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A LOCAL DISCONTINUOUS GALERKIN GRADIENT DISCRETIZATION METHOD FOR LINEAR AND QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. A local weighted discontinuous Galerkin gradient discretization method for solving elliptic equations is introduced. The local scheme is based on a coarse grid and successively improves the solution solving a sequence of local elliptic problems in high gradient regions. Using the gradient discretization framework we prove convergence of the scheme for linear and quasilinear equations under minimal regularity assumptions. The error due to artificial boundary conditions is also analyzed, shown to be of higher order and shown to depend only locally on the regularity of the solution. Numerical experiments illustrate our theoretical findings and the local method's accuracy is compared against the non local approach.

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1. Introduction

Partial differential equations with high contrast are notoriously difficult to solve. In order to capture strong variations of the exact solution in the numerical approximations of the PDE, non uniform grids are usually required. The construction of such grids is often based on an iterative process, where a solution is computed and an a posteriori error estimator is used to indicate the regions where the mesh has to be refined, see [2,4,23,25]. In such approach, the solution is computed on the whole domain at each step, even if the mesh has changed only in a small portion of the domain.

In this paper we propose an algorithm for elliptic PDE, based on a decomposition of the computational domain in local subdomains adapted to the variation of the solution. In each subdomain, only local problems need to be solved and no iterations are needed between subdomains (*e.g.* as in domain decomposition method), as we define artificial boundary conditions and compute the solution only once in each local domain. We concentrate here on the *a priori* error analysis of our scheme, while we postpone the a posteriori error analysis to a companion paper [1]. The local scheme proposed in this paper is more efficient than the classical schemes for elliptic PDEs with strong variations for several reasons.

– For linear problems, when using an iterative solver such as the conjugate gradient (CG) method we have smaller problems to compute on the finer meshes, while the non-local classical schemes need the solution of

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global linear systems with a large number of degrees of freedom (DOF) (recall that the CG method has a convergence rate that is super-linear with respect to the DOF of the system).

– When solving a linear system arising from PDEs with CG methods, preconditioners are usually needed, a usual choice for CG being the incomplete Cholesky (IC) factorization. For non-local schemes, the high contrast of the PDE leads to systems with high condition number (due to mesh and data variations). For the local scheme, as each subdomain involves smaller variations of the solution and data the condition number is smaller, leading to faster convergence of the iterative method.
– Finally for nonlinear problems, in addition to the computational saving described previously, only a nonlinear problem on a coarse global mesh needs to be solved for the local scheme, while the subsequent local problems are linear. The computational saving is therefore significant for such problems.

The local method that we propose in this paper relies on the discontinuous Galerkin discretization, more precisely on the Symmetric Weighted Interior Penalty Galerkin (SWIPG) scheme [8,11]. We consider the elliptic model problem

$$-\nabla\cdot(A\nabla u) = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{in } \partial\Omega, \tag{1.1b}$$

where $\Omega\subset\mathbb{R}^d$ for $d=1,2,3$ is an open bounded polytopal connected set, $u\in H^1_0(\Omega)$ and $f\in H^{-1}(\Omega)$. The matrix A is symmetric, positive definite and can possibly depend on u, since we consider both a linear and a quasilinear case.

The idea of solving local elliptic problems to improve the numerical solution's accuracy is not new in the literature. The first important difference of our scheme with respect to the other methods is that the *a priori* error analysis is performed under minimal regularity assumptions for linear and quasilinear equations, that is, assuming $u\in H^1_0(\Omega)$ and $f\in H^{-1}(\Omega)$. This is achieved by recasting the SWIPG scheme into the Gradient Discretization (GD) framework [10, 12]. The GD method is a framework suitable for studying the *a priori* convergence of various types of diffusion problems: linear and non linear, steady state or transient. For our scheme, the GD framework is convenient to decompose the sources of errors in the local problems. Furthermore, applying the pointwise estimates from [7], we can prove (in some particular cases) that the errors coming from the artificial boundary conditions are of higher order and depend only locally on the regularity of the solution. Finally, we stress out that the GD framework is only used for the analysis, indeed another advantage of the scheme is that it fits very easily in existing codes that use the popular discontinuous Galerkin scheme without needing additional data structures nor additional memory requirements.

One of the first local methods to appear is the Local Defect Correction method (LDC) presented in [16], it is an iterative process that at each step solves a global problem on a coarse mesh and a local problem on a fine mesh. The solution of the global problem provides artificial boundary conditions to the local problem. The solution of the latter is then introduced into the coarse system to estimate its residual. The coarse system is solved again but adding the residual to its right hand side, leading to a more accurate coarse solution and hence better artificial boundary conditions for the next local problem. Two similar methods are the Fast Adaptive Composite grid algorithm [20] and the Multi-Level Adaptive Technique [6]. In [14] it is shown that under reasonable assumptions the three methods lead to the same solution. In their original form the schemes were defined for finite difference methods but finite volumes or finite element versions exist, see [21,26]. Only recently has the LDC scheme been coupled with an a posteriori error estimator in order to automatically select the local domains [5]. The scheme that we propose is different from the LDC method in the sense that it computes only one global solution on the full domain and all the subsequent computations are localized around the singularities.

Since our local scheme computes a solution similar to the one given by the non local method but at a smaller cost, it could in principle be used as a preconditioner for the non local scheme. Related preconditioning strategies are for example the multilevel domain decomposition methods (see, *e.g.* [3, 13]). Numerical investigation such a preconditioning strategy and comparison with multilevel domain decomposition methods is however beyond the aims and scope of this paper.

The paper is organized as follows. In Section 2 we present the Gradient Discretization framework and the Symmetric Weighted Discontinuous Galerkin Gradient Discretization (SWDGGD), which is equivalent to the SWIPG scheme. At the end of the section we introduce a local version of the SWDGGD. In Section 3 we present the local scheme and establish an *a priori* error analysis for linear equations. In Section 4 we introduce the scheme and the *a priori* error analysis for quasilinear equations. Finally, Section 5 provides numerical results and comparison with the classical scheme. The equivalence between the SWDGGD and SWIPG methods is postponed to Appendix A.

2. Notation and preliminary results

Our local scheme is based on the traditional SWIPG scheme but the analysis is done in the GD framework, this allows for minimal assumptions and further generalizations as quasilinear problems. Whence we introduce in Section 2.1 the notation for the GD setting and in Section 2.2 we define a particular GD scheme which is equivalent to SWIPG, their equivalence is shown in Appendix A. The method presented in Section 2.2 is a slight modification of the one proposed in [12], the main difference is that the latter is equivalent to the Symmetric Interior Penalty Galerkin (SIPG) method. We opted for the SWIPG scheme instead of SIPG since it is known to have improved stability in problems with high diffusivity contrasts [11] and also to be suitable for a locally vanishing diffusion [9]. In Sections 2.1 and 2.2 we mainly follow [10] and [12]. In what follows we make the following assumptions on the data for the linear case

Assumption 2.1.

- $\Omega \subset \mathbb{R}^d$ *is an open bounded polytopal domain,*
- $A : \Omega \to \mathbb{R}^{d\times d}$ *is such that* $A(\boldsymbol{x})$ *is a symmetric matrix measurable with respect to* $\boldsymbol{x}$ *and there exists* $\underline{\lambda}, \overline{\lambda} > 0$ *such that it has eigenvalues in* $[\underline{\lambda}, \overline{\lambda}]$*,*
- *the forcing term is* $f \in H^{-1}(\Omega)$*.*

For the quasilinear case, we will assume

Assumption 2.2.

- $\Omega \subset \mathbb{R}^d$ *is an open bounded polytopal domain,*
- $A(\boldsymbol{x}, s) = (a_{ij}(\boldsymbol{x}, s))_{i,j=1}^d$ *is such that* $a_{ij} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is continuous in* $\boldsymbol{x}$ *and Lipschitz continuous in* s*. Furthermore* $A(\boldsymbol{x}, s)$ *is a symmetric matrix with eigenvalues in* $[\underline{\lambda}, \overline{\lambda}]$*,*
- *the forcing term is* $f \in H^{-1}(\Omega)$*.*

For simplicity, the dependence of A on $\boldsymbol{x}$ is left out in our notation. Under Assumption 2.1 the unique weak solution of equation (1.1) is $u \in H^1_0(\Omega)$ such that

$$\int_\Omega A\nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \langle f, v\rangle \qquad \text{for all } v \in H^1_0(\Omega), \tag{2.1}$$

where $\langle \cdot,\cdot\rangle$ denotes the pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Under Assumption 2.2 we have existence and uniqueness of the solution of the quasilinear problem obtained by replacing A by $A(u)$ in equation (2.1). The continuity of A with respect to $\boldsymbol{x}$ is needed to ensure uniqueness of the solution and simplify the presentation in Section 4, but it is not needed by the local scheme.

2.1. The Gradient Discretizazion method

We start by defining the GD method for homogeneous Dirichlet boundary conditions as introduced in [12] along with some of its properties.

Definition 2.3. A gradient discretization method $\mathcal{D}$ for homogeneous Dirichlet boundary conditions is defined by $\mathcal{D} = (X_\mathcal{D}, \Pi_\mathcal{D}, \nabla_\mathcal{D})$, where

(1) the set $X_\mathcal{D}$ is a finite dimensional real vector space,
(2) the reconstruction function $\Pi_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)$ is a linear mapping that reconstructs, from an element in $X_\mathcal{D}$, a function over Ω,
(3) the gradient reconstruction $\nabla_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)^d$ is a linear mapping which reconstructs, from an element of $X_\mathcal{D}$, a gradient over Ω. This gradient reconstruction must be chosen such that $\|\nabla_\mathcal{D} \cdot\|_{L^2(\Omega)^d}$ is a norm on $X_\mathcal{D}$.

Example 2.4. Among others, the conforming $\mathbb{P}_1$ finite element Galerkin method can be written as a GD method. Given a partition of Ω into simplices, let $V_h \subset H^1_0(\Omega)$ be the set of piecewise linear and continuous functions on this partition. Let $\{e_i\}_{i\in I}$ be a basis of V_h, we define $X_\mathcal{D} = \{\phi = (\zeta_i)_{i\in I} : \zeta_i \in \mathbb{R} \text{ for all } i \in I\}$, $\Pi_\mathcal{D}\phi = \sum_{i\in I} \zeta_i e_i$ and $\nabla_\mathcal{D}\phi = \sum_{i\in I} \zeta_i \nabla e_i$. In what follows when we consider sequences $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of gradient discretizations, it is useful to think that each $\mathcal{D}_n$ is associated to a mesh of size h_n with $\lim_{n\to\infty} h_n = 0$.

In the following $(\mathcal{D}_n)_{n\in\mathbb{N}}$ is a sequence of gradient discretizations.

Definition 2.5. If $\mathcal{D}$ is a GD, define $C_\mathcal{D}$ as the norm of $\Pi_\mathcal{D}$:

$$C_\mathcal{D} := \max_{\phi\in X_\mathcal{D}\setminus\{0\}} \frac{\|\Pi_\mathcal{D}\phi\|_{L^2(\Omega)}}{\|\nabla_\mathcal{D}\phi\|_{L^2(\Omega)^d}}.$$

A sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of GD is coercive if there exists $C_\mathrm{p} \in \mathbb{R}_+$ such that $C_{\mathcal{D}_n} \le C_\mathrm{p}$ for all $n \in \mathbb{N}$.

We observe that coercivity implies a kind of Poincaré inequality.

Definition 2.6. If $\mathcal{D}$ is a GD, define $S_\mathcal{D} : H^1_0(\Omega) \to [0,\infty[$ by

$$S_\mathcal{D}(v) := \min_{\phi\in X_\mathcal{D}} \left(\|\Pi_\mathcal{D}\phi - v\|_{L^2(\Omega)} + \|\nabla_\mathcal{D}\phi - \nabla v\|_{L^2(\Omega)^d}\right).$$

A sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of GD is consistent if $\lim_{n\to\infty} S_{\mathcal{D}_n}(v) = 0$ for all $v \in H^1_0(\Omega)$.

Definition 2.7. If $\mathcal{D}$ is a GD, define $W_\mathcal{D} : H_{\mathrm{div}}(\Omega) \to [0,\infty[$ by

$$W_\mathcal{D}(\boldsymbol{v}) = \sup_{\phi\in X_\mathcal{D}\setminus\{0\}} \frac{\left|\int_\Omega (\nabla_\mathcal{D}\phi \cdot \boldsymbol{v} + \Pi_\mathcal{D}\phi\, \nabla\cdot\boldsymbol{v})\mathrm{d}\boldsymbol{x}\right|}{\|\nabla_\mathcal{D}\phi\|_{L^2(\Omega)^d}}.$$

A sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of GD is limit-conforming if $\lim_{n\to\infty} W_{\mathcal{D}_n}(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in H_{\mathrm{div}}(\Omega)$.

The limit conformity of the method implies that the gradient discretization method satisfies asymptotically the Stokes theorem.

Definition 2.8. A sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of GD is compact if, for any sequence $\phi_n \in X_{\mathcal{D}_n}$ such that $(\|\nabla_{\mathcal{D}_n}\phi_n\|_{L^2(\Omega)^d})_{n\in\mathbb{N}}$ is bounded, the sequence $(\Pi_{\mathcal{D}_n}\phi_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

In order to use the GD to solve equation (2.1) it is useful to write $f \in H^{-1}(\Omega)$ as

$$f = f_0 + \sum_{i=1}^d \frac{\partial f_i}{\partial x_i} = f_0 + \nabla\cdot\boldsymbol{F},$$

where $\boldsymbol{x} = (x_1,\dots,x_d) \in \Omega$, $f_0, f_1, \dots, f_d \in L^2(\Omega)$ and $\boldsymbol{F} = (f_1,\dots,f_d)^\top \in L^2(\Omega)^d$. With this notation, equation (2.1) becomes

$$\int_\Omega A\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x} = \int_\Omega (f_0\, v - \boldsymbol{F}\cdot\nabla v)\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } v \in H^1_0(\Omega). \tag{2.2}$$

We next define the Gradient Scheme used to approximate u solution of equation (2.2).

Definition 2.9. For a given gradient discretization $\mathcal{D}$, the Gradient Scheme (GS) for problem (2.2) is defined by: find $\vartheta \in X_{\mathcal{D}}$ such that

$$\int_\Omega A\nabla_{\mathcal{D}}\vartheta \cdot \nabla_{\mathcal{D}}\phi \,\mathrm{d}\boldsymbol{x} = \int_\Omega (f_0\, \Pi_{\mathcal{D}}\phi - \boldsymbol{F}\cdot\nabla_{\mathcal{D}}\phi)\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \phi \in X_{\mathcal{D}}. \tag{2.3}$$

The convergence of the above scheme is given by Theorem 2.10, which is proven in Theorem 2.28 of [10]. Notice that under Assumption 2.1 and $u \in H_0^1(\Omega)$ we have $A\nabla u + \boldsymbol{F} \in H_{\mathrm{div}}(\Omega)$, indeed $-\nabla\cdot(A\nabla u + \boldsymbol{F}) = f_0 \in L^2(\Omega)$.

Theorem 2.10. *Let $\mathcal{D}$ be a GD, then there exists one and only one $\vartheta \in X_{\mathcal{D}}$ solution to equation (2.3) and it satisfies*

$$\|\nabla u - \nabla_{\mathcal{D}}\vartheta\|_{L^2(\Omega)^d} \le \frac{1}{\underline{\lambda}} W_{\mathcal{D}}(A\nabla u + \boldsymbol{F}) + (1+\kappa(A))S_{\mathcal{D}}(u), \tag{2.4}$$

where $\kappa(A) = \overline{\lambda}/\underline{\lambda}$ is the condition number of A.

Corollary 2.11. *If $(\mathcal{D}_n)_{n\in\mathbb{N}}$ is a consistent and limit-conforming sequence of GD and $\vartheta_n \in \mathcal{D}_n$ is a sequence of solutions to equation (2.3), then*

$$\lim_{n\to\infty} \|\nabla u - \nabla_{\mathcal{D}_n}\vartheta_n\|_{L^2(\Omega)^d} = 0.$$

Proof. Follows from equation (2.4) and the definitions of consistency and limit conformity. □

Convergence rates are obtained under stronger regularity hypothesis on the data and the solution, upon the introduction of a mesh and depend on the underlying discretization method. We refer to Corollary 2.17 at the end of Section 2.2 for such results. The compactness hypothesis of Definition 2.8 is needed to establish convergence of the gradient scheme when applied to nonlinear problems.

2.2. The Symmetric Weighted Discontinuous Galerkin Gradient Discretizazion

Inspired from the method proposed in [12] we define the Symmetric Weighted Discontinuous Galerkin GD (SWDGGD).

A polytopal mesh $\mathfrak{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P})$ is defined as follows. $\mathcal{M}$ is a finite family of non empty polytopal open disjoint elements $K \subset \Omega$ such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. We suppose that K is star shaped with respect to an $\boldsymbol{x}_K \in K$ and denote $\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathcal{M}}$. Let $\mathcal{F} = \mathcal{F}_b \cup \mathcal{F}_i$ be the set of faces of the mesh, where $\mathcal{F}_b$, $\mathcal{F}_i$ are the boundary and internal faces, respectively. The set of faces of K is $\mathcal{F}_K = \{\sigma \in \mathcal{F} : \sigma \subset \partial K\}$. For each $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$ we denote by $d_{K,\sigma}$ the orthogonal distance between $\boldsymbol{x}_K$ and σ, hence

$$d_{K,\sigma} = (\boldsymbol{y} - \boldsymbol{x}_K)\cdot \boldsymbol{n}_{K,\sigma} \quad \text{for all } \boldsymbol{y} \in \sigma,$$

where $\boldsymbol{n}_{K,\sigma}$ is the unit vector normal to σ outward to K. We denote by $D_{K,\sigma}$ the cone with vertex $\boldsymbol{x}_K$ and basis σ, that is

$$D_{K,\sigma} = \{\boldsymbol{x}_K + s(\boldsymbol{y} - \boldsymbol{x}_K) : s \in]0,1[,\, \boldsymbol{y} \in \sigma\}.$$

Finally, we define the mesh size and a constant measuring the regularity of the mesh. For $\sigma \in \mathcal{F}$ let $\mathcal{M}_\sigma = \{K \in \mathcal{M} : \sigma \in \mathcal{F}_K\}$ and let h_K be the diameter of $K \in \mathcal{M}$, then

$$\begin{aligned} h_{\mathcal{M}} &= \max\{h_K : K \in \mathcal{M}\},\\ \eta_{\mathfrak{T}} &= \max\left(\left\{\frac{h_T}{h_K} + \frac{h_K}{h_T} : \sigma \in \mathcal{F}_i, \mathcal{M}_\sigma = \{K,T\}\right\} \cup \left\{\frac{h_K}{d_{K,\sigma}} : K \in \mathcal{M}, \sigma \in \mathcal{F}_K\right\}\right.\\ &\qquad\qquad \left. \cup\, \{\#\mathcal{F}_K : K \in \mathcal{M}\}\right), \end{aligned}$$

the term $\{\#\mathcal{F}_K : K \in \mathcal{M}\}$ is needed in Lemma 3.14 of [12] to bound the jumps on the faces of the elements.

Let $V = \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_\ell(K), \forall K \in \mathcal{M}\}$, where $\mathbb{P}_\ell(K)$ is the space of polynomials in K of total degree ℓ. Let $(e_i)_{i\in I}$ be a basis of V such that $\mathrm{supp}(e_i)$ is restricted to one element of $\mathcal{M}$. We set

$$X_\mathcal{D} = \{\phi = (\zeta_i)_{i\in I} : \zeta_i \in \mathbb{R} \text{ for all } i \in I\} \tag{2.5}$$

and define the operator $\Pi_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)$ by

$$\Pi_\mathcal{D}\phi = \sum_{i\in I} \zeta_i e_i. \tag{2.6}$$

For $K \in \mathcal{M}$ we note by $\Pi_{\overline{K}}\phi := \Pi_\mathcal{D}\phi|_{\overline{K}}$ the restriction of $\Pi_\mathcal{D}\phi$ to K extended to $\overline{K}$ and define $\nabla_{\overline{K}}\phi = \nabla\Pi_{\overline{K}}\phi$. Let $\alpha \in]0,1[$ be a user parameter and $\psi : [0,1] \to \mathbb{R}$ such that $\psi(s) = 0$ on $[0,\alpha[$ and $\psi|_{[\alpha,1]} \in \mathbb{P}_{\ell-1}([\alpha,1])$ satisfying

$$\int_\alpha^1 \psi(s)s^{d-1}\mathrm{d}s = 1 \quad \text{and} \quad \int_\alpha^1 (1-s)^i\psi(s)s^{d-1}\mathrm{d}s = 0 \quad \text{for } i = 1,\ldots,\ell-1. \tag{2.7}$$

In the case where $\ell = 1$ we have $\psi(s)|_{[\alpha,1]} = d/(1-\alpha^d)$. This choice of ψ is fundamental to show the equivalence with the SWIPG method, see Appendix A. The discrete gradient $\nabla_\mathcal{D} : X_\mathcal{D} \to L^2(\Omega)^d$ is defined as follows. For $\phi \in X_\mathcal{D}$, $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, we set, for *a.e.* $\boldsymbol{x} \in D_{K,\sigma}$

$$\nabla_\mathcal{D}\phi(\boldsymbol{x}) = \nabla_{\overline{K}}\phi(\boldsymbol{x}) + \psi(s)\frac{[\phi]_{K,\sigma}(\boldsymbol{y})}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma}, \tag{2.8}$$

where $\boldsymbol{x} = \boldsymbol{x}_K + s(\boldsymbol{y} - \boldsymbol{x}_K)$ with $s \in]0,1[$, $\boldsymbol{y} \in \sigma$ and

$$\begin{aligned}
&\text{if } \sigma \in \mathcal{F}_i \text{ with } \sigma = \partial K \cap \partial T \text{ then } [\phi]_{K,\sigma}(\boldsymbol{y}) = \omega_{K,\sigma}(\Pi_{\overline{T}}\phi(\boldsymbol{y}) - \Pi_{\overline{K}}\phi(\boldsymbol{y})),\\
&\text{if } \sigma \in \mathcal{F}_b \text{ with } \sigma = \partial K \cap \partial\Omega \text{ then } [\phi]_{K,\sigma}(\boldsymbol{y}) = 0 - \Pi_{\overline{K}}\phi(\boldsymbol{y}).
\end{aligned}$$

For $\sigma \in \mathcal{F}_b$ with $\sigma = \partial K \cap \partial\Omega$ and $K \in \mathcal{M}$ it is useful to set $\omega_{K,\sigma} = 1$. If instead $\sigma \in \mathcal{F}_i$ with $\sigma = \partial K \cap \partial T$ and $K, T \in \mathcal{M}$ the weights $\omega_{K,\sigma}, \omega_{T,\sigma}$ are two non negative numbers such that

$$\omega_{K,\sigma} + \omega_{T,\sigma} = 1. \tag{2.9}$$

In the original Discontinuous Galerkin GD (DGGD) method introduced in [12] the weights are $(\omega_{K,\sigma}, \omega_{T,\sigma}) = (1/2, 1/2)$ and it is proven that $\mathcal{D} = (X_\mathcal{D}, \Pi_\mathcal{D}, \nabla_\mathcal{D})$, with $X_\mathcal{D}$, $\Pi_\mathcal{D}$, $\nabla_\mathcal{D}$ as in equations (2.5), (2.6) and (2.8), is a GD method. Moreover, any sequence $(\mathcal{D}_n)_{n\in\mathbb{N}}$ of DGGD defined from polytopal meshes $(\mathfrak{T}_n)_{n\in\mathbb{N}}$ with $(\eta_{\mathfrak{T}_n})_{n\in\mathbb{N}}$ bounded and $h_{\mathcal{M}_n} \longrightarrow 0$ is a coercive, consistent, limit-conforming and compact sequence of GD. Thanks to the particular choice of ψ it is possible to show that in the linear case with piecewise constant diffusion the DGGD scheme is equivalent to the well known SIPG method.

In our case we want to be equivalent to the SWIPG method, hence we define the weights as follows. Let $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, we set

$$\delta_{K,\sigma} = \boldsymbol{n}_{K,\sigma}^\top A|_K \boldsymbol{n}_{K,\sigma}.$$

For $\sigma \in \mathcal{F}_i$ such that $\sigma = \partial K \cap \partial T$ with $K, T \in \mathcal{M}$ we define

$$\omega_{K,\sigma} = \frac{\delta_{T,\sigma}}{\delta_{K,\sigma} + \delta_{T,\sigma}}, \qquad \omega_{T,\sigma} = \frac{\delta_{K,\sigma}}{\delta_{K,\sigma} + \delta_{T,\sigma}}\cdot \tag{2.10}$$

Upon changing the constants in Lemma 3.8 of [12] we deduce from Lemma 3.10 of [12] that $\|\nabla_\mathcal{D}\cdot\|_{L^2(\Omega)^d}$ with the choice of weights given by equation (2.10) is a norm on $X_\mathcal{D}$ and hence $\mathcal{D} = (X_\mathcal{D}, \Pi_\mathcal{D}, \nabla_\mathcal{D})$ with $(\omega_{K,\sigma}, \omega_{T,\sigma})$

as in equation (2.10) is a GD method. It can be used to solve diffusion problems with homogeneous boundary conditions as in Definition 2.9. From now on we refer to this method as the Symmetric Weighted DGGD scheme (SWDGGD). Apart from the weights definition, the only difference with respect to the original DGGD method is a factor

$$C_\omega := \max_{K\in\mathcal{M},\,\sigma\in\mathcal{F}_K} \omega_{K,\sigma}^{-1}/2 \tag{2.11}$$

multiplying the constant $C_\mathcal{D}$ of Definition 2.5.

In the foregoing analysis we need the jump semi norm on $X_\mathcal{D}$, defined by

$$|\phi|_J^2 := \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{F}_K} \frac{1}{d_{K,\sigma}} \int_\sigma [\phi]_{K,\sigma}^2(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

We define a stronger version of $S_\mathcal{D}$ which controls the jumps.

Definition 2.12. If $\mathcal{D}$ is a SWDGGD, define $S_{\mathcal{D},J} : H_0^1(\Omega) \to [0,\infty[$ by

$$S_{\mathcal{D},J}(v) := \min_{\phi\in X_\mathcal{D}} (\|\Pi_\mathcal{D}\phi - v\|_{L^2(\Omega)} + \|\nabla_\mathcal{D}\phi - \nabla v\|_{L^2(\Omega)^d} + |\phi|_J).$$

We quote two improved estimates on $S_\mathcal{D}$, $S_{\mathcal{D},J}$ and $W_\mathcal{D}$.

Lemma 2.13. *There exists $C_S > 0$ depending only on $|\Omega|$, α, ℓ, d and ρ such that for all $v \in H^2(\Omega)\cap H_0^1(\Omega)$*

$$S_\mathcal{D}(v) \le C_S h_\mathcal{M}\|v\|_{H^2(\Omega)} \quad \text{and} \quad S_{\mathcal{D},J}(v) \le C_S h_\mathcal{M}\|v\|_{H^2(\Omega)}.$$

The result for $S_{\mathcal{D},J}$ is obtained following the lines of the proof for $S_\mathcal{D}$, which is given in Lemma 3.14 of [12].

Lemma 2.14. *There exists $C_W > 0$ depending only on $|\Omega|$, α, ℓ and d such that for all $\boldsymbol{v} \in H^1(\Omega)^d$*

$$W_\mathcal{D}(\boldsymbol{v}) \le C_W h_\mathcal{M}\|\boldsymbol{v}\|_{H^1(\Omega)^d}.$$

Lemma 2.14 has been proven for the DGGD scheme in Lemma 3.15 of [12]. The proof uses the fact that $(1/2, 1/2)$ is a partition of unity. Thanks to equation (2.9) the same result holds for the SWDGGD method. Next, Theorem 2.15 establishes the asymptotic properties of the SWDGGD schemes.

Theorem 2.15. *Let $(\mathcal{D}_n)_{n\in\mathbb{N}}$ be a sequence of of SWDGGD defined from polytopal meshes $(\mathfrak{T}_n)_{n\in\mathbb{N}}$ with $(\eta_{\mathfrak{T}_n})_{n\in\mathbb{N}}$ bounded and $h_{\mathcal{M}_n} \longrightarrow 0$ for $n\to\infty$. Then it is a coercive, consistent, limit-conforming and compact sequence of GD.*

Proof. Coercivity and compactness are proven as in Lemmas 3.12 and 3.13 of [12]. Consistency follows from Lemma 2.13 and Lemma 2.16 of [10]. Limit-conformity follows from the compactness of the scheme, Lemma 2.14 and Lemma 2.17 of [10]. □

In the SWDGGD scheme the C_p constant depends continuously on C_ω from equation (2.11). We note that, even if C_ω is mesh dependent it can be bounded by terms depending only on A. In the following Lemma we show, by usual density arguments, that even if v is only in $H_0^1(\Omega)$ we have $\lim_{n\to\infty} S_{\mathcal{D}_n,J}(v) = 0$.

Lemma 2.16. *Consider the same assumptions of Theorem 2.15 and $v \in H_0^1(\Omega)$. Then we have $\lim_{n\to\infty} S_{\mathcal{D}_n,J}(v) = 0$.*

Proof. Let $v \in H_0^1(\Omega)$ and $\varepsilon > 0$. Then there exists $v_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\|v - v_\varepsilon\|_{L^2(\Omega)} + \|\nabla v - \nabla v_\varepsilon\|_{L^2(\Omega)^d} \leq \varepsilon$. Let

$$\phi_n = \operatorname*{argmin}_{\phi \in X_{\mathcal{D}_n}} (\|\Pi_{\mathcal{D}_n}\phi - v_\varepsilon\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}_n}\phi - \nabla v_\varepsilon\|_{L^2(\Omega)^d} + |\phi|_J).$$

Hence

$$\begin{aligned} S_{\mathcal{D}_n,J}(v) &\leq \|\Pi_{\mathcal{D}_n}\phi_n - v\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}_n}\phi_n - \nabla v\|_{L^2(\Omega)^d} + |\phi_n|_J \\ &\leq \varepsilon + C_S h_{\mathcal{M}_n}\|v_\varepsilon\|_{H^2(\Omega)}, \end{aligned}$$

using Lemma 2.13 $\lim_{n\to\infty} S_{\mathcal{D}_n,J}(v) \leq \varepsilon$. Since ε is arbitrary the result follows. $\square$

Corollary 2.17 (Of Thm. 2.10)**.** *Let $\mathcal{D}$ be a SWDGGD, under the same assumptions of Theorem 2.10, $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\boldsymbol{F} \in H^1(\Omega)^d$, the solution $\vartheta \in X_{\mathcal{D}}$ to equation (2.3) satisfies*

$$\|\nabla u - \nabla_{\mathcal{D}}\vartheta\|_{L^2(\Omega)^d} \leq h_{\mathcal{M}}\left(\frac{1}{\underline{\lambda}}C_W\|A\nabla u + \boldsymbol{F}\|_{H^1(\Omega)^d} + (1+\kappa(A))C_S\|u\|_{H^2(\Omega)}\right),$$

Proof. Follows from Theorem 2.10 together with Lemmas 2.13 and 2.14. $\square$

2.3. The local weighted discontinuous Galerkin gradient discretization

Let $\{\Omega_k\}_{k=1}^M$ be a sequence of polytopal domains with $\Omega_1 = \Omega$ and $\Omega_k \subset \Omega$. We consider as well a sequence $(\mathfrak{T}_k)_{k=1}^M = ((\mathcal{M}_k, \mathcal{F}_k, \mathcal{P}_k))_{k=1}^M$ of polytopal meshes on Ω and denote $\mathcal{F}_k = \mathcal{F}_{k,b} \cup \mathcal{F}_{k,i}$ with $\mathcal{F}_{k,b}$ and $\mathcal{F}_{k,i}$ the set of boundary and internal faces of $\mathcal{M}_k$. Moreover, $(\mathfrak{T}_k)_{k=1}^M$ satisfies the following.

Assumption 2.18.

(a) *For each $k = 1, \ldots, M$, $\overline{\Omega}_k = \cup_{K \in \mathcal{M}_k,\, K \subset \Omega_k} \overline{K}$.*
(b) *For $k = 1, \ldots, M-1$*
  (i) *$\{K \in \mathcal{M}_{k+1} : K \subset \Omega \setminus \Omega_{k+1}\} = \{K \in \mathcal{M}_k : K \subset \Omega \setminus \Omega_{k+1}\}$,*
  (ii) *if $K, T \in \mathcal{M}_k$ with $K \subset \Omega_{k+1}$, $T \subset \Omega \setminus \Omega_{k+1}$ and $\partial K \cap \partial T \neq \emptyset$ then $K \in \mathcal{M}_{k+1}$,*
  (iii) *if $K \in \mathcal{M}_k$ and $K \subset \Omega_{k+1}$, either $K \in \mathcal{M}_{k+1}$ or K is a union of elements in $\mathcal{M}_{k+1}$.*
(c) *We suppose the existence of $C_{\mathrm{r}} > 0$ such that*
  (i) *for $k = 1, \ldots, M-1$, if $K \in \mathcal{M}_k$ and $\widehat{K} \in \mathcal{M}_{k+1}$ with $\widehat{K} \subset K$ and $\sigma \in \mathcal{F}_K$, $\hat{\sigma} \in \mathcal{F}_{\widehat{K}}$ with $\hat{\sigma} \subset \sigma$ then $d_{K,\sigma} \leq C_{\mathrm{r}} d_{\widehat{K},\hat{\sigma}}$,*
  (ii) *for $k = 1, \ldots, M$, if $\sigma = \partial K \cap \partial T$ with $K, T \in \mathcal{M}_k$, $T \subset \Omega \setminus \Omega_k$ and $K \subset \Omega_k$ then $d_{K,\sigma} \leq C_{\mathrm{r}} d_{T,\sigma}$.*
(d) *It exists $\rho > 0$ such that $\eta_{\mathfrak{T}_k} \leq \rho$ for $k = 1, \ldots, M$.*

The above assumptions on $(\mathfrak{T}_k)_{k=1}^M$ ensure that $\mathfrak{T}_{k+1}$ is a refinement of $\mathfrak{T}_k$ and that this refinement occurs inside the subdomain Ω_{k+1}. Let $\widehat{\mathfrak{T}}_k = (\widehat{\mathcal{M}}_k, \widehat{\mathcal{F}}_k, \widehat{\mathcal{P}}_k)$, with $\widehat{\mathcal{M}}_k = \{K \in \mathcal{M}_k : K \subset \Omega_k\}$, $\widehat{\mathcal{P}}_k = \{\boldsymbol{x}_k \in \mathcal{P}_k : \boldsymbol{x}_k \in \Omega_k\}$ and $\widehat{\mathcal{F}}_k = \widehat{\mathcal{F}}_{k,b} \cup \widehat{\mathcal{F}}_{k,i}$ the set of faces of $\widehat{\mathcal{M}}_k$,with $\widehat{\mathcal{F}}_{k,b}$ and $\widehat{\mathcal{F}}_{k,i}$ the boundary and internal faces of Ω_k, respectively. Condition (a) in Assumption 2.18 assures that $\widehat{\mathfrak{T}}_k$ is a polytopal mesh on Ω_k. (b) guarantees that in $\Omega \setminus \Omega_{k+1}$ and in the neighborhood of $\partial\Omega_{k+1}$ the meshes $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ are equal and that $\mathcal{M}_{k+1}$ is a refinement of $\mathcal{M}_k$ in Ω_{k+1}. (c) and (d) ensure mesh regularity, will permit equivalences between jump norms and make the constant C_S of Lemma 2.13 uniform in k. An example of meshes satisfying Assumption 2.18 is given in Figure 1.

Given $(\mathfrak{T}_k)_{k=1}^M$ we define a sequence $\mathcal{D}_k = (X_{\mathcal{D}_k}, \Pi_{\mathcal{D}_k}, \nabla_{\mathcal{D}_k})$ of SWDGGD. Let

$$V_k = \{v_k \in L^2(\Omega) : v_k|_K \in \mathbb{P}_\ell(K),\, \forall K \in \mathcal{M}_k\} \tag{2.12}$$

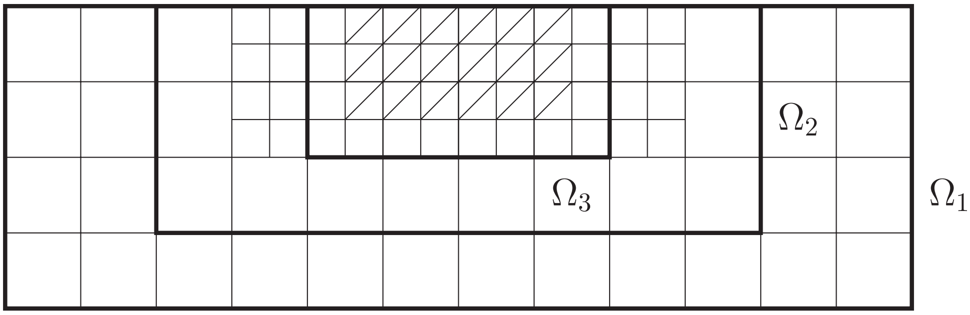


FIGURE 1. Example of possible meshes for three embedded domains Ω_1, Ω_2, Ω_3.

and $(e_{k,i})_{i\in I_k}$ be a basis of V_k such that $\mathrm{supp}(e_{k,i})$ is restricted to one element of $\mathcal{M}_k$. We set

$$X_{\mathcal{D}_k} = \{\phi_k = (\zeta_{k,i})_{i\in I_k} : \zeta_{k,i} \in \mathbb{R} \text{ for all } i \in I_k\}.$$

$\Pi_{\mathcal{D}_k}$ and $\nabla_{\mathcal{D}_k}$ are defined as in equations (2.6), (2.8) and (2.10).

We can write $X_{\mathcal{D}_k} = Y_{\mathcal{D}_k} \oplus Z_{\mathcal{D}_k}$, where $\mathrm{supp}(\Pi_{\mathcal{D}_k}\varphi_k) \subset \Omega_k$ for $\varphi_k \in Y_{\mathcal{D}_k}$ and $\mathrm{supp}(\Pi_{\mathcal{D}_k}\xi_k) \subset \Omega\setminus\Omega_k$ for $\xi_k \in Z_{\mathcal{D}_k}$. For $k=1$ we have $Y_{\mathcal{D}_1} = X_{\mathcal{D}_1}$ and $Z_{\mathcal{D}_1} = \{0\}$. For $k\geq 2$ and $\phi_{k-1}\in X_{\mathcal{D}_{k-1}}$ there exists $\xi_k \in Z_{\mathcal{D}_k}$ such that $\Pi_{\mathcal{D}_{k-1}}\phi_{k-1}\chi_{\Omega\setminus\Omega_k} = \Pi_{\mathcal{D}_k}\xi_k$. By abuse of notation we will denote $\xi_k = \phi_{k-1}\chi_{\Omega\setminus\Omega_k}$, hence $\chi_{\Omega\setminus\Omega_k}$ is seen as an operator from $X_{\mathcal{D}_{k-1}}$ to $Z_{\mathcal{D}_k}$.

In what follows $\Pi_{\widehat{\mathcal{D}}_k}$ is the restriction of $\Pi_{\mathcal{D}_k}$ to $Y_{\mathcal{D}_k}$. Let us define as well a gradient on $Y_{\mathcal{D}_k}$ which will be used to impose inhomogeneous Dirichlet boundary conditions. Let $\varphi_k \in Y_{\mathcal{D}_k}$ and $\xi_k \in Z_{\mathcal{D}_k}$, for $K \in \widehat{\mathcal{M}}_k$, $\sigma \in \mathcal{F}_K$ and $\boldsymbol{x} \in D_{K,\sigma}$ the gradient $\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k(\boldsymbol{x})$ is defined by

$$\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k(\boldsymbol{x}) = \nabla_{\overline{K}}\varphi_k(\boldsymbol{x}) + \psi(s)\frac{[\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y})}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma},$$

where $\boldsymbol{x} = \boldsymbol{x}_K + s(\boldsymbol{y} - \boldsymbol{x}_K)$ and

$$\begin{aligned}
&[\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y}) = [\varphi_k]_{K,\sigma}(\boldsymbol{y}) && \text{if } \sigma \in \widehat{\mathcal{F}}_{k,i} \text{ or } \sigma \in \widehat{\mathcal{F}}_{k,b}\cap\mathcal{F}_{k,b},\\
&[\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y}) = \Pi_{\overline{T}}\xi_k - \Pi_{\overline{K}}\varphi_k && \text{if } \sigma \in \widehat{\mathcal{F}}_{k,b}\setminus\mathcal{F}_{k,b} \text{ with } \sigma = \partial K\cap\partial T\\
& && \text{and } K \in \widehat{\mathcal{M}}_k,\ T \in \mathcal{M}_k\setminus\widehat{\mathcal{M}}_k.
\end{aligned}$$

We will denote $\nabla_{\widehat{\mathcal{D}}_k,0}$ by $\nabla_{\widehat{\mathcal{D}}_k}$.

Theorem 2.19. *The triple $\widehat{\mathcal{D}}_k = (Y_{\mathcal{D}_k}, \Pi_{\widehat{\mathcal{D}}_k}, \nabla_{\widehat{\mathcal{D}}_k})$ is a SWDGGD scheme for each $k = 1,\ldots,M$.*

Proof. We notice that $\widehat{\mathcal{D}}_k$ is the SWDGGD corresponding to the local polytopal mesh $\widehat{\mathfrak{T}}_k$, hence it is a SWDGGD by construction. □

In what follows we will call $\widehat{\mathcal{D}}_k$ the local SWDGGD. Remark that Lemmas 2.13 and 2.14 and Theorem 2.15 are valid if we replace $\mathcal{D}$, Ω, $h_{\mathcal{M}}$ and $\mathfrak{T}$ with $\widehat{\mathcal{D}}_k$, Ω_k, $h_{\widehat{\mathcal{M}}_k}$ and $\widehat{\mathfrak{T}}_k$.

Observe that for $\varphi_k \in Y_{\mathcal{D}_k}$ we have $\nabla_{\widehat{\mathcal{D}}_k}\varphi_k \neq \nabla_{\mathcal{D}_k}\varphi_k$, indeed $\nabla_{\widehat{\mathcal{D}}_k}$ is missing the $\omega_{K,\sigma}$ factor in the jumps at the faces $\sigma \in \widehat{\mathcal{F}}_{k,b}\setminus\mathcal{F}_{k,b}$. Adding the $\omega_{K,\sigma}$ factor in those faces would prevent the limit consistency of $\widehat{\mathcal{D}}_k$.

In what follows $S_{\widehat{\mathcal{D}}_k}$ and $W_{\widehat{\mathcal{D}}_k}$ are the operators defined by Definitions 2.6 and 2.7 but with Ω, $\mathcal{D}$, and $X_{\mathcal{D}}$ replaced by Ω_k, $\widehat{\mathcal{D}}_k$ and $Y_{\mathcal{D}_k}$. We define as well the jump semi norms on $X_{\mathcal{D}_k}$ and $Y_{\mathcal{D}_k}$. For $\phi_k \in X_{\mathcal{D}_k}$ we define

$$|\phi_k|^2_{J(k)} := \sum_{K\in\mathcal{M}_k}\sum_{\sigma\in\mathcal{F}_K}\frac{1}{d_{K,\sigma}}\int_\sigma [\phi_k]_{K,\sigma}(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}$$

and for $\xi_k \in Z_{\mathcal{D}_k}$, $\varphi_k \in Y_{\mathcal{D}_k}$ we set

$$|\varphi_k|^2_{\widehat{J}(k),\xi_k} := \sum_{K\in\widehat{\mathcal{M}}_k} \sum_{\sigma\in\mathcal{F}_K} \frac{1}{d_{K,\sigma}} \int_\sigma [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y})^2 \mathrm{d}\boldsymbol{y}.$$

Since in our local scheme (to be defined in Sect. 3) we solve local elliptic problems with artificial boundary conditions we need a local version of $S_{\mathcal{D}_k,J}$ which measures the error of the method on the boundary.

Definition 2.20. Let $\xi_k \in Z_{\mathcal{D}_k}$ and $\widehat{\mathcal{D}}_k$ be a local SWDGGD, define $S_{\widehat{\mathcal{D}}_k,J,\xi_k} : H^1_0(\Omega) \to [0,\infty[$ by

$$S_{\widehat{\mathcal{D}}_k,J,\xi_k}(v) := \min_{\varphi\in Y_{\mathcal{D}_k}} (\|\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi - \nabla v\|_{L^2(\Omega_k)^d} + |\varphi|_{\widehat{J}(k),\xi_k}).$$

The $L^2(\Omega_k)$ norm is not taken into account in $S_{\widehat{\mathcal{D}}_k,J,\xi_k}$ since our convergence results are in energy and jump norms.

Lemma 2.21. *Let $v \in H^1_0(\Omega) \cap H^2(\Omega)$, then for $k = 1,\dots,M$*

$$\min_{\xi\in Z_{\mathcal{D}_k}} S_{\widehat{\mathcal{D}}_k,J,\xi}(v) \le C_S h_{\widehat{\mathcal{M}}_k} \|v\|_{H^2(\Omega_k)}.$$

Proof. Follows the lines of Lemma 3.14 from [12]. □

In order to provide bounds on $S_{\widehat{\mathcal{D}}_k,J,\xi_k}$ we need an additional norm to measure the error at the interface between subdomains. Let $\phi_k \in X_{\mathcal{D}_k}$, we define

$$|\phi_k|^2_{\partial\Omega_k^-} := \sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}} \sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T} \frac{1}{d_{K,\sigma}} \int_\sigma \Pi_{\overline{T}}\phi_k(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y}.$$

The minus in $|\cdot|_{\partial\Omega_k^-}$ refers to the fact that in the integral the argument lives outside $\widehat{\mathcal{M}}_k$. Later, $|\cdot|_{\partial\Omega_k^+}$ will be defined as well.

Lemma 2.22. *Let $\kappa_k, \xi_k \in Z_{\mathcal{D}_k}$ and $v \in H^1_0(\Omega)$. Then*

$$S_{\widehat{\mathcal{D}}_k,J,\kappa_k}(v) \le S_{\widehat{\mathcal{D}}_k,J,\xi_k}(v) + C_\partial|\kappa_k - \xi_k|_{\partial\Omega_k^-},$$

where $C_\partial = 1 + C_\psi$ and $C_\psi^2 = \int_\alpha^1 \psi(s)^2 s^{d-1}\mathrm{d}s$. If moreover $v \in H^2(\Omega)\cap H^1_0(\Omega)$ we have

$$S_{\widehat{\mathcal{D}}_k,J,\kappa_k}(v) \le C_S h_{\widehat{\mathcal{M}}_k}\|v\|_{H^2(\Omega_k)} + C_\partial|\kappa_k - \xi_k|_{\partial\Omega_k^-} \quad \textit{for} \quad \xi_k = \operatorname*{argmin}_{\xi\in Z_{\mathcal{D}_k}} S_{\widehat{\mathcal{D}}_k,J,\xi}(v).$$

Proof. Let $\kappa_k, \xi_k \in Z_{\mathcal{D}_k}$, $v \in H^1_0(\Omega)$ and $\varphi_k \in Y_{\mathcal{D}_k}$ defined by

$$\varphi_k = \operatorname*{argmin}_{\varphi\in Y_{\mathcal{D}_k}} (\|\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi - \nabla v\|_{L^2(\Omega_k)^d} + |\varphi|_{\widehat{J}(k),\xi_k}),$$

we have

$$\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k - \nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k\|^2_{L^2(\Omega_k)^d} = \sum_{K\in\widehat{\mathcal{M}}_k} \sum_{\sigma\in\mathcal{F}_K\cap\widehat{\mathcal{F}}_{k,b}} \int_{D_{K,\sigma}} \frac{\psi(s)^2}{d^2_{K,\sigma}} ([\varphi_k]_{K,\sigma,\kappa_k}(\boldsymbol{y}) - [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y}))^2 \,\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{x} = \boldsymbol{x}_K + s(\boldsymbol{y} - \boldsymbol{x}_K)$ for $s \in [0,1]$ and $\boldsymbol{y} \in \sigma$. Using the change of variables $\mathrm{d}\boldsymbol{x} = d_{K,\sigma} s^{d-1} ds d\boldsymbol{y}$ yields

$$\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k - \nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k\|^2_{L^2(\Omega_k)^d} = \sum_{K\in\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_K\cap\widehat{\mathcal{F}}_{k,b}}\int_\sigma\int_\alpha^1 \frac{\psi(s)^2}{d_{K,\sigma}^2}([\varphi_k]_{K,\sigma,\kappa_k}(\boldsymbol{y}) - [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y}))^2 d_{K,\sigma}s^{d-1}\,\mathrm{d}s\mathrm{d}\boldsymbol{y}$$
$$= C_\psi^2 \sum_{K\in\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_K\cap\widehat{\mathcal{F}}_{k,b}}\frac{1}{d_{K,\sigma}}\int_\sigma([\varphi_k]_{K,\sigma,\kappa_k}(\boldsymbol{y}) - [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y}))^2\,\mathrm{d}\boldsymbol{y}.$$

If in the above sum $\sigma \in \mathcal{F}_{k,b}$, then $[\varphi_k]_{K,\sigma,\kappa_k} - [\varphi_k]_{K,\sigma,\xi_k} = 0$. Else, if $\sigma \in \mathcal{F}_{k,i}$ there is $T \in \mathcal{M}_k \setminus \widehat{\mathcal{M}}_k$ such that $\sigma = \partial K \cap \partial T$ and

$$[\varphi_k]_{K,\sigma,\kappa_k} - [\varphi_k]_{K,\sigma,\xi_k} = \Pi_{\overline{T}}\kappa_k - \Pi_{\overline{T}}\xi_k,$$

which implies

$$\begin{aligned}
&\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k - \nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k\|^2_{L^2(\Omega_k)^d}\\
&\quad = C_\psi^2 \sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T}\frac{1}{d_{K,\sigma}}\int_\sigma \Pi_{\overline{T}}(\kappa_k - \xi_k)(\boldsymbol{y})^2\,\mathrm{d}\boldsymbol{y} = C_\psi^2|\kappa_k - \xi_k|^2_{\partial\Omega_k^-}.
\end{aligned} \tag{2.13}$$

For the jump term $|\varphi_k|_{\widehat{\mathcal{J}}(k),\kappa_k}$, we have

$$\begin{aligned}
|\varphi_k|^2_{\widehat{\mathcal{J}}(k),\kappa_k} &= |\varphi_k|^2_{\widehat{\mathcal{J}}(k),\xi_k}\\
&\quad + \sum_{K\in\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_K\cap\widehat{\mathcal{F}}_{k,b}}\frac{1}{d_{K,\sigma}}\int_\sigma([\varphi_k]_{K,\sigma,\kappa_k}(\boldsymbol{y})^2 - [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y})^2)\,\mathrm{d}\boldsymbol{y}.
\end{aligned} \tag{2.14}$$

If $\sigma \in \mathcal{F}_{k,b}$ then $[\varphi_k]^2_{K,\sigma,\kappa_k} - [\varphi_k]^2_{K,\sigma,\xi_k} = 0$, else, if $\sigma \in \mathcal{F}_{k,i}$ with $\sigma = \partial T \cap \partial K$, $T \in \mathcal{M}_k \setminus \widehat{\mathcal{M}}_k$ we have

$$\begin{aligned}
[\varphi_k]^2_{K,\sigma,\kappa_k} - [\varphi_k]^2_{K,\sigma,\xi_k} &= ([\varphi_k]_{K,\sigma,\kappa_k} - [\varphi_k]_{K,\sigma,\xi_k})([\varphi_k]_{K,\sigma,\kappa_k} + [\varphi_k]_{K,\sigma,\xi_k})\\
&= (\Pi_{\overline{T}}\kappa_k - \Pi_{\overline{T}}\xi_k)(\Pi_{\overline{T}}\kappa_k - \Pi_{\overline{T}}\xi_k + 2[\varphi_k]_{K,\sigma,\xi_k}).
\end{aligned} \tag{2.15}$$

Using equations (2.14) and (2.15) we obtain

$$\begin{aligned}
|\varphi_k|^2_{\widehat{\mathcal{J}}(k),\kappa_k} &\le |\varphi_k|^2_{\widehat{\mathcal{J}}(k),\xi_k} + \sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T}\frac{1}{d_{K,\sigma}}\int_\sigma \Pi_{\overline{T}}(\kappa_k - \xi_k)(\boldsymbol{y})^2\,\mathrm{d}\boldsymbol{y}\\
&\quad + 2\sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T}\frac{1}{d_{K,\sigma}}\int_\sigma |[\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y})\Pi_{\overline{T}}(\kappa_k - \xi_k)(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y}\\
&\le |\varphi_k|^2_{\widehat{\mathcal{J}}(k),\xi_k} + |\kappa_k - \xi_k|^2_{\partial\Omega_k^-} + 2|\varphi_k|_{\widehat{\mathcal{J}}(k),\xi_k}|\kappa_k - \xi_k|_{\partial\Omega_k^-}\\
&= \left(|\varphi_k|_{\widehat{\mathcal{J}}(k),\xi_k} + |\kappa_k - \xi_k|_{\partial\Omega_k^-}\right)^2.
\end{aligned}$$

Using the above estimate and equation (2.13) we get

$$\begin{aligned}
S_{\widehat{\mathcal{D}}_k,J,\kappa_k}(v) &\le \|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi - \nabla v\|_{L^2(\Omega_k)^d} + |\varphi_k|_{\widehat{\mathcal{J}}(k),\kappa_k}\\
&\le \|\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k - \nabla v\|_{L^2(\Omega_k)^d} + |\varphi_k|_{\widehat{\mathcal{J}}(k),\xi_k}\\
&\quad + (1 + C_\psi)|\kappa_k - \xi_k|_{\partial\Omega_k^-}\\
&= S_{\widehat{\mathcal{D}}_k,J,\xi_k}(v) + (1 + C_\psi)|\kappa_k - \xi_k|_{\partial\Omega_k^-}.
\end{aligned}$$

If moreover $v \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\xi_k = \operatorname{argmin}_{\xi \in Z_{\mathcal{D}_k}} S_{\widehat{\mathcal{D}}_k,J,\xi}(v)$, Lemma 2.21 yields $S_{\widehat{\mathcal{D}}_k,J,\xi_k}(v) \leq C_S h_{\widehat{\mathcal{M}}_k} \|v\|_{H^2(\Omega_k)}$. □

3. The local elliptic scheme

We introduce here our local SWDGGD elliptic scheme before embarking into its *a priori* error analysis.

Set $\vartheta_0 = 0$ and define iteratively $\vartheta_k \in X_{\mathcal{D}_k}$ for $k = 1, \ldots, M$ as

$$\vartheta_k = \hat{\vartheta}_k + \kappa_k, \tag{3.1a}$$

where $\kappa_k \in Z_{\mathcal{D}_k}$ is defined as

$$\kappa_k = \vartheta_{k-1}\chi_{\Omega\setminus\Omega_k} \tag{3.1b}$$

and $\hat{\vartheta}_k \in Y_{\mathcal{D}_k}$ is the solution of the local problem

$$\int_{\Omega_k} A\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k \cdot \nabla_{\widehat{\mathcal{D}}_k}\varphi \,\mathrm{d}\boldsymbol{x} = \int_{\Omega_k} (f_0\, \Pi_{\widehat{\mathcal{D}}_k}\varphi - \boldsymbol{F}\cdot\nabla_{\widehat{\mathcal{D}}_k}\varphi)\,\mathrm{d}\boldsymbol{x} \tag{3.1c}$$

for all $\varphi \in Y_{\mathcal{D}_k}$.

Remember that $\nabla_{\widehat{\mathcal{D}}_k} = \nabla_{\widehat{\mathcal{D}}_k,0}$, hence we use homogeneous boundary conditions for φ. Due to the definition of $\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}$ the inhomogeneous Dirichlet boundary condition κ_k is weakly imposed on $\hat{\vartheta}_k$. We have $\kappa_1 = 0$, hence $\vartheta_1 = \hat{\vartheta}_1 \in X_{\mathcal{D}_1}$. Then, for $k \geq 2$ the scheme (3.1) computes a new local solution $\hat{\vartheta}_k$ on a refined mesh $\widehat{\mathcal{M}}_k$, where the boundary condition is inherited from the previous solution ϑ_{k-1}.

In Section 3.1 we perform the *a priori* error analysis for the local solutions $\hat{\vartheta}_k$ and provide bounds for the errors in the local domains Ω_k. Section 3.2 improves the results of Section 3.1 in a particular case, showing that the error due to artificial boundary conditions is of higher order. Finally, Section 3.3 provides error bounds for the global solution ϑ_k.

3.1. *A priori* error analysis for the local solution

In this section we proceed with the *a priori* analysis of the local elliptic scheme presented in Section 3. Before proving convergence of the scheme we need the following interpolation result.

Lemma 3.1. *Let $\xi_{k-1} \in Z_{\mathcal{D}_{k-1}}$, $\varphi_{k-1} \in Y_{\mathcal{D}_{k-1}}$ and $\xi_k = (\xi_{k-1}+\varphi_{k-1})\chi_{\Omega\setminus\Omega_k} \in Z_{\mathcal{D}_k}$. Then there exists $\varphi_k \in Y_{\mathcal{D}_k}$ such that*

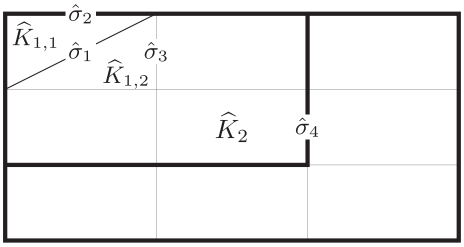
$$\|\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k - \nabla_{\widehat{\mathcal{D}}_{k-1},\xi_{k-1}}\varphi_{k-1}\|_{L^2(\Omega_k)^d} \leq C_{\mathrm{i}}|\varphi_{k-1}|_{\widehat{\mathcal{J}}(k-1),\xi_{k-1}}, \tag{3.2a}$$

$$|\varphi_k|_{\widehat{\mathcal{J}}(k),\xi_k} \leq C_{\mathrm{i}}|\varphi_{k-1}|_{\widehat{\mathcal{J}}(k-1),\xi_{k-1}}, \tag{3.2b}$$

with $C_{\mathrm{i}} = \sqrt{2}C_\psi(1 + C_{\omega,k}^2 C_{\mathrm{r}})^{1/2}$, $C_{\omega,k} = \max_{K\in\mathcal{M}_k,\sigma\in\mathcal{F}_K}\omega_{K,\sigma}^{-1}$, C_ψ from Lemma 2.22 and C_{r} from Assumption 2.18.

Proof. Since $\widehat{\mathcal{M}}_k$ is a refinement of $\widehat{\mathcal{M}}_{k-1}$ in Ω_k, there exists $\varphi_k \in Y_{\mathcal{D}_k}$ such that $\Pi_{\widehat{\mathcal{D}}_k}\varphi_k = \Pi_{\widehat{\mathcal{D}}_{k-1}}\varphi_{k-1}|_{\Omega_k}$. Hence

$$\begin{aligned}
&\|\nabla_{\widehat{\mathcal{D}}_k,\xi_k}\varphi_k - \nabla_{\widehat{\mathcal{D}}_{k-1},\xi_{k-1}}\varphi_{k-1}\|^2_{L^2(\Omega_k)^d}\\
&\qquad\leq 2\sum_{K\in\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_K}\int_{D_{K,\sigma}}\left|\psi(s)\frac{[\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol{y})}{d_{K,\sigma}}\right|^2\mathrm{d}\boldsymbol{x}\\
&\qquad\quad + 2\sum_{\substack{K\in\mathcal{M}_{k-1}\\ K\subset\Omega_k}}\sum_{\sigma\in\mathcal{F}_K}\int_{D_{K,\sigma}}\left|\psi(s)\frac{[\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}(\boldsymbol{y})}{d_{K,\sigma}}\right|^2\mathrm{d}\boldsymbol{x},
\end{aligned}$$



Ω_k and Ω_{k-1} bounded by thick lines ━,
$\hat\sigma_1 \subset K_1 = \widehat K_{1,1} \cup \widehat K_{1,2}$,
$\hat\sigma_2 = \sigma_2$ with $(\hat\sigma_2, \sigma_2) \in \widehat{\mathcal F}_{k,b} \times \widehat{\mathcal F}_{k-1,b}$,
$\hat\sigma_3 \in \widehat{\mathcal F}_{k,i}$,
$\hat\sigma_4 = \sigma_4$ with $(\hat\sigma_4, \sigma_4) \in \widehat{\mathcal F}_{k,b} \times \widehat{\mathcal F}_{k-1,i}$.

FIGURE 2. Example of a situation described in the proof of Lemma 3.1.

since the broken gradients of $\Pi_{\widehat{\mathcal D}_{k-1}}\varphi_{k-1}|_{\Omega_k}$ and $\Pi_{\widehat{\mathcal D}_k}\varphi_k$ cancel each other out. With the change of variables $\mathrm d\boldsymbol x = d_{K,\sigma}s^{d-1}\mathrm ds\mathrm d\boldsymbol y$ we have

$$\int_{D_{K,\sigma}} \left|\psi(s)\frac{[\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol y)}{d_{K,\sigma}}\right|^2 \mathrm d\boldsymbol x = \frac{1}{d_{K,\sigma}}\int_\alpha^1 \psi(s)^2 s^{d-1}\mathrm ds \int_\sigma [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol y)^2 \mathrm d\boldsymbol y$$

and similarly for φ_{k-1}. Using $C_\psi^2 = \int_\alpha^1 \psi(s)^2 s^{d-1}\mathrm ds$ yields

$$\begin{aligned}\|\nabla_{\widehat{\mathcal D}_k,\xi_k}\varphi_k - \nabla_{\widehat{\mathcal D}_{k-1},\xi_{k-1}}\varphi_{k-1}\|^2_{L^2(\Omega_k)^d} &\le 2C_\psi^2 \sum_{K\in\widehat{\mathcal M}_k}\sum_{\sigma\in\mathcal F_K}\frac{1}{d_{K,\sigma}}\int_\sigma [\varphi_k]_{K,\sigma,\xi_k}(\boldsymbol y)^2\mathrm d\boldsymbol y\\ &\quad + 2C_\psi^2 \sum_{\substack{K\in\mathcal M_{k-1}\\ K\subset\Omega_k}}\sum_{\sigma\in\mathcal F_K}\frac{1}{d_{K,\sigma}}\int_\sigma [\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}(\boldsymbol y)^2\mathrm d\boldsymbol y\\ &\le 2C_\psi^2\left(|\varphi_k|^2_{\widehat{\mathcal J}(k),\xi_k} + |\varphi_{k-1}|^2_{\widehat{\mathcal J}(k-1),\xi_{k-1}}\right).\end{aligned}$$

To obtain equation (3.2a) it remains to prove $|\varphi_k|^2_{\widehat{\mathcal J}(k),\xi_k} \le C^2_{\omega,k}C_{\mathrm r}|\varphi_{k-1}|^2_{\widehat{\mathcal J}(k-1),\xi_{k-1}}$. We write $|\varphi_k|^2_{\widehat{\mathcal J}(k),\xi_k}$ as

$$|\varphi_k|^2_{\widehat{\mathcal J}(k),\xi_k} = \sum_{\substack{K\in\mathcal M_{k-1}\\ K\subset\Omega_k}}\sum_{\substack{\widehat K\in\mathcal M_k\\ \widehat K\subset K}}\sum_{\hat\sigma\in\mathcal F_{\widehat K}}\frac{1}{d_{\widehat K,\hat\sigma}}\int_{\hat\sigma}[\varphi_k]_{\widehat K,\hat\sigma,\xi_k}(\boldsymbol y)^2\mathrm d\boldsymbol y.$$

Let K, $\widehat K$ and $\hat\sigma$ be as in the above sum, either $\hat\sigma\subset K$ and so $[\varphi_k]_{\widehat K,\hat\sigma,\xi_k} = 0$ or there exists $\sigma\in\mathcal F_K$ such that $\hat\sigma\subseteq\sigma$. In that latter case, if $(\hat\sigma,\sigma)\in\widehat{\mathcal F}_{k,b}\times\widehat{\mathcal F}_{k-1,b}$ or $\hat\sigma\in\widehat{\mathcal F}_{k,i}$ then $[\varphi_k]_{\widehat K,\hat\sigma,\xi_k} = [\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}$. If instead $(\hat\sigma,\sigma)\in\widehat{\mathcal F}_{k,b}\times\widehat{\mathcal F}_{k-1,i}$ then $[\varphi_k]_{\widehat K,\hat\sigma,\xi_k} = \omega^{-1}_{K,\sigma}[\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}$. See Figure 2 for an illustration of the above cases. Since $\omega^{-1}_{K,\sigma}\ge 1$, we obtain in all cases

$$|[\varphi_k]_{\widehat K,\hat\sigma,\xi_k}| \le \omega^{-1}_{K,\sigma}|[\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}| \le C_{\omega,k}|[\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}|.$$

Furthermore, by Assumption 2.18 we have $d_{K,\sigma}\le C_{\mathrm r}d_{\widehat K,\hat\sigma}$. These considerations together give

$$\sum_{\substack{\widehat K\in\mathcal M_k\\ \widehat K\subset K}}\sum_{\hat\sigma\in\mathcal F_{\widehat K}}\frac{1}{d_{\widehat K,\hat\sigma}}\int_{\hat\sigma}[\varphi_k]_{\widehat K,\hat\sigma,\xi_k}(\boldsymbol y)^2\mathrm d\boldsymbol y \le C^2_{\omega,k}C_{\mathrm r}\sum_{\sigma\in\mathcal F_K}\frac{1}{d_{K,\sigma}}\int_\sigma[\varphi_{k-1}]_{K,\sigma,\xi_{k-1}}(\boldsymbol y)^2\mathrm d\boldsymbol y,$$

hence $|\varphi_k|^2_{\widehat{\mathcal J}(k),\xi_k}\le C^2_{\omega,k}C_{\mathrm r}|\varphi_{k-1}|^2_{\widehat{\mathcal J}(k-1),\xi_{k-1}}$ and equation (3.2a) is proved. In Section 6.1 of [12] it is shown that $C_\psi\ge 1$, hence $C_{\omega,k}C_r^{1/2} < C_i = \sqrt2 C_\psi(1+C^2_{\omega,k}C_r)^{1/2}$ and equation (3.2b) follows. □

The next lemma has been proved for the DGGD scheme in [12] and is valid for the local SWDGGD method as well.

Lemma 3.2. *Let $\widehat{\mathcal{D}}_k$ be a local SWDGGD scheme, then there exists $C_{eq} > 0$ depending only on α, ℓ and d such that*

$$|\varphi_k|_{\widehat{J}(k),0} \leq C_{\mathrm{eq}} \|\nabla_{\widehat{\mathcal{D}}_k} \varphi_k\|_{L^2(\Omega_k)^d} \qquad \text{for all } \varphi_k \in Y_{\mathcal{D}_k}.$$

Proof. Follows the lines of Lemma 3.8 from [12]. □

The next lemma shows that the error of the local solution depends as usual on the regularity of the solution and data but also on the error committed on the artificial boundary condition. The proof is inspired from the one given in [10] leading to equation (2.4) and uses Lemma 2.22.

Lemma 3.3. *Let $u \in H^1_0(\Omega)$ be the exact solution to equation (2.2), $\kappa_k \in Z_{\mathcal{D}_k}$ and $\hat{\vartheta}_k \in Y_{\mathcal{D}_k}$ be solution of equation (3.1c). Then*

$$\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} \leq \frac{1 + C_{\mathrm{eq}}}{\underline{\lambda}} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) + C_A \min_{\xi_k \in Z_{\mathcal{D}_k}} \left(S_{\widehat{\mathcal{D}}_k,J,\xi_k}(u) + C_\partial |\kappa_k - \xi_k|_{\partial\Omega_k^-} \right) \tag{3.3}$$

with $C_A := C_{\mathrm{eq}}(1 + \kappa(A))$ and C_∂ from Lemma 2.22.

Remark 3.4. Observe that Lemma 3.3 is valid for any $\kappa_k \in Z_{\mathcal{D}_k}$ and not only κ_k given by scheme (3.1).

Proof. Since $\widehat{\mathcal{D}}_k$ is a SWDGGD scheme, by Definition 2.7 for any $\boldsymbol{v} \in H_{\mathrm{div}}(\Omega_k)$ and $\psi_k \in Y_{\mathcal{D}_k}$ we have

$$\left| \int_{\Omega_k} (\nabla_{\widehat{\mathcal{D}}_k} \psi_k \cdot \boldsymbol{v} + \Pi_{\widehat{\mathcal{D}}_k} \psi_k \, \nabla \cdot \boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} \right| \leq \|\nabla_{\widehat{\mathcal{D}}_k} \psi_k\|_{L^2(\Omega_k)^d} W_{\widehat{\mathcal{D}}_k}(\boldsymbol{v}).$$

As $-\nabla \cdot (A\nabla u + \boldsymbol{F}) = f_0 \in L^2(\Omega_k)$ we can take $\boldsymbol{v} = A\nabla u + \boldsymbol{F}$ and obtain

$$\left| \int_{\Omega_k} (\nabla_{\widehat{\mathcal{D}}_k} \psi_k \cdot (A\nabla u + \boldsymbol{F}) - \Pi_{\widehat{\mathcal{D}}_k} \psi_k \, f_0) \, \mathrm{d}\boldsymbol{x} \right| \leq \|\nabla_{\widehat{\mathcal{D}}_k} \psi_k\|_{L^2(\Omega_k)^d} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}).$$

Using equation (3.1c) we get

$$\left| \int_{\Omega_k} A(\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k) \cdot \nabla_{\widehat{\mathcal{D}}_k} \psi_k \, \mathrm{d}\boldsymbol{x} \right| \leq \|\nabla_{\widehat{\mathcal{D}}_k} \psi_k\|_{L^2(\Omega_k)^d} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}).$$

Let $\varphi_k \in Y_{\mathcal{D}_k}$, we have

$$\begin{aligned} &\int_{\Omega_k} A(\nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \varphi_k - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k) \cdot \nabla_{\widehat{\mathcal{D}}_k} \psi_k \, \mathrm{d}\boldsymbol{x} \\ &\qquad \leq \|\nabla_{\widehat{\mathcal{D}}_k} \psi_k\|_{L^2(\Omega_k)^d} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) + \int_{\Omega_k} A(\nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \varphi_k - \nabla u) \cdot \nabla_{\widehat{\mathcal{D}}_k} \psi_k \, \mathrm{d}\boldsymbol{x} \\ &\qquad \leq \|\nabla_{\widehat{\mathcal{D}}_k} \psi_k\|_{L^2(\Omega_k)^d} (W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) + \overline{\lambda} \|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \varphi_k - \nabla u\|_{L^2(\Omega_k)^d}). \end{aligned}$$

We choose $\psi_k = \varphi_k - \hat{\vartheta}_k$, since $\nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \varphi_k - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k = \nabla_{\widehat{\mathcal{D}}_k,0}(\varphi_k - \hat{\vartheta}_k) = \nabla_{\widehat{\mathcal{D}}_k}(\varphi_k - \hat{\vartheta}_k)$ we get

$$\underline{\lambda} \|\nabla_{\widehat{\mathcal{D}}_k}(\varphi_k - \hat{\vartheta}_k)\|^2_{L^2(\Omega_k)^d} \leq \int_{\Omega_k} A(\nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \varphi_k - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k) \cdot \nabla_{\widehat{\mathcal{D}}_k}(\varphi_k - \hat{\vartheta}_k) \, \mathrm{d}\boldsymbol{x}$$

and hence

$$\|\nabla_{\widehat{\mathcal{D}}_k}(\varphi_k - \hat{\vartheta}_k)\|_{L^2(\Omega_k)^d} \le \frac{1}{\underline{\lambda}} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) + \kappa(A)\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k - \nabla u\|_{L^2(\Omega_k)^d}. \tag{3.4}$$

This gives

$$\begin{aligned}\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d} &\le \|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k\|_{L^2(\Omega_k)^d} + \|\nabla_{\widehat{\mathcal{D}}_k}(\varphi_k - \hat{\vartheta}_k)\|_{L^2(\Omega_k)^d} \\ &\le \frac{1}{\underline{\lambda}} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) + (1+\kappa(A))\,\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k\|_{L^2(\Omega_k)^d}.\end{aligned} \tag{3.5}$$

Using $|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} \le |\varphi_k|_{\widehat{J}(k),\kappa_k} + |\hat{\vartheta}_k - \varphi_k|_{\widehat{J}(k),0}$, Lemma 3.2 and equation (3.4) yields

$$\begin{aligned}|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} \le |\varphi_k|_{\widehat{J}(k),\kappa_k} &+ \frac{C_{\mathrm{eq}}}{\underline{\lambda}} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) \\ &+ C_{\mathrm{eq}}\kappa(A)\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi_k - \nabla u\|_{L^2(\Omega_k)^d}.\end{aligned} \tag{3.6}$$

Summing equations (3.5) and (3.6) and taking the infimum over $\varphi_k \in Y_{\mathcal{D}_k}$ we get

$$\begin{aligned}\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} \le \frac{1+C_{\mathrm{eq}}}{\underline{\lambda}} & W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) \\ &+ C_{\mathrm{eq}}(1+\kappa(A)) S_{\widehat{\mathcal{D}}_k,J,\kappa_k}(u).\end{aligned}$$

We conclude using Lemma 2.22 and taking the inf over ξ_k. ☐

Lemma 3.5. *Let $((\kappa_k, \hat{\vartheta}_k))_{k=1}^M$ be the sequence defined by the local elliptic scheme (3.1a)–(3.1c). Then for $k \ge 2$*

$$\min_{\xi_k \in Z_{\mathcal{D}_k}} (S_{\widehat{\mathcal{D}}_k,J,\xi_k}(u) + C_\partial|\kappa_k - \xi_k|_{\partial\Omega_k^-}) \le 2C_{\mathrm{i}}\left(\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_{k-1}}\hat{\vartheta}_{k-1} - \nabla u\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}}\right),$$

where C_i is defined in Lemma 3.1.

Proof. Taking $\xi_k = \kappa_k$ we have

$$\min_{\xi_k \in Z_{\mathcal{D}_k}} (S_{\widehat{\mathcal{D}}_k,J,\xi_k}(u) + C_\partial|\kappa_k - \xi_k|_{\partial\Omega_k^-}) \le S_{\widehat{\mathcal{D}}_k,J,\kappa_k}(u).$$

Since $\kappa_k = (\kappa_{k-1} + \hat{\vartheta}_{k-1})\chi_{\Omega\setminus\Omega_k}$ by Lemma 3.1 there exists $\hat{\varphi}_k \in Y_{\mathcal{D}_k}$ satisfying

$$\begin{aligned}\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\varphi}_k - \nabla_{\widehat{\mathcal{D}}_{k-1},\kappa_{k-1}}\hat{\vartheta}_{k-1}\|_{L^2(\Omega_k)^d} &\le C_{\mathrm{i}}|\hat{\vartheta}_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}}, \\ |\hat{\varphi}_k|_{\widehat{J}(k),\kappa_k} &\le C_{\mathrm{i}}|\hat{\vartheta}_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}}\end{aligned}$$

and so

$$\begin{aligned}S_{\widehat{\mathcal{D}}_k,J,\kappa_k}(u) &= \inf_{\varphi \in Y_{\mathcal{D}_k}} \left(\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\varphi - \nabla u\|_{L^2(\Omega_k)^d} + |\varphi|_{\widehat{J}(k),\kappa_k}\right) \\ &\le \|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\varphi}_k - \nabla u\|_{L^2(\Omega_k)^d} + |\hat{\varphi}_k|_{\widehat{J}(k),\kappa_k} \\ &\le \|\nabla_{\widehat{\mathcal{D}}_k,\kappa_{k-1}}\hat{\vartheta}_{k-1} - \nabla u\|_{L^2(\Omega_k)^d} + 2C_{\mathrm{i}}|\hat{\vartheta}_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}}.\end{aligned}$$

☐

Let $\mathcal{H} \subset \mathbb{R}_+$ be a countable set with zero as only accumulation point. For each $h \in \mathcal{H}$ we consider a polytopal mesh sequence $(\mathfrak{T}_{h,k})_{k=1}^M = ((\mathcal{M}_{h,k}, \mathcal{F}_{h,k}, \mathcal{P}_{h,k}))_{k=1}^M$ satisfying Assumption 2.18 with $h = \max_{k=1,\ldots,M} h_{\mathcal{M}_{h,k}}$, where $h_{\mathcal{M}_{h,k}} = \max\{h_K : K \in \mathcal{M}_{h,k}\}$. Let $\mathcal{D}_{h,k}$ and $\widehat{\mathcal{D}}_{h,k}$ be the global and local SWDGGD schemes given by those meshes $\mathfrak{T}_{h,k}$. In the following the index h in $\mathcal{D}_{h,k}$ and $\widehat{\mathcal{D}}_{h,k}$ is left out of notation for the sake of simplicity.

Theorem 3.6. *Let $\mathcal{D}_k$ and $\widehat{\mathcal{D}}_k$ be global and local SWDGGD. Let $((\kappa_k, \hat{\vartheta}_k))_{k=1}^M$ be the sequence defined by the local elliptic scheme (3.1a)–(3.1c) and $u \in H_0^1(\Omega)$ the exact solution to equation (2.2). Then for $k = 1, \ldots, M$*

$$\lim_{h\to 0} \|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{\mathcal{J}}(k),\kappa_k} = 0. \tag{3.7a}$$

If moreover $u \in H_0^1(\Omega) \cap H^2(\Omega)$, the coefficients of A are Lipschitz continuous and $\boldsymbol{F} \in H^1(\Omega)^d$ there exists C_1, C_2, C_3 depending on α, ℓ, d, ρ, C_{r}, $|\Omega|$, A, $\boldsymbol{F}$ and u such that

$$\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{\mathcal{J}}(k),\kappa_k} \leq C_1 h, \tag{3.7b}$$

$$\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{\mathcal{J}}(k),\kappa_k} \leq C_2 h_{\widehat{\mathcal{M}}_k} + C_3 |\kappa_k - \xi_k|_{\partial\Omega_k^-}, \tag{3.7c}$$

where $\xi_k = \operatorname{argmin}_{\xi \in Z_{\mathcal{D}_k}} S_{\widehat{\mathcal{D}}_k,J,\xi}(u)$.

Remark 3.7. The above theorem gives three important results. The first one equation (3.7a) asserts that the numerical solution given by the local scheme (3.1a)–(3.1c) converges to the exact solution even under weak regularity of the solution and data. Assuming more regularity we recover in equation (3.7b) the usual convergence rate. In equation (3.7c) we establish that the error on the local domain depends on the local mesh size and the error committed on the artificial boundary.

Proof of Theorem 3.6. Let

$$E_{\widehat{\mathcal{D}}_k} := \|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k} \hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{\mathcal{J}}(k),\kappa_k}.$$

Since $\kappa_1 = 0$ and $Z_{\mathcal{D}_1} = \{0\}$ by Lemma 3.3 we have

$$E_{\widehat{\mathcal{D}}_1} \leq \frac{1 + C_{\mathrm{eq}}}{\underline{\lambda}} W_{\widehat{\mathcal{D}}_1}(A\nabla u + \boldsymbol{F}) + C_A S_{\widehat{\mathcal{D}}_1,J,0}(u)$$

and by Lemmas 3.3 and 3.5 we have, for $k \geq 2$,

$$E_{\widehat{\mathcal{D}}_k} \leq \frac{1 + C_{\mathrm{eq}}}{\underline{\lambda}} W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) + 2C_{\mathrm{i}} C_A E_{\widehat{\mathcal{D}}_{k-1}}.$$

Let $\alpha = 2C_{\mathrm{i}} C_A$, since $S_{\widehat{\mathcal{D}}_1,J,0}(u) \leq S_{\mathcal{D}_1,J}(u)$ it holds

$$\begin{aligned} E_{\widehat{\mathcal{D}}_k} &\leq \alpha^{k-1} E_{\widehat{\mathcal{D}}_1} + \frac{1 + C_{\mathrm{eq}}}{\underline{\lambda}} \sum_{j=2}^{k} \alpha^{k-j} W_{\widehat{\mathcal{D}}_j}(A\nabla u + \boldsymbol{F}) \\ &\leq C_A \alpha^{k-1} S_{\mathcal{D}_1,J}(u) + \frac{1 + C_{\mathrm{eq}}}{\underline{\lambda}} \sum_{j=1}^{k} \alpha^{k-j} W_{\widehat{\mathcal{D}}_j}(A\nabla u + \boldsymbol{F}). \end{aligned} \tag{3.8}$$

We have thus proved equation (3.7a) thanks to equation (3.8), Lemma 2.16 and the limit conformity of $\widehat{\mathcal{D}}_j$ for $j = 1, \ldots, k$ (we recall that $\widehat{\mathcal{D}}_j$ is a SWDGGD and hence a sequence of $\widehat{\mathcal{D}}_j$ is limit conforming). Under the additional assumptions on the data, from Lemmas 2.13 and 2.14 for $\widehat{\mathcal{D}}_k$ we have

$$\begin{aligned} S_{\mathcal{D}_1,J}(u) &\leq C_S h_{\mathcal{M}_1} \|u\|_{H^2(\Omega)}, \\ W_{\widehat{\mathcal{D}}_k}(A\nabla u + \boldsymbol{F}) &\leq C_W h_{\widehat{\mathcal{M}}_k} \|A\nabla u + \boldsymbol{F}\|_{H^1(\Omega_k)^d} \end{aligned} \tag{3.9}$$

and so

$$E_{\widehat{\mathcal{D}}_k} \leq C_A \alpha^{k-1} C_S h_{\mathcal{M}_1} \|u\|_{H^2(\Omega)} + \frac{1+C_{\text{eq}}}{\underline{\lambda}} \sum_{j=1}^{k} \alpha^{k-1} C_W h_{\widehat{\mathcal{M}}_j} \|A\nabla u + \boldsymbol{F}\|_{H^1(\Omega_j)^d},$$

which implies equation (3.7b) with

$$C_1 := C_A \alpha^{k-1} C_S \|u\|_{H^2(\Omega)} + \frac{1+C_{\text{eq}}}{\underline{\lambda}} \sum_{j=1}^{k} \alpha^{k-1} C_W \|A\nabla u + \boldsymbol{F}\|_{H^1(\Omega_j)^d}.$$

Let $\xi_k = \operatorname{argmin}_{\xi \in Z_{\mathcal{D}_k}} S_{\widehat{\mathcal{D}}_k,J,\xi}(u)$, it holds

$$\min_{\xi \in Z_{\mathcal{D}_k}} (S_{\widehat{\mathcal{D}}_k,J,\xi}(u) + C_\partial |\kappa_k - \xi|_{\partial\Omega_k^-}) \leq S_{\widehat{\mathcal{D}}_k,J,\xi_k}(u) + C_\partial |\kappa_k - \xi_k|_{\partial\Omega_k^-},$$

using Lemma 2.21 we get $S_{\widehat{\mathcal{D}}_k,J,\xi_k}(u) \leq C_S h_{\widehat{\mathcal{M}}_k} \|u\|_{H^2(\Omega_k)}$ and again from Lemma 3.3 and equation (3.9) we obtain the bound equation (3.7c) with

$$C_2 := C_A C_S \|u\|_{H^2(\Omega_k)} + \frac{C_W}{\underline{\lambda}} \|A\nabla u + \boldsymbol{F}\|_{H^1(\Omega_k)^d}, \tag{3.10}$$

where $C_3 = C_A C_\partial$. □

3.2. Improved local estimate

Under stronger conditions and using the pointwise error estimates proved in [7] we can further improve the local estimate equation (3.7c) for $k = 2$.

Let $\boldsymbol{z} \in \Omega$, the weight function $\sigma_{\boldsymbol{z},h}(\boldsymbol{x}) = h/(h + |\boldsymbol{x} - \boldsymbol{z}|)$ and $\|\cdot\|_{W^{2,\infty}_{\boldsymbol{z},h}(\Omega)}$ a weighted Sobolev norm defined by

$$\|v\|_{W^{2,\infty}_{\boldsymbol{z},h}(\Omega)} = \max_{i=0,1,2} |v|_{W^{i,\infty}_{\boldsymbol{z},h}}, \qquad |v|_{W^{i,\infty}_{\boldsymbol{z},h}} = \max_{|\alpha|=i} \|\sigma_{\boldsymbol{z},h} \frac{\partial^\alpha v}{\partial \boldsymbol{x}^\alpha}\|_{L^\infty(\Omega)}.$$

We will use the following lemma, which is a version of Corollary 5.5 from [7].

Lemma 3.8. *Let $A = aI_d$ with $I_d \in \mathbb{R}^{d\times d}$ the identity matrix and $a > 0$. Let $u \in W^{1,\infty}_0(\Omega) \cap W^{2,\infty}(\Omega)$ be solution of equation (2.1) with $f \in L^2(\Omega)$, $\vartheta_1 \in X_{\mathcal{D}_1}$ solution of equation (3.1c). Then there is a constant $C_\infty > 0$ such that for any $\boldsymbol{z} \in \overline{\Omega}$*

$$|u(\boldsymbol{z}) - \Pi_{\mathcal{D}_1}\vartheta_1(\boldsymbol{z})| \leq C_\infty h^2 \log(h^{-1}) \|u\|_{W^{2,\infty}_{\boldsymbol{z},h}(\Omega)}. \tag{3.11}$$

Applying Lemma 3.8 to equation (3.7c) we obtain a better bound on the local error for $k = 2$, as explained in the following theorem.

Theorem 3.9. *Let $u \in W^{1,\infty}_0(\Omega) \cap W^{2,\infty}(\Omega)$ be solution of equation (2.1) with $A = aI_d$ and $f \in L^2(\Omega)$ as in Lemma 3.8. Let $\mathcal{D}_k$ and $\widehat{\mathcal{D}}_k$ be global and local SWDGGD schemes and $((\kappa_k, \hat{\vartheta}_k))_{k=1}^2$ the sequence defined by the local elliptic scheme (3.1a)–(3.1c). Under the assumption that $h \leq C \min_{K\in\mathcal{M}_1} h_K$ with $C > 0$ independent of h, there exists C_4 independent of u and h such that*

$$\begin{aligned} \|\nabla u - \nabla_{\widehat{\mathcal{D}}_2,\kappa_2} \hat{\vartheta}_2\|_{L^2(\Omega_2)^d} &+ |\hat{\vartheta}_2|_{\widehat{J}(2),\kappa_2} \\ &\leq C_2 h_{\widehat{\mathcal{M}}_2} + C_4 \left(h_{\widehat{\mathcal{M}}_2} |u|_{H^2(D_2)} + h^{3/2} \log(h^{-1}) \sup_{\boldsymbol{y}\in\partial\Omega_2\setminus\partial\Omega} \|u\|_{W^{2,\infty}_{\boldsymbol{y},h}(\Omega)} \right), \end{aligned} \tag{3.12}$$

where D_2 is a neighborhood of Ω_2 specified below.

Remark 3.10. Equation (3.12) bounds the error in the local domain Ω_2 and has three terms in the right. From equation (3.10) we see that the first term depends on u and $\boldsymbol{F}$ in Ω_2. The second term depends on u in a small neighborhood of Ω_2. The last term depends on the regularity of u in the whole domain, but it is of higher order and is measured in a weighted norm which weight is $\mathcal{O}(1)$ close to the artificial boundary and $\mathcal{O}(h)$ far from it. Hence the error in Ω_2 depends mostly on the regularity of u and $\boldsymbol{F}$ inside or very close to Ω_2.

Proof. First we observe that equation (3.7c) for $k=2$ is valid with $\xi_2 \in Z_{\mathcal{D}_2}$ such that $\Pi_{\mathcal{D}_2}\xi_2$ is the orthogonal projection of u onto $\Pi_{\mathcal{D}_2}Z_{\mathcal{D}_2}$, indeed even for this choice of ξ_2 we still have $S_{\widehat{\mathcal{D}}_2,J,\xi_2}(u) \leq C_S h_{\widehat{\mathcal{M}}_2}\|u\|_{H^2(\Omega_2)}$. Let $K \in \widehat{\mathcal{M}}_2$, $T \in \mathcal{M}_2 \setminus \widehat{\mathcal{M}}_2$ and $\sigma \in \mathcal{F}_K \cap \mathcal{F}_T$. From Assumption 2.18b we have $K, T \in \mathcal{M}_1$ and Assumption 2.18d implies $h_K \leq \rho h_T$. There exists C_Π ([8], Lem. 1.59) independent of u, T and h_K such that

$$\int_\sigma |\Pi_{\overline{T}}\xi_2 - u|(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} \leq C_\Pi h_K^3 |u|^2_{H^2(T)}.$$

Using Assumption 2.18d we obtain $1/d_{K,\sigma} \leq \rho/h_K$, hence

$$\sum_{\substack{K\in\widehat{\mathcal{M}}_2\\ T\in\mathcal{M}_2\setminus\widehat{\mathcal{M}}_2}} \sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T} \frac{1}{d_{K,\sigma}} \int_\sigma |\Pi_{\overline{T}}\xi_2 - u|(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} \leq C_\Pi \rho h^2_{\widehat{\mathcal{M}}_2} |u|^2_{H^2(D_2)},$$

with $D_2 = \cup_{\{T\in\mathcal{M}_2\setminus\widehat{\mathcal{M}}_2 : \partial T\cap\partial K\neq\emptyset, K\in\widehat{\mathcal{M}}_2\}} T$. From Lemma 3.8 we have

$$\int_\sigma |u - \Pi_{\overline{T}}\kappa_2|(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} = \int_\sigma |u - \Pi_{\overline{T}}\vartheta_1|(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} \leq |\sigma| C_\infty^2 h^4 \log(h^{-1})^2 \sup_{\boldsymbol{y}\in\sigma} \|u\|^2_{W^{2,\infty}_{\boldsymbol{y},h}(\Omega)}.$$

Since $h \leq C \min_{K\in\mathcal{M}_1} h_K$ it follows that $1/d_{K,\sigma} \leq C\rho/h$ and thus

$$\sum_{\substack{K\in\widehat{\mathcal{M}}_2\\ T\in\mathcal{M}_2\setminus\widehat{\mathcal{M}}_2}} \sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T} \frac{1}{d_{K,\sigma}} \int_\sigma |u - \Pi_{\overline{T}}\kappa_2|(\boldsymbol{y})^2 \,\mathrm{d}\boldsymbol{y} \leq |\partial\Omega_2 \setminus \partial\Omega| C_\infty^2 C\rho h^3 \log(h^{-1})^2 \sup_{\boldsymbol{y}\in\partial\Omega_2\setminus\partial\Omega} \|u\|^2_{W^{2,\infty}_{\boldsymbol{y},h}(\Omega)}.$$

Applying a triangle inequality on $|\kappa_2 - \xi_2|_{\partial\Omega_2^-}$ in equation (3.7c) we get equation (3.12). □

3.3. *A priori* error analysis for the global solution

We next study the error on the whole domain Ω of the numerical solution $\vartheta_k \in X_{\mathcal{D}_k}$ defined by our local scheme equation (3.1). The next Lemma 3.11 is the main ingredient for the global error bound.

Lemma 3.11. *Let $u \in H_0^1(\Omega)$ be solution to equation (2.2) and $(\vartheta_k)_{k=1}^M$ be the sequence defined by scheme (3.1a) to (3.1c). Then we have*

$$\begin{aligned} \|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)} \leq\ & C_5(\|\nabla u - \nabla_{\mathcal{D}_{k-1}}\vartheta_{k-1}\|_{L^2(\Omega)^d} + |\vartheta_{k-1}|_{J(k-1)}) \\ & + C_5(\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k}). \end{aligned}$$

where $C_5 = \sqrt{2}(1 + C_\psi)(1 + \sqrt{2}C_{\omega,k})$.

Proof. We have

$$\begin{aligned}\|\nabla u-\nabla_{\mathcal{D}_k}\vartheta_k\|^2_{L^2(\Omega)^d} &= \sum_{K\in\mathcal{M}_k}\sum_{\sigma\in\mathcal{F}_K}\int_{D_{K,\sigma}}|\nabla u(\boldsymbol{x})-\nabla_{\overline{K}}\vartheta_k(\boldsymbol{x})-\psi(s)\frac{[\vartheta_k]_{K,\sigma}(\boldsymbol{y})}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma}|^2\,\mathrm{d}\boldsymbol{x}\\ &= \sum_{T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_T}\int_{D_{T,\sigma}}|\nabla u(\boldsymbol{x})-\nabla_{\overline{T}}\vartheta_k(\boldsymbol{x})-\psi(s)\frac{[\vartheta_k]_{T,\sigma}(\boldsymbol{y})}{d_{T,\sigma}}\boldsymbol{n}_{T,\sigma}|^2\,\mathrm{d}\boldsymbol{x}\\ &\quad+\sum_{K\in\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_K}\int_{D_{K,\sigma}}|\nabla u(\boldsymbol{x})-\nabla_{\overline{K}}\vartheta_k(\boldsymbol{x})-\psi(s)\frac{[\vartheta_k]_{K,\sigma}(\boldsymbol{y})}{d_{K,\sigma}}\boldsymbol{n}_{K,\sigma}|^2\,\mathrm{d}\boldsymbol{x}\\ &= I+II.\end{aligned}$$

For the first term I, we have the following considerations. Let $T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k$, then $T\in\mathcal{M}_{k-1}$ and $\nabla_{\overline{T}}\vartheta_k=\nabla_{\overline{T}}\vartheta_{k-1}$. Let $\sigma\in\mathcal{F}_T$, if $\sigma\notin\widehat{\mathcal{F}}_{k,b}$ then $[\vartheta_k]_{T,\sigma}=[\vartheta_{k-1}]_{T,\sigma}$. If $\sigma\in\widehat{\mathcal{F}}_{k,b}$ then $\sigma=\partial K\cap\partial T$ with $K\in\widehat{\mathcal{M}}_k$ and by Assumption 2.18b $K\in\widehat{\mathcal{M}}_{k-1}$. Using equations (3.1a) and (3.1b) we have

$$\begin{aligned}[\vartheta_k]_{T,\sigma}-[\vartheta_{k-1}]_{T,\sigma}&=\omega_{T,\sigma}(\Pi_{\overline{K}}\hat{\vartheta}_k-\Pi_{\overline{T}}\vartheta_{k-1})-\omega_{T,\sigma}(\Pi_{\overline{K}}\vartheta_{k-1}-\Pi_{\overline{T}}\vartheta_{k-1})\\ &=\omega_{T,\sigma}(\Pi_{\overline{K}}\hat{\vartheta}_k-\Pi_{\overline{K}}\vartheta_{k-1}).\end{aligned}$$

Next, adding and removing $[\vartheta_{k-1}]_{T,\sigma}$ from $[\vartheta_k]_{T,\sigma}$ we get

$$\begin{aligned}I\le 2&\sum_{T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_T}\int_{D_{T,\sigma}}|\nabla u(\boldsymbol{x})-\nabla_{\overline{T}}\vartheta_{k-1}(\boldsymbol{x})-\psi(s)\frac{[\vartheta_{k-1}]_{T,\sigma}(\boldsymbol{y})}{d_{T,\sigma}}\boldsymbol{n}_{T,\sigma}|^2\,\mathrm{d}\boldsymbol{x}\\ &+2\sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_T\cap\mathcal{F}_K}\int_{D_{T,\sigma}}|\psi(s)\omega_{T,\sigma}\frac{(\Pi_{\overline{K}}\hat{\vartheta}_k-\Pi_{\overline{K}}\vartheta_{k-1})(\boldsymbol{y})}{d_{T,\sigma}}|^2\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Since $\omega_{T,\sigma}\le 1$, using a change of variables we have

$$\begin{aligned}I&\le 2\|\nabla u-\nabla_{\mathcal{D}_{k-1}}\vartheta_{k-1}\|^2_{L^2(\Omega\setminus\Omega_k)}\\ &\quad+2C_\psi^2\sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_T\cap\mathcal{F}_K}\frac{1}{d_{T,\sigma}}\int_\sigma(\Pi_{\overline{K}}\hat{\vartheta}_k-\Pi_{\overline{K}}\vartheta_{k-1})(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\\ &=2\|\nabla u-\nabla_{\mathcal{D}_{k-1}}\vartheta_{k-1}\|^2_{L^2(\Omega\setminus\Omega_k)}+2C_\psi^2|\hat{\vartheta}_k-\vartheta_{k-1}|^2_{\partial\Omega_k^+},\end{aligned}$$

where for $\phi_k\in X_{\mathcal{D}_k}$

$$|\phi_k|^2_{\partial\Omega_k^+}:=\sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T}\frac{1}{d_{T,\sigma}}\int_\sigma\Pi_{\overline{K}}\phi_k(\boldsymbol{y})^2\,\mathrm{d}\boldsymbol{y}.$$

For the second term II we have $[\vartheta_k]_{K,\sigma}=[\hat{\vartheta}_k]_{K,\sigma,\kappa_k}$ if $\sigma\in\widehat{\mathcal{F}}_{k,i}$ or $\sigma\in\widehat{\mathcal{F}}_{k,b}\cap\mathcal{F}_{k,b}$ and $[\vartheta_k]_{K,\sigma}=\omega_{K,\sigma}[\hat{\vartheta}_k]_{K,\sigma,\kappa_k}$ if $\sigma\in\widehat{\mathcal{F}}_{k,b}\setminus\mathcal{F}_{k,b}$. Hence

$$\begin{aligned}II\le{}&2\|\nabla u-\nabla_{\widehat{\mathcal{D}}_{k,\kappa_k}}\hat{\vartheta}_k\|^2_{L^2(\Omega_k)^d}\\ &+2\sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_T\cap\mathcal{F}_K}\int_{D_{K,\sigma}}|\psi(s)\frac{(1-\omega_{K,\sigma})[\hat{\vartheta}_k]_{K,\sigma,\kappa_k}(\boldsymbol{y})}{d_{K,\sigma}}|^2\,\mathrm{d}\boldsymbol{x}\end{aligned}$$

$$\leq 2\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|^2_{L^2(\Omega_k)^d} + 2C_\psi^2 \sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}} \sum_{\sigma\in\mathcal{F}_T\cap\mathcal{F}_K} \frac{1}{d_{K,\sigma}}\int_\sigma [\hat{\vartheta}_k]_{K,\sigma,\kappa_k}(\boldsymbol{y})^2\,\mathrm{d}\boldsymbol{y}$$

$$\leq 2\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|^2_{L^2(\Omega_k)^d} + 2C_\psi^2|\hat{\vartheta}_k|^2_{\widehat{J}(k),\kappa_k}.$$

We then obtain

$$\|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|^2_{L^2(\Omega)^d} \leq 2\|\nabla u - \nabla_{\mathcal{D}_{k-1}}\vartheta_{k-1}\|^2_{L^2(\Omega)^d} + 2\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|^2_{L^2(\Omega_k)^d} + 2C_\psi^2|\hat{\vartheta}_k - \vartheta_{k-1}|^2_{\partial\Omega_k^+} + 2C_\psi^2|\hat{\vartheta}_k|^2_{\widehat{J}(k),\kappa_k}. \tag{3.13}$$

Using similar arguments, we have

$$\begin{aligned}
|\vartheta_k|^2_{J(k)} &= \sum_{K\in\mathcal{M}_k}\sum_{\sigma\in\mathcal{F}_K}\frac{1}{d_{K,\sigma}}\int_\sigma[\vartheta_k]_{K,\sigma}(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\\
&\leq \sum_{K\in\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_K}\frac{1}{d_{K,\sigma}}\int_\sigma[\hat{\vartheta}_k]_{K,\sigma,\kappa_k}(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\\
&\quad + 2\sum_{T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}\sum_{\sigma\in\mathcal{F}_T}\frac{1}{d_{T,\sigma}}\int_\sigma[\vartheta_{k-1}]_{T,\sigma}(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\\
&\quad + 2\sum_{\substack{K\in\widehat{\mathcal{M}}_k\\ T\in\mathcal{M}_k\setminus\widehat{\mathcal{M}}_k}}\sum_{\sigma\in\mathcal{F}_K\cap\mathcal{F}_T}\frac{1}{d_{T,\sigma}}\int_\sigma\omega_{T,\sigma}^2(\Pi_{\overline{K}}\hat{\vartheta}_k - \Pi_{\overline{K}}\vartheta_{k-1})(\boldsymbol{y})^2\mathrm{d}\boldsymbol{y}\\
&\leq |\hat{\vartheta}_k|^2_{\widehat{J}(k),\kappa_k} + 2\,|\vartheta_{k-1}|^2_{J(k)} + 2|\hat{\vartheta}_k - \vartheta_{k-1}|^2_{\partial\Omega_k^+}.
\end{aligned} \tag{3.14}$$

Combining equations (3.13) and (3.14) we get

$$\begin{aligned}
\|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)} &\leq \sqrt{2}\Big(\|\nabla u - \nabla_{\mathcal{D}_{k-1}}\vartheta_{k-1}\|_{L^2(\Omega)^d}\\
&\quad + \|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + C_\psi|\hat{\vartheta}_k - \vartheta_{k-1}|_{\partial\Omega_k^+}\\
&\quad + C_\psi|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k}\Big) + \sqrt{2}\left(|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} + |\vartheta_{k-1}|_{J(k)} + |\hat{\vartheta}_k - \vartheta_{k-1}|_{\partial\Omega_k^+}\right)\\
&\leq \sqrt{2}\Big(\|\nabla u - \nabla_{\mathcal{D}_{k-1}}\vartheta_{k-1}\|_{L^2(\Omega)^d} + |\vartheta_{k-1}|_{J(k)} + \|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d}\\
&\quad + (1+C_\psi)|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k}\Big) + \sqrt{2}(1+C_\psi)|\hat{\vartheta}_k - \vartheta_{k-1}|_{\partial\Omega_k^+}.
\end{aligned}$$

Since we easily get

$$|\hat{\vartheta}_k - \vartheta_{k-1}|_{\partial\Omega_k^+} \leq \sqrt{2}C_{\omega,k}(|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} + |\vartheta_{k-1}|_{J(k)})$$

we obtain

$$\begin{aligned}
\|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)} &\leq \sqrt{2}(1+C_\psi)(1+\sqrt{2}C_{\omega,k})\left(\|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_{k-1}\|_{L^2(\Omega)^d} + |\vartheta_{k-1}|_{J(k)}\right)\\
&\quad + \sqrt{2}(1+C_\psi)(1+\sqrt{2}C_{\omega,k})\left(\|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k}\right),
\end{aligned}$$

and the lemma is proved. □

Theorem 3.12. *Let $u \in H_0^1(\Omega)$ be solution of equation (2.2) and $(\vartheta_k)_{k=1}^M$ be the sequence defined by scheme (3.1a) to (3.1c). Then for $k = 1, \dots, M$*

$$\lim_{h\to 0} \|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)} = 0. \tag{3.15}$$

If moreover $u \in H_0^1(\Omega) \cap H^2(\Omega)$, the coefficients of A are Lipschitz continuous and $\boldsymbol{F} \in H^1(\Omega)^d$ there exists C_6 depending on α, ℓ, d, ρ, C_{r}, $|\Omega|$, A, $\boldsymbol{F}$ and u such that

$$\|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)} \le C_6 h. \tag{3.16}$$

Proof. Follows from a recursive argument, Theorem 3.6 and Lemma 3.11. □

4. *A priori* error analysis for quasilinear problems

In this section we analyze our local SWDGGD scheme for a class of non linear problems satisfying Assumption 2.2. For the sake of simplicity we consider $f \in L^2(\Omega)$, but the algorithm and the results can easily be generalized to $f \in H^{-1}(\Omega)$. Under Assumption 2.2 there exists a unique weak solution $u \in H_0^1(\Omega)$ of

$$\int_\Omega A(u)\nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } v \in H_0^1(\Omega). \tag{4.1}$$

The local elliptic scheme for problem equation (4.1) is given as follows. Set $\vartheta_1 \in X_{\mathcal{D}_1}$ a solution of

$$\int_\Omega A(\Pi_{\mathcal{D}_1}\vartheta_1)\nabla_{\mathcal{D}_1}\vartheta_1 \cdot \nabla_{\mathcal{D}_1}\phi_1 \,\mathrm{d}\boldsymbol{x} = \int_\Omega f\, \Pi_{\mathcal{D}_1}\phi_1 \,\mathrm{d}\boldsymbol{x} \tag{4.2a}$$

for all $\phi_1 \in X_{\mathcal{D}_1}$. For $k \ge 2$ we set

$$\vartheta_k = \kappa_k + \hat{\vartheta}_k, \tag{4.2b}$$

where $\kappa_k \in Z_{\mathcal{D}_k}$ is given by

$$\kappa_k = \vartheta_{k-1}\chi_{\Omega\setminus\Omega_k} \tag{4.2c}$$

and $\hat{\vartheta}_k \in Y_{\mathcal{D}_k}$ is solution of

$$\int_{\Omega_k} A(\Pi_{\widehat{\mathcal{D}}_{k-1}}\hat{\vartheta}_{k-1})\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k \cdot \nabla_{\widehat{\mathcal{D}}_k}\varphi_k \,\mathrm{d}\boldsymbol{x} = \int_{\Omega_k} f\, \Pi_{\widehat{\mathcal{D}}_k}\varphi_k \,\mathrm{d}\boldsymbol{x} \tag{4.2d}$$

for all $\varphi_k \in Y_{\mathcal{D}_k}$.

We define again a subset $\mathcal{H} \subset \mathbb{R}_+$ with zero as only accumulation point and for each $h \in \mathcal{H}$ a sequence of meshes $(\mathfrak{T}_{h,k})_{k=1}^M$ satisfying Assumption 2.18 with $h = \max_{k=1,\dots,M} h_{\mathcal{M}_k}$. From $(\mathfrak{T}_{h,k})_{k=1}^M$ we define $(\widehat{\mathfrak{T}}_{h,k})_{k=1}^M$ as explained after Assumption 2.18. We consider the weighted gradient discretization methods $\mathcal{D}_{h,k}$, $\widehat{\mathcal{D}}_{h,k}$ deriving from $\mathfrak{T}_{h,k}$ and $\widehat{\mathfrak{T}}_{h,k}$, as defined in Section 2.3. The following theorem establishes the convergence of the non linear local SWDGGD scheme equation (4.2). The proof is inspired by a result in Chapter 2.1.4 of [10] for global non linear schemes.

Theorem 4.1. *For any $h \in \mathcal{H}$ there exists exactly one $\vartheta_{h,1} \in \mathcal{D}_{h,1}$ solution to equation (4.2a). Moreover, $\Pi_{\mathcal{D}_{h,1}}\vartheta_{h,1}$ converges strongly in $L^2(\Omega)$ to a solution u of equation (4.1) and $\nabla_{\mathcal{D}_{h,1}}\vartheta_{h,1}$ converges strongly in $L^2(\Omega)^d$ to ∇u as $h \to 0$.*

We will prove that the same result holds for $\vartheta_{h,k}$ with $k \ge 2$. We start by proving convergence of the local solutions $\hat{\vartheta}_{h,k}$. For simplicity we drop the index h in what follows.

Theorem 4.2. *Let Assumption 2.2 hold, $((\kappa_k,\hat{\vartheta}_k))_{k=1}^M$ be the sequence given by the local schemes (4.2a)–(4.2d) and $u\in H^1_0(\Omega)$ be solution of equation (4.1). Then for $k=1,\ldots,M$*

$$\lim_{h\to 0}\|\nabla u-\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d}=0, \tag{4.3a}$$

$$\lim_{h\to 0}|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k}=0, \tag{4.3b}$$

where the limit is taken for $h\in\mathcal{H}$.

Proof. We will prove equation (4.3) by recursion. For $k=1$ we easily get equation (4.3a), indeed $\kappa_1=0$, $\hat{\vartheta}_1=\vartheta_1$ and by Theorem 4.1 we get

$$\lim_{h\to 0}\|\nabla u-\nabla_{\widehat{\mathcal{D}}_1,\kappa_1}\hat{\vartheta}_1\|_{L^2(\Omega_k)^d}=\lim_{h\to 0}\|\nabla u-\nabla_{\mathcal{D}_1}\vartheta_1\|_{L^2(\Omega)^d}=0. \tag{4.4}$$

Let $\phi_1\in X_{\mathcal{D}_1}$, we have

$$|\hat{\vartheta}_1|_{\widehat{J}(1),\kappa_1}=|\vartheta_1|_{J(1)}\le|\vartheta_1-\phi_1|_{J(1)}+|\phi_1|_{J(1)}\,.$$

From [12] we infer the existence of a constant C_{eq} depending only on α, ℓ, d such that

$$|\vartheta_1-\phi_1|_{J(1)}\le C_{\mathrm{eq}}\|\nabla_{\mathcal{D}_1}\vartheta_1-\nabla_{\mathcal{D}_1}\phi_1\|_{L^2(\Omega)^d},$$

hence

$$|\hat{\vartheta}_1|_{\widehat{J}(1),\kappa_1}\le C_{\mathrm{eq}}\|\nabla_{\mathcal{D}_1}\vartheta_1-\nabla_{\mathcal{D}_1}\phi_1\|_{L^2(\Omega)^d}+|\phi_1|_{J(1)}\,.$$

Taking $\phi_1=\operatorname{argmin}_{\phi\in X_{\mathcal{D}_1}}(\|\Pi_{\mathcal{D}_1}\phi-u\|_{L^2(\Omega)}+\|\nabla_{\mathcal{D}_1}\phi-\nabla u\|_{L^2(\Omega)^d}+|\phi|_{J(1)})$ we get equation (4.3b) for $k=1$ using the triangle inequality, equation (4.4) and Lemma 2.16.

Let $k\ge 2$ and suppose that equation (4.3) holds for $k-1$. By Lemma 3.1 there exists $\overline{\vartheta}_k\in Y_{\mathcal{D}_k}$ satisfying

$$\begin{aligned}\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k-\nabla_{\widehat{\mathcal{D}}_{k-1},\kappa_{k-1}}\hat{\vartheta}_{k-1}\|_{L^2(\Omega_k)^d}&\le C_{\mathrm{i}}|\hat{\vartheta}_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}},\\ |\overline{\vartheta}_k|_{\widehat{J}(k),\kappa_k}&\le C_{\mathrm{i}}|\hat{\vartheta}_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}}.\end{aligned} \tag{4.5}$$

Let $\tilde{\vartheta}_k\in Y_{\mathcal{D}_k}$ be solution of

$$\int_{\Omega_k}A_{k-1}(\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k+\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k)\cdot\nabla_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x}=\int_{\Omega_k}f\,\Pi_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x}$$

for all $\varphi_k\in Y_{\mathcal{D}_k}$, where $A_{k-1}=A(\Pi_{\widehat{\mathcal{D}}_{k-1}}\hat{\vartheta}_{k-1})$. Since $\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}(\overline{\vartheta}_k+\tilde{\vartheta}_k)=\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k+\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k$ it follows that $\hat{\vartheta}_k=\overline{\vartheta}_k+\tilde{\vartheta}_k$. From equation (4.3) for $k-1$ and equation (4.5) it follows that $\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k\to\nabla u$ strongly in $L^2(\Omega_k)^d$. Thus if $\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\to 0$ strongly in $L^2(\Omega_k)^d$ then $\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\to\nabla u$ strongly in $L^2(\Omega_k)^d$ and whence equation (4.3a) holds for k. From the coercivity of A

$$\begin{aligned}\underline{\lambda}\|\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|^2_{L^2(\Omega_k)^d}&\le\int_{\Omega_k}A_{k-1}\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x}\\ &=\int_{\Omega_k}f\,\Pi_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x}-\int_{\Omega_k}A_{k-1}\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x}\\ &\le\|f\|_{L^2(\Omega_k)}\|\Pi_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|_{L^2(\Omega_k)}+\overline{\lambda}\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k\|_{L^2(\Omega_k)^d}\|\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|_{L^2(\Omega_k)^d}\\ &\le\left(C_{\mathrm{p}}\|f\|_{L^2(\Omega_k)}+\overline{\lambda}\|\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k\|_{L^2(\Omega_k)^d}\right)\|\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|_{L^2(\Omega_k)^d}\end{aligned}$$

and hence $\|\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|_{L^2(\Omega_k)^d}$ is bounded. It follows from the compactness of $\widehat{\mathcal{D}}_k$ and Lemma 2.15 of [10] that there exists $w \in H_0^1(\Omega_k)$ and a subsequence $\mathcal{H}'$ of $\mathcal{H}$ such that $\Pi_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k \to w$ strongly in $L^2(\Omega_k)$ and $\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k \rightharpoonup \nabla w$ weakly in $L^2(\Omega_k)^d$ as $h \to 0$ with $h \in \mathcal{H}'$. We will show that $w = 0$, that the convergence holds for the whole sequence $\mathcal{H}$ and that $\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k$ converges strongly. Let $v \in H_0^1(\Omega_k)$ and

$$\varphi_k = \operatorname{argmin}_{\varphi \in Y_{\mathcal{D}_k}} \left(\|\Pi_{\widehat{\mathcal{D}}_k}\varphi - v\|_{L^2(\Omega_k)} + \|\nabla_{\widehat{\mathcal{D}}_k}\varphi - \nabla v\|_{L^2(\Omega_k)^d} + |\varphi|_{\mathcal{J}(k),0} \right).$$

Since $\widehat{\mathcal{D}}_k$ is a SWDGGD, from Lemma 2.16 we have that $\Pi_{\widehat{\mathcal{D}}_k}\varphi_k \to v$ strongly in $L^2(\Omega_k)$ and $\nabla_{\widehat{\mathcal{D}}_k}\varphi_k \to \nabla v$ strongly in $L^2(\Omega_k)^d$. From equation (4.3a) $\nabla_{\widehat{\mathcal{D}}_{k-1},\kappa_{k-1}}\hat{\vartheta}_{k-1} \to \nabla u$ strongly in $L^2(\Omega_k)^d$, furthermore by coercivity and consistency we can show that $\Pi_{\widehat{\mathcal{D}}_{k-1}}\hat{\vartheta}_{k-1} \to u$ strongly in $L^2(\Omega_k)$ as well. The same holds for $\overline{\vartheta}_k$. Hence by the non-linear strong convergence lemma in Section D.4 of [10] we obtain

$$\begin{aligned} A(\Pi_{\widehat{\mathcal{D}}_{k-1}}\hat{\vartheta}_{k-1})\nabla_{\widehat{\mathcal{D}}_k}\varphi_k &\to A(u)\nabla v \text{ strongly in } L^2(\Omega_k)^d,\\ A(\Pi_{\widehat{\mathcal{D}}_{k-1}}\hat{\vartheta}_{k-1})\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k &\to A(u)\nabla u \text{ strongly in } L^2(\Omega_k)^d. \end{aligned}$$

It follows from weak-strong convergence lemma in Section D.4 of [10] and symmetry of A that

$$\begin{aligned} \int_{\Omega_k} A(u)\nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega_k} \nabla w\cdot A(u)\nabla v\,\mathrm{d}\boldsymbol{x}\\ &= \lim_{h\to 0}\int_{\Omega_k} \nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\cdot A_{k-1}\nabla_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{4.6}$$

where the limit is for $h \in \mathcal{H}'$. On the other hand we have

$$\begin{aligned} \int_{\Omega_k} A_{k-1}\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega_k} f\,\Pi_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x}\\ &\quad - \int_{\Omega_k} A_{k-1}\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

and taking the limit we get

$$\lim_{h\to 0}\int_{\Omega_k} A_{k-1}\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\varphi_k\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_k} fv\,\mathrm{d}\boldsymbol{x} - \int_{\Omega_k} A(u)\nabla u\cdot\nabla v\,\mathrm{d}\boldsymbol{x} = 0. \tag{4.7}$$

Putting together equations (4.6) and (4.7) and using the symmetry of A_{k-1} we obtain

$$\int_{\Omega_k} A(u)\nabla w\cdot\nabla v\,\mathrm{d}\boldsymbol{x} = 0$$

for all $v \in H_0^1(\Omega_k)$ and so $w = 0$. We can repeat the same reasoning for each subsequence of $\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k$ and obtain the same limit $w = 0$, hence $\Pi_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k \to 0$ strongly in $L^2(\Omega_k)$ and $\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k \rightharpoonup 0$ weakly in $L^2(\Omega_k)^d$ for the whole sequence $\mathcal{H}$. Furthermore

$$\begin{aligned} \int_{\Omega_k} A_{k-1}\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x} &= \int_{\Omega_k} f\,\Pi_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x}\\ &\quad - \int_{\Omega_k} A_{k-1}\nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\overline{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

and so

$$\lim_{h\to 0}\int_{\Omega_k} A_{k-1}\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\cdot\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\,\mathrm{d}\boldsymbol{x} = 0,$$

which shows that $\lim_{h\to 0}\|\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|_{L^2(\Omega_k)^d} = 0$ and hence the strong convergence of $\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k$. It rests to show equation (4.3b), we have

$$\begin{aligned}|\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k} &\leq |\overline{\vartheta}_k|_{\widehat{J}(k),\kappa_k} + |\tilde{\vartheta}_k|_{\widehat{J}(k),0}\\ &\leq C_{\mathrm{i}}|\vartheta_{k-1}|_{\widehat{J}(k-1),\kappa_{k-1}} + C_{\mathrm{eq}}\|\nabla_{\widehat{\mathcal{D}}_k}\tilde{\vartheta}_k\|_{L^2(\Omega_k)^d}\end{aligned}$$

and the result follows. □

The next Theorem can be proved with similar arguments as used in Section 3.

Theorem 4.3. *Let Assumption 2.2 hold. Consider $(\vartheta_k)_{k=1}^M$, the sequence given by the local scheme (4.2a)–(4.2d) and $u \in H_0^1(\Omega)$ the solution of equation (4.1). Then for $k = 1,\dots,M$, we have*

$$\lim_{h\to 0}\|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)} = 0,$$

where the limit is taken for $h \in \mathcal{H}$.

5. Numerical experiments

In the following numerical experiments, we will use examples where the subdomains $\{\Omega_k\}_{k=1}^M$ and meshes $\{\mathfrak{T}_k\}_{k=1}^M$ are defined *a priori*. This might be realistic in applications where the location of the singularities or high contrast of the solution are known *a priori*. When such *a priori* knowledge is not available, we should use instead a posteriori error estimators for detecting the local subdomains. This is developed in a companion paper [1].

In what follows $\{\Omega_k\}_{k=1}^M$ will be a sequence of embedded domains but we recall that this is not a requirement. In the examples we consider $f \in L^2(\Omega)$ and denote by $\zeta_k \in X_{\mathcal{D}_k}$ the solution of

$$\int_\Omega A(\Pi_{\mathcal{D}_k}\zeta_k)\nabla_{\mathcal{D}_k}\zeta_k\cdot\nabla_{\mathcal{D}_k}\phi_k\,\mathrm{d}\boldsymbol{x} = \int_\Omega f\,\Pi_{\mathcal{D}_k}\phi_k\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \phi_k \in X_{\mathcal{D}_k}, \tag{5.1}$$

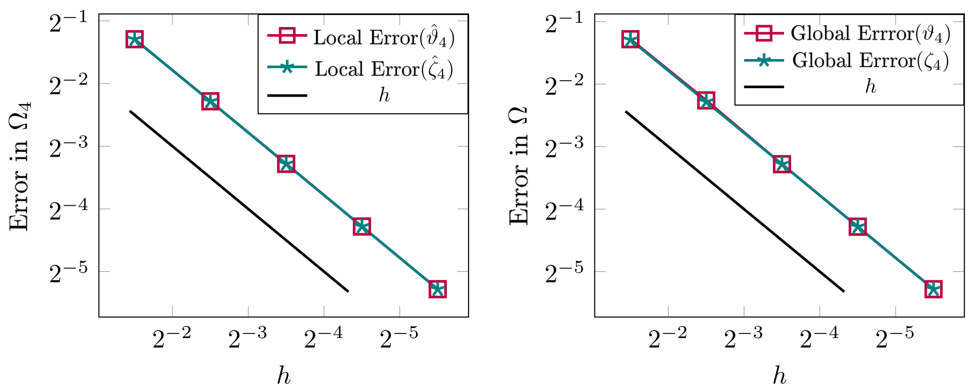
we refer to ζ_k as the classical solution, that is, the one obtained by the usual scheme which solves the equations in the whole domain after each mesh refinement. We can write $\zeta_k = \hat{\zeta}_k + \eta_k$ with $\hat{\zeta}_k \in Y_{\mathcal{D}_k}$ and $\eta_k \in Z_{\mathcal{D}_k}$. We will often compare ϑ_k and $\hat{\vartheta}_k$ the solutions of equation (3.1) or equation (4.2) against ζ_k and $\hat{\zeta}_k$ respectively.

Computational cost. As in our setting, the meshes are defined *a priori* only the most accurate solution ζ_M need to be computed. For the iterative schemes (3.1) and (4.2) instead it is imperative to compute ϑ_k for $k = 1,\dots,M$. If for example a conjugate gradient method is used to solve the linear systems, then the computational cost of the local scheme can be considerably smaller than the classical scheme due to the smaller problems solved on the fine meshes. For nonlinear problems, the local scheme might be faster for any linear solver, as the non linear system is solved only on a coarse mesh (see Sect. 4). This is illustrated in our numerical experiments.

It is useful to define the quantities

$$\begin{aligned}\text{Local Err}(\hat{\vartheta}_k) &:= \|\nabla u - \nabla_{\widehat{\mathcal{D}}_k,\kappa_k}\hat{\vartheta}_k\|_{L^2(\Omega_k)^d} + |\hat{\vartheta}_k|_{\widehat{J}(k),\kappa_k},\\ \text{Global Err}(\vartheta_k) &:= \|\nabla u - \nabla_{\mathcal{D}_k}\vartheta_k\|_{L^2(\Omega)^d} + |\vartheta_k|_{J(k)}.\end{aligned}$$

Similarly we define Local Err$(\hat{\zeta}_k)$ and Global Err(ζ_k) for the local and global error of the classical solutions. The local and cassical schemes have been implemented with the help of the C++ library `libMesh` [18].



(a) *Local error in* Ω_4. (b) *Global error in* Ω.

FIGURE 3. Experiment 5.1: Convergence of the local ϑ_4 and classical ζ_4 solutions letting $h \to 0$.

5.1. Convergence rates

In this example we want to verify the results of Theorem 3.6 (Eqs. (3.7b) and (3.7c)) and of Theorem 3.12 (Eq. (3.16)), hence we consider an example with smooth solution. Let $\Omega = [-1,1] \times [-1,1]$, $A = I_2$ the identity matrix and $f \in L^2(\Omega)$ such that the exact solution is

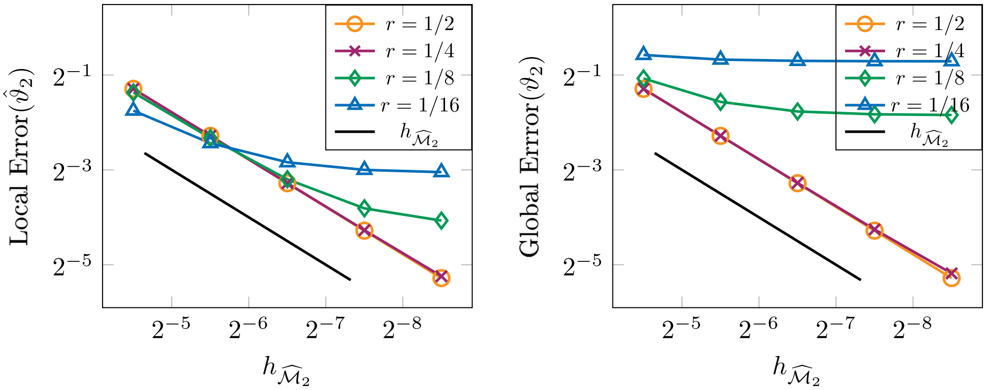
$$u(\boldsymbol{x}) = e^{-120\|\boldsymbol{x}\|_2^2}. \tag{5.2}$$

Let $M = 4$, the local domains are such that $\boldsymbol{x} \in \Omega_k$ if $\|\boldsymbol{x}\|_\infty < (5-k)/4$ for $k = 1, \ldots, 4$.

In the first experiment we want to verify the estimates equations (3.16) and (3.7b), *i.e.* the convergence of the local and global errors with respect to the global mesh size. For a fixed h we consider uniform simplicial meshes $\widehat{\mathcal{M}}_k$ on Ω_k with mesh size $h_{\widehat{\mathcal{M}}_k} = h/2^{k-1}$ and apply the local algorithm equation (3.1), we let $h \to 0$ and verify the convergence rates. From Figures 3a and 3b we see that equations (3.16) and (3.7b) are verified for the local solution ϑ_4. We also see that the classical scheme gives results with the same accuracy as the local scheme, both for the local and the global error. This example also indicates that if the high gradient regions are localized then there is no need of solving the problem in the whole domain after refinements.

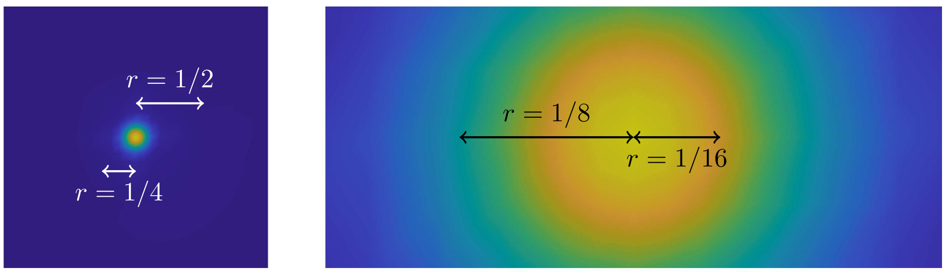
In the next experiment we want to see the influence of the second term (boundary layer term) in the righthand side of equation (3.7c) on Local Error($\hat{\vartheta}_M$). Let $r \in]0,1[$, we set $M = 2$, $\Omega_1 = \Omega$ and $\Omega_2 = [-r,r] \times [-r,r]$. We fix $h_{\mathcal{M}_1} = \sqrt{2}/8$ the mesh size of $\mathcal{M}_1$ and let $h_{\widehat{\mathcal{M}}_2} \to 0$. We plot the results for different values of r (an illustration of this numerical experiment is given in Fig. 5). In Figure 4a we see that when r is large enough the local error scales with the local mesh size. If, instead, r is too small to cover the high gradient regions then the local error saturates very quickly. With $r = 1/8$ we get nice convergence up to $h_{\mathcal{M}_1}/h_{\widehat{\mathcal{M}}_2} = 16$ and with $r = 1/4, 1/2$ we do not see any saturation effects. In Figure 5 we see that $r = 1/16$ is too small to cover the local variations and indeed the local error does not converges. In Figure 4b we plot the total error on Ω. We remark that the error saturates for $r = 1/16, 1/8$. It is interesting to compare the results for $r = 1/8$ in Figures 4a and 4b, in the first one there is a nice convergence while in the second an immediate saturation. This indicates that even if the error outside of Ω_2 is important, it does not propagate quickly into Ω_2. Notice that the results displayed in Figure 4b are not in disagree with equation (3.16) in Theorem 3.12 since in this experiment $h = h_{\mathcal{M}_1}$ is kept constant.

In Figure 6 we plot the results of the same experiment shown in Figure 4 but for ζ_4 instead of ϑ_4. We see that, again, the classical scheme gives similar results.



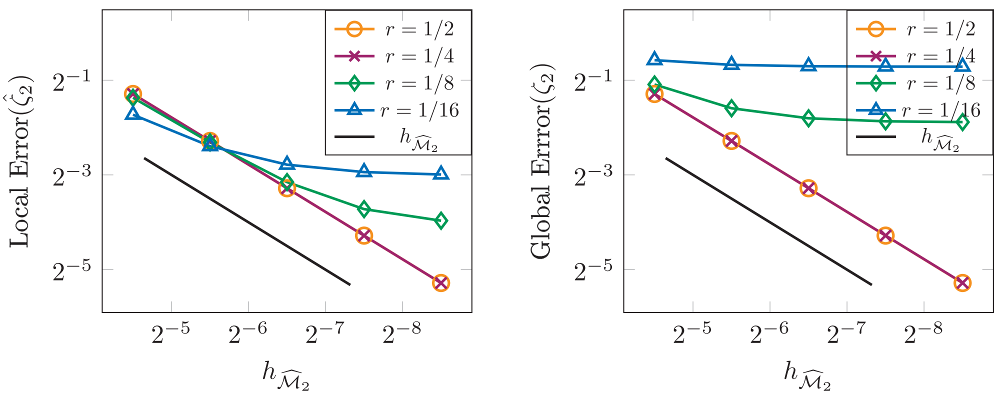
(a) *Local error in* Ω_2. (b) *Global error in* Ω.

FIGURE 4. Experiment 5.1: Effect of the size of Ω_2 on the local solution ϑ_2 when $h_{\widehat{\mathcal{M}}_2} \to 0$.



(a) *Solution in* Ω. (b) *Zoom around the point* $(0,0)$.

FIGURE 5. Experiment 5.1: Solution u from equation (5.2) with the size of domains Ω_2 depending on r.



(a) *Local error in* Ω_2. (b) *Global error in* Ω.

FIGURE 6. Experiment 5.1: Effect of the size of Ω_2 on the classical solution ζ_2 when $h_{\widehat{\mathcal{M}}_2} \to 0$.

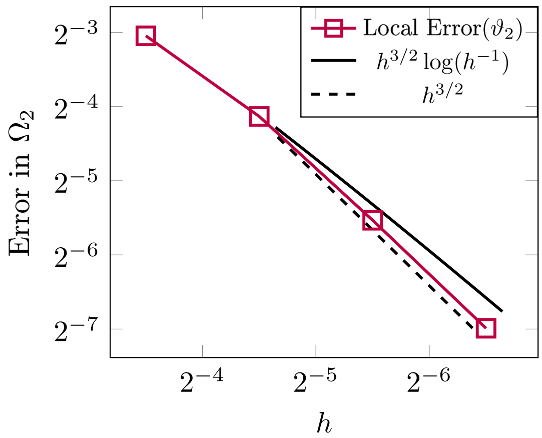


FIGURE 7. Experiment 5.2: Convergence order of artificial boundary conditions error term.

In practice it is desirable to avoid the saturation effects seen in this experiment. For this reason, in [1] we developed a posteriori error estimators for the local scheme. These error estimators are capable of detecting the large error regions and defining the local domains accordingly and can thus mitigate saturation errors.

5.2. Influence of artificial boundary conditions

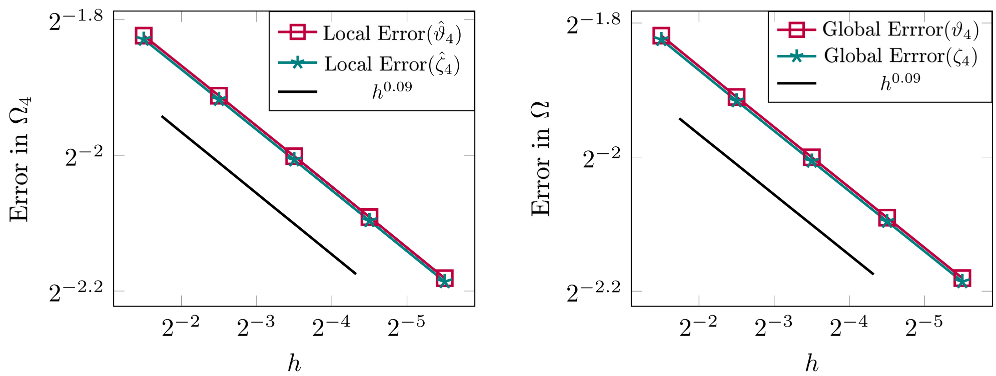
The goal of this experiment is to verify the result of Theorem 3.9, we want to illustrate numerically that the error due to artificial boundary conditions is of higher order as proved in estimate (3.12). We consider the same problem as in Section 5.1 with $M = 2$, $\Omega_1 = \Omega$ and $\Omega_2 = [-r, r] \times [-r, r]$ with $r = 1/16$. We saw previously that with this choice of r the error originating from the artificial boundary conditions dominates the local error in Ω_2. We solve equation (3.1) with different mesh sizes $h = h_{\mathcal{M}_1}$ using $h_{\widehat{\mathcal{M}}_2} = h/2^5$ as local mesh size, with this choice of $h_{\widehat{\mathcal{M}}_2}$ the dominating term in equation (3.12) is the last one, *i.e.* the one in $h^{3/2}\log(h^{-1})$. We measure the local errors in Ω_2 and plot the results in Figure 7. We see that indeed the local error satisfies equation (3.12) and converges even slightly faster than predicted.

5.3. Non regular problem: Discontinuous data

We next want to probe numerically the convergence of our local scheme for a solution only belonging to $H^{1+\varepsilon}(\Omega)$ (for small $\varepsilon > 0$). The convergence is predicted by estimates (3.15) and (3.7a). We consider a problem that has been studied in [17, 22] (in the context of a posteriori error estimators).

Let $\Omega = [-1, 1] \times [-1, 1]$ and consider Problem (1.1) with $f = 0$. We divide the computational domain in four equal parts. Let the tensor be defined as $A(\boldsymbol{x}) = a_1 I_2$ in the 1st and 3rd quadrants and $A(\boldsymbol{x}) = a_2 I_2$ in the 2nd and 4th quadrants. The exact solution is given by $u(r, \theta) = r^\gamma \mu(\theta)$, where

$$\mu(\theta) = \begin{cases} \cos((\pi/2-\sigma)\gamma)\cos((\theta-\pi/2+\rho)\gamma) & \text{if } 0 \le \theta \le \pi/2, \\ \cos(\rho\gamma)\cos((\theta-\pi+\sigma)\gamma) & \text{if } \pi/2 < \theta \le \pi, \\ \cos(\sigma\gamma)\cos((\theta-\pi-\rho)\gamma) & \text{if } \pi < \theta \le 3\pi/2, \\ \cos((\pi/2-\rho)\gamma)\cos((\theta-3\pi/2-\sigma)\gamma) & \text{if } 3\pi/2 < \theta < 2\pi. \end{cases}$$



(a) *Local errors of* $\hat{\vartheta}_4$ *and* $\hat{\zeta}_4$.   (b) *Global errors of* ϑ_4 *and* ζ_4.

FIGURE 8. Experiment 5.3: Convergence of ζ_4 and ϑ_4 letting $h \to 0$.

The parameters γ, ρ, σ and $R := a_1/a_2$ satisfy the following non linear equations

$$\begin{aligned}
&R = -\tan((\pi/2 - \sigma)\gamma)\cot(\rho\gamma),\\
&1/R = -\tan(\rho\gamma)\cot(\sigma\rho),\\
&R = -\tan(\rho\gamma)\cot((\pi/2 - \rho)\gamma),\\
&\max\{0, \pi\gamma - \pi\} < 2\gamma\rho < \min\{\pi\gamma, \pi\},\\
&\max\{0, \pi - \pi\gamma\} < -2\gamma\sigma < \min\{\pi, 2\pi - \pi\gamma\}.
\end{aligned}$$

It is known that $u \in H^{1+\gamma-\epsilon}(\Omega)$ for any $\epsilon > 0$. In this example we choose $\gamma = 0.1$, $\sigma = -19\pi/4$, $\rho = \pi/4$ and $R \approx 161$.

In order to verify the estimates (3.15) and (3.7a), we perform the same experiments as in Section 5.1, shown in Figure 3. We take $M = 4$ and the same domain and mesh sequences. We let $h \to 0$ and show the results in Figure 8. We find a convergence rate of 0.09, which is consistent with the results of [19] and the fact that u is almost in $H^{1.1}(\Omega)$. As was observed in Section 5.1, we see that the two solutions ϑ_4 and ζ_4 have the same errors, both in the local and global domains.

The influence of the term $|\kappa_k - \xi_k|_{\partial\Omega_k^-}$ in equation (3.3) on Local Error($\hat{\vartheta}_M$) is established next, repeating the experiment of Section 5.1, taking Ω_2 depending on $r \in]0,1[$ and letting $h_{\widehat{\mathcal{M}}_2} \to 0$. The results for ϑ_2 and ζ_2 are plotted in Figures 9 and 10 respectively. In contrast to the previous experiment, we do not have any saturation since the error inside the local domain largely dominates.

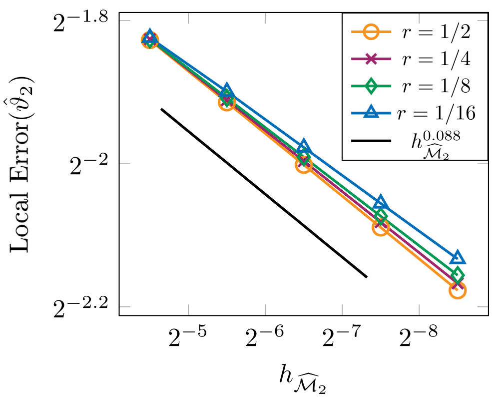
5.4. Computational cost of local *versus* non-local scheme for a linear equation

In this experiment we want to compare the numerical efficiency of the classical and local schemes on a linear equation, by computing a sequence of solutions with each scheme and plotting the accuracy against the cost.

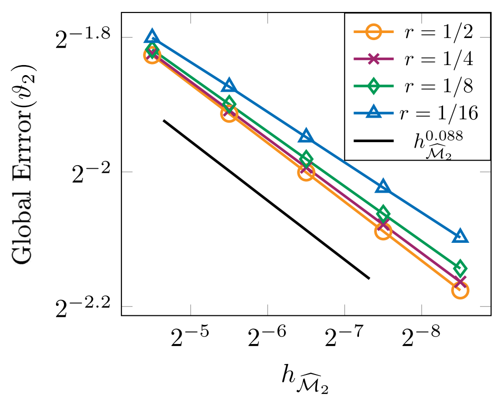
We consider equation (1.1) with $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_2 < 3\pi\}$, a diffusion tensor $A(\boldsymbol{x}) = \varepsilon + 1 - \sin(\|\boldsymbol{x}\|_2)^{100}$ with $\varepsilon = 10^{-3}$ and the force f is 1 if $\boldsymbol{x}$ is the first or third quadrants and -1 else. An illustration of the solution is given in Figure 11. We choose five local domains defined as $\Omega_1 = \Omega$ and

$$\Omega_k = \bigcup_{j=1}^{3} \{\boldsymbol{x} \in \mathbb{R}^2 : |\|\boldsymbol{x}\|_2 - (2j-1)\pi/2| < 2^{2-k}\} \quad \text{for } k = 2, \dots, 5.$$

The meshes $\widehat{\mathcal{M}}_k$ are built so that $h_{\widehat{\mathcal{M}}_1} \approx 0.3$ and for $k = 2, \dots, 5$ we have $h_{\widehat{\mathcal{M}}_k} = h_{\widehat{\mathcal{M}}_{k-1}}/2$.

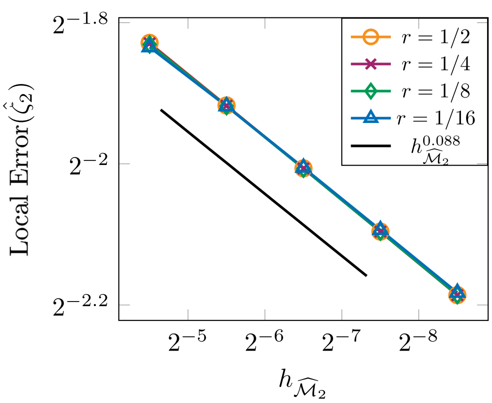


(a) *Local error in* Ω_2.

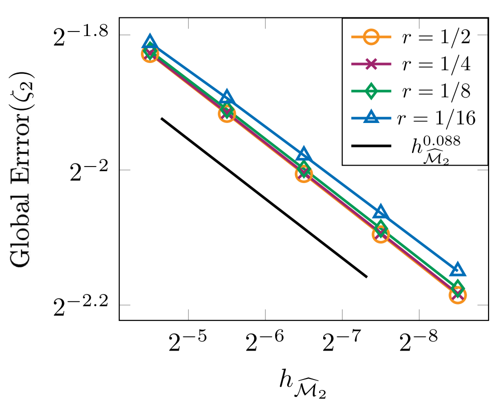


(b) *Global error in* Ω.

FIGURE 9. Experiment 5.3: Effect of the size of Ω_2 on the local solution ϑ_2 when $h_{\widehat{\mathcal{M}}_2} \to 0$.

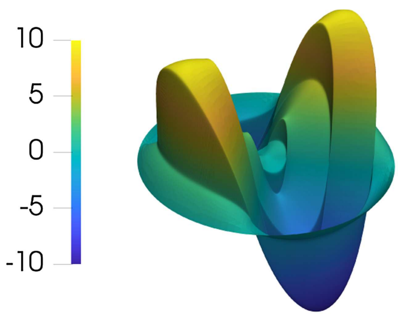


(a) *Local error in* Ω_2.

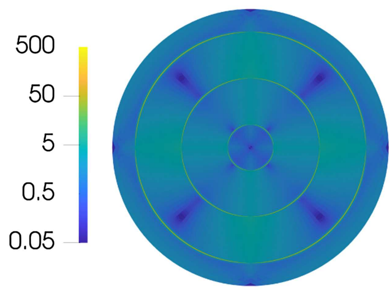


(b) *Global error in* Ω.

FIGURE 10. Experiment 5.3: Effect of the size of Ω_2 on the classical solution ζ_2 when $h_{\widehat{\mathcal{M}}_2} \to 0$.

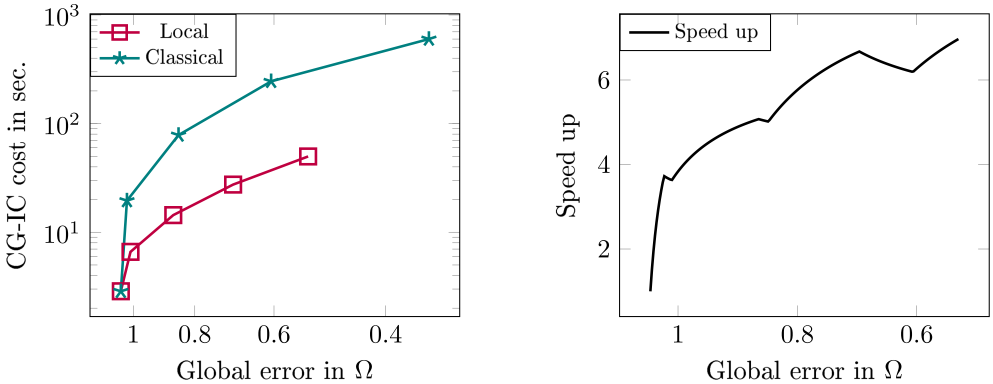


(a) *Solution.*



(b) *Norm of gradient in log color scale.*

FIGURE 11. Experiment 5.4: Solution and norm of the gradient.



(a) *Global error in* Ω *VS CG-IG cost in seconds.* (b) *Speed up in function of the global error in* Ω.

FIGURE 12. Experiment 5.4: Performance comparison in a linear case.

We run the local scheme and at each level we compute the full error and cost of ϑ_k. As a measure of the cost for ϑ_k we take the sum of the time spent solving the linear systems up to level k using the conjugate gradient (CG) method with incomplete Cholesky (IC) factorization as preconditioner. In [15] it is shown that this approach is the most robust and efficient for such problems. Then we run the classical scheme (5.1) on each mesh $\mathcal{M}_k$ and obtain a sequence of solutions ζ_k. For each $k = 1, \ldots, 5$ we compute the full error and cost of ζ_k. The cost is given by the time spent for solving the linear system at level k, where we use again CG with IC as preconditioner. Observe that here the cost is not cumulative as in the local method, since the classical scheme does not need ζ_{k-1} in order to compute ζ_k. In Figure 12a we plot the global error against the cost for both schemes, we see a significant speedup for the local scheme. In Figure 12b we plot the speed up in function of the error, the graph is obtained dividing the two curves seen in Figure 12a.

For linear problems such as in this experiment, the reason for the speed up is not only the reduced number of degrees of freedom but mostly the condition number of the linear system. The classical scheme solves linear systems arising from FE discretizazion on the whole domain, hence the matrix has high variations in its components due to possibly high contrasts in the tensor and the variation in the measure of the different elements. Instead, the local scheme uses matrices built from local discretizations, hence the tensor has milder variations and the elements of the local mesh have uniform size. This leads to matrices with smaller condition number. We see in Figure 13a that the number of degrees of freedom of the two schemes is almost the same, while in Figure 13b it is shown that the condition number of the stiffness matrix is much lower for the local scheme.

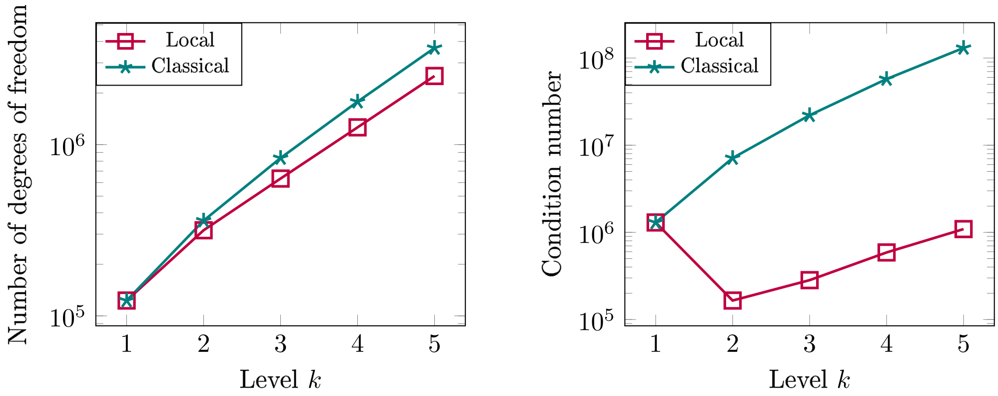
5.5. Quasilinear equation

In our last numerical experiment we want to compare the efficiency of the local and classical methods when solving a quasilinear equation. We consider the stationary Richards equation in pressure head form, given by

$$-\nabla \cdot (A(\boldsymbol{x}, h)\nabla(h - x_2)) = 0. \tag{5.3}$$

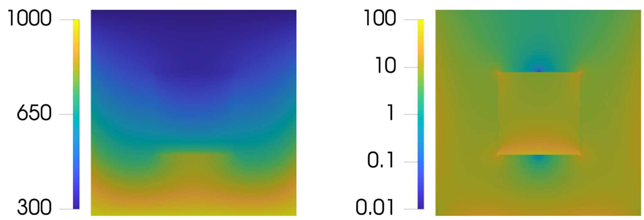
It describes the movement of a fluid in an unsaturated media and can be put in the form of equation (4.1) with the change of variables $u = h - x_2$. We consider $\Omega = [-50, 50] \times [-50, 50]$ and add the Dirichlet condition $g(\boldsymbol{x}) = 10(50 - x_2) + 3(50 + x_2)$. The diffusion tensor is given by $A(\boldsymbol{x}, h) = A_s(\boldsymbol{x})A_r(h)$, where $A_s(\boldsymbol{x})$ is the conductivity in saturated conditions and $A_r(h)$ is the relative conductivity. These latter quantities are defined by

$$A_s(\boldsymbol{x}) = \begin{cases} 10^{-3} & \text{if } \|\boldsymbol{x}\|_\infty \leq 20, \\ 1 & \text{else,} \end{cases} \qquad A_r(h) = \frac{(1 - (ah)^{n-1}(1 + (ah)^n)^{-m})^2}{(1 + (ah)^n)^{m/2}}.$$



(a) *Number of degrees of freedom.* (b) *Condition number of the non preconditioned system.*

FIGURE 13. Experiment 5.4: Properties of the linear systems.



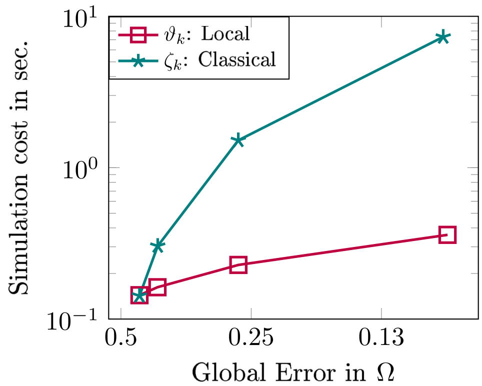
(a) *Solution.* (b) *Norm of gradient in log color scale.*

FIGURE 14. Experiment 5.5: Solution and norm of the gradient for the Richards equation.

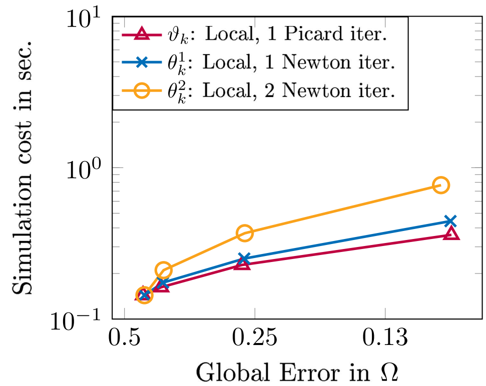
The model $A_r(h)$ has been taken from [24], where $m = 1 - 1/n$ is chosen. The parameters a, n are soil dependent: we choose $a = 1/500$ and $n = 2.68$, which is in the range of real case parameters. Remark that the tensor is discontinuous in $\boldsymbol{x}$ and hence does not satisfy Assumption 2.2. We plot in Figure 14 the reference solution and the norm of its gradient, we see that the gradient is highly discontinuous.

Let $M = 4$, $\Omega_1 = \Omega$ and Ω_k for $k = 2, 3, 4$ defined by $\boldsymbol{x} \in \Omega_k$ if $\|\boldsymbol{x}\|_\infty \leq 20(1 + 2^{-k})$. First, we fix $h_{\widehat{\mathcal{M}}_k} = 100\sqrt{2}/2^{4+k}$ and compute the local solutions ϑ_k given by the local method equation (4.2). At the first level $k = 1$ we need to solve a nonlinear problem on a coarse grid using Newton iterations, where the initial guess is an extrapolation of the Dirichlet condition $g(\boldsymbol{x})$ on the whole domain. In the next levels $k > 1$ the local scheme solves a linear system using the Picard iteration step defined in equation (4.2d). At each level we compute the full error and cost of ϑ_k. As a measure of the cost for ϑ_k we take the sum of the time spent solving the linear and non linear systems up to level k. Since at $k = 1$ we perform a linearization of the system, it is no more symmetric because of the additional term, hence it has to be solved with the GMRES iterative scheme with incomplete LU (ILU) factorization as preconditioner, instead of CG with IC. In the following iterations with $k \geq 2$ we solve a linear system and hence the CG scheme with IC is used.

Then we compute similar solutions with the classical method and compare the costs. For the classical solution we need, for each $k = 1, 2, 3, 4$, to solve equation (5.3) with the Newton method. As initial guess we take again $g(\boldsymbol{x})$ and the Newton iterations are stopped when the error of the classical solution ζ_k is similar to the one of ϑ_k. In about 3 or 4 Newton iterations we obtained errors differing by only about 1%. To measure the cost for ζ_k we consider the time spent in solving the non linear system at level k. The cost here is not cumulative as in the local method but on the other hand the linear systems to solve are not symmetric and the GMRES scheme with ILU preconditioner is used. In Figure 15(a) we plot the error against the cost for this experiment. We see



(a) *Efficiency of local and classical schemes.*



(b) *Efficiency of local scheme and two variants*

FIGURE 15. Experiment 5.5: Performance of classical scheme, local scheme and local scheme with Newton iterations instead of one Picard iteration.

that the local scheme performs much better than the classical scheme in terms of computational cost *versus* accuracy.

Finally we compare the accuracy and cost of solving the local systems equation (4.2d) replacing $\Pi_{\widehat{\mathcal{D}}_{k-1}}\hat{\vartheta}_{k-1}$ with $\Pi_{\widehat{\mathcal{D}}_k}\hat{\vartheta}_k$, *i.e.*, defining nonlinear local problems. These local systems have now to be solved by Newton method and GMRES with ILU. We denote by θ_k^1 the solution where we use one Newton iteration and by θ_k^2 the solution with two Newton iterations. In Figure 15(b) we plot the error against the cost for ϑ_k, θ_k^1 and θ_k^2. We see that one Picard iteration gives very similar error to the one or two Newton iterations but at a smaller cost, thanks to the CG scheme.

6. Conclusion

In this paper we introduced a local scheme for linear and quasilinear elliptic equations. The method does not rely on an iterative procedure and only needs one global solve on a coarse mesh. All subsequent computations are local. The *a priori* error analysis has been performed under weak regularity assumptions thanks to the gradient discretization framework. Numerical experiments have shown the efficiency of the scheme when applied to equations with localized high gradient regions. In a forthcoming paper [1] the a posteriori error analysis of the same scheme will be presented. Thanks to the a posteriori error estimators the local domains can be defined even when the singularities are not known *a priori*. We note that the extension of the local scheme to parabolic problems is also of interest. In particular, we believe that the techniques developed for *a priori* and a posteriori error analysis for the local scheme for elliptic PDEs also allow to analyze local time stepping schemes for parabolic PDEs.

Appendix A. Equivalence to SWIPG scheme

In this appendix we show that the SWDGGD scheme described in Section 2.2 is equivalent to the SWIPG method of [8,11]. In particular we show that equation (2.3) with $\mathcal{D} = (X_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ as defined in Section 2.2 is equivalent to equation 4.63 of [8], in order to do that we follow [12], where the equivalence of a GD to the SIP method is shown. We suppose that $A(\boldsymbol{x}, u) = A(\boldsymbol{x})$ and $A_K := A|_K$ the restriction of A to an element $K \in \mathcal{M}$ is constant, that $f \in L^2(\Omega)$ and hence $f = f_0$.

Starting from equation (2.3) and developing the gradients we get

$$\begin{aligned}\int_\Omega A\nabla_{\mathcal{D}}\vartheta\cdot\nabla_{\mathcal{D}}\phi\,\mathrm{d}\boldsymbol{x} &= \sum_{K\in\mathcal{M}}\int_K A_K\nabla_{\overline{K}}\vartheta\cdot\nabla_{\overline{K}}\phi\,\mathrm{d}\boldsymbol{x}\\ &+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K}\int_{D_{K,\sigma}}\frac{\psi(s)}{d_{K,\sigma}}A_K([\vartheta]_{K,\sigma}(\boldsymbol{y})\nabla_{\overline{K}}\phi(\boldsymbol{x})+[\phi]_{K,\sigma}(\boldsymbol{y})\nabla_{\overline{K}}\vartheta(\boldsymbol{x}))\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\boldsymbol{x}\\ &+\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K}\int_{D_{K,\sigma}}\frac{A_K\boldsymbol{n}_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}}{d_{K,\sigma}^2}\psi(s)^2[\vartheta]_{K,\sigma}(\boldsymbol{y})[\phi]_{K,\sigma}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\\ &= I+II+III.\end{aligned}$$

Since $\boldsymbol{x}=\boldsymbol{x}_K+s(\boldsymbol{y}-\boldsymbol{x}_K)$ for $s\in]0,1[$, $\boldsymbol{y}\in\sigma$ and $\nabla_{\overline{K}}\vartheta\in\mathbb{P}_{\ell-1}(K)^d$ then

$$\nabla_{\overline{K}}\vartheta(\boldsymbol{x})\cdot\boldsymbol{n}_{K,\sigma}=\nabla_{\overline{K}}\vartheta(\boldsymbol{y})\cdot\boldsymbol{n}_{K,\sigma}+\sum_{j=1}^{\ell-1}p_j(\boldsymbol{y})(1-s)^j,$$

with $p_j(\boldsymbol{y})$ polynomials of $\ell-1$ degree in the components of $\boldsymbol{y}$. It follows from equation (2.7) that

$$\int_0^1\nabla_{\overline{K}}\vartheta(\boldsymbol{x})\cdot\boldsymbol{n}_{K,\sigma}s^{d-1}\psi(s)\,\mathrm{d}s=\nabla_{\overline{K}}\vartheta(\boldsymbol{y})\cdot\boldsymbol{n}_{K,\sigma},$$

hence, using the change of variables $\mathrm{d}\boldsymbol{x}=s^{d-1}d_{K,\sigma}\mathrm{d}s\mathrm{d}\boldsymbol{y}$ we get

$$II=\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K}\int_\sigma A_K([\vartheta]_{K,\sigma}(\boldsymbol{y})\nabla_{\overline{K}}\phi(\boldsymbol{y})+[\phi]_{K,\sigma}(\boldsymbol{y})\nabla_{\overline{K}}\vartheta(\boldsymbol{y}))\cdot\boldsymbol{n}_{K,\sigma}\,\mathrm{d}\boldsymbol{y}.$$

For $\sigma\in\mathcal{F}_i$ with $\sigma=\partial K\cap\partial T$ let $\boldsymbol{n}_\sigma=\boldsymbol{n}_{K,\sigma}$ and

$$[\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma=\Pi_{\overline{K}}\vartheta-\Pi_{\overline{T}}\vartheta,\qquad \{\!\{A\nabla\Pi_{\mathcal{D}}\vartheta\}\!\}_{\omega,\sigma}=\omega_{K,\sigma}A|_K\nabla\Pi_{\overline{K}}\vartheta+\omega_{T,\sigma}A|_T\nabla\Pi_{\overline{T}}\vartheta.$$

If $\sigma\in\mathcal{F}_b$ with $\sigma=\partial K\cap\partial\Omega$ let $\boldsymbol{n}_\sigma=\boldsymbol{n}_{K,\sigma}$ and

$$[\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma=\Pi_{\overline{K}}\vartheta,\qquad \{\!\{A\nabla\Pi_{\mathcal{D}}\vartheta\}\!\}_{\omega,\sigma}=A|_K\nabla\Pi_{\overline{K}}\vartheta.$$

It holds $[\vartheta]_{K,\sigma}\cdot\boldsymbol{n}_{K,\sigma}=-\omega_{K,\sigma}[\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma\cdot\boldsymbol{n}_\sigma$ and similarly for ϕ, hence

$$\begin{aligned}II&=-\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K}\int_\sigma\omega_{K,\sigma}([\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma A_K\nabla\Pi_{\overline{K}}\phi+[\![\Pi_{\mathcal{D}}\phi]\!]_\sigma A_K\nabla\Pi_{\overline{K}}\vartheta)\cdot\boldsymbol{n}_\sigma\,\mathrm{d}\boldsymbol{y}\\ &=-\sum_{\sigma\in\mathcal{F}}\int_\sigma([\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma\{\!\{A\nabla\Pi_{\mathcal{D}}\phi\}\!\}_{\omega,\sigma}+[\![\Pi_{\mathcal{D}}\phi]\!]_\sigma\{\!\{A\nabla\Pi_{\mathcal{D}}\vartheta\}\!\}_{\omega,\sigma})\cdot\boldsymbol{n}_\sigma\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

For III, using $C_\psi^2=\int_\alpha^1\psi(s)^2s^{d-1}\mathrm{d}s$ and by the usual change of variables, we obtain

$$\begin{aligned}III&=C_\psi^2\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K}\frac{\delta_{K,\sigma}}{d_{K,\sigma}}\omega_{K,\sigma}^2\int_\sigma[\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma[\![\Pi_{\mathcal{D}}\phi]\!]_\sigma\,\mathrm{d}\boldsymbol{y}\\ &=\sum_{\sigma\in\mathcal{F}}\eta_\sigma\frac{\gamma_\sigma}{h_\sigma}\int_\sigma[\![\Pi_{\mathcal{D}}\vartheta]\!]_\sigma[\![\Pi_{\mathcal{D}}\phi]\!]_\sigma\,\mathrm{d}\boldsymbol{y},\end{aligned}$$

where h_σ is the diameter of σ and γ_σ, η_σ for $\sigma \in \mathcal{F}_i$ are defined by

$$\gamma_\sigma = \frac{2\delta_{K,\sigma}\delta_{T,\sigma}}{\delta_{K,\sigma}+\delta_{T,\sigma}},$$
$$\eta_\sigma = C_\psi^2 \left(\frac{\delta_{K,\sigma}}{d_{K,\sigma}}\omega_{K,\sigma}^2 + \frac{\delta_{T,\sigma}}{d_{T,\sigma}}\omega_{T,\sigma}^2\right)\frac{h_\sigma}{\gamma_\sigma} = C_\psi^2 h_\sigma \left(\frac{\omega_{K,\sigma}}{d_{K,\sigma}} + \frac{\omega_{T,\sigma}}{d_{T,\sigma}}\right)$$

and for $\sigma \in \mathcal{F}_b$ by

$$\gamma_\sigma = \delta_{K,\sigma}, \qquad \eta_\sigma = C_\psi^2 \frac{h_\sigma}{d_{K,\sigma}}\cdot$$

Summing I, II, III we get the equivalence of $\int_\Omega A\nabla_{\mathcal{D}}\vartheta\cdot\nabla_{\mathcal{D}}\phi \,\mathrm{d}\boldsymbol{x}$ and equation (4.64) of [8] with the parameter η_σ chosen as above. Under the additional hypothesis that the mesh sequence satisfies

$$\min\left\{\frac{h_\sigma}{d_{K,\sigma}} : K \in \mathcal{M},\, \sigma \in \mathcal{F}_K\right\} \geq C_{\mathcal{F}} > 0,$$

we have that $\eta_\sigma \geq C_\psi^2 C_{\mathcal{F}}$. Since $C_\psi^2 \geq d/(1-\alpha^d)$, letting $\alpha \to 1$ we can have η_σ as large as desired.

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