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STABILITY OF NUMERICAL METHODS UNDER THE REGIME-SWITCHING JUMP-DIFFUSION MODEL WITH VARIABLE COEFFICIENTS

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Abstract. In this paper we introduce three numerical methods to evaluate the prices of European, American, and barrier options under a regime-switching jump-diffusion model (RSJD model) where the volatility and other parameters are considered as variable coefficients. The prices of the European option, which is one of the financial derivatives, are given by a partial integro-differential equation (PIDE) problem and those of the American option are evaluated by solving a linear complementarity problem (LCP). The proposed methods are constructed to avoid the use of any fixed point iteration techniques at each state of the economy and time step. We analyze the stability of the proposed methods with respect to the discrete ℓ^2-norm in the time and spatial variables. A variety of numerical experiments are carried out to show the second-order convergence of the three numerical methods under the regime-switching jump-diffusion model with variable coefficients.

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1. Introduction

The Black–Scholes model based on the assumption that an underlying asset follows the geometric Brownian motion with a constant volatility can not represent the stylized facts observed in the financial market. In order to overcome these deficiencies, a variety of stochastic models have been proposed such as a local volatility model [6,7], a regime-switching model [18], a jump-diffusion model [13,17], and a stochastic volatility model [9]. In this paper, we are interested in the regime-switching jump-diffusion model (RSJD model) that is the combination of the regime-switching model and the jump-diffusion model. Especially in order to incorporate the volatility smile, the time inhomogeneity, the abrupt regime change, and the deterministic volatility function as exhibited in the financial market, the parameters in the RSJD model are regarded as variable coefficients in the time and spatial variables.

There are a variety of numerical methods to solve the partial integro-differential equation (PIDE) for a European option and the linear complementarity problem (LCP) for an American option under the RSJD model. Costabile *et al.* [5] suggested an explicit formula and a multinomial approach for pricing the European and American options under the RSJD model. The multinomial approach with constant coefficients has the first-order accuracy in the time variable. A radial basis collocation method has been proposed by Bastani *et al.* [2]

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and also has the first-order convergence in the time variable. To keep the second-order accuracy in the time and spatial variables, an implicit method based on three time levels has been developed with constant parameters in [16]. A high order finite element method concerned with an exponential time integration (ETI) has been proposed in [19]. It is required to compute a matrix exponential and the inverse of a dense matrix in the ETI method. Fourier transform methods for pricing contingent claims have been studied in [11,20]. A Fourier space time-stepping (FST) algorithm in [11] is employed with the characteristic exponent of the log return.

The objective of this paper is to develop numerical methods to evaluate the prices of European, American, and barrier options under the RSJD model with variable coefficients. Although there are a number of research papers for pricing financial derivatives, the constant parameters under the RSJD model have been considered and so it is worthwhile to focus on the variable coefficients that enable us to reproduce financial market data comprehensively. We construct three numerical methods, called an implicit method (IM method), a Crank–Nicolson method (CN method), and a 2-step backward differentiation formula (BDF2 method), to solve the PIDE for the European option. It is pointed out in [21] that three implicit methods in which the zeroth derivative term is discretized implicitly yield better stable results as compared with the explicit treatment of the zeroth derivative term when the jump intensity is sufficiently large. According to [21], the three numerical methods we propose under the RSJD model with variable coefficients are designed for the linear system associated with the inverse of a tridiagonal matrix at each state of the economy and time step. So, it can be solved efficiently by using a LU decomposition. We prove that the developed methods are stable with the second-order convergence rate in the discrete ℓ^2-norm. The prices of the barrier option are evaluated by using the proposed methods with an appropriate boundary condition and those of the American option can be computed by the three numerical methods coupled with the operator splitting method in [10].

The rest of this paper is organized in the following way. In Section 2 we introduce the regime-switching jump-diffusion model with variable coefficients to deal with the local volatility with jumps. The PIDE for pricing a European option and the LCP for pricing an American option are described under the regime-switching jump-diffusion model. In Section 3 we propose three numerical methods to solve the PIDE with variable coefficients. In Section 4 it is shown that the three numerical methods are stable with respect to the discrete ℓ^2-norm and have the second-order convergence rate in the time and spatial variables. We then note that it is possible to apply these numerical methods to solve the LCP by combining the operator splitting method in Section 5. Some numerical experiments associated with the European, barrier, and American options are carried out to demonstrate the quadratic convergence in Section 6. Finally we summarize this paper with conclusions in Section 7.

2. Regime-switching jump-diffusion models

In order to incorporate multiple regimes and the behavior of volatility smiles, we consider a regime-switching jump-diffusion model (RSJD model) with variable coefficients. In this section we briefly introduce a RSJD model for an underlying asset S_t on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ in which the financial market allows us to have no arbitrage opportunities. A continuous Markov process X_t for $t \geq 0$ represents a state of an economy in a finite state space of Q-dimensional unit vectors $\mathcal{M} = \{e_1, e_2, \ldots, e_Q\}$ and then the Markov process X_t can be written by

$$X_t = X_0 + \int_0^t \mathcal{A} X_{s-} \mathrm{d}s + M_t,$$

where $\mathcal{A} = (\gamma_{qi})_{Q\times Q}$, called the rate matrix or the $\mathcal{Q}$-matrix, is a generator of the Markov process X_t and M_t is a martingale in the filtration generated by $(X_t)_{t\geq 0}$. A component $\gamma_{ij} \geq 0$ for $i \neq j$ in the generator $\mathcal{A} = (\gamma_{ij})_{Q\times Q}$ is concerned with the probability of $X_t = e_i$ given $X_t = e_j$ and $\gamma_{jj} = -\sum_{i\neq j} \gamma_{ij}$.

A stochastic process of an underlying asset S_t in the risk-neutral world is assumed as follows:

$$\mathrm{d}S_t/S_{t-} = (r_t - d_t - \lambda_t \zeta_t)\mathrm{d}t + \sigma_t \mathrm{d}W_t + \eta_t \mathrm{d}N_t, \tag{2.1}$$

where $r_t = \langle r, X_t\rangle$ is a risk-free interest rate with $r = (r_1, r_2, \dots, r_Q)^{\mathrm{T}}$, $d_t = \langle d, X_t\rangle$ is a continuous dividend rate with $d = (d_1, d_2, \dots, d_Q)^{\mathrm{T}}$, $\sigma_t = \langle \sigma, X_t\rangle$ is a volatility of the stock price S_t with $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_Q)^{\mathrm{T}}$ satisfying a condition $\sigma_i > 0$ for $1 \le i \le Q$, W_t is a Wiener process, $N_t = \langle \mathbf{N}_t, X_t\rangle$ with $\mathbf{N}_t = (N_t^1, N_t^2, \dots, N_t^Q)^{\mathrm{T}}$ is a Poisson process having intensity $\lambda_t = \langle \lambda, X_t\rangle$ at the economic state X_t with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_Q)^{\mathrm{T}}$, $\eta_t = \langle \eta, X_t\rangle$ is a random variable to determine jump sizes from S_{t-} to S_t with $\eta = (\eta_1, \eta_2, \dots, \eta_Q)^{\mathrm{T}}$, and $\zeta_t = \langle \zeta, X_t\rangle$ with $\zeta = E[\eta]$. We enforce the rules that all stochastic processes $W_t, X_t, N_t^1, N_t^2, \dots, N_t^Q$ are mutually independent and the jumps of two processes X_t and $\mathbf{N}_t$ do not occur at the same time almost surely. For more details, see [8,16].

The price of the European option $u(\tau, x, e_i)$ in the risk-neutral RSJD model (2.1) can be evaluated by solving the partial integro-differential equation (PIDE)

$$u_\tau(\tau, x, e_i) - \mathcal{L}u(\tau, x, e_i) = 0 \quad \text{for all } (\tau, x, e_i) \in (0, T] \times \mathbb{R} \times \mathcal{M}, \tag{2.2}$$

where $\mathcal{L}u$ is the integro-differential operator defined by

$$\begin{aligned}\mathcal{L}u(\tau, x, e_i) = {} & \frac{\sigma_i^2}{2} u_{xx}(\tau, x, e_i) + \left(r_i - d_i - \frac{\sigma_i^2}{2} - \lambda_i \zeta_i\right) u_x(\tau, x, e_i) - (r_i + \lambda_i) u(\tau, x, e_i) \\ & + \lambda_i \int_{-\infty}^{\infty} u(\tau, z, e_i) f(z - x, e_i) \mathrm{d}z + \sum_{j=1, j\neq i}^{Q} \gamma_{ji}(u(\tau, x, e_j) - u(\tau, x, e_i)),\end{aligned}$$

$\tau = T - t$ is the time to the expiration date T, $x = \ln(S/S_0)$ is the log price of the underlying asset S with respect to an initial price S_0, $e_i \in \mathcal{M}$ is the economic state, $f(x, e_i)$ is the probability density function of the random variable $\ln(\eta_i + 1)$ at the (i)th state of the economy.

In order to generalize the above PIDE with constant coefficients at each state of the economy, we consider the risk-free interest rate, the dividend rate, and the intensity as functions with respect to the time variable and the volatility parameter and the rate matrix as a function in the time and spatial variables. Then the PIDE (2.2) can be transformed into the following equation:

$$u_\tau^i(\tau, x) - \mathcal{L}u^i(\tau, x) = 0 \tag{2.3}$$

for all $(\tau, x, e_i) \in (0, T] \times \mathbb{R} \times \mathcal{M}$ with an initial payoff function

$$u^i(0, x) = h(x), \tag{2.4}$$

where the integro-differential operator $\mathcal{L}u$ is given by

$$\begin{aligned}\mathcal{L}u^i(\tau, x) = {} & \frac{(\sigma_i(\tau, x))^2}{2} u_{xx}^i(\tau, x) + \alpha_i(\tau, x) u_x^i(\tau, x) + \beta_i(\tau) u^i(\tau, x) \\ & + \lambda_i(\tau) \int_{-\infty}^{\infty} u^i(\tau, z) f_i(z - x) \mathrm{d}z + \langle \mathbf{u}, \mathcal{A}(\tau, x) e_i\rangle\end{aligned} \tag{2.5}$$

with

$$\begin{aligned} u^i(\tau, x) &= u(\tau, x, e_i), \\ \alpha_i(\tau, x) &= r_i(\tau) - d_i(\tau) - \frac{(\sigma_i(\tau, x))^2}{2} - \lambda_i(\tau)\zeta_i, \\ \beta_i(\tau) &= -(r_i(\tau) + \lambda_i(\tau)), \\ f_i(x) &= f(x, e_i), \\ \mathcal{A}(\tau, x) &= (\gamma_{qi}(\tau, x))_{Q\times Q} \end{aligned}$$

and $\mathbf{u}$ is a Q-dimensional column vector of the form $\mathbf{u} = (u^1, u^2, \ldots, u^Q)^{\mathrm{T}}$. We note that the rate matrix $\mathcal{A}(\tau, x)$ is a time homogeneous Markov process if $\mathcal{A}(\tau, x)$ is independent of τ and x. On the other hand, the price of the American option $u^i(\tau, x)$ under the RSJD model (2.1) satisfies the linear complementarity problem (LCP) with variable coefficients

$$\begin{cases} u^i_\tau(\tau, x) - \mathcal{L}u^i(\tau, x) \geq 0, \\ u^i(\tau, x) \geq h(x), \\ \big(u^i_\tau(\tau, x) - \mathcal{L}u^i(\tau, x)\big)\big(u^i(\tau, x) - h(x)\big) = 0 \end{cases} \tag{2.6}$$

for all $(\tau, x, e_i) \in (0, T] \times \mathbb{R} \times \mathcal{M}$, where $\mathcal{L}$ is the integro-differential operator in (2.5).

The payoff functions $h(\cdot)$ of the call and put options are given by

$$h(x) = \max(0, S_0 e^x - K) \quad \text{and} \quad h(x) = \max(0, K - S_0 e^x), \tag{2.7}$$

respectively, where K is the strike price. To solve the PIDE problem (2.3) and (2.4) and the LCP (2.6) numerically, it is needed to truncate the infinite domain $\mathbb{R}$ to the bounded interval $\Gamma = (-X, X)$ with $X > 0$ in the log price x. To set a boundary condition outside the domain Γ in the log price, we use asymptotic behaviors of the European call and put options given by

$$\lim_{x\to-\infty} u^i(\tau, x) = 0 \quad \text{and} \quad \lim_{x\to\infty} \left\{u^i(\tau, x) - \left(S_0 e^{x - \int_0^\tau d_i(s)\mathrm{d}s} - K e^{-\int_0^\tau r_i(s)\mathrm{d}s}\right)\right\} = 0 \tag{2.8}$$

and

$$\lim_{x\to-\infty} \left\{u^i(\tau, x) - \left(K e^{-\int_0^\tau r_i(s)\mathrm{d}s} - S_0 e^{x - \int_0^\tau d_i(s)\mathrm{d}s}\right)\right\} = 0 \quad \text{and} \quad \lim_{x\to\infty} u^i(\tau, x) = 0, \tag{2.9}$$

respectively. On the other hand, asymptotic behaviors of the American call and put options are

$$\lim_{x\to-\infty} u^i(\tau, x) = 0 \quad \text{and} \quad \lim_{x\to\infty} \{u^i(\tau, x) - (S_0 e^x - K)\} = 0 \tag{2.10}$$

and

$$\lim_{x\to-\infty} \{u^i(\tau, x) - (K - S_0 e^x)\} = 0 \quad \text{and} \quad \lim_{x\to\infty} u^i(\tau, x) = 0, \tag{2.11}$$

respectively.

3. Discretization for European options

In this section we formulate three numerical methods, which are based on three time levels, to solve the PIDE problem for the European option

$$u^i_\tau(\tau, x) = \mathcal{L}u^i(\tau, x) \qquad \text{for } (\tau, x, e_i) \in (0, T] \times \Gamma \times \mathcal{M}, \tag{3.1}$$

$$u^i(0, x) = h(x) \qquad \text{for } (x, e_i) \in \Gamma \times \mathcal{M}, \tag{3.2}$$

$$u^i(\tau, x) = g_i(\tau, x) \qquad \text{for } (\tau, x, e_i) \in (0, T] \times \mathbb{R} \setminus \Gamma \times \mathcal{M}, \tag{3.3}$$

where $\mathcal{L}$ is the integro-differential operator in (2.5) and the boundary condition $g_i(\tau, x)$ is given according to the asymptotic behaviors in (2.8) and (2.9).

We make uniform grid points on the truncated domain $(0, T] \times \Gamma$ to discretize the PIDE (3.1). For given two positive integers $N > 0$ and $M > 0$, let $\Delta\tau$ and Δx be grid sizes of time and spatial variables respectively, that is, $\Delta\tau = T/N$ and $\Delta x = 2X/M$. The grid point (τ_n, x_m) is defined by $\tau_n = n\Delta\tau$ and $x_m = -X + m\Delta x$ for $n = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, M$ and we define

$$u^{n,i}_m = u^i(\tau_n, x_m), \quad \sigma^{n,i}_m = \sigma_i(\tau_n, x_m), \quad \alpha^{n,i}_m = \alpha_i(\tau_n, x_m),$$

$$\beta^{n,i} = \beta_i(\tau_n), \quad \lambda^{n,i} = \lambda_i(\tau_n), \quad f^i_{m,j} = f_i(x_j - x_m), \quad \mathcal{A}^n_m = \mathcal{A}(\tau_n, x_m).$$

We first consider the implicit method (IM method) with explicit integration which is similar to that proposed by Kwon and Lee [14]. The discrete equation of the implicit method with explicit integration is

$$\frac{u_m^{n+1,i} - u_m^{n-1,i}}{2\Delta\tau} = \mathcal{L}_\Delta^1 u_m^{n,i}, \tag{3.4}$$

where the discrete integro-differential operator $\mathcal{L}_\Delta^1 u_m^{n,i}$ of the integro-differential operator $\mathcal{L}u_m^{n,i}$ is given by

$$\mathcal{L}_\Delta^1 u_m^{n,i} = \mathcal{D}_\Delta^1(u_m^{n+1,i}, u_m^{n-1,i}) + \mathcal{I}_\Delta u_m^{n,i} + \mathcal{E}_\Delta u_m^{n,i}, \tag{3.5}$$

the discrete differential operator $\mathcal{D}_\Delta^1(u_m^{n+1,i}, u_m^{n-1,i})$ in $\mathcal{L}_\Delta^1 u_m^{n,i}$ is

$$\begin{aligned}\mathcal{D}_\Delta^1(u_m^{n+1,i}, u_m^{n-1,i}) = {} & \frac{(\sigma_m^{n,i})^2}{2}\left(\frac{u_{m+1}^{n+1,i} - 2u_m^{n+1,i} + u_{m-1}^{n+1,i}}{2\Delta x^2} + \frac{u_{m+1}^{n-1,i} - 2u_m^{n-1,i} + u_{m-1}^{n-1,i}}{2\Delta x^2}\right) \\ & + \alpha_m^{n,i}\left(\frac{u_{m+1}^{n+1,i} - u_{m-1}^{n+1,i}}{4\Delta x} + \frac{u_{m+1}^{n-1,i} - u_{m-1}^{n-1,i}}{4\Delta x}\right) + \beta^{n,i}\left(\frac{u_m^{n+1,i}}{2} + \frac{u_m^{n-1,i}}{2}\right),\end{aligned} \tag{3.6}$$

the discrete integral operator $\mathcal{I}_\Delta u_m^{n,i}$ in $\mathcal{L}_\Delta^1 u_m^{n,i}$ is given by using the composite trapezoidal rule with the integral over the domain $\mathbb{R} \setminus \Gamma$ denoted by $R(\tau_n, x_m, e_i, X)$, that is,

$$\mathcal{I}_\Delta u_m^{n,i} = \frac{\lambda^{n,i}\Delta x}{2}\left(u_0^{n,i} f_{m,0}^i + 2\sum_{l=1}^{M-1} u_l^{n,i} f_{m,l}^i + u_M^{n,i} f_{m,M}^i\right) + \lambda^{n,i} R(\tau_n, x_m, e_i, X), \tag{3.7}$$

and the discrete regime-switching operator $\mathcal{E}_\Delta u_m^{n,i}$ in $\mathcal{L}_\Delta^1 u_m^{n,i}$ is given by

$$\mathcal{E}_\Delta u_m^{n,i} = \langle \mathbf{u}_\mathbf{m}^\mathbf{n}, \mathcal{A}_m^n e_i \rangle \tag{3.8}$$

with $\mathbf{u}_\mathbf{m}^\mathbf{n} = (u_m^{n,1}, u_m^{n,2}, \ldots, u_m^{n,Q})^{\mathrm{T}}$.

The next numerical method to solve the PIDE (3.1) is considered as the Crank–Nicolson method (CN method) with extrapolation. The differential terms in $\mathcal{L}u_m^{n+\frac{1}{2},i}$ are approximated by the CN method and the integral and regime-switching terms in $\mathcal{L}u_m^{n+\frac{1}{2},i}$ are discretized by using the extrapolation method. The discrete equation of the CN method with extrapolation is

$$\frac{u_m^{n+1,i} - u_m^{n,i}}{\Delta\tau} = \mathcal{L}_\Delta^2 u_m^{n+\frac{1}{2},i}, \tag{3.9}$$

where the discrete integro-differential operator $\mathcal{L}_\Delta^2 u_m^{n+\frac{1}{2},i}$ of $\mathcal{L}u_m^{n+\frac{1}{2},i}$ is given by

$$\mathcal{L}_\Delta^2 u_m^{n+\frac{1}{2},i} = \mathcal{D}_\Delta^2(u_m^{n+1,i}, u_m^{n,i}) + \mathcal{I}_\Delta\left(\frac{3u_m^{n,i} - u_m^{n-1,i}}{2}\right) + \mathcal{E}_\Delta\left(\frac{3u_m^{n,i} - u_m^{n-1,i}}{2}\right) \tag{3.10}$$

and the discrete differential operator $\mathcal{D}_\Delta^2(u_m^{n+1,i}, u_m^{n,i})$ in $\mathcal{L}_\Delta^2 u_m^{n+\frac{1}{2},i}$ is

$$\begin{aligned}\mathcal{D}_\Delta^2(u_m^{n+1,i}, u_m^{n,i}) = {} & \frac{(\sigma_m^{n+\frac{1}{2},i})^2}{2}\left(\frac{u_{m+1}^{n+1,i} - 2u_m^{n+1,i} + u_{m-1}^{n+1,i}}{2\Delta x^2} + \frac{u_{m+1}^{n,i} - 2u_m^{n,i} + u_{m-1}^{n,i}}{2\Delta x^2}\right) \\ & + \alpha_m^{n+\frac{1}{2},i}\left(\frac{u_{m+1}^{n+1,i} - u_{m-1}^{n+1,i}}{4\Delta x} + \frac{u_{m+1}^{n,i} - u_{m-1}^{n,i}}{4\Delta x}\right) + \beta^{n+\frac{1}{2},i}\left(\frac{u_m^{n+1,i}}{2} + \frac{u_m^{n,i}}{2}\right).\end{aligned} \tag{3.11}$$

The last implicit method to solve the PIDE (3.1) is based on the 2-step backward differentiation formula (BDF2 method) with extrapolation. The discrete equation of the BDF2 method with extrapolation is

$$\frac{1}{\Delta\tau}\left(\frac{3}{2}u_m^{n+1,i} - 2u_m^{n,i} + \frac{1}{2}u_m^{n-1,i}\right) = \mathcal{L}_\Delta^3 u_m^{n+1,i}, \tag{3.12}$$

where the discrete integro-differential operator $\mathcal{L}_\Delta^3 u_m^{n+1,i}$ of $\mathcal{L}u_m^{n+1,i}$ is given by

$$\mathcal{L}_\Delta^3 u_m^{n+1,i} = \mathcal{D}_\Delta^3 u_m^{n+1,i} + \mathcal{I}_\Delta(2u_m^{n,i} - u_m^{n-1,i}) + \mathcal{E}_\Delta(2u_m^{n,i} - u_m^{n-1,i}) \tag{3.13}$$

and the discrete differential operator $\mathcal{D}_\Delta^3 u_m^{n+1,i}$ in $\mathcal{L}_\Delta^3 u_m^{n+1,i}$ is

$$\mathcal{D}_\Delta^3 u_m^{n+1,i} = \frac{(\sigma_m^{n+1,i})^2}{2}\frac{u_{m+1}^{n+1,i} - 2u_m^{n+1,i} + u_{m-1}^{n+1,i}}{\Delta x^2} + \alpha_m^{n+1,i}\frac{u_{m+1}^{n+1,i} - u_{m-1}^{n+1,i}}{2\Delta x} + \beta^{n+1,i}u_m^{n+1,i}. \tag{3.14}$$

Remark 3.1. These finite difference methods are designed to formulate implicit schemes without use of any fixed point iteration techniques at each time step. Moreover, the regime-switching term in the PIDE (3.1) is discretized explicitly so that the prices of the European option at all states of the economy can be obtained in the more efficient way. The above three implicit methods require two initial values at all states of the economy. The vector $u^{1,i}$ on the first time level at the i-state of the economy is evaluated by using the explicit and implicit method in [4].

Remark 3.2. The zeroth derivative term $\beta_i(\tau)u^i(\tau,x)$ in the IM method with explicit integration is discretized implicitly whereas that in [14] is discretized explicitly. It can be observed in [21] that the implicit treatment of the zeroth derivative term $\beta_i(\tau)u^i(\tau,x)$ is more stable and more efficient than is the explicit treatment of the zeroth derivative term in which the finer grid sizes may be needed to maintain the stability.

4. Analysis of stability

In this section we demonstrate the stability of the three numerical methods we have introduced for the European option pricing under the RSJD model with variable coefficients. We make some restrictions, which are similar to those in [1], on the variable coefficients in the PIDE problem (3.1)–(3.3).

(R1) The four variable coefficients σ, α, β, λ, and the rate matrix $\mathcal{A}$ are continuous and sufficiently regular in the time and spatial variables.

(R2) There exist $\underline{\sigma} > 0$ and $\bar{\sigma} > 0$ such that for all $(\tau, x, e_i) \in [0,T] \times \bar{\Gamma} \times \mathcal{M}$

$$0 < \underline{\sigma} < \sigma_i(\tau,x) < \bar{\sigma}.$$

(R3) There exists $C_\sigma > 0$ such that for all $(\tau, x, e_i) \in [0,T] \times \bar{\Gamma} \times \mathcal{M}$

$$\left|\frac{\partial\sigma_i}{\partial x}(\tau,x)\right| < C_\sigma.$$

(R4) There exist $\bar{\alpha} > 0$, $\bar{\beta} > 0$, and $\bar{\lambda} > 0$ such that for all $(\tau, x, e_i) \in [0,T] \times \bar{\Gamma} \times \mathcal{M}$

$$|\alpha_i(\tau,x)| < \bar{\alpha}, \quad |\beta_i(\tau)| < \bar{\beta}, \quad \text{and} \quad |\lambda_i(\tau)| < \bar{\lambda}.$$

(R5) There exists $\bar{\gamma} > 0$ such that for all entries $\gamma_{qi}(\tau,x)$ with $1 \le q, i \le Q$ of the rate matrix $\mathcal{A}(\tau,x)$ and for all $(\tau,x) \in [0,T] \times \bar{\Gamma}$

$$|\gamma_{qi}(\tau,x)| < \bar{\gamma}.$$

In the following theorem it is shown that the IM, CN, and BDF2 methods to solve the PIDE problem are consistent with the second-order in the time and spatial variables.

Theorem 4.1 (Consistency)**.** *Suppose that under the restrictions (R1)–(R5) a function* $v^i(\tau,x) \in C^\infty((0,T]\times\bar{\Gamma})$ *at each* $1 \le i \le Q$ *satisfies the initial and boundary conditions (3.2) and (3.3). Then the local truncation errors for the three numerical methods are the second-order in the time and spatial variables, i.e., in the IM, CN, and BDF2 methods we have*

$$v_\tau^i(\tau_n,x_m) - \mathcal{L}v^i(\tau_n,x_m) - \left(\frac{v_m^{n+1,i}-v_m^{n-1,i}}{2\Delta\tau} - \mathcal{L}_\Delta^1 v_m^{n,i}\right) = O(\Delta\tau^2+\Delta x^2),$$
$$v_\tau^i(\tau_n,x_m) - \mathcal{L}v^i(\tau_n,x_m) - \left(\frac{v_m^{n+1,i}-v_m^{n,i}}{\Delta\tau} - \mathcal{L}_\Delta^2 v_m^{n+\frac{1}{2},i}\right) = O(\Delta\tau^2+\Delta x^2),$$

and

$$v_\tau^i(\tau_n,x_m) - \mathcal{L}v^i(\tau_n,x_m) - \left(\frac{1}{\Delta\tau}\left(\frac{3}{2}v_m^{n+1,i} - 2v_m^{n,i} + \frac{1}{2}v_m^{n-1,i}\right) - \mathcal{L}_\Delta^3 v_m^{n,i}\right) = O(\Delta\tau^2+\Delta x^2)$$

for $(\tau_n,x_m)\in(0,T)\times\Gamma$ *and* $n\ge 1$ *at each state* i *of the economy, respectively.*

Proof. It is obvious by applying the Taylor series expansion, the extrapolation method, and the composite trapezoidal rule to the PIDE problem (3.1)–(3.3). □

Before we analyze the stability, two vector norms $|U^n|_2$ and $\|U^n\|_{\ell^2}$ of the numerical approximations U^n obtained by applying the IM, CN, and BDF2 methods are introduced as below:

$$|U^n|_2 = \sqrt{\frac{1}{\Delta x}\sum_{i=1}^{Q}\left((U_1^{n,i})^2 + \sum_{m=2}^{M-1}(U_m^{n,i}-U_{m-1}^{n,i})^2 + (U_{M-1}^{n,i})^2\right)}$$

and

$$\|U^n\|_{\ell^2} = \sqrt{\Delta x\sum_{i=1}^{Q}\sum_{m=1}^{M-1}(U_m^{n,i})^2},$$

where the column vector U^n on the (n)th time level is arranged in a line with all economic states

$$U^n = (U_1^{n,1},U_2^{n,1},\ldots,U_{M-1}^{n,1},U_1^{n,2},U_2^{n,2},\ldots,U_{M-1}^{n,2},\ldots,U_1^{n,Q},U_2^{n,Q},\ldots,U_{M-1}^{n,Q})^{\mathrm{T}}. \tag{4.1}$$

Note that the discrete ℓ^2-norm $\|U^n\|_{\ell^2}$ is associated with the discrete ℓ^2-inner product $(U^n,V^n)_{\ell^2}$ defined by

$$(U^n,V^n)_{\ell^2} = \Delta x\sum_{i=1}^{Q}\sum_{m=1}^{M-1}U_m^{n,i}V_m^{n,i} \tag{4.2}$$

for any column vector $U^n\in\mathbb{R}^{(M-1)\times Q}$ in (4.1) and any column vector $V^n\in\mathbb{R}^{(M-1)\times Q}$ given by

$$V^n = (V_1^{n,1},V_2^{n,1},\ldots,V_{M-1}^{n,1},V_1^{n,2},V_2^{n,2},\ldots,V_{M-1}^{n,2},\ldots,V_1^{n,Q},V_2^{n,Q},\ldots,V_{M-1}^{n,Q})^{\mathrm{T}}. \tag{4.3}$$

The linear systems derived by the IM, CN, and BDF2 methods are given by

$$\left(I+\frac{\Delta\tau}{2}(A^n-B^n)\right)U^{n+1} = \left(I-\frac{\Delta\tau}{2}(A^n-B^n)\right)U^{n-1} + \Delta\tau(D^n+F^n)U^n + \Delta\tau\rho_1^n, \tag{4.4}$$

$$\left(I+\frac{\Delta\tau}{4}(A^{n+\frac{1}{2}}-B^{n+\frac{1}{2}})\right)U^{n+1} = \left(I-\frac{\Delta\tau}{4}(A^{n+\frac{1}{2}}-B^{n+\frac{1}{2}}-3D^n-3F^n)\right)U^n - \frac{\Delta\tau}{4}\left(D^{n-1}+F^{n-1}\right)U^{n-1} + \Delta\tau\rho_2^{n+\frac{1}{2}}, \tag{4.5}$$

and

$$\left(\frac{3}{2}I + \frac{\Delta\tau}{2}(A^{n+1} - B^{n+1})\right)U^{n+1} = (2I + \Delta\tau(D^n + F^n))U^n - \left(\frac{1}{2}I + \frac{\Delta\tau}{2}(D^{n-1} + F^{n-1})\right)U^{n-1} + \Delta\tau\rho_3^{n+1}, \tag{4.6}$$

respectively, where I is the identity matrix of size $(M-1)\times Q$, the matrix A^n is the block diagonal matrix of size $(M-1)\times Q$

$$A^n = \begin{pmatrix} A^{n,1} & 0 & \cdots & 0 \\ 0 & A^{n,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{n,Q} \end{pmatrix}$$

having diagonal blocks $A^{n,i} := (a_{ml}^{n,i})$ of size $M-1$ for $1 \le i \le Q$ with entries

$$a_{ml}^{n,i} = \begin{cases} -\frac{(\sigma_m^{n,i})^2}{\Delta x^2} + \frac{\alpha_m^{n,i}}{\Delta x} & \text{for } l = m-1 \quad \text{for } 2 \le m \le M-1, \\ \frac{2(\sigma_m^{n,i})^2}{\Delta x^2} & \text{for } l = m \quad \text{for } 1 \le m \le M-1, \\ -\frac{(\sigma_m^{n,i})^2}{\Delta x^2} - \frac{\alpha_m^{n,i}}{\Delta x} & \text{for } l = m+1 \quad \text{for } 1 \le m \le M-2, \\ 0 & \text{otherwise,} \end{cases}$$

two block diagonal matrices B^n and D^n of size $(M-1)\times Q$ are given by

$$B^n = \begin{pmatrix} B^{n,1} & 0 & \cdots & 0 \\ 0 & B^{n,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{n,Q} \end{pmatrix} \quad \text{and} \quad D^n = \begin{pmatrix} D^{n,1} & 0 & \cdots & 0 \\ 0 & D^{n,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{n,Q} \end{pmatrix}$$

having diagonal blocks $B^{n,i}$ of size $M-1$ for $1 \le i \le Q$ of the form

$$B^{n,i} = 2\beta^{n,i}\bar{I}$$

with the identity matrix $\bar{I}$ of size $M-1$ and diagonal blocks $D^{n,i} := (d_{ml}^{n,i})$ of size $M-1$ for $1 \le i \le Q$ with entries

$$d_{ml}^{n,i} = 2\Delta x \lambda^{n,i} f_{m,l}^i \quad \text{for } 1 \le l \le M-1 \quad \text{for } 1 \le m \le M-1,$$

respectively, the square matrix F^n of size $(M-1)\times Q$ is the block matrix with Q row and column partitions

$$F^n = \begin{pmatrix} F_{11}^n & F_{12}^n & \cdots & F_{1Q}^n \\ F_{21}^n & F_{22}^n & \cdots & F_{2Q}^n \\ \vdots & \vdots & \ddots & \vdots \\ F_{Q1}^n & F_{Q2}^n & \cdots & F_{QQ}^n \end{pmatrix}$$

having all blocks F_{iq}^n for $1 \le i, q \le Q$ of the form

$$F_{iq}^n = \begin{pmatrix} 2\gamma_{qi,1}^n & 0 & \cdots & 0 \\ 0 & 2\gamma_{qi,2}^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\gamma_{qi,M-1}^n \end{pmatrix}$$

with $\gamma_{qi,m}^n = \gamma_{qi}(\tau_n, x_m)$ and the column vectors ρ_l^n of size $(M-1)\times Q$ for $l = 1, 2, 3$ are given by

$$\rho_l^n = \begin{pmatrix} \rho_l^{n,1} \\ \rho_l^{n,2} \\ \vdots \\ \rho_l^{n,Q} \end{pmatrix}.$$

In the IM, CN, and BDF2 methods, the entries of all column vectors $\rho_l^{n,i} := (\rho_{l,m}^{n,i})$ of size $M-1$ for $1 \le i \le Q$ and for $l = 1, 2, 3$ are given by

$$\rho_{1,m}^{n,i} = \begin{cases} \frac{1}{2}\left(\frac{(\sigma_1^{n,i})^2}{\Delta x^2} - \frac{\alpha_1^{n,i}}{\Delta x}\right)\left(g_0^{n-1,i} + g_0^{n+1,i}\right) \\ \quad +\Delta x \lambda^{n,i} f_{1,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{1,M}^i g_M^{n,i} + 2\lambda^{n,i} R(\tau_n, x_1, e_i, X) & \text{for } m = 1, \\ \Delta x \lambda^{n,i} f_{m,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{m,M}^i g_M^{n,i} + 2\lambda^{n,i} R(\tau_n, x_m, e_i, X) & \text{for } 2 \le m \le M-2, \\ \frac{1}{2}\left(\frac{(\sigma_{M-1}^{n,i})^2}{\Delta x^2} + \frac{\alpha_{M-1}^{n,i}}{\Delta x}\right)\left(g_M^{n-1,i} + g_M^{n+1,i}\right) \\ \quad +\Delta x \lambda^{n,i} f_{M-1,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{M-1,M}^i g_M^{n,i} & \text{for } m = M-1 \\ \qquad +2\lambda^{n,i} R(\tau_n, x_{M-1}, e_i, X) \end{cases}$$

with $g_m^{n,i} = g(\tau_n, x_m, e_i)$,

$$\rho_{2,m}^{n+\frac{1}{2},i} = \begin{cases} \frac{1}{4}\left(\frac{(\sigma_1^{n+\frac{1}{2},i})^2}{\Delta x^2} - \frac{\alpha_1^{n+\frac{1}{2},i}}{\Delta x}\right)\left(g_0^{n,i} + g_0^{n+1,i}\right) \\ \quad +\frac{3}{4}\left(\Delta x \lambda^{n,i} f_{1,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{1,M}^i g_M^{n,i}\right) & \text{for } m = 1, \\ \qquad -\frac{1}{4}\left(\Delta x \lambda^{n-1,i} f_{1,0}^i g_0^{n-1,i} + \Delta x \lambda^{n-1,i} f_{1,M}^i g_M^{n-1,i}\right) \\ \qquad\quad +\frac{3}{2}\lambda^{n,i} R(\tau_n, x_1, e_i, X) - \frac{1}{2}\lambda^{n-1,i} R(\tau_{n-1}, x_1, e_i, X) \\ \frac{3}{4}\left(\Delta x \lambda^{n,i} f_{m,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{m,M}^i g_M^{n,i}\right) \\ \quad -\frac{1}{4}\left(\Delta x \lambda^{n-1,i} f_{m,0}^i g_0^{n-1,i} + \Delta x \lambda^{n-1,i} f_{m,M}^i g_M^{n-1,i}\right) & \text{for } 2 \le m \le M-2, \\ \qquad +\frac{3}{2}\lambda^{n,i} R(\tau_n, x_m, e_i, X) - \frac{1}{2}\lambda^{n-1,i} R(\tau_{n-1}, x_m, e_i, X) \\ \frac{1}{4}\left(\frac{(\sigma_{M-1}^{n+\frac{1}{2},i})^2}{\Delta x^2} + \frac{\alpha_{M-1}^{n+\frac{1}{2},i}}{\Delta x}\right)\left(g_M^{n,i} + g_M^{n+1,i}\right) \\ \quad +\frac{3}{4}\left(\Delta x \lambda^{n,i} f_{M-1,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{M-1,M}^i g_M^{n,i}\right) \\ \qquad -\frac{1}{4}\left(\Delta x \lambda^{n-1,i} f_{M-1,0}^i g_0^{n-1,i} + \Delta x \lambda^{n-1,i} f_{M-1,M}^i g_M^{n-1,i}\right) & \text{for } m = M-1, \\ \qquad\quad +\frac{3}{2}\lambda^{n,i} R(\tau_n, x_{M-1}, e_i, X) \\ \qquad\qquad -\frac{1}{2}\lambda^{n-1,i} R(\tau_{n-1}, x_{M-1}, e_i, X) \end{cases}$$

and

$$\rho_{3,m}^{n+1,i} = \begin{cases} \frac{1}{2}\left(\frac{(\sigma_1^{n+1,i})^2}{\Delta x^2} - \frac{\alpha_1^{n+1,i}}{\Delta x}\right) g_0^{n+1,i} \\ \quad + \left(\Delta x \lambda^{n,i} f_{1,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{1,M}^i g_M^{n,i}\right) & \text{for } m = 1, \\ \quad -\frac{1}{2}\left(\Delta x \lambda^{n-1,i} f_{1,0}^i g_0^{n-1,i} + \Delta x \lambda^{n-1,i} f_{1,M}^i g_M^{n-1,i}\right) \\ \quad +2\lambda^{n,i} R(\tau_n, x_1, e_i, X) - \lambda^{n-1,i} R(\tau_{n-1}, x_1, e_i, X) \\ \left(\Delta x \lambda^{n,i} f_{m,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{m,M}^i g_M^{n,i}\right) \\ \quad -\frac{1}{2}\left(\Delta x \lambda^{n-1,i} f_{m,0}^i g_0^{n-1,i} + \Delta x \lambda^{n-1,i} f_{m,M}^i g_M^{n-1,i}\right) & \text{for } 2 \le m \le M-2, \\ \quad +2\lambda^{n,i} R(\tau_n, x_m, e_i, X) - \lambda^{n-1,i} R(\tau_{n-1}, x_m, e_i, X) \\ \frac{1}{2}\left(\frac{(\sigma_{M-1}^{n+1,i})^2}{\Delta x^2} + \frac{\alpha_{M-1}^{n+1,i}}{\Delta x}\right) g_M^{n+1,i} \\ \quad + \left(\Delta x \lambda^{n,i} f_{M-1,0}^i g_0^{n,i} + \Delta x \lambda^{n,i} f_{M-1,M}^i g_M^{n,i}\right) \\ \quad -\frac{1}{2}\left(\Delta x \lambda^{n-1,i} f_{M-1,0}^i g_0^{n-1,i} + \Delta x \lambda^{n-1,i} f_{M-1,M}^i g_M^{n-1,i}\right) & \text{for } m = M-1, \\ \quad +2\lambda^{n,i} R(\tau_n, x_{M-1}, e_i, X) \\ \quad -\lambda^{n-1,i} R(\tau_{n-1}, x_{M-1}, e_i, X) \end{cases}$$

respectively. Then we need Lemmas 4.2 and 4.3 to prove the stability in the discrete ℓ^2-norm. Especially Lemma 4.2 is analogous to that in [1] and is a version involving the economic states.

Lemma 4.2. *Suppose that the restrictions (R1)–(R5) hold. For any column vector $U \in \mathbb{R}^{(M-1)\times Q}$ and any two positive constants $\kappa > 0$ and $\omega > 0$, the inequality*

$$(U, A^n U)_{\ell^2} \ge \left(\underline{\sigma}^2 - \frac{2\bar{\sigma}^2}{\kappa} - \frac{\bar{\alpha}}{\omega}\right)|U|_2^2 - \left(\frac{{C_\sigma}^2\kappa}{2} + \bar{\alpha}\omega\right)\|U\|_{\ell^2}^2 \tag{4.7}$$

is satisfied for all $1 \le n \le N$.

Proof. For any column vector $U \in \mathbb{R}^{(M-1)\times Q}$, it can be written by

$$U = (U^1; U^2; \dots; U^Q)$$

with the column vectors U^i of size $M-1$ for $1 \le i \le Q$ of the form $U^i = (U_1^i, U_2^i, \dots, U_{M-1}^i)^{\mathrm{T}}$. By using Lemma 3.1 in [15], there exists $C > 0$ independent of grid sizes $\Delta\tau$ and Δx such that the discrete ℓ^2-inner product $(U, A^n U)_{\ell^2}$ for all $1 \le n \le N$ becomes

$$\begin{aligned}(U, A^n U)_{\ell^2} &= \Delta x \sum_{i=1}^{Q} (U^i)^{\mathrm{T}} A^{n,i} U^i \\ &\ge \left(\underline{\sigma}^2 - \frac{2\bar{\sigma}^2}{\kappa} - \frac{\bar{\alpha}}{\omega}\right)|U|_2^2 - \left(\frac{{C_\sigma}^2\kappa}{2} + \bar{\alpha}\omega\right)\|U\|_{\ell^2}^2,\end{aligned}$$

which is our desired result. ∎

Lemma 4.3. *Let us consider a nonnegative sequence $\{a_n\}_{n\ge 0}$ satisfying*

$$(1 - C\Delta\tau)a_{n+1} \le (1 + C\Delta\tau)a_n + C\Delta\tau a_{n-1} + b \quad \text{for } n \ge 1$$

or

$$(1 - C\Delta\tau)a_{n+1} \le (1 + C\Delta\tau)a_{n-1} + C\Delta\tau a_n + b \quad \text{for } n \ge 1,$$

where C*,* b*, and* $\Delta\tau$ *are three nonnegative constants with the condition* $C\Delta\tau < 1/4$*. Then the* n*-th term* a_n *for* $n \ge 2$ *is bounded by*

$$a_n \le (1 + 4C\Delta\tau)^{n-1} \max(a_0, a_1) + b \sum_{i=1}^{n-1} (1 + 4C\Delta\tau)^i. \tag{4.8}$$

Proof. It can be proved by using the mathematical induction. ☐

Finally we are ready to show that the IM, CN, and BDF2 methods satisfy the stability criterion in the sense of the discrete ℓ^2-norm.

Theorem 4.4 (Stability of the IM method)**.** *Suppose that the restrictions (R1)–(R5) hold and there is a perturbation vector* $E^0 \in \mathbb{R}^{(M-1)\times Q}$ *such that the initial solution vector* U^0 *is perturbed, i.e.,* $V^0 = U^0 + E^0$*. Then the IM method with explicit integration in (4.4) satisfies the stability in the discrete* ℓ^2*-norm with the sufficiently small time grid size* $\Delta\tau < \frac{\underline{\sigma}^2}{8\underline{\sigma}^2(\bar\beta + \bar\lambda + \sqrt{2Q}\bar\gamma) + 4(C_\sigma \bar\sigma + \bar\alpha)^2}$ *described as*

$$\|E^n\|_{\ell^2} \le C \max(\|E^0\|_{\ell^2}, \|E^1\|_{\ell^2}) \quad \text{for } 2 \le n \le N, \tag{4.9}$$

where a positive constant $C > 0$ *is independent of* $\Delta\tau$ *and* Δx*.*

Proof. The perturbed vector V^n is obtained by the linear system for the IM method in (4.4). Then the perturbation vector E^n on the (n)th time levels is $V^n = U^n + E^n$ and the linear system for E^n becomes

$$E^{n+1} - E^{n-1} = -\frac{\Delta\tau}{2}(A^n - B^n)(E^{n+1} + E^{n-1}) + \Delta\tau(D^n + F^n)E^n \quad \text{for } 1 \le n \le N-1.$$

We take the discrete ℓ^2-inner product in (4.2) with $E^{n+1} + E^{n-1}$ on both sides of the above linear system. By applying Lemma 4.2 we have

$$\begin{aligned}
\|E^{n+1}\|_{\ell^2}^2 - \|E^{n-1}\|_{\ell^2}^2 &= -\frac{\Delta\tau\Delta x}{2}(E^{n+1} + E^{n-1})^{\mathrm{T}}(A^n - B^n)(E^{n+1} + E^{n-1}) \\
&\quad + \Delta\tau\Delta x(E^{n+1} + E^{n-1})^{\mathrm{T}}(D^n + F^n)E^n \\
&\le -\frac{\Delta\tau}{2}\left(C_1|E^{n+1} + E^{n-1}|_2^2 - C_2\|E^{n+1} + E^{n-1}\|_{\ell^2}^2\right) \\
&\quad + \Delta\tau\|E^{n+1} + E^{n-1}\|_{\ell^2} \cdot \|D^n + F^n\|_2 \cdot \|E^n\|_{\ell^2},
\end{aligned}$$

where C_1 and C_2 are constants independent of $\Delta\tau$ and Δx given by

$$C_1 = \underline{\sigma}^2 - \frac{2\bar\sigma^2}{\kappa} - \frac{\bar\alpha}{\omega} \quad \text{and} \quad C_2 = \frac{{C_\sigma}^2\kappa}{2} + \bar\alpha\omega + 2\bar\beta$$

and $\|\cdot\|_2$ is the natural matrix norm associated with the discrete ℓ^2-norm. We note that there exist $\kappa > 0$ and $\omega > 0$ such that $C_1 > 0$. From the fact that $\|D^n\|_2 \le 2\bar\lambda$ and $\|F^n\|_2 \le 2\sqrt{2Q}\bar\gamma$, the above inequality with $C_1 > 0$ implies that for $1 \le n \le N-1$

$$\begin{aligned}
\|E^{n+1}\|_{\ell^2}^2 - \|E^{n-1}\|_{\ell^2}^2 &\le (C_2 + C_3)\frac{\Delta\tau}{2}\|E^{n+1} + E^{n-1}\|_{\ell^2}^2 + C_3\frac{\Delta\tau}{2}\|E^n\|_{\ell^2}^2 \\
&\le (C_2 + C_3)\Delta\tau(\|E^{n+1}\|_{\ell^2}^2 + \|E^{n-1}\|_{\ell^2}^2) + C_3\frac{\Delta\tau}{2}\|E^n\|_{\ell^2}^2
\end{aligned}$$

with $C_3 = 2(\bar{\lambda} + \sqrt{2Q}\bar{\gamma})$ and we have

$$(1 - C_4\Delta\tau)\|E^{n+1}\|_{\ell^2}^2 \leq (1 + C_4\Delta\tau)\|E^{n-1}\|_{\ell^2}^2 + C_4\Delta\tau\|E^n\|_{\ell^2}^2,$$

where $C_4 = C_2 + C_3$. By using Lemma 4.3 with $C_4\Delta\tau < 1/4$, we obtain

$$\begin{aligned}\|E^{n+1}\|_{\ell^2}^2 &\leq (1 + 4C_4\Delta\tau)^n \max(\|E^0\|_{\ell^2}^2, \|E^1\|_{\ell^2}^2)\\ &\leq e^{4C_4T} \max(\|E^0\|_{\ell^2}^2, \|E^1\|_{\ell^2}^2).\end{aligned}$$

Thus the inequality (4.9) holds by taking $C = \sqrt{e^{4C_4T}}$. In order to determine the constraint of the time grid size $\Delta\tau$, we consider the region satisfying the condition $C_1 > 0$ with respect to the variables $\kappa > 0$ and $\omega > 0$. Since the inequality (4.9) is satisfied with $C_4\Delta\tau < 1/4$, the minimum value of $C_4 > 0$ on the region can be computed by

$$C_4 \geq \frac{2\underline{\sigma}^2(\bar{\beta} + \bar{\lambda} + \sqrt{2Q}\bar{\gamma}) + (C_\sigma\bar{\sigma} + \bar{\alpha})^2}{\underline{\sigma}^2}$$

and therefore we conclude that Theorem 4.4 holds. □

It can be shown similarly that the CN and BDF2 methods are stable with respect to the discrete ℓ^2-norm in the time and spatial variables.

Theorem 4.5 (Stability of the CN method)**.** *When the initial solution vector U^0 is perturbed, i.e., $V^0 = U^0 + E^0$, then the CN method with extrapolation in (4.5) under the restrictions (R1)–(R5) satisfies the following inequality:*

$$\|E^n\|_{\ell^2} \leq C\max(\|E^0\|_{\ell^2}, \|E^1\|_{\ell^2}) \quad \text{for } 2 \leq n \leq N, \tag{4.10}$$

where a positive constant $C > 0$ is independent of $\Delta\tau$ and Δx and the time grid size $\Delta\tau$ is sufficiently small given by

$$\Delta\tau < \frac{\underline{\sigma}^2}{\underline{\sigma}^2\left\{4\bar{\beta} + 11(\bar{\lambda} + \sqrt{2Q}\bar{\gamma})\right\} + 2(C_\sigma\bar{\sigma} + \bar{\alpha})^2}.$$

Proof. The linear system for the CN method in (4.5) associated with the perturbation vector E^n on the (n)th time level is

$$E^{n+1} - E^n = -\frac{\Delta\tau}{4}(A^{n+\frac{1}{2}} - B^{n+\frac{1}{2}})(E^{n+1} + E^n) + \frac{3\Delta\tau}{4}(D^n + F^n)E^n - \frac{\Delta\tau}{4}\left(D^{n-1} + F^{n-1}\right)E^{n-1}$$

for $1 \leq n \leq N-1$. By taking the discrete ℓ^2-inner product with $E^{n+1} + E^n$ on both sides of the above linear system and using Lemma 4.2, we obtain

$$\begin{aligned}\|E^{n+1}\|_{\ell^2}^2 - \|E^n\|_{\ell^2}^2 = &-\frac{\Delta\tau\Delta x}{4}(E^{n+1} + E^n)^{\mathrm{T}}(A^{n+\frac{1}{2}} - B^{n+\frac{1}{2}})(E^{n+1} + E^n)\\ &+ \frac{3\Delta\tau\Delta x}{4}(E^{n+1} + E^n)^{\mathrm{T}}(D^n + F^n)E^n\\ &- \frac{\Delta\tau\Delta x}{4}(E^{n+1} + E^n)^{\mathrm{T}}\left(D^{n-1} + F^{n-1}\right)E^{n-1}\\ \leq &-\frac{\Delta\tau}{4}\left(C_1|E^{n+1} + E^n|_2^2 - C_2\|E^{n+1} + E^n\|_{\ell^2}^2\right)\\ &+ C_3\frac{\Delta\tau}{8}(4\|E^{n+1} + E^n\|_{\ell^2}^2 + 3\|E^n\|_{\ell^2}^2 + \|E^{n-1}\|_{\ell^2}^2)\\ \leq &\frac{\Delta\tau}{4}(C_2 + 2C_3)\|E^{n+1} + E^n\|_{\ell^2}^2 + C_3\frac{\Delta\tau}{8}(3\|E^n\|_{\ell^2}^2 + \|E^{n-1}\|_{\ell^2}^2),\end{aligned}$$

where $C_1 > 0$, $C_2 > 0$, and $C_3 > 0$ are the constants in the proof of Theorem 4.4. If we put $C_5 = (4C_2+11C_3)/8$, then the above inequality is rewritten by

$$(1 - C_5\Delta\tau)\|E^{n+1}\|^2_{\ell^2} \le (1 + C_5\Delta\tau)\|E^n\|^2_{\ell^2} + C_5\Delta\tau\|E^{n-1}\|^2_{\ell^2}.$$

By applying Lemma 4.3 with $C_5\Delta\tau < 1/4$, the desired result in (4.10) is delivered with the sufficiently small time grid size $\Delta\tau < \frac{\underline{\sigma}^2}{\underline{\sigma}^2\left\{4\bar{\beta}+11(\bar{\lambda}+\sqrt{2Q}\bar{\gamma})\right\}+2(C_\sigma\bar{\sigma}+\bar{\alpha})^2}$ and $C = \sqrt{e^{4C_5T}}$. □

Theorem 4.6 (Stability of the BDF2 method)**.** *Suppose that the restrictions (R1)–(R5) hold. If the initial vector U^0 in the BDF2 method with extrapolation in (4.6) is perturbed by E^0, i.e., $V^0 = U^0 + E^0$, then there exists a constant $C > 0$ independent of the grid sizes $\Delta\tau$ and Δx such that for the sufficiently small time grid size $\Delta\tau < \frac{\underline{\sigma}^2}{8\underline{\sigma}^2\left\{2\bar{\beta}+3(\bar{\lambda}+\sqrt{2Q}\bar{\gamma})\right\}+8(C_\sigma\bar{\sigma}+\bar{\alpha})^2}$ we have*

$$\|E^n\|^2_{\ell^2} + \|2E^n - E^{n-1}\|^2_{\ell^2} \le C\max(5\|E^0\|^2_{\ell^2}, \|E^1\|^2_{\ell^2} + \|2E^1 - E^0\|^2_{\ell^2}) \quad \text{for } 2 \le n \le N. \tag{4.11}$$

Proof. The linear system related to the perturbation vector E^n in the BDF2 method in (4.6) is given by

$$\frac{3}{2}E^{n+1} - 2E^n + \frac{1}{2}E^{n-1} = -\frac{\Delta\tau}{2}(A^{n+1} - B^{n+1})E^{n+1} + \Delta\tau(D^n + F^n)E^n - \frac{\Delta\tau}{2}(D^{n-1} + F^{n-1})E^{n-1}$$

for $1 \le n \le N-1$. By applying the discrete ℓ^2-inner product with E^{n+1} to both sides of the above linear system, we have

$$\begin{aligned}\frac{\Delta x}{2}&(E^{n+1})^{\mathrm{T}}(3E^{n+1} - 4E^n + E^{n-1})\\ &= -\frac{\Delta\tau\Delta x}{2}(E^{n+1})^{\mathrm{T}}(A^{n+1} - B^{n+1})E^{n+1}\\ &\quad+\frac{\Delta\tau\Delta x}{2}(E^{n+1})^{\mathrm{T}}(2(D^n + F^n)E^n - (D^{n-1} + F^{n-1})E^{n-1}).\end{aligned} \tag{4.12}$$

We note in [12] that the following relation holds:

$$\begin{aligned}&2\Delta x(E^{n+1})^{\mathrm{T}}(3E^{n+1} - 4E^n + E^{n-1})\\ &\quad= \|E^{n+1}\|^2_{\ell^2} + \|2E^{n+1} - E^n\|^2_{\ell^2} - \|E^n\|^2_{\ell^2} - \|2E^n - E^{n-1}\|^2_{\ell^2} + \|E^{n+1} - 2E^n + E^{n-1}\|^2_{\ell^2}.\end{aligned} \tag{4.13}$$

By substituting the equality in (4.13) into the equation in (4.12) and using Lemma 4.2, we have

$$\begin{aligned}\|E^{n+1}\|^2_{\ell^2} &+ \|2E^{n+1} - E^n\|^2_{\ell^2} - \|E^n\|^2_{\ell^2} - \|2E^n - E^{n-1}\|^2_{\ell^2}\\ &\le -2\Delta\tau\Delta x(E^{n+1})^{\mathrm{T}}(A^{n+1} - B^{n+1})E^{n+1}\\ &\quad + 2\Delta\tau\Delta x(E^{n+1})^{\mathrm{T}}(2(D^n + F^n)E^n - (D^{n-1} + F^{n-1})E^{n-1})\\ &\le -2\Delta\tau\left(C_1|E^{n+1}|^2_2 - C_2\|E^{n+1}\|^2_{\ell^2}\right) + 2C_3\Delta\tau\|E^{n+1}\|_{\ell^2}(2\|E^n\|_{\ell^2} + \|E^{n-1}\|_{\ell^2})\\ &\le C_6\Delta\tau(\|E^{n+1}\|^2_{\ell^2} + \|2E^{n+1} - E^n\|^2_{\ell^2}) + C_6\Delta\tau(\|E^n\|^2_{\ell^2} + \|2E^n - E^{n-1}\|^2_{\ell^2})\\ &\quad + C_6\Delta\tau(\|E^{n-1}\|^2_{\ell^2} + \|2E^{n-1} - E^{n-2}\|^2_{\ell^2}),\end{aligned}$$

where C_1, C_2, and C_3 are the constants in the proof of Theorem 4.4 and $C_6 = 2C_2 + 3C_3$. This inequality is rewritten by for $1 \le n \le N-1$

$$\begin{aligned}(1 - C_6\Delta\tau)(\|E^{n+1}\|^2_{\ell^2} + \|2E^{n+1} - E^n\|^2_{\ell^2}) &\le (1 + C_6\Delta\tau)(\|E^n\|^2_{\ell^2} + \|2E^n - E^{n-1}\|^2_{\ell^2})\\ &\quad + C_6\Delta\tau(\|E^{n-1}\|^2_{\ell^2} + \|2E^{n-1} - E^{n-2}\|^2_{\ell^2}).\end{aligned}$$

Note that in the case of $n = 1$, we require the vector E^{-1} defined by the zero vector. We finally conclude that the inequality in (4.11) holds by using Lemma 4.3 with $C_6\Delta\tau < 1/4$ and $C = e^{4C_6T}$ and then the time grid size $\Delta\tau$ is bounded by

$$\Delta\tau < \frac{\underline{\sigma}^2}{8\underline{\sigma}^2\left\{2\bar{\beta} + 3(\bar{\lambda} + \sqrt{2Q}\bar{\gamma})\right\} + 8(C_\sigma\bar{\sigma} + \bar{\alpha})^2}.$$

□

5. Operator splitting method for American options

In this section we consider the American option problem, which involves the early exercise boundary that is the curve splitting the continuation and early exercise regions, under the RSJD model with variable coefficients. The LCP (2.6) for the American option pricing is rewritten by using the auxiliary variable ψ

$$\begin{cases} u^i_\tau(\tau, x) - \mathcal{L}u^i(\tau, x) = \psi^i(\tau, x), \\ \psi^i(\tau, x) \geq 0, \quad u^i(\tau, x) \geq h(x), \quad \text{and} \quad \psi^i(\tau, x)\left(u^i(\tau, x) - h(x)\right) = 0 \end{cases} \tag{5.1}$$

for all $(\tau, x, e_i) \in (0, T] \times \mathbb{R} \times \mathcal{M}$, where $\mathcal{L}$ is the integro-differential operator in (2.5). In order to solve the LCP (5.1), we apply the operator splitting method suggested by Ikonen and Toivanen [10] under the Black–Scholes model. The operator splitting method combined with the IM method under the RSJD model with constant coefficients is described in [16]. The advantage of the operator splitting method is coupled with the three proposed methods for the prices of the European option and then the prices of the American option are evaluated efficiently without any fixed point iteration techniques at each economic state and time step.

The operator splitting method consists of two steps. In the first step we compute an intermediate approximation $\widetilde{U}^{n+1,i}_m$ on the $(n+1)$th time level by using the IM method

$$\frac{\widetilde{U}^{n+1,i}_m - U^{n-1,i}_m}{2\Delta\tau} - \mathcal{L}^1_\Delta U^{n,i}_m = \Psi^{n,i}_m \quad \text{for } n \geq 1, \tag{5.2}$$

where Ψ^n_m is an approximate solution of the auxiliary variable ψ^n_m on the (n)th time level. In the case of the CN method, the intermediate approximation $\widetilde{U}^{n+1}_m$ is evaluated by

$$\frac{\widetilde{U}^{n+1,i}_m - U^{n,i}_m}{\Delta\tau} - \mathcal{L}^2_\Delta U^{n+\frac{1}{2},i}_m = \Psi^{n,i}_m \tag{5.3}$$

and the value $\widetilde{U}^{n+1}_m$ by using the BDF2 method is given by

$$\frac{1}{\Delta\tau}\left(\frac{3}{2}\widetilde{U}^{n+1,i}_m - 2U^{n,i}_m + \frac{1}{2}U^{n-1,i}_m\right) - \mathcal{L}^3_\Delta \widetilde{U}^{n+1,i}_m = \Psi^{n,i}_m. \tag{5.4}$$

In the second step it is necessary to satisfy three constraints on the second line in the LCP (5.1). In the IM method, we can find out the values $U^{n+1,i}_m$ and $\Psi^{n+1,i}_m$ by updating the intermediate value $\widetilde{U}^{n+1,i}_m$ in accordance with the procedures described as

$$U^{n+1,i}_m = \max\left(h(x_m), \widetilde{U}^{n+1,i}_m - 2\Delta\tau\Psi^{n,i}_m\right) \quad \text{and} \quad \Psi^{n+1,i}_m = \Psi^{n,i}_m + \frac{U^{n+1,i}_m - \widetilde{U}^{n+1,i}_m}{2\Delta\tau}. \tag{5.5}$$

In the CN method, the numerical solutions $U^{n+1,i}_m$ and $\Psi^{n+1,i}_m$ on the $(n+1)$th time level are computed by

$$U^{n+1,i}_m = \max\left(h(x_m), \widetilde{U}^{n+1,i}_m - \Delta\tau\Psi^{n,i}_m\right) \quad \text{and} \quad \Psi^{n+1,i}_m = \Psi^{n,i}_m + \frac{U^{n+1,i}_m - \widetilde{U}^{n+1,i}_m}{\Delta\tau} \tag{5.6}$$

and in the BDF2 method, the values $U_m^{n+1,i}$ and $\Psi_m^{n+1,i}$ are obtained by

$$U_m^{n+1,i} = \max\left(h(x_m), \widetilde{U}_m^{n+1,i} - \frac{2}{3}\Delta\tau\Psi_m^{n,i}\right) \quad \text{and} \quad \Psi_m^{n+1,i} = \Psi_m^{n,i} + \frac{3}{2}\frac{U_m^{n+1,i} - \widetilde{U}_m^{n+1,i}}{\Delta\tau}. \tag{5.7}$$

6. Numerical experiments

In this section we present a variety of numerical results obtained by using the IM, CN, and BDF2 methods under the RSJD model with variable coefficients. The experiments are carried out with MATLAB on a computer with Intel(R) Core(TM) i7-4500U CPU 1.80 GHz. The three numerical methods are designed to avoid any fixed point iteration techniques at each economic state and time step. We take the initial stock price $S_0 = K$ and the boundary domain $\Gamma = (-2, 2)$ with $X = 2$. The prices of the European and American put options under the regime-switching Merton model are evaluated with three states of the economy. The jump size $\ln(\eta_i + 1)$ in the log price at i-state of the economy has the normal distribution with the mean μ_J^i and standard deviation σ_J^i. Then the integral $R(\tau, x, e_i, X)$ for the European put option is given by

$$R(\tau, x, e_i, X) = Ke^{-r_i\tau}\Phi\left(-\frac{x + X + \mu_J^i}{\sigma_J^i}\right) - S_0 e^{x - d_i\tau + \mu_J^i + \frac{{\sigma_J^i}^2}{2}}\Phi\left(-\frac{x + X + \mu_J^i + {\sigma_J^i}^2}{\sigma_J^i}\right)$$

and in the case of the American put option it becomes

$$R(\tau, x, e_i, X) = K\Phi\left(-\frac{x + X + \mu_J^i}{\sigma_J^i}\right) - S_0 e^{x + \mu_J^i + \frac{{\sigma_J^i}^2}{2}}\Phi\left(-\frac{x + X + \mu_J^i + {\sigma_J^i}^2}{\sigma_J^i}\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. In our simulation the maturity date is $T = 0.25$, the strike price is $K = 100$, and the other corresponding parameters under the regime-switching Merton model are given by

$$r = \begin{pmatrix} 0.03 \\ 0.03 \\ 0.03 \end{pmatrix}, \quad d = \begin{pmatrix} 0.05 \\ 0.05 \\ 0.05 \end{pmatrix}, \quad \mu_J = \begin{pmatrix} -0.95 \\ -0.90 \\ -0.70 \end{pmatrix}, \quad \sigma_J = \begin{pmatrix} 0.35 \\ 0.45 \\ 0.25 \end{pmatrix}, \quad \lambda = \begin{pmatrix} 0.10 \\ 0.30 \\ 0.50 \end{pmatrix},$$

and the $\mathcal{Q}$-matrix $\mathcal{A}$ of the Markov chain is

$$\mathcal{A} = \begin{pmatrix} -3.2 & 1.0 & 3.0 \\ 0.2 & -1.08 & 0.2 \\ 3.0 & 0.08 & -3.2 \end{pmatrix}.$$

The local volatilities at all economic states are given by

$$\sigma_1(\tau, x) = 0.15 + 0.15(0.5 + 2(T - \tau))\frac{(S_0e^x/100 - 1.2)^2}{(S_0e^x/100)^2 + 1.44},$$

$$\sigma_2(\tau, x) = 0.25 + 0.35(0.5 + 3(T - \tau))\frac{(S_0e^x/100 - 1.5)^2}{(S_0e^x/100)^2 + 1.7},$$

$$\sigma_3(\tau, x) = 0.35 - 0.5x + 1.9x^2 - 0.2(T - \tau) + 0.7x(T - \tau).$$

In Table 1, it is shown by using the IM method that the the prices of the European put option at $S = 90$, 100, and 110 and at the second state of the economy converge with the second-order accuracy. The European put option prices at the non-grid points are interpolated by a cubic spline method to keep the second-order

Table 1. The prices of the European put option at the second state of the economy under the regime-switching Merton model by using the IM method. N is the number of time steps and M is the number of spatial steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
25	32	12.547040	–	6.700565	–	4.272312	–
50	64	12.707038	0.159998	7.202881	0.502315	4.508911	0.236599
100	128	12.761998	0.054960	7.312828	0.109948	4.573042	0.064131
200	256	12.776191	0.014193	7.338883	0.026055	4.589256	0.016214
400	512	12.779760	0.003569	7.345326	0.006443	4.593319	0.004063
800	1024	12.780653	0.000893	7.346932	0.001607	4.594336	0.001016
1600	2048	12.780876	0.000223	7.347334	0.000401	4.594590	0.000254

Table 2. The rates of ℓ^2-errors for the prices of the European put option at the second state of the economy under the regime-switching Merton model by using the IM method. N is the number of time steps and M is the number of spatial steps. ϵ is the rate of convergence defined by (6.1).

N	M	$\|U^2(\Delta\tau, \Delta x) - U^2(\Delta\tau/2, \Delta x/2)\|_{\ell^2}$	ϵ
25	32		
		0.186668866336289	
50	64		
		0.043986952977886	2.085
100	128		
		0.010789045294775	2.028
200	256		
		0.002685546645451	2.006
400	512		
		0.000670676309997	2.002
800	1024		
		0.000167625670402	2.000
1600	2048		

accuracy in the spatial variable and the errors are computed by the successive changes of the option prices at each stock price. In order to check the discrete ℓ^2-norm, the ratio ϵ is computed by

$$\epsilon = \log_2 \frac{\|U^i(\Delta\tau, \Delta x) - U^i(\Delta\tau/2, \Delta x/2)\|_{\ell^2}}{\|U^i(\Delta\tau/2, \Delta x/2) - U^i(\Delta\tau/4, \Delta x/4)\|_{\ell^2}}, \tag{6.1}$$

where $U^i(\Delta\tau, \Delta x)$ is the option price on $\tau = T$ at the i-state of the economy. We can observe in Table 2 that the ratio ϵ converges to 2 as the grid points in the time and spatial variables are increased twice and these are our expected results demonstrated in Section 4.

In Tables 3 and 4, the prices of the American put option at the third state of the economy are evaluated by employing the BDF2 method with extrapolation. It is reported that the pointwise errors at $S = 90$, 100, and 110 are decreased with the quadratic convergence in the time and spatial variable and the BDF2 method has the second-order convergence rate with respect to the discrete ℓ^2-norm.

The curves of the European and American put option prices at all states of the economy are plotted in Figure 1. The most important property of the American put option is to find the early exercise boundary and

TABLE 3. The prices of the American put option at the third state of the economy under the regime-switching Merton model by using the BDF2 method combined with the operator splitting method. N is the number of time steps and M is the number of spatial steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
25	32	13.543064	–	7.623522	–	4.985358	–
50	64	13.724032	0.180968	8.092436	0.468914	5.208762	0.223403
100	128	13.773800	0.049768	8.192464	0.100028	5.267481	0.058720
200	256	13.786493	0.012693	8.216120	0.023656	5.282348	0.014867
400	512	13.789676	0.003183	8.221968	0.005848	5.286077	0.003729
800	1024	13.790472	0.000796	8.223426	0.001458	5.287009	0.000933
1600	2048	13.790671	0.000199	8.223790	0.000364	5.287243	0.000233

TABLE 4. The rates of ℓ^2-errors for the prices of the American put option at the third state of the economy under the regime-switching Merton model by using the BDF2 method combined with the operator splitting method. N is the number of time steps and M is the number of spatial steps. ϵ is the rate of convergence defined by (6.1).

N	M	$\Vert U^3(\Delta\tau, \Delta x) - U^3(\Delta\tau/2, \Delta x/2)\Vert_{\ell^2}$	ϵ
25	32		
		0.178671689726253	
50	64		
		0.040212335575868	2.152
100	128		
		0.009866654697158	2.027
200	256		
		0.002457110460194	2.006
400	512		
		0.000614269368532	2.000
800	1024		
		0.000153852136539	1.997
1600	2048		

these curves at all economic states are illustrated in Figure 2. We note that the early exercise boundary is not convex along the time to maturity τ. It is proved in [3] that the exercise boundary is not convex if the dividend rate is slightly larger than the risk-free interest rate in a geometric Brownian motion.

In Figure 3, we present the temporal errors of the IM, CN, and BDF2 methods for the prices of the European and American put options in the grid size $\Delta\tau$. The temporal error is the discrete ℓ^2-norm computed by the successive changes of the put option prices when the grid sizes $\Delta\tau$ and Δx are reduced by half and is given by

$$\text{Temporal error} = \Vert U^i(\Delta\tau, \Delta x) - U^i(\Delta\tau/2, \Delta x/2)\Vert_{\ell^2}.$$

It is shown in Figure 3 that the three numerical methods have the second-order convergence in the discrete ℓ^2-norm with respect to the time and spatial variables as the slopes of the lines are almost 2.

We now turn to take into account the intensity λ_t of the Poisson process N_t about which Salmi and Toivanen [21] have studied three implicit methods under the jump-diffusion model. The intensity λ in the above parameters

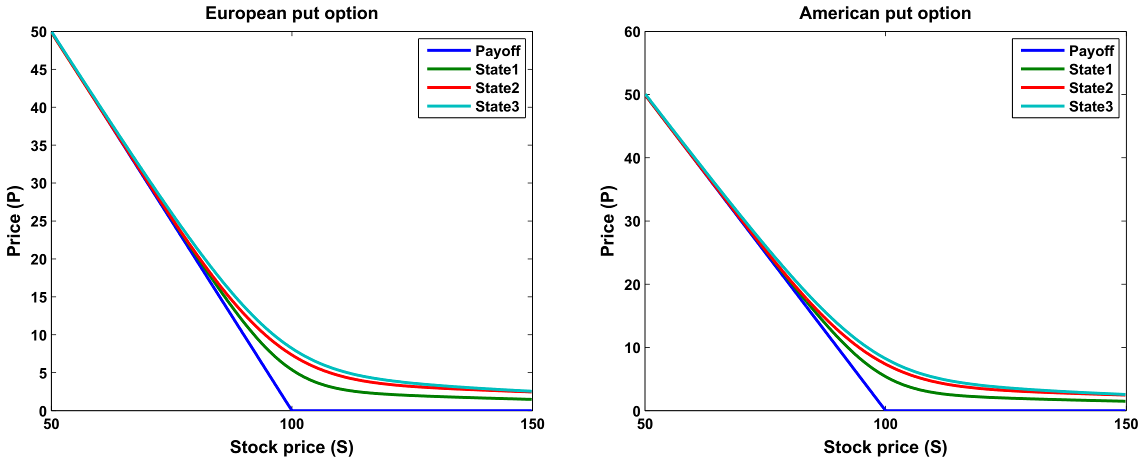

FIGURE 1. The put option price curves obtained with 1600 time and 2048 spatial steps at $\tau = T$ under the regime-switching Merton model. *Left panel*: European option by using the IM method. *Right panel*: American option by using the BDF2 method.

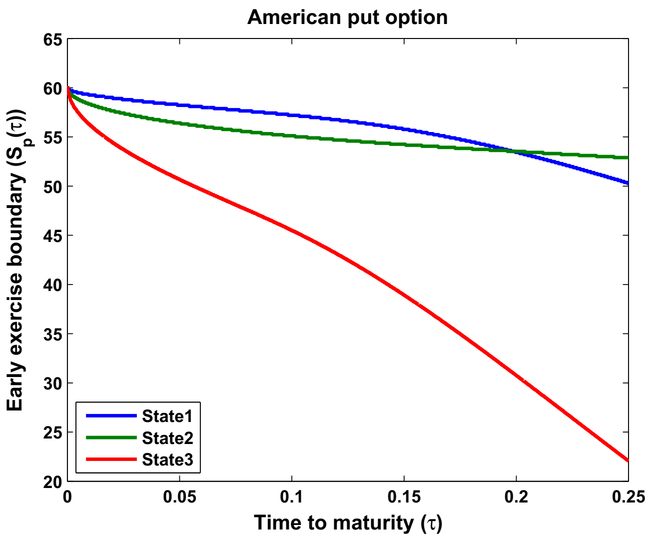

FIGURE 2. The early exercise boundary curves for the American put option obtained with $N = 1600$ and $M = 2048$ under the regime-switching Merton model by using the BDF2 method combined with the operator splitting method.

under the regime-switching Merton model with variable coefficients is replaced with the larger one of the form

$$\lambda = \begin{pmatrix} 10 \\ 30 \\ 50 \end{pmatrix}. \tag{6.2}$$

The three numerical methods are compared with each other in our simulation and then we recognize in Figure 4 that the IM, CN, and BDF2 methods are stable even though the intensity λ is large. On the other hand, we

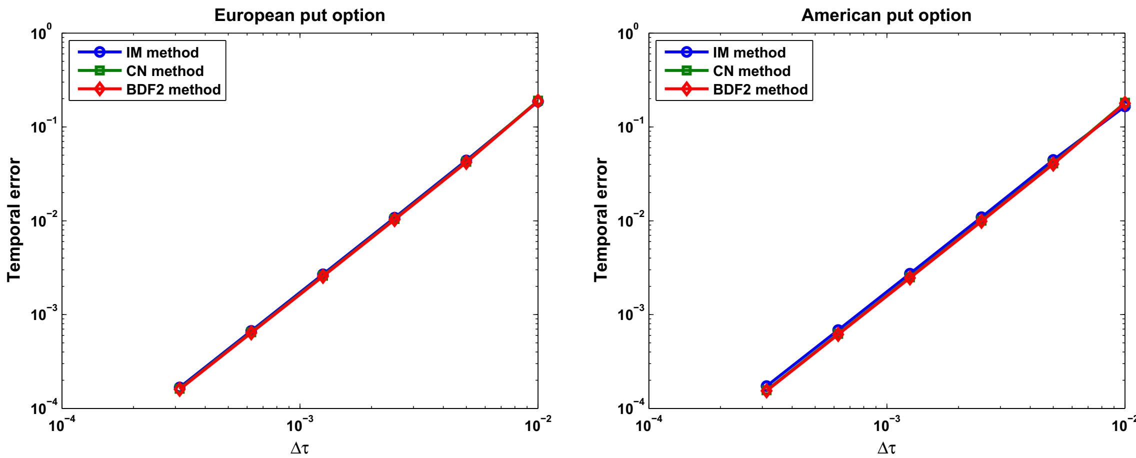

FIGURE 3. Temporal errors of the three numerical methods for the put option under the regime-switching Merton model. *Left panel*: European option at the second state. *Right panel*: American option at the third state.

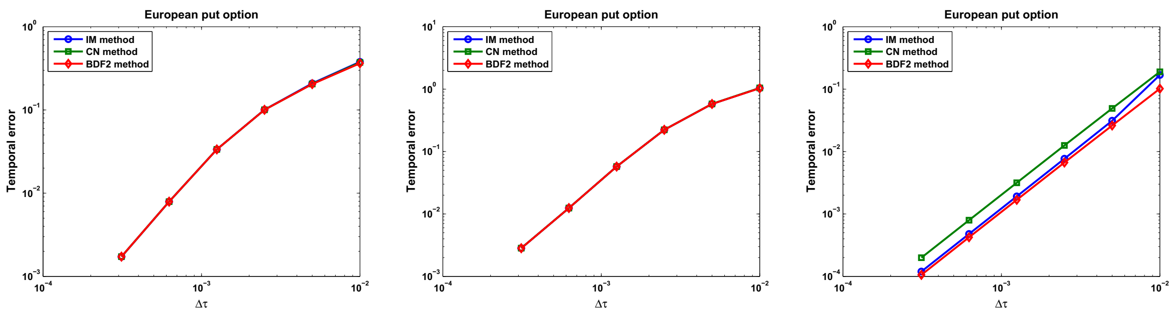

FIGURE 4. Temporal errors of the three numerical methods with $\lambda = (10, 30, 50)^{\mathrm{T}}$ for the European put option under the regime-switching Merton model. *Left panel*: first state. *Center panel*: second state. *Right panel*: third state.

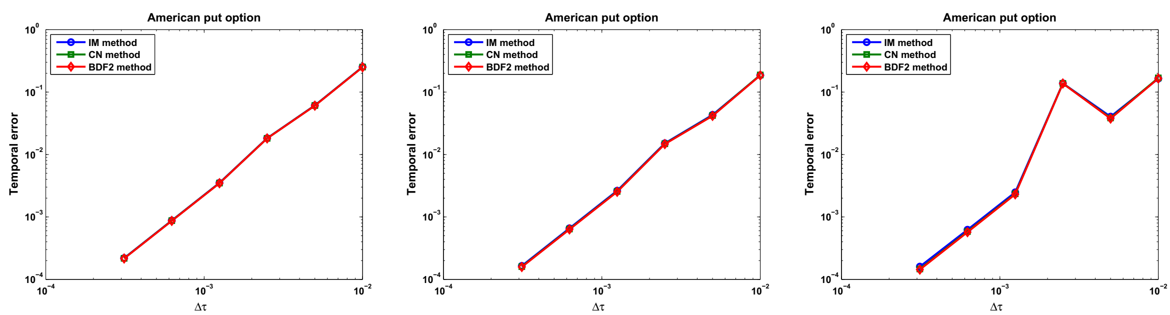

FIGURE 5. Temporal errors of the three numerical methods with the $\mathcal{Q}$-matrix $\mathcal{A}$ in (6.3) for the American put option under the regime-switching Merton model. *Left panel*: first state. *Center panel*: second state. *Right panel*: third state.

TABLE 5. The prices of the up-and-out European call option at the first state of the economy under the regime-switching Merton model by using the CN method. N is the number of time steps and M is the number of spatial steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
25	32	1.034135	–	3.674579	–	8.197534	–
50	64	1.107881	0.073746	4.008503	0.333923	8.519450	0.321916
100	128	1.128039	0.020157	4.085371	0.076868	8.614290	0.094840
200	256	1.133204	0.005165	4.104280	0.018909	8.638654	0.024364
400	512	1.134502	0.001298	4.108988	0.004708	8.644778	0.006124
800	1024	1.134827	0.000325	4.110164	0.001175	8.646310	0.001532
1600	2048	1.134908	0.000081	4.110458	0.000294	8.646694	0.000383

TABLE 6. The prices of the down-and-out European call option at the first state of the economy under the regime-switching Merton model by using the CN method. N is the number of time steps and M is the number of spatial steps.

		$S = 90$		$S = 100$		$S = 110$	
N	M	Value	Error	Value	Error	Value	Error
25	32	1.227158	–	4.413106	–	12.150319	–
50	64	1.259344	0.032187	4.786614	0.373508	12.206459	0.056141
100	128	1.273618	0.014274	4.866686	0.080072	12.228711	0.022252
200	256	1.277632	0.004013	4.885389	0.018703	12.234796	0.006085
400	512	1.278666	0.001034	4.890003	0.004614	12.236344	0.001549
800	1024	1.278926	0.000260	4.891152	0.001150	12.236733	0.000389
1600	2048	1.278991	0.000065	4.891440	0.000287	12.236830	0.000097

consider that the $\mathcal{Q}$-matrix $\mathcal{A}$ is dependent of the underlying asset S given by

$$\mathcal{A} = \begin{pmatrix} -0.8 & 0.6 + 0.2\sin S & 0.7\cos^2 S \\ 0.3 & -1.1 - 0.5\sin S & 0.3\cos^2 S \\ 0.5 & 0.5 + 0.3\sin S & -\cos^2 S \end{pmatrix}. \tag{6.3}$$

Then it can be also observed in Figure 5 that the three numerical methods have the second-order convergence rate in the discrete ℓ^2-norm when the number of the grid points in the time and spatial variables are doubled.

As another numerical experiment, two types of the barrier options are considered in which one is an up-and-out European call option with a barrier level $B = 130$ and the other is a down-and-out European call option with a barrier level $B = 70$. We note that the three numerical methods can be applied with the boundary domain $\Gamma = (X_{\min}, X_{\max})$ even if the value $X_{\min}$ is not equal to $-X_{\max}$. The boundary domain $\Gamma = (-7X, X)$ with $X = \ln(B/K)$ and $B = 130$ is taken to compute the prices of the up-and-out European call option with the corresponding parameters described above. The payoff function of the up-and-out European call option is

$$h(x) = \max(0, S_0 e^x - K) \cdot 1_{\{S_0 e^x < B\}},$$

where 1_A is the indicator function with respect to A and the boundary condition $g_i(\tau, x)$ at the i-state of the economy is given by

$$g_i(\tau, x) = 0 \quad \text{for } x \in \mathbb{R} \setminus \Gamma.$$

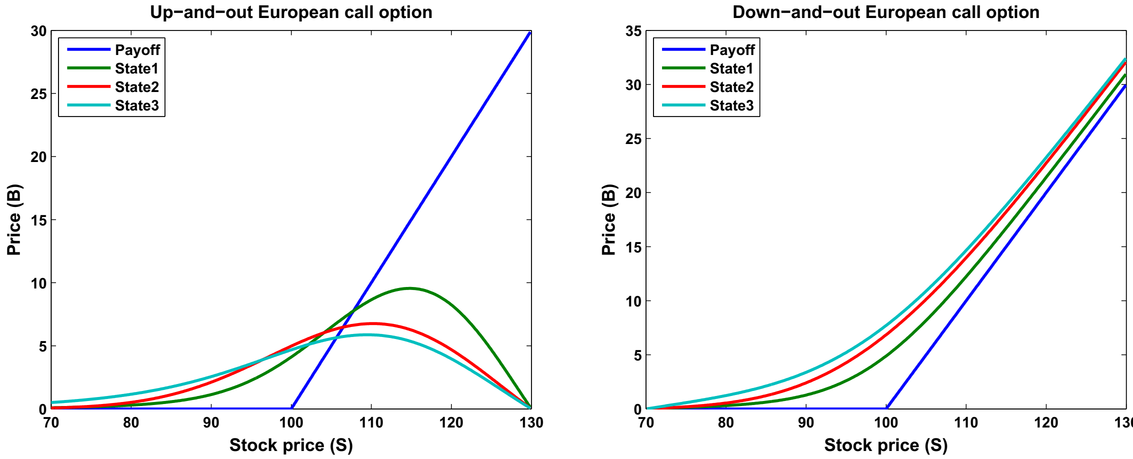

FIGURE 6. The barrier option price curves obtained with 1600 time and 2048 spatial steps at $\tau = T$ under the regime-switching Merton model by using the CN method. *Left panel*: up-and-out European call option. *Right panel*: down-and-out European call option.

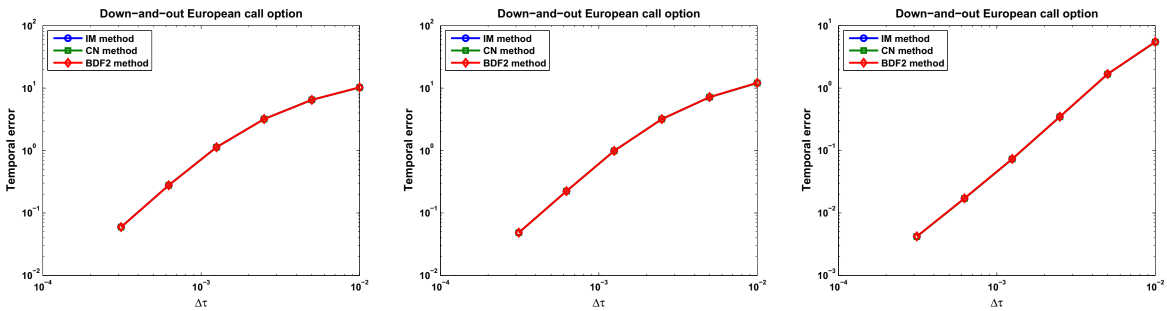

FIGURE 7. Temporal errors of the three numerical methods with $\lambda = (10, 30, 50)^{\mathrm{T}}$ for the down-and-out European call option under the regime-switching Merton model. *Left panel*: first state. *Center panel*: second state. *Right panel*: third state.

In the case of the down-and-out European call option, we take the boundary domain $\Gamma = (-X, 7X)$ with $X = -\ln(B/K)$ and $B = 70$. The payoff function of the down-and-out European call option is

$$h(x) = \max(0, S_0 e^x - K) \cdot 1_{\{S_0 e^x > B\}}$$

and the boundary condition $g_i(\tau, x)$ at the i-state of the economy is given by

$$g_i(\tau, x) = \begin{cases} 0 & \text{for } x \leq X_{\min}, \\ S_0 e^{x - \int_0^\tau d_i(s)\mathrm{d}s} - K e^{-\int_0^\tau r_i(s)\mathrm{d}s} & \text{for } x \geq X_{\max}. \end{cases}$$

The prices of the two barrier options at the first state of the economy are evaluated by employing the CN method with extrapolation in Tables 5 and 6. We can observe that the pointwise errors have the second-order convergence rate in the time and spatial variables. The curves of the two barrier option prices at all states of the economy are plotted in Figure 6. For the given large intensity λ in (6.2), the temporal errors of the three numerical methods for the prices of the down-and-out European call option are presented in Figure 7 and then the second-order accuracy in the discrete ℓ^2-norm is guaranteed with the IM, CN, and BDF2 methods.

7. Conclusion

In this paper the three numerical methods, called IM, CN, and BDF2 methods, are developed to solve the PIDE for the prices of the European and barrier options under the RSJD model with variable coefficients. We extend the proposed methods to evaluate the prices of the American option by combining the operator splitting method. Especially the three numerical methods are designed to avoid any fixed point iteration techniques at each state of the economy and time step and to preserve the second-order convergence in the discrete ℓ^2-norm in the time and spatial variables. The stability of the proposed methods to solve the PIDE with variable coefficients is demonstrated in the discrete ℓ^2-norm. A number of numerical experiments are performed to show the second-order convergence rate under the regime-switching Merton model. The proposed methods require the computational cost of $O(QMN\log_2 M)$ operations as the fast Fourier transform (FFT) is applied to compute the integral over the bounded domain Γ. We can also observe that the IM, CN, and BDF2 methods are stable when the jump intensity λ is large enough under the RSJD model with variable coefficients.

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