ESAIM: M2AN 53 (2019) 1577–1606
https://doi.org/10.1051/m2an/2019021

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

ANALYSIS OF THE ERROR IN AN ITERATIVE ALGORITHM FOR ASYMPTOTIC REGULATION OF LINEAR DISTRIBUTED PARAMETER CONTROL SYSTEMS *

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Abstract. Applications of regulator theory are ubiquitous in control theory, encompassing almost all areas of systems and control engineering. Examples include active noise suppression [Banks *et al.*, Decision and Control, Active Noise Control: Piezoceramic Actuators in Fluid/structure Interaction Models, IEEE, Los Alamitos, CA (1991) 2328-2333], design and control of energy efficient buildings [Borggaard *et al.*, Control, Estimation and Optimization of Energy Efficient Buildings. Riverfront, St. Louis, MO (2009) 837-841.] and control of heat exchangers [Aulisa *et al.*, IFAC-PapersOnLine **49** (2016) 104–109.]. Numerous other examples can be found in [Aulisa and Gilliam, A Practical Guide to Geometric Regulation for Distributed Parameter Systems. Chapman and Hall/CRC, Boca Raton (2015).]. In the geometric approach to asymptotic regulation the main object of interest is a pair of operator equations called the regulator equations, whose solution provides a control solving the tracking/disturbance rejection regulation problem. In this paper we present an iterative algorithm, called the β-iteration method, which is based on the geometric methodology, and delivers accurate control laws for approximate asymptotic regulation. This iterative scheme has been successfully applied to a wide range of linear and nonlinear multi-physics examples and in practice only one or two iterations are usually required to deliver sufficiently accurate results. One drawback to these research efforts is that no proof was given of the convergence of the method. This work contains a detailed analysis of the error in the iterative scheme for a large class of linear distributed parameter systems. In particular we show that the iterative errors converge at a geometric rate. We demonstrate our estimates on three control problems in multi-physics applications.

Mathematics Subject Classification. 93C20, 65Y15.

Received December 11, 2017. Accepted March 13, 2019.

1. Introduction

The geometric theory of asymptotic regulation was first investigated for finite dimensional linear systems by numerous authors during the 1970's and 1980's (*cf.*, Davison [9], Francis and Wonham [13], Francis [12],

Keywords and phrases. Regulator problem, distributed parameter systems, geometric control, dynamic regulator equations, iterative numerical algorithm.

* *The research of the first and third authors was partially supported by the National Science Foundation grant DMS-1412796.*

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Wonham [28]. It was extended to linear distributed parameter systems beginning in the early 1980's, [19,20,25] and 1990's [8].

In the geometric approach the main object of interest is a pair of operator equations called the regulator equations, whose solution provides a control solving the tracking/disturbance rejection regulation problem. Unfortunately, in many practical applications obtaining even approximate solutions of the regulator equations can be extremely challenging. For motivation consider the example (studied in detail in Example 4.3) involving a thermal tracking/disturbance regulation problem in a rectangular room (see Fig. 14) with a boundary airflow/heat source, a boundary exit vent. The control objective is to drive the average temperature of a thin rectangular slab in the room to track a given reference temperature profile. The particular time varying temperature profile considered here is a periodic signal, intended to model a warmer temperature during the day time and a cooler temperature during the night. To further complicate the control problem, we introduce a window as a boundary heat disturbance intended to be a warmer disturbance in daytime and cooler at night. In this two dimensional model we consider the problem of designing a control law capable of tracking the described reference temperature profile, while rejecting the disturbance. Airflow in the room is produced by a steady-state incompressible Navier–Stokes fluid flow that passes through the room. The control is achieved by a distributed source/sink heat flux at the inlet duct. Solving the regulator equations for this example would be rather difficult but as we see in Example 4.3 accurate approximate controls are easily obtained via our β-iteration method.

In this paper we present an iterative scheme, which we call the β-iteration method, for obtaining very accurate approximate solutions of "dynamic regulator equations." The iterative algorithm produces a sequence of increasingly more accurate solutions of the dynamic regulator equations which provides a dynamic model of the attractive invariant subspace at the heart of the classical geometric theory. In this work we are particularly interested in infinite dimensional linear control systems governed by partial differential equations with bounded or unbounded input and disturbance operators and possibly unbounded output operators in the Hilbert state space.

In the case of tracking and disturbance rejection for time independent signals, referred to as set-point control, the regulator equations simplify considerably and can be solved very accurately, without need of iterations, using off the shelf software such as "Comsol" as can be seen in [3]. Furthermore, for piecewise constant time dependent signals a modification of the set-point case together with the β-iteration method considered in this paper produces very accurate results. For example, in [3] and [4] the authors have demonstrated the applicability of the β-iteration method in designing time independent control laws for tracking/disturbance rejection for control problems in a variety of multi-physics applications, including thermal regulation for non-isothermal Navier–Stokes flow. In the work [1,2], the authors have applied this methodology to tracking and disturbance rejection problems, for linear and nonlinear systems, and for general time dependent signals. The numerical results in these works motivate the main point of this work: establishing bounds for the iterative errors and proof of the convergence for the iterative method.

Thus in this work we consider the rigorous analysis of the error encountered at each step in the β iterations and the establishment of a rigorous proof of convergence for the iterative scheme. As we have already pointed out, solving the regulator equations for an infinite dimensional control system can be a very difficult task.

The paper is organized as follows. In Section 2 we present the general setup and describe the class of control systems and the assumptions considered in this work. We note that the class of problems is certainly not most general, even in the linear case, but we feel that the estimates presented below can be modified by the interested reader to cover more general situations. On the other hand, our general assumptions of unboundedness of the input, output and disturbance operators allow application to a wide variety of practical examples that arise in the control of distributed parameter systems, governed by exponentially stable parabolic and damped hyperbolic partial differential equations (see [1]). Also in Section 2 we introduce the dynamic controller, the regularized dynamic controller and β iterative algorithm. Finally in Section 2 we establish some important propositions and lemmas which are needed to prove our main result for the error estimate, namely the main Convergence Theorem 2.17. In Section 3 we present the main results of the error analysis. The proofs of the results stated in Section 3 are given in Section 5. We first show how the errors at each step can be written as a convolution integral and then proceed to derive the estimates. In Section 4 we give some numerical examples that support the theoretical estimates.

2. General setup and assumptions

In this work we are primarily interested in tracking and disturbance rejection problems for linear control systems in the form

$$z_t(t) = Az(t) + B_{\text{in}}v(t) + B_d\, d(t), \tag{2.1}$$

$$z(0) = z_0, \tag{2.2}$$

$$y(t) = Cz(t). \tag{2.3}$$

Here $z(t)$ is the state variable in the infinite dimensional Hilbert space $\mathcal{Z}$ at time t. In the systems (2.1)–(2.3), A is the state operator, B_{in}, B_d and C are the input, disturbance and output operators, respectively, v the control input, d the disturbance, z_0 is the initial data and $y(t)$ is the measured output. The inner product and the induced norm defined on $\mathcal{Z}$ are denoted by $\langle \phi, \psi \rangle$ and $\|\phi\| = \langle \phi, \phi \rangle^{1/2}$, respectively.

Assumption 2.1 (Assumptions on the operator *A*). The operator A is a sectorial operator, with compact resolvent $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$ (the resolvent set of A) and is assumed to generate an exponentially stable analytic semigroup e^{At} in $\mathcal{Z}$. In particular there exist numbers $\omega > 0$ and $M \geq 1$ such that

$$\|\mathrm{e}^{At}\| \leq M\mathrm{e}^{-\omega t}. \tag{2.4}$$

Here the norm on the left is the operator norm. The symbol $\mathcal{L}(W_1, W_2)$ denotes the set of all bounded linear operators from a Hilbert space W_1 to a Hilbert space W_2. When $W = W_1 = W_2$ we write $\mathcal{L}(W)$.

The operator A generates an infinite scale of Banach spaces denoted by $\mathcal{Z}^s$. For $s > 0$, let $\mathcal{Z}^s = \mathrm{Ran}((-A)^{-s})$ with norm $\|\varphi\|_s = \|(-A)^s\varphi\|$, where, as usual, the norm on the right, $\|\cdot\|$, denotes the norm in $\mathcal{Z}$. For $s > 0$ the space $\mathcal{Z}^{-s}$ is the completion of $\mathcal{Z}$ with norm $\|\varphi\|_{-s} = \|(-A)^{-s}\varphi\|$. For a detailed discussion of fractional powers of sectorial operators and properties of the scale of Banach spaces, see [11,14,15,17,22]. In this work we follow the definition of sectorial operator used in [11] which is different than that found, for example, in [15]. In particular, our A would be $-A$ in [15]. Assumption 2.1 implies that the spectrum of A consists of discrete eigenvalues whose only limit point is at infinity and lies in a sector in the left half complex plane.

Assumption 2.2 (Assumptions on input and output operators).

(1) The input space is $\mathcal{U} = \mathbb{R}^k$ and $B_{\text{in}} \in \mathcal{L}(\mathcal{U}, \mathcal{Z}^{-s_b})$ for some $0 \leq s_b < 1$. For a given input vector

$$v(t) = [v_1(t), v_2(t), \ldots, v_k(t)]^T \in \mathcal{U},$$

it follows

$$B_{\text{in}}v(t) = \sum_{j=1}^{k} b_j v_j \text{ with } b_j \in \mathcal{Z}^{-s_b}.$$

For the operator norm the following notation is used

$$\|B_{\text{in}}\|_{\mathcal{L}(\mathcal{U}, \mathcal{Z}^{-s_b})} := \|B_{\text{in}}\|_{-s_b} = \left(\sum_{j=1}^{k} \|(-A)^{-s_b} b_j\|^2 \right)^{1/2}.$$

(2) The disturbance space is $\mathcal{D} = \mathbb{R}^m$ and $B_d \in \mathcal{L}(\mathcal{D}, \mathcal{Z}^{-s_d})$ for some $0 \leq s_d < 1$. For given disturbance vector

$$d(t) = [d_1(t), d_2(t), \ldots, d_m(t)]^T \in \mathcal{D},$$

it follows

$$B_d d(t) = \sum_{j=1}^{m} b_{dj} d_j(t) \text{ with } b_{dj} \in \mathcal{Z}^{-s_d}.$$

For the operator norm the following notation is used

$$\|B_d\|_{\mathcal{L}(\mathcal{D},\mathcal{Z}^{-s_d})} := \|B_d\|_{-s_d} = \left(\sum_{j=1}^{k} \|(-A)^{-s_d} b_{d_j}\|^2 \right)^{1/2}.$$

(3) The output space is $\mathcal{Y} = \mathbb{R}^k$ and $C \in \mathcal{L}(\mathcal{Z}^{s_c}, \mathcal{Y})$ for some $0 \leq s_c < 1$. Therefore, for $\varphi \in \mathcal{Z}^{s_c}$

$$\|C\varphi\|_{\mathcal{Y}} = \|C(-A)^{-s_c}(-A)^{s_c}\varphi\|_{\mathcal{Y}} \leq \|C(-A)^{-s_c}\|_{\mathcal{Y}} \, \|(-A)^{s_c}\varphi\|_{\mathcal{Z}} = \|C(-A)^{-s_c}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \|\varphi\|_{s_c},$$

which implies

$$\|C\|_{\mathcal{L}(\mathcal{Z}^{s_c},\mathcal{Y})} \leq \|C(-A)^{-s_c}\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})}.$$

Again the following notation is used

$$\|C\|_{\mathcal{L}(\mathcal{Z}^{s_c},\mathcal{Y})} := \|C\|_{-s_c}.$$

Remark 2.3. Referring to the notation and terminology in [27] and using Assumptions 2.1 and 2.2 with $s_c + s_b < 1$ and $s_c + s_d < 1$, Theorem 5.7.3 in [27] tells us that the systems (2.1)–(2.3) is an L^1-well-posed linear system in $(\mathcal{Y}, \mathcal{Z}^\gamma, \mathcal{U})$ for $\gamma \in (s_c - 1, -\sigma_b]$ and it is also uniformly regular. A detailed discussion of these points is not needed in this paper, however it precisely corresponds to the L^1-well-posedness that motivates the bounds obtained in the error analysis of the proposed iterative scheme.

Definition 2.4. For $I \subseteq \mathbb{R}^+ = \{t : 0 \leq t < \infty\}$ a fixed interval, the space $C_b^N(I)$ consisting of bounded and N-times continuously differentiable functions with derivatives bounded on I is a Banach space with the norm

$$\|\varphi\|_{I,N} = \max_{0 \leq j \leq N} \left(\sup_{t \in I} |\varphi^{(j)}(t)| \right).$$

Let denote the norm for $\varphi = [\varphi_1, \varphi_2, \ldots, \varphi_k]^T \in C_b^N(I, \mathcal{Y})$ by

$$\|\varphi\|_{I,N,\mathcal{Y}} = \sup_{1 \leq p \leq k} (\|\varphi_p\|_{I,N}).$$

We are particularly interested in supremum norm taken over time intervals of the form $I = [T, \infty)$ for a $T > 0$. This allows us to consider reference and disturbance signals that may have large excursions for some time but eventually settle into a more stable type of oscillation. As an example consider the reference and disturbance signals generated by the nonlinear oscillator in Example 4.2. There are initially large oscillations but for sufficiently large T we have the limit superior of the signals and their derivatives are small oscillations about a limit point.

Assumption 2.5 (Reference signals $y_r(t)$ and disturbances $d(t)$). Let $\mathcal{Y}$ denote the vector space $\mathbb{R}^k$ with norm $\|\cdot\|_{\mathcal{Y}}$, and consider vector valued time dependent reference signals $y_r = [y_{r_1}, y_{r_2}, \ldots, y_{r_k}]^T \in C_b^N(\mathbb{R}^+, \mathcal{Y})$ where $y_{r_j} \in C_b^N(\mathbb{R}^+)$. The same assumptions apply for the disturbances $d(t) = [d_1(t), d_2(t), \ldots, d_m(t)]^T \in C_b^N(\mathbb{R}^+, \mathcal{D})$, with $d_j \in C_b^N(\mathbb{R}^+)$.

The main theoretical Asymptotic Regulation problem found in the literature is described in Problem 2.6.

Problem 2.6 (**Exact Asymptotic Regulation Problem**). *Find a time dependent control law* $v \in C_b(\mathbb{R}^+, \mathcal{U})$*, for the systems* (2.1)–(2.3)*, so that the tracking error, defined as*

$$E(t) = y_r(t) - y(t), \tag{2.5}$$

satisfies

$$\lim_{t\to\infty} \|E(t)\|_{\mathcal{Y}} = 0 \qquad \text{(asymptotic regulation)}, \tag{2.6}$$

for a given reference signal $y_r \in C_b^N(\mathbb{R}^+, \mathcal{Y})$ *and disturbance* $d \in C_b^N(\mathbb{R}^+, \mathcal{D})$*.*

The beauty of geometric regulation methods is that the desired control v solving Problem 2.6 can be obtained by solving a pair of operator equations referred to as the regulator equations. But, computing an explicit control v solving Problem 2.6 can seldom be done exactly and one usually seeks approximate controls that deliver asymptotically small error to within any desired level of accuracy.

This leads us to the more reasonable problem referred to as Practical Asymptotic Regulation.

Problem 2.7 (**Practical Regulation Problem**). *Given a desired approximation error level* $\epsilon > 0$*, find a time dependent control law* $v \in C_b(\mathbb{R}^+, \mathcal{U})$ *such that*

$$\limsup_{t\to\infty} \|E(t)\|_{\mathcal{Y}} \le \epsilon, \tag{2.7}$$

for a given reference signal $y_r \in C_b^N(\mathbb{R}^+, \mathcal{Y})$ *and disturbance* $d \in C_b^N(\mathbb{R}^+, \mathcal{D})$*.*

Remark 2.8. It will be clear later that if the reference and disturbance signals are not infinitely smooth, *i.e.*, they are in C_b^N but not C_b^{N+1}, then at this point our β-iteration method will break down. This means, in that case, there is a lower limit for the value of the constant ϵ. In other words, in order to guarantee that we can choose an arbitrarily small ϵ, using our algorithm, we need the reference and disturbance signals to be in $C_b^\infty([0,\infty))$. Note that in the classical situation (*cf.* [8]) the reference and disturbance signals are assumed to be generated by a finite dimensional, neutrally stable exo-system, in which case the are always in $C_b^\infty([0,\infty))$.

Definition 2.9 (**Transfer function**). The transfer function of the plant is the $k \times k$ matrix valued function

$$G_{\text{in}}(s) = C(sI - A)^{-1}B_{\text{in}}, \tag{2.8}$$

defined for $s \in \mathbb{C}$ in the resolvent set of A. Also define

$$G_{\text{in}} := G_{\text{in}}(0) = C(-A)^{-1}B_{\text{in}}. \tag{2.9}$$

The transfer function plays a central role in linear regulator theory and in the solution of the regulator equations [8, 19, 21, 26]. Notice that $G_{\text{in}} = G_{\text{in}}(0)$, makes sense due to our assumption that A generates an exponentially stable semigroup so $0 \in \rho(A)$. Therefore G is $k \times k$ matrix with entries

$$(G_{\text{in}})_{i,j} = \langle A^{-1}b_j, c_i\rangle \in \mathbb{R},$$

equivalently $(G_{\text{in}})_{i,j} = \langle X_j, c_i\rangle$, with $AX_j = b_j$.

For our algorithm we need the additional assumption

Assumption 2.10 (Invertibility of transfer function). Assume that $s = 0$ is not a transmission zero of the plant which amounts, in this case, to the assumption that the matrix G_{in} is invertible.

A useful simplifying assumption, which is always possible under Assumption 2.10, is

Definition 2.11 (B and B_{in}). From this point on we replace the input operator B_{in} by $B = B_{\text{in}}R$ where R is the nonsingular $k \times k$ matrix satisfying $G_{\text{in}}R = I = RG_{\text{in}}$. In this case we obtain

$$G := C(-A)^{-1}B = I_{k\times k}. \tag{2.10}$$

Replacing B_{in} by B we note that the control $v(t)$ is replaced by $u(t) = R^{-1}v(t)$, and we can rewrite the control systems (2.1)–(2.3) in the following equivalent form

$$z_t(t) = Az(t) + Bu(t) + B_d d(t), \tag{2.11}$$

$$z(0) = z_0, \tag{2.12}$$

$$y(t) = Cz(t). \tag{2.13}$$

Remark 2.12. Although systems (2.1)–(2.3) and systems (2.11)–(2.13) are equivalent, there is an advantage in using the latter in simplifying the formulation of the forthcoming error analysis. In the applications this condition is not needed, but it greatly simplifies the forthcoming theoretical presentation.

2.1. Dynamic Controller (DC)

At the heart of the geometric approach to regulation for linear systems [8, 19, 21, 26] is a pair of operator equations called the regulator equations (see [8, 21]), whose solution delivers the desired control law u in (2.11) (or equivalently, v in (2.1)) solving Problem 2.6. The dynamic form of the regulator equations, which we call the Dynamic Controller, is the dynamic-algebraic system

$$\overline{z}_t(t) = A\overline{z}(t) + B\overline{\gamma}(t) + B_d d(t), \tag{2.14}$$

$$\overline{z}(0) = \overline{z}_0, \tag{2.15}$$

$$C\overline{z}(t) = y_r(t)\ ,\ \forall\ \ t \geq 0, \tag{2.16}$$

with the following three unknowns: (1) the initial condition $\overline{z}_0 \in \mathcal{Z}$, (2) the control $\overline{\gamma}(t)$, and (3) the state variable $\overline{z}(t)$.

Our main assumption in this work is that

Assumption 2.13. For given reference signal, $y_r(t)$, and disturbance, $d(t)$, there exists $\overline{z}_0$, $\overline{\gamma}(t)$ and $\overline{z}(t)$ satisfying (2.14)–(2.16).

Under Assumptions 2.1, 2.2 and 2.13, the exact regulator problem Problem 2.6 is solved by system (2.11)–(2.13) using as control $u = \overline{\gamma}(t)$. To see this assume $\overline{z}_0$, $\overline{\gamma}(t)$, $\overline{z}(t)$ satisfy (2.14)–(2.16) and use the control $u = \overline{\gamma}(t)$ in the systems (2.11)–(2.13). Let $\eta = z - \overline{z}$. Subtracting the two systems it is easy to see that η satisfies

$$\eta_t = A\eta, \tag{2.17}$$

$$\eta(0) = z_0 - \overline{z}_0, \tag{2.18}$$

thus $\eta(t) = \mathrm{e}^{At}\eta(0)$. Then exact asymptotic tracking and disturbance rejection is obtained, *i.e.*,

$$\|y(t) - y_r(t)\|_{\mathcal{Y}} = \|C(z)(t) - C(\overline{z})(t)\|_{\mathcal{Y}} = \|C(\eta)\|_{\mathcal{Y}} = \|C(\mathrm{e}^{At}\eta(0))\|_{\mathcal{Y}} \tag{2.19}$$

$$\leq \|C(-A)^{-s_c}\|\|(-A)^{s_c}\mathrm{e}^{At}\eta(0)\| \leq \|C\|_{-s_c} M_{s_c} \frac{\mathrm{e}^{-\omega t}}{t^{s_c}} \|z_0 - \overline{z}_0\| \xrightarrow{t\to\infty} 0, \tag{2.20}$$

for some $M_{s_c} \geq 1$. In the last inequality we have used a well known result that can be found in (Thm. 1.4.3 of [15]) or (Thm. 6.13 of [22]).

Just as with the classical regulator equations the dynamic controller is difficult to solve and the main contribution of this work is to give an iterative procedure for obtaining increasingly more accurate approximations of the desired control and to prove that, under suitable conditions, these approximate controls lead to a solution of the practical regulation problem in 2.7.

2.2. Iterative scheme and a set-point problem

In order to introduce the iterative scheme let us replace $\overline{z}$, $\overline{\gamma}$ in (2.14)–(2.16) by $\overline{z}_1 = \overline{z}^0 + \overline{z}^1$ and $\overline{\gamma}_1 = \overline{\gamma}^0 + \overline{\gamma}^1$. Then in our first step we solve a set-point regulation problem to track the time independent reference signal $y_r(0)$ and reject the time independent disturbance $d(0)$ using the results from [3]. Namely, we solve the following stationary problem to obtain $\overline{\gamma}^0$ and $\overline{z}^0$

$$0 = A\overline{z}^0 + B\overline{\gamma}^0 + B_d d(0), \tag{2.21}$$

$$y_r(0) = C(\overline{z}^0). \tag{2.22}$$

This is easily accomplished by solving (2.21) for $\overline{z}^0$

$$\overline{z}^0 = (-A)^{-1}B\overline{\gamma}^0 + (-A)^{-1}B_d d(0).$$

Then apply C to both sides and use (2.22) and (2.9) to obtain

$$\overline{\gamma}^0 = y_r(0) + C(A^{-1})B_d d(0).$$

With this we define our first iterative error

$$E_0(t) = y_r(t) - y_r(0). \tag{2.23}$$

and similarly the perturbation with respect to its initial value for the disturbance d

$$\widetilde{d}(t) = d(t) - d(0). \tag{2.24}$$

The solution $\overline{z}^0$ will serve as our desired approximation for the initial condition $\overline{z}_0$ in (2.15).

Substituting, $\overline{z}_1 = \overline{z}^0 + \overline{z}^1$ and $\overline{\gamma}_1 = \overline{\gamma}^0 + \overline{\gamma}^1$ into (2.14)–(2.16) and using (2.21)–(2.22) we see that $\overline{z}^1$ and $\overline{\gamma}^1$ satisfy

$$\overline{z}^1_t(t) = A\overline{z}^1(t) + B\overline{\gamma}^1(t) + B_d\,\widetilde{d}(t), \tag{2.25}$$

$$\overline{z}^1(0) = 0, \tag{2.26}$$

$$C(\overline{z}^1)(t) = E_0(t)\ ,\ \forall\ \ t \geq 0\,. \tag{2.27}$$

where $\widetilde{d}$ is defined in (2.24). Notice that $\widetilde{d}(0) = 0$.

Next we consider solving (2.25)–(2.27) for the control $\overline{\gamma}^1(t)$ by rewriting equation (2.25) as

$$\overline{z}^1(t) = A^{-1}\overline{z}^1_t(t) - A^{-1}B\overline{\gamma}^1(t) - A^{-1}B_d\widetilde{d}(t). \tag{2.28}$$

Then apply C to both sides and use equation (2.27) to obtain

$$E_0(t) = C(\overline{z}^1)(t) = C(A^{-1})\overline{z}^1_t(t) - C(A^{-1})B\overline{\gamma}^1(t) - CA^{-1}B_d\,\widetilde{d}(t). \tag{2.29}$$

In equation (2.29) the condition in (2.10) allows us to write

$$\overline{\gamma}^1(t) = E_0(t) - C(A^{-1})\overline{z}^1_t(t) + C(A^{-1})B_d\widetilde{d}(t), \tag{2.30}$$

but since we don't know $\overline{z}^1_t(t)$ we are no closer to finding $\overline{\gamma}^1$.

Continuing, we substitute (2.30) back into (2.25) in an attempt to obtain a dynamical system that does not contain $\overline{\gamma}^1$. We will show that doing this leads to a singular system. Regularizing this singular system leads to the β-iteration scheme. We have

$$\overline{z}^1_t(t) = A\overline{z}^1(t) + B(E_0(t) - CA^{-1}\overline{z}^1_t(t)) + (I + CA^{-1})B_d\widetilde{d}(t), \tag{2.31}$$

which can be written as

$$(I + BCA^{-1})\overline{z}_t^1(t) = A\overline{z}^1(t) + BE_0(t) + (I + CA^{-1})B_d\widetilde{d}(t), \tag{2.32}$$

which appears to be a dynamic equation not containing $\overline{\gamma}^1$. Unfortunately equation (2.32) is singular, since the operator $(I + BCA^{-1})$ in front of the time derivative term in (2.32) is not invertible. Indeed, $(-BCA^{-1})$ is idempotent due to (2.10) and therefore $(I + BCA^{-1})$ is also idempotent with a nontrivial null space. Our iterative scheme overcomes this difficulty by regularizing equations (2.30) and (2.32) and thereby producing an approximate control $\overline{\gamma}^1(t)$ in (2.30). As a result we do not achieve exact asymptotic tracking. Rather we obtain an approximate tracking control with a nonzero error $E_1(t)$. If the error falls within the prescribed tolerance for Problem 2.7 we stop, otherwise we introduce a new iteration step where, as in the first iteration for $E_0(t)$, the error $E_1(t)$ is used as a new signal to be tracked. This iterative procedure can be repeated to obtain a sequence of more accurate controls and smaller tracking errors till the prescribed tolerance is reached. This is the essence of the β-iteration scheme.

2.3. The Regularized Dynamic Controller (RDC)

The procedure carried out in the previous subsection lies at the heart of the general β iteration scheme which is based on constructing a regularization by introducing a parameter $0 < \beta < 1$, replacing equation (2.30) by

$$\overline{\gamma}^1(t) = E_0(t) - (1-\beta)CA^{-1}\overline{z}_t^1(t) + CA^{-1}B_d\,\widetilde{d}(t). \tag{2.33}$$

Then repeating our attempt to solve for $\overline{z}^1(t)$ using equation (2.33) instead of (2.30) we arrive at

$$(I + (1-\beta)BCA^{-1})\overline{z}_t^1(t) = A\overline{z}(t) + BE_0(t) + (I + CA^{-1})B_d\widetilde{d}(t). \tag{2.34}$$

Using the condition $-CA^{-1}B = I$ from Definition 2.11 it follows that $(I + (1-\beta)BCA^{-1})$ is invertible with inverse

$$\left[I + (1-\beta)BCA^{-1}\right]^{-1} = I - \frac{(1-\beta)}{\beta}(BCA^{-1}). \tag{2.35}$$

Applying $(I + (1-\beta)BCA^{-1})^{-1}$ to both sides of equation (2.34), using formula (2.35), and after a few simple calculations, (2.34) can be written as

$$\overline{z}_t^1(t) = A_\beta\overline{z}^1(t) + \frac{1}{\beta}BE_0(t) + (I + BCA^{-1})B_d\widetilde{d}(t). \tag{2.36}$$

where

$$A_\beta = (A - \zeta BC)\,, \quad \zeta = \frac{(1-\beta)}{\beta}\cdot \tag{2.37}$$

In obtaining formula (2.36) and (2.37) we have used the relations

$$\begin{aligned}
&\left(I - \zeta(BCA^{-1})\right)A = (A - \zeta BC) \equiv A_\beta,\\
&\left(I - \zeta(BCA^{-1})\right)B = \frac{1}{\beta}B,\\
&\left(I - \zeta(BCA^{-1})\right)(I + BCA^{-1}) = I + BCA^{-1},
\end{aligned}$$

which follow from our Assumption 2.11 and the resulting equation (2.10). The following consequences of (2.10) are also used many times throughout the paper

$$CA_\beta^{-1} = \beta CA^{-1}, \quad A_\beta^{-1}B = \beta\, A^{-1}B, \quad CA_\beta^{-1}B = -\beta I. \tag{2.38}$$

Remark 2.14. Under our Assumptions 2.1 and 2.2 it follows from Theorem 3.10.11 in [27] that A_β is a sectorial operator with domain $\mathrm{D}(A_\beta) = \mathrm{D}(A)$ since $P = -\zeta BC \in \mathcal{L}(\mathcal{Z}^{s_c}, \mathcal{Z}^{-s_b})$. This result also shows that A_β generates an analytic semigroup $S_\beta(t)$ on $\mathcal{Z}^\gamma$ for $s_c - 1 \le \gamma \le 1 - s_b$. In particular, under our assumptions of s_b and s_c, this is true for $\gamma = 0$ so $S_\beta(t)$ defines an analytic semigroup in $\mathcal{Z}$.

Furthermore, by Proposition 5.4 in [24], $S_\beta(t)$ is an exponentially stable semigroup for β sufficiently close to 1. In particular, with $\zeta = (1-\beta)/\beta$, for β sufficiently close to 1 there is a constant q so that

$$\|PR(\lambda, A)\|_{-s_b} \le q < 1 \text{ for } \lambda \in \mathbb{C}^+ = \{\lambda \in \mathbb{C} : Re(\lambda) > 0\}.$$

This follows from

$$\|PR(\lambda, A)\|_{-s_b} = \zeta \|BC(-A)^{-s_c}(-A)^{s_c} R(\lambda, A)\|_{-s_b} \le \zeta \|B\|_{-s_b} \|C\|_{-s_c} \sup_{\lambda \in \mathbb{C}^+} \|(-A)^{s_c} R(\lambda, A)\| := q.$$

Here we have

$$\|(-A)^{s_c} R(\lambda, A)\| = \|(-A)^{-1+s_c}(-A)R(\lambda, A)\| \le \|(-A)^{-1+s_c}\| \|AR(\lambda, A)\|$$

and by our assumptions on A, $\|AR(\lambda, A)\|$ is bounded for $\lambda \in \mathbb{C}^+$. In this case the growth bound for A_β satisfies $-\omega_\beta < 0$ and there is an $M_\beta \ge 1$ so that

$$\|S_\beta(t)\| = \|\mathrm{e}^{A_\beta t}\| \le M_\beta \mathrm{e}^{-\omega_\beta t}. \tag{2.39}$$

We have demonstrated that the regularized version of (2.25)–(2.27) can be written as the dynamical system

$$\overline{z}^1_t(t) = A_\beta \overline{z}^1(t) + \frac{1}{\beta} B E_0(t) + \mathcal{D}\widetilde{d}(t), \tag{2.40}$$

$$\overline{z}^1(0) = 0, \tag{2.41}$$

where

$$\mathcal{D} = (I + BCA^{-1})B_d. \tag{2.42}$$

Notice that because of the introduction of the regularization $(1-\beta)$ in equation (2.33) we no longer have $C(\overline{z}^1)(t) = E_0(t)$ for all t, thus we define the resulting difference to be

$$E_1(t) = E_0(t) - C(\overline{z}^1)(t), \tag{2.43}$$

where

$$E_1(t) = E_0(t) - C(\overline{z}^1)(t) = y_r(t) - C(\overline{z}^0)(t) - C(\overline{z}^1)(t) = y_r(t) - C(\overline{z}_1)(t).$$

The dynamical system (2.40), (2.41) forms the basis for our regularization and iteration scheme.

For this scheme, we obtain an explicit estimate for the error $E_n(t)$, obtained at the nth step of the iteration for fairly general smooth bounded reference signals $y_r(t)$ and disturbances $d(t)$.

The above calculations also demonstrate that there is a one-to-one equivalence between the solutions of the two systems

$$\begin{cases} \overline{z}^1_t(t) = A\overline{z}^1(t) + B\overline{\gamma}^1(t) + B_d\, \widetilde{d}(t) \\ \overline{\gamma}^1(t) = E_0(t) - (1-\beta)CA^{-1}\overline{z}^1_t(t) + CA^{-1}B_d\, \widetilde{d}(t) \\ \overline{z}^1(0) = 0, \end{cases} \tag{2.44}$$

and

$$\begin{cases} \overline{z}^1_t(t) = A_\beta \overline{z}^1(t) + \frac{1}{\beta} B E_0(t) + \mathcal{D}\, \widetilde{d}(t) \\ \overline{z}^1(0) = 0. \end{cases} \tag{2.45}$$

This equivalence will be used later in Section 2.4, below.

2.4. The iterative scheme (continued)

In Section 2.2 we obtained $\overline{z}^0$ and $\overline{\gamma}^0$ solving the set-point part of the problem. We also introduced the regularization in Section 2.3 to obtain $\overline{z}^1$ and $\overline{\gamma}^1$, which is the starting point for β-iteration scheme. In particular, for $n = 1$ we defined $\overline{z}_1 = \overline{z}^0 + \overline{z}^1$, and $\overline{\gamma}_1 = \overline{\gamma}^0 + \overline{\gamma}^1$, which provided the first approximations for $\overline{z}$ and $\overline{\gamma}$. For each $n > 1$, the approximate states and controls are defined in the form

$$\overline{z}_n(t) = \sum_{i=0}^{n} \overline{z}^i(t), \quad \overline{\gamma}_n(t) = \sum_{i=0}^{n} \overline{\gamma}^i(t). \tag{2.46}$$

Just as in Sections 2.2 and 2.3, for $n \leq 1$, also for $n > 1$ the state and control, $\overline{z}_n$ and $\overline{\gamma}_n$, are sought to satisfy (2.14) and (2.16). As before, because of regularization, this task can not be achieved exactly. The resulting errors are defined as

$$E_n(t) = y_r(t) - C(\overline{z}_n(t)). \tag{2.47}$$

It is easy to see that the error satisfies the following recursive identity

$$E_n(t) = y_r(t) - C(\overline{z}_{n-1}(t)) - C(\overline{z}^n(t)) = E_{n-1}(t) - C(\overline{z}^n(t)), \text{ for } n > 1. \tag{2.48}$$

Each new iteration is designed to cancel the cumulative error, E_{n-1}, obtained at the previous iteration. But due to the β regularization at the nth step we do not completely achieve this goal. The iterative process can be repeated until the error E_n is small enough, *i.e.*, until the goal set in Problem 2.7 is achieved for a given ϵ.

Referring to the definitions in (2.46), assuming $\overline{z}_j$ and $\overline{\gamma}_j$ satisfy (2.14) for $j = 2, \ldots, n$, since $\overline{z}_j = \overline{z}_{j-1} + \overline{z}^j$ we can repeat the regularization procedure in Section 2.3 to obtain the following regularized system for $\overline{z}^j$.

$$\begin{cases} \overline{z}^j_t(t) = A\overline{z}^j(t) + B\overline{\gamma}^j(t) \\ \overline{\gamma}^j(t) = E_{j-1}(t) - (1-\beta)CA^{-1}\overline{z}^j_t(t) \\ \overline{z}^j(0) = 0. \end{cases} \tag{2.49}$$

Notice that for $j > 1$ the systems no longer contain any contribution from the disturbance term.

The discussion in Section 2.3 shows that system (2.49) can be written equivalently as the dynamical system

$$\begin{cases} \overline{z}^j_t(t) = A_\beta \overline{z}^j(t) + \frac{1}{\beta} B E_{j-1}(t) \\ \overline{z}^j(0) = 0. \end{cases} \tag{2.50}$$

Remark 2.15. Notice that the following condition holds for all n

$$\begin{aligned} E_n(0) = y_r(0) - C(\overline{z}_n(0)) &= y_r(0) - \sum_{j=0}^{n} C(\overline{z}^j(0)) \\ &= \big(y_r(0) - C(\overline{z}^1(0))\big) + \sum_{j=1}^{n} C(\overline{z}^j(0)) = 0\,, \end{aligned} \tag{2.51}$$

because of the zero initial conditions in equation (2.44) and the constraint in (2.22).

We close this section by including the main Convergence Theorem establishing that, under suitable assumptions on the reference and disturbance signals, our β-iteration algorithm solves Problem 2.7.

Definition 2.16. Let

$$K(t) = -CA_\beta^{-1} \mathrm{e}^{A_\beta t} \frac{1}{\beta} B \in \mathcal{L}(\mathcal{U}, \mathcal{Y}), \tag{2.52}$$

$$K_d(t) = -CA_\beta^{-1} \mathrm{e}^{A_\beta t} \mathcal{D} \in \mathcal{L}(\mathcal{D}, \mathcal{Y}), \tag{2.53}$$

where $\mathcal{D}$ is defined in (2.42).

Theorem 2.17. *Let* $\mathcal{L} = \mathcal{L}(\mathcal{U}, \mathcal{Y})$ *and* $\mathcal{L}_d = \mathcal{L}(\mathcal{D}, \mathcal{Y})$ *and define* D *and* D_d *by*

$$D = \int_0^\infty \|K(t)\|_{\mathcal{L}}\, \mathrm{d}t, \qquad D_d = \int_0^\infty \|K_d(t)\|_{\mathcal{L}_d}\, \mathrm{d}t, \quad \mathfrak{D} = \mathfrak{D}_{\mathfrak{d}} D. \tag{2.54}$$

Let

$$C_n = \Big(\limsup_{[0,\infty)} \|y_r^{(n)}(t)\|_{\mathcal{Y}} + \mathfrak{D} \limsup_{[0,\infty)} \|d(t)^{(n)}(t)\|_{\mathcal{D}} \Big). \tag{2.55}$$

Then

$$\limsup_{t>0} \|y_r(t) - C(\overline{z}_n)(t)\|_{\mathcal{Y}} = \limsup_{t>0} \|E_n(t)\|_{\mathcal{Y}} \le D^n\ C_n. \tag{2.56}$$

Moreover, if there exist constants $\overline{\alpha}$ *and* $\overline{A}$ *so that*

$$C_n \le \overline{A}\,\overline{\alpha}^n \quad \textit{and} \quad D\overline{\alpha} < 1,$$

then

$$\limsup_{t>0} \|E_n(t)\|_{\mathcal{Y}} \le \overline{A}(D\overline{\alpha})^n \xrightarrow{n\to\infty} 0.$$

In this case for sufficiently large n *the* β*-iteration solves Problem 2.7 (see Rem. 2.18 below).*

The proof of Theorem 2.17 is given at the end of Section 3.

Remark 2.18. (1) Under our assumptions on C, B and B_d the values of D and D_d in (2.54) are finite. In particular, for values of s including s_c, s_b and s_d in Assumption 2.2, the scale of spaces $\mathcal{Z}^{-s}$ generated by A and the spaces $\mathcal{Z}_\beta^{-s}$ generated by A_β are equal and the norms in these spaces are equivalent (see, *e.g.*, [16], Lem. 9.4.3, p. 443 or [27], Thm. 3.10.11, p. 174). Therefore we can conclude that $C \in \mathcal{L}(\mathcal{Z}^{s_c}, \mathcal{Y})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z}^{-s_b})$ and $D \in \mathcal{L}(\mathcal{D}, \mathcal{Z}^{-s_d})$ if and only if $C \in \mathcal{L}(\mathcal{Z}_\beta^{s_c}, \mathcal{Y})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z}_\beta^{-s_b})$ and $D \in \mathcal{L}(\mathcal{D}, \mathcal{Z}_\beta^{-s_d})$. In the following we denote the norm in $\mathcal{Z}_\beta^s$ by $\|\cdot\|_{\beta,s}$.

(2) In our estimates for the norms of $K(t)$ we will use bounds in the scale of spaces $\mathcal{Z}_\beta^{-s}$.

$$\|K(t)\|_{\mathcal{L}} \le \frac{\|C\|_{\beta,-s_c}\|B\|_{\beta,-s_b}\|(-A_\beta)^{-1+\delta}\|M_\beta\ \mathrm{e}^{-\omega_\beta t}}{\beta} := \widetilde{Q}\,\mathrm{e}^{-\omega_\beta t}, \tag{2.57}$$

so that

$$D = \int_0^\infty \|K(t)\|_{\mathcal{L}}\, \mathrm{d}t \le \widetilde{Q} \int_0^\infty \mathrm{e}^{-\omega_\beta t} \mathrm{d}t = \frac{\widetilde{Q}}{\omega_\beta} := \widetilde{D} < \infty. \tag{2.58}$$

(3) For $K_d(t) = -CA_\beta^{-1}\mathrm{e}^{A_\beta t}\mathcal{D}$ we need to estimate $\mathcal{D} = (I + BCA^{-1})B_d$. Set $s_m = \max\{s_b, s_d\}$ and

$$\|(-A_\beta)^{-s_m}\mathcal{D}\| \le \|(-A_\beta)^{-s_m}B_d\| + \|(-A_\beta)^{-s_m}B\|\|C(-A)^{-s_c}\|\|(-A)^{-1+s_c}B_d\| := \widetilde{R}_{\mathcal{D}} < \infty.$$

Notice that in the last term we have $-1 + s_c < -s_d$ due to our assumption $s_c + s_d < 1$. Set $p = s_c + s_m < 1$ and we have

$$\begin{aligned}\|K_d(t)\|_{\mathcal{L}_d} &= \| - C(-A_\beta)^{-1}\mathrm{e}^{A_\beta t}(I + BCA^{-1})B_d\| \\ &\le \|C(-A_\beta)^{-s_c}\|\ \|(-A_\beta)^{-1+p}\|\ \|\mathrm{e}^{A_\beta t}\|\ \widetilde{R}_{\mathcal{D}} \\ &\le \|C\|_{\beta,-s_c}\widetilde{R}_{\mathcal{D}}\|(-A_\beta)^{-1+p}\|\ M_\beta \mathrm{e}^{-\omega_\beta t} = \widetilde{Q}_d \mathrm{e}^{-\omega_\beta t},\end{aligned} \tag{2.59}$$

where we have defined

$$\widetilde{Q}_d = \|C\|_{\beta,-s_c}\widetilde{R}_{\mathcal{D}} M_\beta \|(-A_\beta)^{-1+p}\|\ . \tag{2.60}$$

Then we have a bound on the L^1 norm of $K_d(t)$ given by

$$D_d = \int_0^\infty \|K_d(t)\|_{\mathcal{L}_d} \mathrm{d}t \le \widetilde{Q}_d \int_0^\infty \mathrm{e}^{-\omega_\beta t} \mathrm{d}t = \frac{\widetilde{Q}_d}{\omega_\beta} = \widetilde{D}_d < \infty. \tag{2.61}$$

(4) For $0 < s < 1$, the definition of $(-A_\beta)^{-s}$ (see, for example, Def. 1.4.1 in [15]) gives

$$\|(-A_\beta)^{-1+s}\| \le \frac{M_\beta}{\omega_\beta^{1-s}}. \tag{2.62}$$

(5) In the special case in which the tracking and disturbance are harmonic signals

$$y_r = m_r + A_r \sin(\alpha_r t), \quad d = m_d + A_d \sin(\alpha_d t), \quad \alpha_r > 0, \quad \alpha_d > 0,$$

the bound on the error $E_n(t)$ given in (2.56) becomes

$$\limsup_{t>0} \|y_r(t) - C(\overline{z}_n)(t)\|_{\mathcal{Y}} = \limsup_{t>0} \|E_n(t)\|_{\mathcal{Y}} \le D^n \left(|A_r|\alpha_r^n + |A_d| \frac{D_d}{D} \alpha_d^n\right).$$

If $\overline{\alpha} = \max\{\alpha_r, \alpha_d\}$ and $\overline{\alpha} D < 1$ then the β-iteration solves Problem 2.7, for sufficiently large n. On the other hand, simple numerical simulations for a one dimensional heat equation (not included in the paper) show that if $\overline{\alpha} D > 1$ then the β-iteration does not converge.

In the following sections we set out to systematically establish Theorem 2.17.

3. Iterative error as a convolution integral

We first show that the error $E_n(t)$ at each iteration can be expressed in terms of a convolution integral.

Error in the first iteration. Recall that the two systems in (2.44) and (2.45) and apply the variation of parameters formula to (2.45) to obtain

$$\overline{z}^1(t) = \int_0^t \mathrm{e}^{A_\beta(t-\tau)} \left(\frac{1}{\beta} B E_0(\tau) + \mathcal{D}\widetilde{d}(\tau)\right) \mathrm{d}\tau. \tag{3.1}$$

Since $E_0(t)$, $\widetilde{d}(t)$ and $\mathrm{e}^{A_\beta t}$ are continuously differentiable we can integrate by parts to obtain

$$\begin{aligned}\overline{z}^1(t) &= (-A_\beta^{-1}) \left(\frac{1}{\beta} B E_0(t) + \mathcal{D}\widetilde{d}(t)\right) + A_\beta^{-1} \mathrm{e}^{A_\beta t} \left(\frac{1}{\beta} B E_0(0) + \mathcal{D}\widetilde{d}(0)\right) \\ &\quad + A_\beta^{-1} \int_0^t \mathrm{e}^{A_\beta(t-\tau)} \left(\frac{1}{\beta} B E_0(\tau) + \mathcal{D}\widetilde{d}(\tau)\right)' \mathrm{d}\tau.\end{aligned}$$

Recalling that $E_0(0) = 0$, $\widetilde{d}(0) = 0$ and applying the output operator C to the previous equation we have

$$\begin{aligned}C(\overline{z}^1(t)) &= C(-A_\beta^{-1}) \left(\frac{1}{\beta} B E_0(t) + \mathcal{D}\widetilde{d}(t)\right) \\ &\quad + C A_\beta^{-1} \int_0^t \mathrm{e}^{A_\beta(t-\tau)} \left(\frac{1}{\beta} B E_0(\tau) + \mathcal{D}\widetilde{d}(\tau)\right)' \mathrm{d}\tau.\end{aligned} \tag{3.2}$$

Notice that

$$\begin{aligned}A_\beta^{-1}\mathcal{D} &= A_\beta^{-1}(I + BCA^{-1})B_d = A^{-1}\left(I - \frac{(1-\beta)}{\beta} BCA^{-1}\right)^{-1} (I + BCA^{-1})B_d \\ &= A^{-1}(I + BCA^{-1})B_d \widetilde{d}(t) = A^{-1}\mathcal{D},\end{aligned} \tag{3.3}$$

where on the last step we have used (2.35). equations. (2.10), (2.38) and (3.3) imply $C\left(A_\beta^{-1}\right)\mathcal{D} = CA^{-1}(I + BCA^{-1}) = 0$ so we have

$$\begin{aligned} C\left(-A_\beta^{-1}\right)\left(\frac{1}{\beta}BE_0(t) + \mathcal{D}\widetilde{d}(t)\right) &= -C\left(A_\beta^{-1}\right)\frac{1}{\beta}BE_0(t) - CA_\beta^{-1}\mathcal{D}\widetilde{d}(t) \\ &= E_0(t) - CA^{-1}(I + BCA^{-1})B_d\widetilde{d}(t) = E_0(t). \end{aligned}$$

Therefore, after rearranging terms and recalling the definition of $E_1(t)$ in (2.43), equation (3.2) can be written, in the form,

$$E_1(t) := E_0(t) - C(\overline{z}^1(t)) = -CA_\beta^{-1}\int_0^t \mathrm{e}^{A_\beta(t-\tau)}\left(\frac{1}{\beta}BE_0'(\tau) + \mathcal{D}(\widetilde{d})'(\tau)\right)\mathrm{d}\tau. \tag{3.4}$$

Error at the nth-iteration with $(n \geq 2)$. To derive a general formula for the error for the $n \geq 2$ iteration, we proceed starting from system (2.50) as done above for the $n = 1$ iteration. Notice that system (2.45) and (2.50) differ only by the indices and the absence of the disturbance, so that by analogy with equation (3.4) we have

$$E_n(t) = E_{n-1}(t) - C(\overline{z}^n(t)) = -CA_\beta^{-1}\int_0^t \mathrm{e}^{A_\beta(t-\tau)}\frac{1}{\beta}BE_{n-1}'(\tau)\,\mathrm{d}\tau, \text{ for } n = 2, 3, \ldots. \tag{3.5}$$

3.1. Error estimates

In this section we present the main estimates for $E_n(t)$ which are given in Theorem 3.8. Several propositions and lemmas in this section have proofs that involve mathematical induction involving several steps whose proofs would detract from understanding the statements of the main results. Therefore the proofs are postponed to Section 5. In this section we will show that the error at the nth iteration can be written as a sum of iterated convolutions that can be estimated explicitly. This is explained in Lemma 3.5.

Using the definitions of K and K_d in Definition 2.16 together with (3.5) we can write

$$E_n(t) = K * E_{n-1}'(t), \quad \text{for} \quad n \geq 2. \tag{3.6}$$

where from (3.4) for $n = 1$ we have

$$E_1(t) = K * E_0'(t) + K_d * (\widetilde{d})'(t) \tag{3.7}$$

Our objective is to estimate the error at the nth β-iteration, $E_n(t)$, in terms of the reference, $y_r(t)$, and disturbance, $d(t)$. To do this we want to express $E_n(t)$ as an n-fold convolution. This can be accomplished using (3.6), (3.7) and some elementary properties of convolutions. The first such properties are contained in the following proposition.

Recall our notation for the input and disturbance spaces $\mathcal{U} = \mathbb{R}^k$ for $K(t)$ or $\mathcal{D} = \mathbb{R}^m$ for $K_d(t)$.

Proposition 3.1. *For any $\varphi \in C_b^\infty(\mathbb{R}^+, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$ or $C_b^\infty(\mathbb{R}^+, \mathcal{L}(\mathcal{D}, \mathcal{Y}))$ we have*

$$(K * \varphi)'(t) = K(t)\varphi(0) + (K * \varphi')(t) \tag{3.8}$$
$$= \varphi(t) + (K' * \varphi)(t), \tag{3.9}$$

$$(K_d * \varphi)'(t) = K_d(t)\varphi(0) + (K_d * \varphi')(t) \tag{3.10}$$
$$= (K_d' * \varphi)(t), \tag{3.11}$$

The two forms of the equations in Proposition 3.1 follow by applying Leibniz rule and using the commutativity of the convolution product, *e.g.*,

$$(K*\varphi)'(t) = K(0)\varphi(t) + (K'*\varphi)(t) = K(t)\varphi(0) + (K*\varphi')(t).$$

The expressions in (3.9) and (3.11) follow from $K(0) = -CA_\beta^{-1}\beta^{-1}B = -CA^{-1}B = I$ and

$$K_d(0) = -\beta CA^{-1}(I + BCA^{-1})B_d = -\beta(CA^{-1} - CA^{-1})B_d = 0.$$

Remark 3.2. The first form of the result in Proposition 3.1 allows us to avoid the appearance of the higher order derivatives of K and K_d by sending the derivatives to φ. The second form is useful to obtain our error estimates involving $y_r(t)$ or more appropriately $E_0(t)$.

Let us introduce the notation $(\mathcal{G}*)^i$ for the i-fold-convolution of $\mathcal{G}$, *i.e.*,

$$(\mathcal{G}*)^i\varphi(t) = \int_0^t \mathcal{G}(t-\tau_1)\int_0^{\tau_1}\mathcal{G}(\tau_1-\tau_2)\dots\int_0^{\tau_{i-1}}\mathcal{G}(\tau_{i-1}-\tau_i)\varphi(\tau_i)\,\mathrm{d}\tau_i\dots\mathrm{d}\tau_2\,\mathrm{d}\tau_1\,.$$

Applying Proposition 3.1 repeatedly in (3.6) and beginning with (3.7) we are able to reduce the complicated expressions to multi-fold convolutions.

Lemma 3.3. *Let us introduce some new notation*

$$H_0(t) = K*K(t) \text{ and } H_n(t) = K*(H_{n-1})'(t) \text{ for } n = 1,2,\dots,$$

and

$$H_{d,0}(t) = K*K_d(t) \text{ and } H_{d,n}(t) = K*(H_{d,n-1})'(t) \text{ for } n = 1,2,\dots,$$

then we have

$$H_n(t) = \sum_{i=0}^{n}\binom{n}{i}K*(K'*)^iK(t), \tag{3.12}$$

$$H_{d,n}(t) = \sum_{i=0}^{n}\binom{n}{i}K*(K'*)^iK_d(t). \tag{3.13}$$

The importance of this lemma is that it allows us to replace the derivative terms obtained in H_n and $H_{d,n}$ by sums involving iterated convolutions. The proof of Lemma 3.3 can be found in Section 5.

Remark 3.4. Since $H_n(t)$ and $H_{d,n}(t)$ are convolutions we have

$$H_n(0) = H_{d,n}(0) = 0.$$

Returning to our objective of reducing $E_n(t)$ to a sum of iterated convolutions that can be estimated explicitly we have the following main lemma.

Lemma 3.5. *Assume K and K_d are as in the Definition 2.16. Let $y_r \in C_b^N(\mathbb{R}^+, \mathcal{Y})$, and $d \in C_b^N(\mathbb{R}^+, \mathcal{D})$ and recall $E_0(t) = y_r(t) - y_r(0)$ and $\widetilde{d}(t) = d(t) - d(0)$.*

$$E_1(t) = K*E_0'(t) + K_d*(\widetilde{d})'(t) \text{ and } E_n(t) = K*E_{n-1}'(t) \text{ for } n \geq 2.$$

For all $n \geq 2$

$$E_n(t) = \sum_{i=1}^{n-1}(K*)^{i-1}\Big(H_{n-(i+1)}E_0^{(i)}(0) + H_{d,n-(i+1)}\widetilde{d}^{(i)}(0)\Big) + (K*)^{n-1}K*E_0^{(n)}(t) + (K*)^{n-1}K_d*\widetilde{d}^{(n)}(t). \tag{3.14}$$

Furthermore, using Lemma 3.3, we can rewrite the error $E_n(t)$ as a sum of iterated convolutions

$$E_n(t) = \sum_{i=1}^{n-1} \sum_{j=0}^{n-(i+1)} \binom{n-(i+1)}{j} (K*)^i * (K'*)^j \left(KE_0^{(i)}(0) + K_d\widetilde{d}^{(i)}(0)\right) + (K*)^{n-1}K * E_0^{(n)}(t) + (K*)^{n-1}K_d * \widetilde{d}^{(n)}(t). \tag{3.15}$$

The proofs of Lemma 3.5 and Proposition 3.7 are given Section 5.

Remark 3.6. In the rest of the paper we will suppress, for ease of notation, the subscripts $\mathcal{L}$ and $\mathcal{L}_d$ for the norms of $\|K(t)\|$ and $\|K_d(t)\|$.

Proposition 3.7. *Assume K and K_d as given in Definition 2.16 with B and C defined in Assumption 2.2. Let $\delta = s_c + s_b$ and let $\beta \in (0,1)$ be chosen so that the operator A_β generates an exponentially stable analytic semigroup in $\mathcal{Z}$. Then,*

$$\|(K*)^i(K'*)^jK(t)\| \leq R_{ij}\ \mathrm{e}^{-\omega_\beta t}\frac{t^{i+j(1-\delta)}\Gamma(1-\delta)^j}{\Gamma(i+1+j(1-\delta))}, \tag{3.16}$$

and

$$\|(K*)^i(K'*)^jK_d(t)\| \leq R_{d,ij}\ \mathrm{e}^{-\omega_\beta t}\frac{t^{i+j(1-\delta)}\Gamma(1-\delta)^j}{\Gamma(i+1+j(1-\delta))}, \tag{3.17}$$

where R_{ij} and $R_{d,ij}$ are defined in (5.14).

Moreover, recalling the definitions of D and D_d in Theorem 2.17, for all $n \geq 1$ the following inequalities hold

$$\|(K*)^n(t)\| \leq (\|K\|*)^n(t) \leq \left(\int_0^t \|K(\tau)\|\,\mathrm{d}\tau\right)^n \leq (\|K\|)^n = D^n, \tag{3.18}$$

and

$$\|(K*)^{n-1}(K_d*)(t)\| \leq (\|K\|*)^{n-1}(\|K_d\|*)\| \leq \left(\int_0^t \|K_d(\tau)\|\,\mathrm{d}\tau\right)\left(\int_0^t \|K(\tau)\|\,\mathrm{d}\tau\right)^{n-1} \leq D_dD^{n-1}. \tag{3.19}$$

Theorem 3.8. *Let $y_r \in C_b^N(\mathbb{R}^+, \mathcal{Y})$, $d \in C_b^N(\mathbb{R}^+, \mathcal{D})$. Assume that A, B and C satisfy Assumptions 2.1, 2.2, and $0 < \beta_0 < 1$ is chosen, as in Remark 2.14, so that for $\beta \in (\beta_0, 1)$, A_β generates an exponentially stable analytic semigroup in $\mathcal{Z}$. Then for any $T \geq 0$ we can write the explicit error $E_n(t)$ at the nth iteration, given in (3.15), as a sum of three terms:*

$$E_n(t) = \mathcal{E}_{1,n}(t) + \mathcal{E}_{2,T,n}(t) + \mathcal{E}_{3,T,n}(t), \tag{3.20}$$

where,

$$\|\mathcal{E}_{1,n}(t) + \mathcal{E}_{2,T,n}(t)\|_{\mathcal{Y}} = g_n(t)\mathrm{e}^{-\overline{\omega}t}, \text{ for some } \overline{\omega} > 0, \tag{3.21}$$

and g_n is a function (whose exact form is given in the proof in Section 5) that is bounded by a polynomial in t. Therefore

$$\limsup_{t>0} \|\mathcal{E}_{1,n}(t) + \mathcal{E}_{2,T,n}(t)\|_{\mathcal{Y}} = 0. \tag{3.22}$$

Further, for any $T > 0$,

$$\sup_{[T,\infty)} \|\mathcal{E}_{3,T,n}\|_{\mathcal{Y}} \leq D^n\left(\sup_{[T,\infty)}\|E_0^{(n)}\|_{\mathcal{Y}} + \frac{D_d}{D}\sup_{[T,\infty)}\|\widetilde{d}^{(n)}\|_{\mathcal{D}}\right), \tag{3.23}$$

where D and D_d are defined in Theorem 2.17.

Explicitly we have

$$\mathcal{E}_{1,n}(t) = \sum_{i=1}^{n-1} \sum_{j=0}^{n-(i+1)} \binom{n-(i+1)}{j} (K*)^i * (K'*)^j (K E_0^{(i)}(0) + K_d \widetilde{d}^{(i)}(0)), \tag{3.24}$$

$$\mathcal{E}_{2,T,n}(t) = \int_0^T K(t-\tau)\Big((K*)^{n-2} K * E_0^{(n)}(\tau) + (K*)^{n-2} K_d * \widetilde{d}^{(n)}(\tau)\Big)\mathrm{d}\tau, \tag{3.25}$$

$$\mathcal{E}_{3,T,n}(t) = \int_T^t K(t-\tau)\Big((K*)^{n-2} K * E_0^{(n)}(\tau) + (K*)^{n-2} K_d * \widetilde{d}^{(n)}(\tau)\Big)\mathrm{d}\tau. \tag{3.26}$$

The proof of Theorem 3.8 is given in Section 5.

Proof of Theorem 2.17. The main result (2.56) in Theorem 2.17 follows immediately from (3.22) and (3.23) in Theorem 3.8 by taking the limit supremum over $t > 0$.

$$\limsup_{t>0} \|E_n(t)\|_{\mathcal{Y}} \le D^n \left(\limsup_{t>0} \|E_0^{(n)}\|_{\mathcal{Y}} + \frac{D_d}{D} \limsup_{t>0} \|\widetilde{d}^{(n)}\|_{\mathcal{D}} \right). \tag{3.27}$$

□

4. Numerical examples

The main objective of this section is to present three numerical examples to demonstrate the utility of the estimates obtained in Section 3. In the first example we consider a one dimensional heat equation with boundary control input, disturbance entering through a Dirac delta source in the interior and a point evaluation operator producing the measured output. Therefore all three are unbounded operators in the Hilbert state space. In the second example the reference and disturbance signals are generated as oscillating solutions of a nonlinear Duffing equation. Because of this nonlinearity it would be quite challenging to find the desired control input using the classical theory of geometric control [1] by solving the regulator equations. On the other hand the use of the β iterative algorithm is straightforward, since its applicability is general and independent on how the reference and disturbance signals are generated. This example also demonstrates a situation that motivates our use of limsup for $t > 0$ due to the fact that reference and disturbance signals, or their derivatives, can have large transient oscillations but eventually settle into a simpler asymptotic behavior. The third example is somewhat more complex involving an example of thermal regulation of a steady Navier–Stokes flow.

Example 4.1. In our first example we consider a plant modeled by a boundary controlled one dimensional heat equation in $\mathcal{Z} = L^2(0,1)$

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + \delta_{0.5} d(t), \tag{4.1}$$

$$z(0,t) = 0, \quad z_x(1,t) = u(t), \tag{4.2}$$

$$y(t) = z(0.75, t), \tag{4.3}$$

where δ_{x_0} is the Dirac delta function supported at x_0. In addition, we supplement (4.1)–(4.3) with an arbitrary initial condition $z(x,0) = \varphi(x)$ with $\varphi \in \mathcal{Z}$.

The system (4.1)–(4.3) can be rewritten in standard systems form as

$$z_t = Az + B_d d(t) + B_{\text{in}} u(t), \tag{4.4}$$

$$y(t) = Cz(t), \tag{4.5}$$

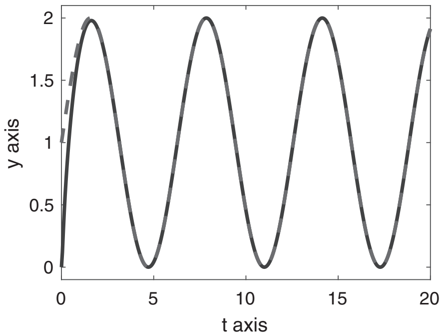

FIGURE 1. $y(t)$, $y_r(t)$.

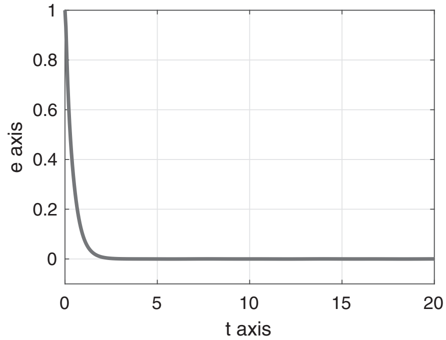

FIGURE 2. Error $E(t)$.

where $B_d = \delta_{0.5}$ and $B_{\text{in}} = \delta_1$, and the operator A is

$$A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad \mathrm{D}(A) = \{\varphi \in H^2(0,1) \,:\, \varphi(0) = 0, \varphi'(1) = 0\}. \tag{4.6}$$

The measured output $y(t)$ in (4.3) is the point evaluation at $x = 0.75$, so C is an unbounded operator in $\mathcal{Z}$. It follows from Sobolev embedding that $C \in \mathcal{L}(H^1(0,1), \mathbb{R})$ where $H^1(0,1) = \mathcal{Z}^{1/2}$.

A is a negative self-adjoint (and hence sectorial operator) in $\mathcal{Z} = L^2(0,1)$ that generates an exponentially stable analytic semigroup in $\mathcal{Z}$. The spectrum of A consists of an infinite set of real negative eigenvalues $\lambda_j = -\mu_j^2$ where $\mu_j = (j - 1/2)\pi$ and a complete set (in $L^2(0,1)$) of orthonormal eigenfunctions $\psi_j(x) = \sqrt{2}\sin(\mu_j x)$ for $j = 1, 2, \ldots$.

It is easy to show using the functional calculus from spectral theory for the Hilbert scale of spaces $\mathcal{Z}^{-s}$ for $s > 0$ that the operator $B_{\text{in}} \in \mathcal{L}(\mathbb{R}, \mathcal{Z}^{-s_b})$, $B_d \in \mathcal{L}(\mathbb{R}, \mathcal{Z}^{-s_d})$ and $C \in \mathcal{L}(\mathcal{Z}^{s_c}, \mathbb{R})$ with $s_c = s_d = s_b = 1/4 + \epsilon$ for any $\epsilon > 0$. Notice, in particular, that $s_c + s_b < 1$ and $s_c + s_d < 1$, for $\epsilon < 1/4$, so the conditions in Assumption 2.2 are satisfied.

Having chosen our value $\beta = 0.3$, we solve the β iterative algorithm, discussed in section 2.4 for four iterations in COMSOL. The reference signal to be tracked is given by $y_r(t) = 1 + \sin(t)$ and the disturbance to be rejected is $d(t) = 1 + 2\sin(2t)$.

In Figure 1 we have depicted $y = C(z)$ and y_r. In Figure 2 we have plotted the resulting error $E(t)$. In Figures 3–6 we have plotted the respective pairs E_{j-1}, and E_j for $j = 1, 2, 3, 4$. Just as our estimates in (3.20) and (3.22) predict, the errors $E_j(t)$ possess larger transients that converge to 0 exponentially in t. Since we are interested in asymptotic tracking, in the following figures we plot pairs of the values of $E_j(t)$, $j = 0, 1, 2, 3, 4$ for $10 < t < 20$ to draw attention to the asymptotic behavior.

Finally in Figure 7 we have plotted both E and E_4 and in Figure 8 we have plotted the resulting control. We note that the maximum deviation of $|E(t) - E_4(t)|$ for $10 \le t \le 20$ is less than 2.2e-11.

We can estimate the errors $E_j(t)$ using Comsol to numerically evaluate D and D_d in (2.56). For this particular example we have

$$A_r = 1, \;\; \alpha_r = 1, \;\; A_d = 2, \;\; \alpha_d = 2,$$

resulting in

$$D = 1.2195\text{e-}01, \quad D_d = 3.9658\text{e-}02, \quad D^j C_j = D^j \left(1 + 2\frac{D_d}{D} 2^j\right) \text{ for } j = 1, \ldots, 4.$$

In Table 1 we display the values of $\limsup \|E_j\|$ compared to the theoretical bounds $D^j C_j$. For all j we notice that the estimates are fairly sharp.

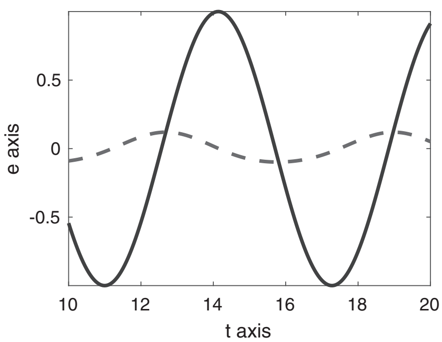

FIGURE 3. $E_0(t)$, $E_1(t)$ (*dashed*).

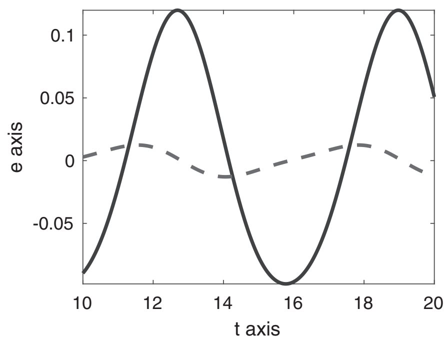

FIGURE 4. $E_1(t)$, $E_2(t)$ (*dashed*).

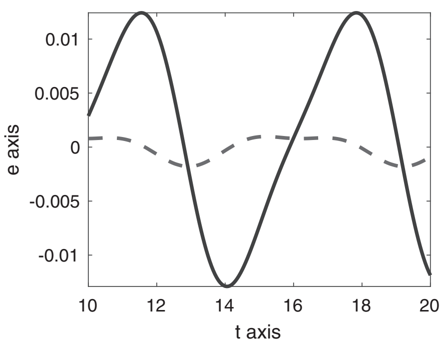

FIGURE 5. $E_2(t)$, $E_3(t)$ (*dashed*).

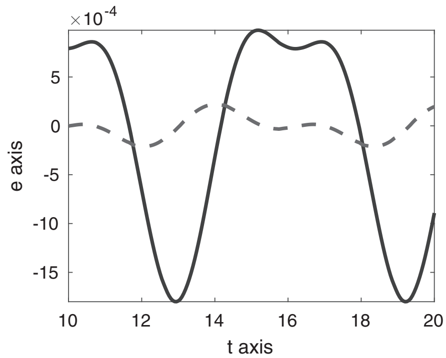

FIGURE 6. $E_3(t)$, $E_4(t)$ (*dashed*).

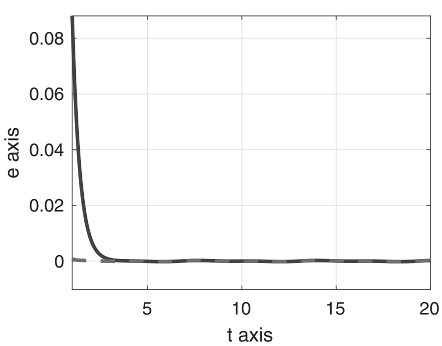

FIGURE 7. $E(t)$, $E_4(t)$ (*dashed*).

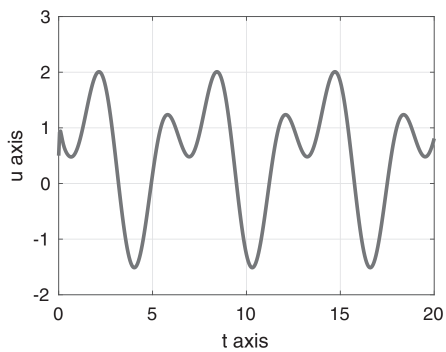

FIGURE 8. Control $u_4(t)$.

TABLE 1. Compare lim sup of E_j and $D^j C_j$.

	$j=1$	$j=2$	$j=3$	$j=4$
$D^j C_j$	2.8058e-01	5.3563e-02	1.1250e-02	2.5228e-03
$\limsup \Vert E_j \Vert$	1.1985e-01	1.2901e-02	1.8003e-03	2.2047e-04

Example 4.2. In our second example we consider again the control system governed by a one dimensional heat equation on $0 \le x \le 1$ for $t \ge 0$, namely

$$\begin{aligned} &\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + B_{\mathrm{in}} u(t) + B_d d(t), \\ &z(0) = z_0, \\ &z(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = 0, \\ &y(t) = Cz(t) = z(1,t). \end{aligned}$$

The operator A in this case is exactly the same as in equation (4.6) in Example 4.1. The operators B_{in} and B_d are distributed forces over the intervals $I_1 = \{x\colon 1/4 \le x \le 1/2\}$ and $I_2 = \{x\colon 1/2 \le x \le 3/4\}$, respectively, given as multiplication operators in terms of the characteristic function χ by $B_{\mathrm{in}} = \chi_{I_1}$ and $B_d = \chi_{I_2}$.

Unlike Example 4.1, in this example the control and disturbance enter through bounded operators in $\mathcal{Z}$ and the reference and disturbance signals are generated by a nonlinear ordinary differential equation

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}t^2} + \frac{3}{20}\frac{\mathrm{d}\omega}{\mathrm{d}t} - \omega + \omega^3 = 0. \tag{4.7}$$

The nonlinear equation (4.7) is a Duffing oscillator which has 3 equilibria given in terms of the pair $(\omega, \mathrm{d}\omega/\mathrm{d}t)$ by $(0,0)$ and $(\pm 1, 0)$: $(0,0)$ is a saddle node and $(\pm 1, 0)$ are sinks. For the initial condition $(\omega(0), \mathrm{d}\omega(0)/\mathrm{d}t) = (0, 1.7)$ the solution converges to the equilibrium $(1,0)$, while it converges to the other equilibrium $(-1,0)$ for the initial condition $(1,1)$. Global stability analysis of this system and proof of the boundedness of trajectories are given in [23]. For our simulation we take the reference signal $y_r(t)$ generated using initial conditions $(0,1.7)$ and the disturbance $d(t)$ is obtained using initial conditions $(1,1)$. These signals are depicted in Figure 9.

In this example we chose $\beta = 0.45$ which produces all real eigenvalues and $\omega_\beta = 8.4475$. We then solve the problem in COMSOL applying the β-iteration algorithm. In Figure 10 the first and second iteration errors E_1, E_2 are depicted. In Figure 11 the second and third iteration errors E_2, E_3 are depicted. And, in Figure 12 we plot the third and fourth iteration errors E_3, E_4. The closed loop system error E is depicted in Figure 13.

For this example it can be shown that for $y_r(t)$ and $\widetilde{d}(t)$ we have $\limsup_{t>0} \left| y_r^{(n)} \right| = 0$ and $\limsup_{t>0} \left| \widetilde{d}^{(n)} \right| = 0$ for $n = 1, 2, \ldots$. Therefore, in order to demonstrate that the β-iteration is actually improving the accuracy of tracking at each iteration and that our estimates for the error are fairly accurate, in our numerical simulations we have fixed a value of $T = 40$ and replaced the limsups in (2.55) of $\|y_r^{(n)}(t)\|_{\mathcal{Y}}$ and $\|d(t)^{(n)}(t)\|_{\mathcal{D}}$ by

$$\sup_{t>T} \|y_r^{(n)}(t)\|_{\mathcal{Y}} \text{ and } \sup_{t>T} \|d(t)^{(n)}(t)\|_{\mathcal{D}}.$$

We computed these values for $n = 1, 2, 3, 4$ and used them in (2.55) to obtain $C_1 =$ 8.3312e-02, $C_2 =$ 1.1781e-01, $C_3 =$ 1.6466e-01, $C_4 =$ 2.7273e-01 which are then used in Table 2. Once again in this example we have also numerically estimated the terms in (2.54) yielding $D =$ 2.1341e-01, $D_d =$ 3.0478e-02. Then in Table 2 we present the comparison of the $\sup_{t>T} \|E_n\|$ and $\sup_{t>T} D^n C_n$. Notice that our estimates for the errors are once again very accurate at each iteration.

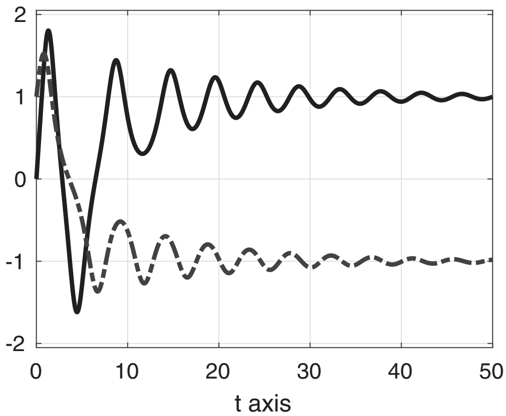

FIGURE 9. $y_r(t)$ (solid) and $d(t)$ (*dashed*).

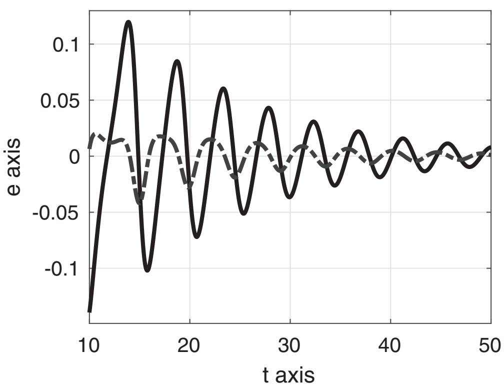

FIGURE 10. E_1 and E_2.

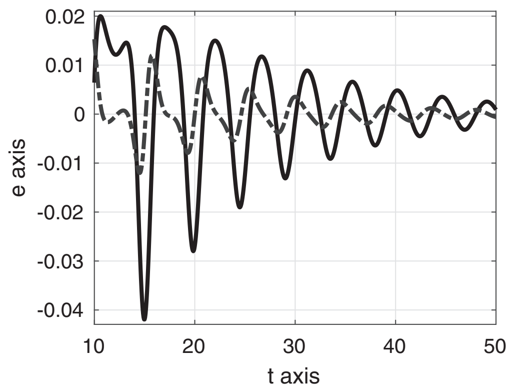

FIGURE 11. E_2 and E_3.

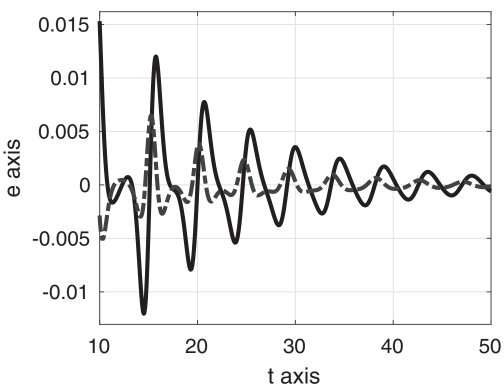

FIGURE 12. E_3 and E_4.

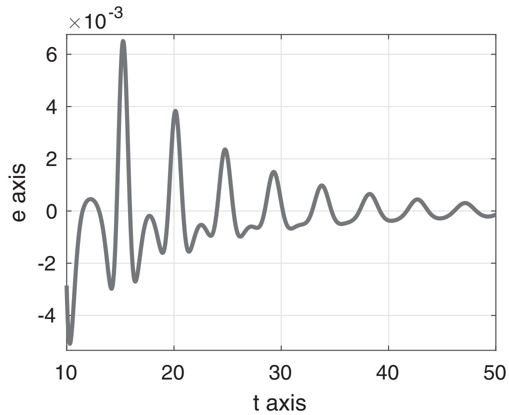

FIGURE 13. Error E.

TABLE 2. Compare lim sup of e_n.

	$n=1$	$n=2$	$n=3$	$n=4$
$D^n C_n$	1.7779e-02	5.3657e-03	1.6004e-03	5.6571e-04
$\sup \|E_n\|$	1.5835e-02	4.8450e-03	1.3744e-03	4.4237e-04

Example 4.3. A steady-state incompressible Navier–Stokes fluid flow passes through a two-dimensional region Ω (shown in Fig. 14) consisting of a main square box with side length 1 unit. In this model we have a square inlet at Ω_1 and outlet at Ω_4 with side length 0.1. The part of the boundary Γ_1 corresponds to the inflow boundary and Γ_2 is the outflow boundary. We consider the rest of the boundaries of Ω, denoted by Γ_w, as insulated walls. Our objective is to control the average temperature of the region Ω_3 to track a desired periodic reference temperature. The control is achieved by controlling the temperature of the inlet flow by heating or cooling the region Ω_1. The region Ω_2 corresponds to a window producing a periodic heat source disturbance to be rejected. This example is motivated by research in the design of energy efficient buildings.

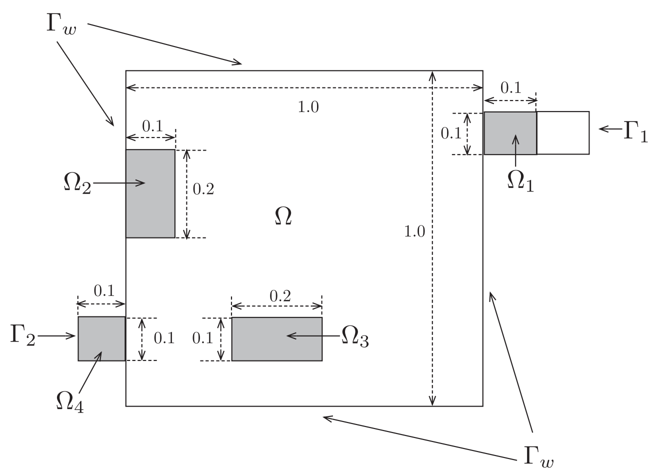

FIGURE 14. Two Dimensional Box domain.

For the mathematical description of the model assume that a parabolic inflow profile $f(s) = 4s(1-s)$ enters through Γ_1, where s is the arc length normalized between 0 and 1 and produces the fluid velocity $\boldsymbol{v}$ in Ω. Homogeneous Dirichlet boundary conditions for the temperature T are considered on Γ_1. On the boundaries Γ_2 and Γ_w, Neumann zero boundary conditions are considered for T. The surface stress vector $\boldsymbol{\tau}$ and heat flux q on Γ are given, respectively, by

$$\boldsymbol{\tau}(\boldsymbol{v}, p) = (pI - \mu[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T]) \cdot \boldsymbol{n},$$
$$q(T) = -\alpha \nabla T \cdot \boldsymbol{n}.$$

Here p, T denote the pressure and the temperature, respectively, $\boldsymbol{v} = [v_1, v_2]^T$ denotes the velocity vector field, and $\boldsymbol{n} = [n_x, n_y]^T$ denotes the outward normal vector on the boundary Γ. Zero stress boundary condition is considered on Γ_2, while we assume zero velocity on Γ_w.

With the above, the model is governed by the following equations

$$\begin{aligned}
&\rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nabla\cdot\left[\mu(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T)\right] + \nabla p = 0,\\
&\nabla\cdot\boldsymbol{v} = 0,\\
&\frac{\partial T}{\partial t} = \alpha\Delta T - \boldsymbol{v}\cdot\nabla T + B_{\text{in}}u + B_d d,\\
&y(t) = CT(t),
\end{aligned}$$

with initial conditions $\boldsymbol{v}(x,0) = 0$, $p(x,0) = 0$, and $T(x,0) = 0$, and boundary conditions

$$\begin{aligned}
&\boldsymbol{v} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad T = 0 \quad \text{on} \quad \Gamma_1, \quad q = 0 \quad \text{on} \quad \Gamma_2 \cup \Gamma_w,\\
&\boldsymbol{\tau} = 0 \quad \text{on} \quad \Gamma_2, \quad \boldsymbol{v} = 0 \quad \text{on} \quad \Gamma_w.
\end{aligned}$$

In this example the sectorial operator A is given by

$$A = \alpha\Delta - \boldsymbol{v}\cdot\nabla,\ \ \mathrm{D}(A) = \left\{\varphi \in H^2(\Omega) : \varphi\big|_{\Gamma_1} = 0, -\alpha\nabla\varphi\cdot\boldsymbol{n}\big|_{\Gamma_2\cup\Gamma_w} = 0\right\}.$$

We consider the operators B_{in} and B_d to be distributed forces over the subdomains Ω_1 and Ω_2, defined by

$$B_{\text{in}} = \chi_{\Omega_1} \qquad \text{and} \qquad B_d = \chi_{\Omega_2},$$

respectively. The measured output $C(T)$ is the average of the temperature on the subdomain Ω_3

$$C(T) = \frac{1}{|\Omega_3|}\int_{\Omega_3} T\,\mathrm{d}s.$$

In this example we have one way coupling between the steady Navier–Stokes equation and the heat equation. In particular, we first solve the Navier–Stokes equation independently from the heat equation then we use the velocity $\boldsymbol{v}$ in the convective term of the heat equation explicitly. Notice that for large enough $\boldsymbol{v}$ (the magnitude of the velocity field) the operator A might be unstable. To avoid instability of the operator A we have selected the thermal diffusivity coefficient α large enough and the magnitude of the velocity $\boldsymbol{v}$ small enough to ensure stability. In our numerical simulations we have chosen $\alpha = 0.5$, $\mu = 0.005$, $\rho = 1$ and $\beta = 0.3$. The reference signal to be tracked and the disturbance to be rejected are chosen as

$$y_r(t) = 0.5 + \sin(0.1t), \quad d(t) = 0.5 + \sin(0.2t).$$

In Figures 15–17 the first four iteration errors are depicted for $50 \le t \le 300$. Figure 18 depicts the error $E_4(t)$ and the closed loop system error $E(t)$ for $50 < t < 300$. Figure 19 depicts the reference $y_r(t)$ and the measured output of the closed loop system $y(t)$.

We can estimate the errors $E_j(t)$ using Comsol and numerically evaluate D and D_d in (2.56). For this particular example we have

$$A_r = 1,\ \alpha_r = 0.1,\ A_d = 1,\ \alpha_d = 0.2,$$

resulting in

$$D = 1.9892\text{e+}00, \quad D_d = 5.3189\text{e-}01, \quad C_n = \left((0.1)^n + 2\frac{D_d}{D}(0.2)^n\right).$$

In Table 3 we display the values of $\limsup \|E_n\|$ compared to the theoretical bounds $D^n C_n$. Once again, for all n, we notice that the estimates are fairly sharp.

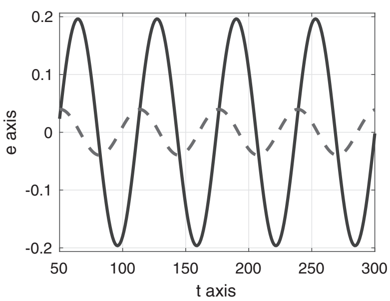

FIGURE 15. E_1, E_2 (*dashed*).

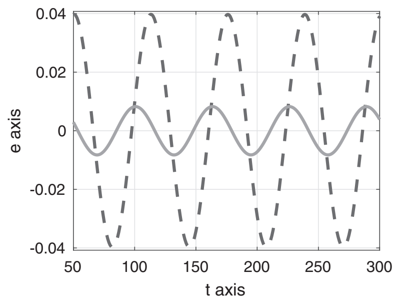

FIGURE 16. E_2 (*dashed*), E_3.

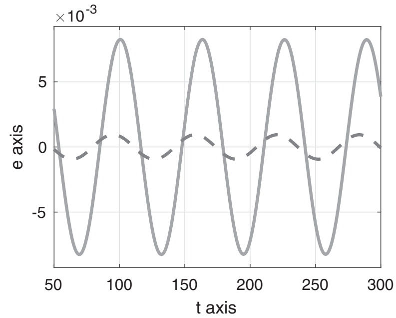

FIGURE 17. E_3, E_4 (*dashed*).

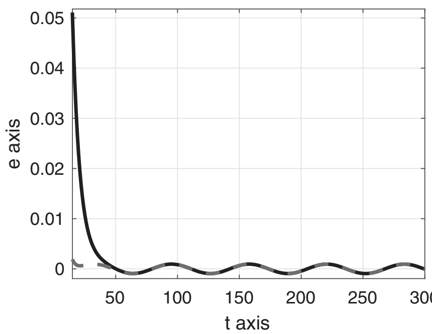

FIGURE 18. error E.

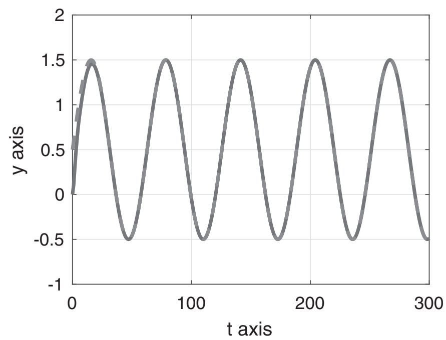

FIGURE 19. y and y_r.

TABLE 3. Compare lim sup of E_n and $D^n C_n$

.

	$n=1$	$n=2$	$n=3$	$n=4$
$D^n C_n$	3.0530e-01	8.1891e-02	2.4708e-02	8.2642e-03
$\limsup \|E_n\|$	1.9636e-01	3.9747e-02	8.2471e-03	9.5529e-04

5. PROOFS OF RESULTS FROM SECTION 3

Proof of Lemma 3.3. We prove this lemma using mathematical induction and only consider the case (3.12) since the details for (3.13) are virtually identical. For $n = 1$, using Proposition 3.1 we have

$$H_1 = K * H_0' = K * (K * K)' = K * K + K * K' * K = \sum_{i=0}^{1} \binom{1}{i} K * (K'*)^i K.$$

Next we assume that the equation (3.12) holds for H_{n-1} and show that it holds for H_n.

$$\begin{aligned} H_n &= K * (H_{n-1})' = K * \Big(\sum_{i=0}^{n-1} \binom{n-1}{i} K * (K'*)^i K \Big)' \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} K * (K * (K'*)^i K)' \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}\binom{n-1}{i}K*(K'*)^iK + \sum_{i=0}^{n-1}\binom{n-1}{i}K*K'*(K'*)^iK \quad \text{(by (3.9))}\\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}K*(K'*)^iK + \sum_{i=0}^{n-1}\binom{n-1}{i}K*(K'*)^{i+1}K\\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}K*(K'*)^iK + \sum_{i=1}^{n}\binom{n-1}{i-1}K*(K'*)^iK \quad \text{(shift indices)}\\
&= K*K + \sum_{i=1}^{n-1}\left(\binom{n-1}{i}+\binom{n-1}{i-1}\right)K*(K'*)^iK + K*(K'*)^nK\\
&= K*K + \sum_{i=1}^{n-1}\binom{n}{i}K*(K'*)^iK + K*(K'*)^nK = \sum_{i=0}^{n}\binom{n}{i}K*(K'*)^iK.
\end{aligned}
$$

On the second to last step we have used Pascal's formula

$$\binom{n-1}{i}+\binom{n-1}{i-1}=\binom{n}{i} \quad \text{for} \quad i=1,\dots,(n-1).$$

□

Proof Lemma 3.5. We first prove formula (3.14) by induction. To simplify our discussion we split formula (3.14) into two parts,

$$E_n(t)=\widetilde{E}_n(t)+\widetilde{E}_{d,n}(t), \tag{5.1}$$

where

$$\widetilde{E}_n(t)=\sum_{i=1}^{n-1}(K*)^{i-1}H_{n-(i+1)}y_r^{(i)}(0)+(K*)^{n-1}K*y_r^{(n)}(t), \tag{5.2}$$

$$\widetilde{E}_{d,n}(t)=\sum_{i=1}^{n-1}(K*)^{i-1}H_{d,n-(i+1)}d^{(i)}(0)+(K*)^{n-1}K_d*d^{(n)}(t). \tag{5.3}$$

Also notice that by definition of the error in equation (3.5) we have $E_{n+1}(t)=K*E_n'(t)$ and therefore

$$
\begin{aligned}
\widetilde{E}_{n+1}(t)&=K*\widetilde{E}_n'(t),\\
\widetilde{E}_{d,n+1}(t)&=K*\widetilde{E}_{d,n}'.
\end{aligned}
$$

We prove the induction step for $\widetilde{E}_n(t)$ only, since the proof for $\widetilde{E}_{d,n}(t)$ is similar.

First we show that equation (5.2) is true for $n=2$, namely, applying Proposition 3.1

$$
\begin{aligned}
\widetilde{E}_2(t)=K*\widetilde{E}_1'(t)&=K*(K*y_r^{(1)}(t))'\\
&=(K*)Ky_r'(0)+K*K*y_r''(t)\\
&=H_0y_r'(0)+K*K*y_r''(t).
\end{aligned}
$$

So equation (5.2) holds for $n=2$. Next we verify the induction step assuming that equation (5.2) is true for $n=k-1$, and show that it is also true $n=k$. For this step we will need the following simple result. Recalling again the definition of H_n in Lemma 3.3, for any two positive integers j and k we have

$$K*\left[(K*)^jH_k\right]'=(K*)^jH_{(k+1)}. \tag{5.4}$$

To see this we use Proposition 3.1 and the fact that $\{(K*)^{j-1}H_k\}(0) = 0$

$$\begin{aligned} K*[(K*)^jH_k]' &= K*[K*\{(K*)^{j-1}H_k\}]' = K*\left[K\{(K*)^{j-1}H_k\}(0) + K*\{(K*)^{j-1}H_k\}'\right] \\ &= (K*)^2\{(K*)^{j-1}H_k\}'. \end{aligned}$$

Repeating the above calculation a total of j times we arrive at

$$K*[(K*)^jH_k]' = (K*)^{j+1}H_k' = (K*)^jH_{k+1}.$$

Returning to our induction proof, we need to establish equation (5.2) with $n = k$, *i.e.*, we need to show

$$\widetilde{E}_k(t) = \sum_{i=1}^{k-1}(K*)^{i-1}\left(H_{k-(i+1)}y_r^{(i)}(0)\right) + (K*)^{k-1}K*y_r^{(k)}(t). \tag{5.5}$$

We assume as our induction hypothesis that equation (5.2) holds with $n = (k-1)$, *i.e.*,

$$\widetilde{E}_{k-1} = \sum_{i=1}^{k-2}(K*)^{i-1}(H_{k-1-(i+1)}\,y_r^{(i)}(0)) + (K*)^{k-2}K*y_r^{(k-1)}(t).$$

So we have

$$\begin{aligned} \widetilde{E}_k(t) &= K*\widetilde{E}_{k-1}'(t) \\ &= K*\Big[\sum_{i=1}^{k-2}(K*)^{i-1}(H_{k-1-(i+1)}\,y_r^{(i)}(0)) + (K*)^{k-2}K*y_r^{(k-1)}(t)\Big]' \\ &= \sum_{i=1}^{k-2}K*\left[(K*)^{i-1}H_{k-1-(i+1)}\right]'y_r^{(i)}(0) + K*\left[(K*)^{k-2}K*y_r^{(k-1)}(t)\right]' \end{aligned}$$

Applying (5.4) to each term in above the sum we have

$$\sum_{i=1}^{k-2}K*\left[(K*)^{i-1}H_{k-1-(i+1)}\right]'y_r^{(i)}(0) = \sum_{i=1}^{k-2}(K*)^{i-1}H_{k-(i+1)}y_r^{(i)}(0).$$

For the remaining term we have

$$\begin{aligned} &K*\left[(K*)^{k-2}K*y_r^{(k-1)}(t)\right]' = K*\left[K*\{(K*)^{k-2}y_r^{(k-1)}\}\right]' \\ &= K*\left[K\{(K*)^{k-2}y_r^{(k-1)}\}(0)\right] + K*K*\{(K*)^{k-2}y_r^{(k-1)}\}' \\ &= (K*)^2\{(K*)^{k-3}K*y_r^{(k-1)}\}' \end{aligned}$$

Repeating the above reduction k times we obtain

$$K*\left[(K*)^{k-2}K*y_r^{(k-1)}(t)\right]' = (K*)^k\{y_r^{(k-1)}\}' = (K*)^ky_r^{(k)}.$$

Combining these results the induction step (5.5) is verified.

Similarly we can prove

$$\widetilde{E}_{d,k}(t) = \sum_{i=1}^{k-1}(K*)^{i-1}\left(H_{d,k-(i+1)}d^{(i)}(0)\right) + (K*)^{k-1}K_d*d^{(k)}(t).$$

and hence equation (3.14) follows.

Using formulas (3.12) and (3.13) for $H_{k-(i+1)}$, $H_{d,k-(i+1)}$ in Lemma 3.3 we immediately obtain formula (3.15). $\square$

Proof Proposition 3.7. First we note that, in general, for any time dependent operator valued $F \in C(\mathbb{R}^+, \mathcal{L}(X))$ where X is a Banach space and $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(X))$

$$\|(F*)^j(t)\| \leq (\|F\|*)^j(t) \tag{5.6}$$

as can be seen by repetition of the case with $j = 2$

$$\|(F * F)(t)\| = \left\|\int_0^t F(t-\tau)F(\tau)\mathrm{d}\tau\right\| \leq \int_0^t \|F(t-\tau)\|\|F(\tau)\|\mathrm{d}\tau = (\|F\| * \|F\|)(t).$$

We use this to estimate the norm of $(K(t)*)^j$ in two ways. It follows immediately from (2.57) that

$$\|(K*)^j(t)K(t)\| \leq \widetilde{Q}^{j+1}[(\mathrm{e}^{-\omega_\beta t}*)^j\mathrm{e}^{-\omega_\beta t}] = \widetilde{Q}^{j+1}\frac{t^j}{j!}, \tag{5.7}$$

and

$$\|(K*)^j(t)K_d(t)\| \leq \widetilde{Q}^j\widetilde{Q}_d[(\mathrm{e}^{-\omega_\beta t}*)^j\mathrm{e}^{-\omega_\beta t}] = \widetilde{Q}^j\widetilde{Q}_d\frac{t^j}{j!}\cdot \tag{5.8}$$

Here we have used the formula

$$(\mathrm{e}^{-\omega_\beta t}*)^j\mathrm{e}^{-\omega_\beta t} = \frac{t^j}{j!},$$

which is easily verified using Laplace transformation of the iterated convolutions followed by partial fractions and inverse Laplace transformation.

Next we consider estimating $K'(t) = -\beta^{-1}C\mathrm{e}^{A_\beta t}B$. For this we appeal to Theorem 1.2.3 in [15], to conclude that for $0 < s < 1$ there exists $M_{\beta,s} \geq 1$, so that

$$\|(-A_\beta)^s\mathrm{e}^{A_\beta t}\| \leq M_{\beta,s}\frac{\mathrm{e}^{-\omega_\beta t}}{t^s}. \tag{5.9}$$

Defining $\delta = s_c + s_b < 1$ and writing the identity operator as $(-A_\beta)^{-s_c}(-A_\beta)^\delta(-A_\beta)^{-s_b}$ we have

$$\begin{aligned} \|K'(t)\| &\leq \|-\beta^{-1}C(-A_\beta)^{-s_c}(-A_\beta)^\delta\mathrm{e}^{A_\beta t}(-A_\beta)^{-s_b}B\| \\ &= \frac{M_{\beta,\delta}\|C\|_{\beta,-s_c}\|B\|_{\beta,-s_b}}{\beta}\,\frac{\mathrm{e}^{-\omega_\beta t}}{t^\delta} := \overline{Q}\,\frac{\mathrm{e}^{-\omega_\beta t}}{t^\delta}. \end{aligned} \tag{5.10}$$

Applying (5.6) to $(K*)^i(K'*)^jK$ using (5.7) to estimate $\|K\|$ and (5.10) to estimate $\|K'\|$ we have

$$\|(K*)^i(K'*)^jK(t)\| \leq \widetilde{Q}^{i+1}\overline{Q}^j(\mathrm{e}^{-\omega_\beta t}*)^i\left(\frac{\mathrm{e}^{-\omega_\beta t}}{t^\delta}*\right)^j\mathrm{e}^{-\omega_\beta t} = \widetilde{Q}^{i+1}\overline{Q}^j\frac{t^{i+j-j\delta}\Gamma(1-\delta)^j}{\Gamma(i+j+1-j\delta)}\,\mathrm{e}^{-\omega_\beta t}. \tag{5.11}$$

On the last step in (5.11) (and (5.13) below) we use

$$\left(\mathrm{e}^{-\omega_\beta t}*\right)^i\left(\frac{\mathrm{e}^{-\omega_\beta t}}{t^\delta}*\right)^j\mathrm{e}^{-\omega_\beta t} = \mathrm{e}^{-\omega_\beta t}\frac{t^{i+j(1-\delta)}\Gamma(1-\delta)^j}{\Gamma(i+1+j(1-\delta))},\ i \geq 1, j \geq 0,\quad 0 \leq \delta < 1, \tag{5.12}$$

which is easily verified using Laplace transformation of the iterated convolutions followed by partial fractions and inverse Laplace transformation.

Similarly repeating the above for $(K*)^i(K'*)^jK_d$ we have

$$\|(K*)^i(K'*)^jK_d(t)\| \leq \widetilde{Q}^i\widetilde{Q}_d\overline{Q}^j(\mathrm{e}^{-\omega_\beta t}*)^i\left(\frac{\mathrm{e}^{-\omega_\beta t}}{t^\delta}*\right)^j\mathrm{e}^{-\omega_\beta t} = \widetilde{Q}^i\widetilde{Q}_d\overline{Q}^j\frac{t^{i+j-j\delta}\Gamma(1-\delta)^j}{\Gamma(i+j+1-j\delta)}\,\mathrm{e}^{-\omega_\beta t}. \tag{5.13}$$

Define

$$R_{ij} = \widetilde{Q}^{i+1}\overline{Q}^{j}, \quad R_{d,ij} = \widetilde{Q}^{i}\widetilde{Q}_d\overline{Q}^{j}. \tag{5.14}$$

To prove (3.18) we derive a second estimate for (5.6) involving the $L^1 := L^1(\mathbb{R}^+, \mathcal{L}(\mathcal{U}, \mathcal{Y}))$ norm of K. Consider the case $j = 2$

$$\begin{aligned}
\|(K*)^2(t)\| \leq (\|K\|*)^2\| &= \int_0^t \|K(t-\tau)\| \left(\int_0^\tau \|K(\tau-\tau_1)\| \,\mathrm{d}\tau_1\right) \mathrm{d}\tau \\
&= \int_0^t \|K(t-\tau)\| \left(\int_0^\tau \|K(\tau_1)\| \,\mathrm{d}\tau_1\right) \mathrm{d}\tau \leq \left(\int_0^t \|K(\tau_1)\| \,\mathrm{d}\tau_1\right) \int_0^t \|K(t-\tau)\| \,\mathrm{d}\tau \\
&= \left(\int_0^t \|K(\tau)\| \,\mathrm{d}\tau\right)^2 \leq (\|K\|_{L^1})^2 = D^2.
\end{aligned}$$

A simple induction argument provides the desired result for an arbitrary n

$$\|(K*)^n(t)\| \leq \left(\int_0^t \|K(\tau)\| \,\mathrm{d}\tau\right)^n \leq \|K\|_{L^1}^n = D^n. \tag{5.15}$$

Then, for (3.19), consider the case $n = 1$

$$\begin{aligned}
\|(K*)^1(K_d*)(t)\| \leq (\|K\|*)^1(\|K_d\|*)\| &= \int_0^t \|K(t-\tau)\| \left(\int_0^\tau \|K_d(\tau-\tau_1)\| \,\mathrm{d}\tau_1\right) \mathrm{d}\tau \\
&= \int_0^t \|K(t-\tau)\| \left(\int_0^\tau \|K_d(\tau_1)\| \,\mathrm{d}\tau_1\right) \mathrm{d}\tau \leq \left(\int_0^t \|K_d(\tau_1)\| \,\mathrm{d}\tau_1\right) \int_0^t \|K(t-\tau)\| \,\mathrm{d}\tau \\
&= \left(\int_0^t \|K_d(\tau)\| \,\mathrm{d}\tau\right) \left(\int_0^t \|K(\tau)\| \,\mathrm{d}\tau\right) \leq (\|K_d\|_{L^1})\,(\|K\|_{L^1}) = D_d D.
\end{aligned}$$

Once again a simple induction argument gives the desired general result in (3.19)

$$\|(K*)^n(K_d*)(t)\| \leq (\|K\|*)^n(\|K_d\|*)\| \leq \left(\int_0^t \|K_d(\tau)\| \,\mathrm{d}\tau\right) \left(\int_0^t \|K(\tau)\| \,\mathrm{d}\tau\right)^n \leq D_d D^n.$$

□

Proof Theorem 3.8. Let us recall, according to equation (3.15) in Lemma 3.5, we can write $E_n(t) = \mathcal{E}_{1,n}(t) + \mathcal{E}_{2,T,n}(t) + \mathcal{E}_{3,T,n}(t)$ where these terms are given explicitly in (3.24)–(3.26), respectively. From Proposition 3.7, with R_{ij}, $R_{d,ij}$ in (5.14) we have

$$\begin{aligned}
\|\mathcal{E}_{1,n}(t)\|_{\mathcal{Y}} \leq & \sum_{i=1}^{n-1} \sum_{j=0}^{n-(i+1)} \binom{n-(i+1)}{j} (\|K\|*)^i(\|K'\|*)^j \\
& \times \left(\|K\|\|E_0(0)\|_{\infty,\mathcal{Y}}^{(n-1)} + \|K_d\|\|\widetilde{d}(0)\|_{\infty,\mathcal{D}}^{(n-1)}\right) \\
\leq & \sum_{i=1}^{n-1} \sum_{j=0}^{n-(i+1)} \binom{n-(i+1)}{j} (R_{ij}\|E_0(0)\|_{\infty,\mathcal{Y}}^{(n-1)} + R_{d,ij}\|\widetilde{d}(0)\|_{\infty,\mathcal{D}}^{(n-1)}) \\
& \times \mathrm{e}^{-\omega_\beta t} \left(\frac{t^{i+j-j\delta}\Gamma(1-\delta)^j}{\Gamma(i+j+1-j\delta)}\right).
\end{aligned}$$

where

$$\|E_0(0)\|_{\infty,\mathcal{Y}}^{(n-1)} = \sup_{1\leq \ell\leq (n-1)} \|E_0^{(\ell)}(0)\|_{\mathcal{Y}}, \quad \|\widetilde{d}(0)\|_{\infty,\mathcal{D}}^{(n-1)} = \sup_{1\leq \ell\leq (n-1)} \|\widetilde{d}^{(\ell)}(0)\|_{\mathcal{D}}.$$

For $i = 1, \ldots, (n-1)$ and $j = 0, \ldots, (n-1) - i$, it is easy to show that

$$t^{i+j(1-\delta)} \leq t^{(n-1)} \quad \text{for all} \quad t > 1.$$

Then using elementary calculus we can show that

$$t^{(n-1)} \mathrm{e}^{-\omega_\beta t} \leq \left(\frac{2(n-1)}{e\omega_\beta}\right)^{(n-1)} \mathrm{e}^{-\omega_\beta t/2} \quad \text{for all} \;\; t > 1.$$

Therefore, for a fixed n,

$$\|\mathcal{E}_{1,n}(t)\|_{\mathcal{Y}} \leq \Psi_n \mathrm{e}^{-\omega_\beta t/2} \xrightarrow{t\to\infty} 0, \tag{5.16}$$

exponentially fast, where

$$N_{i,j} = \left(R_{ij}\|E_0(0)\|_{\infty,\mathcal{Y}}^{(n-1)} + R_{d,ij}\|d(0)\|_{\infty,\mathcal{D}}^{(n-1)}\right),$$

and Ψ_n is the constant

$$\Psi_n = \sum_{i=1}^{n-1} \sum_{j=0}^{n-(i+1)} N_{i,j} \binom{n-(i+1)}{j} \left(\frac{2(n-1)}{e\omega_\beta}\right)^{(n-1)}. \tag{5.17}$$

Recalling that $T > 0$ is fixed but otherwise arbitrary, we turn to (3.25) which we break up into

$$\mathcal{E}_{2,n}(t) = \mathcal{E}_{2,T,n,r}(t) + \mathcal{E}_{2,T,n,d}(t),$$

where

$$\mathcal{E}_{2,T,n,r}(t) = \int_0^T K(t-\tau)(K*)^{n-1} E_0^{(n)}(\tau)\mathrm{d}\tau, \quad \mathcal{E}_{2,T,n,d}(t) = \int_0^T K(t-\tau)(K*)^{n-2} K_d * d^{(n)}(\tau)\mathrm{d}\tau.$$

We will establish estimates for $\mathcal{E}_{2,T,n,r}(t)$ only since those for $\mathcal{E}_{2,T,n,d}(t)$ almost identical. For this we will use (3.18) and we consider

$$\begin{aligned}
\|\mathcal{E}_{2,T,n,r}(t)\| &= \left\|\int_0^T K(t-\tau)(K*)^{n-1} E_0^{(n)}(\tau)\mathrm{d}\tau\right\| \\
&\leq \widetilde{Q} D^{n-1} \left(\int_0^T \mathrm{e}^{-\omega_\beta(t-\tau)}\mathrm{d}\tau\right) \sup_{t\geq 0} \|E_0^{(n)}(t)\|_{\mathcal{Y}} \\
&= \widetilde{Q} D^{n-1} \mathrm{e}^{-\omega_\beta t}(\mathrm{e}^{\omega_\beta T} - 1) \sup_{t\geq 0} \|E_0^{(n)}(t)\|_{\mathcal{Y}},
\end{aligned}$$

which, for fixed T, goes to zero exponentially fast as $t \to \infty$. Repeating almost exactly these same calculations for $\mathcal{E}_{2,T,n,d}(t)$ we obtain

$$\|\mathcal{E}_{2,T,n}(t)\| \xrightarrow{t\to\infty} 0.$$

Finally we consider $\mathcal{E}_{3,T,n}(t)$ and we want to prove the inequality (3.23). To this end let us once again decompose $\mathcal{E}_{3,T,n}(t)$ into two parts corresponding to E_0 and $\widetilde{d}$ which can be treated separately by almost identical means. We define

$$\mathcal{E}_{3,T,n,r}(t) = \int_T^t K(t-\tau)(K*)^{n-1} E_0^{(n)}(\tau)\,\mathrm{d}\tau, \quad \mathcal{E}_{3,T,n,d}(t) = \int_T^t K(t-\tau)(K*)^{n-2} K_d * \widetilde{d}^{(n)}(\tau)\,\mathrm{d}\tau,$$

so that for $t \geq T$

$$\mathcal{E}_{3,T,n}(t) = \mathcal{E}_{3,T,n,r}(t) + \mathcal{E}_{3,T,n,d}(t). \tag{5.18}$$

Just as above we use (3.18) and only consider the terms involving $\mathcal{E}_{3,T,n,r}(t)$ since the estimates for $\mathcal{E}_{3,T,n,d}(t)$ are virtually the same.

$$\begin{aligned}
\|\mathcal{E}_{3,T,n,r}(t)\| &= \left\| \int_T^t K(t-\tau)(K*)^{n-1} E_0^{(n)}(\tau)\,\mathrm{d}\tau \right\| \\
&\le D^{n-1} \left(\int_T^t \|K(t-\tau)\| \mathrm{d}\tau \right) \sup_{t\ge T} \|E_0^{(n)}(t)\|_{\mathcal{Y}} \\
&\le D^{n-1} \left(\int_0^t \|K(t-\tau)\|\,\mathrm{d}\tau \right) \sup_{t\ge T} \|E_0^{(n)}(t)\|_{\mathcal{Y}} \\
&= D^{n-1} \left(\int_0^t \|K(\tau)\|\,\mathrm{d}\tau \right) \sup_{t\ge T} \|E_0^{(n)}(t)\|_{\mathcal{Y}} \le D^n \sup_{t\ge T} \|E_0^{(n)}(t)\|_{\mathcal{Y}}.
\end{aligned} \tag{5.19}$$

Similarly, replacing one K by K_d the same basic calculations show that for $t \ge T$

$$\|\mathcal{E}_{3,T,n,d}(t)\| \le D^n \frac{D_d}{D} \sup_{t\ge T} \|\widetilde{d}^{(n)}(t)\|_{\mathcal{D}}. \tag{5.20}$$

Combining (5.19) and (5.20) gives the desired result for (5.18) in (3.23). Namely, for $t \ge T \ge 0$, we have

$$\|\mathcal{E}_{3,T,n}(t)\| \le D^n \left(\sup_{t\ge T} \|E_0^{(n)}\|_{\mathcal{Y}} + \frac{D_d}{D} \sup_{t\ge T} \|\widetilde{d}^{(n)}\|_{\mathcal{D}} \right).$$

□

Acknowledgements. The authors would like to thank the reviewer for the careful reading of the paper and for the many suggestions which led to consider unbounded control inputs and yielded sharper error estimates.

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