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ON EULER PRECONDITIONED SHSS ITERATIVE METHOD FOR A CLASS OF COMPLEX SYMMETRIC LINEAR SYSTEMS

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Abstract. In this paper, we propose an Euler preconditioned single-step HSS (EP-SHSS) iterative method for solving a broad class of complex symmetric linear systems. The proposed method can be applied not only to the non-singular complex symmetric linear systems but also to the singular ones. The convergence (semi-convergence) properties of the proposed method are carefully discussed under suitable restrictions. Furthermore, we consider the acceleration of the EP-SHSS method by preconditioned Krylov subspace method and discuss the spectral properties of the corresponding preconditioned matrix. Numerical experiments verify the effectiveness of the EP-SHSS method either as a solver or as a preconditioner for solving both non-singular and singular complex symmetric linear systems.

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1. Introduction

We focus on the solution of the following complex system of linear equations

$$Ax \equiv (W + \mathrm{i}T)x = b \tag{1.1}$$

or equivalently

$$\tilde{A}x \equiv (T - \mathrm{i}W)x = -\mathrm{i}b, \tag{1.2}$$

where $W, T \in \mathbb{R}^{n\times n}$ are both symmetric positive semi-definite matrices, $b \in \mathbb{C}^n$ is a given vector, $x \in \mathbb{C}^n$ is an unknown vector and $i = \sqrt{-1}$ is the imaginary unit. Here and in the sequel, we assume $W, T \neq 0$, which implies that A and $\tilde{A}$ are both non-Hermitian matrices.

Complex linear system of the form (1.1) comes from many problems in scientific computing and engineering applications, such as non-linear waves [1], diffuse optical tomography [2], FFT-based solution of certain time-dependent PDEs [18], structural dynamics [23], lattice quantum chromodynamics [25], and so on. For more details about its application, see [3, 4, 6, 26–28, 32, 36, 38–41, 43, 47–51].

When W, T are both symmetric positive semi-definite satisfying $\mathrm{null}(W) \cap \mathrm{null}(T) = \{0\}$ (or at least one of them is symmetric positive definite), then the coefficient matrix A of (1.1) is non-singular. Accordingly, many efficient iterative methods as well as their numerical properties have been proposed for solving non-singular

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complex symmetric linear system (1.1). For example, based on the special structure of the coefficient matrix A and its Hermitian and skew-Hermitian parts, Bai *et al.* [8] designed the modified HSS (MHSS) iterative method

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \mathrm{i}T)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b, \end{cases} \tag{1.3}$$

where α is a given positive constant and I represents the identity matrix. In fact, the iterative scheme (1.3) is also obtained from the splitting

$$A = M_\alpha - N_\alpha,$$

of the coefficient matrix A of the linear system (1.1), where

$$M_\alpha = \frac{1+i}{2\alpha}(\alpha I + W)(\alpha I + T), \quad N_\alpha = \frac{1+i}{2\alpha}(\alpha I + iW)(\alpha I - iT),$$

and the splitting matrix M_α can be chosen as the preconditioner for the Krylov subspace methods. Due to the MHSS method converging unconditionally to the exact solution of non-singular complex symmetric linear system (1.1) and reducing the workload caused by the complex arithmetic for solving the subsystem at each iterate step of HSS [11] method, these methods have received considerable attention and many variants have been subsequently proposed. Such as the preconditioned MHSS (PMHSS) [9] iterative method and the generalized preconditioned MHSS (GPMHSS) [22] iterative method. For more efficient methods for solving non-singular complex linear system (1.1), see [12, 13, 33, 37, 53] and references therein.

Alternatively, let $x = y + \mathrm{i}z$ and $b = p + \mathrm{i}q$ with $y, z, f, g \in \mathbb{R}^n$, then the complex linear system (1.1) can be rewritten as the following two-by-two block real equivalent formulation

$$\mathcal{A}u = \begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}. \tag{1.4}$$

To solve the real equivalent non-singular linear system (1.4), Salkuyeh *et al.* [46] studied the generalized successive overrelaxation (GSOR) iterative method

$$\begin{cases} Wy^{(k+1)} = (1-\alpha)Wy^{(k)} + \alpha T z^{(k)} + \alpha p, \\ Wz^{(k+1)} = -\alpha T y^{(k+1)} + (1-\alpha)Wz^{(k)} + \alpha q, \end{cases} \tag{1.5}$$

where α is a given positive constant. Note that the iterative scheme (1.5) can also come from the splitting of the coefficient matrix $\mathcal{A}$ of the linear system (1.4) as follows

$$\mathcal{A} = \mathcal{M}_\alpha - \mathcal{N}_\alpha,$$

where

$$\mathcal{M}_\alpha = \frac{1}{\alpha}\begin{bmatrix} W & O \\ \alpha T & W \end{bmatrix}, \quad \mathcal{N}_\alpha = \frac{1}{\alpha}\begin{bmatrix} (1-\alpha)W & \alpha T \\ O & (1-\alpha)W \end{bmatrix},$$

and the splitting matrix $\mathcal{M}_\alpha$ can be seen as the preconditioner to accelerate the convergence rate when GMRES is applied to solve the system (1.4). Moreover, Hezari *et al.* [30] presented a preconditioned variant of the GSOR (PGSOR) iterative method, Chen and Ma [21] established the AOR-Uzawa iterative method. For more iterative methods and Krylov subspace methods in real arithmetic for solving non-singular linear system (1.4), see [10, 15, 16, 31, 34] and references therein.

When W, T are both symmetric positive semi-definite satisfying $\mathrm{null}(W) \cap \mathrm{null}(T) \neq \{0\}$, then the coefficient matrix A of (1.1) is singular. For this case, a large variety of efficient iterative methods which are designed for solving non-singular linear systems, could be also efficient for solving singular ones. For example, Bai [5] studied the semi-convergence properties of the HSS method for solving singular, non-Hermitian, and positive semi-definite linear systems, Chen and Liu [20] established the semi-convergence properties of the MHSS method for solving singular complex symmetric linear system (1.1). To further improve the semi-convergence rate, Chao and Chen [19] discussed the semi-convergence properties of the generalized MHSS (GMHSS) iterative method with two parameters, Zeng and Zhang [54] investigated the semi-convergence properties of the Complex-extrapolated MHSS iterative method with a complex relaxation parameter. For more efficient methods for solving singular complex linear system (1.1), see [35, 52] and references therein.

In this paper, we construct an Euler preconditioned single-step HSS (EP-SHSS) iterative method for solving both non-singular and singular complex symmetric linear system (1.1).

The rest of this paper is organized as follows. In Section 2, we establish the EP-SHSS method and its implementations. In Sections 3 and 4, the convergence (semi-convergence) properties of the EP-SHSS method and the spectral properties of the corresponding preconditioned matrix are carefully studied under suitable restrictions. Numerical experiments are given to support our theoretical results and verify the feasibility and effectiveness of the EP-SHSS method either as a solver or as a preconditioner in Section 5. Finally, some concluding remarks are given in Section 6.

2. The EP-SHSS method and its implementations

In this section, we establish the EP-SHSS iterative method as well as iterative-based resulting in an effective EP-SHSS preconditioner for solving (1.1).

First, multiplying the Euler's formula $e^{-\mathrm{i}\theta} = \cos(\theta) - \mathrm{i}\sin(\theta)$ in both sides of the linear system (1.1), we obtain

$$\big[\cos(\theta)W + \sin(\theta)T + \mathrm{i}\big(\cos(\theta)T - \sin(\theta)W\big)\big]x = e^{-\mathrm{i}\theta}b, \tag{2.1}$$

where $\theta \in [2k\pi, \frac{\pi}{2} + 2k\pi]$, and k is an integer. Without loss of generality, we take $\theta \in [0, \frac{\pi}{2}]$. Inspired by the idea of the singe-step HSS (SHSS) [37] method, we establish the following Euler preconditioned SHSS (EP-SHSS) iterative method.

Algorithm 2.1. (The EP-SHSS method) Given arbitrary initial guesses $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \dots$ until the sequence of iterates $\{x^{(k)}\}_{k=0}^{\infty} \in \mathbb{C}^n$ converges, compute the next iterate $x^{(k+1)}$ according to the following procedure

$$\big[\alpha I + \cos(\theta)W + \sin(\theta)T\big]x^{(k+1)} = \big[\alpha I - \mathrm{i}(\cos(\theta)T - \sin(\theta)W)\big]x^{(k)} + e^{-\mathrm{i}\theta}b, \tag{2.2}$$

where $\theta \in [0, \frac{\pi}{2}]$ and α is a positive constant.

Remark 2.1. It is worth mentioning that (2.1) reduces to (1.1) when $\theta = 0$, and (2.1) reduces to (1.2) when $\theta = \frac{\pi}{2}$. Moreover, (2.1) can be regarded as a combination of (1.1) and (1.2), *i.e.*, (2.1) $= \cos(\theta)\cdot$(1.1)$+\sin(\theta)\cdot$(1.2) with $\cos^2(\theta) + \sin^2(\theta) = 1$, which is called Euler-extrapolated technique.

Remark 2.2. It is obvious that the EP-SHSS method reduces to the SHSS method [37] when choosing $\theta = 0$, namely, the first formula of (1.3) (the MHSS method), and the EP-SHSS method reduces to the second formula of (1.3) (the MHSS method) when choosing $\theta = \frac{\pi}{2}$.

Note that the iterative scheme (2.2) can be described using the following standard form

$$x^{(k+1)} = \mathcal{T}_{\alpha,\theta}x^{(k)} + M_{\alpha,\theta}^{-1}b, \ k = 0, 1, 2, \dots,$$

where

$$\mathcal{T}_{\alpha,\theta} = \big[\alpha I + \cos(\theta)W + \sin(\theta)T\big]^{-1}\big[\alpha I - \mathrm{i}\big(\cos(\theta)T - \sin(\theta)W\big)\big] \tag{2.3}$$

is the iteration matrix of the EP-SHSS method, and

$$M_{\alpha,\theta} = e^{\mathrm{i}\theta}\big[\alpha I + \cos(\theta)W + \sin(\theta)T\big].$$

In fact, the iterative scheme (2.2) can also be obtained from the following splitting of the coefficient matrix of the linear system (1.1)

$$A = M_{\alpha,\theta} - N_{\alpha,\theta},$$

where

$$N_{\alpha,\theta} = e^{\mathrm{i}\theta}\big[\alpha I - \mathrm{i}\big(\cos(\theta)T - \sin(\theta)W\big)\big].$$

From (2.2), we see that the above splitting matrix $M_{\alpha,\theta}$ can be seen as a preconditioner, which is called the EP-SHSS preconditioner, to the matrix A. Thus, the preconditioned system takes the form

$$M_{\alpha,\theta}^{-1}Ax = M_{\alpha,\theta}^{-1}b.$$

In every step of the iterative scheme (2.2) or applying the EP-SHSS preconditioner $M_{\alpha,\theta}$ to accelerate the convergence rate of Krylov subspace methods (such as GMRES), it is required to solve a linear system with $M_{\alpha,\theta}$ as the coefficient matrix. That is to say, we need to solve a generalized residual equation of the form $M_{\alpha,\theta}z = r$, where $r, z \in \mathbb{C}^n$ are the current and generalized residual vectors, respectively. Accordingly, we obtain $z = M_{\alpha,\theta}^{-1}r = [\alpha I + \cos(\theta)W + \sin(\theta)T]^{-1}e^{-\mathrm{i}\theta}r$, *i.e.*, it is required to solve a linear system with $\alpha I + \cos(\theta)W + \sin(\theta)T$ as the coefficient matrix. Note that $\alpha I + \cos(\theta)W + \sin(\theta)T$ is symmetric positive definite for any $\theta \in [0, \frac{\pi}{2}]$ and $\alpha > 0$, hence the linear system with coefficient matrix $\alpha I + \cos(\theta)W + \sin(\theta)T$ can be solved exactly by the Cholesky factorization [44] or inexactly by the CG algorithm [29].

3. Convergence and preconditioning properties for non-singular case

In this section, we analyze the convergence of the EP-SHSS method and the spectral properties of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ with respect to the EP-SHSS preconditioner for solving non-singular complex symmetric linear system (1.1).

When A is non-singular, we know that the EP-SHSS method is convergent for every initial guess $x^{(0)}$ if and only if $\rho(\mathcal{T}_{\alpha,\theta}) < 1$, where $\rho(\mathcal{T}_{\alpha,\theta})$ denotes the spectral radius of $\mathcal{T}_{\alpha,\theta}$. Let x be an eigenvector corresponding to the eigenvalue λ of the iteration matrix $\mathcal{T}_{\alpha,\theta}$. Then, we have

$$\big[\alpha I - \mathrm{i}\big(\cos(\theta)T - \sin(\theta)W\big)\big]x = \lambda\big[\alpha I + \cos(\theta)W + \sin(\theta)T\big]x. \tag{3.1}$$

To obtain the convergence of the EP-SHSS method, we first give some lemmas.

Lemma 3.1. *[9] Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. Then A is non-singular if and only if null(W) $\cap$ null$(T) = \{0\}$.*

Lemma 3.2. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a non-singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. If $\theta \in (0, \frac{\pi}{2})$, then the matrices $\cos(\theta)W + \sin(\theta)T$ and $\cos(\theta)T - \sin(\theta)W$ are symmetric positive definite and symmetric, respectively.*

Proof. Since W, T are both symmetric positive semi-definite and $\theta \in (0, \frac{\pi}{2})$, it is easy to verify that the matrices $\cos(\theta)W + \sin(\theta)T$ and $\cos(\theta)T - \sin(\theta)W$ are symmetric positive semi-definite and symmetric, respectively. Thus, for any vector $0 \neq v \in \mathbb{C}^n$, it holds that

$$\cos(\theta)v^*Wv + \sin(\theta)v^*Tv \geqslant 0, \quad \theta \in \left(0, \frac{\pi}{2}\right)\cdot$$

Next, we only need to prove that $\cos(\theta)v^*Wv + \sin(\theta)v^*Tv \neq 0$. If $\cos(\theta)v^*Wv + \sin(\theta)v^*Tv = 0$, then $v^*Wv = v^*Tv = 0$, which leads to $Wv = Tv = 0$, *i.e.*, $0 \neq v = \mathrm{null}(W) \cap \mathrm{null}(T)$. It then follows from Lemma 3.1 that this contradicts to the matrix A is non-singular. Therefore, for any vector $v \neq 0$, $\cos(\theta)v^*Wv + \sin(\theta)v^*Tv > 0$, *i.e.*, $\cos(\theta)W + \sin(\theta)T$ is symmetric positive definite. Hence, the conclusion of this lemma is true. ☐

Lemma 3.3. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a non-singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. If λ is an eigenvalue of the iteration matrix $\mathcal{T}_{\alpha,\theta}$ defined as in (2.3), then $\lambda \neq 1$.*

Proof. If $\lambda = 1$, then it follows from (3.1) that

$$Ax = (W + \mathrm{i}T)x = 0.$$

Since the matrix A is non-singular, we have $x = 0$, which contradicts the assumption that x is an eigenvector of the iteration matrix $\mathcal{T}_{\alpha,\theta}$. Hence, $\lambda \neq 1$. □

The convergence properties of the EP-SHSS method when $\theta = 0$ or $\theta = \frac{\pi}{2}$ have been completely described in [37]. Hence, just the situation that $\theta \in (0, \frac{\pi}{2})$ is considered here. First, the following theorem give the sufficient and necessary conditions for the convergence of the EP-SHSS method.

Theorem 3.1. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a non-singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite, and let $\theta \in (0, \frac{\pi}{2})$ and α be a positive constant. Let x be an eigenvector of the iteration matrix $\mathcal{T}_{\alpha,\theta}$ corresponding to the eigenvalue λ, then*

$$\lambda = \frac{\alpha - \mathrm{i}\tilde{b}}{\alpha + \tilde{a}},$$

where

$$\tilde{a} = \frac{x^*[\cos(\theta)W + \sin(\theta)T]x}{x^*x}, \quad \tilde{b} = \frac{x^*[\cos(\theta)T - \sin(\theta)W]x}{x^*x}. \tag{3.2}$$

Furthermore, for all $\theta \in (0, \frac{\pi}{2})$, the EP-SHSS method is convergent if and only if the parameter α satisfies the following condition

$$\alpha > \max\left\{0, \ \frac{\tilde{b}^2 - \tilde{a}^2}{2\tilde{a}}\right\}. \tag{3.3}$$

Proof. Let (λ, x) be the eigenpair of the iteration matrix $\mathcal{T}_{\alpha,\theta}$, then multiplying both sides of (3.1) from the left by x^*, we have

$$\alpha x^*x - \mathrm{i}x^*\big[\cos(\theta)T - \sin(\theta)W\big]x = \lambda\big[\alpha x^*x + x^*\big(\cos(\theta)W + \sin(\theta)T\big)x\big].$$

Thus, it follows from (3.2) that

$$\lambda = \frac{\alpha - \mathrm{i}\tilde{b}}{\alpha + \tilde{a}}.$$

According to Lemma 3.2, we have $\tilde{a} > 0$. Then

$$|\lambda| = \frac{\sqrt{\alpha^2 + \tilde{b}^2}}{\alpha + \tilde{a}} < 1, \quad i.e., \ \alpha > \frac{\tilde{b}^2 - \tilde{a}^2}{2\tilde{a}}.$$

Thus, under the condition of $\alpha > 0$, we know that the EP-SHSS method is convergent if and only if the parameter α satisfies (3.3). This completes the proof. □

Remark 3.1. According to Theorem 3.1, we know that if there exists a parameter θ such that $\tilde{a} \geqslant |\tilde{b}|$, *i.e.*, it holds

$$|\lambda| = \frac{\sqrt{\alpha^2 + \tilde{b}^2}}{\alpha + \tilde{a}} < \sqrt{\frac{\alpha^2 + \tilde{b}^2}{\alpha^2 + \tilde{a}^2}} \leqslant 1, \quad \forall\, \alpha > 0,$$

then the EP-SHSS method is convergent for any $\alpha > 0$.

Theorem 3.2. *Under the assumption of Theorem 3.1, the optimal parameter α^* of the EP-SHSS method is given by*

$$\alpha_\theta^* = \arg\min_\alpha \Big\{ \frac{\sqrt{\alpha^2+\tilde{b}^2}}{\alpha+\tilde{a}} \Big\} = \frac{\tilde{b}^2}{\tilde{a}},$$

and

$$\rho(\mathcal{T}_{\alpha_\theta^*,\theta}) = \frac{|\tilde{b}|}{\sqrt{\tilde{b}^2+\tilde{a}^2}}. \tag{3.4}$$

Moreover, the quasi-optimal parameter θ^* of the EP-SHSS method by minimizing the spectral radius $\rho(\mathcal{T}_{\alpha_\theta^*,\theta})$ is given by

$$\theta^* = \left\{ \arctan\left(\frac{\mu_{\min}\mu_{\max} - 1 + \sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min}+\mu_{\max}} \right) \in \left(0, \frac{\pi}{2}\right) \right\} \tag{3.5}$$
$$\left(= \left\{ \operatorname{arccot}\left(\frac{1 - \mu_{\min}\mu_{\max} + \sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min}+\mu_{\max}} \right) \in \left(0, \frac{\pi}{2}\right) \right\} \right),$$

where $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest generalized eigenvalues of the matrix pair (W, T), respectively.

Proof. According to Theorem 3.1, through direct computations, we obtain

$$\big[\rho(\mathcal{T}_{\alpha,\theta})\big]' = \frac{\alpha\tilde{a} - \tilde{b}^2}{(\alpha+\tilde{a})^2\sqrt{\alpha^2+\tilde{b}^2}}.$$

We see that $\big[\rho(\mathcal{T}_{\alpha,\theta})\big]' > 0$ for $\alpha > \frac{\tilde{b}^2}{\tilde{a}}$ and $\big[\rho(\mathcal{T}_{\alpha,\theta})\big]' < 0$ for $\alpha < \frac{\tilde{b}^2}{\tilde{a}}$. Hence, the spectral radius $\rho(\mathcal{T}_{\alpha,\theta})$ achieves its minimum at $\alpha_\theta^* = \frac{\tilde{b}^2}{\tilde{a}}$. So, the minimum value of $\rho(\mathcal{T}_{\alpha,\theta})$ with respect to α is given by (3.4).

Next, we minimize the value of $\rho(\mathcal{T}_{\alpha_\theta^*,\theta})$ with respect to θ. Note that the function

$$\rho(\mathcal{T}_{\alpha_\theta^*,\theta}) = \frac{|\tilde{b}|}{\sqrt{\tilde{b}^2+\tilde{a}^2}} = \frac{|\frac{\tilde{b}}{\tilde{a}}|}{\sqrt{(|\frac{\tilde{b}}{\tilde{a}}|)^2+1}},$$

is increasing with respect to $|\frac{\tilde{b}}{\tilde{a}}|$. Hence, $\rho(\mathcal{T}_{\alpha_\theta^*,\theta})$ achieves its minimum at the minimum value of $|\frac{\tilde{b}}{\tilde{a}}|$ with respect to θ. In fact, if μ is an arbitrary generalized eigenvalue of the matrix pair (W, T) and suppose $0 \neq x \in \mathbb{C}^n$ is its corresponding eigenvector, we get $Tx = \mu Wx$, which means that

$$\big[\cos(\theta)W + \sin(\theta)T\big]x = \big[\cos(\theta) + \mu\sin(\theta)\big]Wx.$$

Notice that $\mu \geqslant 0$ and $\theta \in (0, \frac{\pi}{2})$, by Lemma 3.2, we get $\cos(\theta) + \mu\sin(\theta) > 0$ and

$$\big[\cos(\theta)W + \sin(\theta)T\big]^{-1}Wx = \frac{1}{\cos(\theta)+\mu\sin(\theta)}x.$$

Therefore, it follows from (3.2) that

$$\begin{aligned}\frac{\tilde{b}}{\tilde{a}} &= \frac{x^*\big[\sin(\theta)W - \cos(\theta)T\big]x}{x^*\big[\cos(\theta)W + \sin(\theta)T\big]x} \\ &= \frac{\sin(\theta) - \mu\cos(\theta)}{\cos(\theta)+\mu\sin(\theta)}.\end{aligned}$$

Note that

$$\min_{\theta,\mu}\left\{\left|\frac{\tilde{b}}{\tilde{a}}\right|\right\} = \min_{\theta,\mu}\left\{\left|\frac{\sin(\theta)-\mu\cos(\theta)}{\cos(\theta)+\mu\sin(\theta)}\right|\right\} :\equiv \min_\theta\left\{\left|\frac{\sin(\theta)-\mu_{\min}\cos(\theta)}{\cos(\theta)+\mu_{\min}\sin(\theta)}\right|, \left|\frac{\sin(\theta)-\mu_{\max}\cos(\theta)}{\cos(\theta)+\mu_{\max}\sin(\theta)}\right|\right\},$$

here we use the function $f(\mu)=\frac{\sin(\theta)-\mu\cos(\theta)}{\cos(\theta)+\mu\sin(\theta)}$ is decreasing with respect to $\mu\geqslant 0$. Using the same strategy of the HSS method, we know that if θ_* is such a minimum point, by making use of the function $f(\theta)=\frac{\sin(\theta)-\mu\cos(\theta)}{\cos(\theta)+\mu\sin(\theta)}$ is decreasing with respect to $\theta\in(0,\frac{\pi}{2})$, it must satisfy $\mu_{\max}\cos(\theta^*)-\sin(\theta^*)>0$, $\mu_{\min}\cos(\theta^*)-\sin(\theta^*)<0$ and

$$\frac{\sin(\theta^*)-\mu_{\min}\cos(\theta^*)}{\cos(\theta^*)+\mu_{\min}\sin(\theta^*)}=\frac{\mu_{\max}\cos(\theta^*)-\sin(\theta^*)}{\cos(\theta^*)+\mu_{\max}\sin(\theta^*)},\quad \theta^*\in\left(0,\frac{\pi}{2}\right).$$

After some computations, we obtain the results of (3.5). □

Remark 3.2. Based on Theorem 3.2, we get $|\frac{\tilde{b}}{\tilde{a}}|=|\mu|$ for $\theta=0$ and $|\frac{\tilde{b}}{\tilde{a}}|=\frac{1}{|\mu|}$ for $\theta=\frac{\pi}{2}$ when $\mu\neq 1$, and $|\frac{\tilde{b}}{\tilde{a}}|=0$ for $\theta=\frac{\pi}{4}$ when $\mu=1$. So, it must exists an optimal parameter θ^* in (3.5) such that $\min_{\theta^*}\{|\frac{\tilde{b}}{\tilde{a}}|\}<1$ and then the EP-SHSS method is always convergent for any $\alpha>0$ by taking the optimal parameter θ^*.

Additionally, by making use of Theorem 3.1, we derive the following sufficient conditions for the convergence of the EP-SHSS method.

Lemma 3.4. *Under the assumption of Theorem 3.1, the spectral radius $\rho(\mathcal{T}_{\alpha,\theta})$ of the iteration matrix $\mathcal{T}_{\alpha,\theta}$ is bounded by*

$$\delta_{\alpha,\theta}=\frac{\sqrt{\alpha^2+\tilde{\mu}_{\max}^2}}{\alpha+\tilde{\eta}_{\min}},$$

where

$$\tilde{\eta}_{\min}=\min_{\tilde{\eta}_j\in sp\left(\cos(\theta)W+\sin(\theta)T\right)}\{\tilde{\eta}_j\},\ \tilde{\mu}_{\max}=\max_{\tilde{\mu}_j\in sp\left(\cos(\theta)T-\sin(\theta)W\right)}\{|\tilde{\mu}_j|\},\tag{3.6}$$

and $sp(\cdot)$ represents the spectral set of a matrix. Furthermore, for all $\theta\in(0,\frac{\pi}{2})$, the EP-SHSS method is convergent if $\delta_{\alpha,\theta}<1$, or equivalently, the parameter α satisfies

$$\alpha>\max\left\{0,\ \frac{\tilde{\mu}_{\max}^2-\tilde{\eta}_{\min}^2}{2\tilde{\eta}_{\min}}\right\}.\tag{3.7}$$

Proof. Using the Rayleigh-Ritz theorem, we have

$$\tilde{a}=\frac{x^*[\cos(\theta)W+\sin(\theta)T]x}{x^*x}\geqslant\tilde{\eta}_{\min},\quad |\tilde{b}|=\left|\frac{x^*[\cos(\theta)W+\sin(\theta)T]x}{x^*x}\right|\leqslant\tilde{\mu}_{\max}.$$

And it follows from Theorem 3.1 that

$$\rho(\mathcal{T}_{\alpha,\theta})=\max\{|\lambda|\}=\max\{\frac{\sqrt{\alpha^2+\tilde{b}^2}}{\alpha+\tilde{a}}\}\leqslant\frac{\sqrt{\alpha^2+\tilde{\mu}_{\max}^2}}{\alpha+\tilde{\eta}_{\min}}=\delta_{\alpha,\theta}.$$

Moreover, we have the following conclusions.

1) If $|\tilde{\mu}_{\max}|\leqslant\tilde{\eta}_{\min}$, then (3.7) is equivalent to $\alpha>0$, which means that $\delta_{\alpha,\theta}<1$;
2) If $|\tilde{\mu}_{\max}|>\tilde{\eta}_{\min}$, then (3.7) is equivalent to

$$\alpha>\frac{\tilde{\mu}_{\max}^2-\tilde{\eta}_{\min}^2}{2\tilde{\eta}_{\min}}>0,$$

or equivalently,

$$\alpha^2+2\alpha\tilde{\eta}_{\min}+\tilde{\eta}_{\min}^2>\alpha^2+\tilde{\mu}_{\max}^2,$$

which leads to $\delta_{\alpha,\theta}<1$.

Thus, the EP-SHSS method is convergent if $\delta_{\alpha,\theta} < 1$ or equivalently the parameter α satisfies (3.7). This completes the proof. □

Remark 3.3. According to Lemma 3.4, we know that if there exists a parameter $\theta \in (0, \frac{\pi}{2})$ such that $\tilde{\eta}_{\min} \geqslant \tilde{\mu}_{\max}$, then the EP-SHSS method is convergent for any $\alpha > 0$.

Similar to the proof of [37, 53], we obtain the following quasi-optimal parameter α and the corresponding quasi-optimal convergence factor.

Lemma 3.5. *Under the assumption of Theorem 3.1, the quasi-optimal parameter α^*_θ by minimizing the upper bound $\delta_{\alpha,\theta}$ of the spectral radius $\rho(\mathcal{T}_{\alpha,\theta})$ is given by*

$$\tilde{\alpha}^*_\theta = \arg\min_{\alpha} \left\{ \frac{\sqrt{\alpha^2 + \tilde{\mu}^2_{\max}}}{\alpha + \tilde{\eta}_{\min}} \right\} = \frac{\tilde{\mu}^2_{\max}}{\tilde{\eta}_{\min}},$$

and

$$\delta_{\tilde{\alpha}^*_\theta,\theta} = \frac{\tilde{\mu}_{\max}}{\sqrt{\tilde{\eta}^2_{\min} + \tilde{\mu}^2_{\max}}}.$$

Proof. By Theorem 3.2, after direct calculations, we have

$$\alpha^*_\theta = \frac{\tilde{b}^2}{\tilde{a}} \leqslant \frac{\tilde{\mu}^2_{\max}}{\tilde{\eta}_{\min}} := \tilde{\alpha}^*_\theta,$$

and

$$\rho(\mathcal{T}_{\alpha^*_\theta,\theta}) = \frac{|\tilde{b}|}{\sqrt{\tilde{b}^2 + \tilde{a}^2}} \leqslant \frac{\tilde{\mu}_{\max}}{\sqrt{\tilde{\eta}^2_{\min} + \tilde{\mu}^2_{\max}}} = \delta_{\tilde{\alpha}^*_\theta,\theta},$$

here we use the function $f(|\tilde{b}|) = \dfrac{|\tilde{b}|}{\sqrt{(|\tilde{b}|)^2 + \tilde{a}^2}}$ is increasing with respect to $|\tilde{b}| > 0$. □

Remark 3.4. According to Theorem 3.2 and Lemma 3.5, the quasi-optimal parameters of the EP-SHSS method are given by θ^* defined as in (3.5) and $\tilde{\alpha}^*_{\theta^*}$, respectively.

However, to our knowledge, the clustered spectrum of the preconditioned matrix often leads to rapid convergence ([7, 14]) of the GMRES method [45], so possessing the clustering properties of the eigenvalues of the preconditioned matrix $M^{-1}_{\alpha,\theta}A$ plays a key role in estimating the convergence rate of the preconditioned Krylov subspace methods. Thus, we have the following useful results.

Theorem 3.3. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a non-singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite, and let $\theta \in [0, \frac{\pi}{2}]$ and α be a positive constant. Then the eigenvalue ξ of the preconditioned matrix $M^{-1}_{\alpha,\theta}A$ are*

$$\xi = \frac{\tilde{a} + \mathrm{i}\tilde{b}}{\alpha + \tilde{a}},$$

where $\tilde{a}$ and $\tilde{b}$ are defined as in (3.2). Moreover, it holds

$$\frac{\tilde{\eta}_{\min}}{\alpha + \tilde{\eta}_{\min}} \leqslant \Re(\xi) = \frac{\tilde{a}}{\alpha + \tilde{a}} \leqslant \frac{\tilde{\eta}_{\max}}{\alpha + \tilde{\eta}_{\max}} \quad \text{and} \quad \left|\Im(\xi)\right| = \frac{|\tilde{b}|}{\alpha + \tilde{a}} \leqslant \frac{\tilde{\mu}_{\max}}{\alpha + \tilde{\eta}_{\min}}, \tag{3.8}$$

where $\tilde{\eta}_{\min}$ and $\tilde{\eta}_{\max}$ are defined as in (3.6) and $\tilde{\mu}_{\max} = \max\limits_{\tilde{\eta}_j \in sp(\cos(\theta)W + \sin(\theta)T)} \{\tilde{\eta}_j\}$.

Proof. Let ξ be an eigenvalue of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$, by Theorem 3.1, we get

$$\xi = 1 - \lambda = \frac{\tilde{a} + \mathrm{i}\tilde{b}}{\alpha + \tilde{a}}\cdot$$

Based on this, we get

$$\Re(\xi) = \frac{\tilde{a}}{\alpha + \tilde{a}} \quad \text{and} \quad \Im(\xi) = \frac{\tilde{b}}{\alpha + \tilde{a}}\cdot$$

Using the Rayleigh-Ritz theorem again, we obtain the conclusion of (3.8). □

From Theorem 3.3, we obtain the following asymptotic behavior of the eigenvalue ξ of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$.

Corollary 3.1. *It follows from Theorems 3.2 and 3.3 that if α is small enough and take θ^* defined as in (3.5), then for the eigenvalue ξ of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$, it holds*

$$\xi \approx 1 + \mathrm{i}\,\frac{\tilde{b}}{\tilde{a}}$$

and

$$\Re(\xi) \to 1, \quad \left|\Im(\xi)\right| \to \left|\frac{\tilde{b}}{\tilde{a}}\right| \leqslant \frac{\tilde{\mu}_{\max}}{\tilde{\eta}_{\min}} < 1.$$

So, the eigenvalues of $M_{\alpha,\theta}^{-1}A$ are contained within the complex disk centered at $(1,0)$ with radius $r \approx \frac{\tilde{\mu}_{\max}}{\tilde{\eta}_{\min}}$ strictly less 1 when taking $\alpha \to 0_+$ and θ^*, which is a desirable property for Krylov subspace acceleration.

4. Semi-convergence and preconditioning properties for singular case

In this section, we investigate the semi-convergence of the EP-SHSS method and the spectral properties of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ with respect to the EP-SHSS preconditioner for solving singular complex symmetric linear system (1.1).

We now introduce some basic concepts about the semi-convergence of an iterative method, which can be found in [17], p. 198, Lemma 6.9.

Lemma 4.1. *[17] Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a singular matrix, and has the spitting $A = M - N$, where M is a non-singular matrix. We can define an iterative method $x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b$, ($k$ = 0, 1, 3,...), for the systems of linear equations $Ax = b$. The necessary and sufficient conditions for guaranteeing the semi-convergence of this iterative method are as follows*

1. *the elementary divisors of the iteration matrix $\mathcal{T} = M^{-1}N$ associated with its eigenvalue $\lambda = 1$ are linear, i.e., $rank(I - \mathcal{T}) = rank((I - \mathcal{T})^2)$, or equivalently, $index(I - \mathcal{T}) = 1$;*
2. *The pseudo-spectral radius of the iteration matrix $\mathcal{T}$ is less than 1, i.e., $\vartheta(\mathcal{T}) \equiv \max\{|\lambda| : \lambda \in \sigma(\mathcal{T}), \lambda \neq 1\} < 1$, where $\vartheta(\mathcal{T})$ is said to be the semi-convergence factor and $\sigma(\mathcal{T})$ is the spectrum of the matrix $\mathcal{T}$.*

Based on the characteristic of the matrix A, we have the following lemma.

Lemma 4.2. *[5] Assume that $A = W + iT \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. Then $null(A) = null(W) \cap null(T)$.*

According to Lemma 4.2, we immediately obtain the following results.

Lemma 4.3. *Assume that $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite satisfying $null(W) \cap null(T) \neq \{0\}$. Then the matrix $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is singular.*

When A is singular, we know that the iteration matrix $\mathcal{T}_{\alpha,\theta}$ has eigenvalue one, which means that the spectral radius of the iteration matrix cannot be less than one. Hence, we only need to prove two conditions of Lemma 4.1 for the semi-convergence of the EP-SHSS method. By making use of the same as proof strategy in [19], we obtain the following results immediately.

Lemma 4.4. *[19] Assume that $A = W + iT \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. Then there exist an unitary matrix $U \in \mathbb{C}^{n\times n}$ and non-singular matrix $\hat{A} \in \mathbb{C}^{r\times r}$ such that $r = rank(A) = rank(\hat{A})$, which satisfies*

$$A = U \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where $\hat{A} = \hat{W} + \mathrm{i}\hat{T}$ is a non-singular matrix, with $\hat{W}, \hat{T} \in \mathbb{R}^{r\times r}$ being symmetric positive semi-definite satisfying $null(\hat{W}) \cap null(\hat{T}) = \{0\}$.

Theorem 4.1. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite. Then for the iteration matrix $\mathcal{T}_{\alpha,\theta}$ of the EP-SHSS method, it holds $index(I - \mathcal{T}_{\alpha,\theta}) = 1$.*

Proof. According to Lemma 4.4, there exists an unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$W = U \begin{bmatrix} \hat{W} & 0 \\ 0 & 0 \end{bmatrix} U^* \quad and \quad T = U \begin{bmatrix} \hat{T} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where $\hat{W}, \hat{T} \in \mathbb{R}^{r\times r}$ are symmetric positive semi-definite satisfying $\hat{A} = \hat{W} + \mathrm{i}\hat{T} \in \mathbb{C}^{r\times r}$ being the non-singular matrix, *i.e.*, it holds $\mathrm{null}(\hat{W}) \cap \mathrm{null}(\hat{T}) = \{0\}$ with $\hat{W} = \frac{1}{2}(\hat{A} + \hat{A}^*)$, $\mathrm{i}\hat{T} = \frac{1}{2}(\hat{A} - \hat{A}^*)$. Thus, we have

$$\mathcal{T}_{\alpha,\theta} = U \begin{bmatrix} \hat{\mathcal{T}}_{\alpha,\theta} & 0 \\ 0 & I \end{bmatrix} U^*,$$

where $\hat{\mathcal{T}}_{\alpha,\theta} = \left[\alpha I + \cos(\theta)\hat{W} + \sin(\theta)\hat{T}\right]^{-1}\left[\alpha I - \left(\cos(\theta)\hat{T} - \sin(\theta)\hat{W}\right)\right]$. Note that $\hat{\mathcal{T}}_{\alpha,\theta}$ is the iteration matrix of the EP-SHSS method for solving non-singular complex symmetric linear system $\hat{A}\hat{x} = \hat{b}$, then we have

$$I - \mathcal{T}_{\alpha,\theta} = U \begin{bmatrix} I - \hat{\mathcal{T}}_{\alpha,\theta} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

it is straightforward to see that $index\,(I - \mathcal{T}_{\alpha,\theta}) = 1$. This completes the proof. □

Following, we will turn to verify the second condition of Lemma 4.1, *i.e.*, $\vartheta(\mathcal{T}_{\alpha,\theta}) < 1$, or equivalently, $\rho(\hat{\mathcal{T}}_{\alpha,\theta}) < 1$. Let $\hat{\lambda}$ be an eigenvalue of iteration matrix $\hat{\mathcal{T}}_{\alpha,\theta}$ and $\hat{x}$ be the corresponding eigenvector. Then we have $\hat{\mathcal{T}}_{\alpha,\theta}\hat{x} = \hat{\lambda}\hat{x}$ or equivalently

$$[\alpha I - i\big(\cos(\theta)\hat{T} - \sin(\theta)\hat{W}\big)]\hat{x} = \hat{\lambda}[\alpha I + \cos(\theta)\hat{W} + \sin(\theta)\hat{T}]\hat{x}. \tag{4.1}$$

Thus, we need prove $\rho(\hat{\mathcal{T}}_{\alpha,\theta}) < 1$ and the proof strategy is similar to Section 3. According to the proof of Theorem 3.1, the sufficient and necessary conditions for semi-convergence of the EP-SHSS method can be derived and summarize as follows.

Theorem 4.2. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite, and let $\theta \in (0, \frac{\pi}{2})$ and α be a positive constant. Let $\hat{x}$ be an eigenvector of the iteration matrix $\hat{\mathcal{T}}_{\alpha,\theta}$ corresponding to the eigenvalue $\hat{\lambda}$, then*

$$\hat{\lambda} = \frac{\alpha - \mathrm{i}\hat{b}}{\alpha + \hat{a}},$$

where

$$\hat{a} = \frac{\hat{x}^*[\cos(\theta)\hat{W} + \sin(\theta)\hat{T}]\hat{x}}{\hat{x}^*\hat{x}}, \quad \hat{b} = \frac{\hat{x}^*[\cos(\theta)\hat{T} - \sin(\theta)\hat{W}]\hat{x}}{\hat{x}^*\hat{x}}. \tag{4.2}$$

Furthermore, for all $\theta \in (0, \frac{\pi}{2})$, the EP-SHSS method is semi-convergent if and only if the parameter α satisfies the following condition

$$\alpha > \max\left\{0, \ \frac{\hat{b}^2 - \hat{a}^2}{2\hat{a}}\right\}.$$

Proof. The proof is similar to that of Theorem 3.1. □

Theorem 4.3. *Under the assumption of Theorem 4.2, the optimal parameter α_* of the EP-SHSS method is given by*

$$\alpha_\theta^* = \arg\min\left\{\frac{\sqrt{\alpha^2 + \hat{b}^2}}{\alpha + \hat{a}}\right\} = \frac{\hat{b}^2}{\hat{a}},$$

and the optimal semi-convergence factor of the EP-SHSS method is

$$\rho(\hat{\mathcal{T}}_{\alpha_\theta^*,\theta}) = \frac{|\hat{b}|}{\sqrt{\hat{b}^2 + \hat{a}^2}}. \tag{4.3}$$

Moreover, the quasi-optimal parameter θ^* of the EP-SHSS method by minimizing the semi-convergence factor $\rho(\hat{\mathcal{T}}_{\alpha_\theta^*,\theta})$ is given by

$$\theta^* = \left\{\arctan\left(\frac{\mu_{\min}\mu_{\max} - 1 + \sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min} + \mu_{\max}}\right) \in \left(0, \frac{\pi}{2}\right)\right\} \tag{4.4}$$
$$\left(=\left\{\operatorname{arccot}\left(\frac{1 - \mu_{\min}\mu_{\max} + \sqrt{(1+\mu_{\min}^2)(1+\mu_{\max}^2)}}{\mu_{\min} + \mu_{\max}}\right) \in \left(0, \frac{\pi}{2}\right)\right\}\right),$$

where $\mu_{\min}$ and $\mu_{\max}$ are the smallest and largest nonzero generalized eigenvalues of the matrix pair (W, T), respectively.

Proof. See Theorem 3.2. □

Analogously, we derive the following sufficient conditions for the semi-convergence of the EP-SHSS method.

Lemma 4.5. *Under the assumption of Theorem 4.2, the spectral radius $\rho(\hat{\mathcal{T}}_{\alpha,\theta})$ of the iteration matrix $\hat{\mathcal{T}}_{\alpha,\theta}$ is bounded by*

$$\hat{\delta}_{\alpha,\theta} = \frac{\sqrt{\alpha^2 + \hat{\mu}_{\max}^2}}{\alpha + \hat{\eta}_{\min}},$$

where

$$\hat{\eta}_{\min} = \min_{\eta_j \in sp\left(\cos(\theta)W + \sin(\theta)T\right)} \{\eta_j \setminus \{0\}\}, \ \hat{\mu}_{\max} = \max_{\mu_j \in sp\left(\cos(\theta)T - \sin(\theta)W\right)} \{|\mu_j|\}. \tag{4.5}$$

Furthermore, for all $\theta \in (0, \frac{\pi}{2})$, the EP-SHSS method is semi-convergent if $\hat{\delta}_{\alpha,\theta} < 1$, or equivalently, the parameter α satisfies the following condition

$$\alpha > \max\left\{0,\ \frac{\hat{\mu}_{\max}^2 - \hat{\eta}_{\min}^2}{2\hat{\eta}_{\min}}\right\}. \tag{4.6}$$

Proof. Analogous to the proof of Lemma 3.4, hence it is omitted. □

By Lemma 4.5, we obtain the quasi semi-convergence factor of the EP-SHSS method.

Lemma 4.6. *Under the assumption of Theorem 4.2, the quasi-optimal parameter $\hat{\alpha}_\theta^*$ by minimizing the upper bound $\hat{\delta}_{\alpha,\theta}$ of the spectral radius $\rho(\hat{\mathcal{T}}_{\alpha,\theta})$ is given by*

$$\hat{\alpha}_\theta^* = \arg\min\left\{\frac{\sqrt{\alpha^2 + \hat{\mu}_{\max}^2}}{\hat{\alpha} + \hat{\eta}_{\min}}\right\} = \frac{\hat{\mu}_{\max}^2}{\hat{\eta}_{\min}},$$

and

$$\hat{\delta}_{\hat{\alpha}_\theta^*,\theta} = \frac{\hat{\mu}_{\max}}{\sqrt{\hat{\eta}_{\min}^2 + \hat{\mu}_{\max}^2}}.$$

Proof. See Lemma 3.5. □

Remark 4.1. Note that the quasi-optimal parameters of the EP-SHSS method are given by θ^* defined as in (4.4) and $\hat{\alpha}_{\theta^*}^*$, respectively.

Similarly, to estimate the convergence rate of the preconditioned Krylov subspace methods (such as GMRES [44]), we also obtain the clustering property of the eigenvalues of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ in the following.

Theorem 4.4. *Assume that $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a singular matrix, with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semi-definite, and let $\theta \in [0, \frac{\pi}{2}]$ and α be a positive constant. Then the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ has an eigenvalue 1 with multiplicity $n - r$ and the remaining r the eigenvalues are*

$$\hat{\xi} = \frac{\hat{a} + \mathrm{i}\hat{b}}{\alpha + \hat{a}},$$

where $\hat{a}$ and $\hat{a}$ are defined as in (4.2). Moreover, it holds

$$\frac{\hat{\eta}_{\min}}{\alpha + \hat{\eta}_{\min}} \leqslant \Re(\hat{\xi}) = \frac{\hat{a}}{\alpha + \hat{a}} \leqslant \frac{\hat{\eta}_{\max}}{\alpha + \hat{\eta}_{\max}} \quad \text{and} \quad \left|\Im(\hat{\xi})\right| = \frac{|\hat{b}|}{\alpha + \hat{a}} \leqslant \frac{\hat{\mu}_{\max}}{\alpha + \hat{\eta}_{\min}}, \tag{4.7}$$

where $\hat{\eta}_{\min}$ and $\hat{\mu}_{\max}$ are defined as in (4.5), and $\hat{\eta}_{\max} = \max\limits_{\eta_j \in sp\left(\cos(\theta)W + \sin(\theta)T\right)} \{\eta_j\}$.

Proof. Let ξ be an eigenvalue of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$. According to Theorem 4.2 and Lemma 4.4, the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ has an eigenvalue 1 with multiplicity $n - r$ and the remaining r eigenvalues are

$$\hat{\xi} = 1 - \hat{\lambda} = \frac{\hat{a} + i\hat{b}}{\alpha + \hat{a}}.$$

Based on this, we have

$$\Re(\hat{\xi}) = \frac{\hat{a}}{\alpha + \hat{a}} \quad \text{and} \quad \Im(\hat{\xi}) = \frac{\hat{b}}{\alpha + \hat{a}}.$$

So, it is easy to obtain the results (4.7). This completes the proof. □

From Theorem 4.4, we know that the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ has an eigenvalue 1 with multiplicity $n-r$, thus we only require to discuss the asymptotic behavior of the eigenvalue $\hat{\xi}$ of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$ and obtain the following practical corollary.

Corollary 4.1. *It follows from Theorems 4.2 and 4.4 that if α is small enough and take θ^* defined as in (4.4), then for the eigenvalue $\hat{\xi}$ of the preconditioned matrix $M_{\alpha,\theta}^{-1}A$, it holds*

$$\hat{\xi} \approx 1 + \mathrm{i}\,\frac{\hat{b}}{\hat{a}},$$

and

$$\Re(\hat{\xi}) \to 1, \quad \left|\Im(\hat{\xi})\right| \to \left|\frac{\hat{b}}{\hat{a}}\right| \leqslant \frac{\hat{\mu}_{\max}}{\hat{\eta}_{\min}} < 1.$$

So, the eigenvalues of $M_{\alpha,\theta}^{-1}A$ are contained within the complex disk centered at $(1,0)$ with radius $r \approx \frac{\hat{\mu}_{\max}}{\hat{\eta}_{\min}}$ strictly less 1 when choosing $\alpha \to 0_+$ and θ^*, which is also a desirable property for Krylov subspace acceleration.

5. Numerical experiments

In this section, we perform some numerical experiments to illustrate the effectiveness of the EP-SHSS method for solving both non-singular and singular complex symmetric linear system (1.1). We compare the performance of the EP-SHSS method with those of MHSS [8], GSOR [46] and RMHSS [54] methods when they are used as solvers and as preconditioners for the GMRES method, from point of view of the number of iterations (denoted by "IT") and elapsed CPU time in seconds (denoted by "CPU"). In actual computations, we choose the zero vector as an initial guess and the stopping criterion if

$$RES := \frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} < 10^{-6}$$

where $x^{(k)}$ is the current approximate solutions, or the maximum prescribed number of iteration steps $k_{\max} = 600$. All the computations are run in MATLAB R2017a on a personal computer with Intel(R) Core(TM) i5-6500 CPU 3.20 GHz and 16.00GB of RAM memory. In our experiments, the linear sub-systems involved in each step of those methods can be solved effectively by the Cholesky factorization [44].

Example 1. (Structural dynamics)[8] Consider the non-singular system of linear equation (1.1) of the form

$$\left[(-\omega^2 M + K) + \mathtt{i}(\omega C_V + C_H)\right]x = b,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_H = \mu K$ with ν a damping coefficient, $M = I, C_V = 10I$, and K is the five-point centred difference matrix approximating the negative Laplacian operator $L = -\triangle$ with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1] \times [0,1]$ with the mesh-size $h = 1/(m+1)$. The matrix $K \in \mathbb{R}^{n \times n}$ possesses the tensor-product form $K = I \otimes V_m + V_m \otimes I$, with $V_m = h^{-2} tridiag(-1,2,-1) \in \mathbb{R}^{m \times m}$. Hence, K is an $n \times n$ block-tridiagonal matrix, with $n = m^2$. In addition, we set $\omega = \pi, \mu = 0.02$, and the right-hand side vector $b = (1+\mathrm{i})A\mathbf{1}$ with $A = (-\omega^2 M + K) + \mathtt{i}(\omega C_V + C_H)$ and $\mathbf{1}$ being the vector of all entries equal to 1. As before, we normalize coefficient matrix and right-hand side by multiplying both by h^2.

In fact, this non-singular complex symmetric system of linear equations arises in direct frequency domain analysis of an n-degree-of-freedom (n-DOF) linear system. And the equations of motion of an n-DOF linear system can be written in matrix form as

$$M\ddot{q} + (C_V + \frac{1}{\omega}C_H)\dot{q} + Kq = p,$$

TABLE 1. Numerical results of different iterative methods for Example 1.

Method		Grid			
		16×16	32×32	48×48	64×64
	$\|W\|_2/\|T\|_2$	29.5416	42.2993	46.1928	47.7643
MHSS	α^*	0.2153	0.0836	0.0671	0.0431
	IT	34	37	49	52
	CPU	0.129	2.612	20.998	49.821
	RES	7.53e−07	9.78e−07	8.69e−07	9.13e−07
GSOR	α^*	0.4554	0.4567	0.4570	0.4571
	IT	29	27	26	25
	CPU	0.076	0.986	10.621	23.249
	RES	5.77e−07	5.65e−07	6.15e−07	7.42e−07
EP-SHSS	$\alpha^*_{\theta^*}$	5.35e−04	1.54e−04	7.10e−05	4.06e−05
	θ^*	0.6527	0.6470	0.6459	0.6455
	IT	37	40	41	42
	CPU	0.083	0.819	7.815	16.323
	RES	7.39e−07	8.30e−07	9.10e−07	7.88e−07

where M, K, C_V, C_H and ω are the same as above, q is the configuration vector and p is the vector of generalized components of dynamic forces. Thus, it leads to the above non-singular complex symmetric linear system. For more details, see [5, 23].

Example 2. (Helmholtz-type equations)[18, 24] Consider the following non-singular complex Helmholtz equation

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

where σ_1 and σ_2 are real coefficient functions, u satisfies Dirichlet boundary conditions in $D = [0,1] \times [0,1]$ and $\mathrm{i} = \sqrt{-1}$. By discretizing the problem with finite differences on an $m \times m$ grid with mesh size $h = 1/(m+1)$. This leads to the complex linear system of the form

$$\big((K + \sigma_1 I) + \mathrm{i}\sigma_2 I\big)x = b,$$

where $K = I \otimes V_m + V_m \otimes I$ is the discretization of $-\Delta$ by means of centered differences, wherein $V_m = h^{-2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$. The right-hand side vector b is taken as $b = (1 + \mathrm{i})A\mathbf{1}$, with $\mathbf{1}$ being the vector of all entries equal to 1. Moreover, we also normalize the coefficient matrix and in right-hand side of the above equation by etaltiplying both by h^2.

In Tables 1 and 2, we list the numerical results about IT, CPU and RES of the tested methods with respective to different problem sizes for Example 1, *i.e.*, the non-singular complex symmetric linear system. The optimal parameter α for MHSS method is obtained by minimizing the numbers of iteration with respect to each test example and each spatial mesh-size while the optimal parameter α for GSOR method is chosen based on Theorem 2 of [46]. As for the EP-SHSS method, the optimal parameters are chosen based on Theorem 3.1. For better understanding the numerical results of Table 2, Figure 1 depicts the eigenvalues distribution of the corresponding preconditioned matrices with $m = 32$.

TABLE 2. Numerical results of the preconditioned GMRES method for Example 1.

Preconditioner		Grid			
		16×16	32×32	48×48	64×64
	$\|W\|_2/\|T\|_2$	29.5416	42.2993	46.1928	47.7643
I	IT	42	83	123	161
	CPU	1.710	13.101	51.141	113.243
	RES	5.53e−07	7.05e−07	8.36e−07	9.19e−07
	α^*	0.2153	0.0836	0.0671	0.0431
MHSS	IT	12	16	20	22
	CPU	0.131	1.634	10.245	27.417
	RES	7.73e−07	4.99e−07	2.51e−07	5.50e−07
	α^*	0.4554	0.4567	0.4570	0.4571
GSOR	IT	8	8	8	8
	CPU	0.045	0.524	6.147	15.258
	RES	9.78e−08	1.07e−07	1.09e−07	1.10e−07
	$\alpha^*_{\theta^*}$	5.35e−04	1.54e−04	7.10e−05	4.06e−05
	θ^*	0.6527	0.6470	0.6459	0.6455
EP-SHSS	IT	12	12	12	12
	CPU	0.053	0.461	3.157	10.546
	RES	1.14e−07	2.43e−07	2.90e−07	3.11e−07

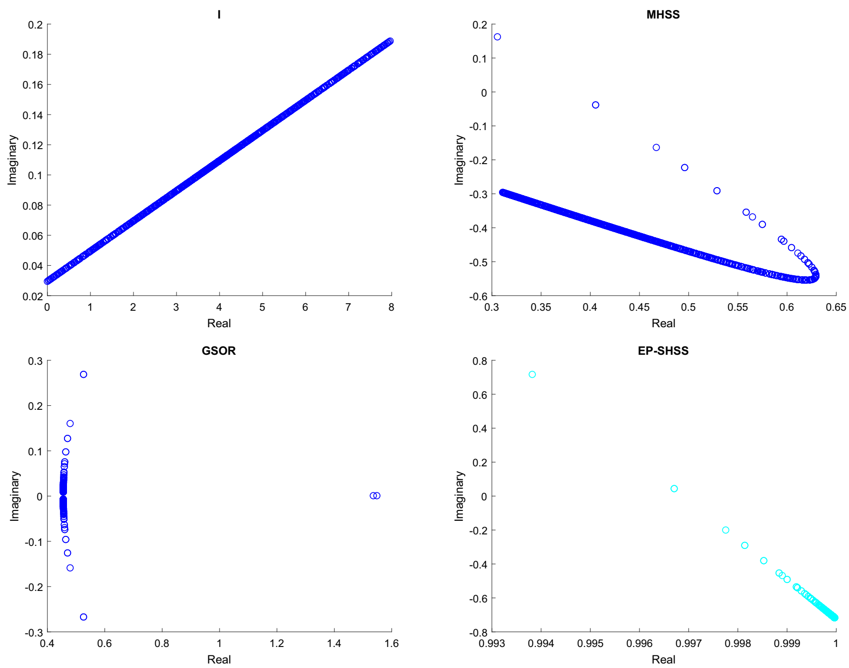

FIGURE 1. Eigenvalues distribution of the preconditioned matrix with $m = 32$ for Example 1.

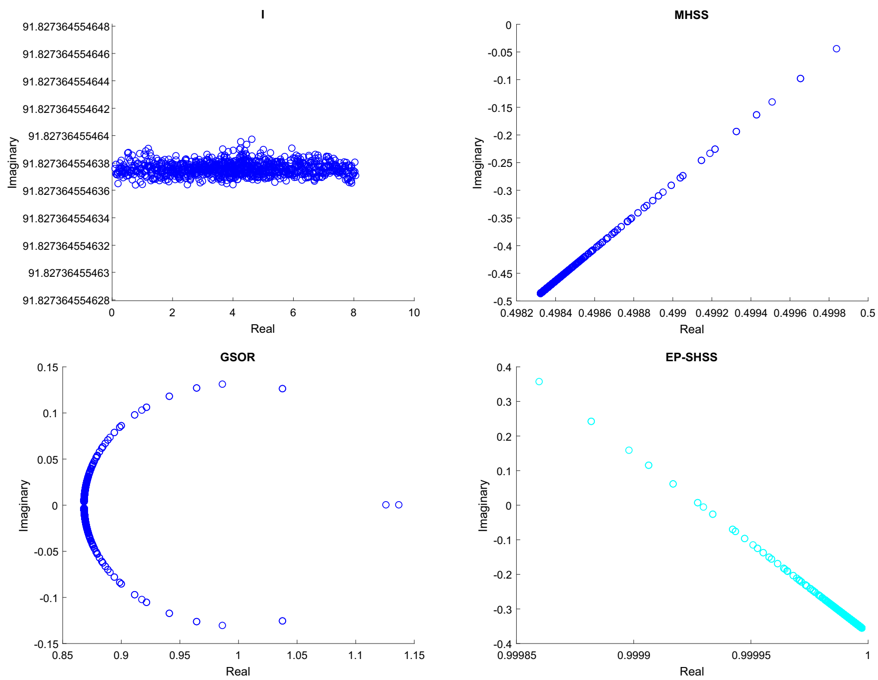

FIGURE 2. Eigenvalues distribution of the preconditioned matrix with $\sigma_1 = \sigma_2 = 10^2$ and $m = 32$ for Example 2.

Example 3. (Second-order differential equation)[5] Consider the singular linear system $Ax = b$ with the coefficient matrix $A = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ being given by

$$W = I \otimes V_c + V_c \otimes I \in \mathbb{R}^{n\times n}, \quad T = \frac{\gamma}{2m}(I \otimes U_c + U_c \otimes I) \in \mathbb{R}^{n\times n},$$

with

$$\begin{aligned} V_c &= V - (e_1 e_m^T + e_m e_1^T) \in \mathbb{R}^{m\times m}, \\ U_c &= U - (e_1 e_{m-1}^T + e_{m-1} e_1^T + e_a e_m^T + e_m e_a^T) \in \mathbb{R}^{m\times m}, \end{aligned}$$

and

$$\begin{aligned} V &= tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}, \\ U &= pentadiag(-1, -1, 4, -1, -1) \in \mathbb{R}^{m\times m}, \\ e_1 &= (1, 0, \dots, 0) \in \mathbb{R}^m, \\ e_{m-1} &= (0, \dots, 0, 1, 0) \in \mathbb{R}^m, \\ e_m &= (0, \dots, 1) \in \mathbb{R}^m, \\ e_a &= (1, 1, 0, \dots, 0) \in \mathbb{R}^m. \end{aligned}$$

The right-hand side vector b is defined as $b = Ax_*$, with $x_* = (1, 2, \dots, n)^\top \in \mathbb{R}^n$.

In fact, the above singular complex symmetric system of linear equations arises in the finite difference discretization with equidistant stepsize $h = 1/m$ of the two-dimensional variable-coefficient second-order differential

TABLE 3. Numerical results of different iterative methods for Example 2.

Method	Grid			32 × 32			
	(σ_1, σ_2)	$(10^2, 10^0)$	$(10^2, 10^1)$	$(10^2, 10^2)$	$(10^2, 10^3)$	$(10^2, 10^4)$	$(10^2, 10^5)$
	$\|W\|_2/\|T\|_2$	8792	879.2	87.92	8.792	0.8792	0.0879
MHSS	α^*	0.0009	0.0091	0.0912	0.9122	9.1223	91.2235
	IT	40	40	36	30	39	40
	CPU	1.181	1.160	1.081	0.873	1.134	1.169
	RES	9.16e−07	7.28e−07	9.09e−07	9.63e−07	7.55e−07	8.80e−07
GSOR	α^*	1.0000	0.9983	0.8685	0.2125	0.0237	0.0024
	IT	2	3	9	81	-	-
	CPU	0.103	0.151	0.276	0.629	-	-
	RES	5.09e−08	1.34e−07	1.72e−07	9.90e−07	-	-
EP-SHSS	$\alpha^*_{\theta^*}$	1.03e−08	1.12e−06	1.89e−05	1.88e−06	3.43e−08	6.39e−07
	θ^*	0.0042	0.0422	0.3536	0.7824	1.2042	1.5263
	IT	3	5	13	58	14	5
	CPU	0.102	0.121	0.182	0.262	0.201	0.111
	RES	5.12e−08	7.53e−08	6.22e−07	9.51e−07	8.06e−07	1.42e−07

equation satisfying the periodic boundary conditions:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \gamma(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}) = -f(x,y), & x, y \in (0,1) \times (0,1); \\ u(x,0) = u(x,1), & x \in (0,1); \\ u(0,y) = u(1,y), & y \in (0,1). \end{cases}$$

with

$$c(x,y) = 2 + \frac{1}{2}\big(\sin(2\pi x) + \sin(2\pi y)\big)$$

a given positive and continuously differentiable function defined on (0,1). Thus, it leads to the above singular complex symmetric matrix A.

By making use of the same strategy in Example 1, Tables 3 and 4 exhibit the numerical results in terms of IT, CPU and RES of the tested methods with respect to the experimental parameters for Example 2, *i.e.*, the non-singular complex Helmholtz equation. For better understanding the numerical results of Table 4, the eigenvalues distribution of the corresponding preconditioned matrices are shown in Figure 3 with $m = 32$ and $\sigma_1 = \sigma_2 = 10^2$.

In order to further testify the effectiveness of the EP-SHSS method, we give a singular linear system in Example 3. Similarly, the results of the tested methods with respect to $\gamma = 1000$ are listed in Tables 5 and 6. In Figure 3, we plot the eigenvalues distribution of the corresponding preconditioned matrices with $m = 32$.

From the numerical results, it is clearly to show that the EP-SHSS method and corresponding preconditioner are less than iteration steps (since the EP-SHSS method in one-step iterative method while others are a two-step iterative methods) and CPU times to achieve the stopping criterion when the (quasi-)optimal parameters are employed. we also see that the eigenvalues distribution of preconditioned matrix $M^{-1}_{\alpha^*_{\theta^*},\theta^*}A$ are quite clustered

TABLE 4. Numerical results of the preconditioned GMRES methods for Example 2.

Preconditioner	Grid			32 × 32			
	(σ_1, σ_2)	$(10^2, 10^0)$	$(10^2, 10^1)$	$(10^2, 10^2)$	$(10^2, 10^3)$	$(10^2, 10^4)$	$(10^2, 10^5)$
	$\|W\|_2/\|T\|_2$	8792	879.2	87.92	8.792	0.8792	0.0879
I	IT	53	56	58	50	18	8
	CPU	0.292	0.323	0.596	2.624	1.532	0.883
	RES	8.03e−07	8.72e−07	9.30e−07	8.62e−07	7.18e−07	5.96e−08
	α^*	0.0009	0.0091	0.0912	0.9122	9.1223	91.2235
MHSS	IT	6	8	14	16	14	8
	CPU	0.117	0.201	0.313	1.806	1.161	0.657
	RES	2.34e−08	3.11e−07	4.40e−07	6.21e−07	2.49e−07	5.96e−08
	α^*	1.0000	0.9983	0.8685	0.2125	0.0237	0.0024
GSOR	IT	2	3	7	22	75	94
	CPU	0.055	0.123	0.269	1.245	12.563	31.512
	RES	6.71e−08	6.13e−08	1.47e−07	6.45e−07	9.77e−07	6.73e−07
	$\alpha^*_{\theta^*}$	1.03e−08	1.12e−06	1.89e−05	1.88e−06	3.43e−08	6.39e−07
	θ^*	0.0042	0.0422	0.3536	0.7824	1.2042	1.5263
EP-SHSS	IT	3	5	11	16	10	5
	CPU	0.064	0.115	0.203	0.883	0.349	0.124
	RES	1.70e−09	5.31e−07	3.58e−07	6.22e−07	4.75e−07	3.47e−07

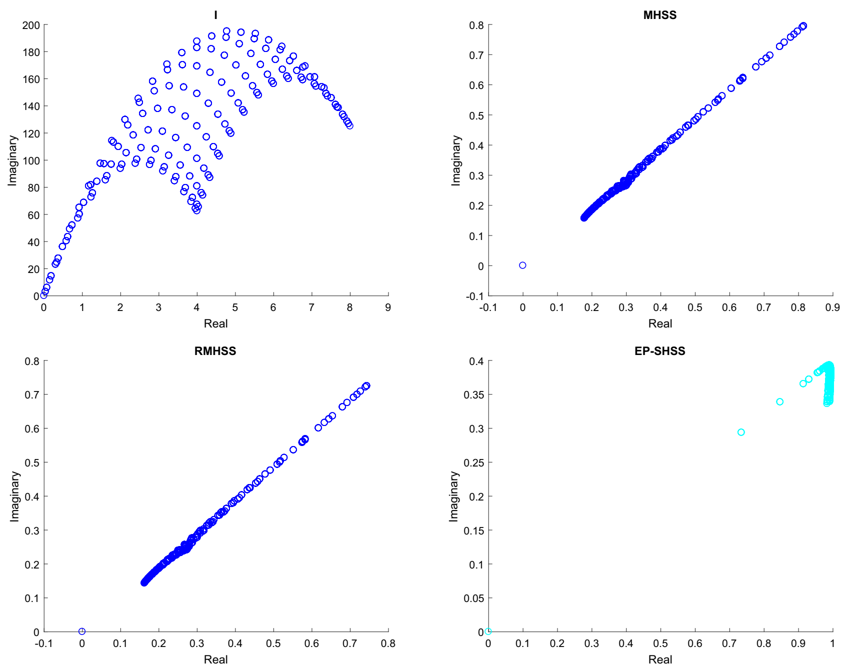

FIGURE 3. Eigenvalues distribution of the preconditioned matrix with $m = 32$ for Example 3.

TABLE 5. Numerical results of different iterative methods for Example 3.

Method		Grid			
		16×16	32×32	48×48	64×64
	$\|W\|_2/\|T\|_2$	0.0207	0.0410	0.0614	0.0820
MHSS	α^*	3.6341	1.6321	1.0632	0.8347
	IT	73	58	57	65
	CPU	0.204	1.627	12.170	33.356
	RES	9.61e−07	9.69e−07	9.30e−07	9.73e−07
RMHSS	α^*	3.6341	1.6321	1.0632	0.8347
	ω^*	0.8856	0.9121	0.9953	1.1241
	IT	51	48	57	61
	CPU	0.112	1.214	9.364	21.384
	RES	8.23e−07	9.47e−07	9.83e−07	8.68e−07
EP-SHSS	$\alpha^*_{\theta^*}$	1.0000	1.0000	1.0000	1.0000
	θ^*	1.1761	1.1776	1.1779	1.1780
	IT	16	15	20	36
	CPU	0.081	0.883	3.541	13.841
	RES	4.22-07	5.70e−07	6.65e−07	7.57e−07

TABLE 6. Numerical results of the preconditioned GMRES method for Example 3.

Preconditioner		Grid			
		16×16	32×32	48×48	64×64
	$\|W\|_2/\|T\|_2$	0.0207	0.0410	0.0614	0.0820
I	IT	16	38	70	98
	CPU	0.514	6.247	34.143	63.741
	RES	2.98e−07	1.31e−07	2.34e−07	5.08e−07
MHSS	α^*	3.6341	1.6321	1.0632	0.8347
	IT	14	20	24	26
	CPU	0.133	1.114	10.143	25.143
	RES	3.39e−07	3.57e−07	6.99e−07	8.43e−07
RMHSS	α^*	3.6341	1.6321	1.0632	0.8347
	ω^*	0.8856	0.9121	0.9953	1.1241
	IT	14	20	24	26
	CPU	0.102	0.984	8.124	19.543
	RES	3.39e−07	3.57e−07	6.99e−07	8.43e−07
EP-SHSS	$\alpha^*_{\theta^*}$	1.0000	1.0000	1.0000	1.0000
	θ^*	1.1761	1.1776	1.1779	1.1780
	IT	6	9	11	14
	CPU	0.056	0.537	2.247	9.587
	RES	5.75e−07	6.06e−07	8.96e−07	1.98e− 07

accord with theoretical analysis. In other words, the numerical results show that the effectiveness of the EP-SHSS method either as a solver or as a preconditioner for solving both non-singular and singular complex symmetric linear system (1.1).

6. Conclusions

In this paper, we establish the EP-SHSS method for solving both non-singular and singular complex symmetric linear systems. We considered the acceleration of the EP-SHSS method by preconditioned Krylov subspace method and studied the spectral properties of the corresponding preconditioned matrix. Numerical results show that the effectiveness of the EP-SHSS method either as a solver or as a preconditioner in terms of the number of iteration steps ("IT") and CPU times ("CPU").

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