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QUASI-OPTIMALITY OF AN ADAPTIVE FINITE ELEMENT METHOD FOR CATHODIC PROTECTION

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Abstract. In this work, we derive a reliable and efficient residual-typed error estimator for the finite element approximation of a 2D cathodic protection problem governed by a steady-state diffusion equation with a nonlinear boundary condition. We propose a standard adaptive finite element method involving the Dörfler marking and a minimal refinement without the interior node property. Furthermore, we establish the contraction property of this adaptive algorithm in terms of the sum of the energy error and the scaled estimator. This essentially allows for a quasi-optimal convergence rate in terms of the number of elements over the underlying triangulation. Numerical experiments are provided to confirm this quasi-optimality.

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1. Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with its boundary Γ consisting of three mutually disjoint parts: $\Gamma := \Gamma_0 \cup \Gamma_A \cup \Gamma_C$, all of which are line segments, such that Γ_0 and $\Gamma_0 \cup \Gamma_A$ are closed while Γ_C is open. This work is concerned with the numerical treatment of the following problem:

$$-\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\sigma \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_0, \quad \sigma \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_A, \quad \sigma \frac{\partial u}{\partial n} = -f(u) \quad \text{on } \Gamma_C, \tag{1.2}$$

where the conductivity σ is assumed to be a piecewise $W^{1,\infty}$ function such that $\sigma_1 \leq \sigma \leq \sigma_2$ a.e. in Ω with two positive constants σ_1 and σ_2, n is the unit outward normal on Γ and $g \in L^2(\Gamma_A)$. The system (1.1) and (1.2) arises in cathodic protection in electrochemistry. In a container Ω occupied by electrolyte, the first boundary condition in (1.2) describes insulation of the surface Γ_0 by painting. The second boundary equation in (1.2) reflects the fact that a current density g on anodes Γ_A induces an electrical potential u in Ω governed by (1.1). The corrosion process on cathodes Γ_C is slowed down through the nonlinear relation f, which depends on the

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electrode material and is given by [16] either

$$f_1(u) = C_1 u + C_2 u^3 \tag{1.3}$$

or the Butler–Volmer function

$$f_2(u) = C_5(e^{C_3 u} - e^{-C_4 u}), \tag{1.4}$$

where C_i, $i = 1 \dots 5$, are all positive constants.

In problems (1.1) and (1.2), the sudden change of the boundary condition from Neumann type on anodes and the insulated part to a nonlinear one on cathodes gives rise to local solution singularities in these regions. Furthermore, internal layers may appear due to the discontinuity of the conductivity. Consequently, the computational efficiency will be compromised if a uniform mesh refinement is employed in the finite element discretization. One remedy in practice is to employ adaptivity techniques featuring local refinement so that numerical results can attain better accuracy with minimum degrees of freedom. The aim of this work is to investigate the computational complexity of an adaptive finite element method (AFEM) for problems (1.1) and (1.2).

A typical adaptive algorithm comprises successive iterations of the following loop:

$$\text{SOLVE} \ \rightarrow \ \text{ESTIMATE} \ \rightarrow \ \text{MARK} \ \rightarrow \ \text{REFINE}. \tag{1.5}$$

That is, SOLVE yields a finite element approximation on the current mesh; ESTIMATE computes the relevant *a posteriori* error estimator; MARK picks some elements to be subdivided; REFINE produces a new finer mesh.

The module ESTIMATE, depending on some computable quantities, *i.e.*, the discrete solution, local mesh size and given problem data, plays an indispensable role in (1.5). Since the seminal work [2], *a posteriori* error estimation for FEMs has been well understood in scientific computing and engineering [1, 27]. As to the mathematical theory of AFEM, *e.g.*, convergence and computational complexity, there have been great developments (see the overview [6, 22] and the references therein) over the past thirty years. For linear elliptic problems, this issue has been investigated at depth [4, 7, 10, 12, 24]. Recently, the analysis has been extended to some nonlinear problems; see [3, 9] for p-Laplacian and [14, 15] for quasi-linear equations.

Recently, we [20] have proposed an AFEM of the form (1.5) for problems (1.1) and (1.2) and proved its plain convergence, namely the H^1-norm error and the sequence of relevant estimators both go to zero as the loop (1.5) proceeds. This work is a continuation of [20], and it is devoted to the complexity of the algorithm. In the AFEM, ESTIMATE, MARK and REFINE use a residual-type *a posteriori* error estimator, Dörfler strategy and the bisection [19, 25], respectively. The main contributions include a contraction property in Theorem 5.1 and the quasi-optimality computational complexity in terms of the number of elements associated with underlying triangulations in Theorem 6.4.

Our analysis is inspired by [12], to first obtain optimal marking for the error estimator, *cf.* Lemma 6.2 and then the optimal decay rate for the energy error plus an oscillation term so that the upper bound of the parameter in Dörfler strategy is independent of the efficiency constant. However, the nonlinear term f on the boundary Γ_C requires a different treatment. First, due to the presence of the nonlinear term f on the boundary Γ_C, the Galerkin orthogonality fails. We employ the energy functional instead of the energy norm as in [9, 15]. By the equivalence of $\mathcal{J}(u_\mathcal{T}) - \mathcal{J}(u)$ and $\|u - u_\mathcal{T}\|_{H^1(\Omega)}$, *cf.* Lemma 4.2, we prove that the adaptive algorithm reduces the sum of energy error and the scaled estimator for any two consecutive iterations. Second, instead of standard arguments for linear problems, we use the generalized Hölder inequality and the stability of solutions to establish a Céa-type lemma for complexity estimate, *cf.* Lemma 4.7.

The remainder of the paper is organized as follows. Section 2 is devoted to the *a posteriori* error analysis. An adaptive algorithm to approximate problems (1.1) and (1.2) is described in Section 3. We prove the convergence of this algorithm by a contraction property in Section 5 after presenting preliminary results in Section 4. Section 6 focuses on the quasi-optimal convergence rate, which is illustrated by two numerical examples in Section 7. The paper is ended with some concluding remarks. Throughout, we adopt standard notation for Sobolev spaces and related norms and semi-norms. Moreover, any generic constant, with or without subscript, is independent of the mesh size and is not necessarily the same at each occurrence.

2. *A posteriori* error analysis

In this section, we shall derive a residual-type error estimator for the finite element approximation of problem (2.1), which forms the basis of our AFEM. To introduce the AFEM, we first recall the variational formulation of problems (1.1) and (1.2): find $u \in H^1(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\Gamma_C} f(u) v \,\mathrm{d}s = \int_{\Gamma_A} g v \,\mathrm{d}s \quad \forall\, v \in H^1(\Omega). \tag{2.1}$$

We refer to [16, 17] for its unique solvability. For f defined in (1.3) and (1.4), it is easy to check that f' is convex and there exists an $\alpha > 0$ such that

$$f'(t) \geq \alpha \quad \forall\, t \in \mathbb{R}, \qquad \text{and} \qquad f(0) = 0. \tag{2.2}$$

Then by the application of the Poincaré inequality, the boundedness of σ and the trace theorem, we arrive at

$$\beta_1 \|v\|^2_{H^1(\Omega)} \leq \int_\Omega \sigma |\nabla v|^2 \mathrm{d}x + \alpha \int_{\Gamma_C} v^2 \mathrm{d}s \leq \beta_2 \|v\|^2_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega), \tag{2.3}$$

where β_1 and β_2 are positive constants depending only on σ, α and Ω.

Utilizing (2.1) with $v := u$, together with the mean value theorem, (2.2) and (2.3), we can obtain

$$\begin{aligned} \beta_1 \|u\|^2_{H^1(\Omega)} &\leq \int_\Omega \sigma |\nabla u|^2 \mathrm{d}x + \alpha \int_{\Gamma_C} u^2 \mathrm{d}s \leq \int_\Omega \sigma |\nabla u|^2 \mathrm{d}x + \int_{\Gamma_C} f(u) u \mathrm{d}s \\ &= \int_{\Gamma_A} g u \mathrm{d}s \leq \|g\|_{L^2(\Gamma_A)} \|u\|_{L^2(\Gamma_A)}. \end{aligned}$$

Then the trace theorem yields the *a priori* estimate to problem (2.1):

$$\|u\|_{H^1(\Omega)} \leq C_{\mathrm{stb}} \|g\|_{L^2(\Gamma_A)} \tag{2.4}$$

with $C_{\mathrm{stb}} > 0$ being a constant depending only on σ, α and Ω.

Furthermore, the trace theorem the continuity of the embedding $H^1(\Omega) \hookrightarrow H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^q(\Gamma)$ in 2D for all $q < \infty$ implies the existence of a constant $C_{\mathrm{imb},q} > 0$ depending on Ω and q, satisfying

$$\|v\|_{L^q(\Gamma_C)} \leq C_{\mathrm{imb},q} \|v\|_{H^1(\Omega)} \quad \forall\, v \in H^1(\Omega). \tag{2.5}$$

Next, we proceed to the discretization. Let $\mathcal{T}$ be a shape-regular conforming triangulation of $\bar{\Omega}$ into a set of disjoint closed triangles such that the coefficient σ is piecewise $W^{1,\infty}$ over $\mathcal{T}_0$. For each element $T \in \mathcal{T}$, we denote its mesh size $h_T := |T|^{\frac{1}{2}}$ and ρ_T the diameter of the largest inscribed ball. We associate each triangulation $\mathcal{T}$ with its shape regular parameter $C_{\mathcal{T}} := \max_{T \in \mathcal{T}} \frac{h_T}{\rho_T}$. Over the mesh $\mathcal{T}$, we consider the usual H^1-conforming finite element space $V_{\mathcal{T}}$, consisting of all piecewise polynomials of degree less than or equal to $m \in \mathbb{N}_+$, *i.e.*,

$$V_{\mathcal{T}} := \{ v \in H^1(\Omega) \mid v|_T \in P_m(T), \forall\, T \in \mathcal{T} \}.$$

Then the discrete problem corresponding to (2.1) reads: find $u_{\mathcal{T}} \in V_{\mathcal{T}}$ such that

$$\int_\Omega \sigma \nabla u_{\mathcal{T}} \cdot \nabla v_{\mathcal{T}} \mathrm{d}x + \int_{\Gamma_C} f(u_{\mathcal{T}}) v_{\mathcal{T}} \mathrm{d}s = \int_{\Gamma_A} g v_{\mathcal{T}} \mathrm{d}s \quad \forall\, v_{\mathcal{T}} \in V_{\mathcal{T}}. \tag{2.6}$$

Similar to the continuous case, the following stability estimate holds

$$\|u_{\mathcal{T}}\|_{H^1(\Omega)} \leq C_{\mathrm{stb}} \|g\|_{L^2(\Gamma_A)}. \tag{2.7}$$

To describe the error estimator, we need a few notation and definitions. The collection of all edges (resp. all interior edges) in $\mathcal{T}$ is denoted by $\mathcal{F}_{\mathcal{T}}$ (resp. $\mathcal{F}_{\mathcal{T}}(\Omega)$) and its restriction on Γ (resp. Γ_0, Γ_A and Γ_C) by $\mathcal{F}_{\mathcal{T}}(\Gamma)$ (resp. $\mathcal{F}_{\mathcal{T}}(\Gamma_0)$, $\mathcal{F}_{\mathcal{T}}(\Gamma_A)$ and $\mathcal{F}_{\mathcal{T}}(\Gamma_C)$). Analogously, $\mathcal{F}_{\mathcal{T}}(T)$ is the collection of all edges in $\mathcal{F}_{\mathcal{T}}$ restricted on the boundary of $T \in \mathcal{T}$. The scalar $h_F := |F|$ stands for the diameter of $F \in \mathcal{F}_{\mathcal{T}}$, which is associated with a fixed normal unit vector n_F in $\bar{\Omega}$ with $n_F = n$ on the boundary $\partial\Omega$. For each $T \in \mathcal{T}$, we denote ω_T as the union of all elements in $\mathcal{T}$ with non-empty intersection with element T. For any $F \in \mathcal{F}_{\mathcal{T}}$, ω_F is the union of two elements that share F. Further, we let

$$C_{\text{ov}} := \max_{\tilde{T} \in \mathcal{T}} \#\{T \in \mathcal{T} : \tilde{T} \subset \omega_T\}.$$

Let $I_{\mathcal{T}}^{sz} : H^1(\Omega) \to V_{\mathcal{T}}^m$ be the Scott-Zhang quasi-interpolation operator over $\mathcal{T}$ [23]. Then for all $v \in H^1(\Omega)$, $T \in \mathcal{T}$ and $F \in \mathcal{F}_{\mathcal{T}}(T) \cap \mathcal{F}_{\mathcal{T}}$, there holds

$$h_T^{-\frac{1}{2}} \|v - I_{\mathcal{T}}^{sz} v\|_{L^2(F)} + h_T^{-1} \|v - I_{\mathcal{T}}^{sz} v\|_{L^2(T)} \le C_I \|\nabla v\|_{L^2(\omega_T)}, \tag{2.8}$$

with C_I a constant depending only on the shape regularity parameter $C_{\mathcal{T}}$.

For any $v_{\mathcal{T}} \in V_{\mathcal{T}}$, we define the residuals on each element $T \in \mathcal{T}$ and each edge $F \in \mathcal{F}_{\mathcal{T}}$ by

$$\begin{aligned} R_T(v_{\mathcal{T}}) &:= \nabla \cdot (\sigma \nabla v_{\mathcal{T}}), \\ J_F(v_{\mathcal{T}}) &:= \begin{cases} [\sigma \nabla v_{\mathcal{T}} \cdot n_F] & \text{for } F \in \mathcal{F}_{\mathcal{T}}(\Omega), \\ \sigma \nabla v_{\mathcal{T}} \cdot n & \text{for } F \in \mathcal{F}_{\mathcal{T}}(\Gamma_0), \\ g - \sigma \nabla v_{\mathcal{T}} \cdot n & \text{for } F \in \mathcal{F}_{\mathcal{T}}(\Gamma_A), \\ f(v_{\mathcal{T}}) + \sigma \nabla v_{\mathcal{T}} \cdot n & \text{for } F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C), \end{cases} \end{aligned}$$

where $[\cdot]$ denotes jumps across interior edges F:

$$[v](x) = \lim_{t \to 0^+} v(x - t n_F) - \lim_{t \to 0^-} v(x + t n_F).$$

Then the local error indicator on any element $T \in \mathcal{T}$ is defined by

$$\begin{aligned} \eta_{\mathcal{T}}^2(v_{\mathcal{T}}, T) := h_T^2 \|R_T(v_{\mathcal{T}})\|_{L^2(T)}^2 &+ \frac{1}{2} \sum_{F \in \mathcal{F}_{\mathcal{T}}(T) \cap \mathcal{F}_{\mathcal{T}}(\Omega)} h_T \|J_F(v_{\mathcal{T}})\|_{L^2(F)}^2 \\ &+ \sum_{F \in \mathcal{F}_{\mathcal{T}}(T) \cap \mathcal{F}_{\mathcal{T}}(\Gamma)} h_T \|J_F(v_{\mathcal{T}})\|_{L^2(F)}^2. \end{aligned} \tag{2.9}$$

The error estimator over a set of elements $\mathcal{M} \subseteq \mathcal{T}$ is

$$\eta_{\mathcal{T}}^2(v_{\mathcal{T}}, \mathcal{M}) := \sum_{T \in \mathcal{M}} \eta_{\mathcal{T}}^2(v_{\mathcal{T}}, T).$$

Similarly, the oscillation term can be defined locally and globally by

$$\text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, T) := h_T^2 \|R_T(v_{\mathcal{T}}) - \bar{R}_T(v_{\mathcal{T}})\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_{\mathcal{T}}(T)} h_T \|J_F(v_{\mathcal{T}}) - \bar{J}_F(v_{\mathcal{T}})\|_{L^2(F)}^2, \tag{2.10}$$

$$\text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, \mathcal{M}) := \sum_{T \in \mathcal{M}} \text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, T). \tag{2.11}$$

Here, $\bar{R}_T(v_{\mathcal{T}})$ is the integral average of $R_T(v_{\mathcal{T}})$ over T if $m = 1$, or the L^2-projection on $P_{m-2}(T)$ if $m \ge 2$. $\bar{J}_F(v_{\mathcal{T}})$ is the L^2-projection of $J_F(v_{\mathcal{T}})$ on $P_{m-1}(F)$ if $F \in \mathcal{F}_{\mathcal{T}} \setminus \mathcal{F}_{\mathcal{T}}(\Gamma_C)$. When $F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C)$, $\bar{J}_F(v_{\mathcal{T}}, g)$ is the L^2-projection on $P_{3m}(F)$ for f in (1.3) and the L^2-projection on $P_{m-1}(F)$ for f in (1.4). To simplify our notations, we denote $\eta_{\mathcal{T}}(v_{\mathcal{T}}) := \eta_{\mathcal{T}}(v_{\mathcal{T}}, \mathcal{T})$ and $\text{osc}_{\mathcal{T}}(v_{\mathcal{T}}) := \text{osc}_{\mathcal{T}}(v_{\mathcal{T}}, \mathcal{T})$.

The following upper and lower bounds on the error estimator were given in [20]. Here we give a more precise estimate with respect to the occurring constant. For completeness, we provide the proof, since a related argument will be used in the proof of Lemma 4.9.

Theorem 2.1 (Reliability)**.** *Let $u \in H^1(\Omega)$ and $u_{\mathcal{T}} \in V_{\mathcal{T}}$ be the solutions to problems* (2.1) *and* (2.6)*, respectively. Then there exists a positive constant $C_{\rm rel} > 0$ depending only on σ, α, Ω and $C_{\mathcal{T}}$ such that*

$$\|u - u_{\mathcal{T}}\|^2_{H^1(\Omega)} \le C_{\rm rel}\ \eta^2_{\mathcal{T}}(u_{\mathcal{T}}).$$

Proof. By (2.3), (2.2), mean value theorem and (2.1) with $v := u - u_{\mathcal{T}}$, we deduce

$$\begin{aligned}\beta_1\|u - u_{\mathcal{T}}\|^2_{H^1(\Omega)} &\le \int_\Omega \sigma|\nabla(u-u_{\mathcal{T}})|^2 {\rm d}x + \alpha\int_{\Gamma_C}(u-u_{\mathcal{T}})^2{\rm d}s\\ &\le \int_\Omega \sigma|\nabla(u-u_{\mathcal{T}})|^2 {\rm d}x + \int_{\Gamma_C}(f(u)-f(u_{\mathcal{T}}))(u-u_{\mathcal{T}}){\rm d}s\\ &= \int_{\Gamma_A} gv{\rm d}s - \int_\Omega \sigma\nabla u_{\mathcal{T}}\cdot\nabla v{\rm d}x - \int_{\Gamma_C} f(u_{\mathcal{T}})v{\rm d}s.\end{aligned} \tag{2.12}$$

The discrete variational equation (2.6) implies

$$\int_{\Gamma_A} gv_{\mathcal{T}}{\rm d}s - \int_\Omega \sigma\nabla u_{\mathcal{T}}\cdot\nabla v_{\mathcal{T}}{\rm d}x - \int_{\Gamma_C} f(u_{\mathcal{T}})v_{\mathcal{T}}{\rm d}s = 0 \quad \forall\ v_{\mathcal{T}} \in V_{\mathcal{T}}.$$

These two estimates together yield

$$\beta_1\|u - u_{\mathcal{T}}\|^2_{H^1(\Omega)} \le \int_{\Gamma_A} g(v-v_{\mathcal{T}}){\rm d}s - \int_\Omega \sigma\nabla u_{\mathcal{T}}\cdot\nabla(v-v_{\mathcal{T}}){\rm d}x - \int_{\Gamma_C} f(u_{\mathcal{T}})(v-v_{\mathcal{T}}){\rm d}s \quad \forall\ v_{\mathcal{T}} \in V_{\mathcal{T}}.$$

By elementwise integration by parts and taking $v_{\mathcal{T}} := I^{sz}_{\mathcal{T}}v$, we obtain

$$\begin{aligned}\beta_1\|u - u_{\mathcal{T}}\|^2_{H^1(\Omega)} &\le \int_{\Gamma_A} g(v-v_{\mathcal{T}}){\rm d}s - \int_\Omega \sigma\nabla u_{\mathcal{T}}\cdot\nabla(v-v_{\mathcal{T}}){\rm d}x - \int_{\Gamma_C} f(u_{\mathcal{T}})(v-v_{\mathcal{T}}){\rm d}s\\ &= \sum_{T\in\mathcal{T}}\Big(\int_T \nabla\cdot(\sigma\nabla u_{\mathcal{T}})(v-v_{\mathcal{T}}){\rm d}x - \frac{1}{2}\sum_{F\in\mathcal{F}_{\mathcal{T}}(T)\cap\mathcal{F}_{\mathcal{T}}(\Omega)}\int_F[\sigma\nabla u_{\mathcal{T}}\cdot n_F](v-v_{\mathcal{T}}){\rm d}s\\ &\quad - \sum_{F\in\mathcal{F}_{\mathcal{T}}(T)\cap\mathcal{F}_{\mathcal{T}}(\Gamma_0)}\int_F \sigma\nabla u_{\mathcal{T}}\cdot n(v-v_{\mathcal{T}}){\rm d}s + \sum_{F\in\mathcal{F}_{\mathcal{T}}(T)\cap\mathcal{F}_{\mathcal{T}}(\Gamma_A)}\int_F(g-\sigma\nabla u_{\mathcal{T}}\cdot n)(v-v_{\mathcal{T}}){\rm d}s\\ &\quad - \sum_{F\in\mathcal{F}_{\mathcal{T}}(T)\cap\mathcal{F}_{\mathcal{T}}(\Gamma_C)}\int_F(f(u_{\mathcal{T}})+\sigma\nabla u_{\mathcal{T}}\cdot n)(v-v_{\mathcal{T}}){\rm d}s\Big).\end{aligned}$$

Then a combination of (2.8) and Young's inequality completes the proof with $C_{\rm rel} := 4C_I^2C_{\rm ov}/\beta_1^2$. □

Theorem 2.2 (Efficiency)**.** *Let $u \in H^1(\Omega)$ and $u_{\mathcal{T}} \in V_{\mathcal{T}}$ be the solutions to problems* (2.1) *and* (2.6)*, respectively. Then there exists a positive constant $C_{\rm eff}$ depending only on σ, α, Ω and $C_{\mathcal{T}}$ such that*

$$C_{\rm eff}\eta^2_{\mathcal{T}}(u_{\mathcal{T}}) \le \|u - u_{\mathcal{T}}\|^2_{H^1(\Omega)} + {\rm osc}^2_{\mathcal{T}}(u_{\mathcal{T}}).$$

3. Adaptive algorithm

Now we present the AFEM for problem (2.1). Let $\mathbb{T}$ be the set of all possible conforming triangulations of $\bar\Omega$ obtained from some initial mesh by successive bisections [19, 21, 25]. The refinement process ensures that all constant depending on the shape regularity of $\mathcal{T} \in \mathbb{T}$ are uniformly bounded by a constant depending only on the initial mesh [22, 26]. $\mathcal{T}_*$ is a called refinement of $\mathcal{T}$ for $\mathcal{T} \in \mathbb{T}$, if $\mathcal{T}_* \in \mathbb{T}$ is produced from $\mathcal{T}$ by a finite number of bisections.

The proposed adaptive algorithm is given below. For each triangulation $\mathcal{T}_k$, $k \in \mathbb{N}_0$, we denote $V_k := V_{\mathcal{T}_k}$, $\eta_k := \eta_{\mathcal{T}_k}$ and $u_k := u_{\mathcal{T}_k}$.

Algorithm 3.1. Given an initial conforming mesh $\mathcal{T}_0$ and a parameter $\theta \in (0,1)$. Set $k := 0$.

1. (SOLVE) Solve the discrete problem (2.6) on $\mathcal{T}_k$ for $u_k \in V_k$.
2. (ESTIMATE) Compute the error estimator $\eta_k(u_k, g)$.
3. (MARK) Mark a subset $\mathcal{M}_k \subset \mathcal{T}_k$ with minimal cardinality such that

$$\eta_k^2(u_k, \mathcal{M}_k) \geq \theta \eta_k^2(u_k). \tag{3.1}$$

4. (REFINE) Refine each $T \in \mathcal{M}_k$ by bisection to get $\mathcal{T}_{k+1}$.
5. Set $k := k+1$ and go to Step 1.

The convergence and quasi optimality of Algorithm 3.1 will be analyzed in Sections 5 and 6. A key ingredient is the so-called closure estimate over the meshes $\{\mathcal{T}_k\}$ [18]:

$$\#\mathcal{T}_k \leq \#\mathcal{T}_0 + C_0 \sum_{j=0}^{k-1} \#\mathcal{M}_j \tag{3.2}$$

with the constant C_0 depending only on $C_{\mathcal{T}_0}$ and $\#\mathcal{T}$ denoting the number of elements in $\mathcal{T}$. This estimate was first proved in [4], Theorem 2.4 with an admissibility condition on reference edges of $\mathcal{T}_0$, then extended to the n-simplex case in [25], Theorem 6.1 and the condition for 2D was removed in [18].

4. Auxiliary results

This section is devoted to several technical lemmas for the convergence analysis of Algorithm 3.1. As is well known, Galerkin orthogonality or Pythagoras property is key to the convergence analysis of the linear problems, which regretfully fails for the nonlinear case. Thus a new equivalent error has to be developed that can play the role of the Galerkin orthogonality property.

First, we introduce the associated functional to (2.1) by

$$\mathcal{J}(v) := \frac{1}{2}\int_\Omega \sigma |\nabla v|^2 \mathrm{d}x + \int_{\Gamma_C} F(v)\mathrm{d}s - \int_{\Gamma_A} gv \ \mathrm{d}s \quad \forall\, v \in H^1(\Omega) \tag{4.1}$$

with $F(t) := \int_0^t f(\tau)d\tau$. Then problem (2.1) is equivalent to the minimization problem [20]:

$$u = \underset{v \in H^1(\Omega)}{\arg\min}\, \mathcal{J}(v).$$

Let $u \in H^1(\Omega)$ and $u_\mathcal{T} \in V_\mathcal{T}$ be solutions to problems (2.1) and (2.6), respectively. The non-negative quantity

$$E(u_\mathcal{T}) := \mathcal{J}(u_\mathcal{T}) - \mathcal{J}(u)$$

is referred to as the energy error throughout this paper.

The next lemma ([17], Lem. 2.1) is useful to handle exponential nonlinearity in (1.4).

Lemma 4.1. *Let $v \in H^1(\Omega)$ and $t > 0$, then $e^{t|v|} \in L^1(\Gamma)$. Moreover, there exists a positive constant $C_{\exp}$, depending only on Ω, satisfying*

$$\int_\Gamma e^{t|v|}\mathrm{d}s \leq 1 + |\Gamma| + e^{C_{\exp} t^2 \|v\|^2_{H^1(\Omega)}} |\Gamma| < \infty.$$

Here, $|\Gamma|$ denotes the measure of Γ.

Next we show the equivalence between $E(u_\mathcal{T})$ and $\|u - u_\mathcal{T}\|^2_{H^1(\Omega)}$.

Lemma 4.2. *Let $u \in H^1(\Omega)$ and $u_\mathcal{T} \in V_\mathcal{T}$ be solutions to problems (2.1) and (2.6), respectively. Then there holds*

$$c_{\text{equ}}\|u - u_\mathcal{T}\|^2_{H^1(\Omega)} \leq E(u_\mathcal{T}) \leq C_{\text{equ}}\|u - u_\mathcal{T}\|^2_{H^1(\Omega)}$$

with positive constants c_{equ} and C_{equ} depending only on σ, α, Ω, Γ, $\|g\|_{L^2(\Gamma_A)}$ and C_j for $j = 1, \ldots, 5$.

Proof. Let $y(t) := \mathcal{J}(w(t))$ for $t \in [0, 1]$, with $w(t) := (1-t)u + tu_\mathcal{T}$. Since $w(0) = u$ is the minimizer of $\mathcal{J}$ over $H^1(\Omega)$, consequently, we can obtain $y'(0) = 0$. In the meanwhile, we can obtain by Taylor's theorem that

$$E(u_\mathcal{T}) = y(1) - y(0) = \int_0^1 y''(t)(1-t)\mathrm{d}t. \tag{4.2}$$

To finish the proof, we need to compute $y''(t)$. In view of $F'(\cdot) = f(\cdot)$, an application of the chain rule implies

$$\frac{\partial}{\partial t}F(w(t)) = f(w(t))w'(t) = f(w(t))(u_\mathcal{T} - u), \quad \frac{\partial^2}{\partial t^2}F(w(t)) = f'(w(t))(u_\mathcal{T} - u)^2,$$

which, together with the identity

$$\frac{\partial^2}{\partial t^2}\Big(\frac{1}{2}\sigma|\nabla w(t)|^2\Big) = \sigma|\nabla(u_\mathcal{T} - u)|^2$$

and the definition (4.1), yields

$$y''(t) = \int_\Omega \sigma|\nabla(u_\mathcal{T} - u)|^2\mathrm{d}x + \int_{\Gamma_C} f'(w(t))(u_\mathcal{T} - u)^2\mathrm{d}s.$$

Noting (4.2), we arrive at

$$E(u_\mathcal{T}) = \int_0^1\int_\Omega \sigma|\nabla(u_\mathcal{T} - u)|^2(1-t)\mathrm{d}x\mathrm{d}t + \int_0^1\int_{\Gamma_C} f'(w(t))(u_\mathcal{T} - u)^2(1-t)\mathrm{d}s\mathrm{d}t.$$

This, combined with (2.2) and (2.3), induces the lower bound with $c_{\text{equ}} := \frac{1}{2}\beta_1$.
In the following we will prove the upper bound. The convexity of f' implies

$$f'(w(t)) = f'((1-t)u + tu_\mathcal{T}) \leq (1-t)f'(u) + tf'(u_\mathcal{T}) \text{ for all } t \in [0, 1].$$

We will discuss the cases $f = f_1$ and $f = f_2$ separately.
For $f := f_1$ in (1.3), Hölder inequality, (2.5), (2.4) and (2.7) imply

$$\begin{aligned}\int_{\Gamma_C} f'(u)(u_\mathcal{T} - u)^2\mathrm{d}s &= C_1\int_{\Gamma_C}(u - u_\mathcal{T})^2\mathrm{d}s + 3C_2\int_{\Gamma_C} u^2(u - u_\mathcal{T})^2\mathrm{d}s\\ &\leq C_1\|u - u_\mathcal{T}\|^2_{L^2(\Gamma_C)} + 3C_2\|u\|^2_{L^4(\Gamma_C)}\|u - u_\mathcal{T}\|^2_{L^4(\Gamma_C)}\\ &\leq \Big(C_1C^2_{\text{imb},2} + 3C_2C^4_{\text{imb},4}C^2_{\text{stb}}\|g\|^2_{L^2(\Gamma_A)}\Big)\|u - u_\mathcal{T}\|^2_{H^1(\Omega)}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_{\Gamma_C} f'(u_\mathcal{T})(u_\mathcal{T} - u)^2\mathrm{d}s &= C_1\int_{\Gamma_C}(u - u_\mathcal{T})^2\mathrm{d}s + 3C_2\int_{\Gamma_C} u^2_\mathcal{T}(u - u_\mathcal{T})^2\mathrm{d}s\\ &\leq (C_1C^2_{\text{imb},2} + 3C_2C^4_{\text{imb},4}C^2_{\text{stb}}\|g\|^2_{L^2(\Gamma_A)})\|u - u_\mathcal{T}\|^2_{H^1(\Omega)}.\end{aligned}$$

Then the upper bound for $f := f_1$ follows from these four estimates above.

For $f := f_2$ in (1.4). We derive by the generalized Hölder inequality, together with Lemma 4.1, (2.4) and (2.5), that

$$\begin{aligned}\int_{\Gamma_C} f'(u)(u_{\mathcal{T}}-u)^2\mathrm{d}s &= C_3C_5\int_{\Gamma_C} e^{C_3u}(u-u_{\mathcal{T}})^2\mathrm{d}s + C_4C_5\int_{\Gamma_C} e^{-C_4u}(u-u_{\mathcal{T}})^2\mathrm{d}s\\ &\le C_3C_5\|e^{C_3u}\|_{L^2(\Gamma_C)}\|u-u_{\mathcal{T}}\|^2_{L^4(\Gamma_C)} + C_4C_5\|e^{-C_4u}\|_{L^2(\Gamma_C)}\|u-u_{\mathcal{T}}\|^2_{L^4(\Gamma_C)}\\ &:= C\|u-u_{\mathcal{T}}\|^2_{L^4(\Gamma_C)} \le CC^2_{\mathrm{imb},4}\|u-u_{\mathcal{T}}\|^2_{H^1(\Omega)}.\end{aligned}$$

A similar argument yields

$$\int_{\Gamma_C} f'(u_{\mathcal{T}})(u_{\mathcal{T}}-u)^2\mathrm{d}s \le CC^2_{\mathrm{imb},4}\|u-u_{\mathcal{T}}\|^2_{H^1(\Omega)}.$$

Finally, we complete the proof after collecting these estimates above. □

Remark 4.3. Note that let $\mathcal{T}_*$ be a refinement of $\mathcal{T}$ and $u_{\mathcal{T}_*}$ be the solution to problem (2.6) over $\mathcal{T}_*$. Then the estimate in Lemma 4.2 still holds should u be replaced with $u_{\mathcal{T}_*}$.

The remaining of this section is concerned with the auxiliary results in the analysis: estimator reduction, Céa's lemma, oscillation perturbation and discrete reliability.

Lemma 4.4 (Estimator reduction)**.** *Let $\mathcal{T}\in\mathbb{T}$, $\mathcal{M}\subset\mathcal{T}$ and $\mathcal{T}_*\in\mathbb{T}$ be obtained from $\mathcal{T}$ by Algorithm 3.1 with $\mathcal{M}$ being the marked set. Let $u_{\mathcal{T}}\in V_{\mathcal{T}}$ and $u_{\mathcal{T}_*}\in V_{\mathcal{T}_*}$ be the solutions to problem (2.6) over $\mathcal{T}$ and $\mathcal{T}_*$ respectively. Then there exists a constant C_{est} depending only on σ, Ω, Γ, $C_{\mathcal{T}}$, m and $\|g\|_{L^2(\Gamma_A)}$ satisfying*

$$\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le (1+\delta)\Big(\eta^2_{\mathcal{T}}(u_{\mathcal{T}}) - \lambda\eta^2_{\mathcal{T}}(u_{\mathcal{T}},\mathcal{M})\Big) + C_{\mathrm{est}}(1+\delta^{-1})\|u_{\mathcal{T}_*}-u_{\mathcal{T}}\|^2_{H^1(\Omega)} \quad \forall\ \delta>0$$

with $\lambda = 1-\frac{1}{\sqrt{2}}$.

Before proceeding to its proof, we need an auxiliary result.

Lemma 4.5 (Local perturbation of estimator)**.** *Let $\mathcal{T}\in\mathbb{T}$. Then there holds*

$$\eta^2_{\mathcal{T}}(v_{\mathcal{T}},T) \le (1+\delta)\eta^2_{\mathcal{T}}(w_{\mathcal{T}},T) + \left(1+\frac{1}{\delta}\right)C_{\mathrm{sym},1}\Big(\|v_{\mathcal{T}}-w_{\mathcal{T}}\|^2_{H^1(\omega_{\mathcal{T}})} + \sum_{F\in\mathcal{F}_{\mathcal{T}}(T)\cap\mathcal{F}_{\mathcal{T}}(\Gamma_C)}\|v_{\mathcal{T}}-w_{\mathcal{T}}\|^2_{L^2(F)}\Big)$$

for all $v_{\mathcal{T}}, w_{\mathcal{T}}\in V^m_{\mathcal{T}}$, $T\in\mathcal{T}$ and any $\delta>0$. Here, the constant $C_{\mathrm{sym},1}$ depends only on $C_{\mathcal{T}}$, m, Ω, Γ, σ, $\|v_{\mathcal{T}}\|_{H^1(\Omega)}$ and $\|w_{\mathcal{T}}\|_{H^1(\Omega)}$.

Proof. Let $e := v_{\mathcal{T}}-w_{\mathcal{T}}$. First, we obtain by the definition (2.9) and Young's inequality

$$\begin{aligned}\eta^2_{\mathcal{T}}(v_{\mathcal{T}},T) \le (1+\delta)\eta^2_{\mathcal{T}}(w_{\mathcal{T}},T) &+ \left(1+\frac{1}{\delta}\right)\Big(h_T^2\|\nabla\cdot(\sigma\nabla e)\|^2_{L^2(T)}\\ &+ \sum_{F\in\mathcal{F}_{\mathcal{T}}(T)} h_T\|J_F(v_{\mathcal{T}})-J_F(w_{\mathcal{T}})\|^2_{L^2(F)}\Big) \quad \forall\ \delta>0.\end{aligned}$$

The inverse estimate indicates for some positive constant C_{inv} depending only on $C_{\mathcal{T}}$ and m, there holds

$$\|\nabla e\|_{L^2(T)} \le C_{\mathrm{inv}}h_T^{-1}\|e\|_{L^2(T)}, \quad \|\nabla e\|_{L^2(F)} \le C_{\mathrm{inv}}h_T^{-1/2}\|\nabla e\|_{L^2(T)}. \tag{4.3}$$

Combining these two estimates, we get

$$\eta_{\mathcal{T}}^2(v_{\mathcal{T}},T) \leq (1+\delta)\eta_{\mathcal{T}}^2(w_{\mathcal{T}},T) + \left(1+\frac{1}{\delta}\right)\Bigg(C_{\mathrm{aux},1}\|e\|_{H^1(\omega_T)}^2 + \sum_{F\in\mathcal{F}_{\mathcal{T}}(T)\cap\mathcal{F}_{\mathcal{T}}(\Gamma_C)} h_T\|f(v_{\mathcal{T}})-f(w_{\mathcal{T}})\|_{L^2(F)}^2\Bigg) \quad \forall\, \delta>0. \tag{4.4}$$

Here, the constant $C_{\mathrm{aux},1}$ depends only on $C_{\mathcal{T}}$, m and σ. Therefore, it suffices to bound the last term. For $f := f_1$ in (1.3), by the mean value theorem, a direct calculation leads to

$$|f(v_{\mathcal{T}})-f(w_{\mathcal{T}})|^2 \leq |e|^2(C_1+3C_2(v_{\mathcal{T}}^2+w_{\mathcal{T}}^2))^2 \leq |e|^2(2C_1^2+36C_2^2(v_{\mathcal{T}}^4+w_{\mathcal{T}}^4)).$$

The inverse estimate in Section 4.5 of [5] gives

$$\|e\|_{L^q(F)} \leq C_{\mathrm{inv},q} h_T^{1/q-1/2}\|e\|_{L^2(F)} \quad \forall\, q>2 \tag{4.5}$$

with the constant $C_{\mathrm{inv},q}$ depending only on $C_{\mathcal{T}}$ and m.

A combination of these two inequalities with $q=6$, together with the generalized Hölder inequality and (2.5), yields

$$\|f(v_{\mathcal{T}})-f(w_{\mathcal{T}})\|_{L^2(F)}^2 \leq \int_F |e|^2(2C_1^2+36C_2^2(v_{\mathcal{T}}^4+w_{\mathcal{T}}^4))\mathrm{d}s \leq C_{\mathrm{aux},2}\|e\|_{L^2(F)}^2,$$

with $C_{\mathrm{aux},2} := 2C_1^2 + 36C_2^2C_{\mathrm{inv},6}^2h_T^{-2/3}C_{\mathrm{imb},6}^4(\|v_{\mathcal{T}}\|_{H^1(\Omega)}^4 + \|w_{\mathcal{T}}\|_{H^1(\Omega)}^4)$. Together with (4.4), this proves the desired result with $C_{\mathrm{sym},1} := \max\{C_{\mathrm{aux},1}, h_T C_{\mathrm{aux},2}\}$.

For $f := f_2$ in (1.4), note that the convexity of $f'(\cdot)$, together with the mean value theorem, implies for some $s\in(0,1)$, there holds

$$\begin{aligned}|f(v_{\mathcal{T}})-f(w_{\mathcal{T}})| &= f'(sv_{\mathcal{T}}+(1-s)w_{\mathcal{T}})|v_{\mathcal{T}}-w_{\mathcal{T}}|\\ &\leq (sf'(v_{\mathcal{T}})+(1-s)f'(w_{\mathcal{T}}))|v_{\mathcal{T}}-w_{\mathcal{T}}|.\end{aligned} \tag{4.6}$$

Consequently, taking square on both sides of (4.6) and employing the definition (1.4) lead to

$$|f(v_{\mathcal{T}})-f(w_{\mathcal{T}})|^2 \leq 4C_5^2|e|^2\Big(C_3^2(e^{2C_3v_{\mathcal{T}}}+e^{2C_3w_{\mathcal{T}}}) + C_4^2(e^{-2C_4v_{\mathcal{T}}}+e^{-2C_4w_{\mathcal{T}}})\Big).$$

Then integrating both sides over F and invoking the generalized Hölder inequality, (4.5) with $q=4$ and Lemma 4.1, we get

$$\begin{aligned}\|f(v_{\mathcal{T}})-f(w_{\mathcal{T}})\|_{L^2(F)}^2 \leq{}& 4C_5^2\Bigg(C_3^2\left(\int_F e^{4C_3v_{\mathcal{T}}}\mathrm{d}s\right)^{1/2} + C_3^2\left(\int_F e^{4C_3w_{\mathcal{T}}}\mathrm{d}s\right)^{1/2}\\ &+C_4^2\left(\int_F e^{-4C_4v_{\mathcal{T}}}\mathrm{d}s\right)^{1/2} + C_4^2\left(\int_F e^{-4C_4w_{\mathcal{T}}}\mathrm{d}s\right)^{1/2}\Bigg)\|v_{\mathcal{T}}-w_{\mathcal{T}}\|_{L^4(F)}^2\\ \leq{}& C_{\mathrm{aux},3}h_T^{-1/2}\|v_{\mathcal{T}}-w_{\mathcal{T}}\|_{L^2(F)}^2.\end{aligned} \tag{4.7}$$

Here

$$\begin{aligned}C_{\mathrm{aux},3} :={}& 4C_{\mathrm{inv},4}^2C_5^2\Big(C_3^2\big(2+2|\Gamma|^{1/2} + (e^{8C_3^2C_{\exp}\|v_{\mathcal{T}}\|_{H^1(\Omega)}^2} + e^{8C_3^2C_{\exp}\|w_{\mathcal{T}}\|_{H^1(\Omega)}^2})|\Gamma|^{1/2}\big)\\ &+ C_4^2\big(2+2|\Gamma|^{1/2} + (e^{8C_4^2C_{\exp}\|v_{\mathcal{T}}\|_{H^1(\Omega)}^2} + e^{8C_4^2C_{\exp}\|w_{\mathcal{T}}\|_{H^1(\Omega)}^2})|\Gamma|^{1/2}\big)\Big).\end{aligned}$$

Therefore, from (4.4) and (4.7), the assertion follows with $C_{\mathrm{sym},1} := \max\{C_{\mathrm{aux},1}, h_T^{1/2}C_{\mathrm{aux},3}\}$. $\square$

Proof of Lemma 4.4. Let $T \in \mathcal{T}_*$. Utilizing Lemma 4.5 with $\mathcal{T} := \mathcal{T}^*$ and $v_\mathcal{T} := u_\mathcal{T}$, $w_\mathcal{T} := u_{\mathcal{T}_*}$, we can derive

$$\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}, T) \le (1+\delta)\eta^2_{\mathcal{T}_*}(u_\mathcal{T}, T) + \left(1 + \frac{1}{\delta}\right) C_{\mathrm{sym},1}\Big(\|u_{\mathcal{T}_*} - u_\mathcal{T}\|^2_{H^1(\omega_T)} + \sum_{F\in\mathcal{F}_\mathcal{T}(T)\cap\mathcal{F}_{\mathcal{T}_*}(\Gamma_C)} \|u_{\mathcal{T}_*} - u_\mathcal{T}\|^2_{L^2(F)}\Big). \tag{4.8}$$

Summing over all elements $T \in \mathcal{T}_*$ in (4.8) leads to

$$\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le (1+\delta)\eta^2_{\mathcal{T}_*}(u_\mathcal{T}) + C_{\mathrm{sym},1}\left(1+\frac{1}{\delta}\right)\Big(C_{\mathrm{ov}}\|u_{\mathcal{T}_*} - u_\mathcal{T}\|^2_{H^1(\Omega)} + \|u_{\mathcal{T}_*} - u_\mathcal{T}\|^2_{L^2(\Gamma_C)}\Big).$$

Now (2.5) implies

$$\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le (1+\delta)\eta^2_{\mathcal{T}_*}(u_\mathcal{T}) + C_{\mathrm{est}}(1+\delta^{-1})\|u_{\mathcal{T}_*} - u_\mathcal{T}\|^2_{H^1(\Omega)} \tag{4.9}$$

with $C_{\mathrm{est}} := C_{\mathrm{sym},1}(C_{\mathrm{ov}} + C^2_{\mathrm{imb},2})$. Thanks to the stability estimate (2.7) for $u_\mathcal{T}$ and $u_{\mathcal{T}_*}$, the positive constant C_{est} depends only on σ, Ω, Γ, C, m $\|g\|_{L^2(\Gamma_A)}$ and C_j for $j = 1, \dots, 5$, we split $\eta^2_{\mathcal{T}_*}(u_\mathcal{T})$ as

$$\eta^2_{\mathcal{T}_*}(u_\mathcal{T}) = \eta^2_\mathcal{T}(u_\mathcal{T}, \mathcal{T}\cap\mathcal{T}_*) + \eta^2_{\mathcal{T}_*}(u_\mathcal{T}, \mathcal{T}_*\backslash\mathcal{T}). \tag{4.10}$$

For each element $T \in \mathcal{T}_*\backslash\mathcal{T}$, there exists a unique $\hat{T} \in \mathcal{T}\backslash\mathcal{T}_*$ such that $T \subset \hat{T}$ and $h_T \le \frac{1}{\sqrt{2}} h_{\hat{T}}$. Then by the definition (2.9), we arrive at

$$\eta^2_{\mathcal{T}_*}(u_\mathcal{T}, \mathcal{T}_*\backslash\mathcal{T}) \le \frac{1}{\sqrt{2}}\eta^2_\mathcal{T}(u_\mathcal{T}, \mathcal{T}\backslash\mathcal{T}_*). \tag{4.11}$$

Finally, a collection of (4.9)–(4.11) and the fact $\mathcal{M} \subseteq \mathcal{T}\backslash\mathcal{T}_*$ yields the assertion. $\square$

Remark 4.6. If we exchange $u_\mathcal{T}$ and $u_{\mathcal{T}_*}$ in (4.8) and then sum up it over $\mathcal{T}\cap\mathcal{T}_*$ instead, a similar argument leads to

$$\eta^2_\mathcal{T}(u_\mathcal{T}, \mathcal{T}\cap\mathcal{T}_*) \le (1+\delta)\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}, \mathcal{T}\cap\mathcal{T}_*) + C_{\mathrm{est}}(1+\delta^{-1})\|u_{\mathcal{T}_*} - u_\mathcal{T}\|^2_{H^1(\Omega)} \quad \forall\, \delta > 0. \tag{4.12}$$

Lemma 4.7 (Céa's lemma)**.** *Let u and $u_\mathcal{T}$ be solutions to problems (2.1) and (2.6) over some mesh $\mathcal{T} \in \mathbb{T}$. Then*

$$\|u - u_\mathcal{T}\|^2_{H^1(\Omega)} \le C_{\mathrm{cea}} \inf_{v_\mathcal{T}\in V_\mathcal{T}} \|u - v_\mathcal{T}\|^2_{H^1(\Omega)}.$$

Here, the constant C_{cea} depends only on σ, α, Ω, Γ, $\|g\|_{L^2(\Gamma_A)}$ and C_j for $j = 1, \dots, 5$.

Proof. We know from (2.12) and the Galerkin orthogonality that for any $v_\mathcal{T} \in V_\mathcal{T}$, there holds

$$\begin{aligned}\beta_1\|u - u_\mathcal{T}\|^2_{H^1(\Omega)} &\le \int_\Omega \sigma|\nabla(u - u_\mathcal{T})|^2 \mathrm{d}x + \int_{\Gamma_C} (f(u) - f(u_\mathcal{T}))(u - u_\mathcal{T})\mathrm{d}s \\ &= \int_\Omega \sigma\nabla(u - u_\mathcal{T})\cdot\nabla(u - v_\mathcal{T})\mathrm{d}x + \int_{\Gamma_C} (f(u) - f(u_\mathcal{T}))(u - v_\mathcal{T})\mathrm{d}s.\end{aligned}$$

We focus on the second term, and claim

$$\Big|\int_{\Gamma_C} (f(u) - f(u_\mathcal{T}))(u - v_\mathcal{T})\mathrm{d}s\Big| \le C_{\mathrm{aux},4}\|u - u_\mathcal{T}\|_{H^1(\Omega)}\|u - v_\mathcal{T}\|_{H^1(\Omega)}.$$

For $f := f_1$ in (1.3), noting $|f(u) - f(u_\mathcal{T})| \leq |u - u_\mathcal{T}|(C_1 + 3C_2(u^2 + u_\mathcal{T}^2))$ on Γ_C, we use the generalized Hölder inequality to get

$$\Big|\int_{\Gamma_C}(f(u) - f(u_\mathcal{T}))(u - v_\mathcal{T})\mathrm{d}s\Big| \leq C_1\|u - u_\mathcal{T}\|_{L^2(\Gamma_C)}\|u - v_\mathcal{T}\|_{L^2(\Gamma_C)} + 3C_2\|u - u_\mathcal{T}\|_{L^4(\Gamma_C)}\|u - v_\mathcal{T}\|_{L^4(\Gamma_C)}\Big(\|u\|^2_{L^4(\Gamma_C)} + \|u_\mathcal{T}\|^2_{L^4(\Gamma_C)}\Big).$$

Thus the claim follows from (2.5), (2.4) and (2.7) with $C_{\text{aux},4} := C_1C^2_{\text{imb},2} + 6C_2C^2_{\text{stb}}C^4_{\text{imb},4}\|g\|^2_{L^2(\Gamma_A)}$. For $f := f_2$ in (1.4), by (4.6), we deduce

$$|f(u) - f(u_\mathcal{T})| \leq C_5|u - u_\mathcal{T}|\Big(C_3e^{C_3u} + C_4e^{-C_4u} + C_3e^{C_3u_\mathcal{T}} + C_4e^{-C_4u_\mathcal{T}}\Big) \quad \text{on } \Gamma_C.$$

Multiplying by $u - v_\mathcal{T}$, integrating over Γ_C and using the generalized Hölder's inequality and Lemma 4.1 yield

$$\begin{aligned}\Big|\int_{\Gamma_C}(f(u) - f(u_\mathcal{T}))(u - v_\mathcal{T})\mathrm{d}s\Big| \leq{}& C_5\|u - u_\mathcal{T}\|_{L^4(\Gamma)}\|u - v_\mathcal{T}\|_{L^4(\Gamma)}\Big(C_3\big(2 + 2|\Gamma|^{1/2}\\ &+ (e^{2C_3^2C_{\exp}\|u\|^2_{H^1(\Omega)}} + e^{2C_3^2C_{\exp}\|u_\mathcal{T}\|^2_{H^1(\Omega)}})|\Gamma|^{1/2}\big) + C_4\big(2 + 2|\Gamma|^{1/2}\\ &+ (e^{2C_4^2C_{\exp}\|u\|^2_{H^1(\Omega)}} + e^{2C_4^2C_{\exp}\|u_\mathcal{T}\|^2_{H^1(\Omega)}})|\Gamma|^{1/2}\big)\Big).\end{aligned}$$

Thereafter, we establish the claim by appealing to (2.5), (2.4) and (2.7) with

$$\begin{aligned}C_{\text{aux},4} := C_5C^2_{\text{imb},4}\Big(&C_3\big(2 + 2|\Gamma|^{1/2} + 2|\Gamma|^{1/2}e^{2C_3^2C_{\exp}C^2_{\text{stb}}\|g\|^2_{L^2(\Gamma_A)}}\big)\\ &+ C_4\big(2 + 2|\Gamma|^{1/2} + 2|\Gamma|^{1/2}e^{2C_4^2C_{\exp}C^2_{\text{stb}}\|g\|^2_{L^2(\Gamma_A)}}\big)\Big).\end{aligned}$$

The desired assertion follows from the claim and Young's inequality with $C_{\text{cea}} := 2(\sigma_2^2/\beta_1^2 + C^2_{\text{aux},4}/\beta_1^2)$. ☐

To establish the oscillation perturbation estimate, we follow the proof of Lemma 4.5 to obtain the local perturbation of oscillation. However, this leads to the issue that the related constant depends on $v_\mathcal{T}$ and $w_\mathcal{T}$ as $C_{\text{sym},1}$ in Lemma 4.5. As a result, the constant in the subsequent convergence rate involves the finite element function $v_\mathcal{T}$, which should be avoided in order to establish the quasi-optimality estimate. To this end, we keep the nonlinear function in the following estimate.

Lemma 4.8 (Oscillation perturbation)**.** *Let $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T}_*$ being a refinement of $\mathcal{T}$. If $v_\mathcal{T} \in V_\mathcal{T}$ and $v_{\mathcal{T}_*} \in V_{\mathcal{T}_*}$, then*

$$\text{osc}^2_\mathcal{T}(v_\mathcal{T}, \mathcal{T}\cap\mathcal{T}_*) \leq 2\text{osc}^2_{\mathcal{T}_*}(v_{\mathcal{T}_*}, \mathcal{T}\cap\mathcal{T}_*) + C_{\text{op}}\|\!|(v_{\mathcal{T}_*} - v_\mathcal{T}, \tilde{f}(v_{\mathcal{T}_*}) - \tilde{f}(v_\mathcal{T}))|\!\|^2_{\mathcal{T}_*},$$

where C_{op} depends only on $C_\mathcal{T}$, m, σ and C_{ov},

$$\tilde{f} := \begin{cases} 0 & \text{if } f = f_1, \\ f_2 & \text{if } f = f_2 \end{cases} \tag{4.13}$$

and a mesh dependent norm $\|\!|(\cdot, \tilde{f}(\cdot))|\!\|_\mathcal{T} := (\|\cdot\|^2_{H^1(\Omega)} + \sum_{F\in\mathcal{F}_\mathcal{T}(\Gamma_C)} h_F\|\tilde{f}(\cdot)\|^2_{L^2(F)})^{\frac{1}{2}}$ over $V_\mathcal{T}$.

Proof. Let $T \in \mathcal{T}\cap\mathcal{T}_*$ and denote $e := v_\mathcal{T} - v_{\mathcal{T}_*}$. We obtain from the definition (2.10) and Young's inequality that

$$\begin{aligned}\text{osc}^2_\mathcal{T}(v_\mathcal{T}, T) \leq{}& (1+\delta)\text{osc}^2_\mathcal{T}(v_{\mathcal{T}_*}, T) + \left(1 + \frac{1}{\delta}\right)\Big(h_T^2\|R_T(e) - \bar{R}_T(e)\|^2_{L^2(T)}\\ &+ \sum_{F\in\mathcal{F}_\mathcal{T}(T)} h_T\|(J_F(v_\mathcal{T}) - J_F(v_{\mathcal{T}_*})) - (\bar{J}_F(v_\mathcal{T}) - \bar{J}_F(v_{\mathcal{T}_*}))\|^2_{L^2(F)}\Big) \quad \forall\, \delta > 0.\end{aligned} \tag{4.14}$$

By the inverse estimate (4.3), we arrive at

$$h_T^2\|R_T(e)-\bar{R}_T(e)\|^2_{L^2(T)}+\sum_{F\in\mathcal{F}_\mathcal{T}(T)\setminus\Gamma_C} h_T\|(J_F(v_\mathcal{T})-J_F(v_{\mathcal{T}_*}))-(\bar{J}_F(v_\mathcal{T})-\bar{J}_F(v_{\mathcal{T}_*}))\|^2_{L^2(F)}$$
$$\le C_{\mathrm{aux},5}\|e\|^2_{H^1(\omega_T)},$$

where $C_{\mathrm{aux},5}$ depends only on $C_\mathcal{T}$, m and σ.

Let $f=f_1$. Since $J_F(v)\in P_{3m}(\mathcal{F}_\mathcal{T}(T)\cap\Gamma_C)$ for all $v\in V_\mathcal{T}$ and $T\in\mathcal{T}$, and $\bar{J}_F(v)$ is the L^2-projection of $J_F(v)$ on $P_{3m}(F)$, we obtain

$$J_F(v)-\bar{J}_F(v)=0 \quad \forall\, v\in V_\mathcal{T} \text{ and } F\in\mathcal{F}_\mathcal{T}(T)\cap\Gamma_C.$$

Furthermore, let $f=f_2$. Noting that $\bar{J}_F(v)$ is the L^2-projection of $J_F(v)$ onto $P_{m-1}(F)$, we have

$$\|(J_F(v_\mathcal{T})-J_F(v_{\mathcal{T}_*}))-(\bar{J}_F(v_\mathcal{T})-\bar{J}_F(v_{\mathcal{T}_*}))\|_{L^2(F)}\le\|J_F(v_\mathcal{T})-J_F(v_{\mathcal{T}_*})\|_{L^2(F)} \quad \forall\, F\in\mathcal{F}_\mathcal{T}(T)\cap\Gamma_C.$$

Plugging those estimates above into (4.14) leads to

$$\begin{aligned}\mathrm{osc}^2_\mathcal{T}(v_\mathcal{T},T)\le(1+\delta)\mathrm{osc}^2_\mathcal{T}(v_{\mathcal{T}_*},T)+\left(1+\frac{1}{\delta}\right)\Big(&C_{\mathrm{aux},5}\|e\|^2_{H^1(\omega_T)}\\ &+2\sum_{F\in\mathcal{F}_\mathcal{T}(T)\cap\Gamma_C}h_T\|\tilde{f}(v_\mathcal{T})-\tilde{f}(v_{\mathcal{T}_*})\|^2_{L^2(F)}\Big)\quad\forall\,\delta>0.\end{aligned}\tag{4.15}$$

In the meanwhile, note that $\mathrm{osc}_{\mathcal{T}_*}(v_\mathcal{T},T)=\mathrm{osc}_\mathcal{T}(v_\mathcal{T},T)$ and $v_\mathcal{T}\in V_{\mathcal{T}_*}$. Then summing over $T\in\mathcal{T}\cap\mathcal{T}_*$ in (4.15) leads to

$$\begin{aligned}\mathrm{osc}^2_\mathcal{T}(v_\mathcal{T},\mathcal{T}\cap\mathcal{T}_*)\le(1+\delta)\mathrm{osc}^2_{\mathcal{T}_*}(v_{\mathcal{T}_*},\mathcal{T}\cap\mathcal{T}_*)+\left(1+\frac{1}{\delta}\right)\Big(&C_{\mathrm{aux},5}C_{\mathrm{ov}}\|v_{\mathcal{T}_*}-v_\mathcal{T}\|^2_{H^1(\Omega)}\\ &+2\sum_{F\in\mathcal{F}_{\mathcal{T}_*}(\Gamma_C)}h_F\|\tilde{f}(v_{\mathcal{T}_*})-\tilde{f}(v_\mathcal{T})\|^2_{L^2(F)}\Big).\end{aligned}$$

The desired result follows by taking $\delta=1$ and $C_{\mathrm{op}}:=2\times\max\{C_{\mathrm{aux},5}C_{\mathrm{ov}},2\}$. $\square$

Lemma 4.9 (Discrete reliability)**.** *Let $\mathcal{T},\mathcal{T}_*\in\mathbb{T}$ with $\mathcal{T}_*$ being a refinement of $\mathcal{T}$ and let $u_\mathcal{T}\in V_\mathcal{T}$, $u_{\mathcal{T}_*}\in V_{\mathcal{T}_*}$ be solutions to problem (2.6) over $\mathcal{T}$ and $\mathcal{T}_*$, respectively. Then there exists $C_{\mathrm{drel}}>0$ depending only on σ, α, Ω and $C_\mathcal{T}$ such that*

$$\|u_{\mathcal{T}_*}-u_\mathcal{T}\|^2_{H^1(\Omega)}\le C_{\mathrm{drel}}\eta^2_\mathcal{T}(u_\mathcal{T},\mathcal{T}\setminus\mathcal{T}_*).$$

Proof. Applying the operator $I^{sz}_\mathcal{T}$ [23] to $v:=u_{\mathcal{T}_*}-u_\mathcal{T}$ and noting $v=I^{sz}_\mathcal{T}v$ on unrefined elements in $\mathcal{T}\cap\mathcal{T}_*$, then by (2.8) the argument of Theorem 2.1 completes the proof with $C_{\mathrm{drel}}:=C_{\mathrm{rel}}$. $\square$

5. Convergence

Now we show each iteration of Algorithm 3.1 reduces a sum of the energy error and a scaled estimator, which implies the convergence of the algorithm.

Theorem 5.1 (Contraction Property)**.** *Let $u\in H^1(\Omega)$ be the solution to problem (2.1) and $\{\mathcal{T}_k,V_k,u_k\}$ be a sequence of meshes, finite element spaces and discrete solutions by Algorithm 3.1. Then there exist constants $\beta>0$ and $0<\mu<1$, depending only on $C_{\mathcal{T}_0}$ and θ, such that*

$$E(u_{k+1})+\beta\eta^2_{k+1}(u_{k+1})\le\mu(E(u_k)+\beta\eta^2_k(u_k)).$$

Proof. By the equality $E(u_{k+1}) := \mathcal{J}(u_{k+1}) - \mathcal{J}(u) = \mathcal{J}(u_k) - \mathcal{J}(u) - (\mathcal{J}(u_k) - \mathcal{J}(u_{k+1}))$ and Lemma 4.4 with $\mathcal{T} = \mathcal{T}_k$ and $\mathcal{T}_* = \mathcal{T}_{k+1}$, we obtain for all $\beta > 0$ that

$$\begin{aligned} E(u_{k+1}) + \beta\eta_{k+1}^2(u_{k+1}) \leq E(u_k) &+ \beta(1+\delta)(\eta_k^2(u_k) - \lambda\eta_k^2(u_k, \mathcal{M}_k)) \\ &- (\mathcal{J}(u_k) - \mathcal{J}(u_{k+1})) + \beta C_{\text{est}}(1+\delta^{-1})\|u_{k+1} - u_k\|_{H^1(\Omega)}^2. \end{aligned}$$

Then by taking $\beta := \dfrac{c_{\text{equ}}}{C_{\text{est}}(1+\delta^{-1})}$, Lemma 4.2 and Remark 4.3 lead to

$$E(u_{k+1}) + \beta\eta_{k+1}^2(u_{k+1}) \leq E(u_k) + \beta(1+\delta)\Big(\eta_k^2(u_k) - \lambda\eta_k^2(u_k, \mathcal{M}_k)\Big),$$

which, together with the marking strategy (3.1), implies

$$E(u_{k+1}) + \beta\eta_{k+1}^2(u_{k+1}) \leq E(u_k) - \beta(1+\delta)\lambda\frac{\theta}{2}\eta_k^2(u_k) + \beta(1+\delta)\left(1 - \lambda\frac{\theta}{2}\right)\eta_k^2(u_k).$$

Now by Theorem 2.1, Lemma 4.2 and the choice $\beta := \dfrac{c_{\text{equ}}}{C_{\text{est}}(1+\delta^{-1})}$, we obtain

$$E(u_{k+1}) + \beta\eta_{k+1}^2(u_{k+1}) \leq \mu_1(\delta)E(u_k) + \mu_2(\delta)\beta\eta_k^2(u_k)$$

with $\mu_1(\delta) := 1 - \frac{\delta c_{\text{equ}}\lambda\theta}{2C_{\text{equ}}C_{\text{rel}}C_{\text{est}}}$ and $\mu_2(\delta) := (1+\delta)(1-\lambda\frac{\theta}{2})$. The proof is completed by choosing $\delta > 0$ small enough such that $0 < \mu := \max(\mu_1(\delta), \mu_2(\delta)) < 1$. ☐

6. Quasi-optimality

Now we give a quasi-optimal convergence rate for Algorithm 3.1. We begin with a generalization of Cea's lemma in Lemma 4.7.

Lemma 6.1. *Let u and $u_{\mathcal{T}}$ be solutions to problems (2.1) and (2.6) over some mesh $\mathcal{T} \in \mathbb{T}$. Then*

$$\begin{aligned} \|u - u_{\mathcal{T}}\|_{H^1(\Omega)}^2 + \operatorname{osc}_{\mathcal{T}}^2(u_{\mathcal{T}}) \leq C_{\text{qs}} \inf_{v_{\mathcal{T}} \in V_{\mathcal{T}}} \Big(&\|u - v_{\mathcal{T}}\|_{H^1(\Omega)}^2 + \operatorname{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}) \\ &+ \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C)} h_F\|\tilde{f}(u) - \tilde{f}(v_{\mathcal{T}})\|_{L^2(F)}^2\Big), \end{aligned} \tag{6.1}$$

$$\inf_{v_{\mathcal{T}} \in V_{\mathcal{T}}} \Big(\|u - v_{\mathcal{T}}\|_{H^1(\Omega)}^2 + \operatorname{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}) + \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C)} h_F\|\tilde{f}(u) - \tilde{f}(v_{\mathcal{T}})\|_{L^2(F)}^2\Big) \leq (1 + C_{\text{aux},6})\|u - u_{\mathcal{T}}\|_{H^1(\Omega)}^2 + \operatorname{osc}_{\mathcal{T}}^2(u_{\mathcal{T}}).$$

Here, the positive constant C_{qs} depends only on $C_{\mathcal{T}}$, m, the problem data and C_{ov} while $C_{\text{aux},6}$ depends only on the problem data.

Proof. Given $v_{\mathcal{T}} \in V_{\mathcal{T}}$. An application of Lemma 4.8 with $v_{\mathcal{T}} := u_{\mathcal{T}}$, $\mathcal{T}_* := \mathcal{T}$ yields

$$\operatorname{osc}_{\mathcal{T}}^2(u_{\mathcal{T}}) \leq 2\operatorname{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}) + C_{\text{op}}\Big(\|u_{\mathcal{T}} - v_{\mathcal{T}}\|_{H^1(\Omega)}^2 + \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C)} h_F\|\tilde{f}(u_{\mathcal{T}}) - \tilde{f}(v_{\mathcal{T}})\|_{L^2(F)}^2\Big).$$

Then by the triangle inequality, we derive

$$\begin{aligned} \operatorname{osc}_{\mathcal{T}}^2(u_{\mathcal{T}}) \leq 2\operatorname{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}) &+ 2C_{\text{op}}\Big(\|u - v_{\mathcal{T}}\|_{H^1(\Omega)}^2 + \|u - u_{\mathcal{T}}\|_{H^1(\Omega)}^2 \\ &+ \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C)} h_F\|\tilde{f}(u) - \tilde{f}(u_{\mathcal{T}})\|_{L^2(F)}^2 + \sum_{F \in \mathcal{F}_{\mathcal{T}}(\Gamma_C)} h_F\|\tilde{f}(u) - \tilde{f}(v_{\mathcal{T}})\|_{L^2(F)}^2\Big). \end{aligned}$$

Combining this with Lemma 4.7, we arrive at

$$\|u-u_{\mathcal{T}}\|^2_{H^1(\Omega)}+\mathrm{osc}^2_{\mathcal{T}}(u_{\mathcal{T}})\le 2\mathrm{osc}^2_{\mathcal{T}}(v_{\mathcal{T}})+\Big(2C_{\rm op}+2C_{\rm op}C_{\rm cea}+C_{\rm cea}\Big)\|u-v_{\mathcal{T}}\|^2_{H^1(\Omega)}$$
$$+2C_{\rm op}\Big(\sum_{F\in\mathcal{F}_{\mathcal{T}}(\Gamma_C)}h_F\|\tilde f(u)-\tilde f(u_{\mathcal{T}})\|^2_{L^2(F)}+\sum_{F\in\mathcal{F}_{\mathcal{T}}(\Gamma_C)}h_F\|\tilde f(u)-\tilde f(v_{\mathcal{T}})\|^2_{L^2(F)}\Big). \tag{6.2}$$

Let $f:=f_1$. Then $\tilde f=0$ by (4.13). This proves the assertion by taking $C_{\rm qs}:=\max\{2,2C_{\rm op}+2C_{\rm op}C_{\rm cea}+C_{\rm cea}\}$. Let $f:=f_2$. Then $\tilde f=f_2$ by (4.13). We can argue as in the first inequality of (4.7) to obtain

$$\|f_2(u)-f_2(u_{\mathcal{T}})\|^2_{L^2(\Gamma_C)}\le 4C_5^2\left(C_3^2\left(\int_{\Gamma_C}e^{4C_3u}\mathrm{d}s\right)^{1/2}+C_3^2\left(\int_{\Gamma_C}e^{4C_3u_{\mathcal{T}}}\mathrm{d}s\right)^{1/2}\right.$$
$$\left.+C_4^2\left(\int_{\Gamma_C}e^{-4C_4u}\mathrm{d}s\right)^{1/2}+C_4^2\left(\int_{\Gamma_C}e^{-4C_4u_{\mathcal{T}}}\mathrm{d}s\right)^{1/2}\right)\|u-u_{\mathcal{T}}\|^2_{L^4(\Gamma_C)}.$$

Then an application of Lemma 4.1, (2.4), (2.7) and (2.5) results in

$$\sum_{F\in\mathcal{F}_{\mathcal{T}}(\Gamma_C)}h_F\|f_2(u)-f_2(u_{\mathcal{T}})\|^2_{L^2(F)}\le C_{\rm aux,6}\|u-u_{\mathcal{T}}\|^2_{H^1(\Omega)}. \tag{6.3}$$

Here,

$$C_{\rm aux,6}:=4C^2_{\rm imb,4}C_5^2\Big(C_3^2\big(2+2|\Gamma|^{1/2}+(e^{8C_3^2C_{\rm exp}C^2_{\rm stb}\|g\|^2_{L^2(\Gamma_A)}}+e^{8C_3^2C_{\rm exp}C^2_{\rm stb}\|g\|^2_{L^2(\Gamma_A)}})|\Gamma|^{1/2}\big)$$
$$+C_4^2\big(2+2|\Gamma|^{1/2}+(e^{8C_4^2C_{\rm exp}C^2_{\rm stb}\|g\|^2_{L^2(\Gamma_A)}}+e^{8C_4^2C_{\rm exp}C^2_{\rm stb}\|g\|^2_{L^2(\Gamma_A)}})|\Gamma|^{1/2}\big)\Big).$$

This, together with Lemma 4.7 and (6.2), yields

$$\|u-u_{\mathcal{T}}\|^2_{H^1(\Omega)}+\mathrm{osc}^2_{\mathcal{T}}(u_{\mathcal{T}})\le 2\mathrm{osc}^2_{\mathcal{T}}(v_{\mathcal{T}})+2C_{\rm op}\sum_{F\in\mathcal{F}_{\mathcal{T}}(\Gamma_C)}h_F\|\tilde f(u)-\tilde f(v_{\mathcal{T}})\|^2_{L^2(F)}$$
$$+\Big(2C_{\rm op}+2C_{\rm op}C_{\rm cea}+C_{\rm cea}+2C_{\rm op}C_{\rm cea}C_{\rm aux,6}\Big)\|u-v_{\mathcal{T}}\|^2_{H^1(\Omega)}.$$

By taking $C_{\rm qs}:=\max\{2,2C_{\rm op}+2C_{\rm op}C_{\rm cea}+C_{\rm cea}+2C_{\rm op}C_{\rm cea}C_{\rm aux,6}\}$, we get the first estimate. The second estimate can be derived from (6.3) directly. $\square$

We refer to the square root of the left hand side of (6.1) as the total error. Therefore, Lemma 6.1 establishes the quasi-optimality of the solution $u_{\mathcal{T}}$ in the sense of the total error. Next, we introduce the approximation class. Let $\mathbb{T}_N\subset\mathbb{T}$ be a subset consisting of all triangulation $\mathcal{T}\in\mathbb{T}$ satisfying $\#\mathcal{T}-\#\mathcal{T}_0\le N$. The approximation class $\mathbb{A}_s$ for $0<s\le m/2$ is defined by

$$\mathbb{A}_s:=\Big\{u\Big|\ |u|_s:=\sup_{N>0}N^s\xi(N;u,\sigma)<+\infty\Big\}$$

with

$$\xi(N;u,\sigma):=\inf_{\mathcal{T}\in\mathbb{T}_N}\inf_{v_{\mathcal{T}}\in V_{\mathcal{T}}}\Big(\|u-v_{\mathcal{T}}\|^2_{H^1(\Omega)}+\mathrm{osc}^2_{\mathcal{T}}(v_{\mathcal{T}})+\sum_{F\in\mathcal{F}_{\mathcal{T}}(\Gamma_C)}h_F\|\tilde f(u)-\tilde f(v_{\mathcal{T}})\|^2_{L^2(F)}\Big)^{1/2}.$$

The upper bound $m/2$ is attained for the uniform refinement if the exact solution to (2.1) is sufficiently smooth.

Then we present fundamental ingredients in the analysis, *i.e.*, the optimal marking and cardinality of $\mathcal{M}_k$. We follow [12] to derive the optimal marking that relates a strict error estimator reduction to Dörfler marking. This type of estimate was first given in [24] for the Poisson equation with an H^1-norm reduction, and then extended in [7] to the total error for linear diffusion-reaction problems. Below, we present a version in terms of the error estimator as in [12] for non-symmetric elliptic problems, the proof of which does not require the efficiency estimate in Theorem 2.2.

Lemma 6.2 (Optimal marking)**.** *Suppose that the marking parameter θ in (3.1) satisfies*

$$\theta \in \left(0, \frac{1}{1+2C_{\text{est}}C_{\text{drel}}}\right). \tag{6.4}$$

Let $\mathcal{T}_* \in \mathbb{T}$ be any refinement of $\mathcal{T} \in \mathbb{T}$ and let $u_{\mathcal{T}} \in V_{\mathcal{T}}$, $u_{\mathcal{T}_*} \in V_{\mathcal{T}_*}$ be solutions to problem (2.6) over $\mathcal{T}$ and $\mathcal{T}_*$, respectively. Define

$$\lambda := \frac{1-(1+2C_{\text{est}}C_{\text{drel}})\theta}{2} \in (0,1).$$

Assume further that

$$\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \lambda \eta^2_{\mathcal{T}}(u_{\mathcal{T}}). \tag{6.5}$$

Then there holds

$$\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) \ge \theta \eta^2_{\mathcal{T}}(u_{\mathcal{T}}).$$

Proof. We get by the estimate (4.12) with $\delta = 1$ in Remark 4.6, (6.5) and the discrete reliability estimate in Lemma 4.9 that

$$\begin{aligned}
\eta^2_{\mathcal{T}}(u_{\mathcal{T}}) &= \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) + \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \cap \mathcal{T}_*) \\
&\le \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) + 2\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}, \mathcal{T} \cap \mathcal{T}_*) + 2C_{\text{est}}\|u_{\mathcal{T}_*} - u_{\mathcal{T}}\|^2_{H^1(\Omega)} \\
&\le \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) + 2\lambda\eta^2_{\mathcal{T}}(u_{\mathcal{T}}) + 2C_{\text{est}}\|u_{\mathcal{T}_*} - u_{\mathcal{T}}\|^2_{H^1(\Omega)} \\
&\le (1+2C_{\text{est}}C_{\text{drel}})\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) + 2\lambda\eta^2_{\mathcal{T}}(u_{\mathcal{T}}).
\end{aligned}$$

A direct calculation leads to

$$\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) \ge \frac{1-2\lambda}{1+2C_{\text{drel}}C_{\text{est}}}\eta^2_{\mathcal{T}}(u_{\mathcal{T}}) = \theta\eta^2_{\mathcal{T}}(u_{\mathcal{T}}).$$

This proves the assertion. ☐

Lemma 6.3 (Cardinality of $\mathcal{M}_k$)**.** *Assume that condition (6.4) holds. Let u be the solution to problem (2.1) and let $\{\mathcal{T}_k, V_k, u_k\}$ be the sequence of meshes, finite element spaces and discrete solutions generated by Algorithm 3.1. If $u \in \mathbb{A}_s$, then with λ in Lemma 6.2, there holds*

$$\#\mathcal{M}_k \le \Big(C_{qs}(C_{rel}+1)/\lambda C_{\text{eff}}\Big)^{1/2s} |u|_s^{1/s} \left(\|u-u_k\|^2_{H^1(\Omega)} + \text{osc}^2_k(u_k)\right)^{-1/2s}.$$

Proof. The proof is similar to Lemma 5.10 of [7]. The assumption $u \in \mathbb{A}_s$ ensures that for

$$\varepsilon^2 := \lambda C_{\text{eff}}\Big(C_{\text{qs}}(C_{\text{rel}}+1)\Big)^{-1}\left(\|u-u_k\|^2_{H^1(\Omega)} + \text{osc}^2_k(u_k)\right) \tag{6.6}$$

with a fixed $k \in \mathbb{N}_+$, there exist a triangulation mesh $\mathcal{T}_\varepsilon \in \mathbb{T}$ and $v_\varepsilon \in V_{\mathcal{T}_\varepsilon}$ such that

$$\#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \le |u|_s^{1/s}\varepsilon^{-1/s}, \quad \|u-v_\varepsilon\|^2_{H^1(\Omega)} + \text{osc}^2_{\mathcal{T}_\varepsilon}(v_\varepsilon) + \sum_{F\in\mathcal{F}_{\mathcal{T}_\varepsilon}(\Gamma_C)} h_F\|\tilde{f}(u) - \tilde{f}(v_{\mathcal{T}_\varepsilon})\|^2_{L^2(F)} \le \varepsilon^2. \tag{6.7}$$

Let $\mathcal{T}_*$ be the smallest common refinement of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_k$, *i.e.*, $\mathcal{T}_* := \mathcal{T}_\varepsilon \oplus \mathcal{T}_k$. Then $\mathcal{T}_*$ is a refinement of $\mathcal{T}_\varepsilon$. Theorem 2.2 and (6.1), combined with two inequalities

$$\operatorname{osc}^2_{\mathcal{T}_*}(v_\varepsilon) \le \operatorname{osc}^2_{\mathcal{T}_\varepsilon}(v_\varepsilon), \qquad \sum_{F\in\mathcal{F}_{\mathcal{T}_*}(\Gamma_C)} h_F\|\tilde f(u)-\tilde f(v_{\mathcal{T}_*})\|^2_{L^2(F)} \le \sum_{F\in\mathcal{F}_{\mathcal{T}_\varepsilon}(\Gamma_C)} h_F\|\tilde f(u)-\tilde f(v_{\mathcal{T}_\varepsilon})\|^2_{L^2(F)}$$

and (6.7), imply

$$\begin{aligned} C_{\mathrm{eff}}\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) &\le \|u-u_{\mathcal{T}_*}\|^2_{H^1(\Omega)} + \operatorname{osc}^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \\ &\le C_{\mathrm{qs}}\Big(\|u-v_\varepsilon\|^2_{H^1(\Omega)} + \operatorname{osc}^2_{\mathcal{T}_\varepsilon}(v_\varepsilon) + \sum_{F\in\mathcal{F}_{\mathcal{T}_\varepsilon}(\Gamma_C)} h_F\|\tilde f(u)-\tilde f(v_{\mathcal{T}_\varepsilon})\|^2_{L^2(F)}\Big) \le C_{\mathrm{qs}}\varepsilon^2. \end{aligned}$$

This, together with Theorem 2.1, the inequality $\operatorname{osc}^2_k(u_k) \le \eta^2_k(u_k)$ and (6.6), gives

$$\eta^2_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \lambda\eta^2_k(u_k).$$

Consequently, the subset $\mathcal{T}_k \setminus \mathcal{T}_*$ satisfies the Dörfler marking strategy owing to Lemma 6.2. But the module MARK in Algorithm 3.1 selects a subset $\mathcal{M}_k \subset \mathcal{T}_k$ with minimal cardinality such that the same property holds, which, together with Lemma 3.7 in [7] (also *cf.* [24]), implies

$$\#\mathcal{M}_k \le \#\mathcal{T}_* - \#\mathcal{T}_k \le \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0. \tag{6.8}$$

Therefore, the assertion readily follows from (6.7) and (6.8). $\square$

Now we are in a position to establish the quasi-optimality of Algorithm 3.1.

Theorem 6.4. *Let u be the solution to problem (2.1) and $\{\mathcal{T}_k, V_k, u_k\}$ be the sequence of meshes, finite element spaces and discrete solutions generated by Algorithm 3.1. If $u \in \mathbb{A}_s$ and the condition (6.4) on the marking paramter θ holds, then*

$$\|u-u_k\|^2_{H^1(\Omega)} + \operatorname{osc}^2_k(u_k) \le C_{\mathrm{qopt}}|u|^2_s(\#\mathcal{T}_k - \#\mathcal{T}_0)^{-2s},$$

where C_{qopt} depends only on the problem data, $C_{\mathcal{T}_0}$, m and μ in Theorem 5.1 but is independent of s or u.

Proof. Let $M := (C_{\mathrm{qs}}(C_{\mathrm{rel}}+1)/\lambda C_{\mathrm{eff}})^{1/2s}|u|_s^{1/s}$. By (3.2) and Lemma 6.3, we deduce

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \le C_0\sum_{j=0}^{k-1}\#\mathcal{M}_j \le C_0 M\sum_{j=0}^{k-1}\Big(\|u-u_j\|^2 + \operatorname{osc}^2_j(u_j)\Big)^{-1/2s}.$$

Since the oscillation term (2.11) is dominated by the global error, it follows from Theorem 2.2 and Lemma 4.2 that

$$c_{\mathrm{equ}}\|u-u_j\|^2_{H^1(\Omega)} + \beta\operatorname{osc}^2_j(u_j) \le \mathcal{J}(u_j) - \mathcal{J}(u) + \beta\eta^2_j(u_j) \le (C_{\mathrm{equ}} + \beta C_{\mathrm{eff}}^{-1})(\|u-u_j\|^2_{H^1(\Omega)} + \operatorname{osc}^2_j(u_j)).$$

On the other hand, Theorem 5.1 implies that

$$\mathcal{J}(u_k) - \mathcal{J}(u) + \beta\eta^2_k(u_k) \le \mu^{k-j}\big(\mathcal{J}(u_j) - \mathcal{J}(u) + \beta\eta^2_j(u_j)\big)$$

for $0 \le j \le k-1$. Now collecting the last three estimates, we arrive at

$$\begin{aligned} \#\mathcal{T}_k - \#\mathcal{T}_0 &\le C_0 M(C_{\mathrm{equ}} + \beta C_{\mathrm{eff}}^{-1})^{1/2s}\sum_{j=0}^{k-1}(\mathcal{J}(u_j) - \mathcal{J}(u) + \beta\eta^2_j(u_j))^{-1/2s} \\ &\le C_0 M(C_{\mathrm{equ}} + \beta C_{\mathrm{eff}}^{-1})^{1/2s}(\mathcal{J}(u_k) - \mathcal{J}(u) + \beta\eta^2_k(u_k))^{-1/2s}\sum_{j=1}^{k}\mu^{j/2s} \\ &\le C_0 C_\theta M(C_{\mathrm{equ}} + \beta C_{\mathrm{eff}}^{-1})^{1/2s}(1/\min(c_{\mathrm{equ}},\beta))^{1/2s}(\|u-u_k\|^2 + \operatorname{osc}^2_k(u_k))^{-1/2s} \end{aligned}$$

with $C_\theta := \mu^{1/2s}/(1-\mu^{1/2s})$ bounding the geometric series. Raising this to the sth power and noting $C_0^s \le C_0^{m/2}$, $C_\theta^s \le 1/(1-\mu^{1/m})^s \le 1/(1-\mu^{1/m})^{m/2}$, we obtain the desired estimate. $\square$

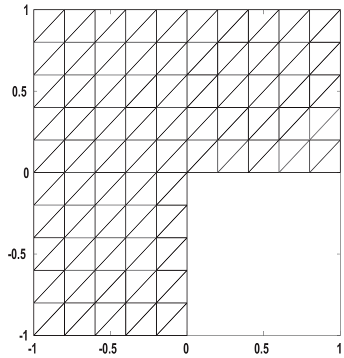


FIGURE 1. Initial uniform triangle mesh with a mesh size $h = 0.2$.

7. NUMERICAL RESULTS

In this section, we present two numerical examples using Algorithm 3.1 with affine elements. The implementation of the algorithm is based on [13]. In the experiments, Ω is an L-shaped domain: $[-1,1]^2\backslash([0,1]\times[-1,0])$, and as in [20], $\sigma = 1$, g on Γ_A and $f(u)$ on Γ_C are set to be the same, *cf.* Examples 7.1 and 7.2 below.

The initial mesh $\mathcal{T}_0$ is a uniform triangulation of the domain, *cf.* Figure 1. At the k^{th} adaptive iteration with triangulation $\mathcal{T}_k$ and $k = 0, 1, 2, \ldots$, we employ the Newton method to obtain the corresponding solution u_k (of the nonlinear system). Specifically, we take the initial guess to be the discrete solution from the previous mesh, *i.e.*,

$$u_k^{(0)} = \begin{cases} 0, & k = 0 \\ u_{k-1}, & k > 0. \end{cases}$$

The stopping criterion for the Newton iteration is

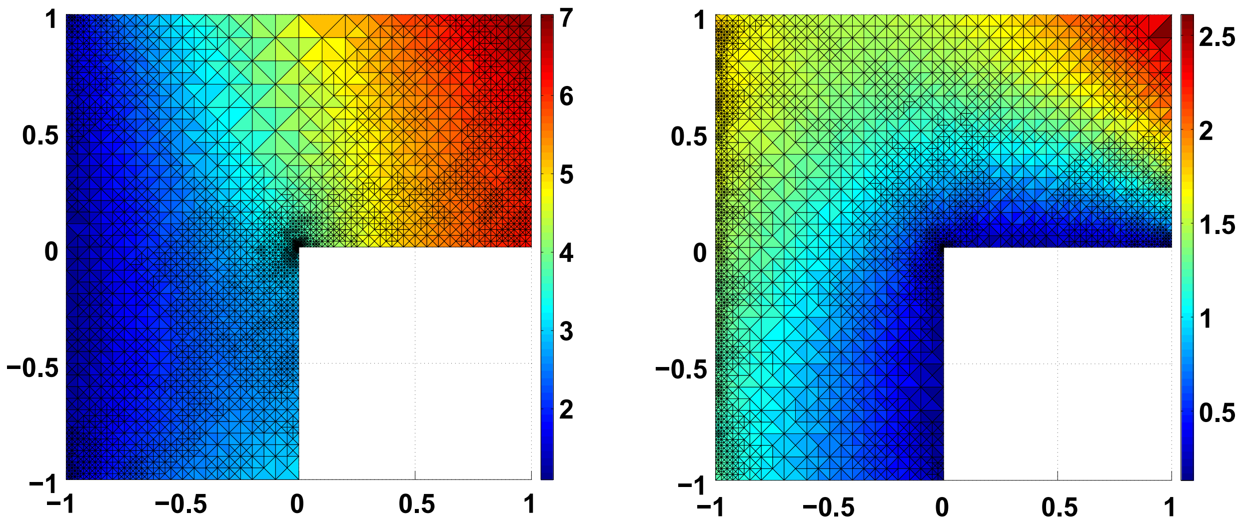
$$\|u_k^{(n)} - u_k^{(n-1)}\|_{H^1(\Omega)} \leq \epsilon$$

with n denoting the Newton iteration number, and ϵ the prescribed accuracy. In the adaptive algorithm, we take $\epsilon = 10^{-7}$. Algorithm 3.1 is terminated once the sum of error indicators $\sum_{T\in\mathcal{T}_k} \eta_k(u_k, T)^2$ falls below a pre-specified threshold tolerance τ. We take $\tau = 10^{-3}$ for both examples below. After obtaining the adaptive solution u_k, we check whether the stopping condition is satisfied. If not, a refinement is carried out for those with large error indicators.

In our simulation, the Dörfler bulk criterion is used to mark elements for refinement, *i.e.*, given $\theta \in (0, 1)$, we look for a minimal set $\mathcal{M}_k \subset \mathcal{T}_k$ satisfying (3.1). In both experiments we show results with $\theta = 0.1$ and $\theta = 0.3$. To refine the mesh, we apply the newest vertex bisection (NVB) refinement [13] and bisect all edges of the elements in $\mathcal{M}_k$ to get a finer mesh $\mathcal{T}_{k+1}$. Alternatively, we can mark only one edge of those elements in $\mathcal{M}_k$ for a refined mesh. Note that a smaller θ yields a more adaptive mesh and a larger iteration number.

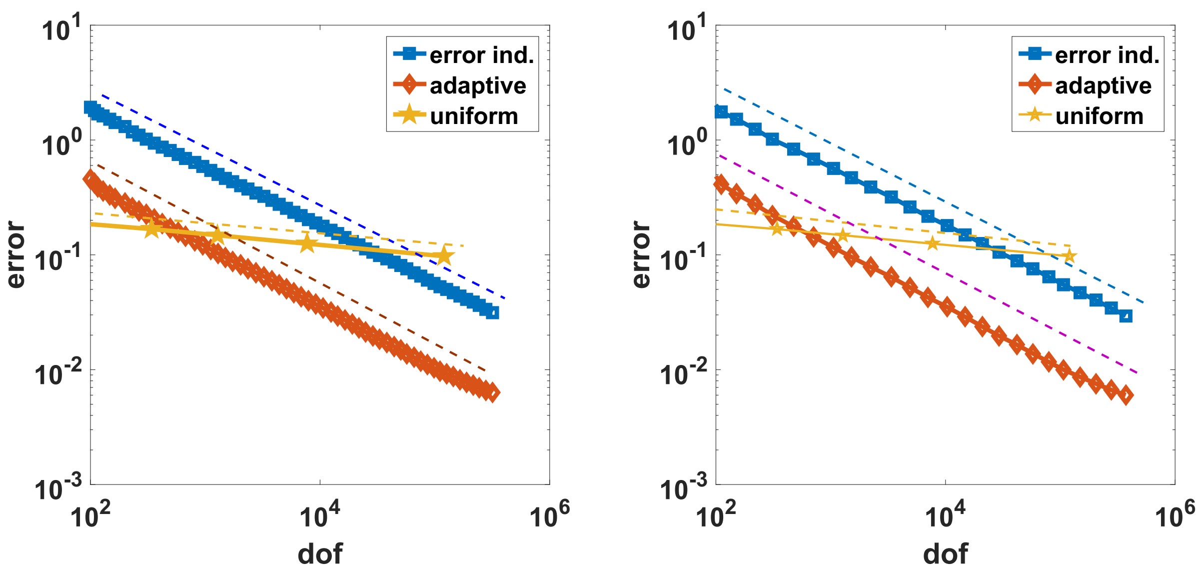
Example 7.1. In the first example, denote $\Gamma_0 = \emptyset$, Γ_C as the left boundary, and Γ_A as the rest of the boundary. We take $g(x, y) = x^2 + y^2$ and $f(u) = u + u^3$.

Due to the nonlinearity of the problem, the exact solution is not available. Hence, we use the solution on a very fine uniform mesh with a mesh size $h = 1/2000$ as the reference solution (and analogously, the Newton method is employed with a much smaller tolerance $\epsilon = 10^{-11}$). The solution on an adaptive mesh is shown in Figure 2A. Since the solution singularity is localized around the re-entrant corner of the domain and the corners where the boundary condition changes, the adaptive algorithm properly refines these regions. In Figure 3A, we observe a convergence rate $O(N^{-0.51})$ for the error estimator, which agrees well with the convergence rate $O(N^{-0.54})$



(A) (B)

FIGURE 2. The adaptive solution with $\theta = 0.1$ and $\tau = 10^{-3}$. *Panel A*: gives the adaptive solution with $k = 25$, d.o.f. $= 3248$ and the $H^1(\Omega)$-relative error is 6.48%; *Panel B*: the adaptive solution with $k = 22$, d.o.f. $= 3070$ and the $H^1(\Omega)$-relative error of 6.27%.



(A) (B)

FIGURE 3. Error estimator and $H^1(\Omega)$-error *versus* d.o.f. with $\tau = 10^{-3}$ for Example 7.1. *Panel A*: the slopes of the dashed lines are -0.51, -0.54 and -0.09 for the indicator, the adaptive refinement $H^1(\Omega)$-error and the uniform refinement $H^1(\Omega)$-error, respectively. *Panel B*: the slopes of the dashed lines are -0.50, -0.53 and -0.09, for the estimator, the adaptive refinement $H^1(\Omega)$-error and the uniform refinement $H^1(\Omega)$-error, respectively.

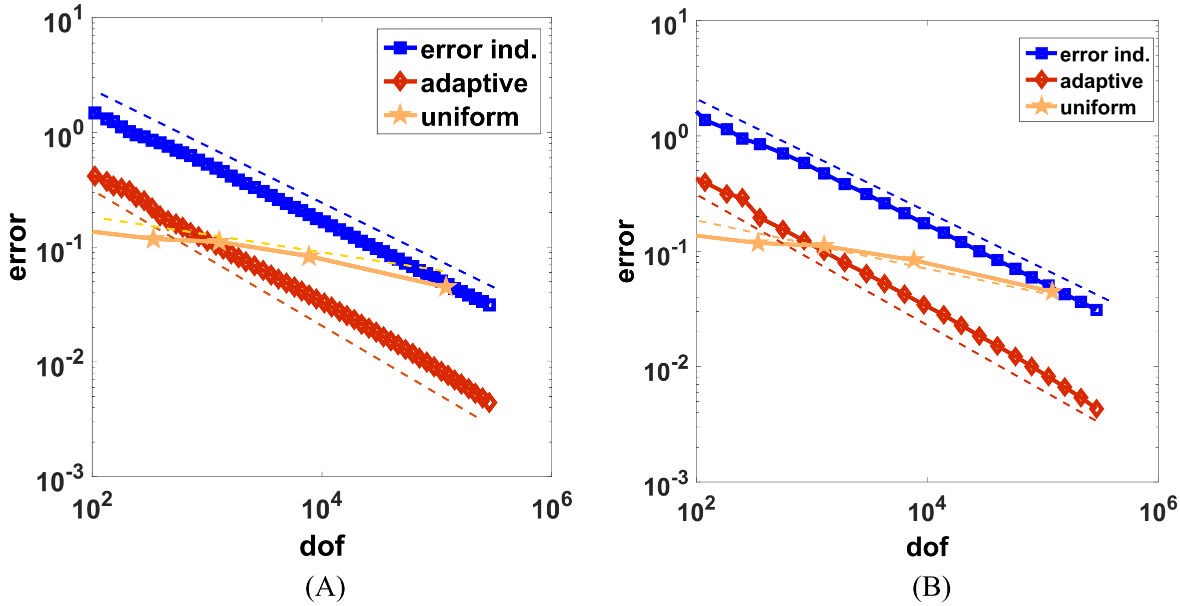


FIGURE 4. Error estimator and $H^1(\Omega)$-error *versus* d.o.f. with $\tau = 10^{-3}$ for Example 7.2. *Panel A*: the slopes of the dashed lines are -0.50, -0.56 and -0.11, for the estimator, the adaptive refinement $H^1(\Omega)$-error and the uniform refinement $H^1(\Omega)$-error, respectively. *Panel B*: the slopes of the dashed lines are -0.51, -0.55 and -0.11, for the estimator, the adaptive refinement $H^1(\Omega)$-error and the uniform refinement $H^1(\Omega)$-error, respectively.

in the $H^1(\Omega)$-norm error of the adaptive solution from Theorem 6.4, numerically verifying the reliability of the estimator. The adaptive algorithm is more efficient than the uniform refinement. Figure 3B displays the convergence history with a larger parameter $\theta = 0.3$. We obtain a smaller iteration number but more degrees of freedom are required over each refinement. The convergence rate of the error estimator and the $H^1(\Omega)$-error are $O(N^{-0.50})$ and $O(N^{-0.53})$, respectively.

In the second example, we consider an oscillatory boundary condition.

Example 7.2. In this example, let $\Gamma_0 = \emptyset$, Γ_C be the boundary segments with the re-entrant corner and Γ_A as the rest of the boundary. We take

$$g(x, y) = \begin{cases} \sin(20y), & \text{on } \Gamma_1 = \{-1\} \times [-1, 1], \\ \sin(x) + \cos(y), & \text{on } \Gamma_A \setminus \Gamma_1. \end{cases} \quad \text{and} \quad f(u) = e^{5u} - e^{-5u}.$$

In Example 7.2, the numerical solution on a fine mesh with a mesh size $h = 1/2000$ and parameter $\epsilon = 10^{-11}$ is taken to be the reference solution. The numerical results for the example are shown in Figures 2B and 4. Due to the oscillatory boundary data, the region close to the left boundary requires adaptive refinement, in addition to the re-entrant corner and the corners where the boundary condition changes. On a very coarse mesh, the oscillatory boundary data is not properly resolved, which leads to a slower decay at the beginning. Nonetheless, as the adaptive procedure proceeds, the convergence of the algorithm is fairly steady, with the estimator decay rate $O(N^{-0.50})$ and the $H^1(\Omega)$ convergence rate $O(N^{-0.56})$ for $\theta = 0.1$. We observe similar convergence rates for $\theta = 0.3$ from Figure 4B.

8. Concluding remark

In this paper, for a 2D variational problem governed by a linear diffusion equation and a nonlinear boundary condition, we have analyzed an adaptive finite element method based on a residual-typed *a posteriori* error estimator and the Dörfler marking. We established a quasi-optimal decay rate in terms of the number of elements for the algorithm, which is confirmed by the numerical experiments. One natural question is to extend the analysis to the 3D case. However, the generalization is not trivial. We need to perform all analysis in a non-Hilbert space $W^{1,3}(\Omega)$, which is required for Lemma 4.1 to hold in 3D [17].

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