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VARIATIONAL METHOD FOR A BACKWARD PROBLEM FOR A TIME-FRACTIONAL DIFFUSION EQUATION

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Abstract. This paper is devoted to solve a backward problem for a time-fractional diffusion equation by a variational method. The regularity of a weak solution for the direct problem as well as the existence and uniqueness of a weak solution for the adjoint problem are proved. We formulate the backward problem into a variational problem by using the Tikhonov regularization method, and obtain an approximation to the minimizer of the variational problem by using a conjugate gradient method. Four numerical examples in one-dimensional and two-dimensional cases are provided to show the effectiveness of the proposed algorithm.

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1. Introduction

Fractional calculus has the natural advantage to describe the hereditary properties of the heterogeneous and the anomalous diffusion process than integer-order models [1,12]. For the last few decades, fractional calculus has been widely applied in many fields, such as biology, physics, hydrology, chemistry and biochemistry, medicine and finance [5, 11, 12, 18, 32]. Time fractional diffusion equations can be used to describe superdiffusion and subdiffusion phenomena [1,13,22]. The direct problems for time fractional diffusion equations have been studied extensively in recent years [6, 8, 10, 12, 15, 20, 31]. However, the inverse problems for time fractional diffusion equations, for examples, to recover initial data or source function or diffusion coefficient and so on by some additional data have not so many works.

In this paper, we consider a backward problem for a time-fractional diffusion equation with the homogeneous Dirichlet boundary condition. Let Ω be a bounded domain in R^d with sufficiently smooth boundary $\partial\Omega$ for $d \le 3$. Consider the following time-fractional diffusion equation:

$$\partial_{0+}^{\alpha} u(x,t) = Lu(x,t) + F(x,t), \quad x \in \Omega, \quad t \in (0,T], \quad 0 < \alpha < 1, \tag{1.1}$$

with the homogeneous Dirichlet condition and initial condition

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \tag{1.2}$$

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$$u(x,0) = \varphi(x), \quad x \in \overline{\Omega}, \tag{1.3}$$

where ∂_{0+}^{α} denotes the Caputo fractional left-sided derivative of order α defined by

$$\partial_{0+}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^{\alpha}}, \quad 0 < t \le T, \quad 0 < \alpha < 1, \tag{1.4}$$

where $\Gamma(\cdot)$ is the Gamma function, $T > 0$ is a fixed final time. And $-L$ is a symmetric uniformly elliptic operator defined on $D(-L) = H^2(\Omega) \cap H_0^1(\Omega)$ given by

$$Lu(x) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) + c(x)u(x), \quad x \in \Omega,$$

and the coefficients satisfy

$$\begin{cases} a_{ij}(x) = a_{ji}(x) \in C^1(\overline{\Omega}), \quad i,j = 1,2,\ldots,d, \\ v \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j, \quad x \in \overline{\Omega}, \quad \xi \in R^d, \quad \text{for a constant} \quad v > 0, \\ c(x) \le 0, \quad x \in \overline{\Omega}, \quad c(x) \in C(\overline{\Omega}). \end{cases}$$

If the functions $\varphi(x)$, $F(x,t)$ are given, then the problem (1.1)–(1.3) is a direct problem. The backward problem in this paper is to recover the initial data $\varphi(x)$ in problem (1.1)–(1.3) from the additional data

$$u(x,T) = h(x), \quad x \in \overline{\Omega}. \tag{1.5}$$

In this paper, we always assume $\varphi \in L^2(\Omega)$, $F(x,t) \in C([0,T]; L^2(\Omega))$, $F_t(x,t) \in L^{\infty}(0,T; L^2(\Omega))$ and $h(x) \in L^2(\Omega)$ unless other specified.

The backward problem for the time fractional diffusion equation in this paper is ill-posed (See Sect. 4), *i.e.* the solution $\varphi(x)$ of the backward problem does not depend continuously on the final time data $h(x)$ and any small perturbation on the given data $h(x)$ may cause large change to the solution $\varphi(x)$. And in practical application, we can not get the exact final time data $h(x)$, we only have the perturbation data $h^{\delta}(x)$ satisfying $||h(x) - h^{\delta}(x)||_{L^2(\Omega)} \le \delta$, and δ is a noise level, thus we need some regularization methods to deal with this ill-posed problem in order to ensure the stability of the numerical solution. In recent years, there have been some papers studied the backward problem for the time fractional diffusion equation. Liu and Yamamoto [9] used a quai-reversibility regularization method to solve the backward problem in one-dimensional case. In [20], Sakamoto and Yamamoto proved the uniqueness of a weak solution $\varphi(x) \in L^2(\Omega)$ for the backward problem under the condition $h(x) \in H^2(\Omega) \cap H_0^1(\Omega)$. Wang and Liu [26] used a truncation method to deal with the backward problem with the Neumann condition. Wang and Wei [25] used the Tikhonov regularization method to deal with the ill-posedness of the backward problem. Ren *et al.* [19] used a projection method to solve the backward problem. Wei and Wang [28] used a modified quasi-boundary value regularization method to solve this ill-posed problem. In [24], Wang and Wei used an iterative regularization method to solve the backward problem. In this paper, we use the Tikhonov regularization method, but finding the minimizer by a conjugate gradient method without using any spectral information of the operator $-L$, therefore we can consider numerical examples in a general domain. The Tikhonov regularization method is well known, however, we have to face new difficulties on its application in solving inverse problems of fractional partial differential equations by gradient-type algorithm since we need new integration by parts formula for deducing the adjoint problem.

Our paper is divided into seven sections. In Section 2, we present some preliminary used in this paper. In Section 3, we prove the regularity of a weak solution for the direct problem (1.1)–(1.3) and the existence and uniqueness of a weak solution for the adjoint problem. In Section 4, we formulate the backward problem into a variational problem and obtain the gradient of the Tikhonov regularization functional. We present the conjugate gradient algorithm in Section 5. Numerical results for four examples in one-dimensional and two-dimensional cases are investigated in Section 6. Finally, we give a conclusion in Section 7.

2. Preliminaries

Let $AC[0,T]$ be the space of functions f which are absolutely continuous on $[0,T]$. Throughout this paper, we use the following definitions and propositions in [7,16] if no other specified.

Definition 2.1. The Mittag-Leffler function is

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in C, \tag{2.1}$$

where $\alpha>0$ and $\beta\in R$ are arbitrary constants.

Proposition 2.2. *Let $0<\alpha<2$ and $\beta\in R$ be arbitrary. We suppose that μ is such that $\pi\alpha/2<\mu<min\{\pi,\pi\alpha\}$. Then there exists a constant $C=C(\alpha,\beta,\mu)>0$ such that*

$$|E_{\alpha,\beta}(z)|\le\frac{C}{1+|z|},\quad \mu\le|\arg(z)|\le\pi. \tag{2.2}$$

Proposition 2.3. *Let $0<\alpha<1$ and $\lambda>0$, then we have*

$$\frac{d}{\mathrm{d}t}E_{\alpha,1}(-\lambda t^\alpha)=-\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha),\quad t>0. \tag{2.3}$$

$$\frac{d}{\mathrm{d}t}[t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)]=t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha),\quad t>0. \tag{2.4}$$

Proposition 2.4. *Let $0<\alpha<1$ and $\lambda>0$, then we have*

$$\frac{d}{\mathrm{d}t}E_{\alpha,\alpha}(-\lambda t^\alpha)=-\lambda t^{\alpha-1}\left((1-\alpha)E_{\alpha,2\alpha}(-\lambda t^\alpha)+E_{\alpha,2\alpha-1}(-\lambda t^\alpha)\right),\quad t>0. \tag{2.5}$$

Proof. By (2.4) in Proposition 2.3, we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}E_{\alpha,\alpha}(-\lambda t^\alpha)&=\frac{d}{\mathrm{d}t}\left(t^{1-\alpha}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\right)\\&=(1-\alpha)t^{-\alpha}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)+t^{1-\alpha}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha)\\&=t^{-1}\left((1-\alpha)E_{\alpha,\alpha}(-\lambda t^\alpha)+E_{\alpha,\alpha-1}(-\lambda t^\alpha)\right)\\&=t^{-1}\left((1-\alpha)\sum_{k=0}^{\infty}\frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k+\alpha)}+\sum_{k=0}^{\infty}\frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k+\alpha-1)}\right)\\&=t^{-1}\Bigg((1-\alpha)\left(\frac{1}{\Gamma(\alpha)}-\lambda t^\alpha\sum_{k=1}^{\infty}\frac{(-\lambda t^\alpha)^{k-1}}{\Gamma(\alpha(k-1)+2\alpha)}\right)\\&\quad+\frac{1}{\Gamma(\alpha-1)}-\lambda t^\alpha\sum_{k=1}^{\infty}\frac{(-\lambda t^\alpha)^{k-1}}{\Gamma\big(\alpha(k-1)+2\alpha-1\big)}\Bigg)\end{aligned}$$

$$= t^{-1}\left(\frac{1-\alpha}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} - (1-\alpha)\lambda t^{\alpha} E_{\alpha,2\alpha}(-\lambda t^{\alpha}) - \lambda t^{\alpha} E_{\alpha,2\alpha-1}(-\lambda t^{\alpha})\right)$$

$$= -\lambda t^{\alpha-1}\left((1-\alpha)E_{\alpha,2\alpha}(-\lambda t^{\alpha}) + E_{\alpha,2\alpha-1}(-\lambda t^{\alpha})\right).$$

□

Proposition 2.5. *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\partial_{0+}^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}), \quad t > 0. \tag{2.6}$$

Proposition 2.6. *(See [17]) For $0 < \alpha < 1$, $t > 0$, we have $0 < E_{\alpha,1}(-t) < 1$. Moreover $E_{\alpha,1}(-t)$ is completely monotonic, that is*

$$(-1)^n \frac{d^n}{\mathrm{d}t^n} E_{\alpha,1}(-t) \geq 0, \quad \forall n \in N. \tag{2.7}$$

Proposition 2.7. *(See [17]) For $0 < \alpha < 1$, $\eta > 0$, we have $0 \leq E_{\alpha,\alpha}(-\eta) \leq \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Proposition 2.8. *(See [27]) Suppose $p(t) \in L^{\infty}(0,T)$, $0 < \alpha < 1$, $\lambda \geq 0$, denote*

$$g(t) = \int_0^t p(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})\mathrm{d}\tau, \quad t \in (0,T], \tag{2.8}$$

and define $g(0) = 0$, then $g(t) \in C[0,T]$.

Proposition 2.9. *(See [30]) For $0 < \alpha < 1$ and $\lambda > 0$, if $q(t) \in AC[0,T]$, we have*

$$\begin{aligned} &\partial_{0+}^{\alpha} \int_0^t q(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})\mathrm{d}\tau \\ &\quad = q(t) - \lambda \int_0^t q(\tau)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})\mathrm{d}\tau, \quad 0 < t \leq T. \end{aligned} \tag{2.9}$$

In particular, if $\lambda = 0$, we have

$$\partial_{0+}^{\alpha} \int_0^t q(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau = \Gamma(\alpha) q(t), \quad 0 < t \leq T. \tag{2.10}$$

Definition 2.10. If $f(t) \in L(0,T)$, then for $\alpha > 0$, the Riemann–Liouville fractional left-sided integral $I_{0+}^{\alpha} f$ and the right-sided integral $I_{T-}^{\alpha} f$ are defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)\mathrm{d}s}{(t-s)^{1-\alpha}}, \quad 0 < t \leq T, \tag{2.11}$$

and

$$I_{T-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)\mathrm{d}s}{(s-t)^{1-\alpha}}, \quad 0 \leq t < T. \tag{2.12}$$

Definition 2.11. If $y(t) \in AC[0,T]$, then for $0 < \alpha < 1$, the Riemann–Liouville fractional left-sided derivative $D_{0+}^{\alpha} y$ and the right-sided derivative $D_{T-}^{\alpha} y$ of order α are defined by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_0^t \frac{y(s)\mathrm{d}s}{(t-s)^{\alpha}} := \frac{d}{\mathrm{d}t}(I_{0+}^{1-\alpha} y)(t), \quad 0 < t \leq T, \tag{2.13}$$

and

$$D_{T-}^{\alpha} y(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_t^T \frac{y(s)\mathrm{d}s}{(s-t)^{\alpha}} := -\frac{d}{\mathrm{d}t}(I_{T-}^{1-\alpha} y)(t), \quad 0 \leq t < T. \tag{2.14}$$

Definition 2.12. If $y(t) \in AC[0,T]$, then for $0 < \alpha < 1$, the Caputo fractional left-sided derivative $\partial_{0+}^{\alpha} y$ and the right-sided derivative $\partial_{T-}^{\alpha} y$ of order α are defined by

$$\partial_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)\mathrm{d}s}{(t-s)^{\alpha}} := (I_{0+}^{1-\alpha} y')(t), \quad 0 < t \le T, \tag{2.15}$$

and

$$\partial_{T-}^{\alpha} y(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{y'(s)\mathrm{d}s}{(s-t)^{\alpha}} := -(I_{T-}^{1-\alpha} y')(t), \quad 0 \le t < T. \tag{2.16}$$

Proposition 2.13. *(See p. 78 in [7]) Let $\lambda \in C$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\mu) \ge 0$, then*

$$\left(I_{a+}^{\alpha} \left[(t-a)^{\beta-1} E_{\mu,\beta}(\lambda(t-a)^{\mu}) \right] \right)(x) = (x-a)^{\alpha+\beta-1} E_{\mu,\alpha+\beta}[\lambda(x-a)^{\mu}], \tag{2.17}$$

and

$$\left(D_{a+}^{\alpha} \left[(t-a)^{\beta-1} E_{\mu,\beta}(\lambda(t-a)^{\mu}) \right] \right)(x) = (x-a)^{\beta-\alpha-1} E_{\mu,\beta-\alpha}[\lambda(x-a)^{\mu}]. \tag{2.18}$$

Proposition 2.14. *Let $0 < \alpha < 1$ and $\lambda > 0$, then*

$$D_{0+}^{\alpha} \left(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) \right) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}), \quad t > 0. \tag{2.19}$$

Proof. By Proposition 2.13, we have

$$\begin{aligned} D_{0+}^{\alpha} \left(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) \right) &= t^{-1} E_{\alpha,0}(-\lambda t^{\alpha}) = t^{-1} \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k)} \\ &= t^{-1} \left(\frac{1}{\Gamma(0)} + \sum_{k=1}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k)} \right) = t^{-1}(-\lambda t^{\alpha}) \sum_{k=1}^{\infty} \frac{(-\lambda t^{\alpha})^{k-1}}{\Gamma\big(\alpha(k-1)+\alpha\big)} \\ &= -\lambda t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha})^k}{\Gamma(\alpha k+\alpha)} = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}). \end{aligned}$$

□

Definition 2.15. Let $0 < \alpha < 1$, the weighted function space $C_{\alpha}[0,T]$ and $C_{\alpha}^{T}[0,T]$ are defined by

$$C_{\alpha}[0,T] = \{ y(t) \ | \ y(t) t^{\alpha} \in C[0,T] \}, \tag{2.20}$$

and

$$C_{\alpha}^{T}[0,T] = \{ y(t) \ | \ y(t)(T-t)^{\alpha} \in C[0,T] \}, \tag{2.21}$$

with the norm

$$||y||_{C_{\alpha}[0,T]} = ||y(t)t^{\alpha}||_{C[0,T]} \text{ and } ||y||_{C_{\alpha}^{T}[0,T]} = ||y(t)(T-t)^{\alpha}||_{C[0,T]} \text{ respectively.}$$

Lemma 2.16. *(See Chap. 2 in [7]) Let $\Re(\alpha) > 0$, $0 \le \Re(\gamma) < 1$. If $\Re(\gamma) \le \Re(\alpha)$, then the fractional integration operators I_{0+}^{α} and I_{T-}^{α} are bounded from $C_{\gamma}[0,T]$ and $C_{\gamma}^{T}[0,T]$ into $C[a,b]$ respectively:*

$$||I_{0+}^{\alpha} f||_{C[a,b]} \le C ||f||_{C_{\gamma}[0,T]} \text{ and } ||I_{T-}^{\alpha} f||_{C[a,b]} \le C ||f||_{C_{\gamma}^{T}[0,T]} ,$$

where $C = C(\alpha, \gamma) > 0$ is a constant.

Definition 2.17. Let $0 < \alpha < 1$, the weighted function space $AC_\alpha[0,T]$ and $AC_\alpha^T[0,T]$ are defined by

$$AC_\alpha[0,T] = \{y(t) \ | \ y(t)t^\alpha \in AC[0,T]\}, \tag{2.22}$$

and

$$AC_\alpha^T[0,T] = \{y(t) \ | \ y(t)(T-t)^\alpha \in AC[0,T]\}. \tag{2.23}$$

Proposition 2.18. *Let $0 < \alpha < 1$. If $v(t) \in AC_{1-\alpha}[0,T]$, then $D_{0+}^\alpha v \in L(0,T)$. If $v(t) \in AC_{1-\alpha}^T[0,T]$, then $D_{T-}^\alpha v \in L(0,T)$.*

Proof. Since $v(t) \in AC_{1-\alpha}[0,T]$, there exists $g(t) \in AC[0,T]$ such that $g(t) = v(t)t^{1-\alpha}$, we have

$$\begin{aligned} D_{0+}^\alpha v(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_0^t \frac{v(s)\mathrm{d}s}{(t-s)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_0^t \frac{g(s)\mathrm{d}s}{(t-s)^\alpha s^{1-\alpha}} \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_0^1 \frac{g(t\tau)\mathrm{d}\tau}{(1-\tau)^\alpha \tau^{1-\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{g'(t\tau)\mathrm{d}\tau}{(1-\tau)^\alpha \tau^{-\alpha}}, \end{aligned}$$

then

$$\begin{aligned} \int_0^T |D_{0+}^\alpha v| \mathrm{d}t &\le \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^1 \frac{|g'(t\tau)|\mathrm{d}\tau}{(1-\tau)^\alpha \tau^{-\alpha}} \mathrm{d}t \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{\mathrm{d}\tau}{(1-\tau)^\alpha \tau^{-\alpha}} \int_0^T |g'(t\tau)| \mathrm{d}t \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{\mathrm{d}\tau}{(1-\tau)^\alpha \tau^{1-\alpha}} \int_0^{\tau T} |g'(s)| \mathrm{d}s \\ &\le \frac{1}{\Gamma(1-\alpha)} \int_0^1 \frac{\mathrm{d}\tau}{(1-\tau)^\alpha \tau^{1-\alpha}} \int_0^T |g'(s)| \mathrm{d}s \\ &= \Gamma(\alpha) ||g'||_{L(0,T)} < +\infty. \end{aligned}$$

Thus we have $D_{0+}^\alpha v \in L(0,T)$. The second result can be proved by the same process, we omit it here. ☐

Proposition 2.19. *Let $0 < \alpha < 1$. Suppose $\omega(t) \in AC[0,T]$, $\omega'(t) \in C_{1-\alpha}[0,T]$, $v(t) \in AC_{1-\alpha}^T[0,T]$, then we have*

$$\int_0^T \partial_{0+}^\alpha \omega(t) \cdot v(t) \mathrm{d}t = \omega(T) I_{T-}^{1-\alpha} v(T) - \omega(0) I_{T-}^{1-\alpha} v(0) + \int_0^T \omega(t) \cdot D_{T-}^\alpha v(t) \mathrm{d}t. \tag{2.24}$$

Proof. Since $\omega'(t) \in C_{1-\alpha}[0,T]$, we know $I_{0+}^{1-\alpha}\omega'(t) \in C[0,T]$. By Proposition 2.18, we know $D_{T-}^\alpha v = -\frac{d}{\mathrm{d}t} I_{T-}^{1-\alpha} v \in L(0,T)$, then $I_{T-}^{1-\alpha} v \in AC[0,T]$, so we have

$$\begin{aligned} \int_0^T \partial_{0+}^\alpha \omega(t) v(t) \mathrm{d}t &= \int_0^T I_{0+}^{1-\alpha} \omega'(t) v(t) \mathrm{d}t = \int_0^T \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\omega'(s)\mathrm{d}s}{(t-s)^\alpha} \right) v(t) \mathrm{d}t \\ &= \int_0^T \omega'(s) \left(\frac{1}{\Gamma(1-\alpha)} \int_s^T \frac{v(t)\mathrm{d}t}{(t-s)^\alpha} \right) \mathrm{d}s = \int_0^T \omega'(t) I_{T-}^{1-\alpha} v(t) \mathrm{d}t \\ &= \omega(t) I_{T-}^{1-\alpha} v(t)|_0^T - \int_0^T \omega(t) \frac{d}{\mathrm{d}t} I_{T-}^{1-\alpha} v(t) \mathrm{d}t \\ &= \omega(T) I_{T-}^{1-\alpha} v(T) - \omega(0) I_{T-}^{1-\alpha} v(0) + \int_0^T \omega(t) \cdot D_{T-}^\alpha v(t) \mathrm{d}t. \end{aligned}$$

Thus the proof is complete. ☐

Lemma 2.20. *(Thm. 2.1 in [20]) Let $0 < \alpha < 1$ and $F = 0$.*

(i). Let $\varphi \in L^2(\Omega)$. Then there exists a unique weak solution $u \in C([0,T];L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H_0^1(\Omega))$ to (1.1)–(1.3) such that $\partial_{0+}^\alpha u \in C((0,T];L^2(\Omega))$. Moreover there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &||u||_{C([0,T];L^2(\Omega))} \le C_1||\varphi||_{L^2(\Omega)}, \\ &||u(\cdot,t)||_{H^2(\Omega)} + ||\partial_{0+}^\alpha u(\cdot,t)||_{L^2(\Omega)} \le C_1 t^{-\alpha}||\varphi||_{L^2(\Omega)}, \quad t > 0, \end{aligned} \tag{2.25}$$

and we have

$$u(x,t) = \sum_{n=1}^{\infty} (\varphi,\phi_n) E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n(x). \tag{2.26}$$

Moreover $u : (0,T] \to L^2(\Omega)$ is analytically extended to a sector $\{z \in C; z \neq 0, |\arg z| < \frac{1}{2}\pi\}$.

(ii). We assume that $\varphi(x) \in H^2(\Omega)\cap H_0^1(\Omega)$. Then the unique weak solution u belongs to $C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_{0+}^\alpha u \in C([0,T];L^2(\Omega)) \cap C((0,T];H_0^1(\Omega))$ and the following inequality holds:

$$||u||_{C([0,T];H^2(\Omega))} + ||\partial_{0+}^\alpha u||_{C([0,T];L^2(\Omega))} \le C_2||\varphi||_{H^2(\Omega)}. \tag{2.27}$$

Lemma 2.21. *(Thm. 2.2(i) in [20]) Let $0 < \alpha < 1$ and let $\varphi = 0$. Let $F \in L^\infty(0,T;L^2(\Omega))$. Then there exists a unique weak solution $u \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ to (1.1)–(1.3) such that $\partial_{0+}^\alpha u \in L^2(\Omega\times(0,T))$. In particular, for any γ satisfying $\gamma > \frac{d}{4} - 1$, we have $u \in C([0,T];D((-L)^{-\gamma}))$, $\lim_{t\to 0} ||u(\cdot,t) - \varphi||_{D((-L)^{-\gamma})} = 0$, and if $d =$ 1, 2, 3, then $\lim_{t\to 0} ||u(\cdot,t) - \varphi||_{L^2(\Omega)} = 0$. Moreover there exists a constant $C_3 > 0$ such that*

$$||u||_{L^2(0,T;H^2(\Omega))} + ||\partial_{0+}^\alpha u||_{L^2(\Omega\times(0,T))} \le C_3||F||_{L^2(\Omega\times(0,T))}, \tag{2.28}$$

and we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t (F(\cdot,\tau),\phi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \mathrm{d}\tau \right) \phi_n(x). \tag{2.29}$$

3. Regularity of the weak solution for the direct problem

In this section, we firstly give a stronger regularity of the weak solution for the direct problem (1.1)–(1.3), then we prove the existence and uniqueness of a weak solution for the adjoint problem which is used in Section 4.

Since $-L$ is a symmetric uniformly elliptic operator, we denote the eigenvalues of $-L$ as λ_n and the corresponding orthonormal eigenfunctions as $\phi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $-L\phi_n = \lambda_n\phi_n$. We know that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$, and the sequence $\{\phi_n\}_{n\in N}$ is orthonormal basis in $L^2(\Omega)$.

Define the Hilbert space $D((-L)^\gamma)$ as

$$D((-L)^\gamma) = \left\{ \psi \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi,\phi_n)|^2 < \infty \right\}$$

with the norm:

$$||\psi||_{D((-L)^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi,\phi_n)|^2 \right)^{\frac{1}{2}},$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ and $\gamma \in R$. And we have $D((-L)^\gamma) \subset H^{2\gamma}(\Omega)$ for $0 < \gamma \le 1$, in particular, $D((-L)^{\frac{1}{2}}) = H_0^1(\Omega)$, $D(-L) = H^2(\Omega)\cap H_0^1(\Omega)$.

Based on the methods in [20], we give a stronger regularity of the weak solution in Lemma 2.20 and Lemma 2.21. We firstly introduce two function spaces $AC([0,T];L^2(\Omega))$ and $C_\alpha([0,T];L^2(\Omega))$ which are given by

$$AC([0,T];L^2(\Omega)) = \{u(x,t) \mid ||u(\cdot,t)||_{L^2(\Omega)} \in AC[0,T]\},$$

and

$$C_\alpha([0,T];L^2(\Omega)) = \{u(x,t) \mid ||u(\cdot,t)||_{L^2(\Omega)} \in C_\alpha[0,T]\}, \quad for \quad 0<\alpha<1.$$

Theorem 3.1. *If $\varphi(x) \in H^2(\Omega)\cap H_0^1(\Omega)$, $F(x,t)\in C([0,T];L^2(\Omega))$, $F_t(x,t)\in L^\infty(0,T;L^2(\Omega))$, and $d<4$, the unique weak solution to (1.1)–(1.3) is given by*

$$u(x,t)=\sum_{n=1}^{\infty}(\varphi,\phi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n(x)+\sum_{n=1}^{\infty}\left(\int_0^t (F(\cdot,\tau),\phi_n)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\right)\phi_n(x). \quad (3.1)$$

Then we can obtain $u\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap AC([0,T];L^2(\Omega))$, $\partial_{0+}^\alpha u\in C([0,T];L^2(\Omega))$ and $\partial_t u \in C_{1-\alpha}([0,T];L^2(\Omega))$. Moreover there exist constants $C_4>0$, $C_5>0$ such that

$$||u||_{C([0,T];H^2(\Omega))}+||\partial_{0+}^\alpha u||_{C([0,T];L^2(\Omega))} \le C_4(||\varphi||_{H^2(\Omega)}+||F||_{C([0,T];L^2(\Omega))}+||F_t||_{L^\infty(0,T;L^2(\Omega))}), \quad (3.2)$$

$$||\partial_t u(\cdot,t)||_{L^2(\Omega)} \le C_5(||\varphi||_{H^2(\Omega)}+||F||_{C([0,T];L^2(\Omega))}+||F_t||_{L^\infty(0,T;L^2(\Omega))})t^{\alpha-1}. \quad (3.3)$$

Proof. Based on Proposition 2.5 and Proposition 2.9, by the separation of variables, we can obtain a formal solution for the direct problem (1.1)–(1.3) as (3.1).

Denote $F_n(t)=(F(\cdot,t),\phi_n)$ and $f_n(t)=\int_0^t F_n(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau$, we have $u(x,t)=\sum_{n=1}^\infty(\varphi,\phi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n(x)+\sum_{n=1}^\infty f_n(t)\phi_n(x)$.

(1). We first verify $u\in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$, $\partial_{0+}^\alpha u\in C([0,T];L^2(\Omega))$.
By Proposition 2.3, it is easy to prove

$$\begin{aligned} f_n(t) &= \int_0^t F_n(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau \\ &= \frac{1}{\lambda_n}\left(F_n(t)-F_n(0)E_{\alpha,1}(-\lambda_n t^\alpha)-\int_0^t F_n'(\tau)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)\mathrm{d}\tau\right), \end{aligned} \quad (3.4)$$

then by Proposition 2.6, we have

$$\begin{aligned} |\lambda_n f_n(t)|^2 &\le 3\left(F_n^2(t)+F_n^2(0)E_{\alpha,1}^2(-\lambda_n t^\alpha)+\left(\int_0^t |F_n'(\tau)E_{\alpha,1}(-\lambda_n(t-\tau)^\alpha)|\mathrm{d}\tau\right)^2\right) \\ &\le 3\left(F_n^2(t)+F_n^2(0)+\left(\int_0^T |F_n'(\tau)|\mathrm{d}\tau\right)^2\right) \\ &\le 3\left(F_n^2(t)+F_n^2(0)+T\int_0^T |F_n'(\tau)|^2\mathrm{d}\tau\right), \end{aligned}$$

thus

$$\begin{aligned} \sum_{n=1}^\infty |\lambda_n f_n(t)|^2 &\le 3\left(\sum_{n=1}^\infty F_n^2(t)+\sum_{n=1}^\infty F_n^2(0)+T\sum_{n=1}^\infty\int_0^T |F_n'(\tau)|^2\mathrm{d}\tau\right) \\ &\le 3\left(||F(\cdot,t)||_{L^2(\Omega)}^2+||F(\cdot,0)||_{L^2(\Omega)}^2+T||F_t||_{L^2(0,T;L^2(\Omega))}^2\right) \\ &\le 6||F||_{C([0,T];L^2(\Omega))}^2+3T^2||F_t||_{L^\infty(0,T;L^2(\Omega))}^2. \end{aligned}$$

Therefore

$$\begin{aligned}||Lu(\cdot,t)||^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}|\lambda_n(\varphi,\phi_n)E_{\alpha,1}(-\lambda_n t^{\alpha})+\lambda_n f_n(t)|^2\\ &\le 2\left(\sum_{n=1}^{\infty}\lambda_n^2|(\varphi,\phi_n)|^2E^2_{\alpha,1}(-\lambda_n t^{\alpha})+\sum_{n=1}^{\infty}|\lambda_n f_n(t)|^2\right)\\ &\le 2(C_4^{'}||\varphi||^2_{H^2(\Omega)}+6||F||^2_{C([0,T];L^2(\Omega))}+3T^2||F_t||^2_{L^{\infty}(0,T;L^2(\Omega))})\\ &\le C_4(||\varphi||^2_{H^2(\Omega)}+||F||^2_{C([0,T];L^2(\Omega))}+||F_t||^2_{L^{\infty}(0,T;L^2(\Omega))}),\end{aligned}$$

thus $Lu(x,t)$ is convergent in $L^2(\Omega)$ uniformly on $t\in[0,T]$. Note that $f_n(t)\in C[0,T]$ by Proposition 2.8, we have $Lu\in C([0,T];L^2(\Omega))$ by Lebesgue theorem and $\partial^{\alpha}_{0+}u\in C([0,T];L^2(\Omega))$ by (1.1), then $u\in C([0,T];H^2(\Omega)\cap H^1_0(\Omega))$ and the estimate (3.2) is true.

(2). We verify $\partial_t u\in C_{1-\alpha}([0,T];L^2(\Omega))$ and $u\in AC([0,T];L^2(\Omega))$.
By Proposition 2.3 and (3.4), we have

$$\begin{aligned}\partial_t u=&-\sum_{n=1}^{\infty}(\varphi,\phi_n)\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\phi_n(x)+\sum_{n=1}^{\infty}F_n(0)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\phi_n(x)\\ &+\sum_{n=1}^{\infty}\left(\int_0^t F_n'(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\mathrm{d}\tau\right)\phi_n(x)\\ &:=I_1+I_2+I_3.\end{aligned}$$

For the series I_1 and I_2, using Proposition 2.7, we have

$$\begin{aligned}t^{1-\alpha}||I_1+I_2||_{L^2(\Omega)} &\le t^{1-\alpha}(||I_1||_{L^2(\Omega)}+||I_2||_{L^2(\Omega)})\\ &=\left(\sum_{n=1}^{\infty}|(\varphi,\phi_n)|^2\lambda_n^2E^2_{\alpha,\alpha}(-\lambda_n t^{\alpha})\right)^{\frac12}+\left(\sum_{n=1}^{\infty}|F_n(0)|^2E^2_{\alpha,\alpha}(-\lambda_n t^{\alpha})\right)^{\frac12}\\ &\le\frac{1}{\Gamma(\alpha)}\left(\sum_{n=1}^{\infty}|(\varphi,\phi_n)|^2\lambda_n^2\right)^{\frac12}+\frac{1}{\Gamma(\alpha)}\left(\sum_{n=1}^{\infty}|(F(\cdot,0),\phi_n)|^2\right)^{\frac12}\\ &\le C_5(||\varphi||_{H^2(\Omega)}+||F||_{C([0,T];L^2(\Omega))}).\end{aligned}$$

Since $\lambda_n\ge Cn^{\frac{2}{d}}, n\in N$ (See [2]) and $d<4$, combining with Propositions 2.3, 2.6 and 2.7, we have

$$\begin{aligned}t^{1-\alpha}||I_3||_{L^2(\Omega)} &= t^{1-\alpha}\left(\sum_{n=1}^{\infty}\left(\int_0^t F_n'(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\mathrm{d}\tau\right)^2\right)^{\frac12}\\ &\le t^{1-\alpha}\left(\sum_{n=1}^{\infty}\sup_{0\le\tau\le T}|F_n'(\tau)|^2\left(\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^{\alpha})\mathrm{d}\tau\right)^2\right)^{\frac12}\\ &\le T^{1-\alpha}\left(\sum_{n=1}^{\infty}||F_t||^2_{L^{\infty}(0,T;L^2(\Omega))}\frac{1}{\lambda_n^2}\right)^{\frac12}\\ &\le C_5||F_t||_{L^{\infty}(0,T;L^2(\Omega))}.\end{aligned}$$

Thus $t^{1-\alpha}||\partial_t u(\cdot,t)||_{L^2(\Omega)}\le C_5(||\varphi||_{H^2(\Omega)}+||F||_{C([0,T];L^2(\Omega))}+||F_t||_{L^{\infty}(0,T;L^2(\Omega))})$, that is $t^{1-\alpha}\partial_t u(x,t)$ is convergent in $L^2(\Omega)$ uniformly on $t\in[0,T]$. Since $F_t(x,t)\in L^{\infty}(0,T;L^2(\Omega))$, we know $\int_0^t F_n'(\tau)(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-$

$\tau)^{\alpha})\mathrm{d}\tau \in C[0,T]$ by Proposition 2.8, thus we have $t^{1-\alpha}||\partial_t u(\cdot,t)||_{L^2(\Omega)} \in C[0,T]$ by Lebesgue theorem. Therefore, $\partial_t u \in C_{1-\alpha}([0,T];L^2(\Omega))$ and $u \in AC([0,T];L^2(\Omega))$, and the the estimate (3.3) is true. Thus the proof is complete. $\square$

In this paper, in order to obtain the gradient of the functional $J(\varphi)$ (See Sect. 4), we have to solve the following adjoint problem

$$\begin{cases} D^{\alpha}_{0+}v(x,t) = Lv(x,t), \quad x \in \Omega, \quad t \in (0,T], \quad 0<\alpha<1, \\ v(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \\ I^{1-\alpha}_{0+}v(x,0) = b(x), \quad x \in \overline{\Omega}. \end{cases} \tag{3.5}$$

Based on the methods in [20], now we prove the existence and uniqueness of the weak solution to (3.5). We introduce a function space $AC_\alpha([0,T];L^2(\Omega))$ which is given by

$$AC_\alpha([0,T];L^2(\Omega)) = \{u(x,t) \mid ||u(\cdot,t)||_{L^2(\Omega)} \in AC_\alpha[0,T]\}, \quad for \quad 0<\alpha<1.$$

Theorem 3.2. *(i). If $b(x) \in L^2(\Omega)$, then there exists a unique weak solution $v \in C_{1-\alpha}([0,T];\ L^2(\Omega)) \cap C((0,T];H^2(\Omega)\cap H^1_0(\Omega))$ to problem (3.5) such that $D^\alpha_{0+}v \in C((0,T];L^2(\Omega))$ and $I^{1-\alpha}_{0+}v \in C([0,T];L^2(\Omega))$, and $\lim_{t\to 0}||I^{1-\alpha}_{0+}v(\cdot,t)-b||_{L^2(\Omega)} = 0$. Moreover, there exist constants $C_6>0$, $C_7>0$, $C_8>0$, such that*

$$\begin{aligned} &||v(\cdot,t)||_{L^2(\Omega)} \le C_6||b||_{L^2(\Omega)}t^{\alpha-1}, \\ &||v(\cdot,t)||_{H^2(\Omega)} + ||D^\alpha_{0+}v(\cdot,t)||_{L^2(\Omega)} \le C_7||b||_{L^2(\Omega)}t^{-1}, \\ &||I^{1-\alpha}_{0+}v||_{C([0,T];L^2(\Omega))} \le C_8||b||_{L^2(\Omega)}, \end{aligned} \tag{3.6}$$

and we have

$$v(x,t) = \sum_{n=1}^{\infty}(b,\phi_n)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x). \tag{3.7}$$

(ii). If $b(x) \in H^2(\Omega)\cap H^1_0(\Omega)$, then the unique weak solution $v \in L(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap AC_{1-\alpha}([0,T];L^2(\Omega))$ such that $D^\alpha_{0+}v \in L(0,T;L^2(\Omega))$. Moreover, there exist constants $C_9>0$, $C_{10}>0$ such that

$$\begin{aligned} &||v||_{L(0,T;H^2(\Omega))} + ||D^\alpha_{0+}v||_{L(0,T;L^2(\Omega))} \le C_9||b||_{H^2(\Omega)}, \\ &||\partial_t(t^{1-\alpha}v(x,t))||_{L(0,T;L^2(\Omega))} \le C_{10}||b||_{H^2(\Omega)}. \end{aligned} \tag{3.8}$$

Proof. (i). Based on Proposition 2.14, by the separation of variables, we can obtain a formal solution for the adjoint problem (3.5) as (3.7).

By Proposition 2.7, we have

$$||v(\cdot,t)||^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}|(b,\phi_n)|^2 t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_n t^\alpha) \le \left(\frac{1}{\Gamma(\alpha)}\right)^2 \sum_{n=1}^{\infty}|(b,\phi_n)|^2 t^{2\alpha-2} \le C_6^2||b||^2_{L^2(\Omega)}t^{2\alpha-2},$$

then $t^{1-\alpha}||v(\cdot,t)||_{L^2(\Omega)} \le C_6||b||_{L^2(\Omega)}$, we have $t^{1-\alpha}||v(\cdot,t)||_{L^2(\Omega)} \in C[0,T]$ by Lebesgue theorem, then $v \in C_{1-\alpha}([0,T];L^2(\Omega))$. By Lemma 2.16, we know that $I^{1-\alpha}_{0+}v \in C([0,T];L^2(\Omega))$, and $||I^{1-\alpha}_{0+}v||_{C([0,T];L^2(\Omega))} \le C||v||_{C_{1-\alpha}([0,T];L^2(\Omega))} = C\big|\big|t^{1-\alpha}||v(\cdot,t)||_{L^2(\Omega)}\big|\big|_{C[0,T]} \le C_8||b||_{L^2(\Omega)}$.

By Proposition 2.2, we have

$$\begin{aligned} ||Lv||^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\lambda_n^2|(b,\phi_n)|^2 t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_n t^\alpha) \le \sum_{n=1}^{\infty}\lambda_n^2|(b,\phi_n)|^2 t^{2\alpha-2}\left(\frac{C_7}{1+\lambda_n t^\alpha}\right)^2 \\ &\le C_7^2 t^{-2}\sum_{n=1}^{\infty}|(b,\phi_n)|^2 \le C_7^2 t^{-2}||b||^2_{L^2(\Omega)} \end{aligned}$$

so $Lv = \sum_{n=1}^{\infty} \lambda_n (b, \phi_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x)$ is convergent in $L^2(\Omega)$ uniformly in $t \in [\delta_0, T]$ with any given $\delta_0 > 0$. Thus we have $Lv \in C((0,T]; L^2(\Omega))$ by Lebesgue theorem and $v \in C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, $D_{0+}^{\alpha} v \in C((0,T]; L^2(\Omega))$ by (3.5).

By Proposition 2.13, we have $||I_{0+}^{1-\alpha} v(\cdot,t) - b||_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |(b,\phi_n)|^2 (E_{\alpha,1}(-\lambda_n t^\alpha) - 1)^2 \le ||b||_{L^2(\Omega)}^2$, it deduces $\lim_{t\to 0} ||I_{0+}^{1-\alpha} v(\cdot,t) - b||_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |(b,\phi_n)|^2 \lim_{t\to 0}(E_{\alpha,1}(-\lambda_n t^\alpha) - 1)^2 = 0$ by Lebesgue theorem.

Now we prove the uniqueness of the weak solution for (3.5). Under the condition $b(x) = 0$, we have to prove that the system (3.5) has only a trivial solution. We set $v(x,t) = \sum_{n=1}^{\infty} (v(\cdot,t),\phi_n)\phi_n(x) := \sum_{n=1}^{\infty} v_n(t)\phi_n(x)$ and take the inner product with $\phi_n(x)$ to the first and third equations in (3.5), if $v(x,t) \in C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, then we have

$$\begin{cases} D_{0+}^{\alpha} v_n(t) = -\lambda_n v_n(t), \quad t \in (0,T], \\ I_{0+}^{1-\alpha} v_n(0) = 0, \end{cases}$$

Due to the existence and uniqueness of the ordinary fractional differential equation (see [7]), we have $v_n(t) = 0, n \in N$. Since $\{\phi_n\}_{n\in N}$ is a complete orthonormal system in $L^2(\Omega)$, we have $v = 0$ in $\Omega \times (0,T]$.

(ii). If $b(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, by Proposition 2.7, we have

$$\begin{aligned} ||Lv||_{L(0,T;L^2(\Omega))} &= \int_0^T \left(\sum_{n=1}^{\infty} \lambda_n^2 |(b,\phi_n)|^2 t^{2\alpha-2} E_{\alpha,\alpha}^2(-\lambda_n t^\alpha) \right)^{\frac{1}{2}} \mathrm{d}t \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^T t^{\alpha-1} \mathrm{d}t \left(\sum_{n=1}^{\infty} \lambda_n^2 |(b,\phi_n)|^2 \right)^{\frac{1}{2}} = \frac{T^\alpha}{\Gamma(\alpha+1)} \left(\sum_{n=1}^{\infty} \lambda_n^2 |(b,\phi_n)|^2 \right)^{\frac{1}{2}} \\ &\le C_9 ||b||_{H^2(\Omega)} < +\infty, \end{aligned}$$

thus we have $v \in L(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ and $D_{0+}^{\alpha} v \in L(0,T; L^2(\Omega))$ by (3.5), and the first estimate of (3.8) is true.

We denote $u(x,t) = t^{1-\alpha} v(x,t) = \sum_{n=1}^{\infty} (b,\phi_n) E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x)$, in order to prove $v \in AC_{1-\alpha}([0,T]; L^2(\Omega))$, we need just to prove $u \in AC([0,T]; L^2(\Omega))$.

By the prove of Theorem 3.1, we have known $u \in C([0,T]; L^2(\Omega))$, combining Proposition 2.4, we have

$$\begin{aligned} \partial_t u &= \sum_{n=1}^{\infty} (b,\phi_n) \partial_t E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x) \\ &= -\sum_{n=1}^{\infty} (b,\phi_n) \lambda_n t^{\alpha-1} ((1-\alpha) E_{\alpha,2\alpha}(-\lambda_n t^\alpha) + E_{\alpha,2\alpha-1}(-\lambda_n t^\alpha))\phi_n(x), \end{aligned}$$

by Proposition 2.2, we have

$$\begin{aligned} \int_0^T ||\partial_t u||_{L^2(\Omega)} \mathrm{d}t &\le C_{10}' \int_0^T \left(\sum_{n=1}^{\infty} |(b,\phi_n)|^2 \lambda_n^2 t^{2\alpha-2} \right)^{\frac{1}{2}} \mathrm{d}t \le C_{10}' ||b||_{H^2(\Omega)} \int_0^T t^{\alpha-1} \mathrm{d}t \\ &\le C_{10} ||b||_{H^2(\Omega)} < +\infty, \end{aligned}$$

that is $\partial_t u \in L(0,T; L^2(\Omega))$, so we have $u \in AC([0,T]; L^2(\Omega))$ and the second estimate of (3.8) is true. Thus the proof is complete. □

Now let $v(x,t)$ satisfy the following problem

$$\begin{cases} D_{T-}^{\alpha} v(x,t) = Lv(x,t), \quad x \in \Omega, \quad t \in [0,T), \quad 0 < \alpha < 1, \\ v(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \\ I_{T-}^{1-\alpha} v(x,T) = b(x), \quad x \in \overline{\Omega}. \end{cases} \tag{3.9}$$

By the similar proof, we can obtain the following theorem.

Theorem 3.3. *(i). If $b(x) \in L^2(\Omega)$, then there exists a unique weak solution $v \in C^T_{1-\alpha}([0,T];\ L^2(\Omega)) \cap C([0,T); H^2(\Omega) \cap H^1_0(\Omega))$ to problem (3.9) such that $D^\alpha_{T-}v \in C([0,T); L^2(\Omega))$ and $I^{1-\alpha}_{T-}v \in C([0,T]; L^2(\Omega))$, and $\lim_{t\to T} ||I^{1-\alpha}_{T-}v(\cdot,t) - b||_{L^2(\Omega)} = 0$. Moreover, there exist constants $C_6 > 0$, $C_7 > 0$, $C_8 > 0$, such that*

$$\begin{aligned}&||v(\cdot,t)||_{L^2(\Omega)} \le C_6||b||_{L^2(\Omega)}(T-t)^{\alpha-1},\\ &||v(\cdot,t)||_{H^2(\Omega)} + ||D^\alpha_{T-}v(\cdot,t)||_{L^2(\Omega)} \le C_7||b||_{L^2(\Omega)}(T-t)^{-1},\\ &||I^{1-\alpha}_{T-}v||_{C([0,T];L^2(\Omega))} \le C_8||b||_{L^2(\Omega)},\end{aligned} \tag{3.10}$$

and we have

$$v(x,t) = \sum_{n=1}^{\infty}(b,\phi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)\phi_n(x). \tag{3.11}$$

(ii). If $b(x) \in H^2(\Omega)\cap H^1_0(\Omega)$, then the unique weak solution $v \in L(0,T; H^2(\Omega)\cap H^1_0(\Omega))\cap AC^T_{1-\alpha}([0,T]; L^2(\Omega))$ such that $D^\alpha_{T-}v \in L(0,T; L^2(\Omega))$. Moreover, there exist constants $C_9 > 0$, $C_{10} > 0$ such that

$$\begin{aligned}&||v||_{L(0,T;H^2(\Omega))} + ||D^\alpha_{T-}v||_{L(0,T;L^2(\Omega))} \le C_9||b||_{H^2(\Omega)},\\ &||\partial_t((T-t)^{1-\alpha}v(x,t))||_{L(0,T;L^2(\Omega))} \le C_{10}||b||_{H^2(\Omega)}.\end{aligned} \tag{3.12}$$

4. Variational method and the gradient of functional

In this section, we firstly show the ill-posedness of the backward problem, then we formulate the backward problem into a variational problem by using the Tikhonov regularization method, and lastly we obtain the gradient of the functional by solving a sensitivity problem and an adjoint problem.

Let $F(x,t) \in C([0,T]; L^2(\Omega))$ such that $F_t(x,t) \in L^\infty(0,T; L^2(\Omega))$ is a fixed function in the direct problem (1.1)–(1.3), denote $g(x) = \sum_{n=1}^\infty \left(\int_0^T (F(\cdot,\tau),\phi_n)(T-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-\tau)^\alpha)\mathrm{d}\tau \right)\phi_n(x)$.

We define a mapping $A : \varphi(x) \in L^2(\Omega) \to u_\varphi(x,T) - g(x) \in L^2(\Omega)$, where $u_\varphi(x,t)$ is the weak solution of problem (1.1)–(1.3) with the initial condition $u(x,0) = \varphi(x) \in L^2(\Omega)$ and the source function $F(x,t) \in C([0,T]; L^2(\Omega))$, $F_t(x,t) \in L^\infty(0,T; L^2(\Omega))$, $g(x)$ is a given function as above. Then the backward problem is to solve the following operator equation

$$(A\varphi)(x) = h(x) - g(x), \quad x \in \Omega. \tag{4.1}$$

From Lemma 2.20 and Theorem 3.1, if $\varphi(x) \in L^2(\Omega)$ and $F(x,t) \in C([0,T]; L^2(\Omega))$ such that $F_t(x,t) \in L^\infty(0,T; L^2(\Omega))$, we have $u_\varphi(x,T) - g(x) \in H^2(\Omega)\cap H^1_0(\Omega)$, and $||u_\varphi(x,T) - g||_{H^2(\Omega)} \le C||\varphi||_{L^2(\Omega)}$, where $C > 0$ is a constant. Because $H^2(\Omega)$ is compactly imbedded into $L^2(\Omega)$, then the operator A is a bounded linear compact operator and the backward problem (4.1) is ill-posed (See [25]).

In order to ensure the stability of numerical solution of the backward problem (4.1), we use the Tikhonov regularization method and define the following Tikhonov regularization functional

$$\begin{aligned}J(\varphi) &= \frac{1}{2}||A\varphi - (h^\delta - g)||^2_{L^2(\Omega)} + \frac{\mu}{2}||\varphi||^2_{L^2(\Omega)} = \frac{1}{2}||u_\varphi(x,T) - h^\delta||^2_{L^2(\Omega)} + \frac{\mu}{2}||\varphi||^2_{L^2(\Omega)}\\ &= \frac{1}{2}\int_\Omega \big(u_\varphi(x,T) - h^\delta(x)\big)^2\mathrm{d}x + \frac{\mu}{2}\int_\Omega \varphi^2(x)\mathrm{d}x,\end{aligned} \tag{4.2}$$

where $\mu > 0$ is a regularization parameter and $h^\delta(x) \in L^2(\Omega)$ is the noisy function of $h(x)$ which satisfies

$$||h^\delta(x) - h(x)||_{L^2(\Omega)} \le \delta. \tag{4.3}$$

Therefore the backward problem (4.1) is transformed into the following variational problem

$$J(\varphi^\delta_\mu) = \min_{\varphi\in L^2(\Omega)} J(\varphi) \tag{4.4}$$

It has been proved in [3] that there exists a uniqueness minimizer φ_μ^δ to (4.4), and the minimizer φ_μ^δ converges to the exact solution φ as δ tends to 0 when $\mu = \mu(\delta)$ is chosen by a suitable choice.

In order to find an approximation to the minimizer φ_μ^δ, we use a conjugate gradient method in this paper, and the key task is to obtain the gradient $J'(\varphi)$ of $J(\varphi)$ in (4.2).

Based on the method in [23,27,29], we solve the gradient $J'(\varphi)$ by constructing a sensitivity problem and an adjoint problem.

Let the initial value $\varphi(x)$ be perturbed by a small amount $\delta\varphi(x) \in L^2(\Omega)$, then the forward solution has a small change $\omega = u_{\varphi+\delta\varphi} - u_\varphi$. From (1.1)–(1.3), we have that ω satisfies

Sensitivity problem:

$$\begin{cases} \partial_{0+}^\alpha \omega(x,t) = L\omega(x,t), \quad x \in \Omega, \quad t \in (0,T], \\ \omega(x,0) = \delta\varphi(x), \quad x \in \overline{\Omega}, \\ \omega(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \end{cases} \tag{4.5}$$

From (4.2),we have

$$\begin{aligned} \delta J(\varphi) &= J(\varphi+\delta\varphi) - J(\varphi) \\ &= \frac{1}{2}\int_\Omega (u_\varphi(x,T) + \omega(x,T) - h^\delta(x))^2 \mathrm{d}x + \frac{\mu}{2}\int_\Omega (\varphi+\delta\varphi)^2 \mathrm{d}x \\ &\quad - \frac{1}{2}\int_\Omega (u_\varphi(x,T) - h^\delta(x))^2 \mathrm{d}x - \frac{\mu}{2}\int_\Omega \varphi^2 \mathrm{d}x \\ &= \int_\Omega (u_\varphi(x,T) - h^\delta(x))\omega(x,T)\mathrm{d}x + \mu\int_\Omega \varphi\delta\varphi \mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega \omega^2(x,T)\mathrm{d}x + \frac{\mu}{2}\int_\Omega \delta\varphi^2 \mathrm{d}x \\ &= \int_\Omega (u_\varphi(x,T) - h^\delta(x))\omega(x,T)\mathrm{d}x + \mu\int_\Omega \varphi\delta\varphi \mathrm{d}x + o(||\omega(x,T)||_{L^2(\Omega)} + ||\delta\varphi||_{L^2(\Omega)}). \end{aligned}$$

By Lemma 2.20, we have $||\omega(x,T)||_{L^2(\Omega)} \le C_1||\delta\varphi||_{L^2(\Omega)}$, then

$$\delta J(\varphi) = \int_\Omega (u_\varphi(x,T) - h^\delta(x))\omega(x,T)\mathrm{d}x + \mu\int_\Omega \varphi\delta\varphi \mathrm{d}x + o(||\delta\varphi||_{L^2(\Omega)}). \tag{4.6}$$

Since $H^2(\Omega) \cap H_0^1(\Omega)$ is dense in $L^2(\Omega)$, then there exist an approximating sequence $\{\delta\varphi_n\}_{n=1}^\infty \subseteq H^2(\Omega) \cap H_0^1(\Omega)$ and $\{h_n^\delta\}_{n=1}^\infty \subseteq H^2(\Omega) \cap H_0^1(\Omega)$ for $\delta\varphi$ and h^δ in $L^2(\Omega)$ respectively such that

$$||\delta\varphi_n - \delta\varphi||_{L^2(\Omega)} \to 0, \quad as \quad n \to +\infty, \tag{4.7}$$

$$||h_n^\delta - h^\delta||_{L^2(\Omega)} \to 0, \quad as \quad n \to +\infty. \tag{4.8}$$

Let ω_n is the weak solution of problem (4.5) with the initial condition $\omega_n(x,0) = \delta\varphi_n(x)$, we have $||\omega_n(x,T) - \omega(x,T)||_{L^2(\Omega)} \le ||\delta\varphi_n - \delta\varphi||_{L^2(\Omega)} \to 0$, as $n \to +\infty$ by Lemma 2.20.

Let $v_n(x,t)$ be an arbitrary function in $L(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap AC_{1-\alpha}^T([0,T];L^2(\Omega))$, ω_n is the weak solution of (4.5) with $\omega_n(x,0) = \delta\varphi_n(x)$, multiply v_n on both sides of the first equation in (4.5), we have

$$\int_0^T \int_\Omega \partial_{0+}^\alpha \omega_n v_n \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega L\omega_n v_n \mathrm{d}x\mathrm{d}t. \tag{4.9}$$

By Proposition 2.19, for the left side of (4.9), we have

$$\int_0^T \int_\Omega \partial_{0+}^\alpha \omega_n v_n \mathrm{d}x\mathrm{d}t = \int_\Omega (\omega_n(x,T) I_{T-}^{1-\alpha} v_n(x,T) - \omega_n(x,0) I_{T-}^{1-\alpha} v_n(x,0))\mathrm{d}x + \int_0^T \int_\Omega \omega_n(x,t) D_{T-}^\alpha v_n(x,t)\mathrm{d}x\mathrm{d}t$$

and for the right side of (4.9), we have

$$\int_0^T \int_\Omega L\omega_n v_n \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \omega_n L v_n \mathrm{d}x\mathrm{d}t + \sum_{i,j=1}^d \int_0^T \int_{\partial\Omega} a_{ij}(x) \frac{\partial \omega_n}{\partial x_j} v_n n^i \mathrm{d}S\mathrm{d}t$$

where $n = (n^1, n^2, \ldots, n^d)$ is the outward unit normal vector of $\partial\Omega$.

We rewrite (4.9) as

$$\begin{aligned} &\int_0^T \int_\Omega \omega_n(x,t)(D_{T-}^\alpha v_n(x,t) - L v_n(x,t))\mathrm{d}x\mathrm{d}t - \sum_{i,j=1}^d \int_0^T \int_{\partial\Omega} a_{ij}(x) \frac{\partial \omega_n}{\partial x_j} v_n n^i \mathrm{d}S\mathrm{d}t \\ &= \int_\Omega (\omega_n(x,0) I_{T-}^{1-\alpha} v_n(x,0) - \omega_n(x,T) I_{T-}^{1-\alpha} v_n(x,T))\mathrm{d}x. \end{aligned} \tag{4.10}$$

In order to obtain the gradient $J'(\varphi)$, we let $v_n(x,t)$ satisfies

$$\begin{cases} D_{T-}^\alpha v_n(x,t) = L v_n(x,t), & x \in \Omega, \quad t \in [0,T), \\ v_n(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ I_{T-}^{1-\alpha} v_n(x,T) = u_\varphi(x,T) - h_n^\delta(x), & x \in \overline{\Omega} \end{cases} \tag{4.11}$$

Since $\delta\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_{\varphi_n}(x,T) - h_n^\delta(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, we know $\omega_n \in AC([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ such that $\frac{\partial \omega_n}{\partial t} \in C_{1-\alpha}([0,T]; L^2(\Omega))$ and $v_n \in AC_{1-\alpha}^T([0,T]; L^2(\Omega)) \cap L(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ by Theorems 3.1 and 3.3, then the above calculation deductions are strict by Proposition 2.19.

Since $\omega_n(x,0) = \delta\varphi_n(x)$, $x \in \overline{\Omega}$ in (4.5), then (4.10) becomes

$$\int_\Omega \omega_n(x,T)(u_\varphi(x,T) - h_n^\delta(x))\mathrm{d}x = \int_\Omega \delta\varphi_n(x) I_{T-}^{1-\alpha} v_n(x,0)\mathrm{d}x. \tag{4.12}$$

Let $v(x,t)$ satisfies the following adjoint problem.

Adjoint problem:

$$\begin{cases} D_{T-}^\alpha v(x,t) = L v(x,t), & x \in \Omega, \quad t \in [0,T), \\ v(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ I_{T-}^{1-\alpha} v(x,T) = u_\varphi(x,T) - h^\delta(x), & x \in \overline{\Omega}. \end{cases} \tag{4.13}$$

Since $u_\varphi(x,T) - h^\delta(x) \in L^2(\Omega)$, we have $v \in C_{1-\alpha}^T([0,T]; L^2(\Omega))$ and $I_{T-}^{1-\alpha} v \in C([0,T]; L^2(\Omega))$ by Theorem 3.3, then

$$||I_{T-}^{1-\alpha} v_n(x,0) - I_{T-}^{1-\alpha} v(x,0)||_{L^2(\Omega)} \le ||I_{T-}^{1-\alpha}(v_n - v)||_{C([0,T];L^2(\Omega))} \le C||h_n^\delta - h^\delta||_{L^2(\Omega)} \to 0, \quad as \quad n \to +\infty.$$

Taking a limit as $n \to +\infty$ in (4.12), note that $||(u_\varphi(x,T) - h_n^\delta(x)) - (u_\varphi(x,T) - h^\delta(x))||_{L^2(\Omega)} \le ||h_n^\delta - h^\delta||_{L^2(\Omega)} \to 0$, as $n \to +\infty$, then we have

$$\int_\Omega \omega(x,T)(u_\varphi(x,T) - h^\delta(x))\mathrm{d}x = \int_\Omega \delta\varphi(x) I_{T-}^{1-\alpha} v(x,0)\mathrm{d}x. \tag{4.14}$$

Together with (4.6), by the definition of gradient operator in $L^2(\Omega)$, we have

$$J'(\varphi) = I_{T-}^{1-\alpha} v(x,0) + \mu\varphi(x), \quad x \in \Omega, \tag{4.15}$$

where $v(x,t)$ satisfies the adjoint problem (4.13).

In order to solve the adjoint problem (4.13) explicitly, we let $\widetilde{v}(x,t) = v(x,T-t)$, and we have

$$\begin{aligned} D_{T-}^{\alpha} v(x,t) &= -\frac{d}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{v(x,s)}{(s-t)^{\alpha}}\mathrm{d}s = -\frac{d}{\mathrm{d}t}\frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{\widetilde{v}(x,T-s)}{(s-t)^{\alpha}}\mathrm{d}s \\ &= \frac{d}{d(T-t)}\frac{1}{\Gamma(1-\alpha)}\int_0^{T-t}\frac{\widetilde{v}(x,\tau)}{((T-t)-\tau)^{\alpha}}\mathrm{d}\tau = D_{0+}^{\alpha}\widetilde{v}(x,T-t), \end{aligned}$$

and

$$\begin{aligned} I_{T-}^{1-\alpha} v(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{v(x,s)}{(s-t)^{\alpha}}\mathrm{d}s = \frac{1}{\Gamma(1-\alpha)}\int_t^T \frac{\widetilde{v}(x,T-s)}{(s-t)^{\alpha}}\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{T-t}\frac{\widetilde{v}(x,\tau)}{((T-t)-\tau)^{\alpha}}\mathrm{d}\tau = I_{0+}^{1-\alpha}\widetilde{v}(x,T-t). \end{aligned}$$

Denote $\tau = T-t$, then $\widetilde{v}(x,\tau)$ satisfies

$$\begin{cases} D_{0+}^{\alpha}\widetilde{v}(x,\tau) = L\widetilde{v}(x,\tau), \quad x \in \Omega, \quad \tau \in (0,T], \\ \widetilde{v}(x,\tau) = 0, \quad x \in \partial\Omega, \quad \tau \in (0,T), \\ I_{0+}^{1-\alpha}\widetilde{v}(x,0) = u_{\varphi}(x,T) - h^{\delta}(x), \quad x \in \overline{\Omega}. \end{cases} \tag{4.16}$$

Thus we solve the problem (4.16) instead of (4.13), and then obtain $v(x,t) = \widetilde{v}(x,T-t)$.

5. Conjugate gradient algorithm

In this section, based on the method in [23, 27, 29], we use a conjugate gradient method to find an approximation to the minimizer of the functional $J(\varphi)$.

We denote $\varphi_k(x)$ be the kth approximate solution to $\varphi(x)$, then the iteration process of conjugate gradient algorithm is given by

$$\varphi_{k+1} = \varphi_k + \beta_k d_k, \quad k = 0,1,2,\ldots, \tag{5.1}$$

where β_k is the step size and d_k is the descent direction in the kth iteration. We obtain d_k as follow:

$$d_0 = -J_0', \quad \text{and} \quad d_k = -J_k' + \gamma_k d_{k-1}, \quad k = 1,2,\ldots, \tag{5.2}$$

where $J_k' = J'(\varphi_k)$, and the conjugate coefficient γ_k is calculated by

$$\gamma_0 = 0, \quad \text{and} \quad \gamma_k = \frac{\int_{\Omega}(J_k')^2\mathrm{d}x}{\int_{\Omega}(J_{k-1}')^2\mathrm{d}x}, \quad k = 1,2,\ldots. \tag{5.3}$$

Now we calculate the step size β_k, we have

$$\begin{aligned} J(\varphi_k + \beta_k d_k) &= \frac{1}{2}\int_{\Omega}(u_{\varphi_k+\beta_k d_k}(x,T) - h^{\delta}(x))^2\mathrm{d}x + \frac{\mu}{2}\int_{\Omega}(\varphi_k + \beta_k d_k)^2\mathrm{d}x \\ &= \frac{1}{2}\int_{\Omega}(u_{\varphi_k}(x,T) + \beta_k\omega_k(x,T) - h^{\delta}(x))^2\mathrm{d}x + \frac{\mu}{2}\int_{\Omega}(\varphi_k + \beta_k d_k)^2\mathrm{d}x, \end{aligned}$$

where $\omega_k(x,t)$ is the weak solution to the sensitivity problem (4.5) with $\delta\varphi(x) = d_k(x)$. Let

$$\frac{dJ(\varphi_k + \beta_k d_k)}{d\beta_k} = \int_\Omega (u_{\varphi_k}(x,T) + \beta_k \omega_k(x,T) - h^\delta(x))\omega_k(x,T)\mathrm{d}x + \mu \int_\Omega (\varphi_k + \beta_k d_k) d_k \mathrm{d}x = 0,$$

then we have

$$\beta_k = -\frac{\int_\Omega (u_{\varphi_k}(x,T) - h^\delta(x))\omega_k(x,T)\mathrm{d}x + \mu \int_\Omega \varphi_k(x) d_k(x)\mathrm{d}x}{\int_\Omega \omega_k^2(x,T)\mathrm{d}x + \mu \int_\Omega d_k^2(x)\mathrm{d}x}. \tag{5.4}$$

Therefore, we have the following **Conjugate Gradient Algorithm** to solve the variational problem (4.4):

1. Initialize $\varphi_0 = 0$, and set $k = 0$;
2. Solve the direct problem (1.1)–(1.3) with the initial value $\varphi(x) = \varphi_k$, and determine the residual $r_k = u_{\varphi_k}(x,T) - h^\delta(x)$;
3. Solve the adjoint problem (4.13) with $u_\varphi(x,T) - h^\delta(x) = r_k$, and obtain the gradient J'_k by (4.15);
4. Calculate the conjugate coefficient γ_k by (5.3) and the descent direction d_k by (5.2);
5. Solve the sensitivity problem (4.5) for ω_k with $\delta\varphi = d_k$;
6. Calculate the step size β_k by (5.4);
7. Update the initial value φ_k by (5.1);
8. Increase k by one and go step 2, repeat the above procedure until a stopping criterion is satisfied.

6. Numerical experiments

In this section, we present some numerical results for four examples in one-dimensional and two-dimensional cases to show the effectiveness of the conjugate gradient algorithm.

The noisy data are generated by adding a random perturbation, *i.e.*

$$h^\delta = h + \varepsilon h \cdot (2rand(size(h)) - 1). \tag{6.1}$$

The corresponding noise level is calculated by $\delta = ||h^\delta - h||_{L^2(\Omega)}$.

In order to show the accuracy of numerical solution, we compute the approximate L^2 error denoted by

$$e_k = ||\varphi_k(x) - \varphi(x)||_{L^2(\Omega)}, \tag{6.2}$$

where $\varphi_k(x)$ is the initial value reconstructed at the kth iteration, and $\varphi(x)$ is the exact solution.

The residual E_k at the kth iteration is given by

$$E_k = ||u_{\varphi_k}(x,T) - h^\delta(x)||_{L^2(\Omega)}. \tag{6.3}$$

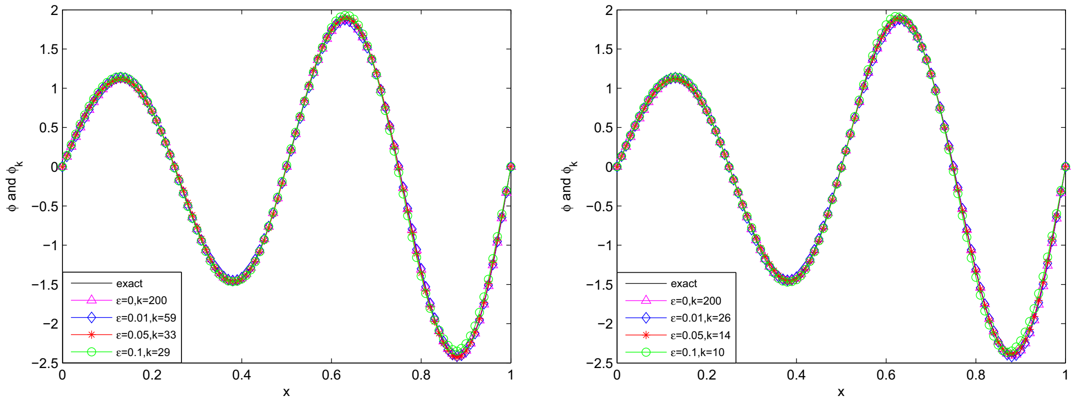
In an iteration algorithm, the most important work is to find a suitable stopping rule. In this paper we use the well-known Morozov's discrepancy principle [14], i.e. we choose k satisfying the following inequality:

$$E_k \le \tau\delta < E_{k-1}, \tag{6.4}$$

where $\tau > 1$ is a constant and can be taken heuristically to be 1.01, as suggested by Hanke and Hansen [4]. If $\delta = 0$, we take $k = 200$.

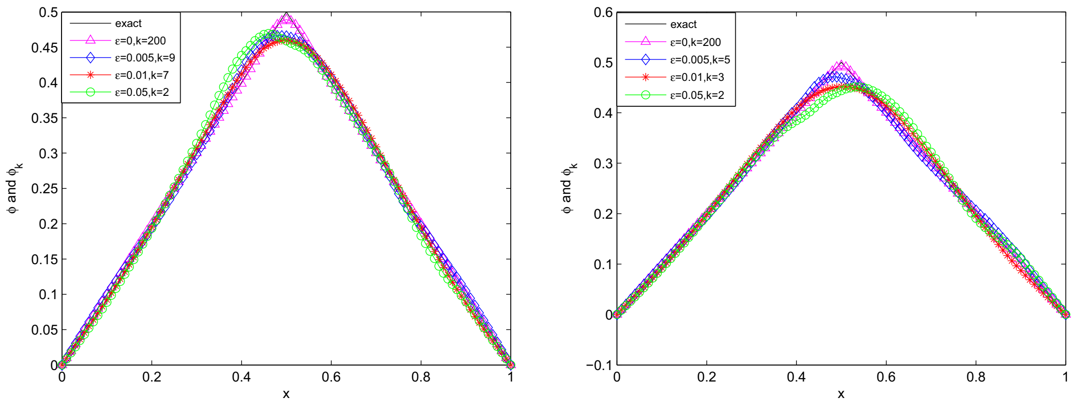
6.1. One-dimensional case

Without the loss of generality, we assume $\Omega = (0,1)$ and $T = 1$. In this case,we take the operator $L = \triangle = \frac{\partial^2}{\partial x^2}$. We solve the direct problem (1.1)–(1.3) and the sensitivity problem (4.5) by using a finite difference scheme given in [15, 21]. We solve the adjoint problem (4.16) by using a truncation method, and we take $\widetilde{v}_N(x,t) = \sum_{n=1}^{N=500} (u_\varphi(x,T) - h^\delta(x), \phi_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x)$ as an approximation to $\widetilde{v}(x,t)$ in problem (4.16). The grid size for time and space variable in the finite difference algorithm are $\Delta t = \frac{1}{100}$ and $\Delta x = \frac{1}{100}$ respectively.



(a) α = 0.3 **(b)** α = 0.7

FIGURE 1. The numerical results for Example 1 for various noise levels with $\mu = 1.0 \times 10^{-6}\delta^{\frac{2}{3}}$.



(a) α = 0.3 **(b)** α = 0.7

FIGURE 2. The numerical results for Example 2 for various noise levels with $\mu = 0$.

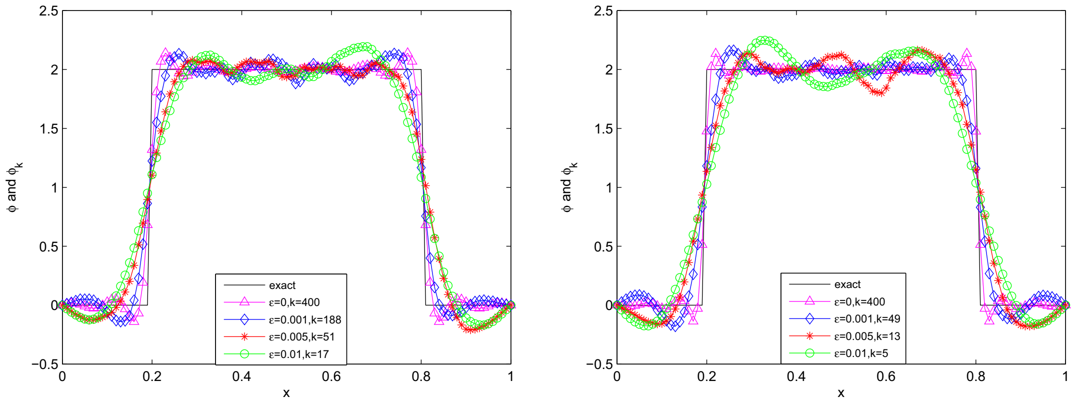
Example 1. In the first example, we test a smooth solution. Let the initial data $\varphi(x) = \sin(4\pi x)e^x$, and the source data $F(x,t) = x^\alpha(1-x)^\alpha \sin(4\pi x)e^{-t}$. The final data $u(x,T)$ is obtained by solving the direct problem (1.1)–(1.3) by using the finite difference scheme.

Example 2. In the second example, we test a nonsmooth solution with a cusp. Let the initial data $\varphi(x) = \frac{1}{2} - |x - \frac{1}{2}|$ and the source term $F(x,t) = \sin(5\pi x) + \cos(5\pi x)e^t$. The final data $u(x,T)$ is obtained by solving the direct problem (1.1)–(1.3) by using the finite difference scheme.

Example 3. We consider a discontinuous example. Let the initial data

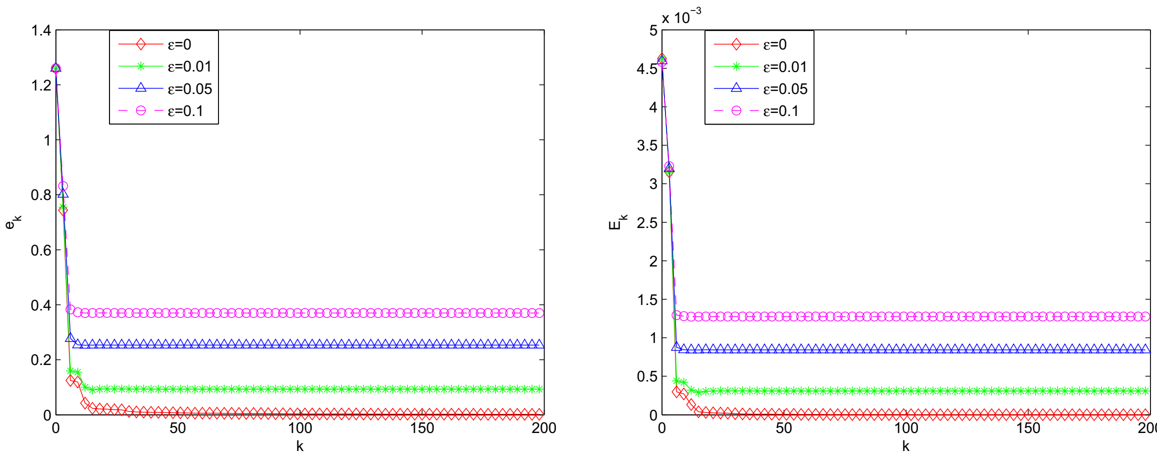
$$\varphi(x) = \begin{cases} 2, \, x \in [0.2, 0.8]; \\ 0, \, x \in [0, 0.2) \cup (0.8, 1]. \end{cases}$$

and the source term $F(x,t) = \sin(10x)e^{x-t} + \cos(10x)$. The final data $u(x,T)$ is obtained by solving the direct problem (1.1)–(1.3) by using the finite difference scheme.



(a) $\alpha = 0.3$ **(b)** $\alpha = 0.7$

FIGURE 3. The numerical results for Example 3 for various noise levels with $\mu = 0$.



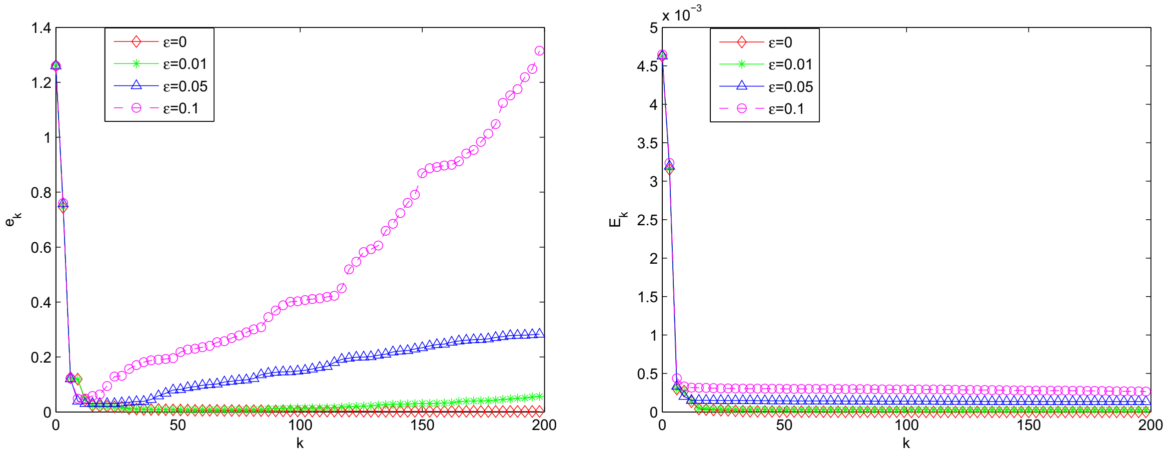
(a) e_k **(b)** E_k

FIGURE 4. The errors e_k and E_k for Example 1 for various noise levels with $\alpha = 0.5$ and $\mu = 1.0 \times 10^{-3}\delta^{\frac{2}{3}}$.

The numerical results for Examples 1–3 by using Morozov's discrepancy principle for various noise levels in the case of $\alpha = 0.3, 0.7$ are shown in Figures 1–3. We take $\mu = 1.0 \times 10^{-6}\delta^{\frac{2}{3}}$ in Example 1 and $\mu = 0$ in Examples 2–3 respectively. We can see that the numerical results are quite accurate to the exact solutions even up to 10% and 1% noise added up in the exact final data $u(x, T) = h(x)$ in Example 1 and Examples 2–3 respectively. We note that the iterative times decrease when α is close to 1, and the numerical results become less accurate if the noise level increase.

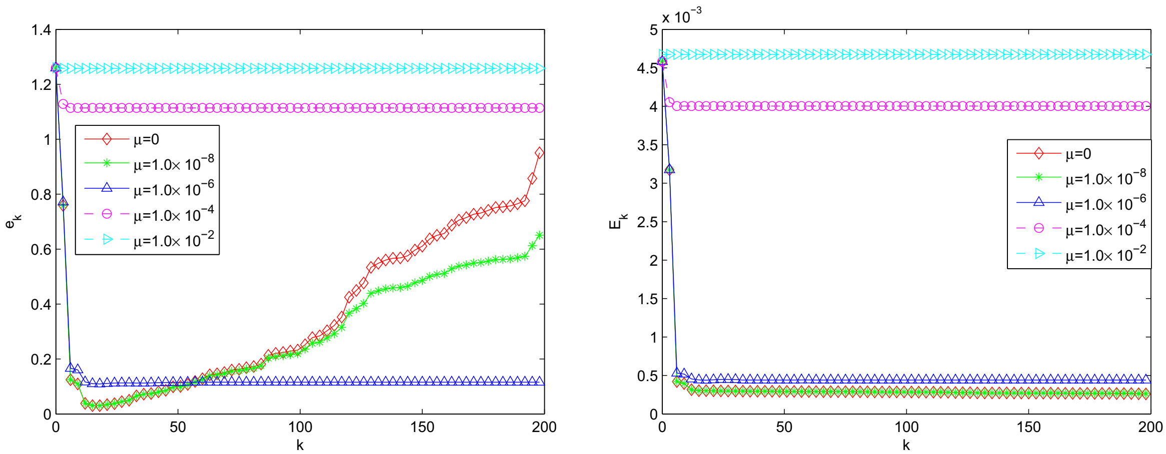
In the following, we investigate the convergence and stability of the proposed algorithm. We also show the effectiveness of the regularization parameter in the Tikhonov regularization functional.

The approximation errors e_k and the residuals E_k for Example 1 with various noise levels are shown in Figure 4 with $\alpha = 0.5$, $\mu = 1.0 \times 10^{-3}\delta^{\frac{2}{3}}$ and Figure 5 with $\alpha = 0.5$, $\mu = 0$ respectively. It can be observed in Figure 4 that if $\mu \neq 0$ the numerical errors decrease quickly and then keep a stable level which are yielded by the Tikhonov regularization. However, if $\mu = 0$ the computing errors show the semi-convergence phenomenon



(a) e_k **(b)** E_k

FIGURE 5. The errors e_k and E_k for Example 1 for various noise levels with $\alpha = 0.5$ and $\mu = 0$.

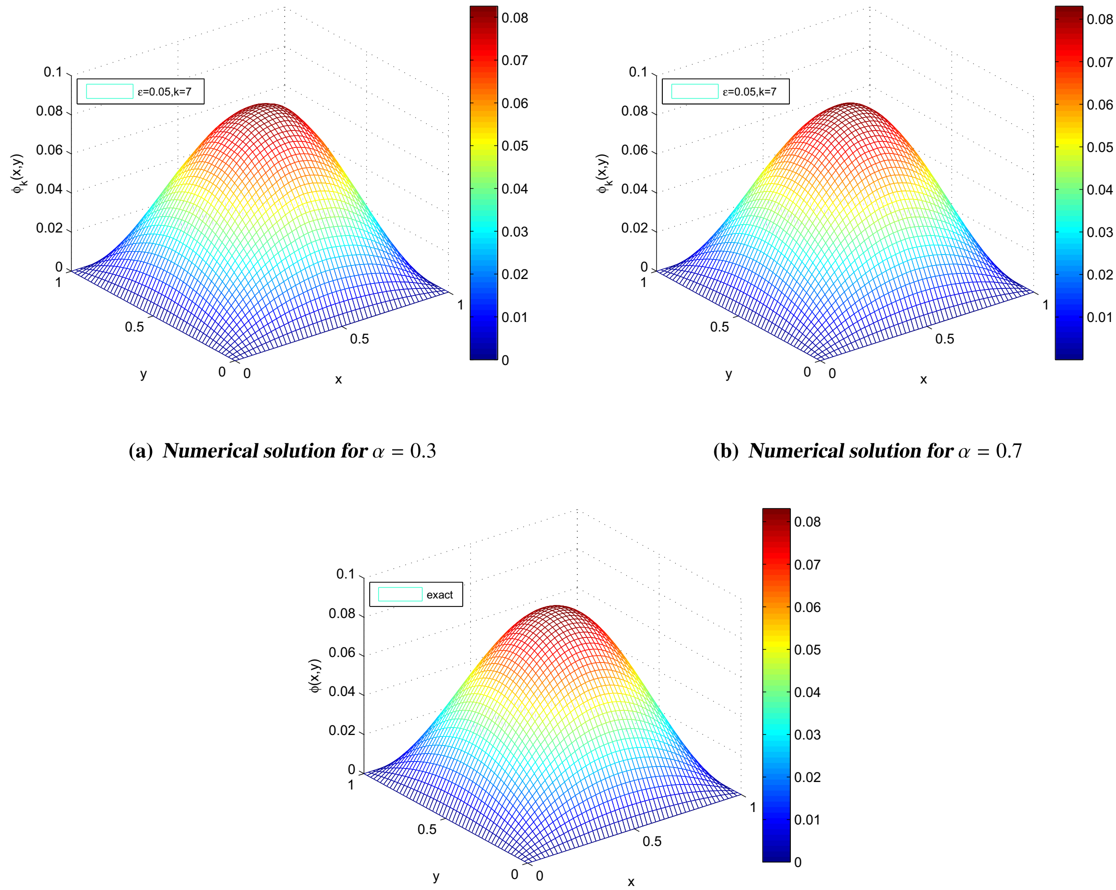


(a) e_k **(b)** E_k

FIGURE 6. The errors e_k and E_k for Example 1 for various μ with $\alpha = 0.5$ and the noise level $\varepsilon = 0.1$.

for a little large noise levels clearly in Figure 5 and the iteration step plays a role of the regularization parameter by Morozov's discrepancy principle in this case. From Figures 4 and 5, we can also see that the approximation errors e_k become smaller as the noise levels decrease, and the numerical results are better when $\mu = 0$ than when $\mu = 1.0 \times 10^{-3}\delta^{\frac{2}{3}}$ at the same noise level.

Next, we investigate the effect of the the regularization parameter. The approximation errors e_k and the residuals E_k for Example 1 with various μ are shown in Figure 6 with $\alpha = 0.5$, $\varepsilon = 0.1$. We can see that the regularization parameter μ is smaller, the numerical result is better by using Morozov's discrepancy principle. But when μ is too small, in particular $\mu = 0$, that means we do not use the Tikhonov regularization, the approximation errors e_k will increase when the iterative steps become large which indicate the discrepancy principle plays a role of regularization in this case. Figure 6(b) indicates that if the regularization parameter μ



(a) ***Numerical solution for*** $\alpha = 0.3$

(b) ***Numerical solution for*** $\alpha = 0.7$

(c) ***Exact solution***

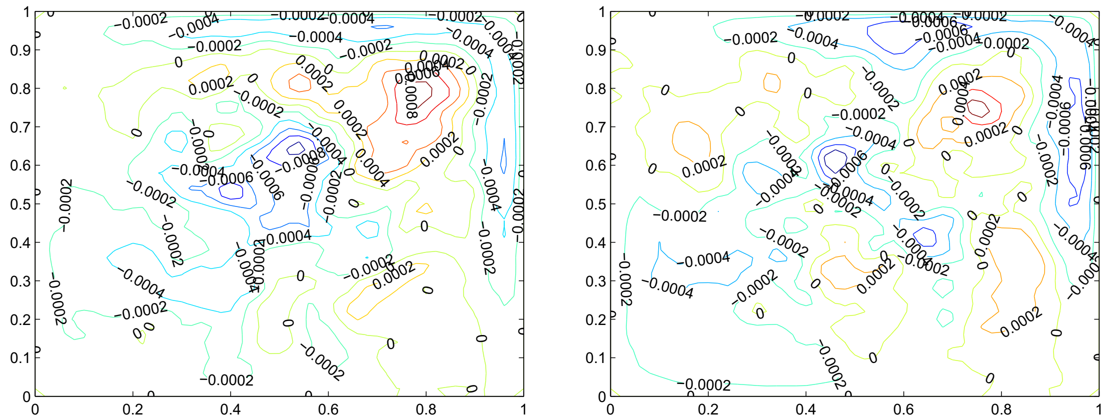
FIGURE 7. The exact solution and numerical results for Example 4 for $\varepsilon = 0.05$ with $\mu = 0$.

is too large, the Morozov's discrepancy principle is fail to use. Since we use Morozov's discrepancy principle, we can take $\mu = 0$ in this paper. In fact, it is a difficult problem to choose an appropriate regularization parameter.

6.2. Two-dimensional case

Denote the coordinate as (x, y). Let $\Omega = (0,1) \times (0,1)$ and $T = 1$. In this case, we take the operator $L = \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We solve the direct problem (1.1)–(1.3) and the sensitivity problem (4.5) by using a finite difference scheme. We solve the adjoint problem (4.16) by using a truncation method, and we take $\widetilde{v}_N(x,t) = \sum_{n=1}^{N=625}(u_\varphi(x,y,T) - h^\delta(x,y), \phi_n)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x,y)$ as an approximation to $\widetilde{v}(x,t)$ in problem (4.16). The grid size for time and space variable in the finite difference algorithm are $\varDelta t = \frac{1}{50}$ and $\varDelta x = \varDelta y = \frac{1}{50}$ respectively.

Example 4. In the last example, we take the initial data $\varphi(x,y) = xy(1-x)(1-y)e^{xy}$ and the source term $F(x,y,t) = xyt(1-x)(1-y)(1-t)$. The final data $u(x,T)$ is obtained by solving the direct problem (1.1)–(1.3) by using the finite difference scheme.



(a) ***Numerical error for*** $\alpha = 0.3$ **(b)** ***Numerical error for*** $\alpha = 0.7$

FIGURE 8. Numerical errors $\varphi_k(x, y) - \varphi(x, y)$ for Example 4.

The numerical results for Example 4 by using Morozov's discrepancy principle in the case of $\alpha = 0.3, 0.7$ are shown in Figure 7 with $\mu = 0$, and the numerical errors are shown in Figure 8. We take the noise level $\varepsilon = 0.05$ in Example 4. We can see that the numerical results are quite accurate to the exact solutions, and the proposed algorithm in this paper is effective.

7. Conclusion

In this paper, we consider a backward problem for a time-fractional diffusion equation. We obtain a stronger regularity of the weak solution for the direct problem, and the existence and uniqueness of a weak solution for the adjoint problem is proved. We formulate the backward problem into a variational problem by using the Tikhonov regularization method and use a conjugate gradient method to find the approximation of the regularized solution. Four numerical examples are provided to show that the proposed method is effective.

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