6a), and the momentum estimate (3.6b) indicates

$$
E_1(r_h) \lesssim h^\beta, \quad \beta = 1 - \frac{\varepsilon+2}{2\gamma} > 0, \text{ provided } \varepsilon < 2(\gamma-1),
$$

for r_h being ϱ_h or $\varrho_h u_{i,h}$, $i = 1, \ldots, d$.

– Secondly, we deal with the error term E_2. In accordance with (2.11), we have

$$
E_2(r_h) \lesssim h \sum_{\sigma\in\mathcal{E}} \int_\sigma |\, [\![\mathbf{u}_h]\!] \cdot \mathbf{n}\, [\![r_h]\!] \,|\, \mathrm{d}S(x)\,\mathrm{d}t.
$$

For r_h being ϱ_h, we further write

$$
\begin{aligned}
E_2(\varrho_h) & \lesssim h \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\mathbf{u}_h]\!]^2 \,\mathrm{d}S(x)\,\mathrm{d}t \right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma [\![\varrho_h]\!]^2 \,\mathrm{d}S(x)\,\mathrm{d}t \right)^{1/2} \\
& \lesssim h h^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h}^2 \,\mathrm{d}S(x)\,\mathrm{d}t \right)^{1/2} \\
& \lesssim h^{3/2} h^{-1/2} \|\varrho_h\|_{L^2L^2} \lesssim h^\beta, \quad \beta = 1 - \frac{\varepsilon+2}{2\gamma} > 0, \text{ as soon as } \varepsilon < 2(\gamma-1).
\end{aligned}
$$

Here we have used Hölder's inequality, (3.5d), (3.6a), and the fact $|\, [\![\varrho_h]\!] \,| < 2\overline{\varrho_h}$.
For r_h being $\varrho_h u_{i,h}$, we get

$$
E_2(\varrho_h \mathbf{u}_h) \lesssim h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\, [\![\mathbf{u}_h]\!] \cdot \mathbf{n}| \;\; |[\![\varrho_h]\!]\, \overline{\mathbf{u}_h} + [\![\mathbf{u}_h]\!]\, \overline{\varrho_h}| \,\mathrm{d}S(x)\,\mathrm{d}t := T_1 + T_2.
$$

To control the residual term T_1 we apply Hölder's inequality, (3.5a), (3.5g), inverse estimate (2.15) and the inequality $|\, [\![\varrho_h]\!] \,| < 2\overline{\varrho_h}$ to obtain

$$
\begin{aligned}
T_1 & \lesssim h \int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma |\, [\![\mathbf{u}_h]\!] \cdot \mathbf{n} | \overline{\varrho_h} | \overline{\mathbf{u}_h} | \,\mathrm{d}S(x)\,\mathrm{d}t \\
& \lesssim h \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h}\, [\![\mathbf{u}_h]\!]^2 \,\mathrm{d}S(x) \right)^{1/2} \left(\int_0^T \sum_{\sigma\in\mathcal{E}} \int_\sigma \overline{\varrho_h} |\overline{\mathbf{u}_h}|^2 \,\mathrm{d}S(x) \right)^{1/2} \\
& \lesssim h^{(1-\varepsilon)/2}.
\end{aligned}
$$

Further, applying (3.5g) we can control the residual term T_2 as

$$T_2 = h \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma | [\![\mathbf{u}_h]\!] \cdot \mathbf{n} | \, [\![\mathbf{u}_h]\!] \, |\overline{\varrho_h} \, \mathrm{d}S(x) \, \mathrm{d}t \lesssim h^{1-\varepsilon}.$$

Therefore, we claim that provided $\varepsilon < 2(\gamma - 1)$ we have

$$E_2(r_h) \lesssim h^\beta, \quad \beta > 0$$

for r_h being ϱ_h or $\varrho_h u_{i,h}$, $i = 1, \dots, d$.

– Next, we consider the error term E_3. Analogously as above, the integration by parts formula (2.10a), Hölder's inequality, and the interpolation error (2.14) yield

$$\begin{aligned} E_3(r_h) &= h^\varepsilon \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma [\![r_h]\!] \, [\![\Pi_\mathcal{T} \phi]\!] \, \mathrm{d}S(x) \, \mathrm{d}t = -h^{\varepsilon+1} \int_0^T \int_\Omega r_h \Delta_h \Pi_\mathcal{T} \phi \, \mathrm{d}x \, \mathrm{d}t \\ &\lesssim h^{\varepsilon+1} \left\| r_h \right\|_{L^1 L^1} \left(\left\| \phi \right\|_{C^2} + h \left\| \phi \right\|_{C^3} \right) \lesssim h^{\varepsilon+1} \left\| r_h \right\|_{L^1 L^1}. \end{aligned}$$

Using the estimates (3.5b) and (3.5c) we can conclude for r_h being ϱ_h or $\varrho_h u_{i,h}$, $i = 1, \dots, d$, that

$$E_3(r_h) \lesssim h^{\varepsilon+1}.$$

– Finally, using the estimates of kinetic energy (3.5a) and momentum (3.5c) together with the interpolation error (2.13) we obtain for r_h being ϱ_h or $\varrho_h u_{i,h}$, $i = 1, \dots, d$ that

$$E_4(r_h) = \int_0^T \int_\Omega r_h \mathbf{u}_h \cdot \left(\nabla_x \phi - \nabla_\mathcal{T} \Pi_\mathcal{D} \big(\Pi_\mathcal{T} \phi \big) \right) \, \mathrm{d}x \, \mathrm{d}t \lesssim h \left\| \phi \right\|_{C^2} \left\| r_h \mathbf{u}_h \right\|_{L^1 L^1} \lesssim h \left\| r_h \mathbf{u}_h \right\|_{L^\infty L^1} \lesssim h.$$

Consequently, we conclude the consistency formulation of the convective terms in both equations (2.3a) and (2.3b), by collecting the above estimates of the four terms E_j, $j = 1, \dots, 4$,

$$\int_\Omega \varrho_h \mathbf{u}_h \cdot \nabla_x \phi \, \mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma F[\varrho_h, \mathbf{u}_h] \, [\![\Pi_\mathcal{T} \phi]\!] \, \mathrm{d}S(x) \lesssim h^{\beta_1}, \tag{4.4a}$$

$$\int_\Omega \varrho_h \mathbf{u}_h \otimes \mathbf{u}_h : \nabla_x \boldsymbol{\phi} \, \mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma F[\varrho_h \mathbf{u}_h, \mathbf{u}_h] \, [\![\Pi_\mathcal{T} \boldsymbol{\phi}]\!] \, \mathrm{d}S(x) \lesssim h^{\beta_2}, \tag{4.4b}$$

for some β_1, $\beta_2 > 0$ provided $0 < \varepsilon < \min\{1, 2(\gamma - 1)\}$.

Remark 4.2. We would like to emphasize that the additional numerical diffusion of order $\mathcal{O}(h^{\varepsilon+1})$ plays a crucial role in order to obtain the consistency of the convective terms, which requires $\varepsilon < \min\{1, 2(\gamma - 1)\}$. Indeed the deriviation of (4.4) requires (3.5g) and uses Lemma 3.1, which benefits from the uniform bounds (3.5h) of the h^ε-terms.

Step 3 – Viscosity terms:
In accordance with (2.12) and (3.5d) we can control the viscosity terms. Indeed, we have

$$\begin{aligned} &\int_0^T \int_\Omega \nabla_\mathcal{E} \mathbf{u}_h : \nabla_x \boldsymbol{\phi} \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \frac{1}{d_\sigma} [\![\mathbf{u}_h]\!] \cdot [\![\Pi_\mathcal{T} \boldsymbol{\phi}]\!] \, \mathrm{d}S(x) \, \mathrm{d}t \\ &= \int_0^T \int_\Omega \nabla_\mathcal{E} \mathbf{u}_h : \left(\nabla_x \boldsymbol{\phi} - \nabla_\mathcal{E} \Pi_\mathcal{T} \boldsymbol{\phi} \right) \mathrm{d}x \, \mathrm{d}t \lesssim \left\| \nabla_\mathcal{E} \mathbf{u}_h \right\|_{L^2 L^2} h \left\| \boldsymbol{\phi} \right\|_{C^2} \lesssim h, \end{aligned} \tag{4.5a}$$

and for the divergence term we get

$$
\begin{aligned}
&\int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h \, \mathrm{div}_h \big(\Pi_{\mathcal{T}} \boldsymbol{\phi}\big) \,\mathrm{d}x - \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h \, \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \mathrm{div}_h \mathbf{u}_h \Big(\mathrm{div}_h \big(\Pi_{\mathcal{T}} \boldsymbol{\phi}\big) - \mathrm{div}_x \boldsymbol{\phi} \Big) \,\mathrm{d}x \,\mathrm{d}t \lesssim \|\mathrm{div}_h \mathbf{u}_h\|_{L^2 L^2} \, h \, \|\boldsymbol{\phi}\|_{C^2} \lesssim h,
\end{aligned}
\tag{4.5b}
$$

by using (3.5e) and (2.13).

Step 4 – Pressure term:
The pressure term can be controlled by using the integration by parts formula (2.9), the interpolation error (2.13), and the estimate (3.5b), *i.e.*,

$$
\begin{aligned}
&\int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{p}_h \mathbf{n} \cdot [\![\Pi_{\mathcal{T}} \boldsymbol{\phi}]\!] \,\mathrm{d}S(x) \,\mathrm{d}t - \int_0^T \int_\Omega p_h \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad = - \int_0^T \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{\Pi_{\mathcal{T}} \boldsymbol{\phi}} \cdot \mathbf{n} [\![p_h]\!] \,\mathrm{d}S(x) \,\mathrm{d}t - \int_0^T \sum_{K \in \mathcal{T}} \int_K p_h \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad = \int_0^T \sum_{K \in \mathcal{T}} p_K \sum_{\sigma \in \mathcal{E}(K)} \int_\sigma \overline{\Pi_{\mathcal{T}} \boldsymbol{\phi}} \cdot \mathbf{n} \,\mathrm{d}S(x) \,\mathrm{d}t - \int_0^T \sum_{K \in \mathcal{T}} \int_K p_h \mathrm{div}_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad = \int_0^T \sum_{K \in \mathcal{T}} \int_K p_h \left(\mathrm{div}_h \big(\Pi_{\mathcal{T}} \boldsymbol{\phi}\big) - \mathrm{div}_x \boldsymbol{\phi} \right) \,\mathrm{d}x \,\mathrm{d}t \lesssim \|p_h\|_{L^\infty L^1} \, h \, \|\boldsymbol{\phi}\|_{C^2} \lesssim h.
\end{aligned}
\tag{4.6}
$$

Collecting the inequalities (4.3)–(4.6) we complete the proof of Theorem 4.1. □

5. Convergence

In this section, we show the main result, the convergence of the numerical solution to the strong solution of the system (1.1) on the lifespan of the latter. To this end we start by introducing the concept of the dissipative measure-valued (DMV) solutions to (1.1). The interested reader may consult [11] for the discussion about the concept of DMV solutions and the DMV–strong uniqueness principle that will be used later in this section.

Definition 5.1 (DMV solution)**.** We say that a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x) \in (0,T) \times \Omega}$,

$$
\mathcal{V}_{t,x} \in L^\infty_{\text{weak}}\Big((0,T) \times \Omega; \mathcal{P}(Q)\Big), \ Q = \Big\{ [\varrho, \mathbf{u}] \ \Big| \ \varrho \in [0,\infty), \ \mathbf{u} \in R^N \Big\},
$$

is *a dissipative measure-valued (DMV) solution* of the Navier–Stokes system in $(0,T) \times \Omega$, with the initial condition $\mathcal{V}_{0,x} \in \mathcal{P}(Q)$ and dissipation defect $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \geq 0$, if the following holds:

–

$$
\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho \rangle \phi(t,\cdot) \,\mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\langle \mathcal{V}_{t,x}; \varrho \rangle \partial_t \phi + \langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \rangle \cdot \nabla_x \phi] \,\mathrm{d}x \,\mathrm{d}t
$$

for any $0 \leq \tau \leq T$ and $\phi \in C^1\big([0,T] \times \Omega\big)$;

–

$$
\begin{aligned}
\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \rangle \boldsymbol{\phi}(t,\cdot) \,\mathrm{d}x \right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega [\langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \rangle \partial_t \boldsymbol{\phi} + \langle \mathcal{V}_{t,x}; \varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \rangle : \nabla_x \boldsymbol{\phi}] \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \int_0^\tau \int_\Omega \mathcal{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\phi} \,\mathrm{d}x \,\mathrm{d}t + \int_0^\tau \langle r^M; \nabla_x \boldsymbol{\phi} \rangle \,\mathrm{d}t
\end{aligned}
$$

for any $0 \leq \tau \leq T$ and $\boldsymbol{\phi} \in C_c^1\big([0,T] \times \Omega; R^d\big)$, where

$$\mathbf{u} = \langle \mathcal{V}_{t,x}; \mathbf{u} \rangle\,,\ \mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^d)),$$
$$\mathcal{S}(\nabla_x \mathbf{u}) = \mu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda \mathrm{div}_x \mathbf{u}\mathbb{I},\ \text{and}\ r^M \in L^1\big(0,T; \mathcal{M}(\Omega)\big);$$

–

$$\left[\int_\Omega \langle \mathcal{V}_{t,x}; \frac{1}{2}\varrho \mathbf{u}^2 + \mathcal{H}(\varrho)\rangle \,\mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathcal{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t + \mathcal{D}(\tau) \leq 0,$$

for a.a. $0 \leq \tau \leq T$. The dissipation defect $\mathcal{D}$ dominates the concentration measure r^M, specifically,

$$\left|\langle r^M(\tau); \phi\rangle\right| \lesssim \xi(\tau)\mathcal{D}(\tau)\left\|\phi\right\|_{C(\Omega)}\,,\ \text{for some}\ \xi \in L^1(0,T).$$

5.1. Convergence to dissipative measure-valued solution

In this subsection, we show that any Young measure generated by a family of numerical solutions is a DMV solution in the sense of Definition in 5.1.

Theorem 5.2. *Let $\{(\varrho_h^k, \mathbf{u}_h^k)\}_{k=1}^{N_T}$ be a family of solutions generated by the numerical scheme (2.3), with $\Delta t \approx h$, $1 < \gamma < 2$, $0 < \varepsilon < \min\{1, 2(\gamma-1)\}$, and the initial data satisfying*

$$\varrho_0 \in L^\gamma(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in L^2(\Omega; R^d).$$

Then any Young measure $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ generated by $(\varrho_h^k, \mathbf{u}_h^k)$ for $h \to 0$ represents a dissipative measure-valued solution of the Navier–Stokes system (1.1) in the sense of Definition 5.1.

Proof. We may use the energy estimates (3.3) to deduce that, at least for suitable subsequences,

$$\begin{aligned}
\varrho_h &\to \varrho \ \text{weakly-(*) in}\ L^\infty(0,T; L^\gamma(\Omega)),\ \varrho \geq 0\\
\mathbf{u}_h &\to \mathbf{u}\ \text{weakly in}\ L^2((0,T)\times\Omega; R^d),\\
\text{where}\ \mathbf{u} &\in L^2(0,T; W^{1,2}(\Omega)),\ \nabla_{\mathcal{E}} \mathbf{u}_h \to \nabla_x \mathbf{u}\ \text{weakly in}\ L^2((0,T)\times\Omega; R^{d\times d}),\\
\varrho_h \mathbf{u}_h &\to \widetilde{\varrho \mathbf{u}}\ \ \text{weakly-(*) in}\ L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)).
\end{aligned}$$

where the superscript "$\sim$" denotes the L^1-weak limit.

Note that, the limit functions satisfy the equation of continuity in the form

$$-\int_\Omega \varrho_0 \phi(0,\cdot)\,\mathrm{d}x = \int_0^T \int_\Omega [\varrho \partial_t \phi + \widetilde{\varrho \mathbf{u}} \cdot \nabla_x \phi]\,\mathrm{d}x\,\mathrm{d}t,\ \ \text{for all}\ \phi \in C_c^\infty([0,\infty)\times\Omega),$$

which can be further rewritten as

$$\left[\int_\Omega \varrho\phi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho \partial_t \phi + \widetilde{\varrho \mathbf{u}} \cdot \nabla_x \phi]\,\mathrm{d}x\,\mathrm{d}t \tag{5.1}$$

for any $0 \leq \tau \leq T$ and any $\phi \in C^\infty([0,T]\times\Omega)$.

In accordance with the weak convergence statement derived in the preceding part, the family $[\varrho_h, \mathbf{u}_h]$ generates a Young measure – a parameterized measure [2, 30]

$$\mathcal{V}_{t,x} \in L^\infty((0,T)\times\Omega; \mathcal{P}([0,\infty)\times R^d))\ \text{for a.e.}\ (t,x)\in(0,T)\times\Omega,\quad \text{with}\quad \mathcal{V}_{0,x} = \delta_{[\varrho_0(x), \mathbf{u}_0(x)]},$$

such that

$$\langle \mathcal{V}_{t,x}, g(\varrho, \mathbf{u})\rangle = \widetilde{g(\varrho, \mathbf{u})}(t,x)\ \text{for a.e.}\ (t,x)\in(0,T)\times\Omega,$$

for any $g \in C([0,\infty) \times R^d)$ such that

$$g(\varrho_h, \mathbf{u}_h) \to \widetilde{g(\varrho, \mathbf{u})} \text{ weakly in } L^1((0,T)\times\Omega).$$

Accordingly, the equation of continuity (5.1) can be written as

$$\left[\int_\Omega \varrho\phi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\phi + \langle \mathcal{V}_{t,x}, \varrho\mathbf{u}\rangle \cdot \nabla_x\phi]\,\mathrm{d}x\,\mathrm{d}t. \tag{5.2}$$

For the consistency formulation of the momentum equation (4.2), we apply a similar treatment. Whence letting $h \to 0$ in (4.2) gives rise to

$$\begin{aligned}\left[\int_\Omega \langle \mathcal{V}_{t,x}; \varrho\mathbf{u}\rangle \cdot \boldsymbol{\phi}(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = &\int_0^\tau \int_\Omega \Big[\langle \mathcal{V}_{t,x}; \varrho\mathbf{u}\rangle \cdot \partial_t\boldsymbol{\phi} + \langle \mathcal{V}_{t,x}; \varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}\rangle : \nabla_x\boldsymbol{\phi}\Big]\,\mathrm{d}x\ \ \mathrm{d}t\\ &- \int_0^\tau \int_\Omega \Big[\mu\nabla_x\mathbf{u} : \nabla_x\boldsymbol{\phi} + (\mu+\lambda)\mathrm{div}_x\mathbf{u}\cdot\mathrm{div}_x\boldsymbol{\phi}\Big]\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega r^M : \nabla_x\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{5.3}$$

for any $0 \le \tau \le T$, $\boldsymbol{\phi} \in C_c^\infty([0,T]\times\Omega; R^d)$ where the *concentration remainder* reads

$$r^M = \{\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}\} - \langle \mathcal{V}_{t,x}; \varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}\rangle \in [L^\infty(0,T;\mathcal{M}(\Omega))]^{d\times d}.$$

Hereafter $\{s\}$ denotes the weak-(*) limit of a sequence s_h in $\mathcal{M}([0,T]\times\Omega))$.

Similarly, letting $h \to 0$ in the energy inequality (3.3) yields

$$\left[\int_\Omega \left\langle \mathcal{V}_{t,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + \mathcal{H}(\varrho)\right\rangle \mathrm{d}x\right]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \left(\mu|\nabla_x\mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}_x\mathbf{u}|^2\right)\mathrm{d}x\,\mathrm{d}t + \mathcal{D}(\tau) \le 0, \tag{5.4}$$

for a.e. $\tau \in [0,T]$, with the *dissipation defect*

$$\begin{aligned}\mathcal{D}(\tau) = &\int_\Omega \left\{\frac{1}{2}\varrho|\mathbf{u}|^2 + \mathcal{H}(\varrho)\right\}\mathrm{d}x - \int_\Omega \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + \mathcal{H}(\varrho)\right\rangle \mathrm{d}x\\ &+ \int_0^\tau\int_\Omega \left\{\mu|\nabla_x\mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}_x\mathbf{u}|^2\right\}\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\int_\Omega \mu|\nabla_x\mathbf{u}|^2 + (\mu+\lambda)|\mathrm{div}_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

satisfying

$$\mathcal{D}(\tau) \ge \liminf_{h\to 0}\left(\int_0^\tau \|\nabla_\mathcal{E}\mathbf{u}_h\|_{L^2}^2\ \mathrm{d}t\right) - \int_0^\tau\int_\Omega |\nabla_x\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{5.5}$$

Note that $\mathcal{D} \ge 0$ is a consequence of ([11], Lem. 2.1) and the non–negativity of the total energy and $\mathcal{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}$. Since $|\varrho_h\mathbf{u}_h\otimes\mathbf{u}_h + p(\varrho_h)\mathbb{I}| \lesssim \varrho_h|\mathbf{u}_h|^2 + \mathcal{H}(\varrho_h)$, we again recall ([11], Lem. 2.1) for

$$F(\varrho,\mathbf{u}) = |\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}|, \quad \text{and} \quad G(\varrho,\mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \mathcal{H}(\varrho)$$

to conclude

$$\int_\Omega 1\,|\mathrm{d}\,r^M| \lesssim \mathcal{D}, \quad \text{for a.e. in } (0,T). \tag{5.6}$$

Collecting (5.2)–(5.6) implies that the Young measure $\{\mathcal{V}_{t,x}\}_{t,x\in(0,T)\times\Omega}$ represents a dissipative measure-valued solution of the Navier–Stokes system (1.1) in the sense of Definition 5.1. Seeing that validity of (5.2) and (5.3) can be extended to the class of test functions from $C^1([0,T]\times\Omega;R^d)$, we have proved Theorem 5.2. $\square$

5.2. Convergence to strong solution

In the previous subsection, we have shown that the numerical solution generates the dissipative measure-valued solution. We admit that the conclusion of Theorem 5.2 is rather weak, also due to the non-uniqueness of Young measure. However, we may directly use the DMV-strong uniqueness principle established in ([11], Thm. 4.1) to obtain convergence to the strong solution as long as it exists.

Theorem 5.3 (Convergence to strong solution)**.** *In addition to the hypotheses of Theorem 5.2, suppose that the Navier–Stokes system (1.1) endowed with the initial data* $(\varrho_0, \mathbf{u}_0)$ *admits a strong solution* $(\varrho, \mathbf{u})$ *belonging to the class*

$$\varrho, \nabla_x \varrho, \mathbf{u}, \nabla_x \mathbf{u} \in C([0,T] \times \Omega), \ \partial_t \mathbf{u} \in L^2\left(0,T; C(\Omega; R^d)\right), \ \varrho > 0.$$

Then

$$\varrho_h \to \varrho\,(\text{strongly})\,\text{in}\,L^\gamma\left((0,T)\times\Omega\right), \ \mathbf{u}_h \to \mathbf{u}\,(\text{strongly})\,\text{in}\,L^2\left((0,T)\times\Omega; R^d\right).$$

Remark 5.4. By strong solution we mean that all generalized derivatives appearing in the equations can be identified with integrable functions, see [11]. In particular, this typically amounts to requiring the functions to be at least Lipschitz continuous.

Indeed, the DMV–strong uniqueness implies that the Young measure generated by the family of numerical solutions coincides at a.a. point (t, x) with the Dirac mass supported by the smooth solution of the problem. In particular, the numerical solutions converge strongly and no oscillations occur.

Remark 5.5. We have constructed solution on a space-periodic domain Ω. When considering a polyhedral domain, the existence of smooth solutions remains open and may be a delicate task. To avoid this problem, one has to approximate a smooth domain by a family of polyhedral domains analogously as in [13]. Note, however, this problem does not occur in the case of periodic domain.

If, in addition, we assume the density is uniformly bounded, meaning independently of the numerical step, the results of Theorems 5.2 and 5.3 may be possibly extended to an unstructured grid. Indeed the only difference of the proof would be showing the consistency of the convective terms in (4.4). The estimate of the error terms $E_1(\varrho_h)$ and $E_1(\varrho_h\mathbf{u}_h)$ could be done without the discrete integration by parts thanks to $L^\infty-$bound on the density. Moreover, in view of the conditional regularity result [32], we obtain the unconditional convergence to the strong solution since the DMV solution with bounded density is regular.

Theorem 5.6 (Convergence with bounded density)**.** *Let* $d = 3$*. In addition to the hypotheses of Theorem 5.2, suppose that*

- *the initial data belong to the class*

$$\varrho_0 \in W^{3,2}(\Omega), \ \mathbf{u}_0 \in W^{3,2}(\Omega; R^d);$$

- *bulk viscosity vanishes, meaning*

$$\lambda + \frac{2}{3}\mu = 0;$$

-

$$\|\varrho_h\|_{L^\infty((0,T)\times\Omega)} \le c$$

uniformly for $h \to 0$*.*

Then

$$\varrho_h \to \varrho\,(\text{strongly})\,\text{in}\,L^q\left((0,T)\times\Omega\right), \ q \ge 1, \ \mathbf{u}_h \to \mathbf{u}\,(\text{strongly})\,\text{in}\,L^2\left((0,T)\times\Omega; R^d\right),$$

$(\varrho, \mathbf{u})$ *is the strong solution to the Navier–Stokes system (1.1) with the initial data* $(\varrho_0, \mathbf{u}_0)$*.*

TABLE 1. Numerical convergence for Experiment 6.1.

h	$\|e_{\nabla_x \mathbf{u}}\|_{L^2(L^2)}$	EOC	$\|e_{\mathbf{u}}\|_{L^2(L^2)}$	EOC	$\|e_{\varrho}\|_{L^1(L^1)}$	EOC	$\|e_{\varrho}\|_{L^\infty(L^\gamma)}$	EOC
1/32	4.21e−02	–	3.43e−03	–	1.24e−03	–	4.28e−02	–
1/64	1.78e−02	1.24	1.39e−03	1.30	4.95e−04	1.32	1.81e−02	1.24
1/128	7.75e−03	1.20	5.88e−04	1.24	2.04e−04	1.28	7.86e−03	1.21
1/256	3.51e−03	1.14	2.59e−04	1.18	8.69e−05	1.23	3.50e−03	1.17

The condition on vanishing bulk viscosity is technical and we refer to [32] for the discussion of its necessity. We point out that Theorem 5.6 guarantees *unconditional* convergence of the scheme without the *a priori* hypothesis of the existence of smooth solution. In other words, uniform boundedness of the numerical densities implies the existence of global smooth solution as long as the initial data are sufficiently regular. It is also worth noting that boundedness of the numerical densities is still a considerably weaker assumptions than the hypothesis made by Jovanović [25].

6. Numerical experiment

In this section we show the numerical performance of scheme (2.3) in two space dimensions. Note that scheme (2.3) is nonlinear, thus we solve it numerically by a fixed-point iteration. For each sub-iteration, we set the time step as $\Delta t = \text{CFL}\frac{h}{(|u|+c)_{\max}}$, where CFL $= 0.3$, $c = \sqrt{\gamma p/\rho}$. We set the viscosity coefficients $\mu = \lambda = 0.01$ and the adiabatic coefficient $\gamma = 1.4$ in all experiments. Moreover, we choose the artificial diffusion $\varepsilon = 0.6$ which satisfies the assumption of $0 < \varepsilon < \min\{1, 2(\gamma-1)\}$.

Experiment 6.1. First we validate the accuracy of the scheme by considering

$$\rho_{\text{ref}} = 2 + \cos(2\pi(x+y)), \quad \mathbf{u}_{\text{ref}} = \left(\frac{\sin(2\pi t)}{2+\cos(2\pi(x+y))}, -\frac{\sin(2\pi t)}{2+\cos(2\pi(x+y))} \right)^T,$$

with the corresponding driving force in the momentum equation.

We compute the relative error e_{ϕ_h} for $\phi \in \{\varrho, \mathbf{u}, \nabla \mathbf{u}\}$ in the corresponding norms, and the experimental order of convergence (EOC), where

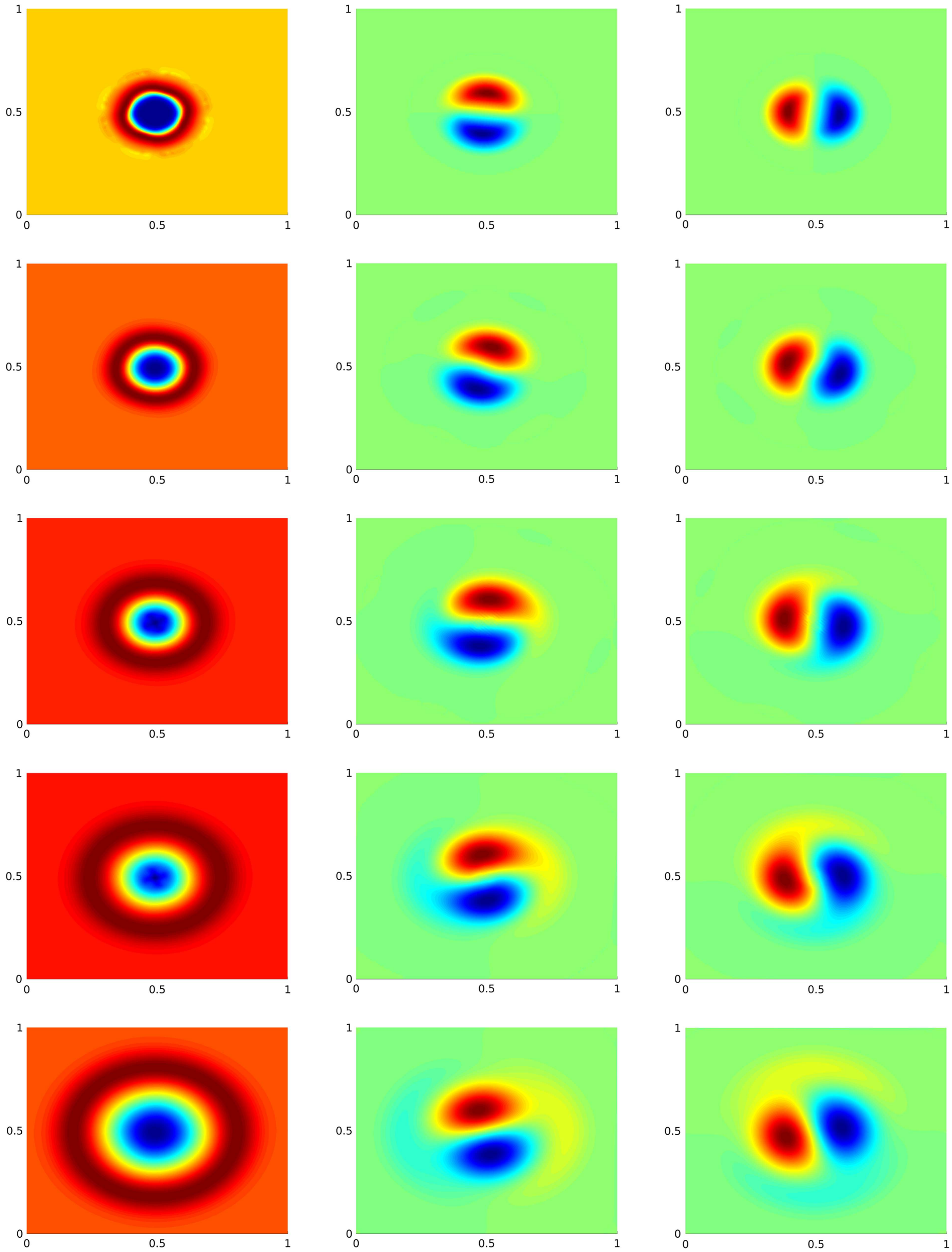
$$e_{\phi_h} = \frac{\|\phi_h - \phi_{\text{ref}}\|}{\|\phi_{\text{ref}}\|}, \quad \text{EOC} = \log_2 \frac{e_{\phi_{2h}}}{e_{\phi_h}},$$

and ϕ_{ref} denotes the reference solution. From the numerical results, we observe the first order of convergence of the scheme, see Table 1.

Experiment 6.2. In this experiment, we simulate the Gresho–vortex flow [5, 23, 29]. The initial state is the vortex of radius $r_0 = 0.2$ located at $(0.5, 0.5)$ with

$$\varrho(0,\mathbf{x}) = 1, \quad \mathbf{u}(0,\mathbf{x}) = \begin{pmatrix} y-0.5 \\ 0.5-x \end{pmatrix} \frac{u_r(r)}{r}, \quad \text{with} \quad u_r(r) = \sqrt{\gamma} \begin{cases} 2r/r_0 & \text{if } 0 \le r < r_0/2, \\ 2(1-r/r_0) & \text{if } r_0/2 \le r < r_0, \\ 0 & \text{if } r \ge r_0, \end{cases}$$

where $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$. We present the evolution of the flow in Figure 2 for the mesh size $h = 1/128$. We can clearly recognize that the solution is in a good agreement with those presented in the literature, see [23]. To further invest the numerical convergence, we present in Table 2 the errors for different mesh parameters and the reference solution is computed at a fine mesh $h = 1/2048$. We observe approximately first order convergence rate.



(a) density ϱ (b) velocity component u_1 (c) velocity component u_2

FIGURE 2. Time evolution of the Gresho–vortex: solution at $t = 0.01, 0.05, 0.1, 0.15, 0.2$ from top to bottom, solution of density and velocity components from left to right.

Table 2. Numerical convergence for Experiment 6.2.

h	$\|e_{\nabla_x \mathbf{u}}\|_{L^2(L^2)}$	EOC	$\|e_{\mathbf{u}}\|_{L^2(L^2)}$	EOC	$\|e_{\varrho}\|_{L^1(L^1)}$	EOC	$\|e_{\varrho}\|_{L^\infty(L^\gamma)}$	EOC
1/32	6.66e−01	–	3.16e−02	–	6.64e−04	–	1.64e−02	–
1/64	3.75e−01	0.83	1.66e−02	0.93	3.60e−04	0.88	8.85e−03	0.89
1/128	1.91e−01	0.97	8.21e−03	1.01	1.80e−04	1.00	4.43e−03	1.00
1/256	9.11e−02	1.07	3.86e−03	1.09	8.51e−05	1.08	2.09e−03	1.08
1/512	3.93e−02	1.21	1.66e−03	1.22	3.66e−05	1.22	8.96e−04	1.22

7. Conclusion

We have studied a finite volume method for the multi-dimensional compressible isentropic Navier–Stokes equations on regular quadrilateral mesh in a periodic domain. Due to the artificial diffusion in the numerical flux function (2.1) we have sufficiently strong *a priori* estimate on jumps of the discrete density. The solutions of the scheme were shown to exist while preserving the positivity of the discrete density. Moreover, we have shown the stability of the scheme by deriving the unconditional balance of the discrete total energy in Theorem 3.3. Furthermore, we have established the consistency formulation provided the artificial diffusion coefficient is large enough, see Theorem 4.1. In addition, we have shown in Theorem 5.2 that the numerical solutions of scheme (2.3) generate a DMV solution of the Navier–Stokes system (1.1). Finally, using the recent result on the DMV–strong uniqueness principle and the conditional regularity result [32], we have proven the convergence to the strong solution assuming only the existence of the latter, *cf.* Theorem 5.3, and the unconditional convergence to regular solution, *cf.* Theorem 5.6. Numerical experiments are also presented to support the theoretical results. To the best of our knowledge, this is the first rigorous result concerning convergence of a finite volume method for the compressible isentropic Navier–Stokes equations in the multi-dimensional setting.

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