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ERROR ANALYSIS OF PML-FEM APPROXIMATIONS FOR THE HELMHOLTZ EQUATION IN WAVEGUIDES

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Abstract. In this paper, we study finite element approximate solutions to the Helmholtz equation in waveguides by using a perfectly matched layer (PML). The PML is defined in terms of a piecewise linear coordinate stretching function with two parameters for absorbing propagating and evanescent components respectively, and truncated with a Neumann condition on an artificial boundary rather than a Dirichlet condition for cutoff modes that waveguides may allow. In the finite element analysis for the PML problem, we have to deal with two difficulties arising from the lack of full regularity of PML solutions and the anisotropic nature of the PML problem with, in particular, large PML damping parameters. Anisotropic finite element meshes in the PML regions depending on the damping parameters are used to handle anisotropy of the PML problem. As a main goal, we establish quasi-optimal *a priori* error estimates, that does not depend on anisotropy of the PML problem (when no cutoff mode is involved), including the exponentially convergent PML error with respect to the width and the strength of PML. The numerical experiments that confirm the convergence analysis will be presented.

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1. Introduction

We consider the time-harmonic wave propagation problem in a semi-infinite waveguide Ω_∞

$$\begin{aligned} \Delta u + k^2 u &= f \quad \text{in } \Omega_\infty, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega_\infty \end{aligned} \tag{1.1}$$

with a radiation condition at infinity, where ν stands for the outward unit normal vector on the boundary $\partial\Omega_\infty$. Here Ω_∞ is a domain such that

$$\begin{aligned} &\Omega_\infty \cap \{(x,y) \in \mathbb{R}\times\mathbb{R}^{d-1} \ : \ x > -\delta\} = (-\delta,\infty)\times\Theta, \\ &\Omega := \Omega_\infty \cap \{(x,y) \in \mathbb{R}\times\mathbb{R}^{d-1} \ : \ x < 0\} \text{ is bounded,} \end{aligned} \tag{1.2}$$

with a constant $\delta > 0$ and a bounded cross-section $\Theta \subset \mathbb{R}^{d-1}$, $d = 2$ or 3 (see Fig. 1), and f is a wave source in $L^2(\Omega_\infty)$ supported in the region for $x < -\delta$. We assume that the waveguide Ω_∞ and the cross-section Θ have

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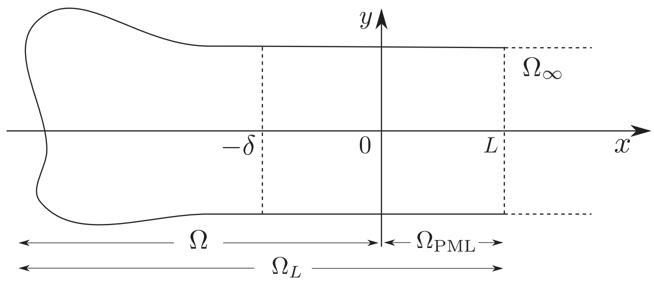


FIGURE 1. Configuration of the waveguide Ω_∞ in $\mathbb{R}^2$.

sufficiently regular boundaries so that the solution u to the problem (1.1) lies in $H^2(\Omega)$, for example, Ω_∞ is a semi-infinite waveguide with smooth boundary or a semi-infinite convex waveguide with a convex cross-section Θ. Also, we assume that k^2 is not an eigenvalue of the waveguide Ω_∞ for well-posedness of the problem. The existence of eigenvalues of a waveguide can be found in *e.g.* [13, 16].

Numerical study for the problem requires domain truncation techniques that can produce approximate solutions in the region of interest not to be seriously contaminated by the reflection resulting from the domain truncation. They include methods such as PML [3], truncated DtN approaches [4, 18, 20], rational function approximations to the DtN operator [14] or complete radiation boundary conditions [19, 23, 28, 29].

In this paper, we study an error analysis for finite element approximations to the waveguide problem truncated *via* a PML with a piecewise linear coordinate stretching. The PML application transforms the Helmholtz equation to an anisotropic partial differential equation with discontinuous coefficients, whose solution lacks the full regularity. The anisotropy and low regularity may decrease convergence rate, however by observing that PML approximations can have the full regularity in each subdomain on which coefficients are constant (which will be verified later) and taking finite element meshes reflecting the anisotropic property of coefficients, we show that finite element PML approximations converge in a quasi-optimal rate, independent of anisotropy of the PML problem, including an exponentially decaying reflection error written as $\mathrm{e}^{-2\sigma_\mu L}$, where σ_μ is a constant associated with the PML strength and L stands for the width of the PML region.

It is out of question that the PML method is a well-analyzed numerical technique for wave propagation problems. One of useful approaches for studying the problem in a Cartesian coordinate configuration is based on the Green's function of the Helmholtz equation [8, 24, 26, 27]. In this approach, we first study well-posedness of an infinite PML problem and then use this result to establish the well-posedness of a truncated PML problem and convergence of approximate solutions. However this approach is not appropriate for studying our case since the Green's function is not available in general semi-infinite waveguides. An alternative way presented in [3, 9, 10] is one that splits the problem into two parts, one of which is posed in only the artificial absorbing layer, PML, and the other is the problem posed only in the physical region. The problem in the physical region is supplemented with a DtN-like operator induced from the problem in PML for an absorbing boundary condition. We follow this approach to study well-posedness and convergence of PML solutions in the physical domain. In addition, we use the series expansion method in terms of cross-sectional eigenfunctions in PML to study a regularity of PML solutions, which is an important ingredient for a quasi-optimal convergence of finite element approximations.

Unlike exterior problems, waveguide problems can involve evanescent modes, and even cutoff modes as well as propagating modes. In order to handle more efficiently evanescent modes and cutoff modes, we modify the standard PML in two aspects. The first one is that PML is truncated with a homogeneous Neumann boundary condition instead of a usual Dirichlet condition. We note that since any coordinate stretching can not make cutoff modes decay exponentially the PML with the homogeneous Dirichlet condition would doom to fail to absorb cutoff modes effectively. However, it can be expected that the Neumann boundary condition behaves the exact radiation condition for cutoff modes in that cutoff modes deformed by any coordinate stretching function have no variation along the axis of waveguides. The second one is that we introduce one more parameter in a coordinate stretching function for reducing reflection of evanescent modes more rapidly, in particular, when wave sources are too close to the absorbing layer or there is slowly decaying evanescent modes. The coordinate stretching function $\tilde{x}$ that will be used for PML is defined as $\tilde{x} = \sigma(x)x$, where σ is a piecewise constant function,

$$\sigma(x) = \begin{cases} 1 & \text{for } x < 0, \\ \sigma_0 = \sigma_r + i\sigma_i & \text{for } x \geq 0 \end{cases} \tag{1.3}$$

with $\sigma_r, \sigma_i > 0$. σ_i is a usual PML parameter responsible for making propagating modes decay exponentially and σ_r is introduced for damping evanescent modes more rapidly. The PML strength σ_μ mentioned earlier to describe the exponential convergence rate is proportional to $|\sigma_0|$ (see (3.2)).

Here we note that this choice of a coordinate stretching function with low order smoothness causes discontinuity of coefficients of the PML problem and so that the problem falls within the realm of interface problems [1, 12, 22, 30]. Since solutions to problems with non-smooth coefficients do not have the full regularity, a low rate of convergence of finite element approximations might occur without special care for finite element spaces. Nonetheless, it is observed in numerical experiments of [8] that standard finite element approximations of the PML problem converge in a desired rate without the full regularity. Even though PML solutions are not regular in the whole computational domain, it turns out that PML solutions can have the full regularity in two separate domains on which coefficients are constant. It allows us to utilize an idea similar to Schatz's argument [31] with the full regularity on each subdomain and derive the quasi-optimal convergence rate of finite element approximations as long as finite element meshes are aligned along the interface between the physical region and PML. It is worth noting that there have been intensive studies on the regularity [6, 12, 22, 30] of solutions of general interface problems (with real coefficients opposed to imaginary coefficients of the PML problem) and finite element applications [1, 5, 7, 11].

Since the convergence rate $\mathrm{e}^{-2\sigma_\mu L}$ of PML solutions depends on the product of the PML strength σ_μ and the PML width L, errors of PML solutions resulting from different pairs (σ_μ, L) of the parameters are all the same as long as $\sigma_\mu L$ is constant. However, it may not be true any more for finite element problems in that the PML problem with large σ_μ has highly anisotropic nature. For finite element problems, we need to avoid using a strong PML strength without special care for meshes in the PML region. In fact, if we set values of σ_μ and L such that $\sigma_\mu L$ is constant and take a shape-regular and quasi-uniform mesh with fixed mesh size, then the anisotropy makes finite element errors worse as σ_μ increases. In this paper, we consider anisotropic meshes reflecting the anisotropic property in the PML region to get rid of the influence of the anisotropy arising from PML. To do this, when the physical domain Ω is decomposed into shape-regular and quasi-uniform meshes with mesh size h, in the PML region we take the mesh size h_{PML} along the axis of the waveguide proportional to $h/|\sigma_0|$ while the cross-sectional mesh size is kept in the same level of h. We note that this approach requires the number of degrees of freedom for finite element approximations to be invariant in order to keep the same reflection errors and finite element errors since the number of grid points along the axis of the waveguide in the PML region is $O(L/h_{\text{PML}}) = O(|\sigma_0|L/h)$. However, by doing so we establish a quasi-optimal *a priori* error estimate, that does not depend on anisotropy of the PML problem (when no cutoff mode involved), including the exponential convergence of PML errors.

The outline of the paper is as follows. In Section 2 as preliminaries, we review the exact radiation condition based on the DtN operator in waveguides and reformulate the PML problem in a variational form. In Section 3 we discuss a selection of the PML parameters and provide some estimates important for the stability and the convergence analysis. In Section 4 we study the problem posed only on PML and introduce an approximate DtN operator induced from it. Section 5 is devoted to an analysis on the problem posed on the physical region Ω and convergence of PML approximate solutions to the radiating solution. In Section 6, the well-posedness of the problem on the whole computational domain and regularity of the solution are investigated. A finite element error analysis will be delivered in Section 7. Finally, numerical experiments illustrating the theory developed in the proceeding sections will be presented in Section 8.

2. Preliminaries

In this section we provide a review on the exact radiation condition based on the Dirichlet-to-Neumann (DtN) operator in waveguides in terms of series expansions and introduce the PML problem in a variational form. For simple presentation, we only deal with the case $d = 2$ throughout the paper and the three-dimensional problem can be analyzed in the same way without any essential change.

Let $\{Y_n\}_{n=0}^{\infty}$ be an orthonormal basis consisting of Neumann eigenfunctions of the transverse negative Laplace operator $-\Delta_y$ in the domain Θ associated with Neumann eigenvalues λ_n^2 such that

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$$

and $\lambda_n \to \infty$ as n tends towards infinity. Clearly, there exists N such that $k \geq \lambda_n$ for $n \leq N$ and $k < \lambda_n$ for $n > N$.

We denote $\Gamma_0 = \{0\} \times \Theta$ and so Γ_0 can be identified with Θ. Let $\dot{H}^s(\Gamma_0)$, $-1 \leq s \leq 2$, be the Sobolev space equipped with the norm

$$\|u\|_{\dot{H}^s(\Gamma_0)} = \left(\sum_{n=0}^{\infty} (1+\lambda_n^2)^s |u_n|^2 \right)^{1/2}$$

for $u = \sum_{n=0}^{\infty} u_n Y_n$. This space of order $-1 \leq s < 3/2$ is identical with the usual fractional Sobolev space $H^s(\Gamma_0)$ of order s obtained by the real interpolation. For $3/2 \leq s \leq 2$, it is thought of as a space of functions which are in the fractional Sobolev space of order s and whose normal derivative vanishes on $\partial\Gamma_0$ [25].

We note that the radiating solution u of the problem has the series representation for $x > 0$,

$$\begin{aligned} u(x,y) &= \sum_{n=0}^{\infty} A_n \mathrm{e}^{i\mu_n x} Y_n(y) \\ &= \sum_{n=0}^{N} A_n \mathrm{e}^{i\mu_n x} Y_n(y) + \sum_{n=N+1}^{\infty} A_n \mathrm{e}^{-\tilde{\mu}_n x} Y_n(y), \end{aligned}$$

where $\mu_n = \sqrt{k^2 - \lambda_n^2}$ with the square root of the negative real axis branch cut and $\mu_n = i\tilde{\mu}_n$ with $\tilde{\mu}_n > 0$, the decay rate of evanescent modes for $n > N$.

Remark 2.1. We assume that k is a fixed positive wavenumber. For a distribution of λ_n, k may coincide with λ_N or may not be equal to any of λ_n, depending on the position of k with respect to λ_n. If $k = \lambda_N$, then the Nth mode is called a cutoff mode. In case that k is not equal to any of λ_n, k may be close to λ_n for some n. In this case the nth mode is a near-cutoff mode with $0 \neq |\mu_n| \ll 1$. As the value of $|\mu_n|$ of near-cutoff modes makes a considerable influence on the stability and the convergence of approximate solutions, some estimates related with the smallest non-zero $|\mu_n|$ will be given in Section 3.

Now, the radiation condition can be characterized in terms of the so-called Dirichlet-to-Neumann(DtN) operator

$$T : \dot{H}^{1/2}(\Gamma_0) \to \dot{H}^{-1/2}(\Gamma_0)$$

defined by

$$T(u) = \sum_{n=0}^{\infty} i\mu_n u_n Y_n \tag{2.1}$$

for $u = \sum_{n=0}^{\infty} u_n Y_n \in \dot{H}^{1/2}(\Gamma_0)$, that is, radiating solutions satisfy the Helmholtz equation with the radiation condition on Γ_0

$$\begin{aligned} \Delta u + k^2 u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \setminus \bar{\Gamma}_0, \\ \frac{\partial u}{\partial \nu} &= T(u) \quad \text{on } \Gamma_0. \end{aligned} \tag{2.2}$$

Denoting the L^2-inner product on a domain $\mathcal{D}$ by $(f,g)_{\mathcal{D}} = \int_{\mathcal{D}} f(x)\bar{g}(x)\mathrm{d}x$, and a duality pairing between $H^{-s}(\mathcal{D}) \times H^s(\mathcal{D})$ for $-1 \le s \le 1$ by $\langle\cdot,\cdot\rangle_{s,\mathcal{D}}$, the variational problem for (2.2) is to find $u \in H^1(\Omega)$ satisfying

$$A_\Omega(u,v) - \langle T(u), v\rangle_{1/2,\Gamma_0} = (f,v)_\Omega \quad \text{for all } v \in H^1(\Omega), \tag{2.3}$$

where

$$A_\Omega(u,v) = (\nabla u, \nabla v)_\Omega - k^2(u,v)_\Omega.$$

We recall the following well-known theorem [4, 17]. The proof is provided to keep this paper self-contained.

Lemma 2.2. *Assume that k^2 is not an eigenvalue of the waveguide Ω_∞. Then there exists a positive constant C such that for $w \in H^1(\Omega)$*

$$\|w\|_{H^1(\Omega)} \le C \sup_{0\ne\phi\in H^1(\Omega)} \frac{|A_\Omega(w,\phi) - \langle T(w),\phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}}. \tag{2.4}$$

Proof. We observe that

$$A_\Omega(w,w) - \langle T(w), w\rangle_{1/2,\Gamma_0} = \|w\|^2_{H^1(\Omega)} - (k^2+1)\|w\|^2_{L^2(\Omega)} - \sum_{n=0}^{\infty} i\mu_n |w_n|^2,$$

for $w \in H^1(\Omega)$ with $w = \sum_{n=0}^\infty w_n Y_n$ on Γ_0, from which Gårding's inequality follows,

$$\begin{aligned}|A_\Omega(w,w) - \langle T(w), w\rangle_{1/2,\Gamma_0}| &\ge \Re(A_\Omega(w,w) - \langle T(w), w\rangle_{1/2,\Gamma_0})\\ &= \|w\|^2_{H^1(\Omega)} - (k^2+1)\|w\|^2_{L^2(\Omega)} + \sum_{n=N+1}^{\infty} \tilde{\mu}_n |w_n|^2\\ &\ge \|w\|^2_{H^1(\Omega)} - (k^2+1)\|w\|^2_{L^2(\Omega)}.\end{aligned}$$

Since k^2 is not an eigenvalue of the waveguide Ω_∞, the Petree-Tartar lemma (see *e.g.*, [15]) leads to the required inf-sup condition (2.4). □

Along the complex stretched contour given by (1.3), the PML solution $\tilde{u}(x,y) = u(\tilde{x},y)$ can be written as

$$\tilde{u}(x,y) = \sum_{n=0}^{N} A_n \mathrm{e}^{i\mu_n\sigma_r x}\mathrm{e}^{-\mu_n\sigma_i x} Y_n(y) + \sum_{n=N+1}^{\infty} A_n \mathrm{e}^{-\tilde{\mu}_n\sigma_r x}\mathrm{e}^{-i\tilde{\mu}_n\sigma_i x} Y_n(y). \tag{2.5}$$

Since the PML solution is a superposition of cutoff modes and evanescent modes whose decay rates are controlled by σ_r and σ_i, it is natural to truncate the infinite domain at $x = L$ (here L is a parameter representing the width of PML and the resulting boundary is denoted by Γ_L) and impose a convenient boundary condition such as a homogeneous Dirichlet or Neumann condition on the fictitious boundary Γ_L. As mentioned in the introduction, the Neumann condition will herein be employed in order to achieve the better performance for cutoff modes.

Let

$$\begin{aligned}\Omega_{\mathrm{PML}} &= (0,L)\times\Theta,\\ \Omega_L &= \Omega_\infty \cap \{(x,y)\in\mathbb{R}\times\mathbb{R}^{d-1} \ : \ x < L\}.\end{aligned}$$

See Figure 1. Here Ω_{PML} is the artificial layer to absorb wave fields propagating into it and Ω_L will serve as the computational domain containing Ω. Denoting $\sigma = \mathrm{d}\tilde{x}/\mathrm{d}x$, the reduced problem supplemented with the homogeneous Neumann boundary condition on Γ_0 is to find the PML solution $\tilde{u}$ satisfying

$$\begin{aligned}\frac{1}{\sigma}\frac{\partial}{\partial x}\frac{1}{\sigma}\frac{\partial}{\partial x}\tilde{u}(x,y) + \Delta_y \tilde{u}(x,y) + k^2\tilde{u}(x,y) = f \quad &\text{in } \Omega_L,\\ \frac{\partial u}{\partial \nu} = 0 \quad &\text{on } \partial\Omega_L.\end{aligned} \tag{2.6}$$

From now on, for a domain $\mathcal{D}$ we will use the weighted norms

$$\|u\|^2_{L^2_\sigma(\mathcal{D})} := (|\sigma|u, u)_{\mathcal{D}},$$
$$\|u\|^2_{H^1_\sigma(\mathcal{D})} := \left(\frac{1}{|\sigma|}\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right)_{\mathcal{D}} + (|\sigma|\nabla_y u, \nabla_y u)_{\mathcal{D}} + (|\sigma|u, u)_{\mathcal{D}}$$

in $L^2(\mathcal{D})$ and $H^1(\mathcal{D})$, respectively. These spaces are denoted by $L^2_\sigma(\mathcal{D})$ and $H^1_\sigma(\mathcal{D})$ to distinguish them from the Sobolev spaces with the usual Sobolev norms.

The truncated problem (2.6) can be reformulated into a variational problem to find $\tilde{u} \in H^1_\sigma(\Omega_L)$ such that

$$A(\tilde{u}, \phi) = (f, \phi)_\Omega \quad \text{for all } \phi \in H^1_\sigma(\Omega_L), \tag{2.7}$$

where $A(\cdot,\cdot)$ is a sesquilinear form defined by

$$A(u, v) = \left(\frac{1}{\sigma}\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{\Omega_L} + \left(\sigma\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_{\Omega_L} - k^2(\sigma u, v)_{\Omega_L} \quad \text{for } u, v \in H^1_\sigma(\Omega_L).$$

We will study the problem (2.7) by splitting it into two sub-problems. One is the problem posed on the absorbing layer Ω_{PML}. This local problem is devoted to defining an approximate DtN operator, which can be substituted for the DtN operator in the problem (2.2) for an approximate radiation condition. The other is the problem on the physical domain Ω, where we are interested in approximate solutions and the convergence analysis will be conducted in this region. Before we proceed to the analysis for the PML problem (2.7), we discuss a selection of the PML parameters and some estimates important for the stability and the convergence analysis in the following section.

3. Parameter selection and near-cutoff modes

In this section, we discuss a selection of the PML parameters and some estimates related with the smallest non-zero $|\mu_n|$. These estimates are important for the stability and the convergence analysis and will be used for norm estimates frequently throughout the paper. We begin with a notational assumption for cutoff modes. Since Fourier coefficients for cutoff modes are linear functions with respect of x, that are different from those (exponential functions) of any other modes and so they need a special care, we will reserve the index N for cutoff modes. Hence, if k^2 does not coincide with any of eigenvalues of $-\Delta_y$ on Θ, then eigenvalues are numbered without the index N, that is, $\ldots < \lambda_{N-1} < \lambda_{N+1} < \ldots$.

As we will see later in the convergence analysis, the reflection of propagating modes with axial frequency μ_n is reduced by a factor $\mathrm{e}^{-2\mu_n\sigma_i L}$ and that of evanescent modes with decay rate $\tilde{\mu}_n$ is decreased by a factor $\mathrm{e}^{-2\tilde{\mu}_n\sigma_r L}$. Thus, the overall reflection error is determined by

$$\min\{\mathrm{e}^{-2\mu_{N-1}\sigma_i L}, \mathrm{e}^{-2\tilde{\mu}_{N+1}\sigma_r L}\}, \tag{3.1}$$

which is related with the reflection error of near-cutoff modes when μ_{N-1} or $\tilde{\mu}_{N+1}$ is very small. In order to describe the impact of near-cutoff modes on the stability and the convergence of approximate solutions, we introduce a constant $\mu_{\min}$ representing the smallest non-zero $|\mu_n|$,

$$\mu_{\min} := \min\{|\mu_n| \ : \ \mu_n \neq 0\},$$

and norm estimates in the rest of the paper will be made with constants involving $\mu_{\min}$.

According to (3.1), the reflection errors of the propagating component and the evanescent component can be kept in the same level when $\mu_{N-1}\sigma_i = \tilde{\mu}_{N+1}\sigma_r$. Thus we will choose the parameters σ_r and σ_i satisfying

$$\mu_{N-1}\sigma_i = \tilde{\mu}_{N+1}\sigma_r > 1.$$

Here we introduce a constant $\sigma_\mu := \mu_{N-1}\sigma_i = \tilde{\mu}_{N+1}\sigma_r$ for the PML strength instead of $|\sigma_0|$. This choice of σ_r and σ_i implies that

$$\sigma_0 = \sigma_\mu \left(\frac{1}{\tilde{\mu}_{N+1}} + \frac{i}{\mu_{N-1}} \right) = |\sigma_0| e^{i\theta_0} \tag{3.2}$$

for a fixed $\theta_0 \in (0, \pi/2)$. Then it can be shown that $|\sigma_0|$ is proportional to σ_μ. More precisely, we have

$$\frac{1}{\mu_{\min}} \sigma_\mu \le |\sigma_0| \le \frac{\sqrt{2}}{\mu_{\min}} \sigma_\mu. \tag{3.3}$$

Also, it holds

$$\begin{aligned} &|\sigma_0| \le \frac{C}{\mu_{\min}} \sigma_r, \qquad |\sigma_0| \le C\sigma_i, \qquad \text{or} \\ &|\sigma_0| \le C\sigma_r, \qquad\quad |\sigma_0| \le \frac{C}{\mu_{\min}} \sigma_i, \end{aligned} \tag{3.4}$$

depending on whether k is closer to λ_{N-1} or λ_{N+1} than the other. From here on we shall use a generic constant C that takes different values at different places but depends only on k^2, the cross-section Θ or the domain Ω but not PML parameters. We further assume that the PML parameters σ_μ and L satisfy

$$\sigma_\mu L \ > \ 1. \tag{3.5}$$

Remark 3.1. In practice, the exact values of μ_{N-1} and $\tilde{\mu}_{N+1}$ may not be available. In this case, we apply a Lanczos algorithm to the cross-sectional domain Θ to compute the eigenvalues nearest k^2 and use this information to find an appropriate σ_0.

Norm estimates in this paper require upper bounds of $|\mu_n|^2(1+\lambda_n^2)^{-1}$ and $|\mu_n|^{-2}(1+\lambda_n^2)$ for $\lambda_n \neq k$, which are also related with $\mu_{\min}$ as reflection errors depend on it. If we work with a wavenumber k such that $k = \lambda_N$, then $\mu_{\min}$ is given by $\min\left\{\sqrt{\lambda_N^2 - \lambda_{N-1}^2}, \sqrt{\lambda_{N+1}^2 - \lambda_N^2}\right\}$, however in case that $k \neq \lambda_n$ for all n but k is close to λ_{N-1} or λ_{N+1} instead, we have

$$\begin{aligned} &\mu_{\min}^2 = k^2 - \lambda_{N-1}^2 \le \frac{1}{2}(\lambda_{N+1}^2 - \lambda_{N-1}^2) \ \text{ and } \ \lambda_{N+1}^2 - k^2 \ge \frac{1}{2}(\lambda_{N+1}^2 - \lambda_{N-1}^2), \quad \text{or} \\ &\mu_{\min}^2 = \lambda_{N+1}^2 - k^2 \le \frac{1}{2}(\lambda_{N+1}^2 - \lambda_{N-1}^2) \ \text{ and } \ k^2 - \lambda_{N-1}^2 \ge \frac{1}{2}(\lambda_{N+1}^2 - \lambda_{N-1}^2). \end{aligned} \tag{3.6}$$

In such a case it can be shown that

$$\frac{|\mu_n|^2}{1+\lambda_n^2} = \frac{|k^2 - \lambda_n^2|}{1+\lambda_n^2} \le C \tag{3.7}$$

$$\frac{1+\lambda_n^2}{|\mu_n|^2} = \frac{1+\lambda_n^2}{|k^2 - \lambda_n^2|} \le \begin{cases} C & \text{for non-near-cutoff modes} \\ C\mu_{\min}^{-2} & \text{for near-cutoff modes} \end{cases}. \tag{3.8}$$

Due to the presence of $\mu_{\min}^{-2}$ in (3.8), it will be found that the convergence of approximate solutions becomes worse if a near-cutoff exists.

4. Problem on Ω_{PML} and approximate DtN operator

In this section we study the PML Helmholtz equation on Ω_{PML} and an approximate DtN operator associated with the local PML problem. Let $\widetilde{H}_\sigma^1(\Omega_{\text{PML}})$ be the subspace of functions in $H_\sigma^1(\Omega_{\text{PML}})$ vanishing on Γ_0. For $g \in \dot{H}^{1/2}(\Gamma_0)$ we consider the variational problem to find $w \in H_\sigma^1(\Omega_{\text{PML}})$ satisfying $w = g$ on Γ_0 and

$$A_{\text{PML}}(w, \phi) = 0 \quad \text{for all } \phi \in \widetilde{H}_\sigma^1(\Omega_{\text{PML}}), \tag{4.1}$$

where $A_{\mathrm{PML}}(\cdot,\cdot)$ is the sesquilinear form on $\widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$ defined by

$$A_{\mathrm{PML}}(u,v) = \left(\frac{1}{\sigma}\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{\Omega_{\mathrm{PML}}} + \left(\sigma\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_{\Omega_{\mathrm{PML}}} - k^2(\sigma u, v)_{\Omega_{\mathrm{PML}}}$$

for $u, v \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$ with the constant $\sigma = \sigma_0$. Clearly, we can observe that $A_{\mathrm{PML}}(\cdot,\cdot)$ is bounded,

$$|A_{\mathrm{PML}}(u,v)| \le C\|u\|_{H^1_\sigma(\Omega_{\mathrm{PML}})}\|v\|_{H^1_\sigma(\Omega_{\mathrm{PML}})}, \tag{4.2}$$

and satisfies

$$A_{\mathrm{PML}}(u,v) = A_{\mathrm{PML}}(\bar{v},\bar{u}) \tag{4.3}$$

for $u, v \in H^1_\sigma(\Omega_{\mathrm{PML}})$. The well-posedness of general source problems associated with the sesquilinear form $A_{\mathrm{PML}}(\cdot,\cdot)$ in $\widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$ is established by the following lemma regarding the coercivity of $A_{\mathrm{PML}}(\cdot,\cdot)$, which can be proved by the similar idea used in [10].

Lemma 4.1. *There exists a positive constant C such that*

$$\|w\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})} \le C\frac{(\sigma_\mu L)^2}{\mu^3_{\min}}|A_{\mathrm{PML}}(w,w)| \tag{4.4}$$

for $w \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$.

Proof. For $w \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$, we have

$$\frac{1}{\sigma_r}\Re(A_{\mathrm{PML}}(w,w)) = \frac{1}{|\sigma_0|^2}\|\frac{\partial w}{\partial x}\|^2_{L^2(\Omega_{\mathrm{PML}})} + \|\frac{\partial w}{\partial y}\|^2_{L^2(\Omega_{\mathrm{PML}})} - k^2\|w\|^2_{L^2(\Omega_{\mathrm{PML}})}, \tag{4.5}$$

$$\frac{1}{\sigma_i}\Im(A_{\mathrm{PML}}(w,w)) = \frac{-1}{|\sigma_0|^2}\|\frac{\partial w}{\partial x}\|^2_{L^2(\Omega_{\mathrm{PML}})} + \|\frac{\partial w}{\partial y}\|^2_{L^2(\Omega_{\mathrm{PML}})} - k^2\|w\|^2_{L^2(\Omega_{\mathrm{PML}})}, \tag{4.6}$$

from which it follows that

$$\frac{1}{\sigma_r}\Re(A_{\mathrm{PML}}(w,w)) - \frac{1}{\sigma_i}\Im(A_{\mathrm{PML}}(w,w)) = \frac{2}{|\sigma_0|^2}\|\frac{\partial w}{\partial x}\|^2_{L^2(\Omega_{\mathrm{PML}})}. \tag{4.7}$$

Since w vanishes on Γ_0, it is easy to see that

$$\|w\|^2_{L^2(\Omega_{\mathrm{PML}})} \le L^2\|\frac{\partial w}{\partial x}\|^2_{L^2(\Omega_{\mathrm{PML}})},$$

which implies from (4.5) that

$$\begin{aligned}\frac{|\sigma_0|}{\sigma_r}\Re(A_{\mathrm{PML}}(w,w)) \ge &\left(\frac{1}{|\sigma_0|} - (k^2+1)|\sigma_0|L^2\right)\|\frac{\partial w}{\partial x}\|^2_{L^2(\Omega_{\mathrm{PML}})} \\ &+ |\sigma_0|\|\frac{\partial w}{\partial y}\|^2_{L^2(\Omega_{\mathrm{PML}})} + |\sigma_0|\|w\|^2_{L^2(\Omega_{\mathrm{PML}})}.\end{aligned} \tag{4.8}$$

Now, it can be shown from (4.7) and (4.8) that for $\gamma = (k^2+1)|\sigma_0|^3L^2/2$

$$\begin{aligned}\|w\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})} &\le \gamma\left(\frac{1}{\sigma_r}\Re(A_{\mathrm{PML}}(w,w)) - \frac{1}{\sigma_i}\Im(A_{\mathrm{PML}}(w,w))\right) + \frac{|\sigma_0|}{\sigma_r}\Re(A_{\mathrm{PML}}(w,w)) \\ &= \frac{\gamma+|\sigma_0|}{\sigma_r}\Re(A_{\mathrm{PML}}(w,w)) - \frac{\gamma}{\sigma_i}\Im(A_{\mathrm{PML}}(w,w)).\end{aligned}$$

By using (3.3) and (3.4) we are led to

$$\|w\|^2_{H^1_\sigma(\Omega_{\rm PML})} \leq C\frac{(\sigma_\mu L)^2}{\mu^3_{\min}}|A_{\rm PML}(w,w)|,$$

which is the required coercivity. □

We consider the regularity result of the source problem with $F \in L^2_\sigma(\Omega_{\rm PML})$ to find $w \in \widetilde{H}^1_\sigma(\Omega_{\rm PML})$ satisfying

$$A_{\rm PML}(w,\phi) = (\sigma_0 F, \phi)_{\Omega_{\rm PML}} \quad \text{for all } \phi \in \widetilde{H}^1_\sigma(\Omega_{\rm PML}) \tag{4.9}$$

with respect to the higher order norm

$$\begin{aligned}\|w\|^2_{H^2_\sigma(\Omega_{\rm PML})} &:= \|w\|^2_{H^1_\sigma(\Omega_{\rm PML})} \\ &\quad + \frac{1}{|\sigma_0|^3}\|\frac{\partial^2 w}{\partial x^2}\|^2_{L^2(\Omega_{\rm PML})} + \frac{1}{|\sigma_0|}\|\frac{\partial^2 w}{\partial x \partial y}\|^2_{L^2(\Omega_{\rm PML})} + |\sigma_0|\|\frac{\partial^2 w}{\partial y^2}\|^2_{L^2(\Omega_{\rm PML})}\end{aligned}$$

by using the method of eigenfunction expansions.

Lemma 4.2. *The solution w to the problem (4.9) satisfies*

$$\|w\|_{H^2_\sigma(\Omega_{\rm PML})} \leq C\frac{(\sigma_\mu L)^2}{\mu^3_{\min}}\|F\|_{L^2_\sigma(\Omega_{\rm PML})}.$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. Let $D := (0,L) \subset \mathbb{R}$ and $\{X_m\}_{m=1}^\infty$ be a complete orthonormal basis for $L^2(D)$ consisting of eigenfunctions of the Sturm–Liouville problem on D,

$$-\frac{\mathrm{d}^2 X_m}{\mathrm{d}x^2} = \zeta_m^2 X_m, \quad X_m(0) = 0, \frac{\mathrm{d}X_m}{\mathrm{d}x}(L) = 0. \tag{4.10}$$

For $F \in C^\infty(\overline{\Omega_{\rm PML}})$, we can find an eigenfunction expansion,

$$F(x,y) = \sum_{n=0}^\infty \sum_{m=1}^\infty F_{n,m} X_m(x) Y_n(y).$$

Since we look for the solution w of the form

$$w(x,y) = \sum_{n=0}^\infty w_n(x) Y_n(y) = \sum_{n=0}^\infty \left(\sum_{m=1}^\infty w_{n,m} X_m(x)\right) Y_n(y),$$

each coefficient $w_{n,m}$ solves the problem

$$\left(\zeta_m^2 + (\lambda_n^2 - k^2)\sigma_0^2\right) w_{n,m} = \sigma_0^2 F_{n,m}.$$

Since w_n satisfies the same boundary conditions as those that X_m does, by the idea used in [25], we have

$$\|\frac{\mathrm{d}^j w_n}{\mathrm{d}x^j}\|^2_{L^2(D)} = \sum_{m=1}^\infty \zeta_m^{2j}|w_{n,m}|^2 \quad \text{for } j = 0,1,2,$$

from which it follows that

$$\begin{aligned}
&\|w\|^2_{H^2_\sigma(\Omega_{\rm PML})}\\
&=\sum_{n=0}^\infty\left(\frac{1}{|\sigma_0|^3}\|\frac{\mathrm{d}^2w_n}{\mathrm{d}x^2}\|^2_{L^2(D)}+(1+\lambda_n^2)\frac{1}{|\sigma_0|}\|\frac{\mathrm{d}w_n}{\mathrm{d}x}\|^2_{L^2(D)}+(1+\lambda_n^2+\lambda_n^4)|\sigma_0|\|w_n\|^2_{L^2(D)}\right)\\
&=\sum_{n=0}^\infty\sum_{m=1}^\infty\frac{\zeta_m^4+\zeta_m^2(1+\lambda_n^2)|\sigma_0|^2+(1+\lambda_n^2+\lambda_n^4)|\sigma_0|^4}{|\sigma_0|^3}\frac{|\sigma_0|^4|F_{n,m}|^2}{|\zeta_m^2+(\lambda_n^2-k^2)\sigma_0^2|^2}\\
&\le C\left(\frac{(\sigma_\mu L)^2}{\mu^3_{\min}}\right)^2\|F\|^2_{L^2_\sigma(\Omega_{\rm PML})}.
\end{aligned}$$

In the inequality, we have used Lemma A.1. Finally, the proof is completed by the density of $C^\infty(\overline{\Omega_{\rm PML}})$ in $L^2_\sigma(\Omega_{\rm PML})$. □

For the problem (4.1) with a boundary condition on Γ_0, the stability constant is improved.

Lemma 4.3. *For $g\in\dot H^s(\Gamma_0)$, $s=1/2$ or $3/2$, the problem (4.1) has a unique solution $w\in H^1_\sigma(\Omega_{\rm PML})$ satisfying*

$$\|w\|_{H^{s+1/2}_\sigma(\Omega_{\rm PML})}\le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|g\|_{\dot H^s(\Gamma_0)}. \tag{4.11}$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. Due to the cylindrical geometry of the domain $\Omega_{\rm PML}$, we can show that there exists a lifting $\tilde g\in H^1_\sigma(\Omega_{\rm PML})$ of g satisfying $\|\tilde g\|_{H^1_\sigma(\Omega_{\rm PML})}\le C\|g\|_{\dot H^{1/2}(\Gamma_0)}$ (see Lem. A.2). Thus, the standard theory of the elliptic problem using (4.2) and Lemma 4.1 shows the existence of a unique solution w satisfying $\|w\|_{H^1_\sigma(\Omega_{\rm PML})}\le C\|g\|_{\dot H^{1/2}(\Gamma_0)}$ with C that may depend on $(\sigma_\mu L)^2\mu_{\min}^{-3}$.

For the improved stability constant, let $g=\sum_{n=0}^\infty g_nY_n\in\dot H^{1/2}(\Gamma_0)$. Since w can be written as

$$w(x,y)=(A_N+B_Nx)Y_N(y)+\sum_{n\ne N}(A_ne^{i\mu_n\sigma_0x}+B_n\mathrm{e}^{-i\mu_n\sigma_0x})Y_n(y) \tag{4.12}$$

in $\Omega_{\rm PML}$, the boundary conditions on Γ_0 and Γ_L require that Fourier coefficients satisfy $A_N=g_N$ and $B_N=0$ for $n=N$, and

$$\begin{aligned}A_n+B_n&=g_n,\\ A_n\mathrm{e}^{i\mu_n\sigma_0L}-B_n\mathrm{e}^{-i\mu_n\sigma_0L}&=0\end{aligned} \tag{4.13}$$

for $n\ne N$. By solving the above equations, we have

$$A_n=\frac{g_n}{1+\mathrm{e}^{2i\mu_n\sigma_0L}}\quad\text{and}\quad B_n=\frac{\mathrm{e}^{2i\mu_n\sigma_0L}g_n}{1+\mathrm{e}^{2i\mu_n\sigma_0L}} \tag{4.14}$$

and see that w is given by

$$w(x,y)=g_NY_N(y)+\sum_{n\ne N}g_nE_n(x)Y_n(y),$$

where

$$E_n(x)=\frac{\mathrm{e}^{i\mu_n\sigma_0x}+\mathrm{e}^{2i\mu_n\sigma_0L}\mathrm{e}^{-i\mu_n\sigma_0x}}{1+\mathrm{e}^{2i\mu_n\sigma_0L}}\cdot$$

The denominator and the numerator of E_n satisfy $|1+\mathrm{e}^{2i\mu_n\sigma_0L}|\ge1-|\mathrm{e}^{-2\sigma_\mu L}|>1/2$ for $\sigma_\mu L>1$ and

$$|\mathrm{e}^{i\mu_n\sigma_0x}\pm\mathrm{e}^{2i\mu_n\sigma_0L}\mathrm{e}^{-i\mu_n\sigma_0x}|^2<\begin{cases}2(\mathrm{e}^{-2\mu_n\sigma_ix}+\mathrm{e}^{-2\mu_n\sigma_i(2L-x)}) & \text{for } n<N,\\ 2(\mathrm{e}^{-2\tilde\mu_n\sigma_rx}+\mathrm{e}^{-2\tilde\mu_n\sigma_r(2L-x)}) & \text{for } n>N,\end{cases}$$

respectively.

Now based on these two inequalities, we shall prove that

$$|\sigma_0|^{1-2\ell}\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\|^2_{L^2(D)} \le \frac{C}{\mu_{\min}}(1+\lambda_n^2)^{\ell-1/2} \quad \text{for } \ell = 0,1,2. \tag{4.15}$$

To this end, letting σ_n represent σ_i for $n < N$ and σ_r for $n > N$, we show that

$$\begin{aligned}|\sigma_0|^{1-2\ell}\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\|^2_{L^2(D)} &\le C|\sigma_0|^{1-2\ell}\int_0^L |\mu_n\sigma_0|^{2\ell}\mathrm{e}^{-2|\mu_n|\sigma_n x}\mathrm{d}x\\ &= C|\mu_n|^{2\ell}|\sigma_0|\frac{1}{2|\mu_n|\sigma_n}(1-\mathrm{e}^{-2|\mu_n|\sigma_n L}) \le C\frac{|\mu_n|^{2\ell}|\sigma_0|}{|\mu_n|\sigma_n}.\end{aligned}$$

We first consider the estimate for $n = N \pm 1$. By using $\sigma_n|\mu_n| \ge \sigma_\mu$ and (3.3) for $\ell = 0$ and by using (3.4) and (3.7) for $\ell = 1,2$, it is obtained that

$$|\sigma_0|^{1-2\ell}\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\|^2_{L^2(D)} \le \begin{cases} C\dfrac{|\sigma_0|}{\sigma_\mu}(1+\lambda_n^2)^{1/2}(1+\lambda_n^2)^{-1/2} \le \dfrac{C}{\mu_{\min}}(1+\lambda_n^2)^{-1/2} & \text{for } \ell = 0,\\ C\dfrac{|\sigma_0|}{\sigma_n}|\mu_n|^{2\ell-1} \le \dfrac{C}{\mu_{\min}}(1+\lambda_n^2)^{\ell-1/2} & \text{for } \ell = 1,2.\end{cases}$$

On the other hand, for $n \ne N, N\pm 1$, we employ (3.8) for $\ell = 0$ and (3.7) for $\ell = 1,2$ together with (3.4) to obtain

$$|\sigma_0|^{1-2\ell}\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\|^2_{L^2(D)} \le C\frac{|\sigma_0|}{\sigma_n}|\mu_n|^{2\ell-1} \le \frac{C}{\mu_{\min}}(1+\lambda_n^2)^{\ell-1/2}.$$

Hence it follows that

$$\begin{aligned}\|w\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})} &= |\sigma_0|L(1+\lambda_N^2)|g_N|^2\\ &\quad + \sum_{n\ne N}\left((1+\lambda_n^2)|\sigma_0|\|E_n\|^2_{L^2(D)} + \frac{1}{|\sigma_0|}\|\frac{\mathrm{d}E_n}{\mathrm{d}x}\|^2_{L^2(D)}\right)|g_n|^2\\ &\le C\frac{\sigma_\mu L}{\mu_{\min}}\|g\|^2_{\dot{H}^{1/2}(\Gamma_0)},\end{aligned} \tag{4.16}$$

which leads to (4.11) for $s = 1/2$.

Similarly, for $s = 3/2$ it holds that

$$\begin{aligned}\|w\|^2_{H^2_\sigma(\Omega_{\mathrm{PML}})} &= \|w\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})} + \lambda_N^4(|\sigma_0|L)|g_N|^2\\ &\quad + \sum_{n\ne N}\left(\frac{1}{|\sigma_0|^3}\|\frac{\mathrm{d}^2E_n}{\mathrm{d}x^2}\|^2_{L^2(D)} + \lambda_n^2\frac{1}{|\sigma_0|}\|\frac{\mathrm{d}E_n}{\mathrm{d}x}\|^2_{L^2(D)} + \lambda_n^4|\sigma_0|\|E_n\|^2_{L^2(D)}\right)|g_n|^2\\ &\le C\frac{\sigma_\mu L}{\mu_{\min}}\|g\|^2_{\dot{H}^{3/2}(\Gamma_0)},\end{aligned}$$

which completes the proof. $\square$

Remark 4.4. Due to the symmetric property (4.3) of $A_{\mathrm{PML}}(\cdot,\cdot)$, the results of Lemmas 4.2 and 4.3 hold for the adjoint problems.

We introduce an approximate DtN operator $T_{\mathrm{PML}} : \dot{H}^{1/2}(\Gamma_0) \to \dot{H}^{-1/2}(\Gamma_0)$ as follows: for $g \in \dot{H}^{1/2}(\Gamma_0)$ we define

$$T_{\mathrm{PML}}(g) = \frac{1}{\sigma_0}\frac{\partial w}{\partial x}$$

on Γ_0, where $w \in H^1_\sigma(\Omega_{\mathrm{PML}})$ is the solution to the problem (4.1) with $w = g$ on Γ_0. By considering the coefficients A_n and B_n in (4.14) and ones for $n = N$ it can be shown that

$$T_{\mathrm{PML}}(g) = \sum_{n=0}^{\infty} i\mu_n \frac{1 - \mathrm{e}^{2i\mu_n\sigma_0 L}}{1 + \mathrm{e}^{2i\mu_n\sigma_0 L}} g_n Y_n. \tag{4.17}$$

Here we notice that T_{PML} gives the exact radiation condition for cutoff modes since $\mu_N = 0$.

The following lemma shows the convergence of the approximate DtN operator.

Lemma 4.5. *With the assumption* (3.5)*, for* $u \in \dot{H}^{1/2}(\Gamma_0)$

$$\|(T - T_{\mathrm{PML}})(u)\|_{\dot{H}^{-1/2}(\Gamma_0)} \le Ce^{-2\sigma_\mu L}\|u\|_{\dot{H}^{1/2}(\Gamma_0)}.$$

Proof. Let $u = \sum_{n=0}^{\infty} u_n Y_n \in \dot{H}^{1/2}(\Gamma_0)$. By subtracting (4.17) from (2.1), and by invoking (3.7) and using the inequality $x/(1-x) < 2x$ for $0 < x < 1/2$, we can see that

$$\begin{aligned}
\|(T - T_{\mathrm{PML}})(u)\|^2_{\dot{H}^{-1/2}(\Gamma_0)} &= \sum_{n=0}^{\infty}(1+\lambda_n^2)^{-1/2}|\mu_n|^2|u_n|^2\left|\frac{2\mathrm{e}^{2i\mu_n\sigma_0 L}}{1+\mathrm{e}^{2i\mu_n\sigma_0 L}}\right|^2 \\
&\le C\Bigg(\sum_{n=0}^{N-1}(1+\lambda_n^2)^{1/2}|u_n|^2\left(\frac{\mathrm{e}^{-2\mu_n\sigma_i L}}{1-\mathrm{e}^{-2\mu_n\sigma_i L}}\right)^2 \\
&\quad + \sum_{n=N+1}^{\infty}(1+\lambda_n^2)^{1/2}|u_n|^2\left(\frac{\mathrm{e}^{-2\tilde{\mu}_n\sigma_r L}}{1-\mathrm{e}^{-2\tilde{\mu}_n\sigma_r L}}\right)^2\Bigg) \\
&\le Ce^{-4\sigma_\mu L}\|u\|^2_{\dot{H}^{1/2}(\Gamma_0)},
\end{aligned}$$

which completes the proof. □

Remark 4.6. As noticed in (4.17), the approximate DtN based on the PML with the Neumann boundary condition on Γ_L gives the exact radiation condition for cutoff modes. In contrast, when the Dirichlet condition on Γ_L is used instead of the Neumann condition, the approximate DtN operator is given by

$$T_{\mathrm{PML}}(u) = \frac{-u_N}{\sigma_0 L}Y_N + \sum_{n\neq N} i\mu_n \frac{1 + \mathrm{e}^{2i\mu_n\sigma_0 L}}{1 - \mathrm{e}^{2i\mu_n\sigma_0 L}} u_n Y_n$$

for $u = \sum_{n=0}^{\infty} u_n Y_n$. This approximate condition also produces the exponential convergence result for non-cutoff modes, however it has only linear convergence for cutoff modes with respect to $1/(\sigma_\mu L)$.

5. Problem on Ω and convergence of approximate PML solutions

In this section, we analyze the well-posedness of the problem on the domain Ω supplemented with the boundary condition based on the approximate DtN operator to find $\tilde{u}$ solving

$$\begin{aligned}
\Delta\tilde{u} + k^2\tilde{u} &= f \quad \text{in } \Omega, \\
\frac{\partial \tilde{u}}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \setminus \bar{\Gamma}_0, \\
\frac{\partial \tilde{u}}{\partial \nu} &= T_{\mathrm{PML}}(\tilde{u}) \quad \text{on } \Gamma_0.
\end{aligned} \tag{5.1}$$

The convergence of approximate solutions on Ω will be established as well.

This problem is reformulated into a weak problem to find $\tilde{u} \in H^1(\Omega)$ such that

$$A_\Omega(\tilde{u}, \phi) - \langle T_{\mathrm{PML}}(\tilde{u}), \phi\rangle_{1/2,\Gamma_0} = (f, \phi)_\Omega \quad \text{for all } \phi \in H^1(\Omega). \tag{5.2}$$

We first study the inf-sup conditions of the sesquilinear form corresponding to the problem (5.2)

Lemma 5.1. *Assume that k^2 is not an eigenvalue of the waveguide Ω_∞. Then there exists a positive constant $M > 1$ such that for $\sigma_\mu L > M$ it holds that for $u \in H^1(\Omega)$*

$$\|u\|_{H^1(\Omega)} \le C \sup_{0 \ne \phi \in H^1(\Omega)} \frac{|A_\Omega(u, \phi) - \langle T_{\mathrm{PML}}(u), \phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}} \tag{5.3}$$

and

$$\|u\|_{H^1(\Omega)} \le C \sup_{0 \ne \phi \in H^1(\Omega)} \frac{|A_\Omega(\phi, u) - \langle T_{\mathrm{PML}}(\phi), u\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}}. \tag{5.4}$$

Proof. As $A_\Omega(u, \phi) - \langle T_{\mathrm{PML}}(u), \phi\rangle_{1/2,\Gamma_0} = A_\Omega(\bar{\phi}, \bar{u}) - \langle T_{\mathrm{PML}}(\bar{\phi}), \bar{u}\rangle_{1/2,\Gamma_0}$, it suffices to prove the first inf-sup condition (5.3). We start with the inf-sup condition (2.4) of the sesquilinear form of the Helmholtz equation with the exact radiation condition based on the DtN operator. By using the convergence of the approximate DtN operator of Lemma 4.5 we have

$$\begin{aligned}
\|u\|_{H^1(\Omega)} &\le C \sup_{0 \ne \phi \in H^1(\Omega)} \left(\frac{|A_\Omega(u, \phi) - \langle T_{\mathrm{PML}}(u), \phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}} + \frac{|\langle (T - T_{\mathrm{PML}})(u), \phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}} \right) \\
&\le C \left(\sup_{0 \ne \phi \in H^1(\Omega)} \frac{|A_\Omega(u, \phi) - \langle T_{\mathrm{PML}}(u), \phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}} + \mathrm{e}^{-2\sigma_\mu L} \|u\|_{H^1(\Omega)} \right)
\end{aligned}$$

for large $\sigma_\mu L$ satisfying (3.5). Now, by choosing $\sigma_\mu L$ large enough so that $\mathrm{e}^{-2\sigma_\mu L} < 1/(2C)$, we can have the desired inf-sup condition (5.3). □

The well-posedness of the problem (5.2) and exponential convergence of approximate PML solutions in Ω with increasing $\sigma_\mu L$ will be presented in the following theorem.

Theorem 5.2. *Let M be the constant given in Lemma 5.1. Then for $\sigma_\mu L > M$ the problem (5.2) admits a unique solution $\tilde{u} \in H^1(\Omega)$ satisfying*

$$\|\tilde{u}\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)}. \tag{5.5}$$

In addition, for the exact solution $u \in H^1(\Omega)$ to the problem (2.3) it holds that

$$\|u - \tilde{u}\|_{H^1(\Omega)} \le C e^{-2\sigma_\mu L} \|f\|_{L^2(\Omega)}$$

for $\sigma_\mu L > M$.

Proof. The unique existence of a solution satisfying (5.5) is an immediate consequence of the inf-sup conditions in Lemma 5.1. For the exponential convergence, we use Lemma 5.1 to show that

$$\begin{aligned}
\|u - \tilde{u}\|_{H^1(\Omega)} &\le C \sup_{0 \ne \phi \in H^1(\Omega)} \frac{|A_\Omega(u - \tilde{u}, \phi) - \langle T_{\mathrm{PML}}(u - \tilde{u}), \phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}} \\
&= C \sup_{0 \ne \phi \in H^1(\Omega)} \frac{|\langle (T - T_{\mathrm{PML}})(u), \phi\rangle_{1/2,\Gamma_0}|}{\|\phi\|_{H^1(\Omega)}}.
\end{aligned}$$

Now, the convergence result in Lemma 4.5 of the operator T_{PML}, a trace inequality and the stability of the problem (2.3) yield that

$$\|u - \tilde{u}\|_{H^1(\Omega)} \le C e^{-2\sigma_\mu L} \|u\|_{H^1(\Omega)} \le C e^{-2\sigma_\mu L} \|f\|_{L^2(\Omega)},$$

which completes the proof. □

6. Problem on Ω_L and regularity

In this section, we study the problem (2.7) on the whole computational domain Ω_L and regularity of solutions to the problem when $f \in L^2(\Omega)$ is supported for $x < -\delta$. We observe that two problems (2.7) and (5.2) are equivalent in the sense of the next lemma, due to the fact that

$$\tilde{v}^- = \tilde{v}^+ \quad \text{and} \quad \frac{\partial \tilde{v}^-}{\partial x} = \frac{1}{\sigma_0}\frac{\partial \tilde{v}^+}{\partial x} \quad \text{on } \Gamma_0$$

as traces from Ω and Ω_{PML}, respectively, for a solution $\tilde{v}$ to the problem (2.7).

Lemma 6.1. *Let M be the constant given in Lemma 5.1 and assume that $\sigma_\mu L > M$. If $\tilde{v} \in H^1_\sigma(\Omega_L)$ is a solution to the problem (2.7), then $\tilde{v}|_\Omega \in H^1(\Omega)$, the restriction of $\tilde{v}$ to Ω, is a solution to the problem (5.2). On the other hand, if $\tilde{u} \in H^1(\Omega)$ is a solution to the problem (5.2), then the extension $\tilde{v}$ of $\tilde{u}$ defined by*

$$\tilde{v} = \begin{cases} \tilde{u} & \text{in } \Omega, \\ w & \text{in } \Omega_{\mathrm{PML}}, \end{cases}$$

where w is the solution to (4.1) with $w = \tilde{u}$ on Γ_0, is a solution to the problem (2.7).

Lemma 6.1 and Theorem 5.2 yield the well-posedness of the problem (2.7).

Lemma 6.2. *Let M be the constant given in Lemma 5.1 and assume that $\sigma_\mu L > M$. Then the problem (2.7) admits a unique solution $\tilde{v}$ satisfying*

$$\|\tilde{v}\|_{H^1_\sigma(\Omega_L)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|f\|_{L^2(\Omega)}.$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. The unique existence of solutions is a consequence of Lemma 6.1 and Theorem 5.2. For the stability estimate, we use Lemma 6.1 and Theorem 5.2 again to show that

$$\|\tilde{v}|_\Omega\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{6.1}$$

By using Lemma 4.3, a trace inequality and (6.1) we can also show that

$$\|\tilde{v}|_{\Omega_{\mathrm{PML}}}\|_{H^1_\sigma(\Omega_{\mathrm{PML}})} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|\tilde{v}\|_{\dot{H}^{1/2}(\Gamma_0)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|\tilde{v}|_\Omega\|_{H^1(\Omega)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|f\|_{L^2(\Omega)}. \tag{6.2}$$

Finally, combining (6.1) and (6.2) shows that

$$\|\tilde{v}\|^2_{H^1_\sigma(\Omega_L)} = \|\tilde{v}|_\Omega\|^2_{H^1(\Omega)} + \|\tilde{v}|_{\Omega_{\mathrm{PML}}}\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})} \le C\frac{\sigma_\mu L}{\mu_{\min}}\|f\|^2_{L^2(\Omega)},$$

which completes the proof. □

We will present the regularity of solutions to the problem (2.7). The Helmholtz equation deformed by PML with the piecewise linear coordinate stretching function (1.3) has discontinuous coefficients. A regularity result in such a case can be found in *e.g.*, [6,21], which shows that there is s_0 with $0 < s_0 < 1/2$ such that if $0 < s < s_0$, then the solution u is in $H^{1+s}(\Omega_L)$ for $f \in (H^{1-s}(\Omega_L))^*$, the dual space of $H^{1-s}(\Omega_L)$. This regularity result is not sufficient to obtain the desired convergence rate of finite element approximations, however, in case that

$f \in L^2_\sigma(\Omega_L)$ is supported in the region for $x < -\delta$, the approximate PML solution can have the higher regularity in the local subdomains Ω and Ω_{PML}. To show this, we define a Sobolev space

$$X = \{u \in H^1_\sigma(\Omega_L) \ : \ u|_\Omega \in H^2(\Omega),\ u|_{\Omega_{\mathrm{PML}}} \in H^2_\sigma(\Omega_{\mathrm{PML}})\}$$

with the norm

$$\|u\|_X = (\|u\|^2_{H^2(\Omega)} + \|u\|^2_{H^2_\sigma(\Omega_{\mathrm{PML}})})^{1/2}.$$

Lemma 6.3. *Let M be the constant given in Lemma 5.1 and assume that $\sigma_\mu L > M$. If $f \in L^2_\sigma(\Omega_L)$ is supported in the region for $x < -\delta$, then the solution $\tilde{u}$ to the problem (2.7) is in X and satisfies*

$$\|\tilde{u}\|_X \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|f\|_{L^2(\Omega)}. \tag{6.3}$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. Let $\Omega_\delta = \Omega \cap \{(x,y) \in \mathbb{R}^d \ : \ x < -\delta\}$ and $\Omega_{\delta,0} = \Omega \cap \{(x,y) \in \mathbb{R}^d \ : \ -\delta < x < 0\}$. The common boundary of these two domains is denoted by $\Gamma_\delta = \{x = -\delta\} \times \Theta$.

By Lemma 6.1 the solution $\tilde{u}$ solves the problem (5.2). In addition, since Ω_δ is away from the boundary Γ_0 on which the non-standard boundary condition is imposed, $\tilde{u}$ is in $H^2(\Omega_\delta)$ and satisfies

$$\|\tilde{u}\|_{H^2(\Omega_\delta)} \le C\|f\|_{L^2(\Omega)}. \tag{6.4}$$

Also, we can find a series expression of the solution $\tilde{u}$ in $\Omega_{\delta,0}$, as in (4.12), satisfying two boundary conditions $\tilde{u}|_{\Omega_{\delta,0}} = \tilde{u}|_{\Omega_\delta}$ on Γ_δ and $T_{\mathrm{PML}}(\tilde{u}|_{\Omega_{\delta,0}}) = 0$ on Γ_0. Indeed, by following the computation similar to that used for (4.14) and invoking the formula (4.17) of the operator T_{PML}, it is easy to show that $\tilde{u}$ in $\Omega_{\delta,0}$ can be written as

$$\tilde{u}(x,y) = \tilde{u}_N Y_N(y) + \sum_{n \ne N} \frac{\mathrm{e}^{i\mu_n(x+\delta)} + \mathrm{e}^{2i\mu_n\sigma_0 L}\mathrm{e}^{-i\mu_n(x-\delta)}}{1 + \mathrm{e}^{2i\mu_n(\sigma_0 L+\delta)}}\tilde{u}_n Y_n(y)$$

for $\tilde{u} = \sum_{n=0}^\infty \tilde{u}_n Y_n \in \dot{H}^{3/2}(\Gamma_\delta)$. Let $E_n(x)$, $n \ne N$, be the coefficient of the variable x in each mode,

$$E_n(x) = \frac{\mathrm{e}^{i\mu_n\delta}(\mathrm{e}^{i\mu_n x} + \mathrm{e}^{2i\mu_n\sigma_0 L}\mathrm{e}^{-i\mu_n x})}{1 + \mathrm{e}^{2i\mu_n(\sigma_0 L+\delta)}}$$

whose numerator satisfies

$$|\mathrm{e}^{i\mu_n\delta}(\mathrm{e}^{i\mu_n x} \pm \mathrm{e}^{2i\mu_n\sigma_0 L}\mathrm{e}^{-i\mu_n x})|^2 \le \begin{cases} 4(1+\lambda_n^2)^{1/2-1/2} \le C(1+\lambda_n^2)^{-1/2}, & \text{for } n < N, \\ 2\mathrm{e}^{-2\tilde{\mu}_n\delta}(\mathrm{e}^{-2\tilde{\mu}_n x} + \mathrm{e}^{2\tilde{\mu}_n x}), & \text{for } n > N, \end{cases}$$

and its denominator complies with $|1 + \mathrm{e}^{2i\mu_n(\sigma_0 L+\delta)}| > 1/2$ since $\sigma_\mu L > 1$. By using similar computations that we have done for (4.15) we shall show that

$$\left\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\right\|^2_{L^2(-\delta,0)} \le C(1+\lambda_n^2)^{\ell-1/2} \quad \text{for } \ell = 0,1,2 \tag{6.5}$$

and for all $n \ne N$.

For $n < N$, by using (3.7) it can be easily shown that

$$\left\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\right\|^2_{L^2(-\delta,0)} \le C\mu_n^{2\ell}\int_{-\delta}^0 (1+\lambda_n^2)^{-1/2}\mathrm{d}x \le C(1+\lambda_n^2)^{\ell-1/2} \qquad \text{for } \ell = 0,1,2. \tag{6.6}$$

For $n > N$, we begin with the simple computation

$$\int_{-\delta}^{0} \mathrm{e}^{-2\tilde{\mu}_n\delta}(\mathrm{e}^{-2\tilde{\mu}_n x} + \mathrm{e}^{2\tilde{\mu}_n x})\mathrm{d}x = \frac{1}{\tilde{\mu}_n}(1 - \mathrm{e}^{-2\tilde{\mu}_n\delta}). \tag{6.7}$$

For $n = N+1$ we apply $1 - \mathrm{e}^{-2\tilde{\mu}_n\delta} \leq 2\tilde{\mu}_n\delta$ to (6.7) and use (3.7) to have

$$\begin{aligned}\left\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\right\|^2_{L^2(-\delta,0)} &\leq C\tilde{\mu}_n^{2\ell}\int_{-\delta}^{0} \mathrm{e}^{-2\tilde{\mu}_n\delta}(\mathrm{e}^{-2\tilde{\mu}_n x} + \mathrm{e}^{2\tilde{\mu}_n x})\mathrm{d}x \\ &\leq C\delta\tilde{\mu}_n^{2\ell}(1+\lambda_n^2)^{1/2}(1+\lambda_n^2)^{-1/2} \leq C(1+\lambda_n^2)^{\ell-1/2}.\end{aligned}$$

Also, since $1 - \mathrm{e}^{-2\tilde{\mu}_n x} \leq 1$, for $n > N+1$ we utilize (3.8) for $\ell = 0$ and (3.7) for $\ell = 1, 2$ to obtain

$$\begin{aligned}\left\|\frac{\mathrm{d}^\ell E_n}{\mathrm{d}x^\ell}\right\|^2_{L^2(-\delta,0)} &\leq C\tilde{\mu}_n^{2\ell}\int_{-\delta}^{0} \mathrm{e}^{-2\tilde{\mu}_n\delta}(\mathrm{e}^{-2\tilde{\mu}_n x} + \mathrm{e}^{2\tilde{\mu}_n x})\mathrm{d}x \\ &\leq C\mu_n^{2\ell-1} \leq C(1+\lambda_n^2)^{\ell-1/2}.\end{aligned}$$

Therefore it follows from (6.5) that

$$\begin{aligned}\|\tilde{u}\|^2_{H^2(\Omega_{\delta,0})} \leq C\Bigg(&(1+\lambda_N^2)^2\delta|\tilde{u}_N|^2 \\ &+ \sum_{n\neq N}\left[(1+\lambda_n^2)^2\|E_n\|^2_{L^2(-\delta,0)} + (1+\lambda_n^2)\left\|\frac{\mathrm{d}E_n}{\mathrm{d}x}\right\|^2_{L^2(-\delta,0)} + \left\|\frac{\mathrm{d}^2E_n}{\mathrm{d}x^2}\right\|^2_{L^2(-\delta,0)}\right]|\tilde{u}_n|^2\Bigg) \\ \leq C\|\tilde{u}\|^2_{\dot{H}^{3/2}(\Gamma_\delta)}&.\end{aligned} \tag{6.8}$$

Using it together with a trace inequality given by [25], Theorem 2.13 and (6.4) yields that

$$\|\tilde{u}\|_{H^2(\Omega_{\delta,0})} \leq C\|\tilde{u}\|_{\dot{H}^{3/2}(\Gamma_\delta)} \leq C\|\tilde{u}\|_{H^2(\Omega_\delta)} \leq C\|f\|_{L^2(\Omega)}. \tag{6.9}$$

Noting that $\tilde{u}$ is in H^2 in the vicinity of Γ_δ, combining (6.4) and (6.9) shows that

$$\|\tilde{u}\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Furthermore, the H^2_σ-estimate in Ω_{PML} is obtained as follows

$$\begin{aligned}\|\tilde{u}\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} &\leq C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|\tilde{u}\|_{\dot{H}^{3/2}(\Gamma_0)} \\ &\leq C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|\tilde{u}\|_{H^2(\Omega)} \leq C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|f\|_{L^2(\Omega)}.\end{aligned}$$

by using Lemma 4.3 and a trace inequality, which completes the proof. □

7. Finite element analysis

We investigate the solvability of the finite element problem pertaining to the problem (2.7) and the convergence of finite element approximations. To describe finite element approximations, let $\mathcal{T}$ be a finite element mesh of the domain Ω_L consisting of quadrilaterals. Here we assume that the finite element mesh of the domain is aligned with the interface Γ_0. In case when a large PML strength σ_μ is employed for a fixed computational domain Ω_L to obtain an approximate solution with exponentially small error in the continuous level as shown in Theorem 5.2, the problem becomes anisotropic in the PML layer and so special finite element meshes, having different mesh sizes in different directions, are required to capture the anisotropic behavior in $\Omega_{\rm PML}$ in discrete levels. To do this, we keep quasi-uniform and shape-regular meshes in Ω, whose maximal diameter is denoted by h, however we take rectangular meshes in $\Omega_{\rm PML}$, whose mesh size along the axis of the waveguide (the direction of the exponential decay of solutions) is reduced by a scale factor $O(1/|\sigma_0|)$ compared with the mesh size h along the perpendicular direction. We also denote a continuous piecewise bilinear finite element space by V_h. Then the finite element approximation $\tilde u_h \in V_h$ is a solution to the finite dimensional problem

$$A(\tilde u_h, \phi_h) = (f, \phi_h)_\Omega \quad \text{for all } \phi_h \in V_h. \tag{7.1}$$

Let $u \in H^1(\Omega)$ and $\tilde u \in H^1_\sigma(\Omega_L)$ be the solutions to the problems (2.3) and (2.7), respectively. The error $e = u - \tilde u_h$ of the PML-FEM approximation $\tilde u_h$ consists of the PML approximate error, $u - \tilde u$, controlled by $\mathrm{e}^{-2\sigma_\mu L}$ and finite element error, $\tilde u - \tilde u_h$, depending on h. Let us denote the finite element error by $\tilde{\mathrm{e}} = \tilde u - \tilde u_h$ and we shall estimate $\tilde{\mathrm{e}}$ in two regions Ω and $\Omega_{\rm PML}$, separately.

We first notice that the nodal interpolation I_h satisfies the following error estimates.

Lemma 7.1. *For every rectangular element τ of $\mathcal{T}$ contained in $\Omega_{\rm PML}$, the interpolation error satisfies*

$$\|v - I_h(v)\|^2_{L^2(\tau)} \le Ch^2 \left(\frac{1}{|\sigma_0|^2} \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\tau)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\tau)} \right)$$

for $v \in H^1_\sigma(\Omega_{\rm PML})$, and

$$\left\|\frac{\partial}{\partial x}(v - I_h(v))\right\|^2_{L^2(\tau)} \le Ch^2 \left(\frac{1}{|\sigma_0|^2} \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2_{L^2(\tau)} + \left\|\frac{\partial^2 v}{\partial xy}\right\|^2_{L^2(\tau)} + |\sigma_0|^2 \left\|\frac{\partial^2 v}{\partial y^2}\right\|^2_{L^2(\tau)} \right),$$
$$\left\|\frac{\partial}{\partial y}(v - I_h(v))\right\|^2_{L^2(\tau)} \le Ch^2 \left(\frac{1}{|\sigma_0|^4} \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2_{L^2(\tau)} + \frac{1}{|\sigma_0|^2}\left\|\frac{\partial^2 v}{\partial xy}\right\|^2_{L^2(\tau)} + \left\|\frac{\partial^2 v}{\partial y^2}\right\|^2_{L^2(\tau)} \right)$$

for $v \in H^2_\sigma(\Omega_{\rm PML})$.

Proof. It suffices to verify the error estimates for an affine transformation $F : \hat\tau \to \tau$ defined by

$$F(\hat x, \hat y) = \left(\frac{h}{|\sigma_0|}\hat x + p, \quad h\hat y + q \right),$$

for constants p and q, where $\hat\tau = (0,1)\times(0,1)$ is a reference element. As proofs for three inequalities are all similar, we provide the proof for the second estimate. Let $\hat v = v \circ F$ and $\hat I_h$ be the nodal interpolation operator on $\hat\tau$. Since $\mathrm{d}x/\mathrm{d}\hat x = h/|\sigma_0|$, $\mathrm{d}y/\mathrm{d}\hat y = h$ and $|\det(D(F))| = h^2/|\sigma_0|$, by transforming the integral on τ to one on $\hat\tau$, using the Bramble-Hilbert lemma and transforming back to the element τ, we obtain that

$$\begin{aligned}
\|\frac{\partial}{\partial x}(v - I_h(v))\|^2_{L^2(\tau)} &= \int_{\hat\tau} \frac{|\sigma_0|^2}{h^2} |\frac{\partial}{\partial \hat x}(\hat v - \hat I_h(\hat v))|^2 \frac{h^2}{|\sigma_0|}\mathrm{d}\hat x \mathrm{d}\hat y \\
&\le C|\sigma_0| \int_{\hat\tau} \left(\left|\frac{\partial^2 \hat v}{\partial \hat x^2}\right|^2 + \left|\frac{\partial^2 \hat v}{\partial \hat x \hat y}\right|^2 + \left|\frac{\partial^2 \hat v}{\partial \hat y^2}\right|^2 \right) \mathrm{d}\hat x \mathrm{d}\hat y \\
&\le C|\sigma_0| \int_{\tau} \left(\frac{h^4}{|\sigma_0|^4} \left|\frac{\partial^2 v}{\partial x^2}\right|^2 + \frac{h^4}{|\sigma_0|^2} \left|\frac{\partial^2 v}{\partial xy}\right|^2 + h^4 \left|\frac{\partial^2 v}{\partial y^2}\right|^2 \right) \frac{|\sigma_0|}{h^2}\mathrm{d}x \mathrm{d}y \\
&\le Ch^2 \left(\frac{1}{|\sigma_0|^2} \left\|\frac{\partial^2 v}{\partial x^2}\right\|^2_{L^2(\tau)} + \left\|\frac{\partial^2 v}{\partial xy}\right\|^2_{L^2(\tau)} + |\sigma_0|^2 \left\|\frac{\partial^2 v}{\partial y^2}\right\|^2_{L^2(\tau)} \right),
\end{aligned}$$

which completes the proof of the second error estimate. ∎

Lemma 7.2. *The interpolation error satisfies*

$$\|v - I_h(v)\|_{H^1_\sigma(\Omega_{\mathrm{PML}})} \le Ch\|v\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} \tag{7.2}$$

for $v \in H^2_\sigma(\Omega_{\mathrm{PML}})$ *and*

$$\|v - I_h(v)\|_{H^1_\sigma(\Omega_L)} \le Ch\|v\|_X \tag{7.3}$$

for $v \in X$.

Proof. We begin by noting that

$$\begin{aligned}\|v - I_h(v)\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})} &= \frac{1}{|\sigma_0|}\left\|\frac{\partial}{\partial x}(v - I_h(v))\right\|^2_{L^2(\Omega_{\mathrm{PML}})} \\ &\quad + |\sigma_0|\left\|\frac{\partial}{\partial y}(v - I_h(v))\right\|^2_{L^2(\Omega_{\mathrm{PML}})} + |\sigma_0|\|v - I_h(v)\|^2_{L^2(\Omega_{\mathrm{PML}})}.\end{aligned} \tag{7.4}$$

Using Lemma 7.1 for each elements contained in Ω_{PML} and summing up all terms with an appropriate weight $|\sigma_0|$ or $1/|\sigma_0|$ give (7.2) for $v \in H^2_\sigma(\Omega_{\mathrm{PML}})$.

Also, since $\|v - I_h(v)\|^2_{H^1_\sigma(\Omega_L)} = \|v - I_h(v)\|^2_{H^1(\Omega)} + \|v - I_h(v)\|^2_{H^1_\sigma(\Omega_{\mathrm{PML}})}$ for $v \in X$, the interpolation error (7.3) follows from (7.2) and the standard interpolation error estimate for quasi-uniform meshes. $\square$

The solvability and quasi-optimal convergence of finite element solutions associated with indefinite problems such as the Helmholtz equation can be found in [31] based on the Aubin-Nitsche duality argument and regularity of solutions. Since solutions of the PML problem on Ω_L lack the full regularity, we need to modify this approach by focusing on each subdomain where solutions have the full regularity. We first investigate the L^2-bound of the error $\tilde{\mathrm{e}} = \tilde{u} - \tilde{u}_h$ on Ω and Ω_{PML} with respect to the H^1_σ-error.

Lemma 7.3. *Let M be the constant given in Lemma 5.1 and assume that $\sigma_\mu L > M$. Also assume that the problem (7.1) has a solution $\tilde{u}_h$. Then it holds that*

$$\|\tilde{\mathrm{e}}\|_{L^2(\Omega)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}(h + \mathrm{e}^{-2\sigma_\mu L})\|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)}.$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. Let $w \in H^1(\Omega)$ be the solution to the problem

$$A_\Omega(\phi, w) - \langle T(\phi), w\rangle_{1/2,\Gamma_0} = (\phi, \tilde{\mathrm{e}})_\Omega \quad \text{for all } \phi \in H^1(\Omega) \tag{7.5}$$

and $\tilde{w} \in H^1(\Omega_{\mathrm{PML}})$ be the solution to the problem

$$A_{\mathrm{PML}}(\phi, \tilde{w}) = 0 \quad \text{for all } \phi \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}}) \tag{7.6}$$

with $\tilde{w} = w$ on Γ_0. The existence of a unique solution to the problem (7.5) is an obvious result since $\bar{w}$ is the radiating solution generated by the source function $\bar{\tilde{\mathrm{e}}}$ noting that

$$A_\Omega(u, \phi) - \langle T(u), \phi\rangle_{1/2,\Gamma_0} = A_\Omega(\bar{\phi}, \bar{u}) - \langle T(\bar{\phi}), \bar{u}\rangle_{1/2,\Gamma_0}.$$

One for (7.6) is addressed in Remark 4.4.

The following two facts are of importance for the L^2-error analysis. The first ingredient is the full regularity of w and $\tilde{w}$,

$$\begin{aligned}\|w\|_{H^2(\Omega)} &\le C\|\tilde{\mathrm{e}}\|_{L^2(\Omega)}, \\ \|\tilde{w}\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} &\le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|w\|_{\dot{H}^{3/2}(\Gamma_0)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|w\|_{H^2(\Omega)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}\|\tilde{\mathrm{e}}\|_{L^2(\Omega)}\end{aligned}$$

due to the the regularity of the radiating solution and Remark 4.4. It allows us to have that for $w_h \in V_h$ which is the linear interpolation of w in Ω and $\tilde{w}$ in $\Omega_{\rm PML}$, that is $w_h|_\Omega = I_h(w)$ and $w_h|_{\Omega_{\rm PML}} = I_h(\tilde{w})$,

$$\|w - w_h\|_{H^1(\Omega)} \le Ch\|\tilde{\mathrm{e}}\|_{L^2(\Omega)}, \tag{7.7}$$

$$\|\tilde{w} - w_h\|_{H^1_\sigma(\Omega_{\rm PML})} \le Ch\|\tilde{w}\|_{H^2_\sigma(\Omega_{\rm PML})} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2} h\|\tilde{\mathrm{e}}\|_{L^2(\Omega)}. \tag{7.8}$$

The second ingredient is that $\tilde{w}$ satisfies

$$A_{\rm PML}(\psi, \tilde{w}) + \langle T_{\rm PML}(\psi), \tilde{w}\rangle_{1/2,\Gamma_0} = 0 \quad \text{for all } \psi \in H^1_\sigma(\Omega_{\rm PML}). \tag{7.9}$$

Indeed, since the solution $\tilde{w}$ to the problem (7.6) satisfies

$$A(\bar{\tilde{w}}, \bar{\phi}) = 0 \quad \text{for all } \phi \in \widetilde{H}^1_\sigma(\Omega_{\rm PML}),$$

we have

$$0 = A_{\rm PML}(\bar{\tilde{w}}, \bar{\psi}) + \langle \frac{1}{\sigma}\frac{\partial \bar{\tilde{w}}}{\partial x}, \bar{\psi}\rangle_{1/2,\Gamma_0} = A_{\rm PML}(\bar{\tilde{w}}, \bar{\psi}) + \langle T_{\rm PML}(\bar{\tilde{w}}), \bar{\psi}\rangle_{1/2,\Gamma_0}$$

for $\psi \in H^1_\sigma(\Omega_{\rm PML})$, which is equivalent to (7.9) by interchanging two arguments with complex conjugate.

For the L^2-error estimate, we begin by taking $\phi = \tilde{\mathrm{e}}$ in (7.5) to obtain

$$\begin{aligned}\|\tilde{\mathrm{e}}\|^2_{L^2(\Omega)} &= A_\Omega(\tilde{\mathrm{e}}, w) - \langle T(\tilde{\mathrm{e}}), w\rangle_{1/2,\Gamma_0}\\ &= A_\Omega(\tilde{\mathrm{e}}, w) - \langle T_{\rm PML}(\tilde{\mathrm{e}}), w\rangle_{1/2,\Gamma_0} + \langle (T_{\rm PML} - T)(\tilde{\mathrm{e}}), w\rangle_{1/2,\Gamma_0}.\end{aligned}$$

Here by adding and subtracting the term

$$I_2 := A_\Omega(\tilde{\mathrm{e}}, w_h) - \langle T_{\rm PML}(\tilde{\mathrm{e}}), w_h\rangle_{1/2,\Gamma_0},$$

we are led to $\|\tilde{\mathrm{e}}\|^2_{L^2(\Omega)} = I_1 + I_2$, where

$$I_1 := A_\Omega(\tilde{\mathrm{e}}, w - w_h) - \langle T_{\rm PML}(\tilde{\mathrm{e}}), (w - w_h)\rangle_{1/2,\Gamma_0} + \langle (T_{\rm PML} - T)(\tilde{\mathrm{e}}), w\rangle_{1/2,\Gamma_0}.$$

Now, we shall estimate I_1 and I_2. For I_1, we use (7.7) and Lemma 4.5 to obtain

$$|I_1| \le C(h + \mathrm{e}^{-2\sigma_\mu L})\|\tilde{\mathrm{e}}\|_{H^1(\Omega)}\|\tilde{\mathrm{e}}\|_{L^2(\Omega)}. \tag{7.10}$$

For I_2, the Galerkin orthogonality,

$$A(\tilde{\mathrm{e}}, w_h) = A_\Omega(\tilde{\mathrm{e}}, w_h) + A_{\rm PML}(\tilde{\mathrm{e}}, w_h) = 0,$$

leads to $A_\Omega(\tilde{\mathrm{e}}, w_h) = -A_{\rm PML}(\tilde{\mathrm{e}}, w_h)$, from which and (7.9) it then follows that

$$\begin{aligned}I_2 &= -A_{\rm PML}(\tilde{\mathrm{e}}, w_h) - \langle T_{\rm PML}(\tilde{\mathrm{e}}), w_h\rangle_{1,2,\Gamma_0}\\ &= A_{\rm PML}(\tilde{\mathrm{e}}, \tilde{w} - w_h) + \langle T_{\rm PML}(\tilde{\mathrm{e}}), (\tilde{w} - w_h)\rangle_{1/2,\Gamma_0}.\end{aligned}$$

Consequently, by Lemma A.3 for a trace inequality of functions in $H^1_\sigma(\Omega_{\rm PML})$ (in Appendix A) and (7.8), it can be concluded that

$$|I_2| \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2} h\|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_{\rm PML})}\|\tilde{\mathrm{e}}\|_{L^2(\Omega)}. \tag{7.11}$$

Combining (7.10) and (7.11) results in

$$\|\tilde{\mathrm{e}}\|_{L^2(\Omega)} \le C\left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2}(h + \mathrm{e}^{-2\sigma_\mu L})\|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)},$$

which completes the proof.

□

Lemma 7.4. *Let M be the constant given in Lemma 5.1 and assume that $\sigma_\mu L > M$. Also assume that the problem (7.1) has a solution $\tilde{u}_h$. Then it holds that*

$$\|\tilde{\mathrm{e}}\|_{L^2_\sigma(\Omega_{\mathrm{PML}})} \le C\frac{(\sigma_\mu L)^{5/2}}{\mu_{\min}^{7/2}}(h + \mathrm{e}^{-2\sigma_\mu L})\|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)}. \tag{7.12}$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. Let $\tilde{\mathrm{e}}_r \in H^1_\sigma(\Omega_L)$ be the piecewise function such that $\tilde{\mathrm{e}}_r|_\Omega = \tilde{\mathrm{e}}$ and $\tilde{\mathrm{e}}_r|_{\Omega_{\mathrm{PML}}}$ is the solution to the problem

$$A_{\mathrm{PML}}(\tilde{\mathrm{e}}_r, \phi) = 0 \quad \text{for all } \phi \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}}) \tag{7.13}$$

with $\tilde{\mathrm{e}}_r = \tilde{\mathrm{e}}$ on Γ_0. Denoting the zero extension of $(\tilde{\mathrm{e}} - \tilde{\mathrm{e}}_r)|_{\Omega_{\mathrm{PML}}}$ to Ω by $\tilde{\mathrm{e}}_0$, $\tilde{\mathrm{e}}$ can be decomposed into $\tilde{\mathrm{e}} = \tilde{\mathrm{e}}_r + \tilde{\mathrm{e}}_0$. Here we claim that $\tilde{\mathrm{e}}_0$ has a certain Galerkin orthogonality in $\widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$, that is,

$$A_{\mathrm{PML}}(\tilde{\mathrm{e}}_0, \phi_h) = 0 \tag{7.14}$$

for all $\phi_h \in V_h$ satisfying $\phi_h = 0$ in Ω. Indeed, we begin with the Galerkin orthogonality of $\tilde{\mathrm{e}}$ in $H^1_\sigma(\Omega_L)$,

$$A(\tilde{\mathrm{e}}, \phi_h) = A_\Omega(\tilde{\mathrm{e}}, \phi_h) + A_{\mathrm{PML}}(\tilde{\mathrm{e}}, \phi_h) = 0 \quad \text{for all } \phi_h \in V_h.$$

For any $\phi_h \in V_h$ vanishing in Ω, the definition of $\tilde{\mathrm{e}}_r$ results in

$$0 = A(\tilde{\mathrm{e}}, \phi_h) = A_{\mathrm{PML}}(\tilde{\mathrm{e}}_r + \tilde{\mathrm{e}}_0, \phi_h) = A_{\mathrm{PML}}(\tilde{\mathrm{e}}_0, \phi_h).$$

Now, we shall estimate $\tilde{\mathrm{e}}_0$. Let $w_0 \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$ be the solution to the problem

$$A_{\mathrm{PML}}(\phi, w_0) = (\sigma_0\phi, \tilde{\mathrm{e}}_0)_{\Omega_{\mathrm{PML}}} \quad \text{for all } \phi \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}}). \tag{7.15}$$

By Lemma 4.2 and Remark 4.4, we note that

$$\|w_0\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} \le C\frac{(\sigma_\mu L)^2}{\mu_{\min}^3}\|\tilde{\mathrm{e}}_0\|_{L^2_\sigma(\Omega_{\mathrm{PML}})}. \tag{7.16}$$

Since $\|w_0 - I_h(w_0)\|_{H^1_\sigma(\Omega_{\mathrm{PML}})} \le Ch\|w_0\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} \le C(\sigma_\mu L)^2\mu_{\min}^{-3}h\|\tilde{\mathrm{e}}_0\|_{L^2_\sigma(\Omega_{\mathrm{PML}})}$, setting $\phi = \tilde{\mathrm{e}}_0$ in (7.15), we have

$$\begin{aligned}\|\tilde{\mathrm{e}}_0\|^2_{L^2_\sigma(\Omega_{\mathrm{PML}})} &= |A_{\mathrm{PML}}(\tilde{\mathrm{e}}_0, w_0)| = |A_{\mathrm{PML}}(\tilde{\mathrm{e}}_0, w_0 - I_h(w_0))| \\ &\le C\frac{(\sigma_\mu L)^2}{\mu_{\min}^3}h\|\tilde{\mathrm{e}}_0\|_{H^1_\sigma(\Omega_{\mathrm{PML}})}\|\tilde{\mathrm{e}}_0\|_{L^2_\sigma(\Omega_{\mathrm{PML}})}\end{aligned}$$

and hence

$$\|\tilde{\mathrm{e}}_0\|_{L^2_\sigma(\Omega_{\mathrm{PML}})} \le C\frac{(\sigma_\mu L)^2}{\mu_{\min}^3}h\|\tilde{\mathrm{e}}_0\|_{H^1_\sigma(\Omega_{\mathrm{PML}})}. \tag{7.17}$$

On the other hand, for the estimate of $\tilde{\mathrm{e}}_r$, let $w_r \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$ be the solution to the problem

$$A_{\mathrm{PML}}(\phi, w_r) = (\sigma_0\phi, \tilde{\mathrm{e}}_r)_{\Omega_{\mathrm{PML}}} \quad \text{for all } \phi \in \widetilde{H}^1_\sigma(\Omega_{\mathrm{PML}})$$

satisfying

$$\|w_r\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} \le C\frac{(\sigma_\mu L)^2}{\mu_{\min}^3}\|\tilde{\mathrm{e}}_r\|_{L^2_\sigma(\Omega_{\mathrm{PML}})} \tag{7.18}$$

by Lemma 4.2. Here we notice that w_r solves the problem

$$-\nabla \cdot \bar{H} \nabla w_r - k^2 \bar{\sigma}_0 w_r = \bar{\sigma}_0 \tilde{\mathrm{e}}_r \quad \text{in } \Omega_{\mathrm{PML}}, \tag{7.19}$$

where $H = \mathrm{diag}(\sigma_0^{-1}, \sigma_0)$. Multiplying (7.19) by $\tilde{\mathrm{e}}_r$ and integration by parts gives

$$\left\langle \tilde{\mathrm{e}}_r, \frac{1}{\bar{\sigma}_0} \frac{\partial w_r}{\partial x} \right\rangle_{1/2,\Gamma_0} = (\sigma_0 \tilde{\mathrm{e}}_r, \tilde{\mathrm{e}}_r)_{\Omega_{\mathrm{PML}}} \tag{7.20}$$

due to (7.13).

Now we assert that there exists $\tilde{w}_r \in H^2(\Omega)$ satisfying boundary conditions

$$\tilde{w}_r = 0, \quad \frac{\partial \tilde{w}_r}{\partial x} = \frac{1}{\bar{\sigma}_0} \frac{\partial w_r}{\partial x} \quad \text{on } \Gamma_0, \quad \frac{\partial \tilde{w}_r}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \bar{\Gamma}_0 \tag{7.21}$$

and

$$\|\tilde{w}_r\|_{H^2(\Omega)} \leq C \|w_r\|_{H^2_\sigma(\Omega_{\mathrm{PML}})}. \tag{7.22}$$

Once we have it and denote $g := -\Delta \tilde{w}_r - k^2 \tilde{w}_r$ in $L^2(\Omega)$, we obtain

$$A_\Omega(\tilde{\mathrm{e}}, \tilde{w}_r) - \left\langle \tilde{\mathrm{e}}, \frac{\partial \tilde{w}_r}{\partial x} \right\rangle_{1/2,\Gamma_0} = (\tilde{\mathrm{e}}, g)_\Omega. \tag{7.23}$$

Since $\tilde{\mathrm{e}} = \tilde{\mathrm{e}}_r$ and $\partial \tilde{w}_r / \partial x = \bar{\sigma}_0^{-1} \partial w_r / \partial x$ on Γ_0, from (7.20) and (7.23) it can be shown that

$$\|\tilde{\mathrm{e}}_r\|^2_{L^2_\sigma(\Omega_{\mathrm{PML}})} \leq |A_\Omega(\tilde{\mathrm{e}}, \tilde{w}_r)| + |(\tilde{\mathrm{e}}, g)_\Omega|.$$

By the Galerkin orthogonality of $A_\Omega(\tilde{\mathrm{e}}, I_h(\tilde{w}_r)) = 0$, (7.22), (7.18) and Lemma 7.3, we can further show that

$$\begin{aligned}
\|\tilde{\mathrm{e}}_r\|^2_{L^2_\sigma(\Omega_{\mathrm{PML}})} &\leq |A_\Omega(\tilde{\mathrm{e}}, \tilde{w}_r - I_h(\tilde{w}_r))| + |(\tilde{\mathrm{e}}, g)_\Omega| \\
&\leq C\big(h\|\tilde{\mathrm{e}}\|_{H^1(\Omega)} + \|\tilde{\mathrm{e}}\|_{L^2(\Omega)}\big)\|\tilde{w}_r\|_{H^2(\Omega)} \\
&\leq C \frac{(\sigma_\mu L)^{5/2}}{\mu_{\min}^{7/2}} (h + \mathrm{e}^{-2\sigma_\mu L}) \|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)} \|\tilde{\mathrm{e}}_r\|_{L^2_\sigma(\Omega_{\mathrm{PML}})}.
\end{aligned}$$

Then (7.12) follows from (7.17) and the above inequality.

It remains to show the existence of $\tilde{w}_r$ satisfying (7.21) and (7.22). We define ζ by a function of the form

$$\zeta(x, y) = \sum_{n=0}^{\infty} A_n \mathrm{e}^{\alpha_n x} Y_n(y) + \sum_{n=0}^{\infty} B_n \mathrm{e}^{2\alpha_n x} Y_n(y)$$

with $\alpha_n = (1 + \lambda_n^2)^{1/2}$ for $-\delta < x < 0$ such that

$$A_n + B_n = 0, \qquad \alpha_n(A_n + 2B_n) = \gamma_n \qquad \text{for } n \geq 0,$$

where γ_n is the nth Fourier coefficient of $\bar{\sigma}_0^{-1} \partial w_r / \partial x$ on Γ_0. A direct computation similar to that used for (6.5) and (6.8) gives

$$\|\zeta\|_{H^2(\Omega_{\delta,0})} \leq C \left\| \frac{1}{\bar{\sigma}_0} \frac{\partial w_r}{\partial x} \right\|_{\dot{H}^{1/2}(\Gamma_0)} = C \left\| \frac{1}{\sigma_0} \frac{\partial w_r}{\partial x} \right\|_{\dot{H}^{1/2}(\Gamma_0)}$$

and hence Lemma A.3 for a trace inequality of functions in $H^2_\sigma(\Omega_{\mathrm{PML}})$ (in Appendix A) completes to show

$$\|\zeta\|_{H^2(\Omega_{\delta,0})} \leq C \|w_r\|_{H^2_\sigma(\Omega_{\mathrm{PML}})}.$$

Finally, by using a cutoff function χ, which is one for $-\delta/2 < x < 0$ and vanishes for $x < -\delta$, the zero extension of $\chi\zeta|_{\Omega_{\delta,0}}$ to Ω_δ can fulfill all conditions for $\tilde{w}_r$, which completes the proof. $\square$

Lemmas 7.3 and 7.4 show that there exists a constant $C_{L^2_\sigma} > 0$ such that

$$\|\tilde{\mathrm{e}}\|^2_{L^2_\sigma(\Omega_L)} \leq C_{L^2_\sigma} \frac{(\sigma_\mu L)^5}{\mu^7_{\min}} (h^2 + \mathrm{e}^{-4\sigma_\mu L}) \|\tilde{\mathrm{e}}\|^2_{H^1_\sigma(\Omega_L)} \tag{7.24}$$

for sufficiently large $\sigma_\mu L$.

We are now in a position to present the H^1_σ-error estimate, of which the proof follows standard Schatz's argument applied to Gårding's inequality

$$C_1 \|\tilde{\mathrm{e}}\|^2_{H^1_\sigma(\Omega_L)} - C_2 \|\tilde{\mathrm{e}}\|^2_{L^2_\sigma(\Omega_L)} \leq |A(\tilde{\mathrm{e}}, \tilde{\mathrm{e}})| = |A(\tilde{\mathrm{e}}, \tilde{u} - I_h(\tilde{u}))|. \tag{7.25}$$

for some positive constants C_1 and C_2.

Theorem 7.5. *Let u be the solution to the problem (2.3). Then, there exist positive constants $M_0 > M$ and $h_0 > 0$ such that for $\sigma_\mu L > M_0$ and $0 < h < h_0$, the problem (7.1) has a unique solution $\tilde{u}_h$ satisfying*

$$\|u - \tilde{u}_h\|_{H^1(\Omega)} \leq C \left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2} (h + \mathrm{e}^{-2\sigma_\mu L}) \|f\|_{L^2(\Omega)}.$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.

Proof. We apply (7.24) to Gårding's inequality (7.25) to show

$$\left(C_1 - C_2 C_{L^2_\sigma} \frac{(\sigma_\mu L)^5}{\mu^7_{\min}} (h^2 + \mathrm{e}^{-4\sigma_\mu L})\right) \|\tilde{\mathrm{e}}\|^2_{H^1_\sigma(\Omega_L)} \leq C \|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)} \|\tilde{u} - I_h(\tilde{u})\|_{H^1_\sigma(\Omega_L)}.$$

For such M_0 and h_0 satisfying $C_2 C_{L^2_\sigma} (\sigma_\mu L)^5 \mu^{-7}_{\min} \mathrm{e}^{-4M_0} < C_1/4$ and $C_2 C_{L^2_\sigma} (\sigma_\mu L)^5 \mu^{-7}_{\min} h_0^2 < C_1/4$, by Lemmas 7.2 and 6.3 we can show that for $M > M_0$ and $0 < h < h_0$ it holds that

$$\|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)} \leq Ch \|\tilde{u}\|_X \leq C \left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2} h \|f\|_{L^2(\Omega)}. \tag{7.26}$$

Finally, by using Theorem 5.2 we have

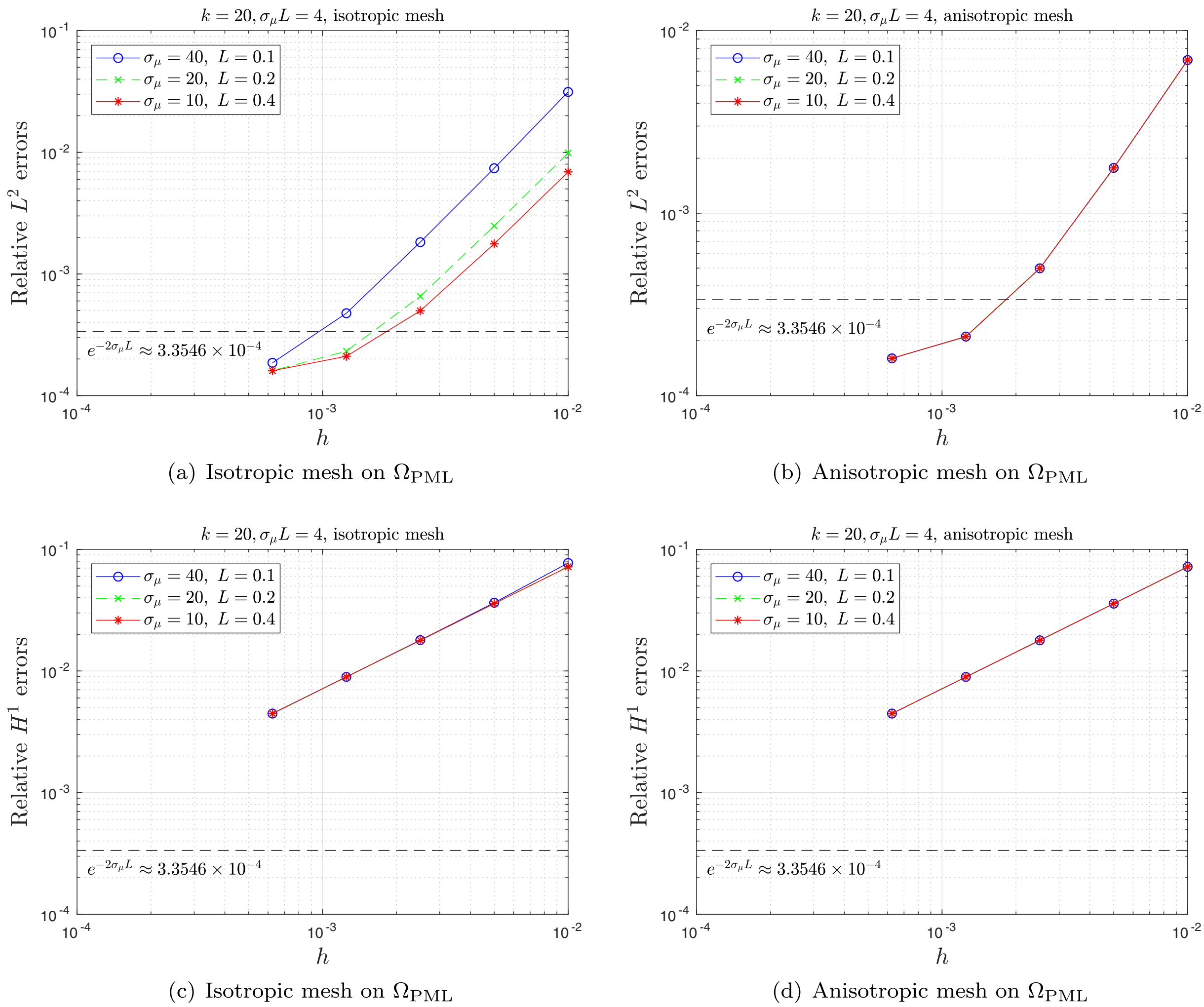
$$\begin{aligned} \|u - \tilde{u}_h\|_{H^1(\Omega)} &\leq \|u - \tilde{u}\|_{H^1(\Omega)} + \|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)} \\ &\leq C \left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2} (h + \mathrm{e}^{-2\sigma_\mu L}) \|f\|_{L^2(\Omega)}. \end{aligned}$$

This error estimate shows the uniqueness of solutions to the linear problem (4.1) in the finite dimensional space V_h by setting $f = 0$, and thus the proof is completed. ☐

Theorem 7.6. *Let u be the solution to the problem (2.3). Also let M_0 and h_0 be defined as in Theorem 7.5. Then, for $\sigma_\mu L > M_0$ and $0 < h < h_0$, it holds that*

$$\|u - \tilde{u}_h\|_{L^2(\Omega)} \leq C \frac{\sigma_\mu L}{\mu_{\min}} (h^2 + \mathrm{e}^{-2\sigma_\mu L}) \|f\|_{L^2(\Omega)}.$$

Here $\sigma_\mu L$ is not involved if there exists no cutoff mode.



(a) Isotropic mesh on Ω_{PML}

(b) Anisotropic mesh on Ω_{PML}

(c) Isotropic mesh on Ω_{PML}

(d) Anisotropic mesh on Ω_{PML}

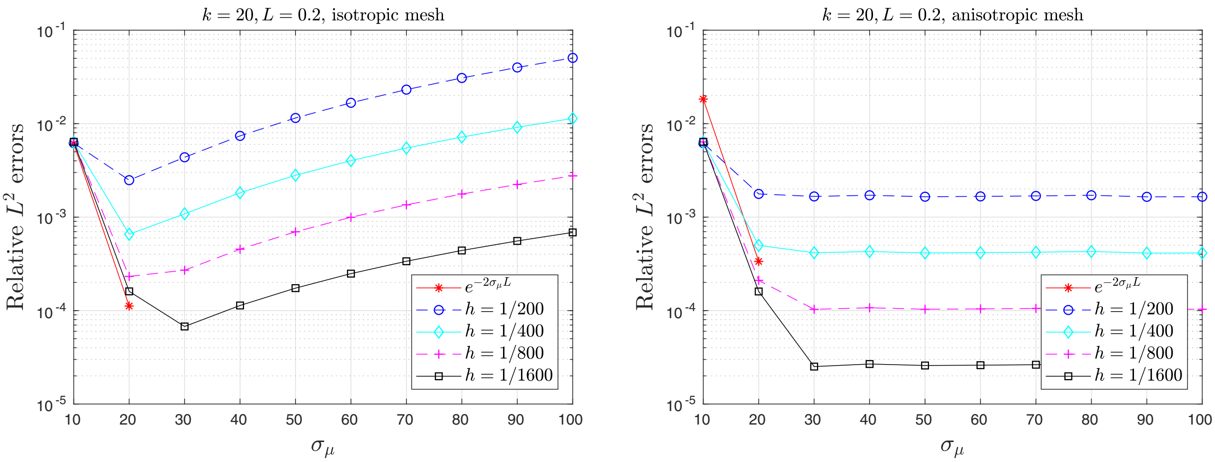
FIGURE 2. Relative L^2- and H^1-errors *vs.* h with $k = 20$ and $\sigma_\mu L = 4$.

Proof. From Lemma 7.3, (7.26) and a Cauchy–Schwarz inequality, we see that

$$\begin{aligned}\|\tilde{\mathrm{e}}\|_{L^2(\Omega)} &\leq C \left(\frac{\sigma_\mu L}{\mu_{\min}}\right)^{1/2} (h + \mathrm{e}^{-2\sigma_\mu L}) \|\tilde{\mathrm{e}}\|_{H^1_\sigma(\Omega_L)} \leq C \frac{\sigma_\mu L}{\mu_{\min}} (h + \mathrm{e}^{-2\sigma_\mu L}) h \|f\|_{L^2(\Omega)} \\ &\leq C \frac{\sigma_\mu L}{\mu_{\min}} (h^2 + \mathrm{e}^{-4\sigma_\mu L}) \|f\|_{L^2(\Omega)}.\end{aligned}$$

Now, Theorem 5.2 and the above inequality result in

$$\begin{aligned}\|e\|_{L^2(\Omega)} &\leq \|u - \tilde{u}\|_{L^2(\Omega)} + \|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega)} \\ &\leq C \mathrm{e}^{-2\sigma_\mu L} \|f\|_{L^2(\Omega)} + C \frac{\sigma_\mu L}{\mu_{\min}} (h^2 + \mathrm{e}^{-4\sigma_\mu L}) \|f\|_{L^2(\Omega_L)} \\ &\leq C \frac{\sigma_\mu L}{\mu_{\min}} (h^2 + \mathrm{e}^{-2\sigma_\mu L}) \|f\|_{L^2(\Omega_L)},\end{aligned}$$



(a) Isotropic mesh on Ω_{PML} (b) Anisotropic mesh on Ω_{PML}

FIGURE 3. Relative L^2-errors *vs.* PML strength σ_μ with $k = 20$, $L = 0.2$.

which establishes the desired L^2-error estimate.

□

8. NUMERICAL EXPERIMENTS

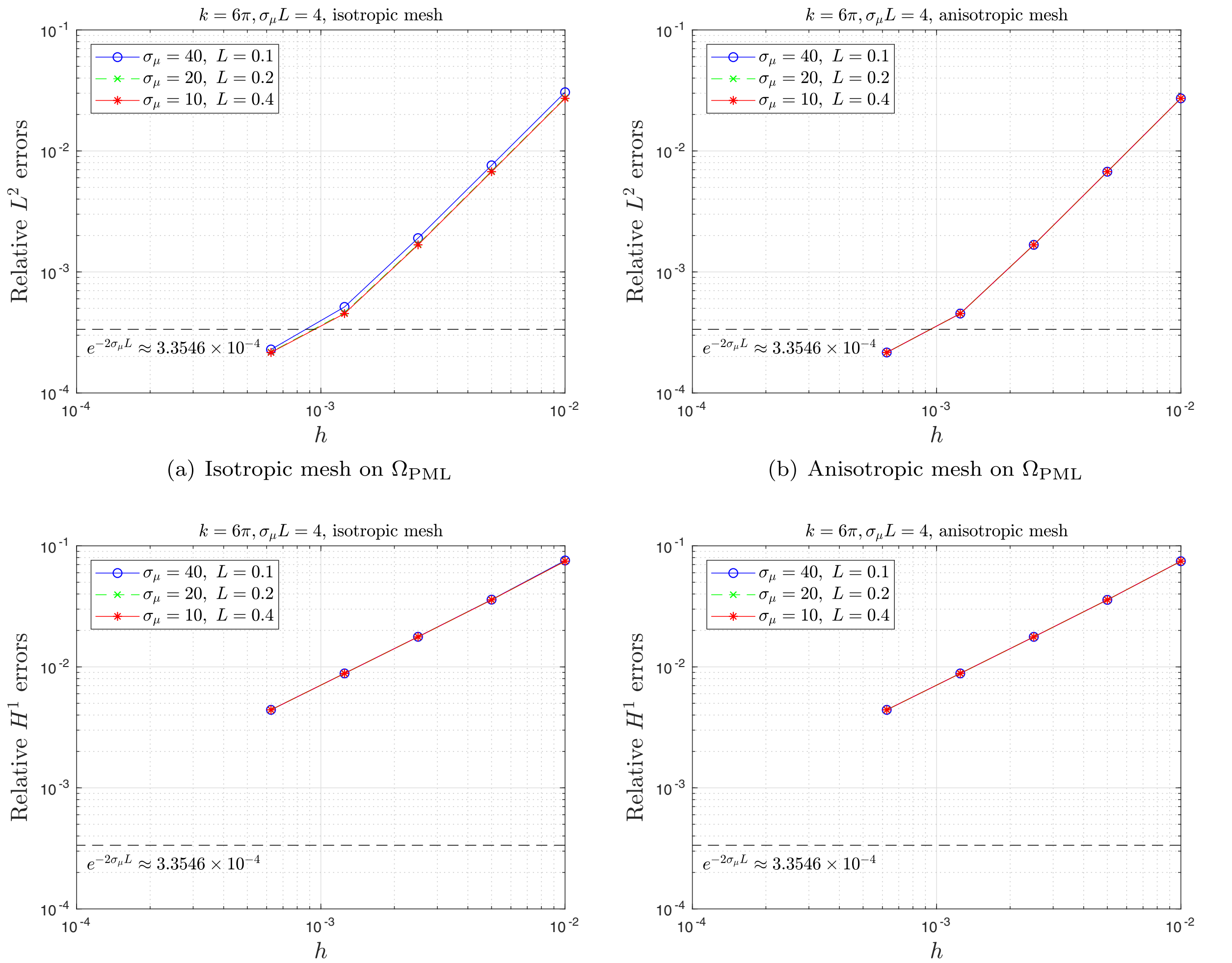
In this section, we provide numerical examples demonstrating the convergence theory developed in the preceding sections. For a simple example with a known exact solution, we consider the boundary value problem in $\Omega = (0, 0.2) \times \Theta \subset \mathbb{R}^2$ with an interval $\Theta = (0, 1) \subset \mathbb{R}$

$$\begin{aligned} \Delta u + k^2 u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } (0, 0.2) \times \partial\Theta, \qquad u = u^{ex} \quad \text{on } \{0\} \times \Theta \end{aligned}$$

with the radiation condition on $\{0.2\} \times \Theta$, where the exact solution is given by

$$u^{ex}(x, y) = \sum_{n=0}^{2N-1} \frac{\mathrm{e}^{i\mu_n x}}{N+1} Y_n(y).$$

The radiation condition on $\{0.2\} \times \Theta$ is replaced by PML with width L, that is, $\Omega_{\mathrm{PML}} = (0.2, 0.2+L) \times \Theta$. We compute bilinear finite element approximations on quadrilateral decompositions of Ω_L by using finite element library `deal.II` [2]. We take $k = 20$ and 6π in the example. For $k = 20$, there is neither cutoff mode nor near-cutoff mode in the solution as $6\pi < 20 < 7\pi$ and $\mu_{\min} = \sqrt{k^2 - 6\pi} \approx 1.0726$. $N = 6$ is the largest index for which $\mathrm{e}^{i\mu_N x} Y_N(y)$ is a propagating mode so that it has 7 propagating modes, $0 \le n \le N$, and 5 evanescent modes, $N + 1 \le n < 2N$. On the other hand, the case of $k = 6\pi$ represents an example with a cutoff mode, $N = 6$. To illustrate errors for a case including near-cutoff modes, the wavenumber will be $k = 6\pi + \varepsilon$ with $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ and 10^{-4}. Numerical tests are conducted to investigate



(a) Isotropic mesh on Ω_{PML} (b) Anisotropic mesh on Ω_{PML}

(c) Isotropic mesh on Ω_{PML} (d) Anisotropic mesh on Ω_{PML}

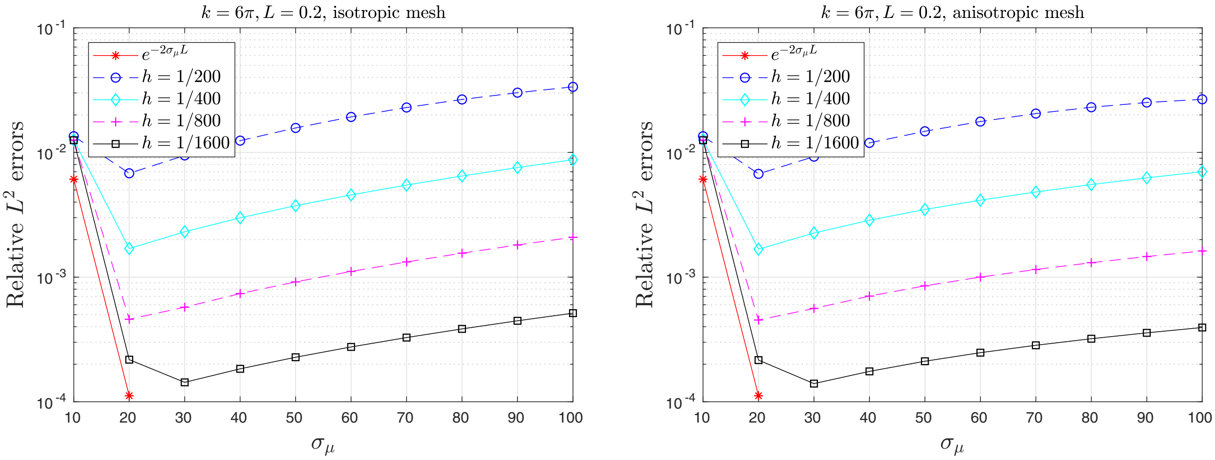
FIGURE 4. Relative L^2- and H^1-errors *vs.* h with $k = 6\pi$ and $\sigma_\mu L = 4$.

(i) Relative L^2- and H^1-errors *vs.* h with $\sigma_\mu L$ fixed,
(ii) Relative L^2-errors *vs.* σ_μ with L fixed,
(iii) Relative L^2-errors *vs.* L with σ_μ fixed.
(iv) Relative L^2-errors *vs.* $\mu_{\min}$ with σ_μ and L fixed when k approaches 6π.

Here we set the PML strength σ_μ from which σ_0 in the coordinate stretching function is determined by the formula (3.2) with

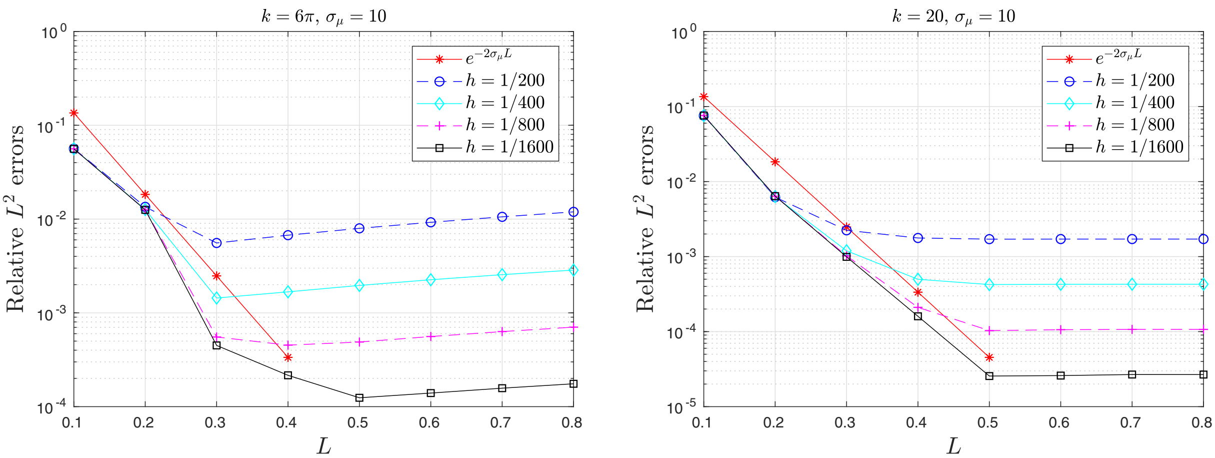
$$\begin{aligned} \mu_N &= \sqrt{k^2 - (6\pi)^2}, \qquad \tilde{\mu}_{N+1} = \sqrt{(7\pi)^2 - k^2} \qquad \text{for } k = 20, \\ \mu_{N-1} &= \sqrt{k^2 - (5\pi)^2}, \qquad \tilde{\mu}_{N+1} = \sqrt{(7\pi)^2 - k^2} \qquad \text{for } k = 6\pi. \end{aligned}$$

The first test for $k = 20$ is to see errors in PML-FEM approximate solutions as h decreases with $\sigma_\mu L$ fixed. To do this, we choose different pairs of PML parameters $(\sigma_\mu, L) = (40, 0.1), (20, 0.2)$, and $(10, 0.4)$ with $\sigma_\mu L = 4$ so that the reflection error is estimated to be $\mathrm{e}^{-2\sigma_\mu L} = \mathrm{e}^{-8} \approx 3.3546 \times 10^{-4}$. The relative L^2- and H^1-errors



(a) Isotropic mesh on $\Omega_{\rm PML}$ (b) Anisotropic mesh on $\Omega_{\rm PML}$

FIGURE 5. Relative L^2-errors *vs.* PML strength σ_μ with $k = 6\pi$, $L = 0.2$.



(a) Isotropic mesh, $k = 6\pi$ (b) Isotropic mesh, $k = 20$

FIGURE 6. Relative L^2-errors *vs.* PML width L with $\sigma_\mu = 10$.

reported in Figure 2 show that PML-FEM approximate solutions converge in a quasi-optimal rate as h decreases, $h = 1/100, 1/200, 1/400, 1/800$ and $h = 1/1600$ until reflection errors are dominant. When anisotropic meshes are used in $\Omega_{\rm PML}$, the errors are independent of the PML parameter σ_μ as in Figures 2b and 2d. In contrast, when isotropic meshes are used in $\Omega_{\rm PML}$, the errors (L^2-errors, in particular) are worse as the problem is more anisotropic with increasing σ_μ. See Figure 2a. Since it is desirable to have smaller degrees of freedom in numerical computations, one might want to take a small size PML, $\Omega_{\rm PML}$, with isotropic decomposition and choose large σ_μ to compensate small L instead. However, we observe that highly anisotropic nature of the problem with large σ_μ deteriorates accuracy of finite element approximations. By doing anisotropic mesh refinement in $\Omega_{\rm PML}$ (so that degrees of freedoms of problems for three pairs of parameters are all same), we can keep the same accuracy of approximate solutions.

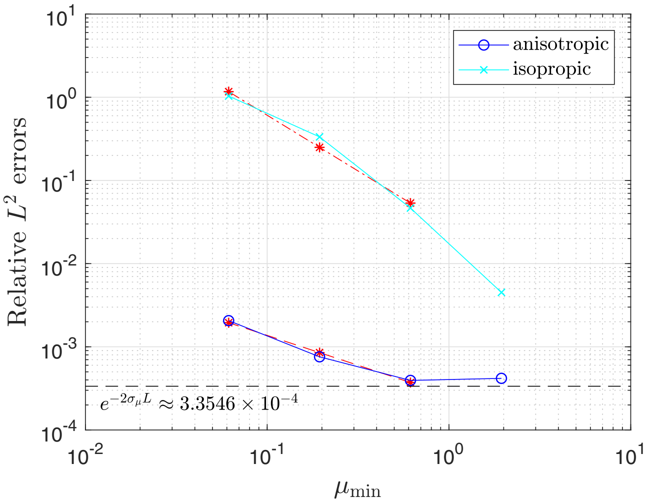


FIGURE 7. Relative L^2-errors *vs.* $\mu_{\min}$ with $L = 0.2$ and $\sigma_\mu = 20$.

TABLE 1. Magnitude of $\sigma_0 = \sigma_\mu(\frac{1}{\tilde{\mu}_{N+1}} + \frac{i}{\mu_{N-1}})$ for $\sigma_\mu = 20$.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\mu_{\min}$	1.9442	0.6141	0.1942	0.0614
$\lvert\sigma_0\rvert$	10.44	32.62	103.02	325.74

The second test for $k = 20$ is to see errors in PML-FEM approximate solutions as σ_μ increases with L fixed. Here we set $L = 0.2$ and increase σ_μ from 10 to 100. The results are depicted in Figure 3. As mentioned in the preceding paragraph, whereas errors in finite element approximations with isotropic meshes in $\Omega_{\rm PML}$ grow worse with increasing σ_μ due to the anisotropic nature, finite element approximations with anisotropic mesh refinement in $\Omega_{\rm PML}$ have the same level of errors after mesh errors are prevalent. Noting that *a priori* information on an optimal PML parameter σ_μ for given h and L is not available in general, it is an advantage of the anisotropic mesh refinement that finite element errors do not increase with increasing σ_μ opposed to ones obtained from isotropic meshes. We also obtain the similar results for $k = 6\pi$, for which a cutoff mode is involved, as see in Figures 4 and 5. One important difference of the results for $k = 6\pi$ from the case for $k = 20$ is that since the stability constant depends on $\sigma_\mu L$, errors in PML-FEM approximate solutions get larger with increasing σ_μ even though anisotropic mesh refinement is applied to the layer $\Omega_{\rm PML}$ as shown in Figure 5b.

The third test is to see errors in PML-FEM approximate solutions as L increases with σ_μ fixed. Here we set $\sigma_\mu = 10$ and take isotropic meshes on the whole computational domain Ω_L. The results for both $k = 6\pi$ and $k = 20$ are shown in Figure 6. They show that errors in PML-FEM approximate solutions decay in the exponential rate as expected until mesh errors are dominant. In addition, errors in approximate solutions for $k = 6\pi$ in Figure 6a are found to be worse as L increases due to the dependency of the stability constant on $\sigma_\mu L$.

The last numerical test is in regard to the error behavior associated with near-cutoff modes. The wavenumber k is chosen such that $k = 6\pi + \varepsilon$ with $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ and 10^{-4}. In this test, we set the finite element mesh $h = 1/400$ and the PML parameters are chosen to be $L = 0.2$ and $\sigma_\mu = 20$, by taking into account that errors in approximate solutions decrease at the same rate as reflection errors until $\sigma_\mu = 20$ (when $L = 0.2$) with ignorable mesh errors according to Figures 3 and 5. The reflection errors for fixed $L = 0.2$ and $\sigma_\mu = 20$ are all the same for all ε, $\mathrm{e}^{-2\sigma_\mu L} \approx 3.3546 \times 10^{-4}$. The results are reported in Figure 7, which shows that errors in approximate solutions grow approximately as $\mu_{\min}^{-0.71}$ (dashed red line) for anisotropic meshes and as $\mu_{\min}^{-1.31}$ (dash-dot red line) for isotropic meshes with $\mu_{\min}$ approaching zero. Here the constant σ_0 becomes large with small ε as seen in Table 1. It results in the fast growth of errors in approximate solutions for isotropic meshes that can not treat anisotropy of the problem properly. In contrast, applying anisotropic mesh refinements provides errors in approximate solutions that grow slowly as $\mu_{\min} \to 0^+$, in this particular example, slower than $\mu_{\min}^{-1}$ in the theory.

Appendix A.

Lemma A.1. *There exists a positive constant C such that for any $n \neq N$ and $m \in \mathbb{N}$*

$$\left(\frac{\zeta_m^4 + \zeta_m^2(1+\lambda_n^2)|\sigma_0|^2 + (1+\lambda_n^2+\lambda_n^4)|\sigma_0|^4}{|\zeta_m^2 + \sigma_0^2(\lambda_n^2 - k^2)|^2}\right)^{1/2} < \frac{C}{\mu_{\min}^3}. \tag{A.1}$$

For $n = N$, it holds that

$$\left(\frac{\zeta_m^4 + \zeta_m^2(1+\lambda_n^2)|\sigma_0|^2 + (1+\lambda_n^2+\lambda_n^4)|\sigma_0|^4}{|\zeta_m^2|^2}\right)^{1/2} < C(\sigma_\mu L)^2. \tag{A.2}$$

Proof. It is clear that

$$\begin{aligned}
&\zeta_m^4 + \zeta_m^2(1+\lambda_n^2)|\sigma_0|^2 + (1+\lambda_n^2+\lambda_n^4)|\sigma_0|^4 \\
&\quad \le \zeta_m^4 + 2\zeta_m^2(1+\lambda_n^2)|\sigma_0|^2 + (1+2\lambda_n^2+\lambda_n^4)|\sigma_0|^4 = (\zeta_m^2 + (1+\lambda_n^2)|\sigma_0|^2)^2.
\end{aligned}$$

Also, if $\tilde{\zeta}_m^2$ are eigenvalues of the Sturm–Liouville problem (4.10) with $L = 1$, then it holds that $\zeta_m^2 = \tilde{\zeta}_m^2/L^2$ and $\tilde{\zeta}_m > C$ for $m \in \mathbb{N}$. Thus, it suffices to show that

$$C_{n,m} := \frac{\tilde{\zeta}_m^2 + (1+\lambda_n^2)|\sigma_0|^2 L^2}{|\tilde{\zeta}_m^2 - \mu_n^2\sigma_0^2 L^2|} \le \begin{cases} C(\mu_{\min})^{-3} & \text{if } n \neq N, \\ C(\sigma_\mu L)^2 & \text{if } n = N \end{cases} \tag{A.3}$$

for $m \in \mathbb{N}$.

We note that the angle between $\tilde{\zeta}_m^2$ and $\mu_n^2\sigma_0^2 L^2$ in $\mathbb{C}$ is given by $2\theta_0$ for $n < N$ and $\pi - 2\theta_0$ for $n > N$ by (3.2). One can easily show that the denominator of $C_{n,m}$ for $n \neq N$ is bounded below by

$$\begin{aligned}
|\tilde{\zeta}_m^2 - \mu_n^2\sigma_0^2 L^2| &= (\tilde{\zeta}_m^4 + |\mu_n\sigma_0 L|^4 \pm 2\tilde{\zeta}_m^2|\mu_n\sigma_0 L|^2\cos 2\theta_0)^{1/2} \\
&\ge \left(\frac{1 - |\cos 2\theta_0|}{2}\right)^{1/2}(\tilde{\zeta}_m^2 + |\mu_n\sigma_0 L|^2).
\end{aligned}$$

Since $|\cos 2\theta_0| < 1$, in general $((1 - |\cos(2\theta_0)|)/2)^{1/2}$ is bounded below away from zero but it may be close to zero when $\theta_0 \approx 0$ or $\pi/2$. When $0 < \theta_0 \ll 1$ (or $0 < \pi/2 - \theta_0 \ll 1$, in this case we work with $\alpha_0 = \pi/2 - \theta_0$ instead of θ_0), it holds that

$$1 - |\cos 2\theta_0| = 1 - \cos 2\theta_0 \ge \frac{1}{2!}(2\theta_0)^2 - \frac{1}{4!}(2\theta_0)^4.$$

In such a case, since $C\mu_{\min} \le \tilde{\mu}_{N+1}/\mu_{N-1} = \tan(\theta_0) \le C\theta_0$, it can be obtained

$$\left(\frac{1 - |\cos 2\theta_0|}{2}\right)^{1/2} \ge \theta_0\sqrt{1 - \frac{1}{3}\theta_0^2} \ge C\theta_0 \ge C\mu_{\min}. \tag{A.4}$$

Therefore, by using (A.4) and (3.8) we can derive that

$$|\tilde{\zeta}_m^2 - \mu_n^2\sigma_0^2 L^2| \geq C\mu_{\min}^3(\tilde{\zeta}_m^2 + (1+\lambda_n^2)|\sigma_0|^2 L^2),$$

from which (A.3) for $n \neq N$ immediately follows.

On the other hand, when $n = N$, we have $\mu_n = 0$ and $\lambda_n = k$, and hence

$$C_{n,m} = \frac{\tilde{\zeta}_m^2 + (1+k^2)|\sigma_0|^2 L^2}{\tilde{\zeta}_m^2}.$$

In this case we note that $|\sigma_0| \leq C|\sigma_\mu|$ instead of the second inequality in (3.3), since $\mu_{\min}$ can be considered as constant. Therefore, by using the fact that $\tilde{\zeta}_m > C$ and $\sigma_\mu L > 1$ in (3.5), it can be shown that

$$C_{n,m} \leq C(\sigma_\mu L)^2,$$

which completes the proof. □

It is a well-known theory that trace operators from usual Sobolev spaces are bounded and have continuous right inverses. The following lemmas are devoted to verifying the same results independent of σ_0 for the weighted Sobolev space $H_\sigma^1(\Omega_{\mathrm{PML}})$. More precisely, we will study on liftings in $H_\sigma^1(\Omega_{\mathrm{PML}})$ of functions in $\dot{H}^{1/2}(\Gamma_0)$ and trace inequalities for functions in $H_\sigma^1(\Omega_{\mathrm{PML}})$ or $H_\sigma^2(\Omega_{\mathrm{PML}})$.

Lemma A.2. *For $g \in \dot{H}^{1/2}(\Gamma_0)$, there exists $\phi \in H_\sigma^1(\Omega_{\mathrm{PML}})$ such that $\phi|_{\Gamma_0} = g$ and*

$$\|\phi\|_{H_\sigma^1(\Omega_{\mathrm{PML}})} \leq C\|g\|_{\dot{H}^{1/2}(\Gamma_0)}.$$

Proof. Let $g = \sum_{n=0}^\infty g_n Y_n$ in $\dot{H}^{1/2}(\Gamma_0)$. We define

$$\phi(x,y) = \sum_{n=0}^\infty g_n e^{-|\sigma_0|\alpha_n x} Y_n(y) := \sum_{n=0}^\infty \phi_n(x) Y_n(y) \qquad \text{for } (x,y) \in \Omega_{\mathrm{PML}}$$

with $\alpha_n = (1+\lambda_n^2)^{1/2}$. A straightforward computation shows that

$$\begin{aligned} |\sigma_0| \int_0^L |\phi_n(x)|^2 \mathrm{d}x &= |g_n|^2 |\sigma_0| \int_0^L \mathrm{e}^{-2|\sigma_0|\alpha_n x}\mathrm{d}x \leq \frac{|g_n|^2}{2}(1+\lambda_n^2)^{-1/2}, \\ \frac{1}{|\sigma_0|}\int_0^L \left|\frac{d}{\mathrm{d}x}\phi_n(x)\right|^2 \mathrm{d}x &\leq \frac{|g_n|^2}{|\sigma_0|}\int_0^L |\sigma_0\alpha_n|^2 \mathrm{e}^{-2|\sigma_0|\alpha_n x}\mathrm{d}x \leq \frac{|g_n|^2}{2}(1+\lambda_n^2)^{1/2}. \end{aligned}$$

By using Fubini's theorem and the monotone convergence theorem, we achieve the inequality independent of σ_0

$$\begin{aligned} \|\phi\|_{H_\sigma^1(\Omega_{\mathrm{PML}})}^2 &= \int_0^L \sum_{n=0}^\infty \left(\frac{1}{|\sigma_0|}\left|\frac{d\phi_n}{\mathrm{d}x}(x)\right|^2 + |\sigma_0|(1+\lambda_n^2)|\phi_n(x)|^2 \right)\mathrm{d}x \\ &\leq C\sum_{n=0}^\infty (1+\lambda_n^2)^{1/2}|g_n|^2 = C\|g\|_{\dot{H}^{1/2}(\Gamma_0)}^2, \end{aligned}$$

which completes the proof. □

Lemma A.3. *The following trace inequalities hold,*

$$\|\phi\|_{\dot{H}^{1/2}(\Gamma_0)} \le C\|\phi\|_{H^1_\sigma(\Omega_{\mathrm{PML}})} \quad \text{for } \phi \in H^1_\sigma(\Omega_{\mathrm{PML}}) \tag{A.5}$$

$$\|\phi\|_{\dot{H}^{3/2}(\Gamma_0)} \quad \text{and} \quad \|\frac{1}{\sigma_0}\frac{\partial \phi}{\partial x}\|_{\dot{H}^{1/2}(\Gamma_0)} \le C\|\phi\|_{H^2_\sigma(\Omega_{\mathrm{PML}})} \quad \text{for } \phi \in H^2_\sigma(\Omega_{\mathrm{PML}}). \tag{A.6}$$

Proof. We will prove (A.6) as (A.5) can be proved in the same way. To this end, we let $\widetilde{\Omega}_\infty = \mathbb{R}\times\Theta$ be an infinite waveguide and define an extension $\tilde{\phi}$ of ϕ to $(-L/2,0)\times\Theta$ by

$$\tilde{\phi}(x,y) = \begin{cases} 3\phi(-x,y) - 2\phi(-2x,y) & \text{for } -L/2 < x \le 0, \\ \phi(x,y) & \text{for } 0 < x < L. \end{cases}$$

The construction of $\tilde{\phi}$ implies that $\tilde{\phi}$ and $\partial\tilde{\phi}/\partial x$ are continuous across Γ_0. We introduce a cutoff function $\chi(x)$ defined by one for $|x| < |\sigma_0|L/4$ and zero for $|x| > |\sigma_0|L/2$, that satisfies

$$\|\chi\|_{L^\infty(\mathbb{R})} \le 1, \quad \left\|\frac{\mathrm{d}\chi}{\mathrm{d}x}\right\|_{L^\infty(\mathbb{R})} \le \frac{C}{|\sigma_0|L} \le C, \quad \left\|\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2}\right\|_{L^\infty(\mathbb{R})} \le \frac{C}{|\sigma_0|^2L^2} \le C.$$

Then the zero extension $v(x,y)$ of $\chi(|\sigma_0|x)\tilde{\phi}(x,y)$ to $\widetilde{\Omega}_\infty$ satisfies

$$\|v\|_{H^2_\sigma(\widetilde{\Omega}_\infty)} \le C\|\phi\|_{H^2_\sigma(\Omega_{\mathrm{PML}})}, \tag{A.7}$$

where

$$\begin{aligned}\|v\|^2_{H^2_\sigma(\widetilde{\Omega}_\infty)} := \frac{1}{|\sigma_0|^3}\left\|\frac{\partial^2 v}{\partial x^2}\right\|^2_{L^2(\widetilde{\Omega}_\infty)} + \frac{1}{|\sigma_0|}\left(\left\|\frac{\partial^2 v}{\partial xy}\right\|^2_{L^2(\widetilde{\Omega}_\infty)} + \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(\widetilde{\Omega}_\infty)}\right) \\ + |\sigma_0|\left(\left\|\frac{\partial^2 v}{\partial y^2}\right\|^2_{L^2(\widetilde{\Omega}_\infty)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2(\widetilde{\Omega}_\infty)} + \|v\|^2_{L^2(\widetilde{\Omega}_\infty)}\right).\end{aligned}$$

Therefore, since $v=\phi$ and $\partial v/\partial x = \partial\phi/\partial x$ on Γ_0 and we have the inequality (A.7), it suffices to prove

$$\|v\|_{\dot{H}^{3/2}(\Gamma_0)} \quad \text{and} \quad \left\|\frac{1}{\sigma_0}\frac{\partial v}{\partial x}\right\|_{\dot{H}^{1/2}(\Gamma_0)} \le C\|v\|_{H^2_\sigma(\widetilde{\Omega}_\infty)} \tag{A.8}$$

for establishing (A.6).

Now, we first observe that

$$v(x,y) = \sum_{n=0}^\infty v_n(x)Y_n(y) = \sum_{n=0}^\infty \left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{v}_n(\xi)\mathrm{e}^{-ix\xi}\mathrm{d}\xi\right)Y_n(y), \tag{A.9}$$

where $v_n(x) = \int_\Theta v(x,y)Y_n(y)\mathrm{d}y$ and $\hat{v}_n$ is the Fourier transform of v_n. By invoking Fubini's theorem, the monotone convergence theorem and Plancherel's theorem, we have

$$\begin{aligned}\|v\|^2_{H^2_\sigma(\widetilde{\Omega}_\infty)} &= \int_{\mathbb{R}}\sum_{n=0}^\infty\left(\frac{1}{|\sigma_0|^3}\left|\frac{\mathrm{d}^2 v_n}{\mathrm{d}x^2}(x)\right|^2 + \frac{1+\lambda_n^2}{|\sigma_0|}\left|\frac{\mathrm{d}v_n}{\mathrm{d}x}(x)\right|^2 + |\sigma_0|(1+\lambda_n^2+\lambda_n^4)|v_n(x)|^2\right)\mathrm{d}x \\ &\ge C\sum_{n=0}^\infty\int_{\mathbb{R}}\frac{1}{|\sigma_0|}\left(\frac{\xi^2}{|\sigma_0|} + |\sigma_0|(1+\lambda_n^2)\right)^2|\hat{v}_n(\xi)|^2\mathrm{d}\xi.\end{aligned} \tag{A.10}$$

On the other hand, the nth Fourier coefficient of the trace of v on Γ_0 can be written as

$$(v|_{\Gamma_0})_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{v}_n(\xi)\mathrm{d}\xi$$

and the Schwarz inequality yields that

$$|(v|_{\Gamma_0})_n|^2 \leq \frac{1}{2\pi}\left(\int_{\mathbb{R}}|\sigma_0|\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)^{-2}\mathrm{d}\xi\right)\left(\int_{\mathbb{R}}\frac{1}{|\sigma_0|}\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)^{2}|\hat{v}_n(\xi)|^2\mathrm{d}\xi\right). \quad \text{(A.11)}$$

By the change of variables $t = \xi/(|\sigma_0|\sqrt{1+\lambda_n^2})$, the first integration term in (A.11) can be evaluated independently of σ_0 as

$$\begin{aligned}\int_{\mathbb{R}}|\sigma_0|\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)^{-2}\mathrm{d}\xi &= \int_{\mathbb{R}}\frac{|\sigma_0|^3}{(|\sigma_0|^2(1+\lambda_n^2)+\xi^2)^2}\mathrm{d}\xi \\ &= (1+\lambda_n^2)^{-3/2}\int_{\mathbb{R}}\frac{1}{(1+t^2)^2}dt = \frac{\pi}{2}(1+\lambda_n^2)^{-3/2},\end{aligned} \quad \text{(A.12)}$$

which shows from (A.11) that

$$(1+\lambda_n^2)^{3/2}|(v|_{\Gamma_0})_n|^2 \leq \frac{1}{4}\int_{\mathbb{R}}\frac{1}{|\sigma_0|}\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)^{2}|\hat{v}_n(\xi)|^2\mathrm{d}\xi. \quad \text{(A.13)}$$

Finally, it then follows from (A.13) and (A.10) that

$$\begin{aligned}\|v\|^2_{\dot{H}^{3/2}(\Gamma_0)} &= \sum_{n=0}^{\infty}(1+\lambda_n^2)^{3/2}|(v|_{\Gamma_0})_n|^2 \\ &\leq C\sum_{n=0}^{\infty}\int_{\mathbb{R}}\frac{1}{|\sigma_0|}\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)^{2}|\hat{v}_n(\xi)|^2\mathrm{d}\xi \leq C\|v\|^2_{H^2_\sigma(\widetilde{\Omega}_\infty)},\end{aligned}$$

which completes the proof of the first inequality of (A.8).

From (A.9), we have

$$\left(\frac{\partial v}{\partial x}|_{\Gamma_0}\right)_n = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}-i\xi\hat{v}_n(\xi)\mathrm{d}\xi$$

and hence the Cauchy–Schwarz inequality and the same change of variables used in (A.12) show again

$$\begin{aligned}\left|\frac{1}{\sigma_0}(\frac{\partial v}{\partial x}|_{\Gamma_0})_n\right|^2 &\leq \frac{1}{2\pi}\left(\int_{\mathbb{R}}\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)^{-1}\mathrm{d}\xi\right)\left(\int_{\mathbb{R}}\left(\frac{\xi^2}{|\sigma_0|}+|\sigma_0|(1+\lambda_n^2)\right)\frac{\xi^2}{|\sigma_0|^2}|\hat{v}_n(\xi)|^2\mathrm{d}\xi\right) \\ &\leq \frac{1}{2}(1+\lambda_n)^{-1/2}\int_{\mathbb{R}}\left(\frac{|\xi|^4}{|\sigma_0|^3}+\frac{(1+\lambda_n^2)\xi^2}{|\sigma_0|}\right)|\hat{v}_n(\xi)|^2\mathrm{d}\xi.\end{aligned}$$

It then follows that

$$\left\|\frac{1}{\sigma_0}\frac{\partial v}{\partial x}\right\|_{\dot{H}^{1/2}(\Gamma_0)} \leq C\|v\|_{H^2_\sigma(\widetilde{\Omega}_\infty)}$$

which proves the second inequality of (A.8). □

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