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PENALTY METHOD WITH CROUZEIX–RAVIART APPROXIMATION FOR THE STOKES EQUATIONS UNDER SLIP BOUNDARY CONDITION *

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Abstract. The Stokes equations subject to non-homogeneous slip boundary conditions are considered in a smooth domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$). We propose a finite element scheme based on the nonconforming P1/P0 approximation (Crouzeix–Raviart approximation) combined with a penalty formulation and with reduced-order numerical integration in order to address the essential boundary condition $u \cdot n_{\partial\Omega} = g$ on $\partial\Omega$. Because the original domain Ω must be approximated by a polygonal (or polyhedral) domain Ω_h before applying the finite element method, we need to take into account the errors owing to the discrepancy $\Omega \neq \Omega_h$, that is, the issues of domain perturbation. In particular, the approximation of $n_{\partial\Omega}$ by $n_{\partial\Omega_h}$ makes it non-trivial whether we have a discrete counterpart of a lifting theorem, i.e., continuous right inverse of the normal trace operator $H^1(\Omega)^N \to H^{1/2}(\partial\Omega)$; $u \mapsto u \cdot n_{\partial\Omega}$. In this paper we indeed prove such a discrete lifting theorem, taking advantage of the nonconforming approximation, and consequently we establish the error estimates $O(h^\alpha + \epsilon)$ and $O(h^{2\alpha} + \epsilon)$ for the velocity in the H^1- and L^2-norms respectively, where $\alpha = 1$ if $N = 2$ and $\alpha = 1/2$ if $N = 3$. This improves the previous result [T. Kashiwabara *et al.*, *Numer. Math.* **134** (2016) 705–740] obtained for the conforming approximation in the sense that there appears no reciprocal of the penalty parameter ϵ in the estimates.

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1. Introduction

This work is continuation of [16] and we consider the same PDEs as there, that is, the slip boundary value problem of the Stokes equations in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ as follows:

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$$\begin{cases} u - \nu\Delta u + \nabla p = f & \text{in} \quad \Omega, \\ \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ u \cdot n = g & \text{on} \quad \Gamma := \partial\Omega, \\ (\mathbb{I} - n \otimes n)\sigma(u,p)n = \tau & \text{on} \quad \Gamma. \end{cases} \tag{1.1}$$

As in [16], $\nu > 0$ is a viscosity constant, n means the outer unit normal to Γ, and $\sigma(u,p) := -p\mathbb{I} + \nu(\nabla u + (\nabla u)^\top)$ denotes the stress tensor. We impose the compatibility condition between $(1.1)_2$ and $(1.1)_3$ by

$$\int_\Gamma g \,\mathrm{d}s = 0. \tag{1.2}$$

Remark 1.1. The first term of $(1.1)_1$ is added in order to ensure coercivity of the problem, (*cf.* [16], Rem. 1.1). Therefore, strictly speaking, we deal with the Brinkman equations (see e.g. [14]) rather than the Stokes equations. However, the methods and results of this paper can be extended to the Stokes case if a Dirichlet boundary condition is imposed on a part of $\partial\Omega$, as discussed in our related papers [21, 22].

Before explaining the goals of the present paper, let us review the results of [16]. Since the original domain Ω has a curved boundary, we need to approximate it by a polygonal or polyhedral domain Ω_h to invoke the finite element method, where we construct meshes, build finite element spaces, and define variational formulations. In case of the slip boundary problem, however, one has to be careful in setting a test function space. In fact, imposing the constraint $v_h \cdot n_h = 0$ at each degree of freedom on Γ_h (n_h being the outer unit normal to Γ_h), which seems natural at first glance, would result in a variational crime. Several strategies to overcome it are proposed *e.g.* in [1, 12, 17, 20].

In [16], we proposed to apply a penalty method, in the P1/P1 finite element approximation, to the essential boundary condition $(1.1)_3$. At the continuous level, this strategy means replacing $(1.1)_3$ by

$$\frac{1}{\epsilon}(u \cdot n - g) + \sigma(u,p)n \cdot n = 0 \quad \text{on} \quad \Gamma,$$

where $\epsilon > 0$ is a very small constant (note that this formally recovers $(1.1)_3$ as $\epsilon \to 0$). Employing reduced-order numerical integration for the penalty term, we derived the error bound $O(h + \epsilon^{1/2} + h^{2\alpha}/\epsilon^{1/2})$ for the H^1- and L^2-norms of velocity and pressure, respectively. Here h denotes the discretization parameter, and the number α is given by $\alpha = 1$ if $N = 2$ and $\alpha = 1/2$ if $N = 3$. In particular, the optimal rate of convergence $O(h)$ was achieved by choosing $\epsilon = O(h^2)$ in the two-dimensional case. This strategy was then extended to the stationary Navier–Stokes equations in [21] and to the non-stationary Stokes equations in [22].

The first goal of the present paper is to improve the error bound mentioned above. In fact, the rate $O(h+\epsilon^{1/2}+h^{2\alpha}/\epsilon^{1/2})$ is not optimal because it is known that the penalty method admits the optimal rate of convergence $O(h + \epsilon)$ for polygonal or polyhedral domains, *i.e.*, when $\Omega = \Omega_h$ (see [9]). We show that the nonconforming P1/P0 approximation (also known as the Crouzeix–Raviart approximation, see [8, 11]) for smooth domains, combined with the penalty method and with reduced-order numerical integration, leads to the rate $O(h^\alpha + \epsilon)$, where the meaning of α is the same as above. Therefore, for the two-dimensional case we establish the optimal rate $O(h + \epsilon)$ even when $\Omega \neq \Omega_h$. Moreover, we also provide the L^2-error estimate for velocity, giving the rate of convergence $O(h^{2\alpha} + \epsilon)$, which was not available in [16].

The key point of our approach is that, in the Crouzeix–Raviart approximation, the degrees of freedom for velocity (namely, the midpoints of edges or the barycenters of faces) agree with those of n_h on the boundary Γ_h. This fact enables us to prove a discrete counterpart to the inf-sup condition

$$C\|\mu\|_{H^{-1/2}(\Gamma)} \le \sup_{v \in H^1(\Omega)} \frac{\int_\Gamma (v \cdot n)\mu \,\mathrm{d}s}{\|v\|_{H^1(\Omega)}} \qquad \forall \mu \in H^{-1/2}(\Gamma),$$

which was not available for the P1/P1 approximation in [16]. This follows from a discrete counterpart of a lifting theorem, more precisely, a stability estimate concerning a continuous right inverse of the trace operator in the normal direction:

$$H^1(\Omega)^N \to H^{1/2}(\Gamma); \quad v \mapsto v|_\Gamma \cdot n.$$

We emphasize, however, that such a discrete lifting theorem in Ω_h is completely non-trivial since n_h, which is only piecewise constant on Γ_h, has jump discontinuities and thus fails to belong to $H^{1/2}(\Gamma_h)^N$. Similarly, the trace of a nonconforming P1 function v_h to the boundary does not necessarily admit $H^{1/2}$-regularity (*cf.* [3], Appendix). To overcome those difficulties, we introduce a discrete version of the $H^{1/2}(\Gamma_h)$-norm and combine it with the so called *enriching operator* (*cf.* [7], Appendix B) to reduce the nonconforming approximation to the conforming one, which is a basic strategy to prove the discrete lifting theorem.

The second goal of the present paper is to provide, in case of nonconforming approximations, a framework to address the errors owing to the discrepancy $\Omega \neq \Omega_h$, which we refer to as *domain perturbation*. To the best of our knowledge, there are very few studies in the literature dealing with the issues of domain perturbation when nonconforming approximations, including discontinuous Galerkin methods, are involved. However, nonconforming approximations in the situation of domain perturbation is important when considering interfacial transmission problems (an example is the Stokes–Darcy problem, see *e.g.* [3, 18]). In fact, for such problems it is natural to encounter physical jump discontinuities in normal or tangential directions along curved interfaces, which could be treated by the use of nonconforming approximations. In future work, we would like to extend the techniques developed in this paper to interface problems in dealing with domain perturbation.

The rest of this paper is organized as follows. In Section 2 we introduce variational formulation, triangulation, and finite element spaces. We also propose our finite element scheme and state the main results. In Section 3, auxiliary lemmas relating to the discrete $H^{1/2}$-norm and to domain perturbation estimates are stated. Some of their proofs will be given in Appendices. After establishing discrete well-posedness in Section 4, we derive the H^1- and L^2-error estimates (for velocity) in Sections 5 and 6, respectively. We give a numerical example in Section 7 to confirm the theoretical result. Throughout this paper, C will denote a generic constant which may depend only on Ω, N, and ν unless otherwise stated.

2. Preliminaries and main theorem

2.1. Function spaces and variational forms

Throughout this paper, we adopt the standard notion of Lebesgue and Sobolev spaces. To state a variational formulation for (1.1), we set

$$V = H^1(\Omega)^N, \quad Q = L^2(\Omega), \quad \mathring{V} = H^1_0(\Omega)^N, \quad \mathring{Q} = L^2_0(\Omega),$$

and

$$V_n = \{v \in V \,:\, v \cdot n = 0 \text{ on } \Gamma\}.$$

Next, for a domain $G \subset \mathbb{R}^N$ we define bilinear forms as follows:

$$\begin{aligned} a_G(u, v) &= (u, v)_G + \frac{\nu}{2}(\mathbb{E}(u), \mathbb{E}(v))_G, \\ b_G(p, v) &= -(p, \operatorname{div} v)_G, \\ c_{\partial G}(\lambda, \mu) &= (\lambda, \mu)_{\partial G}, \end{aligned}$$

where $\mathbb{E}(u) := \nabla u + (\nabla u)^\top$ and $(\cdot, \cdot)_G$ denotes the inner product of $L^2(G)$.

The weak form for (1.1) now reads as follows: find $(u, p) \in V \times \mathring{Q}$ satisfying $u \cdot n = g$ on Γ and

$$\begin{cases} a(u, v) + b(p, v) = (f, v)_\Omega + (\tau, v)_\Gamma & \forall v \in V_n, \\ \qquad b(q, u) = 0 & \forall q \in \mathring{Q}, \end{cases} \tag{2.1}$$

where we have employed the abbreviations $a := a_\Omega$ and $b := b_\Omega$. Defining the Lagrange multiplier $\lambda := -\sigma(u,p)n\cdot n \in H^{-1/2}(\Gamma) =: \Lambda$, one sees that (u,p,λ) satisfies

$$\begin{cases} a(u,v) + b(p,v) + c(\lambda, v\cdot n) = (f,v)_\Omega + (\tau, v)_\Gamma & \forall v \in V, \\ b(q,u) = 0 & \forall q \in Q, \\ c(\mu, u\cdot n - g) = 0 & \forall \mu \in \Lambda, \end{cases} \tag{2.2}$$

where c means $c_{\partial\Omega}$. The well-posedness of (1.1) (or (2.1), (2.2)) is well known *e.g.* in [2]; in particular, if $f \in L^2(\Omega)$, $g \in H^{3/2}(\Gamma)$, and $\tau \in H^{1/2}(\Gamma)^N$, then there exists a unique solution such that $u \in H^2(\Omega)^N$ and $p \in H^1(\Omega) \cap L^2_0(\Omega)$.

2.2. Triangulations

Let $\{\mathcal{T}_h\}_{h\downarrow 0}$ be a regular family of triangulations of a polyhedral domain Ω_h, which is assigned the *mesh size* $h > 0$. Namely, we assume that:

(H1) each $T \in \mathcal{T}_h$ is a closed N-simplex such that $h_T := \operatorname{diam} T \le h$;
(H2) $\Omega_h = \bigcup_{T\in\mathcal{T}_h} T$;
(H3) the intersection of any two distinct elements is empty or consists of their common face of dimension $\le N-1$;
(H4) there exists a constant $C > 0$, independent of h, such that $\rho_T \ge C h_T$ for all $T \in \mathcal{T}_h$ where ρ_T denotes the diameter of the inscribed ball of T.

Moreover, we denote by $\mathcal{E}_h$ the set of the edges or faces, that is,

$$\mathcal{E}_h = \{e \subset \overline{\Omega}_h \,:\, e \text{ is an } (N-1)\text{-dimensional face of some } T \in \mathcal{T}_h\}.$$

The sets of the interior and boundary edges are denoted by $\mathring{\mathcal{E}}_h$ and $\mathcal{E}_h^\partial$ respectively, namely,

$$\mathcal{E}_h^\partial = \{e \in \mathcal{E}_h \,:\, e \subset \Gamma_h\}, \qquad \mathring{\mathcal{E}}_h = \mathcal{E}_h \setminus \mathcal{E}_h^\partial.$$

We assume that Ω_h approximates Ω in the following sense:

(H5) the vertices of every $e \in \mathcal{E}_h^\partial$ lie on $\Gamma = \partial\Omega$.

Throughout this paper, we confine ourselves to the case where $0 < h \ll 1$ is sufficiently small, which will not be emphasized below.

The set of vertices and that of midpoints of edges are defined as

$$\mathcal{V}_h = \{p \in \overline{\Omega}_h \,:\, p \text{ is a vertex of some } T \in \mathcal{T}_h\}, \qquad \mathcal{M}_h = \{m_e \in \overline{\Omega}_h \,:\, e \in \mathcal{E}_h\},$$

where m_e means the midpoint (barycenter) of $e \in \mathcal{E}_h$. We introduce a broken Sobolev space by

$$H^1(\mathcal{T}_h) = \{v \in L^2(\Omega_h) \,:\, v|_T \in H^1(T)\ \forall T \in \mathcal{T}_h\}.$$

To describe jump discontinuities across interior edges, for $v \in H^1(\mathcal{T}_h)$ we define

$$[v](x) := \lim_{s\to 0+} (v(x + s n_e) - v(x - s n_e)), \quad x \in e \in \mathring{\mathcal{E}}_h,$$

where n_e is a unit normal vector to e. For $e \in \mathring{\mathcal{E}}_h$ (resp. $e \in \mathcal{E}_h^\partial$), there exists a unique element $T_e^\pm \in \mathcal{T}_h$ (resp. $T \in \mathcal{T}_h$) such that $m_e \pm s n_e \in T_e^\pm$ with sufficiently small $s > 0$ (resp. $m_e \in T_e$).

Remark 2.1. There are two choices for the direction of n_e. In this paper, we suppose that each $e \in \mathring{\mathcal{E}}_h$ is given an arbitrary orientation, which determines the direction of n_e. Note that, given a vector function v, the jump term $[v\cdot n_e](x)$ is well defined regardless of the orientation.

2.3. Crouzeix–Raviart element

For each $T \in \mathcal{T}_h$ we denote by $P_k(T)$ the space of the polynomial functions of degree up to k defined in T. In the Crouzeix–Raviart element, velocity and pressure are approximated by nonconforming P1 and P0 functions, respectively. Thereby we introduce

$$\begin{aligned} V_h &= \{v_h \in H^1(\mathcal{T}_h)^N \,:\, v_h|_T \in P_1(T)^N \;\forall T \in \mathcal{T}_h, \quad [v_h](m_e) = \tfrac{1}{|e|}\textstyle\int_e [v_h]\,\mathrm{d}s = 0 \;\forall e \in \mathring{\mathcal{E}}_h\}, \\ Q_h &= \{q_h \in L^2(\Omega_h) \,:\, v_h|_T \in P_0(T) \;\forall T \in \mathcal{T}_h\}, \end{aligned}$$

where $|e|$ stands for the $(N-1)$-dimensional measure of e. We will also utilize the conforming P1 finite element space, that is,

$$\overline{V}_h = \{v_h \in C(\overline{\Omega}_h)^N \,:\, v_h|_T \in P_1(T)^N \;\forall T \in \mathcal{T}_h\}.$$

The nodal basis functions of V_h and $\overline{V}_h$ are denoted by $\{\phi_e\}_{e\in\mathcal{E}_h}$ and $\{\bar\phi_p\}_{p\in\mathcal{V}_h}$ respectively, where $\phi_e \in V_h$ and $\bar\phi_p \in \overline{V}_h$ are defined by the conditions

$$\phi_e(x) = \begin{cases} 1 & \text{if } x = m_e, \\ 0 & \text{if } x \neq m_e, e \in \mathcal{E}_h, \end{cases} \qquad \bar\phi_p(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & \text{if } x \neq p, x \in \mathcal{V}_h. \end{cases}$$

It follows from ([10], Thm. 3.1.2) and regularity of meshes that

$$\begin{aligned} &\|\phi_e\|_{H^m(T)} \le C h_e^{N/2-m}, && e \in \mathcal{E}_h,\, T \in \mathcal{T}_h,\, e \cap T \neq \emptyset, \\ &\|\phi_e\|_{H^m(e')} \le C h_e^{(N-1)/2-m}, && e, e' \in \mathcal{E}_h,\, e \cap e' \neq \emptyset, \end{aligned}$$

where $h_e := \operatorname{diam} e$, and the quantities dependent only on a fixed reference element (e.g. unit simplex) are combined into generic constants C. Similar estimates also hold for nodal basis functions $\bar\phi_p$ of $\overline{V}_h$, provided that the vertex p belongs to $T \in \mathcal{T}_h$ or $e' \in \mathcal{E}_h$.

Approximate spaces for $\mathring{V}$ and $\mathring{Q}$ are given as

$$\mathring{V}_h = \{v_h \in V_h \,:\, v_h(m_e) = 0 \;\forall e \in \mathcal{E}_h^\partial\}, \qquad \mathring{Q}_h = Q_h \cap \mathring{Q}.$$

We note, however, that $v_h \in \mathring{V}_h$ does not imply $v_h|_{\Gamma_h} \equiv 0$. We equip V_h and Q_h with the norms

$$\|v_h\|_{V_h} = \Big(\|v_h\|^2_{L^2(\Omega_h)} + \sum_{T\in\mathcal{T}_h} \|\nabla v_h\|^2_{L^2(T)}\Big)^{1/2}, \qquad \|q_h\|_{Q_h} = \|q_h\|_{L^2(\Omega_h)}.$$

To describe Lagrange multipliers defined on Γ_h, we set

$$\begin{aligned} \Lambda_h &= \{\mu_h \in L^2(\Gamma_h) \,:\, \mu_h|_e \in P_0(e) \;\forall e \in \mathcal{E}_h^\partial\}, \\ \overline{\Lambda}_h &= \{\mu_h \in C(\Gamma_h) \,:\, \mu_h|_e \in P_1(e) \;\forall e \in \mathcal{E}_h^\partial\}. \end{aligned}$$

An interpolation operator $\Pi_h : H^1(\Omega_h)^N \to V_h$ is defined by $\Pi_h v(m_e) = \frac{1}{|e|}\int_e v\,\mathrm{d}s$ for $e \in \mathcal{E}_h$. It is known (see [11]) that

$$\begin{aligned} \|v - \Pi_h v\|_{L^2(T)} + h_T\|\nabla(v - \Pi_h v)\|_{L^2(T)} &\le C h_T^2 \|\nabla^2 v\|_{L^2(T)}, && T \in \mathcal{T}_h,\; v \in H^2(T), \\ \|v - \Pi_h^\partial v\|_{H^{-1/2}(e)} &\le C h_e \|\nabla v\|_{L^2(T_e)}, && e \in \mathcal{E}_h^\partial,\; v \in H^1(T_e). \end{aligned}$$

For convenience, we also define an analogue of Π_h restricted to the boundary, namely, we define $\Pi_h^\partial : L^2(\Gamma_h) \to \Lambda_h$ by $\Pi_h^\partial v(m_e) = \frac{1}{|e|}\int_e v\,\mathrm{d}s$ for all $e \in \mathcal{E}_h^\partial$.

The continuity at the midpoints ensures that

$$\sum_{e\in\mathring{\mathcal{E}}_h} h_e^{-1} \|[v_h]\|^2_{L^2(e)} \le C \sum_{T\in\mathcal{T}_h} \|\nabla v_h\|^2_{L^2(T)} \quad \forall v_h \in V_h. \tag{2.3}$$

In fact, since $[v_h](m_e) = 0$ for $e \in \mathring{\mathcal{E}}_h$ and ∇v_h is piecewise constant, we have

$$\begin{aligned}\frac{1}{h_e}\int_e |[v_h]|^2\,\mathrm{d}s &\le \frac{1}{2h_e}\left(\int_e \left|v_h|_{T_e^+} - v_h(m_e)\right|^2 \mathrm{d}s + \int_e \left|v_h|_{T_e^-} - v_h(m_e)\right|^2 \mathrm{d}s\right)\\ &\le \frac{|e|h_e}{2}(\|\nabla v_h\|^2_{L^\infty(T_e^+)} + \|\nabla v_h\|^2_{L^\infty(T_e^-)}) \le C\sum_{T\in\mathcal{T}_h(e)} \|\nabla v_h\|^2_{L^2(T)},\end{aligned}$$

which after the summation for $e \in \mathring{\mathcal{E}}_h$ proves (2.3). Hence $\|\cdot\|_{V_h}$ is equivalent to $|\!|\!|\cdot|\!|\!|_{V_h}$ given by

$$|\!|\!|v_h|\!|\!|_{V_h} = \Big(\|v_h\|^2_{L^2(\Omega_h)} + \sum_{T\in\mathcal{T}_h} \|\nabla v_h\|^2_{L^2(T)} + \sum_{e\in\mathring{\mathcal{E}}_h} h_e^{-1}\|[v_h]\|^2_{L^2(e)}\Big)^{1/2}, \quad v_h \in V_h,$$

which often appears in discontinuous Galerkin methods.

Adding up the trace inequality $\|v\|_{L^2(e)} \le C\|v\|^{1/2}_{L^2(T_e)}\|v\|^{1/2}_{H^1(T_e)}$ for $e \in \mathcal{E}_h^\partial$ yields

$$\|v\|_{L^2(\Gamma_h)} \le C\|v\|^{1/2}_{L^2(\Omega_h)}\|v\|^{1/2}_{V_h}, \quad v \in H^1(\mathcal{T}_h),$$

where the constant C depends only on a reference element.

An interpolation operator for pressure is defined as the projector $R_h : Q \to Q_h$, that is, $(R_h p - p, q_h)_{\Omega_h} = 0$ for all $p \in Q$ and $q_h \in Q_h$. Then we have (see [6], Lem. 12.4.3)

$$\|R_h p - p\|_{Q_h} \le Ch\|\nabla p\|_{L^2(\Omega_h)}, \qquad p \in H^1(\Omega_h).$$

We also note that $R_h(\mathring{Q}) \subset \mathring{Q}_h$.

2.4. FE scheme with penalty and main theorem

We propose a finite element approximate problem to (1.1) as follows: choose $\epsilon > 0$ and find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{cases} a_h(u_h, v_h) + b_h(p_h, v_h) + \dfrac{1}{\epsilon}c_h(u_h\cdot n_h - \tilde{g}, v_h\cdot n_h) + j_h(u_h, v_h) = (\tilde{f}, v_h)_{\Omega_h} + (\tilde{\tau}, v_h)_{\Gamma_h} & \forall v_h \in V_h,\\ b_h(q_h, u_h) = 0 & \forall q_h \in Q_h. \end{cases} \tag{2.4}$$

Here, we are making use of an extension operator $P : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^N)$ satisfying the stability condition $\|Pv\|_{W^{m,p}(\mathbb{R}^N)} \le C\|v\|_{W^{m,p}(\Omega)}$, where the constant C depends only on N, Ω, m, and p. If this is combined with a stable lifting operator (continuous right inverse of the trace operator) $L : W^{m-1/p,p}(\Gamma) \to W^{m,p}(\Omega)\,(m \ge 1)$, one can also consider extensions from Γ to $\mathbb{R}^N$. In the following, all of such extensions are simply denoted by $\tilde{f}$, $\tilde{g}$, $\tilde{\tau}$, etc.

Remark 2.2. The way of extensions may be arbitrary as far as they satisfy the stability conditions mentioned above. In particular, P or L has no effect on the rate of convergence in Theorems 2.6 and 2.7, whereas the constants C appearing there will depend on the choice of them.

The bilinear forms in (2.4) are defined by

$$\begin{aligned} a_h(u, v) &= \sum_{T\in\mathcal{T}_h}\Big((u, v)_T + \frac{\nu}{2}(\mathbb{E}(u), \mathbb{E}(v))_T\Big), && u, v \in H^1(\mathcal{T}_h),\\ b_h(p, v) &= -\sum_{T\in\mathcal{T}_h}(p, \operatorname{div} v)_T, && p \in Q, v \in H^1(\mathcal{T}_h),\\ c_h(\lambda, \mu) &= (\Pi_h^\partial \lambda, \Pi_h^\partial \mu)_{\Gamma_h}, && \lambda, \mu \in L^2(\Gamma_h),\\ j_h(u, v) &= \sum_{e\in\mathring{\mathcal{E}}_h}\frac{\gamma}{h_e}([u], [v])_e, && u, v \in H^1(\mathcal{T}_h), \end{aligned}$$

where γ is a stabilization parameter, which one can choose to be any positive constant.

Remark 2.3. For $u_h, v_h \in V_h$, we see that $c_h(u_h \cdot n_h, v_h \cdot n_h)$ agrees with the midpoint (barycenter) formula applied to $(u_h \cdot n_h, v_h \cdot n_h)_{\Gamma_h}$. In this sense, reduced-order numerical integration is applied to the penalty term.

The main results of this paper are the well-posedness and error estimates to (2.4) stated as follows.

Theorem 2.4. *There exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ of (2.4). Moreover, it satisfies*

$$\|u_h\|_{V_h} + \|\mathring{p}_h\|_{Q_h} \le C\big(\|f\|_{L^2(\Omega)} + \|\tau\|_{H^{1/2}(\Gamma)} + (1 + h\epsilon^{-1/2})\|g\|_{H^{3/2}(\Gamma)}\big), \tag{2.5}$$

$$|k_h| \le C\big(\|f\|_{L^2(\Omega)} + \|\tau\|_{H^{1/2}(\Gamma)} + (1 + h^2\epsilon^{-1})\|g\|_{H^{3/2}(\Gamma)}\big), \tag{2.6}$$

where $k_h := (p_h, 1)_{\Omega_h}/|\Omega_h|$ and $\mathring{p}_h := p_h - k_h \in \mathring{Q}_h$.

Remark 2.5. (i) If $g = 0$, the terms involving ϵ^{-1} do not appear.
(ii) Even if $g \neq 0$, $\|u_h\|_{V_h}$ becomes independent of $\epsilon \le 1$ in the end as a consequence of Theorem 2.6.

Theorem 2.6. *Let $(u, p) \in H^2(\Omega)^N \times H^1(\Omega)$ be the solution of (1.1) and $(u_h, p_h) \in V_h \times Q_h$ be that of (2.4). Then we obtain*

$$\|\tilde{u} - u_h\|_{V_h} + \|\tilde{p} - \mathring{p}_h\|_{Q_h} \le C(h^\alpha + \epsilon)(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Gamma)} + \|\tau\|_{H^{1/2}(\Gamma)}),$$

where $\alpha = 1$ if $N = 2$ and $\alpha = 1/2$ if $N = 3$.

Theorem 2.7. *Under the same assumption as in the previous theorem, we obtain*

$$\|\tilde{u} - u_h\|_{L^2(\Omega_h)} \le C(h^{2\alpha} + \epsilon)(\|f\|_{L^2(\Omega)} + \|g\|_{H^{3/2}(\Gamma)} + \|\tau\|_{H^{1/2}(\Gamma)}).$$

The proofs of Theorems 2.4–2.7 will be given in Sections 4–6, respectively.

3. Auxiliary Lemmas

3.1. Discrete $H^{1/2}$-norm

It is well known that there exists a continuous right inverse of the trace operator $H^1(\Omega)^N \to H^{1/2}(\Gamma)$; $v \mapsto (v \cdot n)|_\Gamma$, which we call a *lifting operator* with respect to the normal component. We need its analogue in the Crouzeix–Raviart element case. However, since functions having jump discontinuities do not belong to $H^{1/2}$, we devise a discrete $H^{1/2}(\Gamma_h)$-norm for $\mu_h \in \Lambda_h$ as follows:

$$\|\mu_h\|_{1/2,\Lambda_h} = \Big(\|E_h^\partial \mu_h\|^2_{H^{1/2}(\Gamma_h)} + \sum_{e\in\mathcal{E}_h^\partial} \sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{N-2}|\mu_h(m_e) - \mu_h(m_{e'})|^2 + h\|\mu_h\|^2_{L^2(\Gamma_h)}\Big)^{1/2}.$$

Here, $E_h^\partial : \Lambda_h \to \overline{\Lambda}_h$ is a kind of *enriching operators* (*cf.* [7], Appendix B) defined by

$$E_h^\partial \mu_h = \sum_{p\in\mathcal{V}_h(\Gamma_h)} \Big(\frac{1}{\#\mathcal{E}_h^\partial(p)} \sum_{e\in\mathcal{E}_h^\partial(p)} \mu_h(m_e)\Big)\bar{\phi}_p,$$

where $\mathcal{V}_h(\Gamma_h) = \mathcal{V}_h \cap \Gamma_h$, $\mathcal{E}_h^\partial(p) = \{e \in \mathcal{E}_h^\partial \,:\, p \in e\}$ means the boundary elements sharing the vertex p, and $\bar{\phi}_p \in \bar{V}_h$ is a nodal basis of the conforming P1 functions given in Section 2. Note that, as a result of the regularity of meshes, the number of elements $\#\mathcal{E}_h^\partial(p)$ is bounded independently of p and h. Moreover, $\mathcal{E}_h^\partial(e) = \{e' \in \mathcal{E}_h^\partial \,:\, e \cap e' \neq \emptyset\}$ denotes the neighboring boundary edges around e.

The discrete $H^{1/2}$-norm is compatible with the usual $H^{1/2}$-norm as follows.

Lemma 3.1. *If $\mu \in H^{1/2}(\Gamma_h)$, then*

$$\|\Pi_h^\partial \mu\|_{1/2,\Lambda_h} \le C\|\mu\|_{H^{1/2}(\Gamma_h)}.$$

We also state discrete $H^{1/2}$-stability when n_h is involved.

Lemma 3.2. *Let $\mu \in H^{1/2}(\Gamma_h)$, $v \in H^{1/2}(\Gamma_h)^N$, and $A \in H^{1/2}(\Gamma_h)^{N^2}$ be scalar, vector, and matrix functions respectively. Then we have*

$$\begin{aligned} \|(\Pi_h^\partial \mu) n_h\|_{1/2,\Lambda_h} &\le C\|\mu\|_{H^{1/2}(\Gamma_h)}, \\ \|(\Pi_h^\partial v)\cdot n_h\|_{1/2,\Lambda_h} &\le C\|v\|_{H^{1/2}(\Gamma_h)}, \\ \|(\Pi_h^\partial A) n_h \cdot n_h\|_{1/2,\Lambda_h} &\le C\|A\|_{H^{1/2}(\Gamma_h)}. \end{aligned}$$

The proofs of Lemmas 3.1 and 3.2 will be given in Appendices A.1 and A.2, respectively.

3.2. Discrete lifting theorems with respect to the normal component

Let us state a first version of discrete lifting theorems.

Lemma 3.3. *For all $\mu_h \in \Lambda_h$ we obtain*

$$C\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|\mu_h\|^2_{L^2(e)}\Big)^{1/2} \le \sup_{v_h\in V_h} \frac{c_h(\mu_h, v_h\cdot n_h)}{\|v_h\|_{V_h}}. \tag{3.1}$$

Proof. Define $v_h \in V_h$ by $v_h = \sum_{e\in\mathcal{E}_h^\partial} h_e\mu_h(m_e)n_h(m_e)\phi_e$. Then we see that $c_h(\mu_h, v_h\cdot n_h) = \sum_{e\in\mathcal{E}_h^\partial} h_e\|\mu_h\|^2_{L^2(e)}$ and that

$$\|v_h\|^2_{V_h} = \sum_{e\in\mathcal{E}_h^\partial} h_e^2|\mu_h(m_e)n_h(m_e)|^2\|\phi_e\|^2_{H^1(T_e)} \le \sum_{e\in\mathcal{E}_h^\partial} h_e^2\|\mu_h\|^2_{L^\infty(e)} \times Ch_e^{N-2} \le C\sum_{e\in\mathcal{E}_h^\partial} h_e\|\mu_h\|^2_{L^2(e)},$$

where we have used a local inverse inequality $\|\mu_h\|_{L^\infty(e)} \le Ch_e^{(1-N)/2}\|\mu_h\|_{L^2(e)}$. Combining the two relations, we obtain the desired inf-sup condition. □

We need a more refined discrete lifting theorem than the one above.

Lemma 3.4. *For $\mu_h \in \Lambda_h$ there exists $v_h \in V_h$ satisfying $(v_h\cdot n_h)(m_e) = \mu_h(m_e)$ for all $e \in \mathcal{E}_h^\partial$, together with the stability estimate*

$$\|v_h\|_{V_h} \le C\|\mu_h\|_{1/2,\Lambda_h}. \tag{3.2}$$

The proof of this lemma will be given in Appendix A.3.

Corollary 3.5. *For all $\mu_h \in \Lambda_h$ we obtain*

$$C\|\mu_h\|_{-1/2,\Lambda_h} \le \sup_{v_h\in V_h} \frac{c_h(\mu_h, v_h\cdot n_h)}{\|v_h\|_{V_h}}. \tag{3.3}$$

Proof. By the definition of the dual norm, there exists $\lambda_h \in \Lambda_h$ such that $\|\mu_h\|_{-1/2,\Lambda_h} = \frac{c_h(\mu_h,\lambda_h)}{\|\lambda_h\|_{1/2,\Lambda_h}}$. We apply Lemma 3.4 to λ_h to obtain some $v_h \in V_h$ such that $v_h\cdot n_h = \lambda_h$ at all m_e's lying on Γ_h and $\|v_h\|_{V_h} \le C\|\lambda_h\|_{1/2,\Lambda_h}$. It is now immediate to deduce (3.3). □

3.3. Estimates on the boundary-skin layer

Let us introduce a tubular neighborhood of Γ with width $\delta > 0$ by $\Gamma(\delta) = \{x \in \mathbb{R}^N \,:\, \operatorname{dist}(x, \Gamma) < \delta\}$. For sufficiently small $\delta_0 > 0$, we know that (see [13], Sect. 14.6) there holds a unique decomposition $\Gamma(\delta_0) \ni x = \bar{x} + tn(\bar{x})$ with $\bar{x} \in \Gamma$. The maps $\pi : \Gamma(\delta_0) \to \Gamma$; $x \mapsto \bar{x}$ and $d : \Gamma(\delta_0) \to \mathbb{R}$; $x \mapsto t$ imply the orthogonal projection to Γ and the signed-distance function, respectively. We fix a bounded smooth domain $\tilde{\Omega}$ that contains $\Omega \cup \Gamma(\delta_0)$.

If the mesh size h is sufficiently small, we proved in ([16], Sect. 8) that $\pi|_{\Gamma_h} : \Gamma_h \to \Gamma$ is a homeomorphism and that $|d(x)| \le Ch_e^2 =: \delta_e$ for $x \in e \in \mathcal{E}_h^\partial$. Then the following *boundary-skin estimates* are obtained:

$$\Big| \int_{\pi(e)} f \,\mathrm{d}s - \int_e f \circ \pi \,\mathrm{d}s \Big| \le C\delta_e \|f\|_{L^1(e)}, \qquad f \in L^1(e), \tag{3.4}$$

$$\|f - f \circ \pi\|_{L^p(e)} \le C\delta_e^{1-1/p} \|\nabla f\|_{L^p(\pi(e,\delta_e))}, \qquad f \in W^{1,p}(\pi(e,\delta_e)), \tag{3.5}$$

$$\|f\|_{L^p(\pi(e,\delta_e))} \le C\delta_e^{1/p} \|f\|_{L^p(\pi(e))} + C\delta_e \|\nabla f\|_{L^p(\pi(e,\delta_e))}, \qquad f \in W^{1,p}(\pi(e,\delta_e)), \tag{3.6}$$

where $p \in [1, \infty]$ and $\pi(e, \delta_e) := \{\bar{x} + tn(\bar{x}) \in \mathbb{R}^2 \,:\, \bar{x} \in \pi(e),\ |t| < \delta_e\}$ denotes a tubular neighborhood of $\pi(e) \subset \Gamma$. As a version of (3.6), we also have (see [15], Lem. A.1)

$$\|f\|_{L^p((\Omega_h \setminus \Omega) \cap \pi(e,\delta_e))} \le C\delta_e^{1/p} \|f\|_{L^p(e)} + C\delta_e \|\nabla f\|_{L^p((\Omega_h \setminus \Omega) \cap \pi(e,\delta_e))}.$$

Adding up the estimates above for $e \in \mathcal{E}_h^\partial$, we obtain corresponding global estimates on boundary-skin layers. In particular one has

$$\|v\|_{L^2(\Omega_h \setminus \Omega)} \le Ch \|v\|_{H^1(\Omega_h)} \quad \forall v \in H^1(\Omega_h). \tag{3.7}$$

Here we present its version in case of a nonconforming approximation. For the proof, see Section A.4.

Lemma 3.6. *For all $v \in V_h + H^1(\Omega_h)^N$ we obtain*

$$\|v\|_{L^2(\Omega_h \setminus \Omega)} \le Ch \|\!|v|\!\|_{V_h}.$$

3.4. Interpolation estimates for $u \cdot n = g$

Although the approximability of n_h to n is only $O(h)$ on Γ_h, at the midpoints of edges it is improved to $O(h^2)$ for $N = 2$ as result of super-convergence. This was a key observations in [16] to deal with errors caused by discretization of $u \cdot n = g$; this idea, however, demanded the assumption of the $W^{2,\infty}$-regularity for velocity u. Here we present a different approach which only requires $u \in H^2(\Omega)^N$, taking advantage of the divergence-free condition.

Lemma 3.7. *Let $u \in H^2(\Omega)^N$ satisfy $\operatorname{div} u = 0$. Then for $e \in \mathcal{E}_h^\partial$ we have*

$$\left| \int_e u \cdot n_h \,\mathrm{d}s - \int_{\pi(e)} u \cdot n \,\mathrm{d}s \right| \le \begin{cases} Ch_e^{9/2} \|\nabla^2 \tilde{u}\|_{L^2(\pi(e,\delta_e))} & \text{if} \quad N = 2, \\ Ch_e^3 \|\tilde{u}\|_{H^2(\tilde{\Omega})} & \text{if} \quad N = 3. \end{cases}$$

Proof. We set $D := \pi(e, \delta_e) \cap \Omega$, $D_h := \pi(e, \delta_e) \cap \Omega_h$, and introduce "reminder boundaries" of D and D_h by $R = \partial D \setminus \pi(e)$ and $R_h = \partial D_h \setminus e$. Then it follows from the divergence theorem that

$$\int_e \tilde{u} \cdot n_h \,\mathrm{d}s - \int_{\pi(e)} u \cdot n \,\mathrm{d}s = \int_{D_h \setminus D} \operatorname{div} \tilde{u} \,\mathrm{d}x - \Big(\int_{R_h} \tilde{u} \cdot \nu_h \,\mathrm{d}s - \int_R \tilde{u} \cdot \nu \,\mathrm{d}s \Big) =: I_1 + I_2,$$

where ν and ν_h denote the outer unit normals to R and R_h, respectively.

When $N = 2$, $I_2 = 0$ since $R_h = R$. By (3.6), we have (note that $\operatorname{div} u = 0$ on Γ)

$$|I_1| \le \|\operatorname{div} \tilde{u}\|_{L^1(\pi(e,\delta_e))} \le C\delta_e \|\nabla \operatorname{div} \tilde{u}\|_{L^1(\pi(e,\delta_e))} \le C\delta_e |\pi(e,\delta_e)|^{1/2} \|\nabla \operatorname{div} \tilde{u}\|_{L^2(\pi(e,\delta_e))},$$

which combined with $|\pi(e,\delta_e)|^{1/2} \le Ch_e^{N-1}\delta_e$ implies the desired estimate.

When $N=3$, denoting by $L_e = \{\bar{x} + tn(\bar{x}) \,:\, \bar{x} \in \partial\pi(e),\ |t| \le \delta_e\}$ the lateral boundary of $\pi(e,\delta_e)$, we obtain

$$|I_2| \le |L_e| \|\tilde{u}\|_{L^\infty(\tilde{\Omega})} \le Ch_e\delta_e \|\tilde{u}\|_{H^2(\tilde{\Omega})},$$

where we have used Sobolev's embedding theorem. Since the estimate of I_2 dominates that of I_1, the desired result follows. □

Remark 3.8. If the extension satisfies $\operatorname{div}\tilde{u} = 0$ in $\tilde{\Omega}$, then the error becomes zero for $N = 2$.

We apply the above lemma to estimate the error $\tilde{u}\cdot n_h - \tilde{g}$ on Γ_h.

Lemma 3.9. *Let $u \in H^2(\Omega)^N$ and $g \in H^{3/2}(\Gamma)$ satisfy $\operatorname{div} u = 0$ and $u \cdot n = g$. Then for $e \in \mathcal{E}_h^\partial$ we have*

$$\|\Pi_h^\partial(\tilde{u}\cdot n_h - \tilde{g})\|_{L^2(e)}^2 \le \begin{cases} Ch_e^4(\|\tilde{g}\|_{L^2(e)}^2 + \|\nabla\tilde{g}\|_{L^2(e)}^2 + \|\nabla^2\tilde{g}\|_{L^2(\pi(e,\delta_e))}^2) + Ch_e^8\|\nabla^2\tilde{u}\|_{L^2(\pi(e,\delta_e))}^2 & (N=2),\\ Ch_e^4(\|\tilde{g}\|_{L^2(e)}^2 + \|\nabla\tilde{g}\|_{L^2(e)}^2 + \|\nabla^2\tilde{g}\|_{L^2(\pi(e,\delta_e))}^2) + Ch_e^4\|\tilde{u}\|_{H^2(\tilde{\Omega})}^2 & (N=3).\end{cases}$$

Proof. Observe that

$$\begin{aligned}\|\Pi_h^\partial(\tilde{u}\cdot n_h - \tilde{g})\|_{L^2(e)}^2 &= |e|^{-1}\left|\int_e \tilde{u}\cdot n_h\,\mathrm{d}s - \int_e \tilde{g}\,\mathrm{d}s\right|^2\\ &\le Ch_e^{1-N}\Big(\Big|\int_e \tilde{u}\cdot n_h\,\mathrm{d}s - \int_{\pi(e)} u\cdot n\,\mathrm{d}s\Big|^2 + \Big|\int_{\pi(e)} g\,\mathrm{d}s - \int_e \tilde{g}\,\mathrm{d}s\Big|^2\Big).\end{aligned}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}\Big|\int_{\pi(e)} g\,\mathrm{d}s - \int_e \tilde{g}\,\mathrm{d}s\Big| &\le \Big|\int_{\pi(e)} g\,\mathrm{d}s - \int_e g\circ\pi\,\mathrm{d}s\Big| + \int_e |g\circ\pi - \tilde{g}|\,\mathrm{d}s \le C\delta_e\|\tilde{g}\|_{L^1(e)} + C\|\nabla\tilde{g}\|_{L^1(\pi(e,\delta_e))}\\ &\le C\delta_e(\|\tilde{g}\|_{L^1(e)} + \|\nabla\tilde{g}\|_{L^1(e)} + \|\nabla^2\tilde{g}\|_{L^1(\pi(e,\delta_e))})\\ &\le Ch_e^{(N+3)/2}(\|\tilde{g}\|_{L^2(e)} + \|\nabla\tilde{g}\|_{L^2(e)} + \|\nabla^2\tilde{g}\|_{L^2(\pi(e,\delta_e))}).\end{aligned}$$

Combining these with Lemma 3.7, we conclude the desired estimates. □

Remark 3.10. If (u,p) is a solution of (1.1), then adding up the result of Lemma 3.9 for $e \in \mathcal{E}_h^\partial$ yields

$$\|\Pi_h^\partial(\tilde{u}\cdot n_h - \tilde{g})\|_{L^2(\Gamma_h)} \le Ch^{2\alpha}\|u\|_{H^2(\Omega)}, \tag{3.8}$$

$$\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e^{-1}\|\Pi_h^\partial(\tilde{u}\cdot n_h - \tilde{g})\|_{L^2(e)}^2\Big)^{1/2} \le Ch^{2\alpha-1/2}\|u\|_{H^2(\Omega)}, \tag{3.9}$$

where α is the same as in Theorem 2.6 and we have used $\sum_{e\in\mathcal{E}_h^\partial} h_e^2 \le C$ in case $N = 3$.

4. Well-posedness of the Approximate Problem

We adopt the following discrete version of Korn's inequality proved in [5] (see also [8, p. 993]):

$$C\|v_h\|_{V_h}^2 \le a_h(v_h,v_h) + j_h(v_h,v_h) \quad \forall v_h \in V_h. \tag{4.1}$$

In addition, it is known that an inf-sup condition is valid for b_h (see [4, Sect. 8.4.4]):

$$C\|q_h\|_{Q_h} \le \sup_{v_h\in\mathring{V}_h} \frac{b_h(q_h,v_h)}{\|v_h\|_{V_h}} \quad \forall q_h \in \mathring{Q}_h. \tag{4.2}$$

Remark 4.1. The positive constants C appearing above depend on the $C^{0,1}$-regularity of the domain Ω_h, which is independent of h if it is sufficiently small.

Proof of Theorem 2.4. Because the problem is linear and finite-dimensional, it suffices to show the *a priori* estimate (2.5) assuming the existence of a solution (u_h, p_h) of (2.4). Since v_h vanishes at m_e's on Γ_h and $b_h(1, v_h) = 0$ for $v_h \in \mathring{V}_h$, it follows from the inf-sup condition (4.2) that

$$C\|\mathring{p}_h\|_{Q_h} \le \sup_{v_h\in\mathring{V}_h} \frac{b_h(\mathring{p}_h, v_h)}{\|v_h\|_{V_h}} = \sup_{v_h\in\mathring{V}_h} \frac{(\tilde{f}, v_h)_{\Omega_h} + (\tilde{\tau}, v_h)_{\Gamma_h} - a_h(u_h, v_h) - j_h(u_h, v_h)}{\|v_h\|_{V_h}}$$
$$\le C(\|\tilde{f}\|_{L^2(\Omega_h)} + \|\tilde{\tau}\|_{L^2(\Gamma_h)} + \|u_h\|_{V_h}).$$

Next, by Lemmas 3.4 and 3.2 there exists $w_h \in V_h$ such that $w_h \cdot n_h = -1$ at m_e's on Γ_h and $\|w_h\|_{V_h} \le C$. Taking $v_h = w_h$ in $(2.4)_1$ and noting that $b_h(1, w_h) = -(1, w_h \cdot n_h)_{\Gamma_h} = |\Gamma_h|$, we obtain

$$k_h|\Gamma_h| = (\tilde{f}, w_h)_{\Omega_h} + (\tilde{\tau}, w_h)_{\Gamma_h} - a_h(u_h, w_h) - b_h(\mathring{p}_h, w_h) - \frac{1}{\epsilon} c_h(u_h \cdot n_h - \tilde{g}, 1) - j_h(u_h, w_h).$$

This, together with $c_h(u_h \cdot n_h, 1) = -b_h(1, u_h) = 0$, gives an estimate for k_h:

$$|k_h| \le C\big(\|\tilde{f}\|_{L^2(\Omega_h)} + \|\tilde{\tau}\|_{L^2(\Gamma_h)} + \|u_h\|_{V_h} + \|\mathring{p}_h\|_{Q_h} + \frac{1}{\epsilon}|c_h(\tilde{g}, 1)|\big),$$

where, by the definition of Π_h^∂, by the compatibility condition (1.2) and by (3.4)–(3.5), we have

$$|c_h(\tilde{g}, 1)| = |(\Pi_h^\partial \tilde{g}, 1)_{\Gamma_h}| = \left|\int_{\Gamma_h} \tilde{g}\, \mathrm{d}s - \int_\Gamma g\, \mathrm{d}s\right| \le Ch^2\|\tilde{g}\|_{H^2(\tilde{\Omega})}.$$

In conclusion, the pressure can be estimated as

$$\|p_h\|_{Q_h} \le C(\|\tilde{f}\|_{L^2(\Omega_h)} + \|\tilde{\tau}\|_{L^2(\Gamma_h)} + \epsilon^{-1}h^2\|\tilde{g}\|_{H^2(\tilde{\Omega})} + \|u_h\|_{V_h}). \tag{4.3}$$

Finally, making use of the discrete Korn's inequality (4.1) and taking $v_h = u_h$ in $(2.4)_1$ give

$$C\|u_h\|_{V_h}^2 \le a_h(u_h, u_h) + j_h(u_h, u_h) + \frac{1}{\epsilon}\|u_h \cdot n_h - \Pi_h^\partial \tilde{g}\|_{L^2(\Gamma_h)}^2$$
$$= (\tilde{f}, u_h)_{\Omega_h} + (\tilde{\tau}, u_h)_{\Gamma_h} - \frac{1}{\epsilon} c_h(u_h \cdot n_h - \tilde{g}, \tilde{g}). \tag{4.4}$$

To address the third term on the last line, we find from Lemmas 3.4 and 3.2 some $z_h \in V_h$ such that $z_h \cdot n_h = \Pi_h^\partial \tilde{g}$ at m_e's on Γ_h and $\|z_h\|_{V_h} \le C\|\tilde{g}\|_{H^{1/2}(\Gamma_h)}$. Letting now $v_h = z_h$ in $(2.4)_1$ one gets

$$\big|\frac{1}{\epsilon} c_h(u_h \cdot n_h - \Pi_h^\partial \tilde{g}, \Pi_h^\partial \tilde{g})\big| = \big|(\tilde{f}, z_h)_{\Omega_h} + (\tilde{\tau}, z_h)_{\Gamma_h} - a_h(u_h, z_h) - b_h(p_h, z_h)\big|$$
$$\le C(\|\tilde{f}\|_{L^2(\Omega_h)} + \|\tilde{\tau}\|_{L^2(\Gamma_h)} + \|u_h\|_{V_h} + \|p_h\|_{Q_h})\|\tilde{g}\|_{H^{1/2}(\Gamma_h)}. \tag{4.5}$$

Combining the estimates (4.3)–(4.5), performing an absorbing argument, and using the stability of extensions, we conclude (2.5). $\square$

5. H^1-error estimate

Let us introduce a discrete Lagrange multiplier by $\lambda_h := \frac{1}{\epsilon}\Pi_h^\partial(u_h \cdot n_h - \tilde{g}) \in \Lambda_h$. An easy but important fact is that if (u_h, p_h) solves (2.4), then (u_h, p_h, λ_h) satisfies the following three-field formulation:

$$\begin{cases} a_h(u_h, v_h) + b_h(p_h, v_h) + c_h(\lambda_h, v_h \cdot n_h) + j_h(u_h, v_h) = (\tilde{f}, v_h)_{\Omega_h} + (\tilde{\tau}, v_h)_{\Gamma_h} & \forall v_h \in V_h, \\ b_h(q_h, u_h) = 0 & \forall q_h \in Q_h, \\ c_h(\mu_h, u_h \cdot n_h - \tilde{g}) = \epsilon c_h(\mu_h, \lambda_h) & \forall \mu_h \in \Lambda_h, \end{cases} \tag{5.1}$$

which will be compared with (2.2) in the subsequent arguments.

5.1. Consistency error estimate

Since $\Omega \neq \Omega_h$ and a nonconforming element is employed, the consistency (i.e. the Galerkin orthogonality relation) does not hold exactly. However, it is still valid in an asymptotic sense with respect to $h \to 0$. To see this, we introduce a functional $\mathrm{Res}(v)$ by

$$\mathrm{Res}(v) := (\tilde{u} - \nu\Delta\tilde{u} - \nu\nabla \operatorname{div}\tilde{u} + \nabla\tilde{p} - \tilde{f}, v)_{\Omega_h\setminus\Omega} + \sum_{e\in\mathring{\mathcal{E}}_h} (\sigma(\tilde{u},\tilde{p})n_e, [v])_e \\ + (\sigma(\tilde{u},\tilde{p})n_h, v)_{\Gamma_h} - (\tilde{\tau} - \tilde{\lambda}n_h, v)_{\Gamma_h} + (\tilde{\lambda}, (\Pi_h^\partial v - v)\cdot n_h)_{\Gamma_h}, \tag{5.2}$$

which is well-defined for $v \in H^1(\mathcal{T}_h)^N$. The next lemma shows that $\mathrm{Res}(v)$ describes the residual of the consistency and that it is of $O(h)$.

Lemma 5.1. *Let $(u,p,\lambda) \in H^2(\Omega)^N \times H^1(\Omega) \times H^{1/2}(\Gamma)$ be the solution of (2.2) and $(u_h, p_h, \lambda_h) \in V_h \times Q_h \times \Lambda_h$ be that of (2.4).*

(i) For $v_h \in V_h$ we have

$$a_h(\tilde{u} - u_h, v_h) + b_h(\tilde{p} - p_h, v_h) + c_h(\tilde{\lambda} - \lambda_h, v_h\cdot n_h) - j_h(u_h, v_h) = \mathrm{Res}(v_h). \tag{5.3}$$

(ii) For $v \in V_h + H^1(\Omega_h)^N$ we obtain

$$|\mathrm{Res}(v)| \le Ch(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)})|\!|\!|v|\!|\!|_{V_h}.$$

Remark 5.2. (i) As an easy consequence of (5.3) we have

$$a_h(\tilde{u} - u_h, v_h) + b_h(\tilde{p} - p_h, v_h) - j_h(u_h, v_h) = \mathrm{Res}(v_h) \qquad \forall v_h \in \mathring{V}_h.$$

(ii) Noting that $b_h(k_h, v_h) + c_h(k_h, v_h\cdot n_h) = 0$, where k_h is given in Theorem 2.4, one has

$$a_h(\tilde{u} - u_h, v_h) + b_h(\tilde{p} + k_h - p_h, v_h) + c_h(\tilde{\lambda} + k_h - \lambda_h, v_h\cdot n_h) - j_h(u_h, v_h) = \mathrm{Res}(v_h).$$

Since $R_h\tilde{p} - \tilde{p}$ and $\Pi_h^\partial\tilde{\lambda} - \tilde{\lambda}$ are orthogonal to the functions in Q_h and to those in Λ_h respectively, this in particular implies

$$a_h(\tilde{u} - u_h, v_h) + b_h(R_h(\tilde{p} + k_h) - p_h, v_h) + c_h(\Pi_h^\partial(\tilde{\lambda} + k_h) - \lambda_h, v_h\cdot n_h) - j_h(u_h, v_h) = \mathrm{Res}(v_h).$$

Proof of Lemma 5.1. (i) Integration by parts together with $(5.1)_1$ shows that the left-hand side of (5.3) equals

$$\sum_{T\in\mathcal{T}_h}\Big((\tilde{u} - \nu\Delta\tilde{u} - \nu\nabla\operatorname{div}\tilde{u} + \nabla\tilde{p}, v_h)_T + (\sigma(\tilde{u},\tilde{p})n_{\partial T}, v_h)_{\partial T}\Big) + c_h(\tilde{\lambda}, v_h\cdot n_h) - (\tilde{f}, v_h)_{\Omega_h} - (\tilde{\tau}, v_h)_{\Gamma_h} \\ = (\tilde{u} - \nu\Delta\tilde{u} - \nu\nabla\operatorname{div}\tilde{u} + \nabla\tilde{p} - \tilde{f}, v_h)_{\Omega_h\setminus\Omega} + \sum_{e\in\mathring{\mathcal{E}}_h}(\sigma(\tilde{u},\tilde{p})n_e, [v_h])_e + (\sigma(\tilde{u},\tilde{p})n_h, v_h)_{\Gamma_h} \\ - (\tilde{\tau}, v_h)_{\Gamma_h} + (\tilde{\lambda}, \Pi_h^\partial v_h\cdot n_h)_{\Gamma_h}.$$

Since $-(-\tilde{\lambda}n_h, v_h)_{\Gamma_h} + (\tilde{\lambda}, -v_h\cdot n_h)_{\Gamma_h} = 0$, this implies (5.3).

(ii) For simplicity we abbreviate $C(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)})$ as $C(u,p)$. The first term of $\mathrm{Res}(v)$ is bounded by

$$|(\tilde{u} - \nu\Delta\tilde{u} - \nu\nabla\operatorname{div}\tilde{u} + \nabla\tilde{p} - \tilde{f}, v)_{\Omega_h\setminus\Omega}| \le C(u,p)\|v\|_{L^2(\Omega_h\setminus\Omega)} \le C(u,p)h|\!|\!|v|\!|\!|_{V_h},$$

where we have used (3.6). For the second term, since $\int_e [v]\,\mathrm{d}s = 0$ for $e \in \mathring{\mathcal{E}}_h$, we have

$$\begin{aligned}
\Big| \sum_{e\in\mathring{\mathcal{E}}_h} \big(\sigma(\tilde u,\tilde p)n_e,[v]\big)_e \Big| &= \Big| \sum_{e\in\mathring{\mathcal{E}}_h} \big((\sigma(\tilde u,\tilde p) - \Pi^e\sigma(\tilde u,\tilde p))n_e,[v]\big)_e \Big| \\
&\le \Big(\sum_{e\in\mathring{\mathcal{E}}_h} h_e \|\sigma(\tilde u,\tilde p) - \Pi^e\sigma(\tilde u,\tilde p)\|^2_{L^2(e)} \Big)^{1/2} \Big(\sum_{e\in\mathring{\mathcal{E}}_h} h_e^{-1}\|[v]\|^2_{L^2(e)} \Big)^{1/2} \\
&\le C\Big(\sum_{e\in\mathring{\mathcal{E}}_h} h_e^2 \|\sigma(\tilde u,\tilde p)\|^2_{H^{1/2}(e)} \Big)^{1/2} \Big(\sum_{e\in\mathring{\mathcal{E}}_h} h_e^{-1}\|[v]\|^2_{L^2(e)} \Big)^{1/2} \\
&\le C(u,p)h|\!|\!|v|\!|\!|_{V_h},
\end{aligned}$$

where Π^e denotes the orthogonal projector from $L^2(e)$ onto $P_0(e)$. To address the third and forth terms we observe that

$$\sigma(\tilde u,\tilde p)n_h - \tilde\tau + \tilde\lambda n_h = \big(\sigma(\tilde u,\tilde p)(\mathbb{I} - n_h\otimes n_h)n_h - \tilde\tau\big) + \big(\sigma(\tilde u,\tilde p)n_h\cdot n_h + \tilde\lambda\big)n_h =: F_\tau + F_n.$$

Recalling that $\sigma(u,p)(\mathbb{I} - n\otimes n)n = \tau$ on Γ, one has

$$F_\tau = \sigma(\tilde u,\tilde p)(\mathbb{I} - n_h\otimes n_h)n_h - \big(\sigma(u,p)(\mathbb{I} - n\otimes n)n\big)\circ\pi + \tau\circ\pi - \tilde\tau,$$

which, combined with the estimates

$$\|\sigma(\tilde u,\tilde p) - \sigma(u,p)\circ\pi\|_{L^2(\Gamma_h)} \le C(u,p)h, \quad \|\tilde\tau - \tau\circ\pi\|_{L^2(\Gamma_h)} \le C(u,p)h, \quad \|n_h - n\circ\pi\|_{L^\infty(\Gamma_h)} \le Ch,$$

yields $\|F_\tau\|_{L^2(\Gamma_h)} \le C(u,p)h$. Similarly we have $\|F_n\|_{L^2(\Gamma_h)} \le C(u,p)h$. Therefore,

$$|(F_\tau + F_n, v)_{\Gamma_h}| \le C(u,p)h\|v\|_{L^2(\Gamma_h)} \le C(u,p)h|\!|\!|v|\!|\!|_{V_h}.$$

Finally, the last term of $\mathrm{Res}(v)$ is estimated by

$$\begin{aligned}
\big|\big(\tilde\lambda, (\Pi_h^\partial v - v)\cdot n_h\big)_{\Gamma_h}\big| &= \Big| \sum_{e\in\mathcal{E}_h^\partial} \big(\tilde\lambda - \Pi_h^\partial\tilde\lambda, (\Pi_h^\partial v - v)\cdot n_h\big) \Big| \\
&\le \Big(\sum_{e\in\mathcal{E}_h^\partial} \|\tilde\lambda - \Pi_h^\partial\tilde\lambda\|^2_{L^2(e)} \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h^\partial} \|\Pi_h^\partial v - v\|^2_{L^2(e)} \Big)^{1/2} \\
&\le C\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|\tilde\lambda\|^2_{H^{1/2}(e)} \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|v\|^2_{H^{1/2}(e)} \Big)^{1/2} \\
&\le C(u,p)h|\!|\!|v|\!|\!|_{V_h}.
\end{aligned}$$

Collecting the estimates above concludes $|\mathrm{Res}(v)| \le C(u,p)h|\!|\!|v|\!|\!|_{V_h}$. □

5.2. Proof of Theorem 2.6

In view of the regularity property, stability of extension operators and interpolation estimates, it suffices to prove that

$$|\!|\!|\Pi_h\tilde u - u_h|\!|\!|_{V_h} + \|R_h\tilde p - \mathring{p}_h\|_{Q_h} \le C(h+\epsilon)(\|\tilde u\|_{H^2(\tilde\Omega)} + \|\tilde p\|_{H^1(\tilde\Omega)}). \tag{5.4}$$

In what follows, we abbreviate the quantity $C(\|\tilde u\|_{H^2(\tilde\Omega)} + \|\tilde p\|_{H^1(\tilde\Omega)})$ just as $C(u,p)$, and we set $v_h := \Pi_h\tilde u$, $q_h := R_h\tilde p + k_h = R_h(\tilde p + k_h)$, and $\mu_h := \Pi_h^\partial\tilde\lambda + k_h = \Pi_h^\partial(\tilde\lambda + k_h)$, where k_h is given in Theorem 2.4.

We start from Korn's inequality (4.1) and (5.3) to find that

$$
\begin{aligned}
C\|\!|v_h - u_h|\!\|_{V_h}^2 &\le a_h(v_h - u_h, v_h - u_h) + j_h(v_h - u_h, v_h - u_h)\\
&= a_h(\tilde{u} - u_h, v_h - u_h) - j_h(u_h, v_h - u_h) + a_h(v_h - \tilde{u}, v_h - u_h) + j_h(v_h - \tilde{u}, v_h - u_h)\\
&= \mathrm{Res}(v_h - u_h) - b_h(q_h - p_h, v_h - u_h) - c_h(\mu_h - \lambda_h, (v_h - u_h)\cdot n_h)\\
&\quad + a_h(v_h - \tilde{u}, v_h - u_h) + j_h(v_h - \tilde{u}, v_h - u_h)\\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

By Lemma 5.1 and by the boundedness of a_h and j_h, one has $|I_1 + I_4 + I_5| \le C(u,p)h\|\!|v_h - u_h|\!\|_{V_h}$. For I_2, since $\operatorname{div} u = 0$ in Ω, it follows that

$$
\begin{aligned}
I_2 &= b_h(q_h - p_h, v_h - \tilde{u}) + b_h(q_h - p_h, \tilde{u})\\
&\le \sum_{T\in\mathcal{T}_h} \|q_h - p_h\|_{L^2(T)} C h_T \|\nabla\tilde{u}\|_{L^2(T)} + |(q_h - p_h, \operatorname{div}\tilde{u})_{\Omega_h\setminus\Omega}|\\
&\le C(u,p)h\|q_h - p_h\|_{L^2(\Omega_h)}.
\end{aligned}
$$

For I_3, it follows from (3.9) that

$$
\begin{aligned}
I_3 &= -c_h(\mu_h - \lambda_h, \tilde{u}\cdot n_h - \tilde{g}) + \epsilon c_h(\mu_h - \lambda_h, \mu_h) - \epsilon\|\mu_h - \lambda_h\|_{L^2(\Gamma_h)}^2\\
&\le \sum_{e\in\mathcal{E}_h^\partial} \|\mu_h - \lambda_h\|_{L^2(e)} \|\Pi_h^\partial(\tilde{u}\cdot n_h - \tilde{g})\|_{L^2(e)} + \epsilon\|\mu_h - \lambda_h\|_{-1/2,\Lambda_h}\|\mu_h\|_{1/2,\Lambda_h}\\
&\le C(u,p)h^{2\alpha-1/2}\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|\mu_h - \lambda_h\|_{L^2(e)}^2\Big)^{1/2} + C(u,p)(\epsilon + h^2)\|\mu_h - \lambda_h\|_{-1/2,\Lambda_h},
\end{aligned}
\tag{5.5}
$$

where we have estimated μ_h, using Lemma 3.1 and (2.6), by

$$
\|\mu_h\|_{1/2,\Lambda_h} \le \|\Pi_h^\partial\tilde{\lambda}\|_{1/2,\Lambda_h} + C|k_h| \le C(u,p)(1 + h^2\epsilon^{-1}).
$$

The errors for $\mu_h - \lambda_h$ in (5.5) are bounded by the use of (3.1) and (3.3) as

$$
\begin{aligned}
&C\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|\mu_h - \lambda_h\|_{L^2(e)}^2\Big)^{1/2} + C\|\mu_h - \lambda_h\|_{-\frac{1}{2},\Lambda_h}\\
&\quad\le \sup_{v_h\in V_h}\frac{c_h(\mu_h - \lambda_h, v_h\cdot n_h)}{\|v_h\|_{V_h}} = \sup_{v_h\in V_h}\frac{\mathrm{Res}(v_h) - a_h(\tilde{u} - u_h, v_h) - b_h(q_h - p_h, v_h) + j_h(u_h, v_h)}{\|v_h\|_{V_h}}\\
&\quad\le C(u,p)h + C\|\!|v_h - u_h|\!\|_{V_h} + C\|q_h - p_h\|_{Q_h}.
\end{aligned}
\tag{5.6}
$$

To estimate $\|q_h - p_h\|_{Q_h}$, we notice that

$$
q_h - p_h = R_h\tilde{p} - \mathring{p}_h = R_h\mathring{\tilde{p}} - \mathring{p}_h + \frac{1}{|\Omega_h|}(\tilde{p}, 1)_{\Omega_h},
$$

where the relation $(p,1)_\Omega = 0$ combined with (3.6) gives

$$
|(\tilde{p},1)_{\Omega_h}| = |(\tilde{p},1)_{\Omega_h\setminus\Omega} - (p,1)_{\Omega\setminus\Omega_h}| \le \|\tilde{p}\|_{L^1(\Gamma(\delta))} \le Ch^2\|\tilde{p}\|_{W^{1,1}(\tilde{\Omega})}.
$$

On the other hand, by the inf-sup condition (4.2),

$$
\begin{aligned}
C\|R_h\mathring{\tilde{p}} - \mathring{p}_h\|_{Q_h} &\le \sup_{v_h\in\mathring{V}_h}\frac{b_h(R_h\mathring{\tilde{p}} - \mathring{p}_h, v_h)}{\|v_h\|_{V_h}} = \sup_{v_h\in\mathring{V}_h}\frac{b_h(\tilde{p} - p_h, v_h)}{\|v_h\|_{V_h}}\\
&= \sup_{v_h\in\mathring{V}_h}\frac{\mathrm{Res}(v_h) - a_h(\tilde{u} - u_h, v_h) + j_h(u_h, v_h)}{\|v_h\|_{V_h}}\\
&\le C(u,p)h + C\|\!|v_h - u_h|\!\|_{V_h}.
\end{aligned}
\tag{5.7}
$$

Therefore, we obtain $\|q_h - p_h\|_{Q_h} \le C(u,p)h + C|||v_h - u_h|||_{V_h}$, which concludes

$$\begin{aligned}|I_2| + |I_3| &\le C(u,p)\big(h + h^{2\alpha-1/2} + \epsilon\big)\big(C(u,p)h + |||v_h - u_h|||_{V_h}\big)\\ &\le C(u,p)\big(h^\alpha + \epsilon\big)\big(C(u,p)h + |||v_h - u_h|||_{V_h}\big),\end{aligned}$$

where we note that $\max\{h, h^{2\alpha-1/2}\} \le h^\alpha$ by definition of α.

Combining the estimates above, we deduce that

$$|||v_h - u_h|||_{V_h}^2 \le C(u,p)^2(h^\alpha + \epsilon)^2 + C(u,p)(h^\alpha + \epsilon)|||v_h - u_h|||_{V_h},$$

from which (5.4) follows. This completes the proof of Theorem 2.6.

Remark 5.3. As for error estimation of the Lagrange multiplier, from (5.6) we have

$$\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|\varPi_h^\partial\tilde\lambda + k_h - \lambda_h\|_{L^2(e)}^2\Big)^{1/2} + \|\varPi_h^\partial\tilde\lambda + k_h - \lambda_h\|_{-1/2,\varLambda_h} \le C(u,p)(h^\alpha + \epsilon). \tag{5.8}$$

This combined with (2.6) especially implies the stability

$$\Big(\sum_{e\in\mathcal{E}_h^\partial} h_e\|\lambda_h\|_{L^2(e)}^2\Big)^{1/2} + \|\lambda_h\|_{-1/2,\varLambda_h} \le C(u,p)(1 + \epsilon^{-1}h^2). \tag{5.9}$$

From Remark 2.5(i), the dependency of ϵ^{-1} may be omitted if $g = 0$.

6. L^2-ERROR ESTIMATE

For the L^2-error analysis we need another consistency error estimate as follows:

Lemma 6.1. *In addition to the hypotheses of Lemma 5.1, let $w \in H^2(\varOmega)^N$ satisfy $\operatorname{div} w = 0$ in $\varOmega$ and $w\cdot n = 0$ on $\varGamma$. Then we obtain*

$$|\mathrm{Res}(w)| \le Ch^{2\alpha}(\|u\|_{H^2(\varOmega)} + \|p\|_{H^1(\varOmega)})\|w\|_{H^2(\varOmega)}.$$

Proof. We introduce a signed integration over the boundary-skin layer by

$$(f,g)'_{\varOmega_h\triangle\varOmega} := (f,g)_{\varOmega_h\setminus\varOmega} - (f,g)_{\varOmega\setminus\varOmega_h}.$$

Then it follows from integration by parts that for all $v \in H^1(\tilde\varOmega)^N$

$$(\sigma(\tilde u,\tilde p)n_h, v)_{\varGamma_h} - (\sigma(u,p)n, v)_\varGamma = \frac{\nu}{2}(\mathbb{E}(\tilde u), \mathbb{E}(\tilde v))'_{\varOmega_h\triangle\varOmega} - (\tilde p, \operatorname{div} v)'_{\varOmega_h\triangle\varOmega} + (\nu\varDelta\tilde u + \nu\nabla\operatorname{div}\tilde u - \nabla\tilde p, v)'_{\varOmega_h\triangle\varOmega}.$$

Substituting this formula into (5.2), recalling $\sigma(u,p)n = \tau - \lambda n$ on $\varGamma$, and noting that $[v] = 0$ on each $e \in \mathring{\mathcal{E}}_h$, we obtain

$$\begin{aligned}\mathrm{Res}(v) = (\tilde u - \tilde f, v)'_{\varOmega_h\triangle\varOmega} &+ \frac{\nu}{2}(\mathbb{E}(\tilde u), \mathbb{E}(v))'_{\varOmega_h\triangle\varOmega} - (\tilde p, \operatorname{div} v)'_{\varOmega_h\triangle\varOmega}\\ &+ (\tau, v)_\varGamma - (\tilde\tau, v)_{\varGamma_h} - (\lambda, v\cdot n)_\varGamma + c_h(\tilde\lambda, v\cdot n_h).\end{aligned} \tag{6.1}$$

We now take $v = \tilde w$ and apply (3.4)–(3.6) to see that all the terms but the last one on the right-hand side of (6.1) can be bounded by $Ch^2(\|u\|_{H^2(\varOmega)} + \|p\|_{H^1(\varOmega)})\|w\|_{H^2(\varOmega)}$. The last term is then estimated by (3.8), which completes the proof. $\square$

Proof of Theorem 2.7. In what follows, we abbreviate the quantity $C(\|\tilde{u}\|_{H^2(\tilde{\Omega})} + \|\tilde{p}\|_{H^1(\tilde{\Omega})})$ just as $C(u,p)$. Let $\varphi \in C_0^\infty(\Omega_h)^N$ be such that $\|\varphi\|_{L^2(\Omega_h)} = 1$ and estimate $(\tilde{u} - u_h, \varphi)_{\Omega_h}$. Let $(w, r) \in H^2(\Omega)^N \times H^1(\Omega)$ be the solution of the following dual problem (φ is extended by 0 outside Ω_h):

$$\begin{cases} w - \nu \Delta w + \nabla r = \varphi & \text{in} \quad \Omega, \\ \operatorname{div} w = 0 & \text{in} \quad \Omega, \\ w \cdot n = 0 & \text{on} \quad \Gamma, \\ (\mathbb{I} - n \otimes n)\sigma(w, r)n = 0 & \text{on} \quad \Gamma. \end{cases}$$

Then we see that $\|w\|_{H^2(\Omega)} + \|r\|_{H^1(\Omega)} \le C$. Setting $w_h := \Pi_h \tilde{w}$, we find from integration by parts and from (5.3) that

$$\begin{aligned}
(\tilde{u} - u_h, \varphi)_{\Omega_h} &= (\tilde{u} - u_h, \varphi)_{\Omega_h \cap \Omega} + (\tilde{u} - u_h, \varphi)_{\Omega_h \setminus \Omega} \\
&= (\tilde{u} - u_h, \tilde{w} - \nu \Delta \tilde{w} - \nu \nabla \operatorname{div} \tilde{w} + \nabla \tilde{r})_{\Omega_h} - (\tilde{u} - u_h, \tilde{w} - \nu \Delta \tilde{w} - \nu \nabla \operatorname{div} \tilde{w} + \nabla \tilde{r} - \varphi)_{\Omega_h \setminus \Omega} \\
&= a_h(\tilde{u} - u_h, \tilde{w}) - \sum_{e \in \mathring{\mathcal{E}}_h} ([\tilde{u} - u_h], \sigma(\tilde{w}, \tilde{r}) n_e)_e - (\tilde{u} - u_h, \sigma(\tilde{w}, \tilde{r}) n_h)_{\Gamma_h} \\
&\quad - (\tilde{u} - u_h, \tilde{w} - \nu \Delta \tilde{w} - \nu \nabla \operatorname{div} \tilde{w} + \nabla \tilde{r} - \varphi)_{\Omega_h \setminus \Omega} \\
&= a_h(\tilde{u} - u_h, \tilde{w} - w_h) + \sum_{e \in \mathring{\mathcal{E}}_h} ([u_h], \sigma(\tilde{w}, \tilde{r}) n_e)_e - (\tilde{u} - u_h, \sigma(\tilde{w}, \tilde{r}) n_h)_{\Gamma_h} \\
&\quad + \operatorname{Res}(w_h) - b_h(R_h \tilde{p} + k_h - p_h, w_h) - c_h(\tilde{\lambda} + k_h - \lambda_h, w_h \cdot n_h) + j_h(u_h, w_h) \\
&\quad - (\tilde{u} - u_h, \tilde{w} - \nu \Delta \tilde{w} - \nu \nabla \operatorname{div} \tilde{w} + \nabla \tilde{r} - \varphi)_{\Omega_h \setminus \Omega} \\
&= a_h(\tilde{u} - u_h, \tilde{w} - w_h) + \sum_{e \in \mathring{\mathcal{E}}_h} ([u_h], \sigma(\tilde{w}, \tilde{r}) n_e)_e - (\tilde{u} - u_h, \sigma(\tilde{w}, \tilde{r}) n_h)_{\Gamma_h} \\
&\quad - (\tilde{u} - u_h, \tilde{w} - \nu \Delta \tilde{w} - \nu \nabla \operatorname{div} \tilde{w} + \nabla \tilde{r} - \varphi)_{\Omega_h \setminus \Omega} - j_h(u_h, \tilde{w} - w_h) \\
&\quad + \operatorname{Res}(w_h) - b_h(R_h \tilde{p} + k_h - p_h, \tilde{w}) - (\Pi_h^\partial \tilde{\lambda} + k_h - \lambda_h, \Pi_h^\partial \tilde{w} \cdot n_h)_{\Gamma_h} \\
&=: \sum_{i=1}^{8} I_i,
\end{aligned}$$

where we made use of the fact that $b_h(q_h, \Pi_h \tilde{w}) = b_h(q_h, \tilde{w})$ for $q_h \in Q_h$ in the fifth equality.

Let us bound each term of $I_1, \dots, I_9$. By interpolation estimates and Lemma 3.6, one has $|I_1 + I_2 + I_4 + I_5| \le C(u,p) h \|\tilde{u} - u_h\|_{V_h}$. It follows from Lemma 6.1 and Lemma 5.1(ii) that

$$|I_6| \le |\operatorname{Res}(\tilde{w})| + |\operatorname{Res}(\tilde{w} - w_h)| \le C(u,p) h^{2\alpha} \|w\|_{H^2(\Omega)} + C(u,p) h \|\tilde{w} - w_h\|_{V_h} \le C(u,p) h^{2\alpha}.$$

For I_7, the pressure error estimate obtained in Theorem 2.6 gives

$$|I_7| = |(R_h \tilde{p} + k_h - p_h, \operatorname{div} \tilde{w})_{\Omega_h \setminus \Omega}| \le \|R_h \tilde{p} + k_h - p_h\|_{Q_h} C h \|\tilde{w}\|_{H^2(\Omega_h)} \le C(u,p) h^2 + C h \|\tilde{u} - u_h\|.$$

For I_8, as a result of (5.8) and (3.9) we have

$$\begin{aligned}
|I_8| &\le \Big(\sum_{e \in \mathcal{E}_h^\partial} h_e \|\Pi_h^\partial \tilde{\lambda} + k_h - \lambda_h\|_{L^2(e)}^2 \Big)^{1/2} \times C h^{2\alpha - 1/2} \|\tilde{w}\|_{H^2(\tilde{\Omega})} \\
&\le C(u,p)(h + \|\tilde{u} - u_h\|_{V_h}) h^{2\alpha - 1/2}.
\end{aligned}$$

It remains to estimate I_3. Setting $\tilde{\mu} := \sigma(\tilde{w}, \tilde{r})n_h \cdot n_h$ and $\mu_h := \Pi_h^\partial \tilde{\mu}$, we obtain

$$\begin{aligned}
-I_3 &= (\tilde{u} - u_h, (\mathbb{I} - n_h \otimes n_h)\sigma(\tilde{w}, \tilde{r})n_h)_{\Gamma_h} + ((\tilde{u} - u_h) \cdot n_h, \tilde{\mu})_{\Gamma_h} =: I_{31} + ((\tilde{u} - u_h) \cdot n_h, \tilde{\mu})_{\Gamma_h} \\
&= I_{31} + ((\tilde{u} - u_h) \cdot n_h, \tilde{\mu} - \mu_h)_{\Gamma_h} + ((\Pi_h^\partial \tilde{u} - u_h) \cdot n_h, \mu_h)_{\Gamma_h} \\
&= I_{31} + ((\tilde{u} - u_h) \cdot n_h, \tilde{\mu} - \mu_h)_{\Gamma_h} + (\Pi_h^\partial (\tilde{u} \cdot n_h - \tilde{g}), \mu_h)_{\Gamma_h} - \epsilon c_h(\lambda_h, \mu_h) \\
&=: I_{31} + I_{32} + I_{33} + I_{34}.
\end{aligned}$$

Since $(\mathbb{I} - n \otimes n)\sigma(w, r)n = 0$ on Γ, $\|n \circ \pi - n_h\|_{L^\infty(\Gamma_h)} \leq Ch$, and $\|\sigma(\tilde{w}, \tilde{r}) - \sigma(w, r) \circ \pi\|_{L^2(\Gamma_h)} \leq C\delta^{1/2}\|\nabla\sigma(\tilde{w}, \tilde{r})\|_{L^2(\Gamma(\delta))}$, we have

$$|I_{31}| \leq Ch\|\tilde{u} - u_h\|_{L^2(\Gamma_h)} \leq Ch\|\tilde{u} - u_h\|_{L^2(\Omega_h)}^{1/2}\|\tilde{u} - u_h\|_{V_h}^{1/2}.$$

For I_{32} we get

$$|I_{32}| \leq C\|\tilde{u} - u_h\|_{L^2(\Gamma_h)}\|\sigma(\tilde{w}, \tilde{r}) - \Pi_h^\partial \sigma(\tilde{w}, \tilde{r})\|_{L^2(\Gamma_h)} \leq Ch^{1/2}\|\tilde{u} - u_h\|_{L^2(\Omega_h)}^{1/2}\|\tilde{u} - u_h\|_{V_h}^{1/2}.$$

By (3.8), $|I_{33}| \leq C(u, p)h^{2\alpha}\|\mu\|_{L^2(\Gamma_h)} \leq C(u, p)h^{2\alpha}$. From (5.9) and Lemma 3.2 it follows that

$$|I_{34}| \leq \epsilon\|\lambda_h\|_{-1/2,\Lambda_h}\|\mu_h\|_{1/2,\Lambda_h} \leq C(u, p)(\epsilon + h^2)\|\sigma(\tilde{w}, \tilde{r})\|_{H^{1/2}(\Gamma_h)} \leq C(u, p)(\epsilon + h^2).$$

Consequently,

$$|I_3| \leq Ch^{1/2}\|\tilde{u} - u_h\|_{L^2(\Omega_h)}^{1/2}\|\tilde{u} - u_h\|_{V_h}^{1/2} + C(u, p)(h^{2\alpha} + \epsilon).$$

Recalling φ is arbitrary, collecting the estimates above, and substituting the result of the H^1-error estimate, we deduce that

$$\begin{aligned}
\|\tilde{u} - u_h\|_{L^2(\Omega_h)} &\leq Ch|\!|\!|\tilde{u} - u_h|\!|\!|_{V_h} + (C(u, p)h + C|\!|\!|\tilde{u} - u_h|\!|\!|_{V_h})h^{2\alpha-1/2} \\
&\quad + Ch^{1/2}\|\tilde{u} - u_h\|_{L^2(\Omega_h)}^{1/2}\|\tilde{u} - u_h\|_{V_h}^{1/2} + C(u, p)(h^{2\alpha} + \epsilon) \\
&\leq \frac{1}{2}\|\tilde{u} - u_h\|_{L^2(\Omega_h)} + C(u, p)h(h^\alpha + \epsilon) + C(u, p)h^{2\alpha-1/2}(h^\alpha + \epsilon) + C(u, p)(h^{2\alpha} + \epsilon),
\end{aligned}$$

which concludes $\|\tilde{u} - u_h\|_{L^2(\Omega_h)} \leq C(u, p)(h^{2\alpha} + \epsilon)$. □

7. Numerical results

In this section, we present numerical results using the proposed scheme (2.4) in two- and three- dimensional cases to validate our theoretical results. The same test problems as in [16] are considered. In the following, we set $\nu = 1$ and use unstructured meshes. All computations here were done with FEniCS [19].

7.1. Two-dimensional case

We consider the problem (1.1) where the domain Ω is the unit disk, i.e., $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. The data f, g, and τ are chosen so that the exact solution is

$$\begin{aligned}
u(x, y) &= (-y(x^2 + y^2), x(x^2 + y^2))^\top, \\
p(x, y) &= 8xy.
\end{aligned}$$

We set the parameters as $\epsilon = 0.1h^2$ and $\gamma = 2$. Table 1 shows the history of convergence for the velocity and pressure. We observe that our method achieves optimal orders in all cases, which is in full agreement with Theorem 2.6 with $\alpha = 1$.

TABLE 1. Convergence history in the two-dimensional case.

	$\|u-u_h\|_{L^2(\Omega_h)}$		$\|u-u_h\|_{H^1(\Omega_h)}$		$\|p-p_h\|_{L^2(\Omega_h)}$	
h	Error	Order	Error	Order	Error	Order
0.1734	3.85E−02	–	2.49E−01	–	2.48E−01	–
0.0857	9.59E−03	1.97	1.17E−01	1.07	1.21E−01	1.02
0.0459	2.53E−03	2.13	5.94E−02	1.09	6.21E−02	1.06
0.0232	6.46E−04	2.00	2.98E−02	1.01	3.13E−02	1.00

TABLE 2. Convergence history in the three-dimensional case.

	$\|u-u_h\|_{L^2(\Omega_h)}$		$\|u-u_h\|_{H^1(\Omega_h)}$		$\|p-p_h\|_{L^2(\Omega_h)}$	
h	Error	Order	Error	Order	Error	Order
0.1853	8.62E−02	–	7.88E−01	–	5.39E−01	–
0.0959	4.72E−02	0.87	4.08E−01	0.95	3.02E−01	0.84
0.0679	3.42E−02	0.79	2.86E−01	0.87	2.19E−01	0.79
0.0500	2.56E−02	1.01	2.12E−01	1.04	1.65E−01	1.00

7.2. Three-dimensional case

In this example, the problem with $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$ is considered. The data f, g and τ are chosen so that the exact solution becomes

$$\begin{aligned} u(x,y,z) &= (10x^2yz(y-z), 10xy^2z(z-x), 10xyz^2(x-y))^\top, \\ p(x,y,z) &= 10xyz(z+y+z). \end{aligned}$$

We set $\epsilon = 0.1h$ and $\gamma = 5$. The history of convergence is displayed in Table 2. From the result, we see that all the orders seem to be one. The order of the L^2 error of velocity coincides with Theorem 2.7 where $\alpha = 1/2$. On the other hand, the H^1 and L^2 errors of velocity and pressure, respectively, converge with the optimal order, which is faster than expected in Theorem 2.7.

It is noted that Krylov linear solvers, such as GMRES and BiCGSTAB methods, fail to solve the resulting system of linear equations when ϵ is very small. We do not here present the numerical result because it is similar to that shown in ([16], Tab. 3).

APPENDIX A. PROOFS OF LEMMAS IN SECTION 3

A.1. Proof of Lemma 3.1

In view of the definition of $\|\cdot\|_{1/2,\Lambda_h}$, the lemma is reduced to:

$$\|E_h^\partial \Pi_h^\partial \mu\|_{H^{1/2}(\Gamma_h)} \le C\|\mu\|_{H^{1/2}(\Gamma_h)}, \tag{A.1}$$

$$\sum_{e\in\mathcal{E}_h^\partial}\sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{N-2}|\Pi_h^\partial\mu(m_{e'}) - \Pi_h^\partial\mu(m_e)|^2 \le C\|\mu\|_{H^{1/2}(\Gamma_h)}, \tag{A.2}$$

which will be established in the following Steps 1 and 2 respectively.

Step 1. It is sufficient to show that the operator $E_h^\partial \Pi_h^\partial$ is stable in $L^2(\Gamma_h)$ and $H^1(\Gamma_h)$. Then (A.1) follows by interpolation. To this end, for $x \in e \in \mathcal{E}_h^\partial$ we calculate

$$E_h^\partial \Pi_h^\partial \mu(x) = \sum_{p\in\mathcal{V}_h(e)} \left(\frac{1}{\#\mathcal{E}_h^\partial(p)} \sum_{e'\in\mathcal{E}_h^\partial(p)} \frac{1}{|e'|}\int_{e'} \mu \,\mathrm{d}s \right) \bar\phi_p(x) =: \sum_{p\in\mathcal{V}_h(e)} A_p \bar\phi_p(x).$$

The Hölder and Cauchy–Schwarz inequalities give

$$|A_p| \le C \sum_{e' \in \mathcal{E}_h^\partial(p)} |e'|^{-1/2} \|\mu\|_{L^2(e')} \le C h_e^{-(N-1)/2} \|\mu\|_{L^2(\triangle_e)},$$

where $\triangle_e := \bigcup \mathcal{E}_h^\partial(e)$ stands for a macro element of e. Hence we obtain

$$\|E_h^\partial \Pi_h^\partial \mu\|_{L^2(e)}^2 = \int_e \Big| \sum_{p \in \mathcal{V}_h(e)} A_p \bar\phi_p \Big|^2 \,\mathrm{d}s \le C \max_{p \in \mathcal{V}_h(e)} |A_p|^2 \max_{p \in \mathcal{V}_h(e)} \|\bar\phi_p\|_{L^2(e)}^2 \le C \|\mu\|_{L^2(\triangle_e)}^2,$$

which, after the summation for $e \in \mathcal{E}_h^\partial$, implies the L^2-stability.

For the H^1-stability, noting that $\sum_{p \in \mathcal{V}_h(e)} \bar\phi_p(x) = 1$ for $x \in e \in \mathcal{E}_h^\partial$, we have

$$\nabla_e E_h^\partial \Pi_h^\partial \mu(x) = \sum_{p \in \mathcal{V}_h(e)} \left(\frac{1}{\#\mathcal{E}_h^\partial(p)} \sum_{e' \in \mathcal{E}_h^\partial(p)} \frac{1}{|e'|} \int_{e'} (\mu - \theta) \,\mathrm{d}s \right) \nabla_e \bar\phi_p(x) \qquad \forall \theta \in P_0(\triangle_e).$$

where the ∇_e means the surface gradient along e. By a calculation similar to the one above, we get

$$\|\nabla_e E_h^\partial \Pi_h^\partial \mu\|_{L^2(e)}^2 \le C h_e^{-2} \|\mu - \theta\|_{L^2(\triangle_e)}^2 \qquad \forall \theta \in P_0(\triangle_e). \tag{A.3}$$

Now the Bramble–Hilbert theorem yields $\inf_{\theta \in P_0(\triangle_e)} \|\mu - \theta\|_{L^2(\triangle_e)} \le C h_e |\mu|_{H^1(\triangle_e)}$ (see the remark below for more details). Therefore, the H^1-stability is obtained, and, as we noticed earlier, this proves (A.1).

Step 2. We notice that

$$\begin{aligned}
|\Pi_h^\partial \mu(m_e) - \Pi_h^\partial \mu(m_{e'})|^2 &= \left| \frac{1}{|e||e'|} \int_{e \times e'} \big(\mu(x) - \mu(y)\big) \,\mathrm{d}s(x)\mathrm{d}s(y) \right|^2 \\
&\le \frac{1}{|e||e'|} \int_{e \times e'} |\mu(x) - \mu(y)|^2 \,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\le C h_e^{-2(N-1)} \times C h_e^N \int_{e \times e'} \frac{|\mu(x) - \mu(y)|^2}{|x-y|^N} \,\mathrm{d}s(x)\mathrm{d}s(y) \\
&\le C h_e^{2-N} \|\mu\|_{H^{1/2}(\triangle_e)}^2,
\end{aligned}$$

where we have used $|x - y| \le C h_e$ for $x \in e \in \mathcal{E}_h^\partial$ and $y \in e' \in \mathcal{E}_h^\partial(e)$. Then (A.2) follows by taking summation for e.

Remark A.1. The Bramble–Hilbert theorem used after (A.3) may be justified as follows. Adopting the notation of local coordinates introduced in [16], Section 8, we may assume that Δ_e is contained in some local coordinate neighborhood U such that $\Gamma_h \cap U$ admits a graph representation $(y', \phi_h(y'))$. Let $B : \mathbb{R}^N \to \mathbb{R}^{N-1};\ (y', y_N) \mapsto y'$ denote the projection to the base set and $\Delta'_e := B(\Delta_e)$. We find that the norms $\|f\|_{L^2(\Delta_e)}$ and $\|\nabla_{\Gamma_h} f\|_{L^2(\Delta_e)}$ are equivalent to $\|f'\|_{L^2(\Delta'_e)}$ and $\|\nabla_{y'} f'\|_{L^2(\Delta'_e)}$ respectively, where $f' = f'(y')$ refers to the local coordinate representation of a function f given on Γ_h, and ∇_{Γ_h} is the surface gradient along Γ_h. Then the desired inequality is reduced to show

$$\inf_{\theta' \in P_0(\Delta'_e)} \|\mu' - \theta'\|_{L^2(\Delta'_e)} \le C h_e \|\nabla_{y'} \mu'\|_{L^2(\Delta'_e)},$$

which indeed follows from Lemma 4.3.8 of [6] together with the regularity of the meshes (note that diam $\Delta'_e \le C h_e$ and that Δ'_e is star-shaped with respect to the inscribed ball of e', whose radius is greater than ρ_{T_e}).

A.2. Proof of Lemma 3.2

Below we only prove the scalar case, since the other two cases may be treated similarly. We first notice, from Lemma 3.1, that $\|\varPi_h^\partial(\mu\, n\circ\pi)\|_{1/2,\varLambda_h} \le C\|\mu\, n\circ\pi\|_{H^{1/2}(\varGamma_h)} \le C\|\mu\|_{H^{1/2}(\varGamma_h)}$. Hence it remains to deal with $\|\varPi_h^\partial(\mu(n_h - n\circ\pi))\|_{1/2,\varLambda_h}$, and, in view of the definition of $\|\cdot\|_{1/2,\varLambda_h}$, it suffices to show the following:

$$\|E_h^\partial \varPi_h^\partial(\mu(n_h - n\circ\pi))\|_{H^{1/2}(\varGamma_h)} \le Ch^{1/2}\|\mu\|_{L^2(\varGamma_h)}, \tag{A.4}$$

$$\sum_{e\in\mathcal{E}_h^\partial}\sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{N-2}\left|\frac{1}{|e|}\int_e \mu(n_h - n\circ\pi)\,\mathrm{d}s - \frac{1}{|e'|}\int_{e'}\mu(n_h - n\circ\pi)\,\mathrm{d}s\right|^2 \le Ch\|\mu\|^2_{L^2(\varGamma_h)}. \tag{A.5}$$

Estimate (A.4) follows by interpolation if we establish

$$\|E_h^\partial \varPi_h^\partial(\mu(n_h - n\circ\pi))\|_{L^2(\varGamma_h)} \le Ch\|\mu\|_{L^2(\varGamma_h)},$$
$$\|E_h^\partial \varPi_h^\partial(\mu(n_h - n\circ\pi))\|_{H^1(\varGamma_h)} \le C\|\mu\|_{L^2(\varGamma_h)}.$$

In fact, for $x\in e\in\mathcal{E}_h^\partial$ we have

$$E_h^\partial \varPi_h^\partial[\mu(n_h - n\circ\pi)](x) = \sum_{p\in\mathcal{V}_h(e)}\frac{1}{\#\mathcal{E}_h^\partial(p)}\Big(\sum_{e'\in\mathcal{E}_h^\partial(p)}\frac{1}{|e|}\int_e \mu(n_h - n\circ\pi)\,\mathrm{d}s\Big)\bar\phi_p(x).$$

Noting that $\|n_h - n\circ\pi\|_{L^\infty(e)} \le Ch_e$, we obtain

$$\begin{aligned}\|E_h^\partial \varPi_h^\partial(\mu(n_h - n\circ\pi))\|^2_{L^2(e)} &\le C\sum_{e'\in\mathcal{E}_h^\partial(e)}|e|^{-1}\|\mu\|^2_{L^2(e)}h_e^2 \times \sup_{p\in\mathcal{V}_h(e)}\|\bar\phi_p\|^2_{L^2(e)}\\ &\le C\sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{3-N}\|\mu\|^2_{L^2(e)}\times h_e^{N-1} \le Ch^2\|\mu\|^2_{L^2(\triangle_e)},\end{aligned}$$

which, after the summation for $e\in\mathcal{E}_h^\partial$, implies the L^2-estimate. One can obtain the H^1-estimate in a similar way, and thus (A.4) is proved.

Finally, a direct computation shows that the left-hand side of (A.5) is bounded by

$$C\sum_{e\in\mathcal{E}_h^\partial}\sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{N-2}\big(|e|^{-1}\|\mu\|^2_{L^2(e)}h_e^2 + |e'|^{-1}\|\mu\|^2_{L^2(e')}h_{e'}^2\big) \le Ch\|\mu\|^2_{L^2(\varGamma_h)}.$$

This completes the proof of Lemma 3.2.

A.3. Proof of Lemma 3.4

By the standard lifting theorem, there exists a linear operator $L_h : H^{1/2}(\varGamma_h)^N \to H^1(\varOmega_h)^N$ such that $(L_h\psi)|_{\varGamma_h} = \psi$ and $\|L_h\psi\|_{H^1(\varOmega_h)} \le C\|\psi\|_{H^{1/2}(\varGamma_h)}$. We then define $v_h\in V_h$ by

$$v_h(m_e) = \begin{cases}\left[\varPi_h L_h E_h^\partial(\mu_h n_h)\right](m_e) & \text{for } e\in\mathring{\mathcal{E}}_h,\\ (\mu_h n_h)(m_e) & \text{for } e\in\mathcal{E}_h^\partial.\end{cases}$$

It is clear that $v_h\cdot n_h = \mu_h$ at all m_e's lying on $\varGamma_h$. We prove (3.2) in the following three steps.

Step 1. Let us show

$$\|v_h - \varPi_h L_h E_h^\partial(\mu_h n_h)\|_{V_h} \le C\|\mu_h\|_{1/2,\varLambda_h}. \tag{A.6}$$

Observe that

$$v_h - \varPi_h L_h E_h^\partial(\mu_h n_h) = \sum_{e\in\mathcal{E}_h^\partial}\left[\mu_h n_h - \varPi_h L_h E_h^\partial(\mu_h n_h)\right](m_e)\phi_e.$$

By the definitions of $\varPi_h$, L_h, and E_h^∂, for $e \in \mathcal{E}_h^\partial$ we obtain

$$\left[\varPi_h L_h E_h^\partial(\mu_h n_h)\right](m_e) = \frac{1}{|e|}\int_e L_h E_h^\partial(\mu_h n_h)\,\mathrm{d}s = \frac{1}{|e|}\int_e E_h^\partial(\mu_h n_h)\,\mathrm{d}s$$
$$= \frac{1}{|e|}\int_e \sum_{p\in\mathcal{V}_h(e)} \frac{1}{\#\mathcal{E}_h^\partial(p)} \sum_{e'\in\mathcal{E}_h^\partial(p)} (\mu_h n_h)(m_{e'})\bar{\phi}_p\,\mathrm{d}s.$$

Therefore, noting that $\sum_{p\in\mathcal{V}_h(e)}\bar{\phi}_p \equiv 1$, we deduce

$$v_h - \varPi_h L_h E_h^\partial(\mu_h n_h) = \sum_{e\in\mathcal{E}_h^\partial}\left[\frac{1}{|e|}\sum_{p\in\mathcal{V}_h(e)}\frac{1}{\#\mathcal{E}_h^\partial(p)}\sum_{e'\in\mathcal{E}_h^\partial(p)}\Big((\mu_h n_h)(m_e) - (\mu_h n_h)(m_{e'})\Big)\int_e \bar{\phi}_p\,\mathrm{d}s\right]\phi_e$$
$$=: \sum_{e\in\mathcal{E}_h^\partial} A_e\phi_e,$$

where the coefficient A_e can be estimated, using $|n_h(m_e) - n_h(m_{e'})| \le Ch_e$, by

$$|A_e| \le C\sum_{e'\in\mathcal{E}_h^\partial(e)}|\mu_h(m_e) - \mu_h(m_{e'})| + Ch_e\sum_{e'\in\mathcal{E}_h^\partial(e)}|\mu_h(m_{e'})|.$$

Then we conclude that

$$\|v_h - \varPi_h L_h E_h^\partial(\mu_h n_h)\|_{V_h}^2 = \sum_{T\in\mathcal{T}_h}\Big\|\sum_{e\in\mathcal{E}_h^\partial}A_e\phi_e\Big\|_{H^1(T)}^2 = \sum_{e\in\mathcal{E}_h^\partial}\|A_e\phi_e\|_{H^1(T_e)}^2 \le C\sum_{e\in\mathcal{E}_h^\partial}|A_e|^2 h_e^{N-2}$$
$$\le C\sum_{e\in\mathcal{E}_h^\partial}\sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{N-2}|\mu_h(m_e) - \mu_h(m_{e'})|^2 + C\sum_{e\in\mathcal{E}_h^\partial}\sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^N|\mu_h(m_{e'})|^2.$$

The last term on the right-hand side can be bounded by $h\|\mu_h\|_{L^2(\varGamma_h)}^2$ and this proves (A.6).
Step 2. The stability properties of $\varPi_h$ and L_h imply

$$\|\varPi_h L_h E_h^\partial(\mu_h n_h)\|_{V_h} \le C\|L_h E_h^\partial(\mu_h n_h)\|_{H^1(\Omega_h)} \le C\|E_h^\partial(\mu_h n_h)\|_{H^{1/2}(\varGamma_h)}.$$

Furthermore, by $n\circ\pi \in W^{1,\infty}(\varGamma_h)$ and by the definition of $\|\cdot\|_{1/2,\varLambda_h}$, one has

$$\|(E_h^\partial\mu_h)n\circ\pi\|_{H^{1/2}(\varGamma_h)} \le C\|E_h^\partial\mu_h\|_{H^{1/2}(\varGamma_h)} \le C\|\mu_h\|_{1/2,\varLambda_h}.$$

Therefore, to establish (3.2) it remains to prove

$$\|E_h^\partial(\mu_h n_h) - (E_h^\partial\mu_h)n\circ\pi\|_{H^{1/2}(\varGamma_h)} \le Ch^{1/2}\|\mu_h\|_{L^2(\varGamma_h)}.$$

This estimate follows from interpolation between $L^2(\varGamma_h)$ and $H^1(\varGamma_h)$ if we prove

$$\|E_h^\partial(\mu_h n_h) - (E_h^\partial\mu_h)n\circ\pi\|_{L^2(\varGamma_h)} \le Ch\|\mu_h\|_{L^2(\varGamma_h)}, \tag{A.7}$$
$$\|E_h^\partial(\mu_h n_h) - (E_h^\partial\mu_h)n\circ\pi\|_{H^1(\varGamma_h)} \le C\|\mu_h\|_{L^2(\varGamma_h)}. \tag{A.8}$$

Step 3. Let us prove (A.7) and (A.8). By the definition of E_h^∂, for $x\in e\in\mathcal{E}_h^\partial$ we calculate

$$[E_h^\partial(\mu_h n_h) - (E_h^\partial\mu_h)n\circ\pi](x) = \sum_{p\in\mathcal{V}_h(e)}\frac{1}{\#\mathcal{E}_h^\partial(p)}\sum_{e'\in\mathcal{E}_h^\partial(p)}\mu_h(m_{e'})\big(n_h(m_{e'}) - n\circ\pi(x)\big)\bar{\phi}_p(x).$$

Therefore,

$$\begin{aligned}\|E_h^\partial(\mu_h n_h) - (E_h^\partial \mu_h) n\circ\pi\|_{L^2(e)}^2 &\le C \sum_{e'\in\mathcal{E}_h^\partial(e)} |\mu_h(m_{e'})|^2 \sup_{x\in e} |n_h(m_{e'}) - n\circ\pi(x)|^2 \sup_{p\in\mathcal{V}_h(e)} \|\bar\phi_p\|_{L^2(e)}^2 \\ &\le C \sum_{e'\in\mathcal{E}_h^\partial(e)} h_e^{N+1} |\mu_h(m_{e'})|^2 \le Ch^2 \sum_{e'\in\mathcal{E}_h^\partial(e)} \|\mu_h\|_{L^2(e')}^2,\end{aligned}$$

where we have used the fact $|n_h(m_{e'}) - n\circ\pi(x)| \le Ch_e$. Adding the above estimates for $e \in \mathcal{E}_h^\partial$ yields (A.7). Estimate (A.8) can be proved similarly, and this completes the proof of Lemma 3.4.

A.4. Proof of Lemma 3.6

It suffices to prove $\|v + v_h\|_{L^2(\Omega_h\setminus\Omega)} \le Ch\|v + v_h\|_{V_h}$ for all $v \in H^1(\Omega_h)^N$ and $v_h \in V_h$. We define an enriching operator $E_h : V_h \to \overline{V}_h$ by

$$E_h v_h = \sum_{p\in\mathcal{V}_h} \Big(\frac{1}{\#\mathcal{T}_h(p)} \sum_{T\in\mathcal{T}_h(p)} v_h|_T(p)\Big)\bar\phi_p,$$

where $\mathcal{T}_h(p) := \{T \in \mathcal{T}_h : p \in T\}$ means the elements that share the vertex p. In view of (3.7) we have

$$\begin{aligned}\|v + v_h\|_{L^2(\Omega_h\setminus\Omega)} &\le \|v + E_h v_h\|_{L^2(\Omega_h\setminus\Omega)} + \|v_h - E_h v_h\|_{L^2(\Omega_h\setminus\Omega)} \\ &\le Ch\|v + E_h v_h\|_{H^1(\Omega_h)} + \|v_h - E_h v_h\|_{L^2(\Omega_h\setminus\Omega)} \\ &\le Ch\|v + v_h\|_{V_h} + Ch\|v_h - E_h v_h\|_{V_h} + \|v_h - E_h v_h\|_{L^2(\Omega_h\setminus\Omega)}. \qquad (A.9)\end{aligned}$$

Below we estimate the second and third terms in the right-hand side.

Since v_h and $E_h v_h$ are linear for $x \in T \in \mathcal{T}_h$ we obtain the expression

$$v_h(x) - E_h v_h(x) = \sum_{p\in\mathcal{V}_h(T)} \Big(\frac{1}{\#\mathcal{T}_h(p)} \sum_{T'\in\mathcal{T}_h(p)} (v_h|_T - v_h|_{T'})(p)\Big)\bar\phi_p(x),$$

where $\mathcal{V}_h(T) := \mathcal{V}_h \cap T$ means the vertices of T. Here, discontinuity at p can be estimated by that across edges near p, that is, $\big|(v_h|_T - v_h|_{T'})(p)\big| \le \sum_{e\in\mathring{\mathcal{E}}_h(p)} \|[v_h]\|_{L^\infty(e)}$ where $\mathring{\mathcal{E}}_h(p) = \{e \in \mathring{\mathcal{E}}_h : p \in e\}$ stands for the interior edges sharing the vertex p (*cf.* [5], p. 1073). Therefore,

$$\begin{aligned}\|\nabla(v_h - E_h v_h)\|_{L^2(T)}^2 &\le C \sum_{e\in\mathring{\mathcal{E}}_h(T)} \|[v_h]\|_{L^\infty(e)}^2 \sup_{p\in\mathcal{V}_h(T)} \|\nabla\bar\phi_p\|_{L^2(T)}^2 \\ &\le C \sum_{e\in\mathring{\mathcal{E}}_h(T)} h_e^{-N+1} \|[v_h]\|_{L^2(e)}^2 \times Ch_T^{N-2} \\ &\le C \sum_{e\in\mathring{\mathcal{E}}_h(T)} h_e^{-1} \|[v_h]\|_{L^2(e)}^2,\end{aligned}$$

where $\mathring{\mathcal{E}}_h(T) = \{e \in \mathring{\mathcal{E}}_h : e \subset T\}$ means the faces of T that are inside Ω_h. $\|v_h - E_h v_h\|_{L^2(T)}$ can be estimated in a similar manner, and adding these estimates for $T \in \mathcal{T}_h$ yields

$$\|v_h - E_h v_h\|_{V_h} \le C\Big(\sum_{e\in\mathring{\mathcal{E}}_h} h_e^{-1} \|[v_h]\|_{L^2(e)}^2\Big)^{1/2}. \qquad (A.10)$$

For the third term one has

$$\|v_h - E_h v_h\|_{L^2(\Omega_h\setminus\Omega)}^2 \le \sum_{e\in\mathcal{E}_h^\partial} |T_e \setminus \Omega| \|v_h - E_h v_h\|_{L^\infty(T_e)}^2,$$

where $|T_e \setminus \Omega|$ denotes the N-dimensional measure of $T_e \setminus \Omega$ and is bounded by $Ch_e^{N-1}\delta_e$. It follows that

$$\|v_h - E_h v_h\|^2_{L^\infty(T_e)} \leq C \sum_{e' \in \mathring{\mathcal{E}}_h(T_e)} \|[v_h]\|^2_{L^\infty(e')} \sup_{p \in \mathcal{V}_h(T_{e'})} \|\bar{\phi}_p\|^2_{L^\infty(T_e)} \leq C \sum_{e' \in \mathring{\mathcal{E}}_h(T_e)} h_{e'}^{-N+1} \|[v_h]\|^2_{L^2(e')}.$$

We thus obtain

$$\|v_h - E_h v_h\|_{L^2(\Omega_h \setminus \Omega)} \leq C\Big(h\delta \sum_{e \in \mathring{\mathcal{E}}_h} h_e^{-1} \|[v_h]\|^2_{L^2(e)}\Big)^{1/2} \leq Ch^{3/2}\Big(\sum_{e \in \mathring{\mathcal{E}}_h} h_e^{-1} \|[v_h]\|^2_{L^2(e)}\Big)^{1/2}. \tag{A.11}$$

Combining (A.9)–(A.11) and noting that $[v] = 0$ on each $e \in \mathring{\mathcal{E}}_h$, we conclude the desired estimate.

Remark A.2. Lemma 3.6 holds for general discontinuous P1 functions as well, because we did not use the continuity at midpoints in the proof.

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