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STABILITY ANALYSIS AND BEST APPROXIMATION ERROR ESTIMATES OF DISCONTINUOUS TIME-STEPPING SCHEMES FOR THE ALLEN–CAHN EQUATION

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Abstract. Fully-discrete approximations of the Allen–Cahn equation are considered. In particular, we consider schemes of arbitrary order based on a discontinuous Galerkin (in time) approach combined with standard conforming finite elements (in space). We prove that these schemes are unconditionally stable under minimal regularity assumptions on the given data. We also prove best approximation *a-priori* error estimates, with constants depending polynomially upon $(1/\epsilon)$ by circumventing Gronwall Lemma arguments. The key feature of our approach is a carefully constructed duality argument, combined with a boot-strap technique.

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1. Introduction

The Allen–Cahn equation is a parameter dependent parabolic semi-linear PDE of the form

$$\begin{cases} u_t - \Delta u + \dfrac{1}{\epsilon^2}(u^3 - u) = f & \text{in } (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\Gamma, \\ u(0,x) = u_0 & \text{in } \Omega; \end{cases} \tag{1.1}$$

here, Ω denotes a bounded domain in $\mathbb{R}^d$, $d = 2, 3$ with Lipschitz boundary Γ, u_0 and f denote the initial data and the forcing term, respectively. The principal difficulty involved, concerns the parameter $0 < \epsilon < 1$ which is very small and, typically comparable to the size of the time and space discretization parameters, τ, h respectively. The Allen–Cahn equation was introduced in [2] as the simplest phase field model.

The numerical analysis of any potential scheme is also significantly complicated due to the structrural properties involved. For instance, we note that the natural norms $\|.\|_{L^\infty[0,T;L^2(\Omega)]}$, $\|.\|_{L^2[0,T;H^1(\Omega)]}$ associated to the weak solution of our problem and imposed by its structure, scale differently in terms of the parameter ϵ, compared to the $\|.\|_{L^4[0,T;L^4(\Omega)]}$ norm that naturally arises from the nonlinear term. In addition,

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the presence of $L^2[0,T;L^2(\Omega)]$ norm with the "wrong sign" poses a substantial difficulty in the analysis as well as in the numerical analysis of fully-discrete schemes for such problem. Classical techniques based on Gronwall's type inequalities typically fail, since they introduce constants depending on quantities of $\exp(1/\epsilon)$. This problem was first circumvented in the works [3, 10, 17] through the development of uniform bounds of the principal eigenvalue of the linearized Allen–Cahn operator (spectral estimate), *i.e.*, bounds for the quantity $\inf_{0\neq v\in H^1(\Omega)} \frac{\|\nabla v\|^2_{L^2(\Omega)}+\epsilon^{-2}\left((3u^2-1)v,v\right)}{\|v\|^2_{L^2(\Omega)}}$ which are available when the Allen–Cahn equation describes a smooth evolution of a developing interface.

Based on the above idea, for the numerical analysis of the implicit Euler scheme, in [25], the first *a-priori* bounds were established in various norms with constants that depend upon $(1/\epsilon)$ in a polynomial fashion. For instance, for a discrete analog of the energy norm, an estimate of order $\tau+h$ with constant depending upon $1/\epsilon^3$, when the data $\|\nabla u_0\|_{L^2(\Omega)}, \|\Delta u_0\|_{L^2(\Omega)}, \lim_{s\to 0^+}\|\nabla u_t(s)\|_{L^2(\Omega)} \leq C$ and the spacial and the temporal discretization parameters satisfy $\tau+h^2\leq C\epsilon^7$ and $h|\ln h|^{1/2}\leq\epsilon^3$ when $d=2$ and $\tau+h^2\leq\epsilon^{13}$, and $h\leq\epsilon^6$, when $d=3$, respectively. One key idea involved, among others, in the numerical analysis of [25], was the construction of a discrete approximation of the spectral estimate.

Ideas based on the spectral estimate and its approximation, were further used in order to obtain *a-posteriori* error bounds in [26, 32], while various *a-priori* and *a-posteriori* estimates also based on discretized versions of the principal eigenvalue operator where obtained in the works of [8, 9, 27]. In [24] a fully-implicit scheme using the symmetric interior penalty discontinuous Galerkin (in space) method was considered and error estimates were established with sharp polynomial dependence upon $1/\epsilon$. The key idea involved in [24] was the construction of a suitable discrete approximation of the spectral estimate in presence of discontinuous (in space) spaces.

In [37], semi-implicit schemes of first order were studied, and conditional stability estimates were presented for semi-discrete (in time) approximations. In addition [37], a second order semi-implicit, semi-discrete in time scheme which is conditionally stable was also considered. In [4] a second order convergent in time scheme for the Cahn–Hilliard equation with a source term, was studied. We refer the reader to [6, 7, 16] for earlier works regarding numerical analysis of the Cahn–Hilliard equation and of the coupled Allen–Cahn, Cahn–Hilliard system respectively. Finally in the earlier work of [11] convergence of numerical solutions of a discretized Allen–Cahn equation was established. An overwiew of available *a-priori* and *a-posteriori* error bounds related to the Allen–Cahn equation can be found in [5]. Extensive numerical studies of various numerical schemes for the Allen–Cahn equation are presented in [31, 44]. For various results regarding discountinuous time-stepping schemes for nonlinear parabolic PDEs, we refer the reader to [20–22, 40, 41].

1.1. Main results

Our goal is to provide rigorous stability analysis and convergence for a general class of fully-discrete schemes under minimal regularity assumptions for any choice of τ, h, ϵ as well as to prove best approximation *a-priori* error estimates, in a suitable neigborhood of convergence, when τ, h, ϵ are chosen appopriately. The schemes considered here are discontinuous (in time) and conforming in space. In particular, for quasi-uniform in time partition, $\{t^i\}_{i=0,\ldots,N}$ of $[0,T]$, and for conforming finite element subspace $U_h\subset H^1(\Omega)$, we seek fully-discrete solution (here denoted by u_h) such that

$$u_h\in\mathcal{U}_h\equiv\{w_h\in L^2[0,T;H^1(\Omega)] : w_h|_{(t^{n-1},t^n]}\in\mathcal{P}_k[t^{n-1},t^n;U_h]\}.$$

Here $\mathcal{P}_k[t^{n-1},t^n;U_h]$ denotes the space of polynomials of degree k or less having values in U_h. Our analysis includes high order schemes in both space and time. The motivation for using the discontinuous (in time) Galerkin approach relies in its robust performance in a vast area of problems. The key feature of discontinuous time stepping Galerkin schemes is their ability to mimic the stability properties of the corresponding continuous system without requiring additional regularity on the given data. Indeed, we prove that the fully-discrete solution, computed by using discontinuous Galerkin (in time) and conforming finite elements in space of arbitrary order (in time and space), denoted by u_h, satisfies the following unconditional stability estimates:

$$\|u_h\|_{L^2[0,T;L^2(\Omega)]} \leq C, \quad \text{and} \quad \|u_h\|_{L^\infty[0,T;L^2(\Omega)]} + \|u_h\|_{L^2[0,T;H^1(\Omega)]} \leq \frac{C}{\epsilon},$$

where C denotes a constant depending on the domain Ω, the norms of $\|u_0\|_{L^2(\Omega)}$ and $\|f\|_{L^2[0,T;(H^1(\Omega))^*]}$ and the polynomial degree in time, but it is independent of τ, h, ϵ. The above stability estimates and a compactness argument tailored for discontinuous Galerkin time-stepping schemes by Walkington [41], allows us to deduce the strong convergence in $L^p[0,T;L^2(\Omega)]$ norms, for $1 \leq p < \infty$, which implies convergence of such schemes without using discrete variations/approximations of the spectral estimate.

In addition, using the stability estimates, and within a neighborhood of the established convergence, we prove the following best approximation error estimate,

$$\|\text{error}\|_X \leq \frac{C}{\epsilon^3}(\|u\|^2_{L^\infty[0,T;H^1(\Omega)]} + \|u\|^2_{L^2[0,T;H^2(\Omega)]})\|\text{best approximation error}\|_X,$$

where $X = L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$, and C denotes an algebraic constant depending only upon data, and it is independent of τ, h, ϵ. For the above best approximation error estimate we require that

- if $u \in L^4[0,T;H^2(\Omega)]$, $u_t \in L^4[0,T;L^2(\Omega)]$ then τ, h satisfy
  (1) $\ln(\frac{T}{\tau})(\tau + h^2) \leq \frac{C\epsilon^4}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 3$,
  (2) $\ln(\frac{T}{\tau})(\tau + h^2) \leq \frac{C\epsilon^{7/2}}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 2$,
  (3) $\ln(\frac{T}{\tau})(\tau + h^2) \leq \frac{C\epsilon^3}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 2$, $k = 0, 1$,

or

- if $u \in L^2[0,T;H^2(\Omega)]$, $u_t \in L^2[0,T;L^2(\Omega)]$, then τ, h satisfy,
  (1) $(\tau^{1/2} + h)^{3/2} \leq \frac{C\epsilon^4}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d = 3$,
  (2) $(\tau^{1/2} + h)^{3/2} \leq \frac{C\epsilon^{7/2}}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d = 2$,
  (3) $(\tau^{1/2} + h)^{3/2} \leq \frac{C\epsilon^3}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d = 2$, $k = 0, 1$.

In both cases C depends only upon the domain (independent of ϵ, h, τ). The above estimate states that within the neighborhood of convergence the error is as good as the approximation properties of the underlying subspaces, and the regularity of the solution will allow it to be.

1.2. Our approach

For the stability analysis, instead of focusing on the uniform bounds of the principle eigenvalue of the linearized elliptic part of the Allen–Cahn operator, we define the following auxiliary (almost dual) linearized pde, with appropriate scaling (and positive sign) in the $L^2[0,T;L^2(\Omega)]$-norm. In particular, with right-hand side $u \in L^2[0,T;L^2(\Omega)]$, and zero terminal data $\phi(T) = 0$, we seek $\phi \in L^2[0,T;H^1(\Omega)] \cap L^\infty[0,T;L^2(\Omega)]$ satisfying

$$-\phi_t - \Delta\phi + \frac{1}{\epsilon^2}u^2\phi + \frac{1}{\epsilon^2}\phi = u, \quad \text{in } (0,T) \times \Omega, \qquad \frac{\partial\phi}{\partial n} = 0 \quad \text{on } (0,T) \times \Gamma.$$

The key ingredient in our stability analysis is the construction of the fully-discrete space-time approximation of the above linearized equation with an appropriately scaled $L^2[0,T;L^2(\Omega)]$ part, based on the discontinuous time-stepping Galerkin formulation. This auxiliary space-time projection effectively allows the application of a duality argument, to recover first the unconditional stability with respect to $L^2[0,T;L^2(\Omega)]$ norm, and then a boot-strap argument to recover the unconditional stability in $L^2[0,T;H^1(\Omega)]$, $L^4[0,T;L^4(\Omega)]$, and $L^\infty[0,T;L^2(\Omega)]$ norms. For the later we employ the techniques developed in [12, 13, 41], in a way to avoid the use of Gronwall's type arguments. The discrete compactness argument of Walkington [41], then allows to rigorously pass to the limit to prove convergence. We note that the case of zero Dirichlet boundary data can be also considered in an identical way. The use of parabolic duality was initiated in [35], for the derivation of semi-discrete in space estimates for general linear parabolic PDEs, using the smoothing property (see also [40], Chap. 12 and references within for related results in the context of discontinuous time-stepping methods).

For the best approximation error estimate, within a neighborhood of the established convergence, we employ a similar strategy and the stability estimates in crucial way. To separate the difficulties due to the nonlinear structure from the ones involving the different scaling (in terms of ϵ) of various norms, we derive estimates in three steps:

(1) We define an auxiliary space-time linear parabolic projection that exhibits best approximation error estimates. The auxiliarly space-time parabolic projection u_p is defined as the discnotinuous time stepping solution of a linear parabolic pde with right hand side $u_t - \Delta u$, and appropriate initial data, and using the result of Section 2 from [12] and a proper duality argument we obtain best approximation estimates for the difference between $u - u_p$. In addition, we employ a crucial optimal estimate in $L^4[0,T;L^2(\Omega)]$ by Leykekham and Vexler [34], Corollary 4, which is applicable when $u \in L^4[0,T;H^2(\Omega)]$ and $u_t \in L^4[0,T;L^2(\Omega)]$.

(2) We use a duality argument, combined with the previously developed stability estimates to obtain the key preliminary estimate for the $L^2[0,T;L^2(\Omega)]$ norm without using Gronwall type arguments, with constants depending polynomially upon $1/\epsilon$. To achieve this, first we employ the discrete compactness argument of Walkington [41] to recover strong convergence in $L^4[0,T;L^2(\Omega)]$ to guarantee that the error $u_h - u$ is small enough, for small enough discretization parameters τ, h. Then, we define the space-time discontinuous Galerkin approximation ψ_h of the weak solution of the problem,

$$-\psi_t - \Delta\psi + \frac{1}{\epsilon^2}(3u^2 - 1)\psi = u_h - u_p, \quad \psi(T) = 0, \quad \frac{\partial\psi}{\partial n}|_{(0,T)\times\Gamma} = 0$$

and we prove various key stability estimates for ψ_h, with the help of the spectral estimate. We note that unlike previous works, we do *not* contruct an explicit discrete approximation of the spectral estimate.

(3) Then, we recover the full rate in the $L^2[0,T;H^1(\Omega)]$ norm *via* a boot-strap argument and the estimate at arbitrary time-points via the techniques developed by Chrysafinos and Walkington [12,13,41] to obtain the symmetric structure of the best-approximation error estimate. The boot-stap argument is performed in a way to avoid the use of Gronwall type arguments.

The remaining of the paper is organized as follows: in Section 2, we present the necessary notation, and some preliminary estimates for weak solutions of the Allen–Cahn equation. In Section 3, after defining the fully-discrete discontinuous Galerkin scheme, we present the basic stability estimates, which allow us to establish unconditional estimates in $L^\infty[0,T;L^2(\Omega)]$ and to prove strong convergence in Section 4. Finally in Section 5, we prove best-approximation estimates with constants depending polynomially upon $1/\epsilon$ and apply these results to obtain convergence rates.

2. Preliminaries

2.1. Notation

Let U denote a Banach space. Typically, $U \equiv H^s(\Omega), 0 < s \in \mathbb{R}$, where $H^s(\Omega)$ denotes the standard Sobolev (Hilbert) spaces (see for instance [23,43]). We denote by $H^0(\Omega) \equiv L^2(\Omega)$. Finally, we use the notation $\langle .,. \rangle$ for the duality pairing of $(H^1(\Omega))^*, H^1(\Omega)$ and $(.,.)$ for the standard L^2 inner product, where $(H^1(\Omega))^*$ is the dual space of $H^1(\Omega)$. We denote the time-space spaces by $L^p[0,T;U], L^\infty[0,T;U]$, endowed with norms:

$$\|w\|_{L^p[0,T;U]} = \Big(\int_0^T \|w\|_U^p \mathrm{d}t\Big)^{\frac{1}{p}}, \quad \|w\|_{L^\infty[0,T;U]} = \operatorname{esssup}_{t\in[0,T]} \|w\|_U.$$

The set of all continuous functions $v : [0,T] \to U$, is denoted by $C[0,T;U]$ with norm $\|w\|_{C[0,T;U]} = \max_{t\in[0,T]} \|w(t)\|_U$. For the definition of spaces $H^s[0,T;U]$, we refer the reader to [23,43]. Throughout this work we will use the following space for the solution u of (1.1),

$$X = L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$$

with associated norm $\|w\|_X^2 = \|w\|^2_{L^\infty[0,T;L^2((\Omega)]} + \|w\|^2_{L^2[0,T;H^1(\Omega)]}$. The bilinear form related to our problem is defined by

$$a(w_1,w_2) = \int_\Omega \nabla w_1 \nabla w_2 \mathrm{d}x \qquad \forall\, w_1,w_2 \in H^1(\Omega),$$

which implies the corresponding coercivity condition

$$a(w,w) = \|\nabla w\|^2_{L^2(\Omega)} \qquad \forall\, w \in H^1(\Omega).$$

We close this preliminary section, by recalling Young's inequality and Landyzeskaya–Gagliardo–Nirenberg interpolation inequalities.
Young's Inequality: For any $a,b \geq 0$ any $\delta > 0$, and $s_1, s_2 > 1$

$$ab \leq \delta a^{s_1} + C(s_1,s_2)\delta^{-\frac{s_2}{s_1}} b^{s_2}, \qquad \text{where } (1/s_1) + (1/s_2) = 1.$$

Landyzeshkayka-Gagliardo-Nirenberg Interpolation Inequalities: There exist constant $C > 0$ depending only upon the domain such that, for all $u \in H^1(\Omega)$,

$$\begin{aligned}
\|u\|_{L^4(\Omega)} &\leq C\|u\|^{1/2}_{L^2(\Omega)}\|u\|^{1/2}_{H^1(\Omega)}, &&\text{when } d=2,\\
\|u\|_{L^3(\Omega)} &\leq C\|u\|^{1/2}_{L^2(\Omega)}\|u\|^{1/2}_{H^1(\Omega)}, &&\text{when } d=3,\\
\|u\|_{L^4(\Omega)} &\leq C\|u\|^{1/4}_{L^2(\Omega)}\|u\|^{3/4}_{H^1(\Omega)}, &&\text{when } d=3.
\end{aligned}$$

2.2. Weak formulation and regularity of the Allen–Cahn equation

The following weak formulation of (1.1) will be used subsequently. Let $f \in L^2[0,T;(H^1(\Omega))^*]$ and $u_0 \in L^2(\Omega)$. Then, for all $w \in H^1(\Omega)$ and for *a.e.* $t \in (0,T]$, we seek $u \in L^2[0,T;H^1(\Omega)] \cap H^1[0,T;(H^1(\Omega))^*]$ such that

$$\langle u_t, w\rangle + a(u,w) + (1/\epsilon^2)\langle u^3 - u, w\rangle = \langle f, w\rangle, \quad \text{and} \quad (u(0),w) = (u_0,w).$$

Since, our schemes are based on the discontinuous time-stepping framework, a suitable space-time weak formulation can be written as follows: we seek $u \in L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$, satisfying,

$$\begin{aligned}
&(u(T),w(T)) + \int_0^T \Big(-\langle u, w_t\rangle + a(u,w) + \frac{1}{\epsilon^2}(u^3 - u, w)\Big)\mathrm{d}t\\
&= (u_0, w(0)) + \int_0^T \langle f, w\rangle \mathrm{d}t
\end{aligned} \tag{2.1}$$

for all $w \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))^*)$. It is clear that, using straightforward techniques, (see for instance [39, 43]), one can easily prove the existence of a weak solution $u \in L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$ which satisfies the following estimate

$$\|u\|_X \leq C_\epsilon\Big(\|f\|_{L^2[0,T;(H^1(\Omega))^*]} + \|u_0\|_{L^2(\Omega)}\Big),$$

where C_ϵ depends on Ω, and the parameters ϵ and T.

The following Lemma quantifies the dependence upon ϵ of various norms.

Lemma 2.1. *Suppose that $f \in L^2[0,T;(H^1(\Omega))^*]$ and $u_0 \in L^2(\Omega)$. Then, there exists a constant C, independent of ϵ, such that:*

$$\|u\|_{L^2[0,T;L^2(\Omega)]} + \|u\|^2_{L^4[0,T;L^4(\Omega)]} \leq C\left(T^{1/2} + \epsilon(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2[0,T;(H^1(\Omega))^*]})\right),$$
$$\|u\|_{L^\infty[0,T;L^2(\Omega)]} + \|u\|_{L^2[0,T;H^1(\Omega)]} \leq \frac{C}{\epsilon}.$$

Suppose that for any $\sigma \geq 0$,

$$\|f\|^2_{L^2[0,T;L^2(\Omega)]} \quad \text{and} \quad \|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon^2}\|(u_0^2-1)^2\|_{L^1(\Omega)} \leq \frac{C}{\epsilon^{2\sigma}}. \tag{2.2}$$

Then, there exists a constant C (indpendent of ϵ) such that:

$$\|u\|_{L^2[0,T;H^2(\Omega)]} \leq \frac{C}{\epsilon^{\sigma+1}}, \qquad \|u\|_{L^\infty[0,T;H^1(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]} \leq \frac{C}{\epsilon^\sigma}. \tag{2.3}$$

Proof. For the first estimate, we use the following auxiliary backward in time linear parabolic pde. Let u be the solution of (2.1). Given, right hand side $u \in L^2[0,T;L^2(\Omega)]$, boundary data $\frac{\partial\phi}{\partial n} = 0$, and terminal data $\phi(T) = 0$, we seek $\phi \in L^2[0,T;H^1(\Omega)] \cap H^1[0,T;(H^1(\Omega))^*]$ such that, for all $w \in L^2[0,T;H^1(\Omega)] \cap H^1[0,T;(H^1(\Omega))^*]$,

$$\int_0^T \left((\phi, w_t) + a(\phi,w) + \frac{1}{\epsilon^2}(u^2\phi, w) + \frac{1}{\epsilon^2}(\phi,w)\right)\mathrm{d}t + (\phi(0), w(0)) = \int_0^T (u,w)\mathrm{d}t. \tag{2.4}$$

It is clear that setting $w = \phi$ in (2.4) we obtain the following bound:

$$\begin{aligned}&\frac{1}{2}\|\phi(0)\|_{L^2(\Omega)]} + \|\nabla\phi\|_{L^2[0,T;L^2(\Omega)]} + \frac{1}{\epsilon}\|\phi u\|_{L^2[0,T;L^2(\Omega)]} + \frac{1}{2\epsilon}\|\phi\|_{L^2[0,T;L^2(\Omega)]}\\ &\leq \frac{\epsilon}{2}\|u\|_{L^2[0,T;L^2(\Omega)]}.\end{aligned} \tag{2.5}$$

Note that the above estimate easily implies $\|\phi\|_{L^2[0,T;H^1(\Omega)]} \leq C\epsilon\|u\|_{L^2[0,T;L^2(\Omega)]}$, with C an algebraic constant independent of ϵ. Now, we employ a "duality" argument. Integrating by parts in time (2.1), and setting $w = \phi$ into the resulting equation, we obtain:

$$\int_0^T \left(\langle u_t, \phi\rangle + a(u,\phi) + \frac{1}{\epsilon^2}(u^3 - u, \phi)\right)\mathrm{d}t = \int_0^T \langle f, \phi\rangle \mathrm{d}t. \tag{2.6}$$

Setting $w = u$ into (2.4) and subtracting the resulting equality from (2.6) we derive:

$$\int_0^T \|u\|^2_{L^2(\Omega)}\mathrm{d}t = \frac{2}{\epsilon^2}\int_0^T (\phi, u)\mathrm{d}t + \int_0^T \langle f,\phi\rangle\mathrm{d}t + (\phi(0), u(0)). \tag{2.7}$$

Note that using Hölder's inequality, and the stability estimates, equation (2.7) implies that

$$\begin{aligned}\|u\|^2_{L^2[0,T;L^2(\Omega)]} &\leq \frac{2}{\epsilon^2}\int_0^T |\Omega|^{1/2}\|\phi u\|_{L^2(\Omega)}\mathrm{d}t\\ &\quad + \|f\|_{L^2[0,T;(H^1(\Omega))^*]}\|\phi\|_{L^2[0,T;H^1(\Omega)]} + \|\phi(0)\|_{L^2(\Omega)}\|u(0)\|_{L^2(\Omega)}\\ &\leq \frac{2}{\epsilon^2}|\Omega|^{1/2}T^{1/2}\|\phi u\|_{L^2[0,T;L^2(\Omega)]}\\ &\quad + C(\|f\|_{L^2[0,T;(H^1(\Omega))^*]} + \|u(0)\|_{L^2(\Omega)})\epsilon\|u\|_{L^2[0,T;L^2(\Omega)]}\\ &\leq \frac{2}{\epsilon^2}|\Omega|^{1/2}T^{1/2}\frac{\epsilon^2}{2}\|u\|_{L^2[0,T;L^2(\Omega)]}\\ &\quad + C(\|f\|_{L^2[0,T;(H^1(\Omega))^*]} + \|u(0)\|_{L^2(\Omega)})\epsilon\|u\|_{L^2[0,T;L^2(\Omega)]},\end{aligned}$$

which implies the desired estimate on $\|u\|_{L^2[0,T;L^2(\Omega)]}$. Returning back to (2.1), setting $w = u$, and using the bound on $\|u\|_{L^2[0,T;L^2(\Omega)]}$, we obtain the first estimate. For the second estimate, we set $w = u_t$, and we observe,

$$\int_0^T \left(\|u_t\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon^2}\|(u^2-1)^2\|_{L^1(\Omega)}\right)\right)\mathrm{d}t = \int_0^T (f, u_t)\mathrm{d}t.$$

The estimate now follows by standard algebra. The estimate on $\|\Delta u\|_{L^2[0,T;L^2(\Omega)]}$ follows using standard techniques. □

Remark 2.2.

(1) If more regularity is available, then we can quantify the dependence upon $1/\epsilon$ in other norms (see for instance [25], Prop. 1). In addition to (2.2), if the initial data satisfy, for some $\tilde{\sigma} \geq 0$, $\|\Delta u_0 - \frac{1}{\epsilon^2}(u_0^3 - u_0)\|_{L^2(\Omega)} \leq C\epsilon^{-\tilde{\sigma}}$ with constant C independent of ϵ, then,

$$\|u\|_{L^\infty[0,T;H^2(\Omega)]} + \|u_t\|_{L^\infty[0,T;L^2(\Omega)]} \leq C\epsilon^{\min\{-\sigma-1,-\tilde{\sigma}\}},$$
$$\|\nabla u_t\|_{L^2[0,T;L^2(\Omega)]} \leq C\epsilon^{\min\{-\sigma-1,-\tilde{\sigma}\}}.$$

(2) We point out that the regularity bound on $\frac{1}{4\epsilon^2}\|(u_0^2-1)^2\|_{L^1(\Omega)} \leq \frac{C}{\epsilon^{2\sigma}}$, for $\sigma \geq 0$, is essential in order to obtain (2.3). For example, if only $\|u_0\|_{H^1(\Omega)} \leq C$ is assumed then the dependence upon $\frac{1}{\epsilon}$ deteriorates to:

$$\|u\|_{L^\infty[0,T;H^1(\Omega)]} + \|u_t\|_{L^2[0,T;L^2(\Omega)]} + \|u\|_{L^2[0,T;H^2(\Omega)]} \leq \frac{C}{\epsilon^2}.$$

For the stability analysis of the fully-discrete schemes, enhanced regularity assumptions, such as $u \in L^\infty[0,T;H^2(\Omega)] \cap H^1[0,T;H^1(\Omega)]$ are not necessary. For the error estimates, the constants will depend upon the norms of $\|u\|_{L^\infty[0,T;H^1(\Omega)]}$, $\|u_t\|_{L^2[0,T;L^2(\Omega)]}$ and $\|u\|_{L^2[0,T;H^2(\Omega)]}$.

3. The fully-discrete scheme

3.1. Discontinuous Galerkin time-stepping

For the discretization of the Allen–Cahn model we employ a discontinuous Galerkin time-stepping approach, combined with standard conforming finite elements in space. Approximations will be constructed on a partition $0 = t^0 < t^1 < \ldots < t^N = T$ of $[0,T]$. On each interval of the form $(t^{n-1}, t^n]$ of length $\tau_n = t^n - t^{n-1}$, a subspace U_h of $H^1(\Omega)$ is specified for all $n = 1,\ldots,N$ and it is assumed that each U_h satisfies the classical approximation theory results (see *e.g.*, [14]), on regular meshes. In particular, we assume that there exists an integer $\ell \geq 1$ and a constant $c > 0$ (independent of the mesh-size parameter h) such that if $w \in H^{l+1}(\Omega)$,

$$\inf_{w_h \in U_h} \|w - w_h\|_{H^s(\Omega)} \leq Ch^{l+1-s}\|w\|_{H^{l+1}(\Omega)}, \qquad 0 \leq l \leq \ell, \qquad s = -1, 0, 1.$$

We also assume that the partition is quasi-uniform in time, *i.e.*, there exists a constant $0 < \theta \leq 1$ such that $\theta\tau \leq \min_{n=1,\ldots N} \tau_n$, where $\tau = \max_{n=1,\ldots,N} \tau_n$. We seek approximate solutions which belong to the space

$$\mathcal{U}_h = \{w_h \in L^2[0,T;H^1(\Omega)] : w_h|_{(t^{n-1},t^n]} \in \mathcal{P}_k[t^{n-1},t^n;U_h]\}.$$

Here $\mathcal{P}_k[t^{n-1},t^n;U_h]$ denotes the space of polynomials of degree k or less, having values in U_h. By convention, the functions of $\mathcal{U}_h$ are left continuous with right limits and hence we will subsequently write w^n_{h-} for $w_h(t^n) = w_h(t^n_-)$, and w^n_{h+} for $w_h(t^n_+)$. Note that, we have also used the following notational abbreviation, $w_h \equiv w_{h,\tau}$, $\mathcal{U}_h \equiv \mathcal{U}_{h,\tau}$ etc., since for the stability analysis we will not impose any restriction involving τ, and h. The jump at t^n will be denoted as $[w_h^n] = w_{h+}^n - w_{h-}^n$. The fully discrete system is defined as follows: We seek $u_h \in \mathcal{U}_h$ such that for every $w_h \in \mathcal{U}_h$ and for $n = 1, \ldots, N$,

$$(u_{h-}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} \Big(- \langle u_h, w_{ht} \rangle + a(u_h, w_h) + (1/\epsilon^2)(u_h^3 - u_h, w_h) \Big) \mathrm{d}t$$
$$= (u_{h-}^{n-1}, w_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \langle f, w_h \rangle \mathrm{d}t. \tag{3.1}$$

Recall that f, u_0 are given data, and u^0 denotes approximations of u_0. In our case, we will define $u^0 = P_h u^0$, where P_h denotes the standard L^2 projection, *i.e.*, $P_h : L^2(\Omega) \to U_h$, defined by $(P_h v - v, w_h) = 0, \quad \forall\, w_h \in U_h$.

Remark 3.1. For any $\epsilon > 0$, existence and uniqueness of discontinuous Galerkin approximations of (3.1) can be proved easily (even for more complicated nonlinearities) due to finite dimensionality of the problem. For several results regarding discontinuous time-stepping schemes, with linear and semi-linear terms, we refer [1, 15, 18–20, 30, 36, 40, 41] and the references within.

3.2. The basic estimate using duality

We begin by developing a stability estimate via duality for the $L^2[0,T;L^2(\Omega)]$ norm. For this purpose, we define a backward in time parabolic problem with right hand side $u_h \in L^2[0,T;L^2(\Omega)]$ with an enhanced $L^2[0,T;L^2(\Omega]$ term and zero terminal data. In particular, for right hand side $u_h \in L^2[0,T;L^2(\Omega)]$, and terminal data $\phi_{h+}^N = 0$, we seek $\phi_h \in \mathcal{U}_h$ such that for all $w_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$, and for $n = N, \ldots, 1$,

$$-(\phi_{h+}^n, w_{h-}^n) + \int_{t^{n-1}}^{t^n} \big((\phi_h, w_{ht}) + a(\phi_h, w_h) + (1/\epsilon^2)\langle u_h^2 \phi_h, w_h \rangle \big)$$
$$+ \int_{t^{n-1}}^{t^n} (1/\epsilon^2)(\phi_h, w_h)\mathrm{d}t + (\phi_{h+}^{n-1}, w_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} (u_h, w_h)\mathrm{d}t. \tag{3.2}$$

Note that is easy to prove existence at partition points as well as in $L^2[0,T;H^1(\Omega)]$, due to the signs of the inner products $(1/\epsilon^2)(u_h^2 \phi_h, w_h)$ and $(1/\epsilon^2)(\phi_h, w_h)$. Given, $u_h \in \mathcal{U}_h$, it is obvious that $\phi_h \in \mathcal{U}_h$ is unique. In Section 4.2, we will also prove that $u_h \in L^\infty[0,T;L^2(\Omega)]$.

Lemma 3.2. *Let $f \in L^2[0,T;(H^1(\Omega))^*]$, $u_0 \in L^2(\Omega)$, and $u_h \in \mathcal{U}_h$ are the solutions of (3.1), (3.2) respectively. Then, there exists a constant $C > 0$, depending only upon the domain Ω, T, but is independent of ϵ, such that:*

$$\|u_h\|_{L^2[0,T;L^2(\Omega)]} \leq C \left(T^{1/2} + \epsilon(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2[0,T;(H^1(\Omega))^*]}) \right)$$

In addition, the following estimates hold: for all $n = 1, \ldots, N$

$$\|u_{h-}^n\|_{L^2(\Omega)}^2 + \|u_h\|_{L^2[0,T;H^1(\Omega)]}^2 + (1/\epsilon^2)\|u_h\|_{L^4[0,T;L^4(\Omega)]}^4 + \sum_{i=1}^N \|[u_h^i]\|_{L^2(\Omega)}^2$$
$$\leq (C/\epsilon^2) \left(\|u_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2[0,T;(H^1(\Omega))^*]}^2 \right).$$

where C is a constant depending only upon Ω, T.

Proof. Setting $w_h = \phi_h$, into (3.2), using Young's inequality to bound

$$\int_{t^{n-1}}^{t^n} (u_h, \phi_h)\mathrm{d}t \leq (1/2\epsilon^2) \int_{t^{n-1}}^{t^n} \|\phi_h\|_{L^2(\Omega)}^2 \mathrm{d}t + (\epsilon^2/2) \int_{t^{n-1}}^{t^n} \|u_h\|_{L^2(\Omega)}^2 \mathrm{d}t,$$

and adding the resulting terms, we derive the following estimate: for all $n = N, \dots, 1$

$$\begin{aligned}&\|\phi_{h+}^{n-1}\|^2_{L^2(\Omega)} + \|\nabla\phi_h\|^2_{L^2[0,T;L^2(\Omega)]} + (1/\epsilon^2)\|\phi_h u_h\|^2_{L^2[0,T;L^2(\Omega)]}\\ &+ (1/2\epsilon^2)\|\phi_h\|^2_{L^2[0,T;L^2(\Omega)]} \le (\epsilon^2/2)\|u_h\|^2_{L^2[0,T;L^2(\Omega)]}.\end{aligned} \tag{3.3}$$

The above estimate also implies that $\|\phi_h\|_{L^2[0,T;H^1(\Omega)]} \le \frac{\epsilon}{\sqrt{2}}\|u_h\|_{L^2[0,T;L^2(\Omega)]}$, when $\epsilon < 1/2$. Now, setting $w_h = u_h$ into (3.2), we have

$$\begin{aligned}&-(\phi_{h+}^n, u_{h-}^n) + \int_{t^{n-1}}^{t^n} \big((\phi_h, u_{ht}) + a(u_h,\phi_h) + (1/\epsilon^2)\langle u_h^2\phi_h, u_h\rangle + (1/\epsilon^2)(\phi_h,u_h)\big)\,\mathrm{d}t\\ &+(\phi_{h+}^{n-1}, u_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Integrating by parts in time, we deduce,

$$\begin{aligned}&-(\phi_{h+}^n, u_{h-}^n) + (\phi_{h-}^n, u_{h-}^n) + \int_{t^{n-1}}^{t^n} (-\langle\phi_{ht}, u_h\rangle + a(\phi_h,u_h))\,\mathrm{d}t\\ &+\int_{t^{n-1}}^{t^n} \big((1/\epsilon^2)\langle u_h^2\phi_h, u_h\rangle + (1/\epsilon^2)(\phi_h,u_h)\big)\,\mathrm{d}t = \int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned} \tag{3.4}$$

Setting $w_h = \phi_h$ into (3.1), we obtain,

$$\begin{aligned}&(u_{h-}^n, \phi_{h-}^n) + \int_{t^{n-1}}^{t^n} \big(-\langle u_h, \phi_{ht}\rangle + a(u_h,\phi_h) + (1/\epsilon^2)\langle u_h^3 - u_h, \phi_h\rangle\big)\,\mathrm{d}t\\ &= (u_{h-}^{n-1}, \phi_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \langle f, \phi_h\rangle\mathrm{d}t.\end{aligned} \tag{3.5}$$

Subtracting (3.5) from (3.4), and noting that the terms $(1/\epsilon^2)\int_{t^{n-1}}^{t^n}\int_\Omega u_h^3\phi_h dxdt$ cancel, we arrive at

$$\begin{aligned}&(\phi_{h+}^n, u_{h-}^n) - (u_{h-}^{n-1}, \phi_{h+}^{n-1}) + \int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)}\mathrm{d}t\\ &= (1/\epsilon^2)\int_{t^{n-1}}^{t^n}(\phi_h,u_h)\mathrm{d}t - \int_{t^{n-1}}^{t^n}\langle f,\phi_h\rangle\mathrm{d}t + (1/\epsilon^2)\int_{t^{n-1}}^{t^n}(u_h,\phi_h)\mathrm{d}t.\end{aligned} \tag{3.6}$$

First, we treat the terms involving $(1/\epsilon^2)$ constants. Using Young's inequality with appropriate $\delta_1 > 0$ (to be determined later), we deduce,

$$\begin{aligned}(2/\epsilon^2)\int_{t^{n-1}}^{t^n} |(\phi_h,u_h)|\,\mathrm{d}t &\le (2/\epsilon^2)\int_{t^{n-1}}^{t^n} |\Omega|^{1/2}\|\phi_h u_h\|_{L^2(\Omega)}\mathrm{d}t\\ &\le (2/\epsilon^2)\tau_n^{1/2}|\Omega|^{1/2}\left(\int_{t^{n-1}}^{t^n}\|\phi_h u_h\|^2_{L^2(\Omega)}\mathrm{d}t\right)^{1/2}\\ &\le (2\delta_1/\epsilon^2)\tau_n|\Omega| + (1/2\delta_1\epsilon^2)\int_{t^{n-1}}^{t^n}\|\phi_h u_h\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Similarly, using Young's inequality with appropriate $\delta_2 > 0$, we obtain

$$\int_{t^{n-1}}^{t^n} |\langle f,\phi_h\rangle|\mathrm{d}t \le (\delta_2/\epsilon^2)\int_{t^{n-1}}^{t^n}\|\phi_h\|^2_{H^1(\Omega)}\mathrm{d}t + (\epsilon^2/4\delta_2)\int_{t^{n-1}}^{t^n}\|f\|^2_{(H^1(\Omega))^*}\mathrm{d}t.$$

Substituting the last two inequalities into (3.6), summing the resulting inequalities, using that $\phi_+^N \equiv 0$ by definition, and rearranging terms, we obtain

$$\begin{aligned}\|u_h\|^2_{L^2[0,T;L^2(\Omega)]} &\leq \|u_h^0\|_{L^2(\Omega)}\|\phi^0_{h+}\|_{L^2(\Omega)} + (\delta_2/\epsilon^2)\|\phi_h\|^2_{L^2[0,T;H^1(\Omega)]}\\ &\quad + (\epsilon^2/4\delta_2)\|f\|^2_{L^2[0,T;(H^1(\Omega))^*]} + (2\delta_1/\epsilon^2)\sum_{n=1}^N \tau_n|\Omega| + (1/2\delta_1\epsilon^2)\|\phi_h u_h\|^2_{L^2[0,T;L^2(\Omega)]}\\ &\leq (\delta_3/\epsilon^2)\|\phi^0_{h+}\|^2_{L^2(\Omega)} + (\epsilon^2/4\delta_3)\|u_h^0\|^2_{L^2(\Omega)} + (\delta_2/\epsilon^2)\|\phi_h\|^2_{L^2[0,T;H^1(\Omega)]}\\ &\quad + (\epsilon^2/4\delta_2)\|f\|^2_{L^2[0,T;(H^1(\Omega))^*]} + (2\delta_1/\epsilon^2)\sum_{n=1}^N \tau_n|\Omega| + (1/2\delta_1\epsilon^2)\|\phi_h u_h\|^2_{L^2[0,T;L^2(\Omega)]}.\end{aligned}$$

Using the previous bounds on $\|\phi^0_{h+}\|_{L^2(\Omega)}$, $\|\phi_h\|_{L^2[0,T;H^1(\Omega)]}$, $(1/\epsilon)\|\phi_h\|_{L^2[0,T;L^2(\Omega)]}$, and $(1/\epsilon)$ $\|\phi_h u_h\|_{L^2[0,T;L^2(\Omega)]}$, in terms of $\|u_h\|_{L^2[0,T;L^2(\Omega)]}$ via (3.3) and choosing $\delta_1 = 2\epsilon^2$, $\delta_2 = \delta_3 = 1/4$, to hide the resulting terms on the left, we obtain

$$\|u_h\|_{L^2(0,T;L^2(\Omega)]} \leq C\left(T^{1/2} + \epsilon\left(\|u_h^0\|_{L^2(\Omega)} + \|f\|_{L^2[0,T;(H^1(\Omega))^*]}\right)\right),$$

with C an algebraic constant, depending only upon $|\Omega|$. Setting $w_h = u_h$, in (3.1) respectively and using Young's inequalities we obtain:

$$\begin{aligned}&(1/2)\|u^n_{h-}\|^2_{L^2(\Omega)} - (1/2)\|u^{n-1}_{h-}\|^2_{L^2(\Omega)} + (1/2)\|[u_h^{n-1}]\|^2_{L^2(\Omega)}\\ &\quad + \int_{t^{n-1}}^{t^n}\left(\|\nabla u_h\|^2_{L^2(\Omega)} + (1/\epsilon^2)\|u_h\|^4_{L^4(\Omega)}\right)\mathrm{d}t\\ &\leq (1/\epsilon^2)\int_{t^{n-1}}^{t^n}\|u_h\|^2_{L^2(\Omega)}\mathrm{d}t + \int_{t^{n-1}}^{t^n}(1/C)\|f\|^2_{(H^1(\Omega))^*}\mathrm{d}t. \end{aligned}\tag{3.7}$$

The second estimate follows by summation and the previously developed estimate on $L^2[0,T;L^2(\Omega)]$. □

Remark 3.3. It is evident that the key estimate with respect the dependence upon $(1/\epsilon)$ concerns the term $(1/\epsilon^2)\int_{t^{n-1}}^{t^n}\int_\Omega u_h w_h \mathrm{d}x\mathrm{d}t$ which has the wrong sign and not the term $(1/\epsilon^2)\int_{t^{n-1}}^{t^n}\int_\Omega u_h^3 w_h\mathrm{d}x\mathrm{d}t$ which is positive when setting $w_h = u_h$. For this reason the estimate of (3.1) does not lead to an estimate, with bounds independent of $\exp(1/\epsilon)$ when using Gronwall type arguments even for the lowest order scheme. To the contrary, the duality argument of Lemma 3.2, leads to polynomial dependence upon $(1/\epsilon)$, without imposing any condition between τ, h, and under minimal regularity assumptions. The key question regarding the stability at arbitrary time-points, *i.e.* in $L^\infty[0,T;L^2(\Omega)]$, will be considered next.

4. Estimates at arbitrary time-points and convergence under minimal regularity

We will employ the theory of the approximation of discrete characteristic functions (see *e.g.*, [12,13,41]), which was used to develop estimates at arbitrary time points for linear and nonlinear parabolic PDEs, including the Navier–Stokes equations. The main advantage of this approach is that the proof does not require any additional regularity, apart from the one needed to guarantee the existence of a weak solution. In addition, we will be able to obtain stability estimates without assuming any explicit dependence upon τ and h. A key feature of our analysis is that we are able to include high order time-stepping schemes.

4.1. Preliminaries: Approximation of discrete characteristic functions

Ideally, to obtain a stability estimate at arbitrary $t \in (t^{n-1}, t^n]$, we would like to substitute $u_h = \chi_{[t^{n-1},t)} u_h$ into the discrete equations (3.1). However, this choice is not available in the discrete setting, since $\chi_{[t^{n-1},t)} u_h$ is not a member of $\mathcal{U}_h$, unless t coincides with a partition point. Therefore, approximations of such functions need to be constructed; this is done in Section 2.3 of [12]. For completeness, we state the main results. The approximations are constructed on the interval $(0, \tau)$, and they are invariant under translations. For fixed (but arbitrary) $t \in (0, \tau)$ let $p \in \mathcal{P}_k(0, \tau)$, and denote the discrete approximation of $\chi_{[0,t)} p$ by the polynomial $\tilde{p} \in \mathcal{P}_k(0, \tau)$ with, $\tilde{p}(0) = p(0)$ which satisfies

$$\int_0^\tau \tilde{p} q = \int_0^t p q \qquad \forall\, q \in \mathcal{P}_{k-1}(0, \tau).$$

To motivate the above construction we simply observe that for $q = p'$ we obtain $\int_0^\tau p' \tilde{p} = \int_0^t p p' = \frac{1}{2}(p^2(t) - p^2(0))$.

It is clear that this construction can be extended to approximations of $\chi_{[0,t)} u$ for $u \in \mathcal{P}_k[0, \tau; U]$ where U is a linear space. Note that if $u \in \mathcal{P}_k[0, \tau; U]$ then it can be written as $u = \sum_{i=0}^k p_i(t) u_i$ where $p_i \in \mathcal{P}_k[0, \tau]$ and $u_i \in U$. The discrete approximation of $\chi_{[0,t)} u$ in $\mathcal{P}_k[0, \tau; U]$ is then defined by $\tilde{u} = \sum_{i=0}^k \tilde{p}_i(t) u_i$, and if U is a semi-inner product space, we deduce,

$$\tilde{u}(0) = u(0), \qquad \text{and} \int_0^\tau (\tilde{u}, w)_U = \int_0^t (u, w)_U \quad \forall w \in \mathcal{P}_{k-1}[0, \tau; U].$$

It remains to quote the main results from [12, 13, 41].

Proposition 4.1. *Suppose that U is a (semi) inner product space. Then, the mapping $\sum_{i=0}^k p_i(t) u_i \to \sum_{i=0}^k \tilde{p}_i(t) u_i$ on $\mathcal{P}_k[0, \tau; U]$ is continuous in $\|.\|_{L^2[0,\tau;U]}$. In particular,*

$$\|\tilde{u}\|_{L^2[0,\tau;U]} \le C_k \|u\|_{L^2[0,\tau;U]}, \qquad \|\tilde{u} - \chi_{[0,t)} u\|_{L^2[0,\tau;U]} \le C_k \|u\|_{L^2[0,\tau;U]}$$

where C_k is a constant depending on k.

A standard calculation gives an explicit formula of $\tilde{u} = \rho(s) z$, when we choose $u(s) = z \in U$ to be constant (see *e.g.*, [13]).

Lemma 4.2. *Fix $t \in [0, \tau]$ and let $\rho \in \mathcal{P}_k[0, \tau]$ characterized by*

$$\rho(0) = 1, \qquad \int_0^\tau \rho q = \int_0^t q, \qquad q \in \mathcal{P}_{k-1}[0, \tau].$$

Then,

$$\rho(s) = 1 + (s/\tau) \sum_{i=0}^{k-1} c_i \hat{p}_i(s/\tau), \qquad c_i = \int_{t/\tau}^1 \hat{p}_i(\eta) \mathrm{d}\eta,$$

where $\{\hat{p}_i\}_{i=0}^{k-1}$ is an orthonormal basis of $\mathcal{P}_{k-1}[0, 1]$ in the (weighted) space $L^2_w[0, 1]$ having inner product

$$(\hat{p}, \hat{q}) = \int_0^1 \eta \hat{p}(\eta) \hat{q}(\eta) \mathrm{d}\eta.$$

In particular, $\|\rho\|_{L^\infty(0,\tau)} \le C_k$, where C_k is independent of $t \in [0, \tau]$.

4.2. The main stability estimate at arbitrary time points

Now, we are ready to state the main stability result at arbitrary time points which plays a key role to the derivation of the best approximation estimates below. We emphasize that the time-discretization parameter τ is chosen independent of h and the dependence of the stability constant upon $1/\epsilon$ is polynomial.

Proposition 4.3. *Suppose that $f \in L^2[0,T;(H^1(\Omega))^*]$, $u_0 \in L^2(\Omega)$, and let u_h be the approximate solution by the discontinuous time-stepping scheme. Then, there exists constant $C > 0$ depending on Ω, C_k and T, but not ϵ, such that*

$$\|u_h\|_{L^\infty[0,T;L^2(\Omega)]} \leq (C/\epsilon).$$

Proof. Recall that setting $w_h = u_h$, in (3.1), adding the term $\int_{t^{n-1}}^{t^n} \|u_h\|^2_{L^2(\Omega)}\mathrm{d}t$ on both sides, using Young's inequality to bound $\int_{t^{n-1}}^{t^n} |\langle f, u_h\rangle|\mathrm{d}t \leq (1/2)\int_{t^{n-1}}^{t^n}(\|f\|^2_{(H^1(\Omega))^*} + \|u_h\|^2_{H^1(\Omega)})\mathrm{d}t$, and the fact that $\epsilon < 1$, we easily obtain

$$\begin{aligned}
&(1/2)\|u^n_{h-}\|^2_{L^2(\Omega)} - (1/2)\|u^{n-1}_{h-}\|^2_{L^2(\Omega)} + (1/2)\|[u^{n-1}_h]\|^2_{L^2(\Omega)} \\
&\quad + \int_{t^{n-1}}^{t^n}\Big(\|u_h\|^2_{H^1(\Omega)} + (1/\epsilon^2)\|u_h\|^4_{L^4(\Omega)}\Big)\,\mathrm{d}t \\
&\quad \leq (2/\epsilon^2)\int_{t^{n-1}}^{t^n}\|u_h\|^2_{L^2(\Omega)}\mathrm{d}t + (1/2)\int_{t^{n-1}}^{t^n}\|f\|^2_{(H^1(\Omega))^*}\mathrm{d}t. \qquad (4.1)
\end{aligned}$$

In order to avoid the use of a Gronwall type argument, we will need to estimate the term $(1/\epsilon^2)\int_{t^{n-1}}^{t^n}\|u_h\|^2_{L^2(\Omega)}$ using the approximation of the discrete characteristic. We employ properties of the discrete characteristic and its approximation by following the technique of [13] and the stability estimates of Lemma 3.2. For fixed $t \in [t^{n-1}, t^n)$ and $z_h \in U_h$ we substitute $w_h(s) = z_h\rho(s)$ into (3.1), where $\rho(s) \in \mathcal{P}_k[t^{n-1}, t^n]$ is constructed similarly to Lemma 4.2, *i.e.*,

$$\rho(t^{n-1}) = 1, \qquad \int_{t^{n-1}}^{t^n}\rho q = \int_{t^{n-1}}^{t} q, \qquad q \in \mathcal{P}_{k-1}[t^{n-1}, t^n].$$

Now, it is easy to see that with this particular choice of w_h,

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n}(u_{ht}, w_h)\mathrm{d}s + (u^{n-1}_{h+} - u^{n-1}_{h-}, w^{n-1}_{h+}) \\
&\quad = \int_{t^{n-1}}^{t}(u_{ht}, z_h)\mathrm{d}s + (u^{n-1}_{h+} - u^{n-1}_{h-}, \rho(t^{n-1})z_h) = (u_h(t) - u^{n-1}_{h-}, z_h).
\end{aligned}$$

Hence, integration by parts in time of (3.1), and the above computation imply

$$\begin{aligned}
(u_h(t) - u^{n-1}_{h-}, z_h) &= -\int_{t^{n-1}}^{t^n}\big(a(u_h, z_h\rho) + (1/\epsilon^2)(u_h^3 - u_h, z_h\rho)\big)\,\mathrm{d}s + \int_{t^{n-1}}^{t^n}\langle f, z_h\rho\rangle\mathrm{d}s \\
&\leq C_k\Big[\int_{t^{n-1}}^{t^n}\|\nabla u_h\|_{L^2(\Omega)}\|\nabla z_h\|_{L^2(\Omega)}\mathrm{d}s + \int_{t^{n-1}}^{t^n}\|f\|_{(H^1(\Omega))^*}\|z_h\|_{H^1(\Omega)}\mathrm{d}s \\
&\quad + (1/\epsilon^2)\int_{t^{n-1}}^{t^n}\big(\|u_h^3\|_{L^{4/3}(\Omega)}\|z_h\|_{L^4(\Omega)} + \|u_h\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)}\big)\,\mathrm{d}s\Big],
\end{aligned}$$

where we have used Lemma 4.2 to bound $\|\rho\|_{L^\infty(t^{n-1},t^n)} \leq C_k$ with C_k denoting a constant depending only on k, Ω. Note also that $z_h \in U_h$ and is independent of s, hence the above inequality leads to

$$\begin{aligned}
(u_h(t) - u^{n-1}_{h-}, z_h) &\leq C_k\left[\int_{t^{n-1}}^{t^n}\big(\|u_h\|_{H^1(\Omega)} + \|f\|_{(H^1(\Omega))^*}\big)\mathrm{d}s\right]\|z_h\|_{H^1(\Omega)} \\
&\quad + C_k(1/\epsilon^2)\Big(\left[\int_{t^{n-1}}^{t^n}\|u_h\|^3_{L^4(\Omega)}\mathrm{d}s\right]\|z_h\|_{L^4(\Omega)} + \left[\int_{t^{n-1}}^{t^n}\|u_h\|_{L^2(\Omega)}\mathrm{d}s\right]\|z_h\|_{L^2(\Omega)}\Big).
\end{aligned}$$

Here we have used the fact $\|u_h^3\|_{L^{4/3}(\Omega)} = \|u_h\|^3_{L^4(\Omega)}$. Setting $z_h = u_h(t)$ (for the previously fixed $t \in [t^{n-1}, t^n)$), using Hölder's inequality, and integrating in time the resulting inequality, we obtain,

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|u_h(t)\|^2_{L^2(\Omega)}\mathrm{d}t &\le \|u_{h-}^{n-1}\|_{L^2(\Omega)}\tau_n^{1/2}\|u_h(t)\|_{L^2[t^{n-1},t^n;L^2(\Omega)]} \\ &+ C_k\tau_n^{1/2}\Big(\|u_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]} + \|f\|_{L^2[t^{n-1},t^n;(H^1(\Omega))^*]}\Big)\int_{t^{n-1}}^{t^n}\|u_h(t)\|_{H^1(\Omega)}\mathrm{d}t \\ &+ C_k\tau_n^{1/4}(1/\epsilon^2)\left(\|u_h\|^3_{L^4[t^{n-1},t^n;L^4(\Omega)]}\right)\int_{t^{n-1}}^{t^n}\|u_h(t)\|_{L^4(\Omega)}\mathrm{d}t \\ &+ C_k\tau_n^{1/2}(1/\epsilon^2)\|u_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\int_{t^{n-1}}^{t^n}\|u_h(t)\|_{L^2(\Omega)}\mathrm{d}t. \end{aligned} \tag{4.2}$$

Using appropriately Hölder's inequalities, we obtain that $\int_{t^{n-1}}^{t^n}\|u_h\|_{L^4(\Omega)}\mathrm{d}t \le \tau_n^{3/4}\|u_h\|_{L^4[t^{n-1},t^n;L^4(\Omega)]}$, and $\int_{t^{n-1}}^{t^n}\|u_h\|_{H^1(\Omega)}\mathrm{d}t \le \tau_n^{1/2}\|u_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}$. Thus, using Young's inequalities we deduce (with different C_k),

$$\begin{aligned}(1/2)\int_{t^{n-1}}^{t^n}\|u_h(t)\|^2_{L^2(\Omega)}\mathrm{d}t &\le (\tau_n/2)\|u_{h-}^{n-1}\|^2_{L^2(\Omega)} \\ &+ C_k\tau_n\left(\|u_h\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]} + \|f\|^2_{L^2[t^{n-1},t^n;(H^1(\Omega))^*]}\right) \\ &+ C_k\tau_n(1/\epsilon^2)\left(\|u_h\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]} + \|u_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\right). \end{aligned} \tag{4.3}$$

Now, using an inverse estimate, $\|u_h\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \le (C_k/\tau_n)\int_{t^{n-1}}^{t^n}\|u_h\|^2_{L^2(\Omega)}$, we obtain,

$$\begin{aligned}\|u_h\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \le C_k\Big[&\|u_{h-}^{n-1}\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]} + \|f\|^2_{L^2[t^{n-1},t^n;(H^1(\Omega))^*]} \\ &+ (1/\epsilon^2)\left(\|u_h\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]} + \|u_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\right)\Big].\end{aligned}$$

The proof now follows by simply substituting the bounds of (3.2). □

Remark 4.4. The above theorem states that the discontinuous Galerkin discretization inherits the stability estimates of the weak formulation under minimal regularity assumptions on the given data. This is an important asset related to the discontinuous (in time) Galerkin formulation. We emphasize that we do *not* assume that $\|u_0\|_{L^\infty(\Omega)} \le 1$.

4.3. Convergence under minimal regularity assumptions

We quote a discrete compactness argument of Walkington (see [41], Thm. 3.1) which allows to recover strong convergence in an appropriate norm, and pass the limit through the nonlinear term. The compactness argument combined with the stability estimates of Lemma 3.2 and Proposition 4.3, imply the convergence of the space-time approximations under minimal regularity assumptions.

The compactness argument concerns numerical approximations of solutions $u : [0, T] \to U$ of general evolution equations of the form

$$u_t + A(u) = f(u), \qquad u(0) = u_0, \tag{4.4}$$

where U is a Banach space and each term of the equation takes values in U^*. Both $A(u) = A(t, u)$ and $f(u) = f(t, u)$ may depend upon t and are allowed to be nonlinear, however, in our setting only $f(u) \equiv -(1/\epsilon^2)(u^3 - u)$ contains nonlinear terms. Suppose that $U \subset H \subset U^*$ (with continuous embeddings) form the standard evolution triple, *i.e.*, the pivot space H is a Hilbert space. The numerical schemes approximate the weak form of (4.4), *i.e.*,

$$\langle u_t, w\rangle + a(u, w) = \langle f(u), w\rangle, \qquad \forall\, w \in U \tag{4.5}$$

where $a : U \times U \to \mathbb{R}$ is defined by $a(u,w) = (A(u),w)$. Set $F(u) \equiv f(u) - A(u)$. Then the following theorem ([41], Thm. 3.1) establishes the compactness property of the discrete approximation.

Theorem 4.5. *Let H be a Hilbert space, U be a Banach space and $U \subset H \subset U^*$ be dense and compact embeddings. Fix an integer $k \geq 0$ and let $1 \leq p, q < \infty$. Let $h > 0$ be the mesh parameter, and let $\{t^i\}_{i=0}^N$ denote a quasi-uniform partition of $[0,T]$. Let $U_h \subset U$ denote standard finite element spaces. Assume that*

(1) *For each $h, \tau > 0$, $u_h \in \{u_h \in L^p[0,T;U] \quad | \quad u_h|_{(t^{n-1},t^n)} \in \mathcal{P}_k[t^{n-1},t^n;U_h]\}$ and on each interval, satisfies*

$$\int_{t^{n-1}}^{t^n} \langle u_{ht}, w_h \rangle \mathrm{d}t + (u_{h+}^{n-1} - u_{h-}^{n-1}, w_{h+}^{n-1}) = \int_{t^{n-1}}^{t^n} \langle F(u_h), w_h \rangle \mathrm{d}t$$

for every $w_h \in \mathcal{P}_k[t^{n-1},t^n;U_h]$.

(2) *$\{u_h\}_{h>0}$ is bounded in $L^p[0,T;U]$ and $\{\|F(u_h)\|_{L^q[0,T;U^*]}\}_{h>0}$ is also bounded.*

Then,

(1) *If $p > 1$ then $\{u_h\}_{h>0}$ is compact in $L^r[0,T;H]$ for $1 \leq r < 2p$.*

(2) *If $1 \leq (1/p) + (1/q) < 2$, and $\sum_{i=1}^N \|[u_h^i]\|_H^2 < C$ is bounded independent of h, then $\{u_h\}_{h>0}$ is compact in $L^r[0,T;H]$ for $1 \leq r < 2/((1/p) + (1/q) - 1)$.*

Proof. See [41], Theorem 3.1. □

We will use the above result to obtain strong convergence of the discrete Allen–Cahn equation to the continuous one. The lack of any meaningful regularity for the discrete time derivative due to the presence of discontinuities, requires special attention since the Aubin–Lions compactness argument is not directly applicable.

Theorem 4.6. *Let $f \in L^2[0,T;(H^1(\Omega))^*]$, $u_0 \in L^2(\Omega)$, and $\epsilon < 1$ be a given parameter. Let $\{t^i\}_{i=0}^N$ denote a quasi-uniform partition of $[0,T]$. Suppose that the assumptions of Proposition 4.3 hold, and let $\tau, h \to 0$. Then, the following convergence results hold:*

$$u_h \to u \text{ weakly in } L^2[0,T;H^1(\Omega)], \quad u_h \to u \text{ weakly-* in } L^\infty[0,T;L^2(\Omega)],$$

and

$$u_h \to u \quad \text{ strongly in } L^r[0,T;L^2(\Omega)], \text{ for every } 1 \leq r < \infty.$$

In addition u is a weak solution of the Allen–Cahn equation.

Proof. We follow a similar line of argument with Section 6 of [41]. The stability estimates of Lemma 3.2 and Proposition 4.3, imply (passing to a subsequence if necessary) there exists u such that $u_h \to u$ weakly in $L^2[0,T;H^1(\Omega)]$ and weakly-* in $L^\infty[0,T;L^2(\Omega)]$. We note that $\{u_h\}_{h,\tau}$ is bounded independent of τ, h, ϵ in $L^2[0,T;L^2(\Omega)]$ and $L^4[0,T;L^4(\Omega)]$. It remains to obtain strong convergence. For this purpose, fix $U = H^1(\Omega)$, $H = L^2(\Omega)$, and $F(u) = \Delta u - (1/\epsilon^2)\left(u^3 - u\right) - f$. It is easy to show that $F(u_h) \in L^{4/3}[0,T;(H^1(\Omega))^*]$. Indeed, $u_h \in L^2[0,T;H^1(\Omega)] \cap L^4[0,T;L^4(\Omega)]$, and $u_h \in L^\infty[0,T;L^2(\Omega)]$ clearly imply that $u_h^3 \in L^{4/3}[0,T;(H^1(\Omega))^*]$ by standard interpolation theorems. The remaining terms can be handled easily. Note also that $\sum_{i=1}^N \|[u_h^i]\|_{L^2(\Omega)}^2 \leq C/\epsilon$ where C is independent of τ, h. Therefore, using the Theorem 4.5, we obtain the desired strong convergence. Choose $w_h \in C[0,T;U_h] \cap \mathcal{U}_h$, with $w_h(T) = 0$. Then, summing equations (3.1) from $n = 1$ to $n = N$, we deduce that

$$\begin{aligned}
&(u_h(T), w_h(T)) + \int_0^T \left(-\langle u_h, w_{ht}\rangle + a(u_h, w_h) + (1/\epsilon^2)\langle u_h^3 - u_h, w_h\rangle\right) \mathrm{d}t \\
&= \int_0^T \langle f, w_h \rangle \mathrm{d}t + (u^0, w_h(0)).
\end{aligned}$$

Note that we may pass the limit through the linear terms due to the stability estimates on u_h and the fact that $w_h \in C[0,T;U_h] \cap \mathcal{U}_h$. The semi-linear term can be treated by the strong convergence. Indeed, using Holder's inequality, Landyzeskaya-Gagliardo-Nirenberg interpolation inequality,

$$
\begin{aligned}
\int_0^T \left|\langle u_h^3 - u^3, w_h\rangle\right| \mathrm{d}t &\le \int_0^T \left|\langle (u_h - u)(u_h^2 + u^2 + u_h u), w_h\rangle\right| \mathrm{d}t \\
&\le C \int_0^T \|u_h - u\|_{L^3(\Omega)} (\|u_h\|^2_{L^4(\Omega)} + \|u\|^2_{L^4(\Omega)}) \|w_h\|_{L^6(\Omega)} \mathrm{d}t \\
&\le C \|w_h\|_{C[0,T;H^1(\Omega)]} \int_0^T \|u_h - u\|^{1/2}_{L^2(\Omega)} \|u_h - u\|^{1/2}_{H^1(\Omega)} (\|u_h\|^2_{L^4(\Omega)} + \|u\|^2_{L^4(\Omega)}) \mathrm{d}t \\
&\le C \|w_h\|_{C[0,T;H^1(\Omega)]} \|u_h - u\|^{1/2}_{L^2[0,T;L^2(\Omega)]} \|u_h - u\|^{1/2}_{L^2[0,T;H^1(\Omega)]} \\
&\quad \times (\|u_h\|_{L^4[0,T;L^4(\Omega)]} + \|u\|_{L^4[0,T;L^4(\Omega)]})^2.
\end{aligned}
$$

A standard density argument, now completes the proof. □

The unconditional stability estimates and the above convergence result, validate the use of discontinuous Galerkin time-stepping schemes of order $k \ge 1$. In particular, for any $\alpha > 0$ there exist $\tilde{h}, \tilde{\tau}$ such that, for every $\tau \le \tilde{\tau}$ and $h \le \tilde{h}$, we obtain, $\|u_h - u\|_{L^4[0,T;L^2(\Omega)]} \le \alpha$. For the error estimates, we will choose to work with τ, h (chosen independently) such that, for (τ, h) satisfying $\tau \le \tilde{\tau}$, $h \le \tilde{h}$

$$
\begin{cases}
\|u_h - u\|_{L^4[0,T;L^2(\Omega)]} \le \delta\epsilon^4, & \text{when } d = 3, \\
\|u_h - u\|_{L^4[0,T;L^2(\Omega)]} \le \delta\epsilon^{7/2}, & \text{when } d = 2, \\
\|u_h - u\|_{L^4[0,T;L^2(\Omega)]} \le \delta\epsilon^3, & \text{when } d = 2,\ k = 0, 1,
\end{cases}
\tag{4.6}
$$

where $\delta > 0$ (to be chosen later) is independent of ϵ. Note that due to the unconditional stability in $L^4[0,T;L^4(\Omega)]$ with bounds independent of ϵ, $\tilde{\tau}, \tilde{h}$ can be chosen independent of ϵ. We conclude this Section by a short remark regarding the computation of such discrete solution.

Remark 4.7. It is expected, that at least for moderate values of the papemeter ϵ, even when $\tau \approx h$, the computation of the fully-discrete solution follows by standard techniques. However, when using high order schemes, due to the large and non-symmetric structure of the associated system, special attention is necessary. For specialized preconditioners for high-order discontinuous Galerkin schemes, we refer the reader to the recent work [38].

5. Error estimates

5.1. Preliminary estimates

The following projections related to discontinuous Galerkin time-stepping schemes will be used.

Definition 5.1.

(1) The projection $\mathcal{P}_n^{\mathrm{loc}} : C[t^{n-1}, t^n; L^2(\Omega)] \to \mathcal{P}_k[t^{n-1}, t^n; U_h]$ satisfies $(\mathcal{P}_n^{\mathrm{loc}} w)^n = P_h w(t^n)$, and

$$
\int_{t^{n-1}}^{t^n} (w - \mathcal{P}_n^{\mathrm{loc}} w, w_h) = 0, \quad \forall\, w_h \in \mathcal{P}_{k-1}[t^{n-1}, t^n; U_h].
$$

In the above definition, we have used the convention $(\mathcal{P}_n^{\mathrm{loc}} w)^n \equiv (\mathcal{P}_n^{\mathrm{loc}} w)(t^n)$, and $P_h : L^2(\Omega) \to U_h$ is the orthogonal L^2 projection operator onto $U_h \subset H^1(\Omega)$.

(2) The projection $\mathcal{P}_h^{\rm loc} : C[0,T;L^2(\Omega)] \to \mathcal{U}_h$ satisfies

$$\mathcal{P}_h^{\rm loc} w \in \mathcal{U}_h \text{ and } (\mathcal{P}_h^{\rm loc} w)|_{(t^{n-1},t^n]} = \mathcal{P}_n^{\rm loc}(w|_{[t^{n-1},t^n]}).$$

In the following Lemma, we collect several results regarding optimal rates of convergence for the above projection (see *e.g.*, [13]).

Lemma 5.2. *Let $U_h \subset H^1(\Omega)$, and $\mathcal{P}_h^{\rm loc}$ defined in Section 3.1 and Definition 5.1 respectively. Then, for all $w \in L^2[0,T;H^{l+1}(\Omega)] \cap H^{k+1}[0,T;L^2(\Omega)]$ there exists constant $C \geq 0$ independent of h, τ such that*

$$\begin{aligned}
\|w - \mathcal{P}_h^{\rm loc} w\|_{L^2[0,T;L^2(\Omega)]} &\leq C\big(h^{l+1}\|w\|_{L^2[0,T;H^{l+1}(\Omega)]} + \tau^{k+1}\|w^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]}\big),\\
\|w - \mathcal{P}_h^{\rm loc} w\|_{L^2[0,T;H^1(\Omega)]} &\leq C\big(h^{l}\|w\|_{L^2[0,T;H^{l+1}(\Omega)]} + (\tau^{k+1}/h)\|w^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]}\big),\\
\|w - \mathcal{P}_h^{\rm loc} w\|_{L^\infty[0,T;L^2(\Omega)]} &\leq C\big(h^{l+1}\|w\|_{L^\infty[0,T;H^{l+1}(\Omega)]} + \tau^{k+1}\|w^{(k+1)}\|_{L^\infty[0,T;L^2(\Omega)]}\big).
\end{aligned}$$

Let $k = 0, l = 1$, and $w \in L^2[0,T;H^2(\Omega)] \cap H^1[0,T;L^2(\Omega)]$. Then, there exists constant $C \geq 0$ independent of h, τ such that,

$$\begin{aligned}
\|w - \mathcal{P}_h^{\rm loc} w\|_{L^\infty[0,T;L^2(\Omega)]} + \|w - \mathcal{P}_h^{\rm loc} w\|_{L^2[0,T;H^1(\Omega)]} &\leq C\big(h\|w\|_{L^2[0,T;H^2(\Omega)]}\\
&+\tau^{1/2}(\|w_t\|_{L^2[0,T;L^2(\Omega)]} + \|w\|_{L^2[0,T;H^2(\Omega)]})\big).
\end{aligned}$$

Remark 5.3. If more regularity (in time) is available then the above estimates can be improved. In particular, if $w \in L^2[0,T;H^{l+1}(\Omega)] \cap H^{k+1}[0,T;H^1(\Omega)]$, then we obtain,

$$\|w - \mathcal{P}_h^{\rm loc} w\|_{L^2[0,T;H^1(\Omega)]} \leq C\big(h^{l}\|w\|_{L^2[0,T;H^{l+1}(\Omega)]} + \tau^{k+1}\|w^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]}\big).$$

The fully-discrete Galerkin orthogonality can be written as follows: Subtracting (3.1) from (2.1), we obtain for every $w_h \in \mathcal{U}_h$ and for $n = 1, \ldots, N$,

$$\begin{aligned}
(e^n_-, w^n_{h-}) &+ \int_{t^{n-1}}^{t^n} (-\langle e, w_{ht}\rangle + a(e, w_h))\, \mathrm{d}t \\
&+ (1/\epsilon^2)\int_{t^{n-1}}^{t^n} \big((u_h^3 - u^3, w_h) - (u_h - u, w_h)\big)\, \mathrm{d}t = (e^{n-1}_-, w^{n-1}_{h+})
\end{aligned} \tag{5.1}$$

where $e = u_h - u$ denotes the error. We will split the error as $e = (u_h - u_p) + (u_p - u) \equiv e_h + e_p$, where u_p is the discontinuous Galerkin solution of a linear parabolic pde with right hand side $u_t - \Delta u$, and initial data $u_{p0} = P_h u_0$, *i.e.*, for every $w_h \in \mathcal{U}_h$ and for $n = 1, \ldots, N$, $u_p \in \mathcal{U}_h$ is the solution of,

$$\begin{aligned}
(u^n_{p-}, w^n_{h-}) &+ \int_{t^{n-1}}^{t^n} \Big(-\langle u_p, w_{ht}\rangle + a(u_p, w_h)\Big)\mathrm{d}t \\
&= (u^{n-1}_{p+}, w^{n-1}_{+}) + \int_{t^{n-1}}^{t^n} \langle u_t - \Delta u, w_h\rangle \mathrm{d}t.
\end{aligned} \tag{5.2}$$

Integrating by parts the last term on the right-hand side, using the fact that $\frac{\partial u}{\partial n} = 0$, we obtain the orthogality condition: for $n = 1, \ldots, N$, and $w_h \in \mathcal{U}_h$

$$(e^n_{p-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \Big(-\langle e_p, w_{ht}\rangle + a(e_p, w_h)\Big)\mathrm{d}t = (e^{n-1}_{p+}, w^{n-1}_{+}). \tag{5.3}$$

The following best approximation estimate under minimal regularity assumptions that bound the error $e_p = u_p - u$ in terms of the local projections of Definition 5.1 is straightforward application of Theorems 2.2 and 2.3

from [12]:

$$\|e_p\|_{L^\infty[0,T;L^2(\Omega)]} + \|e_p\|_{L^2[0,T;H^1(\Omega)]} \le C\Big(\|P_h u(0) - u(0)\|_{L^2(\Omega)} + \|u - \mathcal{P}_h^{\mathrm{loc}} u\|_{L^\infty[0,T;L^2(\Omega)]} + \|u - \mathcal{P}_h^{\mathrm{loc}} u\|_{L^2[0,T;H^1(\Omega)]}\Big), \tag{5.4}$$

where C is a constant depending upon Ω and the constant C_k of Proposition 4.1. In addition,

$$\|u_p\|_{L^\infty[0,T;H^1(\Omega)]} \le C(\|u_0\|_{H^1(\Omega)} + \|u_t - \Delta u\|_{L^2[0,T;L^2(\Omega)]}). \tag{5.5}$$

by Theorem 4.10 of [13]. Another key ingredient will be an optimal estimate in $L^4[0,T;L^2(\Omega)]$ by Leykekham and Vexler (see [34], Cor. 4), which states that if the solution u satisfies $u \in L^4[0,T;H^2(\Omega)]$, $u_t \in L^4[0,T;L^2(\Omega)]$, then there exists a constant C independent of τ, h such that,

$$\|e_p\|_{L^4[0,T;L^2(\Omega)]} \le C \ln\left(\frac{T}{\tau}\right)(\tau + h^2)\left(\|u_t\|_{L^4[0,T;L^2(\Omega)]} + \|u\|_{L^4[0,T;H^2(\Omega)]}\right). \tag{5.6}$$

Returning back to the orthogonality condition (5.1) and using (5.3) we obtain, the following relation for $e_h = u_h - u_p$: For all $w_h \in \mathcal{U}_h$ and for $n = 1, \dots, N$,

$$(e^n_{h-}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} (-\langle e_h, w_{ht}\rangle + a(e_h, w_h))\,\mathrm{d}t + (1/\epsilon^2)\int_{t^{n-1}}^{t^n} \big((u_h^3 - u^3, w_h) - (u_h - u, w_h)\big)\,\mathrm{d}t = (e^{n-1}_{h-}, w^{n-1}_{h+}). \tag{5.7}$$

Adding and subtracting the term u_p^3 in the nonlinear term, we equivalently obtain,

$$\begin{aligned}(e^n_{h-}, w^n_{h-}) &+ \int_{t^{n-1}}^{t^n} (-\langle e_h, w_{ht}\rangle + a(e_h, w_h))\,\mathrm{d}t - (e^{n-1}_{h-}, w^{n-1}_{h+})\\ &+ (1/\epsilon^2)\int_{t^{n-1}}^{t^n} \big((u_h^3 - u_p^3, w_h) - (e_h, w_h)\big)\,\mathrm{d}t\\ &+ (1/\epsilon^2)\int_{t^{n-1}}^{t^n} \big(u_p^3 - u^3, w_h) - (e_p, w_h)\big)\,\mathrm{d}t = 0.\end{aligned} \tag{5.8}$$

Our focus is to bound e_h in terms of e_p without introducing constants that depend exponentially upon $1/\epsilon$. To simplify the presentation, we will denote by $C_\infty = \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}$, and we note that if in addition to (2.2), $u_0 \in L^\infty(\Omega)$, with norm bounded independent of ϵ then C_∞ is also bounded independent of ϵ.

We first recall the spectral estimate of [17], which states that if u is solution of (1.1) then there exists a positive constant C_s independent of ϵ such that,

$$\inf_{\phi \in H^1(\Omega), \phi \neq 0} \frac{\|\nabla\phi\|^2_{L^2(\Omega)} + (1/\epsilon^2)\big((3u^2 - 1)\phi, \phi\big)}{\|\phi\|^2_{L^2(\Omega)}} \ge -C_s. \tag{5.9}$$

We follow the approach presented in Section 3. In particular, given right hand side $e_h \in L^\infty[0,T;L^2(\Omega)]$, and terminal data $\psi^N_{h+} = 0$, we seek $\psi_h \in \mathcal{U}_h$ such that for all $w_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$, and for all $n = N, \dots, 1$,

$$\begin{aligned}&-(\psi^n_{h+}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \big((\psi_h, w_{ht}) + a(\psi_h, w_h)\big)\mathrm{d}t + (\psi^{n-1}_{h+}, w^{n-1}_{h+})\\ &+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \big(3u^2\psi_h, w_h\big)\,\mathrm{d}t - \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} (\psi_h, w_h)\mathrm{d}t = \int_{t^{n-1}}^{t^n} (e_h, w_h)\mathrm{d}t.\end{aligned} \tag{5.10}$$

Note that despite the fact that the above pde is a linearized analog of the Allen–Cahn equation, the spectral estimate can be applied directly to obtain a preliminary bound on the $\|.\|_{L^2[0,T;H^1(\Omega)]}$ norm and at arbitrary time-points, when $k = 0, 1$.

Lemma 5.4. *Let $e_h \in L^2[0,T;L^2(\Omega)]$, and $u \in L^4[0,T;L^4(\Omega)]$ with bounds independent of ϵ. Then, for $\tau_n \leq C_k \frac{\epsilon^2}{\|u\|^2_{L^\infty[0,T;L^\infty(\Omega)]}}$, $\psi_h \in \mathcal{U}_h$ satisfies for all $n = N, \ldots, 1$,*

$$\|\psi^{n-1}_{h+}\|_{L^2(\Omega)} + \|\psi_h\|_{L^2[0,T;L^2(\Omega)]} + \|u\psi_h\|_{L^2[0,T;L^2(\Omega)]} + \epsilon\|\psi_h\|_{L^2[0,T;H^1(\Omega)]} \leq C\|e_h\|_{L^2[0,T;L^2(\Omega)]},$$
$$\|\psi_h\|_{L^\infty[0,T;L^2(\Omega)]} \leq \frac{C}{\epsilon}\|e_h\|_{L^2[0,T;L^2(\Omega)]}.$$

where the constants C depend only upon C_s, the domain, the constant C_k of Lemma 4.2 and the data f, u_0 (through the norms of $\|u\|_{L^4[0,T;L^4(\Omega)]}$), and are independent of τ, h, ϵ. In addition, there exists a costant C depending upon C_s, the domain, the constant C_k of Lemma 4.2, and the norm $\|u\|_{L^\infty[0,T;L^\infty(\Omega)]}$ such that,

$$\|\psi_h\|_{L^\infty[0,T;H^1(\Omega)]} + \|\Delta_h\psi_h\|_{L^2[0,T;L^2(\Omega)]} \leq \frac{C_\infty}{\epsilon^2}\|e_h\|_{L^2[0,T;L^2(\Omega)]}.$$

Here $\Delta_h\psi_h \in \mathcal{U}_h$ denotes a discrete approximation of $\Delta\psi$, defined by, $(\Delta_h\psi_h(.), w_h) = a(\psi_h(.), w_h) + (\psi_h(.), w_h)$, for all $w_h \in U_h$ and for every $t \in (t^{n-1}, t^n]$.

Proof.
Step 1. *Stability estimates in $L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$*: We rewrite (5.10) as follows:

$$\begin{aligned}&-(\psi^n_{h+}, w^n_{h-}) + \int_{t^{n-1}}^{t^n} \big((\psi_h, w_{ht}) + \epsilon^2 a(\psi_h, w_h)\big)\mathrm{d}t + (\psi^{n-1}_{h+}, w^{n-1}_{h+})\\ &+(1-\epsilon^2)\left(\int_{t^{n-1}}^{t^n} a(\psi_h, w_h)\mathrm{d}t + \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \big(3(u^2-1)\psi_h, w_h\big)\,\mathrm{d}t\right)\\ &+\int_{t^{n-1}}^{t^n} ((3u^2-1)\psi_h, w_h)\mathrm{d}t = \int_{t^{n-1}}^{t^n} (e_h, w_h)\mathrm{d}t. \end{aligned} \tag{5.11}$$

Setting $w_h = \psi_h$ into (5.11) and using the spectral estimate (5.9) we deduce,

$$\begin{aligned}\frac{1}{2}\|\psi^{n-1}_{h+}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\psi^n_{h+}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[\psi^n_h]\|^2_{L^2(\Omega)} + \epsilon^2\int_{t^{n-1}}^{t^n}\|\nabla\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t&\\ - (1-\epsilon^2)C_s\int_{t^{n-1}}^{t^n}\|\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t + 3\int_{t^{n-1}}^{t^n}\|u\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t&\\ \leq \frac{3}{2}\int_{t^{n-1}}^{t^n}\|\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{2}\int_{t^{n-1}}^{t^n}\|e_h\|^2_{L^2(\Omega)}\mathrm{d}t.&\end{aligned}$$

Hence, using standard algebra we obtain,

$$\begin{aligned}&\frac{1}{2}\|\psi^{n-1}_{h+}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\psi^n_{h+}\|^2_{L^2(\Omega)} + \frac{1}{2}\|[\psi^n_h]\|^2_{L^2(\Omega)} + \epsilon^2\int_{t^{n-1}}^{t^n}\|\nabla\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t\\ &\quad + 3\int_{t^{n-1}}^{t^n}\|u\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t \leq C(C_s)\int_{t^{n-1}}^{t^n}\|\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{2}\int_{t^{n-1}}^{t^n}\|e_h\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned} \tag{5.12}$$

where the constant $C(C_s)$ depends on C_s but it is independent of ϵ. For low order schemes $k = 0, 1$, a standard Gronwall Lemma provides the estimates at arbitrary time points, as well as the estimate for $\|\nabla\psi_h\|^2_{L^2[0,T;L^2(\Omega)]}$.

For higher order schemes, we proceed using the technique of Section 4, based on the approximation of the discrete characteristic. Hence, following exactly the same approach as in Proposition 4.3, for fixed $t \in (t^{n-1}, t^n)$, we obtain with $z_h \in U_h$ independent of t, and ρ defined as in Lemma 4.2 (suitably modified to handle the backwards in time problem)

$$\begin{aligned}(\psi_h(t) - \psi_{h+}^n, z_h) &= -\int_{t^{n-1}}^{t^n} \big(a(\psi_h, z_h\rho) + (1/\epsilon^2)((3u^2-1)\psi_h, z_h\rho)\big)\,\mathrm{d}s + \int_{t^{n-1}}^{t^n} (e_h, z_h\rho)\mathrm{d}s \\ &\leq C_k\Big[\int_{t^{n-1}}^{t^n} \|\nabla\psi_h\|_{L^2(\Omega)}\|\nabla z_h\|_{L^2(\Omega)}\mathrm{d}s + \int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)}\mathrm{d}s \\ &\quad + (1/\epsilon^2)\int_{t^{n-1}}^{t^n} \big(\|u\psi_h\|_{L^2(\Omega)}\|u\|_{L^\infty(\Omega)}\|z_h\|_{L^2(\Omega)} + \|\psi_h\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)}\big)\,\mathrm{d}s\Big],\end{aligned}$$

where C_k is the constant of Lemma 4.2. Since, $z_h \in U_h$ is independent of t, we deduce,

$$\begin{aligned}(\psi_h(t) - \psi_{h+}^n, z_h) \leq C_k\Big[&\|\nabla z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n} \|\nabla\psi_h\|_{L^2(\Omega)}\mathrm{d}s + \|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n} \|e_h\|_{L^2(\Omega)}\mathrm{d}s \\ &+ \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\frac{1}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n} \|u\psi_h\|_{L^2(\Omega)}\mathrm{d}s + \frac{1}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n} \|\psi_h\|_{L^2(\Omega)}\mathrm{d}s\Big].\end{aligned}$$

Therefore, setting $z_h = \psi_h(t)$ (for the previously fixed t), integrating with respect to time, and using Hölder's and Young's inequalities, we deduce (with different C_k),

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|\psi_h(t)\|^2_{L^2(\Omega)}\mathrm{d}t &\leq C_k\tau_n\|\psi^n_{h+}\|^2_{L^2(\Omega)} + C_k\tau_n\|\nabla\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} \\ &\quad + C_k\tau_n\|e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|\psi_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]} \\ &\quad + \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\frac{C_k\tau_n}{\epsilon^2}\|u\psi_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|\psi_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]} \\ &\quad + \frac{C_k}{\epsilon^2}\tau_n\|\psi_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|\psi_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}. \\ &\leq C_k\tau_n\|\psi^n_{h+}\|^2_{L^2(\Omega)} + C_k\tau_n(\|\nabla\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}) \\ &\quad + \frac{C_k\tau_n}{\epsilon^2}\|u\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^\infty(\Omega)]}\frac{C_k\tau_n}{\epsilon^2}\|\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} \\ &\quad + \frac{C_k\tau_n}{\epsilon^2}\|\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + C_k\tau_n\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}.\end{aligned}$$

The proof is now completed using standard techniques. We choose τ_n small enough to hide the $L^2[t^{n-1}, t^n; L^2(\Omega)]$-norm on the left, i.e, $\tau_n \leq (\epsilon^2/4C_k\|u\|^2_{L^\infty[0,T;L^\infty(\Omega)]})$, and $\tau_n \leq (\epsilon^2/4C_k)$ to obtain,

$$\begin{aligned}(1/4)\int_{t^{n-1}}^{t^n} \|\psi_h(t)\|^2_{L^2(\Omega)}\mathrm{d}t &\leq C_k\tau_n\|\psi^n_{h+}\|^2_{L^2(\Omega)} + C_k\tau_n\|\nabla\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} \\ &\quad + \frac{C_k\tau_n}{\epsilon^2}\|u\psi_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + C_k\tau_n\|e_h\|^2_{L^2[t^{n-1},t^n;:^2(\Omega)]}.\end{aligned} \tag{5.13}$$

Then we substitute the resulting bound into (5.12), and hide the terms involving $\|u\psi_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ and $\|\nabla\psi_h\|_{L^2[0,T;L^2(\Omega)]}$ on the left. We note that the worst dependence on ϵ, is $\tau_n \leq \frac{\epsilon^2}{C_k\|u\|^2_{L^\infty[0,T;L^\infty(\Omega)]}}$. Summing the resulting inequalities using a standard Gronwall Lemma, we obtain the first estimate. Returning back to (5.13), diving by τ_n and using an inverse in time estimate, $\|\psi\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \leq \frac{C_k}{\tau_n}\int_{t^{n-1}}^{t^n} \|\psi_h(t)\|^2_{L^2(\Omega)}\mathrm{d}t$ we obtain the second estimate.

Step 2. *Stability estimates in* $L^\infty[0,T;H^1(\Omega)]$: The proof is essentially contained in Theorem 4.10 of [13]. For completeness we describe the main arguments. By definition of $\Delta_h\psi_h$, and since $\psi_h \in \mathcal{P}_k[t^{n-1},t^n;U_h]$, we also have that $\Delta_h\psi_h \in \mathcal{P}_k[t^{n-1},t^n;U_h]$. Setting $w_h = \psi_{ht}$, and $w_h = \Delta_h\psi_h$ we deduce,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\psi_h\|^2_{L^2(\Omega)} + \|\psi_h\|^2_{L^2(\Omega)}) = (\Delta_h\psi_h, \psi_{ht}), \text{ and}$$
$$a(\psi_h, \Delta_h\psi_h) + (\psi_h, \Delta_h\psi_h) = \|\Delta_h\psi_h\|^2_{L^2(\Omega)}.$$

Hence, setting $\Delta_h\psi_h$ into (5.10), substituting the last two equalities and using standard algebra we obtain,

$$\begin{aligned}(1/2)\|\psi^{n-1}_{h+}\|^2_{H^1(\Omega)} &+ (1/2)\|[\psi^{n-1}_h]\|^2_{H^1(\Omega)}\\ &+ \int_{t^{n-1}}^{t^n}\|\Delta_h\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t - \int_{t^{n-1}}^{t^n}(\psi_h, \Delta_h\psi_h)\mathrm{d}t\\ &+ \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}(3u^2\psi_h, \Delta_h\psi_h)\mathrm{d}t - \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}(\psi_h, \Delta_h\psi_h)\mathrm{d}t\\ &= (1/2)\|\nabla\psi^n_{h+}\|^2_{H^1(\Omega)} + \int_{t^{n-1}}^{t^n}(e_h, \Delta_h\psi_h)\mathrm{d}t. \end{aligned} \tag{5.14}$$

Note that $\int_{t^{n-1}}^{t^n}(\psi_h, \Delta_h\psi_h)\mathrm{d}t \le \frac{1}{4}\int_{t^{n-1}}^{t^n}\|\Delta_h\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t + \int_{t^{n-1}}^{t^n}\|\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t$, and

$$\begin{aligned}&\Big|\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}(3u^2\psi_h, \Delta_h\psi_h) - (\psi_h, \Delta_h\psi_h)\mathrm{d}t\Big|\\ &\le (1/4)\int_{t^{n-1}}^{t^n}\|\Delta_h\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{C^2_\infty}{\epsilon^4}\int_{t^{n-1}}^{t^n}\|u\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{\epsilon^4}\int_{t^{n-1}}^{t^n}\|\psi_h\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}$$

Substituting the above inequality into (5.14) and summing the resulting inequalities, and using the bounds $\|u\psi_h\|_{L^2[0,T;L^2(\Omega)} \le C\|e_h\|_{L^2[0,T;L^2(\Omega)]}$ and $\|\psi_h\|_{L^2[0,T;L^2(\Omega)]} \le C\|e_h\|_{L^2[0,T;L^2(\Omega)]}$, we deduce that,

$$\|\psi^{n-1}_{h+}\|^2_{H^1(\Omega)} + \|\Delta_h\psi_h\|^2_{L^2[0,T;L^2(\Omega)]} \le \frac{C^2_\infty + 1}{\epsilon^4}\|e_h\|^2_{L^2[0,T;L^2(\Omega)]},$$

which is the desired estimate. The stability bound in $L^\infty[0,T;H^1(\Omega)]$ follows directly from the above technique when $k = 0, 1$. For higher order schemes we refer the reader to Theorem 4.10 of [13]. □

Now, we are ready to prove the following bound, which will allow us to apply a bootstrap argument. Using an appropriate duality argument, we avoid the use of Gronwall type inequalities. We note that the temporal/spacial restriction in terms of ϵ is stated in terms of the available regularity. Applying the results of Lemma 2.1 and Remark 2.2 we can quantify this dependence only upon data.

Proposition 5.5. *Let* τ, h, ϵ, *satisfy* $\tau \le \tilde{\tau}$, $h \le \tilde{h}$ *(where* $\tilde{\tau}, \tilde{h}$ *defined in (4.6)) and let the assumptions of Lemma 5.4 hold. In addition, suppose that*

- *if* $u \in L^4[0,T;H^2(\Omega)]$, $u_t \in L^4[0,T;L^2(\Omega)]$ *then* τ, h *satisfy*
  (1) $\ln(\frac{T}{\tau})(\tau + h^2) \le \frac{\delta C\epsilon^4}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 3$,
  (2) $\ln(\frac{T}{\tau})(\tau + h^2) \le \frac{\delta C\epsilon^{7/2}}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 2$,
  (3) $\ln(\frac{T}{\tau})(\tau + h^2) \le \frac{\delta C\epsilon^3}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 2$, $k = 0, 1$,

or

- *if* $u \in L^2[0,T;H^2(\Omega)]$, $u_t \in L^2[0,T;L^2(\Omega)]$, *then* τ, h *satisfy,*

(1) $(\tau^{1/2}+h)^{3/2} \leq \frac{\delta C\epsilon^4}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d=3$,

(2) $(\tau^{1/2}+h)^{3/2} \leq \frac{\delta C\epsilon^{7/2}}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d=2$,

(3) $(\tau^{1/2}+h)^{3/2} \leq \frac{\delta \epsilon^3}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d=2$, $k=0,1$,

where $\delta>0$ an algebraic constant (to be chosen later) with constant C depending only upon the domain (independent of ϵ, h, τ). Then, there exists a constant $C>0$ independent of τ, h, ϵ, such that the following estimate holds:

$$\begin{aligned}\|e_h\|_{L^2[0,T;L^2(\Omega)]} \leq C\Big(&\frac{1}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]}+\|u\|^2_{L^\infty[0,T;L^6(\Omega)]})\|e_p\|_{L^2[0,T;H^1(\Omega)]}\\ &+\frac{1}{\epsilon^2}\|e_p\|_{L^2[0,T;L^2(\Omega)]}+\|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\\ &+C\delta(\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]}+\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]})\Big).\end{aligned}$$

Proof. Setting $w_h=e_h$ into (5.10), and using integration by parts in time, we obtain: for all $n=N,\dots,1$,

$$\begin{aligned}&-(\psi^n_{h+},e^n_{h-})+(\psi^n_{h-},e^n_{h-})+\int_{t^{n-1}}^{t^n}-(\psi_{ht},e_h)\mathrm{d}t+\int_{t^{n-1}}^{t^n}a(\psi_h,e_h)\mathrm{d}t\\ &+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\big((3u^2-1)\psi_h,e_h\big)\,\mathrm{d}t=\int_{t^{n-1}}^{t^n}\|e_h\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}\tag{5.15}$$

Setting $w_h=\psi_h$ into (5.8), we deduce for all $n=1,\dots,N$,

$$\begin{aligned}(e^n_{h-},\psi^n_{h-})+&\int_{t^{n-1}}^{t^n}(-(e_h,\psi_{ht})+a(e_h,\psi_h))\,\mathrm{d}t-(e^{n-1}_{h-},\psi^{n-1}_{h+})\\ &+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\big((e_h(u_h^2+u_p^2+u_hu_p),\psi_h)-(e_h,\psi_h)\big)\,\mathrm{d}t\\ &+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\big((e_p(u_p^2+u^2+u_pu),\psi_h)-(e_p,\psi_h)\big)\,\mathrm{d}t=0.\end{aligned}\tag{5.16}$$

Subtracting (5.16) from (5.15), and rearranging terms, we obtain, for each $n=1,\dots,N$,

$$\begin{aligned}\int_{t^{n-1}}^{t^n}\|e_h\|^2_{L^2(\Omega)}\mathrm{d}t=&-(\psi^n_{h+},e^n_{h-})+(e^{n-1}_{h-},\psi^{n-1}_{h+})\\ &-\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\big((e_p(u_p^2+u^2+u_pu),\psi_h)-(e_p,\psi_h)\big)\,\mathrm{d}t\\ &-\frac{1}{\epsilon^2}\int_{t_{n-1}}^{t_n}\big((u_h^2+u_p^2+u_hu_p-3u^2)e_h,\psi_h\big)\,\mathrm{d}t,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\int_{t^{n-1}}^{t^n}\|e_h\|^2_{L^2(\Omega)}\mathrm{d}t=&-(\psi^n_{h+},e^n_{h-})+(e^{n-1}_{h-},\psi^{n-1}_{h+})\\ &-\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\big((e_p(u_p^2+u^2+u_pu),\psi_h)-(e_p,\psi_h)\big)\,\mathrm{d}t\\ &-\frac{1}{\epsilon^2}\int_{t_{n-1}}^{t_n}\big(((u_h^2-u^2)+(u_p^2-u^2)+u_hu_p-u^2)e_h,\psi_h\big)\,\mathrm{d}t.\end{aligned}\tag{5.17}$$

First, note adding and subtracting u_p^2 in the term $u_h^2 - u^2$, using the relation,

$$u_h u_p - u^2 = (u_h - u_p + u_p)u_p - u^2 \equiv (u_h - u_p)u_p + u_p^2 - u^2$$

and substituting the resulting relation into (5.17) we arrive at:

$$\begin{aligned}\int_{t^{n-1}}^{t^n} \|e_h\|^2_{L^2(\Omega)} \mathrm{d}t = &-(\psi^n_{h+}, e^n_{h-}) + (e^{n-1}_{h-}, \psi^{n-1}_{h+})\\ &- \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \big((e_p(u_p^2 + u^2 + u_p u), \psi_h) - (e_p, \psi_h)\big)\,\mathrm{d}t\\ &- \frac{1}{\epsilon^2}\int_{t_{n-1}}^{t_n} \big(\big((u_h^2 - u_p^2) + 3(u_p^2 - u^2) + (u_h - u_p)u_p\big)\, e_h, \psi_h\big)\,\mathrm{d}t. \end{aligned} \tag{5.18}$$

Summing the equalities (5.18), noting that $e^0_{h-} = 0 = \phi^N_{h+}$, and using Hölder's and Young's inequalities, and the identity $a^2 - b^2 = (a-b)(a+b)$, we obtain,

$$\begin{aligned}\int_0^T \|e_h\|^2_{L^2(\Omega)} \mathrm{d}t \leq &\frac{C}{\epsilon^2}\int_0^T \|e_p\|_{L^6(\Omega)}(\|u_p^2\|_{L^3(\Omega)} + \|u^2\|_{L^3(\Omega)})\|\psi_h\|_{L^2(\Omega)}\mathrm{d}t\\ &+ \frac{1}{\epsilon^2}\int_0^T \|e_p\|_{L^2(\Omega)}\|\psi_h\|_{L^2(\Omega)}\mathrm{d}t\\ &+ \frac{2}{\epsilon^2}\int_0^T (\|e_h u_h\|_{L^2(\Omega)} + \|e_h u_p\|_{L^2(\Omega)})\|e_h\|_{L^2(\Omega)}\|\psi_h\|_{L^\infty(\Omega)}\mathrm{d}t\\ &+ \frac{3}{\epsilon^2}\int_0^T \|e_p\|_{L^2(\Omega)}\|e_h u_p\|_{L^2(\Omega)}\|\psi_h\|_{L^\infty(\Omega)}\mathrm{d}t\\ &+ \frac{3}{\epsilon^2}\int_0^T \|u\|_{L^\infty(\Omega)}\|e_p\|_{L^2(\Omega)}\|e_h\|_{L^2(\Omega)}\|\psi_h\|_{L^\infty(\Omega)}\mathrm{d}t. \end{aligned} \tag{5.19}$$

For $d = 3$, we employ the inequality $\|\psi_h\|_{L^\infty(\Omega)} \leq C\|\nabla\psi_h\|^{1/2}_{L^2(\Omega)}\|\Delta_h\psi_h\|^{1/2}_{L^2(\Omega)}$ (which can be proved in an identical way as in [29], p. 298) to get

$$\begin{aligned}\int_0^T \|e_h\|^2_{L^2(\Omega)} \mathrm{d}t \leq &\frac{C}{\epsilon^2}\int_0^T \|e_p\|_{L^6(\Omega)}(\|u_p^2\|_{L^3(\Omega)} + \|u^2\|_{L^3(\Omega)})\|\psi_h\|_{L^2(\Omega)}\mathrm{d}t\\ &+ \frac{1}{\epsilon^2}\int_0^T \|e_p\|_{L^2(\Omega)}\|\psi_h\|_{L^2(\Omega)}\mathrm{d}t\\ &+ \frac{2}{\epsilon^2}\|\psi_h\|^{1/2}_{L^\infty[0,T;H^1(\Omega)]}\int_0^T (\|e_h u_h\|_{L^2(\Omega)} + \|e_h u_p\|_{L^2(\Omega)})\|e_h\|_{L^2(\Omega)}\|\Delta_h\psi_h\|^{1/2}_{L^2(\Omega)}\mathrm{d}t\\ &+ \frac{3}{\epsilon^2}\|\psi_h\|^{1/2}_{L^\infty[0,T;H^1(\Omega)]}\int_0^T \|e_p\|_{L^2(\Omega)}\|e_h u_p\|_{L^2(\Omega)}\|\Delta_h\psi_h\|^{1/2}_{L^2(\Omega)}\mathrm{d}t\\ &+ \frac{3}{\epsilon^2}\|\psi_h\|^{1/2}_{L^\infty[0,T;H^1(\Omega)]}\int_0^T \|u\|_{L^\infty(\Omega)}\|e_p\|_{L^2(\Omega)}\|e_h\|_{L^2(\Omega)}\|\Delta_h\psi_h\|^{1/2}_{L^2(\Omega)}\mathrm{d}t. \end{aligned}$$

Therefore, using the stability bounds of ψ_h of Lemma 5.4, *i.e.*, $\|\psi_h\|_{L^2[0,T;L^2(\Omega)]} \le \|e_h\|_{L^2[0,T;L^2(\Omega)]}$, $\|\psi_h\|_{L^\infty[0,T;L^6(\Omega)]} \le \frac{C}{\epsilon^2}\|e_h\|_{L^2[0,T;L^2(\Omega)]}$, and $\|\Delta_h\psi_h\|_{L^2[0,T;L^2(\Omega)]} \le \frac{C}{\epsilon^2}\|e_h\|_{L^2[0,T;L^2(\Omega)]}$ to deduce,

$$\begin{aligned}\int_0^T \|e_h\|_{L^2(\Omega)}^2 \mathrm{d}t \le{}& \frac{C}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})\|e_p\|_{L^2[0,T;L^6(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{1}{\epsilon^2}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{C}{\epsilon^4}(\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]})\|e_h\|_{L^4[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{C}{\epsilon^4}\|e_p\|_{L^4[0,T;L^2(\Omega)]}\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{C}{\epsilon^4}\|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^4[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}.\end{aligned}$$

Note due to the Theorem 4.6 there exists $\tilde{\tau}$, $\tilde{h}$ such that $\|e\|_{L^4[0,T;L^2(\Omega)]} \le \delta\epsilon^4$ for every $\tau \le \tilde{\tau}$ and $h \le \tilde{h}$. Hence, using (5.4), and the improved estimate in $L^4[0,T;L^2(\Omega)]$, (see *e.g.*, [34]), we obtain that

$$\begin{aligned}\|e_h\|_{L^4[0,T;L^2(\Omega)]} &\le \delta\epsilon^4 + \|e_p\|_{L^4[0,T;L^2(\Omega)]}.\\ &\le \delta\epsilon^4 + C\ln\left(\frac{T}{\tau}\right)\left(\tau+h^2\right)\left(\|u\|_{L^4[0,T;H^2(\Omega)]} + \|u_t\|_{L^4[0,T;L^2(\Omega)]}\right) \le 2\delta\epsilon^4,\end{aligned}$$

provided that $\ln\left(\frac{T}{\tau}\right)\left(\tau+h^2\right) \le \frac{\delta C\epsilon^4}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$. Substituting the above bound, we deduce,

$$\begin{aligned}\int_0^T \|e_h\|_{L^2(\Omega)}^2 \mathrm{d}t \le{}& \frac{C}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})\|e_p\|_{L^2[0,T;L^6(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{1}{\epsilon^2}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \delta(\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]})\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}.\end{aligned}$$

The estimate for the three dimensional case now follows by standard algebra. For $d = 2$, we note that $\|\psi_h\|_{L^\infty(\Omega)} \le C\|\psi_h\|^{1/2}_{L^2(\Omega)}\|\Delta_h\psi_h\|^{1/2}_{L^2(\Omega)}$ (see [29]), hence using the stability bounds of Lemma 5.4, and in particular the fact that $\|\psi_h\|_{L^\infty[0,T;L^2(\Omega)]} \le \frac{C}{\epsilon}\|e_h\|_{L^2[0,T;L^2(\Omega)]}$, we deduce from (5.19),

$$\begin{aligned}\int_0^T \|e_h\|_{L^2(\Omega)}^2 \mathrm{d}t \le{}& \frac{C}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})\|e_p\|_{L^2[0,T;L^6(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{1}{\epsilon^2}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{C}{\epsilon^{7/2}}(\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|_{L^2[0,T;L^2(\Omega)]})\|e_h\|_{L^4[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{C}{\epsilon^{7/2}}\|e_p\|_{L^4[0,T;L^2(\Omega)]}\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\ &+ \frac{C}{\epsilon^{7/2}}\|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^4[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}.\end{aligned}$$

The proof now follows using similar arguments. Indeed, choosing $\tilde{\tau}, \tilde{h}$ to guarantee, $\|e\|_{L^4[0,T;L^2(\Omega)]} \le \delta\epsilon^{7/2}$, for $\tau \le \tilde{\tau}$, $h \le \tilde{h}$, and noting that

$$\|e_p\|_{L^4[0,T;L^2(\Omega)]} \le C\ln\left(\frac{T}{\tau}\right)\left(\tau+h^2\right)\left(\|u\|_{L^4[0,T;H^2(\Omega)]} + \|u_t\|_{L^4[0,T;L^2(\Omega)]}\right) \le C\delta\epsilon^{7/2}$$

provided that $\ln\left(\frac{T}{\tau}\right)\left(\tau+h^2\right) \leq \frac{C\delta\epsilon^{7/2}}{\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]}}$, we derive the desired estimate. Finally, we turn our attention to the case where $k=0,1$ and $d=2$. Then, we note that Lemma 5.4, implies that $\|\psi_h\|_{L^\infty[0,T;L^2(\Omega)]} \leq C$, where C is independent of ϵ, τ, h. As a consequense, we deduce from (5.19),

$$\begin{aligned}
\int_0^T \|e_h\|^2_{L^2(\Omega)}\mathrm{d}t \leq{}& \frac{C}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]}+\|u\|^2_{L^\infty[0,T;L^6(\Omega)]})\|e_p\|_{L^2[0,T;L^6(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\
&+\frac{1}{\epsilon^2}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\
&+\frac{C}{\epsilon^3}(\|e_hu_h\|_{L^2[0,T;L^2(\Omega)]}+\|e_hu_p\|_{L^2[0,T;L^2(\Omega)]})\|e_h\|_{L^4[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\
&+\frac{C}{\epsilon^3}\|e_p\|_{L^4[0,T;L^2(\Omega)]}\|e_hu_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}\\
&+\frac{C}{\epsilon^3}\|u\|_{L^\infty[0,T;L^\infty(\Omega)]}\|e_p\|_{L^2[0,T;L^2(\Omega)]}\|e_h\|_{L^4[0,T;L^2(\Omega)]}\|e_h\|_{L^2[0,T;L^2(\Omega)]}.
\end{aligned}$$

Therefore, we derive the desired estimate, provided that $\tilde{\tau}, \tilde{h}$ are chosen to guarantee, $\|e\|_{L^4[0,T;L^2(\Omega)]} \leq \delta\epsilon^3$, for $\tau \leq \tilde{\tau}$, $h \leq \tilde{h}$, and similarly $\ln\left(\frac{T}{\tau}\right)\left(\tau+h^2\right) \leq \frac{C\delta\epsilon^3}{\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]}}$.

Finally if less regularity is available, *i.e.*, $u \in L^2[0,T;H^2(\Omega)]$, $u_t \in L^2[0,T;L^2(\Omega)]$ then using the bound $\|e_p\|_{L^4[0,T;L^2(\Omega)]} \leq C\|e_p\|^{1/2}_{L^\infty[0,T;L^2(\Omega)]}\|e_p\|^{1/2}_{L^2[0,T;L^2(\Omega)]}$, we easily deduce the restrictions,

(1) $(\tau^{1/2}+h)^{3/2} \leq \frac{\delta C\epsilon^4}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d=3$,
(2) $(\tau^{1/2}+h)^{3/2} \leq \frac{\delta C\epsilon^{7/2}}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d=2$.
(3) $(\tau^{1/2}+h)^{3/2} \leq \frac{\delta\epsilon^3}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d=2$, $k=0,1$,

which completes the proof. □

Remark 5.6. There are many ways to further quantify the restriction between the temporal and the spatial discretiation parameters. If we apply the results of Lemma 2.1 and Remark 2.2, with $\sigma=0$, and $\tilde{\sigma}=1$, we deduce that

$$\|u\|_{L^4[0,T;H^2(\Omega)]} \leq C\|u\|^{1/2}_{L^\infty[0,T;H^2(\Omega)]}\|u\|^{1/2}_{L^2[0,T;H^2(\Omega)]} \leq C/\epsilon.$$

Therefore, the temporal and spatial parameter discretization restriction can be expressed as,

(1) $\ln(T/\tau)(\tau+h^2) \leq \delta C\epsilon^5$, when $d=3$,
(2) $\ln(T/\tau)(\tau+h^2) \leq \delta C\epsilon^{9/2}$, when $d=2$,
(3) $\ln(T/\tau)(\tau+h^2) \leq \delta C\epsilon^4$, when $d=2$, $k=0,1$.

In a similar way, when $u \in L^2[0,T;H^2(\Omega)]$, $u_t \in L^2[0,T;L^2(\Omega)]$ then using the results of Lemma 2.1, with $\sigma=0$, we deduce,

(1) $(\tau^{1/2}+h)^{3/2} \leq \delta C\epsilon^5$, when $d=3$,
(2) $(\tau^{1/2}+h)^{3/2} \leq \delta C\epsilon^{9/2}$, when $d=2$.
(3) $(\tau^{1/2}+h)^{3/2} \leq \delta C\epsilon^4$, when $d=2$, $k=0,1$.

where $\delta > 0$ is a positive algebraic constant independent of τ, h, ϵ (to be chosen later).

We note also that the assumptions of Proposition 5.5 can be replaced by the more general assumption $\|e_p\|_{L^4[0,T;L^2(\Omega)]} \leq C\delta\epsilon^4$. Since $e_p = u_p - u$ refers to the standard error related to discontinuous Galerkin approximation of a linear parabolic pde, with right hand side $u_t - \Delta u$. Therefore, from (5.4) and Lemma 5.2, for instance, we may derive the following restriction when $d = 3$

$$\|e_p\|_{L^4[0,T;L^2(\Omega)]} \leq C\left(\frac{\tau^{k+1}}{h}\|u^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]} + h^l\|u\|_{L^2[0,T;H^{l+1}(\Omega)]}\right)^{1/2} \times\left(\tau^{k+1}\|u^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]} + h^{l+1}\|u\|_{L^2[0,T;H^{l+1}(\Omega)]}\right)^{1/2} \leq C\delta\epsilon^4.$$

5.2. Best approximation error estimates

Now, we are ready to proceed with the main estimate, using a boot-strap argument.

Theorem 5.7. *Let τ, h, ϵ, satisfy $\tau \leq \tilde{\tau}$, $h \leq \tilde{h}$ (where $\tilde{\tau}, \tilde{h}$ defined in (4.6)) and let the assumptions of Lemma 5.4 hold. In addition, suppose that*

- *if $u \in L^4[0,T;H^2(\Omega)]$, $u_t \in L^4[0,T;L^2(\Omega)]$ then τ, h satisfy*
  (1) $\ln(\frac{T}{\tau})(\tau + h^2) \leq \frac{C\epsilon^4}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 3$,
  (2) $\ln(\frac{T}{\tau})(\tau + h^2) \leq \frac{C\epsilon^{7/2}}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 2$,
  (3) $\ln(\frac{T}{\tau})(\tau + h^2) \leq \frac{C\epsilon^3}{(\|u\|_{L^4[0,T;H^2(\Omega)]}+\|u_t\|_{L^4[0,T;L^2(\Omega)]})}$, when $d = 2$, $k = 0, 1$,

or

- *if $u \in L^2[0,T;H^2(\Omega)]$, $u_t \in L^2[0,T;L^2(\Omega)]$, then τ, h satisfy,*
  (1) $(\tau^{1/2} + h)^{3/2} \leq \frac{C\epsilon^4}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d = 3$,
  (2) $(\tau^{1/2} + h)^{3/2} \leq \frac{C\epsilon^{7/2}}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d = 2$,
  (3) $(\tau^{1/2} + h)^{3/2} \leq \frac{C\epsilon^3}{(\|u\|_{L^2[0,T;H^2(\Omega)]}+\|u_t\|_{L^2[0,T;L^2(\Omega)]})}$, when $d = 2$, $k = 0, 1$,

where C depending only upon the domain (independent of ϵ, h, τ). Then, there exists a constant (still) denoted by C depending only upon Ω, and C_k but independent of τ, h, ϵ, such that, for all $n = 1, \ldots, N$,

$$\|e_h\|^2_{L^2[0,T;H^1(\Omega)]} + (1/\epsilon^2)(\|e_h u_h\|^2_{L^2[0,T;L^2(\Omega)]} + \|e_h u_p\|^2_{L^2[0,T;L^2(\Omega)]}) + \|e^n_{h-}\|^2_{L^2(\Omega)} + (1/\epsilon^2)\|e_h\|^4_{L^4[0,T;L^4(\Omega)]} + \sum_{i=1}^{N-1}\|[e^i_h]\|^2_{L^2(\Omega)} \leq C(1/\epsilon^6)\Big((\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2\|e_p\|^2_{L^2[0,T;H^1(\Omega)]} + \|e_p\|^2_{L^2[0,T;L^2(\Omega)]}\Big).$$

Suppose also that (2.2) holds, with $\sigma = 0$ when $k \geq 1$. Then, there exists a constant C depending only upon Ω, and C_k such that

$$\|e_h\|^2_{L^\infty[0,T;L^2(\Omega)]} \leq C(1/\epsilon^6)\Big((\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2\|e_p\|^2_{L^2[0,T;H^1(\Omega)]} + \|e_p\|^2_{L^2[0,T;L^2(\Omega)]}\Big).$$

Proof. Step 1: Estimate at partition points and in $L^2[0,T;H^1(\Omega)]$. Since, we have already obtained a bound on $\|e_h\|_{L^2[0,T;L^2(\Omega)]}$ with constant depending polynomially upon $1/\epsilon$, we may return to the orthogonality condition

(5.8) and set $w_h = e_h$. Then, for every $n = 1, \ldots, N$, we have:

$$
\begin{aligned}
\frac{1}{2}\|e^n_{h-}\|^2_{L^2(\Omega)} &+ \int_{t^{n-1}}^{t^n} \|\nabla e_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{2}\|[e_h^{n-1}]\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2\epsilon^2}\int_{t^{n-1}}^{t^n} (\|e_h u_h\|^2_{L^2(\Omega)} + \|e_h u_p\|^2_{L^2(\Omega)})\mathrm{d}t + \|[e_h^{n-1}]\|^2_{L^2(\Omega)} \\
&\le \frac{1}{2}\|e^{n-1}_{h-}\|^2_{L^2(\Omega)} + \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \|e_h\|^2_{L^2(\Omega)}\mathrm{d}t \\
&\quad + \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \left(|(e_p(u_p^2 + u^2 + u_p u), e_h)| + |(e_p, e_h)|\right)\mathrm{d}t. \qquad (5.20)
\end{aligned}
$$

It remains to bound the last two terms: First, we note that Hölder's and Young's inequalities imply

$$
\begin{aligned}
&\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} |(e_p(u_p^2 + u^2 + u_p u), e_h)|\mathrm{d}t \\
&\quad \le \frac{C}{\epsilon^2}\int_{t^{n-1}}^{t^n} \|e_p\|_{L^6(\Omega)}(\|u_p\|^2_{L^6(\Omega)} + \|u\|^2_{L^6(\Omega)})\|e_h\|_{L^2(\Omega)}\mathrm{d}t \\
&\quad \le \frac{C}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2\int_{t^{n-1}}^{t^n} \|e_p\|^2_{H^1(\Omega)}\mathrm{d}t + \frac{C}{\epsilon^2}\int_{t^{n-1}}^{t^n} \|e_h\|^2_{L^2(\Omega)}\mathrm{d}t.
\end{aligned}
$$

Substituting the last inequality into (5.20) and summing the resulting inequalities we obtain

$$
\begin{aligned}
\frac{1}{2}\|e^N_{h-}\|^2_{L^2(\Omega)} &+ \int_0^T \|\nabla e_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{2}\sum_{i=1}^N \|[e_h^{i-1}]\|^2_{L^2(\Omega)} \\
&\quad + (1/2\epsilon^2)\int_0^T (\|e_h u_h\|^2_{L^2(\Omega)} + \|e_h u_p\|^2_{L^2(\Omega)})\mathrm{d}t \\
&\le \frac{C}{\epsilon^2}\int_0^T \|e_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{C}{\epsilon^2}\int_0^T \|e_p\|^2_{L^2(\Omega}\mathrm{d}t \\
&\quad + \frac{C}{\epsilon^2}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2\int_0^T \|e_p\|^2_{H^1(\Omega)}\mathrm{d}t
\end{aligned}
$$

It remains to replace the term $(1/\epsilon^2)\int_0^T \|e_h\|^2_{L^2(\Omega)}\mathrm{d}t$ using Proposition 5.5. First, note that the bound of Proposition 5.5 implies that:

$$
\begin{aligned}
\frac{1}{2}\|e^N_{h-}\|^2_{L^2(\Omega)} &+ \int_0^T \|\nabla e_h\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{1}{2}\sum_{i=1}^N \|[e_h^{i-1}]\|^2_{L^2(\Omega)} \\
&\quad + (1/2\epsilon^2)\int_0^T (\|e_h u_h\|^2_{L^2(\Omega)} + \|e_h u_p\|^2_{L^2(\Omega)})\mathrm{d}t \\
&\le \frac{C}{\epsilon^6}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2\int_0^T \|e_p\|^2_{H^1(\Omega)}\mathrm{d}t \\
&\quad + \frac{\delta C}{\epsilon^2}\int_0^T (\|e_h u_h\|^2_{L^2(\Omega)} + \|e_h u_p\|^2_{L^2(\Omega)})\mathrm{d}t \\
&\quad + C\left(\frac{1}{\epsilon^6} + \frac{1}{\epsilon^2}\|u\|^2_{L^\infty[0,T;L^\infty(\Omega)]}\right)\int_0^T \|e_p\|^2_{L^2(\Omega)}\mathrm{d}t.
\end{aligned}
$$

Now, noting that we may choose δ aiming to hide the $\|e_h u_h\|_{L^2[0,T;L^2(\Omega)]}$, and $\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}$ on the left hand-side to get the first estimate. Note that the worst dependence upon $1/\epsilon$ in front of $\|e_p\|^2_{L^2[0,T;L^2(\Omega)]}$ is C/ϵ^6, since $C\left(\frac{1}{\epsilon^6}+\frac{1}{\epsilon^2}\|u\|^2_{L^\infty[0,T;L^\infty(\Omega)]}\right)\le C/\epsilon^6$. It is clear that the bounds on $\|e_u u_h\|_{L^2[0,T;L^2(\Omega)}$ and on $\|e_h u_p\|_{L^2[0,T;L^2(\Omega)]}$ imply a similar estimate for $\|e_h\|_{L^4[0,T;L^4(\Omega)]}$, since

$$(1/\epsilon^2)\int_0^T \|e_h\|^4_{L^4(\Omega)}\mathrm{d}t \le (2/\epsilon^2)\int_0^T\int_\Omega |e_h|^2(|u_h|^2+|u_p|^2)\mathrm{d}x\mathrm{d}t$$
$$\le (2/\epsilon^2)\int_0^T (\|e_h u_h\|^2_{L^2(\Omega)}+\|e_h u_p\|^2_{L^2(\Omega)})\mathrm{d}t.$$

The estimate at partition points follows in standard way, summing the equations (5.20) from $i=1$ to $i=n$, and using the previous bounds.

Step 2: Estimates at arbitrary time points. We use similar ideas as in the proof of Proposition 4.3. For fixed $t\in[t^{n-1},t^n)$ and $z_h\in U_h$ we set $w_h(s)=z_h\rho(s)$ into (5.8), with $\rho(s)\in\mathcal{P}_k[t^{n-1},t^n]$ such that

$$\rho(t^{n-1})=1,\qquad \int_{t^{n-1}}^{t^n}\rho q=\int_{t^{n-1}}^{t} q,\qquad q\in\mathcal{P}_{k-1}[t^{n-1},t^n].$$

From Lemma 4.2 we deduce that $\|\rho\|_{L^\infty}\le C_k$, with C_k independent of t, and

$$\int_{t^{n-1}}^{t^n}\langle e_{ht},w_h\rangle\mathrm{d}s+(e^{n-1}_{h+}-e^{n-1}_{h-},w^{n-1}_{h+})$$
$$=\int_{t^{n-1}}^{t}\langle e_{ht},z_h\rangle\mathrm{d}s+(e^{n-1}_{h+}-e^{n-1}_{h-},\rho(t^{n-1})z_h)=(e_h(t)-e^{n-1}_{h-},z_h).$$

Therefore, integrating by parts (in time), (5.8), setting $w_h(s)=z_h\rho(s)$, using the above equality, and standard algebra, we obtain:

$$\begin{aligned}(e_h(t)-e^{n-1}_{h-},z_h)\le C_k\Big[&\int_{t^{n-1}}^{t^n}\int_\Omega|\nabla e_h||\nabla z_h|\mathrm{d}x\mathrm{d}s\\&+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\int_\Omega\left(|e_h|(|u_h|^2+|u_p|^2)|z_h|+|e_h||z_h|\right)\mathrm{d}x\mathrm{d}s\\&+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\int_\Omega\left(|e_p|(|u_p|^2+|u|^2)|z_h|+|e_p||z_h|\right)\mathrm{d}x\mathrm{d}s\Big].\end{aligned}\tag{5.21}$$

Adding and subtracting u_p,u, and using standard algebra, we may bound

$$\int_{t^{n-1}}^{t^n}\int_\Omega|e_h|(|u_h|^2+|u_p|^2)|z_h|\mathrm{d}x\mathrm{d}s\le C\int_{t^{n-1}}^{t^n}\int_\Omega(|e_h|^3+|e_h||u_p-u|^2+|e_h||u|^2)|z_h|\mathrm{d}x\mathrm{d}s.$$

Hence, using Hölder's inequality into (5.21) we derive

$$\begin{aligned}\langle e_h(t)-e^{n-1}_{h-},z_h\rangle\le C_k\Big[&\int_{t^{n-1}}^{t^n}\|\nabla e_h\|_{L^2(\Omega)}\|\nabla z_h\|_{L^2(\Omega)}+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\|e_h\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)}\mathrm{d}s\\&+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\left(\|e_h\|^3_{L^4(\Omega)}\|z_h\|_{L^4(\Omega)}+\|e_h\|_{L^4(\Omega)}\|e_p^2\|_{L^2(\Omega)}\|z_h\|_{L^4(\Omega)}\right)\mathrm{d}t\\&+\frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n}\|e_h\|_{L^6(\Omega)}\|u^2\|_{L^3(\Omega)}\|z_h\|_{L^2(\Omega)}\mathrm{d}s\end{aligned}$$

$$+ \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \|e_p\|_{L^6(\Omega)}\|u_p^2 + u^2 + u_p u\|_{L^3(\Omega)}\|z_h\|_{L^2(\Omega)}\mathrm{d}s$$
$$+ \frac{1}{\epsilon^2}\int_{t^{n-1}}^{t^n} \|e_p\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)}\mathrm{d}s\Big]. \tag{5.22}$$

Noting that z_h is independent of t, and standard algebra implies that

$$\langle e_h(t) - e_{h-}^{n-1}, z_h\rangle \le C_k\Big[\|\nabla z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|\nabla e_h\|_{L^2(\Omega)}\mathrm{d}s + \frac{1}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|e_h\|_{L^2(\Omega)}\mathrm{d}s$$
$$+ \frac{1}{\epsilon^2}\|z_h\|_{L^4(\Omega)}\int_{t^{n-1}}^{t^n}\|e_h\|^3_{L^4(\Omega)}\mathrm{d}s + \frac{1}{\epsilon^2}\|z_h\|_{L^4(\Omega)}\int_{t^{n-1}}^{t^n}\|e_h\|_{L^4(\Omega)}\|e_p\|^2_{L^4(\Omega)}\mathrm{d}s$$
$$+ \frac{1}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|e_h\|_{L^6(\Omega)}\|u\|^2_{L^6(\Omega)}\mathrm{d}s$$
$$+ \frac{1}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|e_p\|_{L^6(\Omega)}(\|u_p\|^2_{L^6(\Omega)} + \|u\|^2_{L^6(\Omega)})\mathrm{d}s$$
$$+ \frac{1}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\int_{t^{n-1}}^{t^n}\|e_p\|_{L^2(\Omega)}\mathrm{d}s\Big]. \tag{5.23}$$

Using once more Hölder's inequality and the fact that $u, u_p \in L^\infty[0,T;H^1(\Omega)]$, we deduce with different constant C_k (independent of ϵ):

$$\langle e_h(t) - e_{h-}^{n-1}, z_h\rangle \le C_k\Big[\|\nabla z_h\|_{L^2(\Omega)}\tau_n^{1/2}\|\nabla e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}$$
$$+ \frac{\tau_n^{1/2}}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\|e_h\|_{L^2[t^{n-1};t^n;L^2(\Omega)]} + \frac{\tau_n^{1/4}}{\epsilon^2}\|z_h\|_{L^4(\Omega)}\|e_h\|^3_{L^4[t^{n-1},t^n;L^4(\Omega)]}$$
$$+ \frac{\tau_n^{1/4}}{\epsilon^2}\|z_h\|_{L^4(\Omega)}\|e_h\|_{L^4[t^{n-1},t^n;L^4(\Omega)]}\|e_p\|^2_{L^4[0,T;L^4(\Omega)]}$$
$$+ \frac{\tau_n^{1/2}}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\|e_h\|_{L^2[t^{n-1},t^n;L^6(\Omega)]}\|u\|^2_{L^\infty[0,T;L^6(\Omega)]}$$
$$+ \frac{\tau_n^{1/2}}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\|e_p\|_{L^2[t^{n-1};t^n;H^1(\Omega)]}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})$$
$$+ \frac{\tau_n^{1/2}}{\epsilon^2}\|z_h\|_{L^2(\Omega)}\|e_p\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big].$$

Setting $z_h = e_h(t)$ and integrating with respect to time, using Hölder's inequallity to bound $\int_{t^{n-1}}^{t^n}\|e_h(t)\|_{L^4(\Omega)}\mathrm{d}t \le \tau^{3/4}\|e_h\|_{L^4[t^{n-1},t^n;L^4(\Omega)]}$, and standard calculations, we derive,

$$\int_{t^{n-1}}^{t^n}\|e_h(t)\|^2_{L^2(\Omega)}\mathrm{d}t \le \|e_{h-}^{n-1}\|_{L^2(\Omega)}\tau_n^{1/2}\|e_h(t)\|_{L^2[t^{n-1},t^n;L^2(\Omega)]} \tag{5.24}$$
$$+ C_k\Big[\tau_n\|\nabla e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]} + \frac{\tau_n}{\epsilon^2}\|e_h\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]}$$
$$+ \frac{\tau_n}{\epsilon^2}\|e_h\|^2_{L^4[t^{n-1};t^n;L^4(\Omega)]}\|e_p\|^2_{L^4[t^{n-1},t^n;L^4(\Omega)]}$$
$$+ \frac{\tau_n}{\epsilon^2}\|e_h\|_{L^2[t^{n-1};t^n;L^2(\Omega)]}\|e_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}\|u\|^2_{L^\infty[0,T;L^6(\Omega)]}$$
$$+ \frac{\tau_n}{\epsilon^2}\|e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|e_p\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]})$$
$$+ \frac{\tau_n}{\epsilon^2}\|e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|e_p\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\Big].$$

For the first term of the left hand side, using Young's inequality, we obtain:

$$\begin{aligned}&\|e_{h-}^{n-1}\|_{L^2(\Omega)}\tau_n^{1/2}\|e_h(t)\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\\&\quad\le \frac{1}{4}\|e_h(t)\|^2_{L^2[t^{n-1},t^n;lL^2(\Omega)]}+C\tau_n\|e_{h+}^{n-1}\|^2_{L^2(\Omega)}.\end{aligned}$$

For the fourth term, we note that using Young's inequality, we obtain

$$\begin{aligned}&\frac{\tau_n}{\epsilon^2}\|e_h\|^2_{L^4[t^{n-1},t^n;L^4(\Omega)]}\|e_p\|^2_{L^4[t^{n-1},t^n;L^4(\Omega)]}\\&\quad\le \frac{\tau_n}{\epsilon^2}\|e_h\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]}+\frac{\tau_n}{\epsilon^2}\|e_p\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]}.\end{aligned}$$

For the fifth term, we note that $\|u\|_{L^\infty[0,T;H^1(\Omega)]}\le C$, where C is a constant independent of ϵ, due to assumption (2.2) with $\sigma=0$. Therefore,

$$\begin{aligned}&\frac{\tau_n}{\epsilon^2}\|e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|e_h\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}\|u\|^2_{L^\infty[0,T;L^6(\Omega)]}\\&\quad\le \frac{\tau_n}{\epsilon^4}\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\tau_n\|e_h\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}\\&\quad\le \frac{\tau_n}{\epsilon^4}\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\tau_n(\|\nabla e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}).\end{aligned}$$

For the last two terms, using similar algebra, we deduce,

$$\begin{aligned}&\frac{\tau_n}{\epsilon^2}\|e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|e_p\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]}+\|u\|^2_{L^\infty[0,T;L^6(\Omega)]})\\&\quad\le \frac{\tau_n}{\epsilon^4}\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\\&\qquad+\tau_n\|e_p\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}\times(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]}+\|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2,\\&\frac{\tau_n}{\epsilon^2}\|e_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\|e_p\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}\\&\quad\le \frac{\tau_n}{\epsilon^4}\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\tau_n\|e_p\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}.\end{aligned}$$

Note that choosing $\frac{C_k\tau_n}{\epsilon^4}\le\frac{1}{8}$, we may hide all $\|e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ of (5.24) on the left. Hence, dividing by τ_n the resulting inequality and using an inverse estimate in time, we arrive at,

$$\begin{aligned}\|e_h\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}\le C_k\Big(&\|e_{h+}^{n-1}\|^2_{L^2(\Omega)}+\|\nabla e_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}+\frac{1}{\epsilon^2}\|e_h\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]}\\&+\|e_p\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}(\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]}+\|u\|^2_{L^\infty[0,T;L^6(\Omega)]})^2+\|e_p\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}\\&+\frac{1}{\epsilon^2}\|e_p\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]}\Big).\end{aligned}$$

Now, note that

$$\|e_p\|^4_{L^4[t^{n-1},t^n;L^4(\Omega)]}\le(\|u_p\|^2_{L^\infty[t^{n-1},t^n;L^4(\Omega)]}+\|u\|^2_{L^\infty[t^{n-1},t^n;H^1(\Omega)]})\|e_p\|^2_{L^2[t^{n-1},t^n;H^1(\Omega)]}.$$

Hence, the desired estimate follows by replacing the bounds of $\|e_{h+}^{n-1}\|^2_{L^2(\Omega)}$, $\frac{1}{\epsilon^2}\|e_h\|^4_{L^4[0,T;L^4(\Omega)]}$, $\|\nabla e_h\|^2_{L^2[0,T;L^2(\Omega)]}$. ∎

Remark 5.8. The estimate at arbitrary time points results in a best-approximation result by using triangle inequality. In addition, the dependence of the constant upon $\frac{1}{\epsilon}$ doesn't deteriorate further, despite the fact that we treat schemes of arbitrary order, provided that the natural assumption $\frac{1}{\epsilon^2}\|(u_0^2-1)^2\|_{L^1(\Omega)}\le C$ holds (which corresponds to $\sigma=0$ in (2.2)).

The best approximation estimate now follows by triangle inequality. In the remaining we present our results by selecting $\sigma = 0$ in (2.2) and $\tilde{\sigma} = 1$ (see Lem. 2.1 and Rem. 2.2).

Theorem 5.9. *Suppose that the assumptions of Theorem 5.7 hold. Then, there exists a constant* $\mathbf{C}$ *depending only upon* Ω, C_k *and* $\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]}$ *but independent of* ϵ, *and the such that,*

$$\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon^3)\left(\|e_p\|_{L^2[0,T;H^1(\Omega)]} + \|e_p\|_{L^\infty[0,T;L^2(\Omega)]}\right).$$

If in addition $u \in L^2[0,T;H^{l+1}(\Omega)]$, $u^{(k+1)} \in L^\infty[0,T;L^2(\Omega)]$ *there exists a positive constant* C *that depends only upon* Ω, C_k *and it is independent of* h, τ, ϵ, *such that*

$$\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq C(1/\epsilon^5)\big(h^l\|u\|_{L^2[0,T;H^{l+1}(\Omega)]} + \tau^{k+1}\|u^{(k+1)}\|_{L^\infty[0,T;L^2(\Omega)]}\big).$$

Proof. Under the assumptions of Theorem 5.7 the rates of convergence follow by the estimates on e_p in $L^2[0,T;H^1(\Omega)]$, and $L^\infty[0,T;L^2(\Omega)]$ norms using Lemma 5.2 in equation (5.4), since $\|u_p\|_{L^\infty[0,T;H^1(\Omega)]} + \|u\|_{L^\infty[0,T;H^1(\Omega)]} \leq C/\epsilon$ by (5.5). □

Proposition 5.10. *Let* $k = 0$, $l = 1$. *If* $u \in L^2[0,T;H^2(\Omega)] \cap H^1[0,T;L^2(\Omega)]$ *suppose that* τ, h *satisfy* $\tau^{1/2} + h \leq C\epsilon^{8/3}$ *for* $d = 2$, *(see Rem. 5.6) and* $\tau^{1/2} + h \leq C\epsilon^{10/3}$ *for* $d = 3$ *(see also Rem. 5.6). Then there exists a positive constant* $\mathbf{C}$ *depending only upon* Ω, C_k *and* $\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]}$ *but independent of* h, τ, ϵ *such that,*

$$\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon^3)(\tau^{1/2} + h).$$

If $u \in L^4[0,T;H^2(\Omega)]$, $u_t \in L^4[0,T;L^2(\Omega)]$ *suppose that* τ, h *satisfy* $\ln(\frac{T}{\tau})(\tau + h^2) \leq C\epsilon^4$ *when* $d = 2$ *(see Rem. 5.6) and* $\ln(\frac{T}{\tau})(\tau + h^2) \leq C\epsilon^5$ *when* $d = 3$ *(see Rem. 5.6). Then, there exists a positive constant* $\mathbf{C}$ *depending only upon* Ω, C_k *and* $\|u_p\|^2_{L^\infty[0,T;L^6(\Omega)]} + \|u\|^2_{L^\infty[0,T;L^6(\Omega)]}$ *but independent of* h, τ, ϵ *such that,*

$$\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon^3)(\tau^{1/2} + h).$$

If in addition $u_t \in L^2[0,T;H^1(\Omega)]$, *then we obtain, under the above assumption,*

$$\|e\|_{L^2[0,T;H^1(\Omega)]} + \|e\|_{L^\infty[0,T;L^2(\Omega)]} \leq \mathbf{C}(1/\epsilon^3)(\tau + h).$$

Proof. The estimates concerning the lowest order scheme follow directly from Theorem 5.7, and the approximation properties of e_p in $L^2[0,T;H^1(\Omega)]$ and $L^2[0,T;L^2(\Omega)]$ norms, when $u \in L^2[0,T;H^2(\Omega)] \cap H^1[0,T;H^1(\Omega)]$. When $u \in L^2[0,T;H^2(\Omega)] \cap H^1[0,T;L^2(\Omega)]$ then the time step and spacial discretization size restrictions are replaced by the ones of Remark 5.6. □

A remark regarding a discrete analog of the energy conservation property follows.

Remark 5.11. Given initial data $u_0 \in H^1(\Omega)$, and zero forcing term $f = 0$, it is well known that the solution of (1.1) satisfies, for any $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \|u_t(t)\|^2_{L^2(\Omega)} = 0 \tag{5.25}$$

where $E(t)$ denotes the associated energy *i.e.*, for *a.e.* $t \in (0,T]$,

$$E(t) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(u^2-1)^2\right)\mathrm{d}x.$$

It is clear that the discrete solution of (3.1) does not possess any meangingful regularity for u_{ht}, due to the discontinuities in time and hence (5.25) is not valid by simply replacing u by u_h. However, for any $t \in (t^{n-1}, t^n]$, we may formally rewrite (5.25) as,

$$(t - t^{n-1})\frac{\mathrm{d}}{\mathrm{d}t}E(t) + (t - t^{n-1})\|u_t(t)\|^2_{L^2(\Omega)} = 0$$

and hence integrating with respect to time and using integration parts in time,

$$\tau_n E(t^n) - \int_{t^{n-1}}^{t^n} E(t)\mathrm{d}t + \int_{t^{n-1}}^{t^n} (t - t^{n-1})\|u_t(t)\|^2_{L^2(\Omega)} = 0. \tag{5.26}$$

It is clear now that the above equality (5.26) is well defined, for discontinuous (in time) schemes, and (at least formally) we may replace u by any $u_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$.

We observe that integrating by parts (in time), (3.1) and setting $v_h = (t - t^{n-1})u_{ht} \in \mathcal{P}_k[t^{n-1}, t^n; U_h]$, with $k \geq 1$, we obtain

$$\begin{aligned}&\int_{t^{n-1}}^{t^n} (t - t^{n-1})\|u_{ht}\|^2_{L^2(\Omega)}\mathrm{d}t \\ &\qquad + \int_{t^{n-1}}^{t^n} \left((t - t^{n-1}) \frac{d}{\mathrm{d}t} \left(\|\nabla u_h(t)\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon^2}\|(u_h^2 - 1)^2\|_{L^1(\Omega)} \right) \right) \mathrm{d}t = 0,\end{aligned}$$

which implies (after integration by parts in time for the second integral)

$$\begin{aligned}&\int_{t^{n-1}}^{t^n} (t - t^{n-1})\|u_{ht}\|^2_{L^2(\Omega)}\mathrm{d}t + \tau_n \left(\|\nabla u^n_{h-}\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon^2}\|((u^2)^n_{h-} - 1)^2\|_{L^1(\Omega)} \right) \\ &- \int_{t^{n-1}}^{t^n} \left(\|\nabla u_h(t)\|^2_{L^2(\Omega)} + \frac{1}{4\epsilon^2}\|(u_h^2 - 1)^2\|_{L^1(\Omega)} \right) \mathrm{d}t = 0.\end{aligned}$$

Hence, we have shown that the discrete solution constructed by (3.1) actually satisfies a discrete local analog of the energy equality. It remains to prove that $\|\nabla u^n_{h-}\|_{L^2(\Omega)}$ and $\|u^n_{h-}\|_{L^4(\Omega)}$ are also bounded, independent of τ, h, which is easily obtained by using the results of Theorem 5.7 and an inverse estimate. Indeed, recall that under the assumptions of Theorem 5.7, we deduce, for any $u \in L^2[0, T; H^2(\Omega)] \cap H^1[0, T; H^1(\Omega)]$ for any $\tau \leq Ch$,

$$\|u_h\|_{L^\infty[0,T;H^1(\Omega)]} \leq C\frac{1}{h}\|u_h - u_p\|_{L^\infty[0,T;L^2(\Omega)]} + C\|u_p\|_{L^\infty[0,T;H^1(\Omega)]} \leq \mathbf{C}(1/\epsilon^2)\left(\frac{\tau}{h} + 1\right).$$

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