ESAIM: M2AN 53 (2019) 105–144
https://doi.org/10.1051/m2an/2018069

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

STABILITY ANALYSIS AND ERROR ESTIMATES OF ARBITRARY LAGRANGIAN–EULERIAN DISCONTINUOUS GALERKIN METHOD COUPLED WITH RUNGE–KUTTA TIME-MARCHING FOR LINEAR CONSERVATION LAWS

Lingling Zhou[1], Yinhua Xia[2,*] and Chi-Wang Shu[3]

Abstract. In this paper, we discuss the stability and error estimates of the fully discrete schemes for linear conservation laws, which consists of an arbitrary Lagrangian–Eulerian discontinuous Galerkin method in space and explicit total variation diminishing Runge–Kutta (TVD-RK) methods up to third order accuracy in time. The scaling arguments and the standard energy analysis are the key techniques used in our work. We present a rigorous proof to obtain stability for the three fully discrete schemes under suitable CFL conditions. With the help of the reference cell, the error equations are easy to establish and we derive the quasi-optimal error estimates in space and optimal convergence rates in time. For the Euler-forward scheme with piecewise constant elements, the second order TVD-RK method with piecewise linear elements and the third order TVD-RK scheme with polynomials of any order, the usual CFL condition is required, while for other cases, stronger time step restrictions are needed for the results to hold true. More precisely, the Euler-forward scheme needs $\tau \leq \rho h^2$ and the second order TVD-RK scheme needs $\tau \leq \rho h^{\frac{4}{3}}$ for higher order polynomials in space, where τ and h are the time and maximum space step, respectively, and ρ is a positive constant independent of τ and h.

Mathematics Subject Classification. 65M60, 35L65, 65M12.

Received March 8, 2018. Accepted November 2, 2018.

1. Introduction

In this paper, we consider the stability analysis and error estimates of an arbitrary Lagrangian–Eulerian discontinuous Galerkin (ALE-DG) method coupled with Runge–Kutta time-marching schemes for one-dimensional linear conservation laws

$$
\begin{aligned}
&u_t + (\beta u)_x = 0, \quad (x,t) \in [a,b] \times (0,T],\\
&u(x,0) = u_0(x), \quad x \in [a,b]
\end{aligned}
\tag{1.1}
$$

Keywords and phrases. Arbitrary Lagrangian–Eulerian discontinuous Galerkin method, Runge–Kutta methods, stability, error estimates, conservation laws.

[1] School of Mathematics and Information Science, Henan Polytechnic University, 454003 Jiaozuo, Henan, PR China.
[2] School of Mathematical Sciences, University of Science and Technology of China, 230026 Hefei, Anhui, PR China.
[3] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.
*Corresponding author: yhxia@ustc.edu.cn

with the periodic boundary condition. Here β is a constant. We only pay attention to the smooth solution of (1.1).

The discontinuous Galerkin (DG) method is a class of finite element methods, in which the basis functions are completely discontinuous, piecewise polynomials. Reed and Hill introduced the first DG method to solve the neutron equation [19] and later, Cockburn *et al.* extended the method to Runge–Kutta DG (RKDG) for nonlinear conservation laws in a series of papers [2,4–6]. The DG method has a wide range of applications owing to some advantages like parallelization capability, the strong stability and high-order accuracy, and so on. We refer to [3, 7, 8, 12, 20] and the references therein for more references of the DG method.

For the theoretical analysis of the fully discrete DG method, Zhang *et al.* have done a lot of work for conservation laws [22–26], where the time discretization is the explicit second or third order total variation diminishing Runge–Kutta (TVD-RK) method. For the smooth solutions, they obtained (quasi)-optimal error estimates for both the second and third order TVD-RK time-marching schemes with periodic boundary conditions and suitable CFL conditions. The stability of the third order TVD-RK (TVD-RK3) was shown in [25]. They also considered the inflow boundary condition as well as the discontinuous initial data [22,26]. Moreover, Burman *et al.* [10] analyzed the explicit RK schemes in combination with stabilized finite element methods for first-order linear partial differential equation systems and established sub-optimal error estimates for smooth solutions, which presented a unified analysis for several high-order symmetrically stabilized finite element methods encountered in the literature. We refer to [15, 21] for the energy analysis, which is the main technique for all work listed above.

However, all the analysis listed above are considered on the static grids. The ALE-DG method discussed here is a moving mesh DG method and the grid moving methodology belongs to the class of arbitrary Lagrangian–Eulerian (ALE) methods [9], which allows the motion of the mesh to be like either the Lagrangian or the Eulerian description of motion and should satisfy the geometric conservation law (GCL). The significance of the GCL has been analyzed by Guillard and Farhat [11]. There have been works about the implementation and applications of the ALE-DG method in the literature, *e.g.*, [14,16–18]. Klingenberg *et al.* developed an ALE-DG method for one-dimensional conservation laws [13], where local affine linear mappings connecting the cells for the current and next time level are defined and yield the time-dependent approximation space. They showed that the ALE-DG method satisfies the GCL for any Runge–Kutta scheme and is efficient for the conservation law. They also showed that the ALE-DG method shares many good properties of the DG method defined on static grids, *e.g.*, the L^2 stability, the local maximum principle, high order accuracy, and so on.

The main purpose of our work is to study the stability and the error estimates for the ALE-DG method combined with the explicit Runge–Kutta time-marching schemes, in which the Euler-forward, the second order TVD-RK (TVD-RK2) and TVD-RK3 methods are considered. Compared with the work on the static grids, our analysis is similar but more technical. Owing to the time-dependent functional space, the scaling arguments play an important role in this work. With the energy estimates, we prove that all three fully discrete schemes are stable under suitable CFL conditions. More precisely, for the Euler-forward scheme with P^0 (piecewise constant) elements, the TVD-RK2 scheme with P^1 (piecewise linear) elements and the TVD-RK3 approach with polynomials of any order in space, the usual CFL condition is needed, while the Euler-forward scheme with P^k elements for $k \geq 1$ requires $\tau \leq \rho h^2$ and the TVD-RK2 approach with P^k elements for $k \geq 2$ needs $\tau \leq \rho h^{\frac{4}{3}}$ for the results to hold true. Here τ and h are the time and maximum spatial mesh sizes, respectively, and ρ is a positive constant independent of τ and h. To best understand the error equations, we reformulate the equation (1.1) in terms of a suitable coordinate transformation. Then we proceed to obtain quasi-optimal error estimates in space and optimal convergence rates in time under the same CFL condition as the stability. To the best of our knowledge, the above results are the first for high order ALE methods with minimum smoothness assumptions on mesh movements (only assuming uniform Lipschitz continuity of the mesh movements) and without the need of remapping.

The organization of our paper is as follows. In Section 2, we list some notations adopted throughout the paper. The semi-discrete ALE-DG scheme for the linear conservation law is given in Section 3, where we also show some properties of the scheme. Section 4 presents the stability of the ALE-DG scheme in combination with the explicit RK time-marching methods up to third order. The error estimates for the three corresponding fully discrete schemes are proven in Section 5. We conclude our results in Section 6.

2. Notations

In this section, we will introduce some notations adopted throughout the paper.

2.1. Notations for the distribution of the mesh

Let $\Omega = [a, b]$. In order to describe the semi-discrete ALE-DG scheme of equation (1.1), we first introduce some notations for the distribution of the mesh. Assume that the mesh generating points $\left\{x^n_{j-\frac{1}{2}}\right\}_{j=1}^{N}$ are given at any time level t_n, $n = 0, \ldots, M$, and the points $x^n_{j-\frac{1}{2}}$ and $x^{n+1}_{j-\frac{1}{2}}$ are connected by time-dependent straight lines

$$x_{j-\frac{1}{2}}(t) := x^n_{j-\frac{1}{2}} + \omega_{j-\frac{1}{2}}(t - t_n), \quad \forall t \in [t_n, t_{n+1}], \tag{2.1}$$

where

$$\omega_{j-\frac{1}{2}} := \frac{x^{n+1}_{j-\frac{1}{2}} - x^n_{j-\frac{1}{2}}}{t_{n+1} - t_n}. \tag{2.2}$$

Note that for any time t, the first point $x_{\frac{1}{2}}(t)$ and the last point $x_{N+\frac{1}{2}}(t)$ stay the same for compactly supported problems and could move with the same speed $\frac{\mathrm{d}}{\mathrm{d}t}x_{\frac{1}{2}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x_{N+\frac{1}{2}}(t)$ for periodic boundary problems. We provide an example to show the distribution of the ALE mesh in Figure 1. The straight lines (2.1) provide the time-dependent cells

$$K_j(t) := [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)], \quad \forall t \in [t_n, t_{n+1}] \quad \textit{and} \quad j = 1, \ldots, N.$$

The length of each cell $K_j(t)$ is denoted by $\Delta_j(t) := x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t)$. Moreover, we set $h(t) := \max\limits_{1 \le j \le N} \Delta_j(t)$ and $h := \max\limits_{t \in [0,T]} h(t)$. We assume that the mesh is quasi-uniform in the sense that $h \le C\Delta_j(t)$ for $j = 1, 2, \ldots, N$,

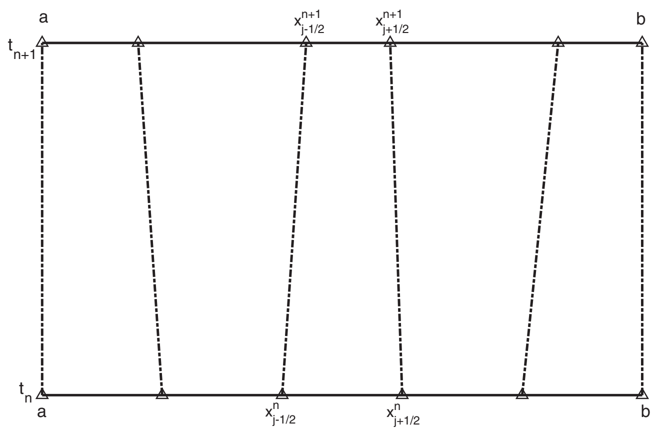

Figure 1. An example of the ALE mesh.

where C is a positive constant and independent of h. In addition, the grid velocity field for all $t \in [t_n, t_{n+1}]$ and $x \in K_j(t)$ is defined by

$$\omega(x,t) = \omega_{j+\frac{1}{2}} \frac{x - x_{j-\frac{1}{2}}(t)}{\Delta_j(t)} + \omega_{j-\frac{1}{2}} \frac{x_{j+\frac{1}{2}}(t) - x}{\Delta_j(t)}, \tag{2.3}$$

and the weak derivative of ω with respect to x is given by

$$\partial_x(\omega(x,t)) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{\Delta_j(t)} = \frac{\Delta_j'(t)}{\Delta_j(t)},\ (x,t) \in K_j(t) \times [t_n, t_{n+1}]. \tag{2.4}$$

Note that $\partial_x(\omega(x,t))$ is independent of x. The quasi-uniformity assumption for the meshes implies that $\Delta_j(t)$ satisfies the following property, for all $t \in [t_n, t_{n+1}]$, $n = 0, \ldots, M-1$,

$$\Delta_j(t) = (\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}})(t - t_n) + \Delta_j(t_n) > 0. \tag{2.5}$$

In addition, we assume that $\omega(x,t)$ satisfies the following properties:

(ω1): There exists a constant $C_w \geq 0$, independent of h, such that,

$$\max_{(x,t)\in[a,b]\times[0,T]} |\omega(x,t)| \leq C_w; \tag{2.6}$$

(ω2): There exists a constant $C_{wx} \geq 0$, independent of h, such that,

$$\max_{(x,t)\in[a,b]\times[0,T]} |\partial_x(\omega(x,t))| \leq C_{wx}. \tag{2.7}$$

For any $K_j(t)$, we define the following time-dependent linear mapping

$$\chi_j : [-1,1] \longrightarrow K_j(t), \quad \chi_j(\xi,t) = \frac{\Delta_j(t)}{2}(\xi+1) + x_{j-\frac{1}{2}}(t), \tag{2.8}$$

which yields a characterization of the grid velocity

$$\partial_t(\chi_j(\xi,t)) = \omega(\chi_j(\xi,t),t), \quad \forall(\xi,t) \in [-1,1] \times [t_n, t_{n+1}].$$

For simplicity, we denote $K_j^n \equiv K_j(t_n)$ and $\Delta_j^n \equiv \Delta_j(t_n)$, for any $n = 1, \ldots, M$.

2.2. Notations for function space and norms

For any $t \in [t_n, t_{n+1}]$, the finite element space is defined by

$$V_h(t) := \{v \in L^2(\Omega) : v(\chi_j(\cdot,t)) \in P^k([-1,1]), \quad j = 1,2,\ldots,N\},$$

where $P^k([-1,1])$ denotes the space of polynomials of degree at most k on $[-1,1]$. We denote the inner product over the interval $K_j(t)$ and the associated norm by

$$(v,r)_{K_j(t)} = \int_{K_j(t)} vr\mathrm{d}x, \quad \|v\|_{K_j(t)} = \sqrt{(v,v)_{K_j(t)}}.$$

We also use the usual notations of Sobolev space. Let $H^s(D)$ be the Sobolev space on sub-domain $D \subset \Omega$, which is equipped with the norm $\|\cdot\|_{H^s(D)}$ for any integer $s \geq 0$. Then we define the broken Sobolev space

$$H_h^1(t) := \{v : v(\chi_j(\cdot,t)) \in H^1([-1,1]), \quad j = 1,2,\ldots,N\},$$

which contains the finite element space. Moreover, the left and right limits of v at the point $x_{j-\frac{1}{2}}(t)$ are denoted by $v^-_{j-\frac{1}{2}}$ and $v^+_{j-\frac{1}{2}}$, respectively, where

$$v^{\pm}_{j-\frac{1}{2}} = \lim_{\varepsilon\to 0^+} v(x_{j-\frac{1}{2}}(t) \pm \varepsilon, t).$$

Thus the cell average and the jump at the point $x_{j-\frac{1}{2}}(t)$ are defined by

$$\{\!\!\{v\}\!\!\}_{j-\frac{1}{2}} = \frac{1}{2}\left(v^+_{j-\frac{1}{2}} + v^-_{j-\frac{1}{2}}\right), \quad [\![v]\!]_{j-\frac{1}{2}} = v^+_{j-\frac{1}{2}} - v^-_{j-\frac{1}{2}}.$$

Summing over all the elements, we denote

$$(v,r) = \sum_{j=1}^{N}(v,r)_{K_j(t)}, \quad \|v\|^2 = \sum_{j=1}^{N}\|v\|^2_{K_j(t)}, \quad [\![v]\!]^2 = \sum_{j=1}^{N}[\![v]\!]^2_{j-\frac{1}{2}}.$$

Let $\mathit{\Gamma}_h(t)$ be the union of all elements interface points and define the L^2-norm on $\mathit{\Gamma}_h(t)$ by

$$\|v\|_{\mathit{\Gamma}_h(t)} = \left[\sum_{j=1}^{N}\left(|v^+_{j-\frac{1}{2}}|^2 + |v^-_{j+\frac{1}{2}}|^2\right)\right]^{1/2}.$$

2.3. Notations for coordinate transformation

In the following, we will introduce some notations for coordinate transformations, which are often used in our stability analysis. For simplicity, we only consider the uniform partition of the time interval $[0,T]$, namely $\{t_n = n\tau\}_{n=0}^{M}$ with the time step τ and $M\tau = T$. With the time-dependent linear mapping (2.8), we have, for three different time stages t_n, $t_{n+\frac{1}{2}} = t_n + \frac{\tau}{2}$, and t_{n+1},

$$[-1,1] \longmapsto K^n_j, \qquad \chi_j(\xi,t_n) = \frac{\Delta^n_j}{2}(\xi+1) + x^n_{j-\frac{1}{2}}, \tag{2.9}$$

$$[-1,1] \longmapsto K^{n+\frac{1}{2}}_j, \qquad \chi_j(\xi,t_{n+\frac{1}{2}}) = \frac{\Delta^{n+\frac{1}{2}}_j}{2}(\xi+1) + x^{n+\frac{1}{2}}_{j-\frac{1}{2}}, \tag{2.10}$$

$$[-1,1] \longmapsto K^{n+1}_j, \qquad \chi_j(\xi,t_{n+1}) = \frac{\Delta^{n+1}_j}{2}(\xi+1) + x^{n+1}_{j-\frac{1}{2}}. \tag{2.11}$$

Thus $\forall\phi \in V_h(t_n)$, $\varphi \in V_h(t_{n+\frac{1}{2}})$, and $\psi \in V_h(t_{n+1})$, define

$$\hat{\phi}\Big(\chi_j(\cdot,t_{n+1})\Big) := \phi\Big(\chi_j(\cdot,t_n)\Big), \qquad \bar{\phi}\Big(\chi_j(\cdot,t_{n+\frac{1}{2}})\Big) := \phi\Big(\chi_j(\cdot,t_n)\Big), \tag{2.12}$$

$$\hat{\varphi}\Big(\chi_j(\cdot,t_{n+1})\Big) := \varphi\Big(\chi_j(\cdot,t_{n+\frac{1}{2}})\Big), \qquad \tilde{\varphi}\Big(\chi_j(\cdot,t_n)\Big) := \varphi\Big(\chi_j(\cdot,t_{n+\frac{1}{2}})\Big), \tag{2.13}$$

$$\tilde{\psi}\Big(\chi_j(\cdot,t_n)\Big) := \psi\Big(\chi_j(\cdot,t_{n+1})\Big), \qquad \bar{\psi}\Big(\chi_j(\cdot,t_{n+\frac{1}{2}})\Big) := \psi\Big(\chi_j(\cdot,t_{n+1})\Big). \tag{2.14}$$

Moreover, from (2.4) and (2.5), we get

$$\frac{\Delta_j^n}{\Delta_j^{n+1}} = 1 - s_2 > 0, \qquad \frac{\Delta_j^{n+1}}{\Delta_j^n} = 1 + s_1 > 0, \tag{2.15}$$

$$\frac{\Delta_j^n}{\Delta_j^{n+\frac{1}{2}}} = 1 - \frac{s_3}{2} > 0, \qquad \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^n} = 1 + \frac{s_1}{2} > 0, \tag{2.16}$$

$$\frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^{n+1}} = 1 - \frac{s_2}{2} > 0, \qquad \frac{\Delta_j^{n+1}}{\Delta_j^{n+\frac{1}{2}}} = 1 + \frac{s_3}{2} > 0, \tag{2.17}$$

where

$$s_1 = \tau\omega_x(t_n), \quad s_2 = \tau\omega_x(t_{n+1}), \quad s_3 = \tau\omega_x(t_{n+\frac{1}{2}}), \tag{2.18}$$

and $\omega_x(t) \equiv \partial_x\omega(x,t)$ is given by (2.4). Note that

$$\frac{\Delta_j^n}{\Delta_j^{n+1}} \cdot \frac{\Delta_j^{n+1}}{\Delta_j^n} = 1, \quad \frac{\Delta_j^n}{\Delta_j^{n+\frac{1}{2}}} \cdot \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^n} = 1, \quad \frac{\Delta_j^{n+\frac{1}{2}}}{\Delta_j^{n+1}} \cdot \frac{\Delta_j^{n+1}}{\Delta_j^{n+\frac{1}{2}}} = 1,$$

we have

$$s_1 = s_2 + s_1 s_2, \quad s_1 = s_3 + \frac{s_1 s_3}{2}, \quad s_3 = s_2 + \frac{s_2 s_3}{2}. \tag{2.19}$$

In the end, we present some properties. For any function $v_h^n \in V_h(t_n)$, the scaling arguments and the assumption (2.7) of ω_x indicate that,

$$\|v_h^n\|^2_{K_j^n} = \frac{\Delta_j^n}{\Delta_j^{n+1}}\|\widehat{v_h^n}\|^2_{K_j^{n+1}} = (1 - s_2)\|\widehat{v_h^n}\|^2_{K_j^{n+1}} \le (1 + C_{wx}\tau)\|\widehat{v_h^n}\|^2_{K_j^{n+1}}, \tag{2.20}$$

$$\|\widehat{v_h^n}\|^2_{K_j^{n+1}} = \frac{\Delta_j^{n+1}}{\Delta_j^n}\|v_h^n\|^2_{K_j^n} = (1 + s_1)\|v_h^n\|^2_{K_j^n} \le (1 + C_{wx}\tau)\|v_h^n\|^2_{K_j^n}. \tag{2.21}$$

Similarly, we also have

$$\|\overline{v_h^n}\|^2_{K_j^{n+\frac{1}{2}}} \le (1 + \frac{C_{wx}}{2}\tau)\|\widehat{v_h^n}\|^2_{K_j^{n+1}}, \quad \|\widehat{v_h^n}\|^2_{K_j^{n+1}} \le (1 + \frac{C_{wx}}{2}\tau)\|\overline{v_h^n}\|^2_{K_j^{n+\frac{1}{2}}}. \tag{2.22}$$

Remark 2.1. The introduction of coordinate transformations (2.9)–(2.11) is to make the presentations simplified and clear. With the help of them, the relation between ϕ, $\hat{\phi}$ and $\bar{\phi}$ in (2.12), the representations of the same function at different time stages, is easy to be understood. Moreover, it is straightforward to obtain properties (2.20)–(2.22), which are frequently used in our analysis.

2.4. Projections and inverse properties

In this section, we will present two types of projections. The L^2 projection P_h and Gauss-Radau projections $P_h^{\pm}$ into $V_h(t)$, which are often used to derive the quasi-optimal and optimal L^2 error bounds of the DG method. For a function $u \in L^2(\Omega)$, the L^2 projection is defined by

$$(P_h u, v)_{K_j(t)} = (u, v)_{K_j(t)}, \quad \forall v \in V_h(t). \tag{2.23}$$

For $k \geq 1$ and $v(\chi(\cdot,t)) \in P^{k-1}([-1,1])$, the Gauss-Radau projections are defined by

$$\begin{aligned}(P_h^- u, v)_{K_j(t)} &= (u,v)_{K_j(t)}, \quad P_h^- u\left(x_{j+\frac12}^-(t)\right) = u\left(x_{j+\frac12}^-(t)\right),\\ (P_h^+ u, v)_{K_j(t)} &= (u,v)_{K_j(t)}, \quad P_h^+ u\left(x_{j-\frac12}^+(t)\right) = u\left(x_{j-\frac12}^+(t)\right).\end{aligned} \tag{2.24}$$

Let $Q_h u$ be either $P_h u$ or $P_h^{\pm} u$. Suppose $u \in H^{k+1}(\Omega)$, then by a standard scaling argument, it is easy to show (*c.f.* [1]) for both projections that

$$\|\eta\| + h^{1/2}\|\eta\|_{\Gamma_h(t)} + h\|\partial_x \eta\| \leq Ch^{k+1}, \tag{2.25}$$

where $\eta = Q_h u - u$ and the positive constant C depends on u and its derivatives, but it is independent of h. Finally, we present the well-known inverse properties of the finite element space $V_h(t)$. For any $v \in V_h(t)$, there exists positive constants μ_1 and μ_2, independent of v and h, such that

$$h\|v_x\| \leq \mu_1 \|v\|, \quad h^{\frac12}\|v\|_{\Gamma_h(t)} \leq \mu_2 \|v\|. \tag{2.26}$$

In the following, we denote $\mu = \max\{\mu_1, \mu_2^2\}$. For more details of the inverse property, we refer the reader to [1].

3. Semi-discrete ALE-DG method

3.1. ALE-DG scheme

To derive the semi-discrete ALE-DG method, we first list the following lemma, which has been proven in [13].

Lemma 3.1. *Let u be a sufficiently smooth function in any cell $K_j(t)$. Then for all $v \in V_h(t)$, there holds the transport equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,v)_{K_j(t)} = (\partial_t u, v)_{K_j(t)} + (\partial_x(\omega u), v)_{K_j(t)}, \quad \forall j = 1, \ldots, N. \tag{3.1}$$

Next, multiply the equation (1.1) by a test function $v \in V_h(t)$ and apply the integration by parts as well as the transport equation (3.1), we obtain the semi-discrete ALE-DG method for arbitrary $K_j(t)$, $t \in [t_n, t_{n+1}]$: find $u_h \in V_h(t)$ such that for all test functions $v \in V_h(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_h, v)_{K_j(t)} = (g(\omega, u_h), v_x)_{K_j(t)} - \hat{g}(\omega, u_h)_{j+\frac12} v_{j+\frac12}^- + \hat{g}(\omega, u_h)_{j-\frac12} v_{j-\frac12}^+, \tag{3.2}$$

where $g(\omega, u_h) = (\beta - \omega)u_h$ and the numerical flux $\hat{g}(\omega, u_h)_{j-\frac12}$ can be chosen as the Lax-Friedrichs flux, for $j = 1, \ldots, N$,

$$\hat{g}(\omega, u_h)_{j-\frac12} = (\beta - \omega_{j-\frac12})\{\!\{u_h\}\!\}_{j-\frac12} - \frac{\alpha}{2}[\![u_h]\!]_{j-\frac12}, \quad \alpha = \max_{\Omega \times [t_n, t_{n+1}]} |\beta - \omega|. \tag{3.3}$$

For simplicity, we define the ALE-DG spatial operator $\mathcal{A}$ as

$$\mathcal{A}(v,r)(t) = \sum_{j=1}^{N} \mathcal{A}(v,r)_{K_j(t)}, \quad \forall v, r \in H_h^1(t), \tag{3.4}$$

where

$$\mathcal{A}(v,r)_{K_j(t)} = -\Big((\beta - \omega)v, r_x\Big)_{K_j(t)} + \hat{g}(\omega, v)_{j+\frac12} r_{j+\frac12}^- - \hat{g}(\omega, v)_{j-\frac12} r_{j-\frac12}^+, \tag{3.5}$$

and $\hat{g}(\omega, v)_{j-\frac12}$ is the Lax-Friedrichs flux defined by (3.3). Then by the above notations, the semi-discrete ALE-DG scheme (3.2) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_h, v)_{K_j(t)} = -\mathcal{A}(u_h, v)_{K_j(t)}, \quad \forall v \in V_h(t).$$

3.2. The properties of the ALE-DG scheme

In this subsection, we shall present some properties of the operator $\mathcal{A}$ defined by (3.4), which implies the properties of the ALE-DG spatial discretization.

Lemma 3.2 (Boundedness of the operator $\mathcal{A}$)**.** *Suppose $\mathcal{A}$ is defined by (3.4), then for any v, $r \in V_h(t)$ and $t \in [t_n, t_{n+1}]$, we have*

$$|\mathcal{A}(v,r)(t)| \leq 3\alpha\mu h^{-1}\|v\|\|r\|, \tag{3.6}$$

$$|\mathcal{A}(v,r)(t)| \leq \left(\alpha\|v_x\| + C_{wx}\|v\| + \sqrt{2}\alpha\mu h^{-\frac{1}{2}}[\![v]\!]\right)\|r\|. \tag{3.7}$$

Moreover, for the piecewise linear case, i.e., $k=1$ in the finite element space $V_h(t)$, there holds

$$|\mathcal{A}(v, r - P_h^0 r)(t)| \leq \left((\mu+1)C_{wx}\|v\| + \sqrt{2}\alpha\mu h^{-\frac{1}{2}}[\![v]\!]\right)\|r - P_h^0 r\|, \tag{3.8}$$

where $P_h^0 r$ denotes the L^2 projection of r onto the piecewise constant finite element space.

Proof. By the periodic boundary condition, we first obtain

$$\mathcal{A}(v,r)(t) = -\Big((\beta-\omega)v, r_x\Big) - \sum_{j=1}^{N} \hat{g}(\omega, v)_{j+\frac{1}{2}}[\![r]\!]_{j+\frac{1}{2}}. \tag{3.9}$$

The definition (3.3) yields,

$$|\hat{g}(\omega,v)_{j+\frac{1}{2}}| \leq \alpha(|v^+_{j+\frac{1}{2}}| + |v^-_{j+\frac{1}{2}}|).$$

Then sum over all j to get

$$\sum_{j=1}^{N} \hat{g}(\omega,v)^2_{j+\frac{1}{2}} \leq 2\sum_{j=1}^{N}\alpha^2\left(|v^+_{j+\frac{1}{2}}|^2 + |v^-_{j+\frac{1}{2}}|^2\right) = 2\alpha^2\|v\|^2_{\Gamma_h(t)}. \tag{3.10}$$

In addition, we have the following estimates

$$\sum_{j=1}^{N}[\![r]\!]^2_{j+\frac{1}{2}} \leq 2\sum_{j=1}^{N}\left(|r^+_{j+\frac{1}{2}}|^2 + |r^-_{j+\frac{1}{2}}|^2\right) = 2\|r\|^2_{\Gamma_h(t)}, \tag{3.11}$$

$$\sum_{j=1}^{N}\{\!\{r\}\!\}^2_{j+\frac{1}{2}} \leq \frac{1}{2}\sum_{j=1}^{N}\left(|r^+_{j+\frac{1}{2}}|^2 + |r^-_{j+\frac{1}{2}}|^2\right) = \frac{1}{2}\|r\|^2_{\Gamma_h(t)}. \tag{3.12}$$

Thus we can obtain the first inequality (3.6),

$$\begin{aligned} |\mathcal{A}(v,r)(t)| &\leq \alpha\|v\|\|r_x\| + \left(\sum_{j=1}^{N}\hat{g}(\omega,v)^2_{j+\frac{1}{2}}\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N}[\![r]\!]^2_{j+\frac{1}{2}}\right)^{\frac{1}{2}} \\ &\leq \alpha\mu_1 h^{-1}\|v\|\|r\| + 2\alpha\|v\|_{\Gamma_h(t)}\|r\|_{\Gamma_h(t)} \\ &\leq 3\alpha\mu h^{-1}\|v\|\|r\|. \end{aligned}$$

Here we use the Cauchy–Schwarz inequality as well as the inverse property (2.26). To obtain the second inequality (3.7), we integrate (3.5) by parts and sum over all j,

$$\begin{aligned}\mathcal{A}(v,r)(t) &= \Big((\beta-\omega)v_x, r\Big) + \Big(\partial_x(\beta-\omega)v, r\Big) \\ &\quad + \sum_{j=1}^{N}(\beta-\omega_{j+\frac12})[\![v]\!]_{j+\frac12}\{\!\{r\}\!\}_{j+\frac12} + \sum_{j=1}^{N}\frac{\alpha}{2}[\![v]\!]_{j+\frac12}[\![r]\!]_{j+\frac12} \\ &= -\omega_x(t)(v,r) + \mathcal{B}(v,r)(t),\end{aligned} \tag{3.13}$$

where

$$\mathcal{B}(v,r)(t) = \Big((\beta-\omega)v_x, r\Big) + \sum_{j=1}^{N}(\beta-\omega_{j+\frac12})[\![v]\!]_{j+\frac12}\{\!\{r\}\!\}_{j+\frac12} + \sum_{j=1}^{N}\frac{\alpha}{2}[\![v]\!]_{j+\frac12}[\![r]\!]_{j+\frac12}.$$

Here we use the fact that the quantity $\omega_x(t)$ defined by (2.4) only depends on t. By (3.12) and the similar arguments to estimate (3.6), we get

$$\begin{aligned}|\mathcal{B}(v,r)(t)| &\le \alpha\|v_x\|\|r\| + \sqrt{2}\alpha[\![v]\!]\|r\|_{\Gamma_h(t)} \\ &\le \Big(\alpha\|v_x\| + \sqrt{2}\alpha\mu h^{-\frac12}[\![v]\!]\Big)\|r\|,\end{aligned} \tag{3.14}$$

which yields the desired result (3.7),

$$|\mathcal{A}(v,r)(t)| \le \Big(\alpha\|v_x\| + C_{wx}\|v\| + \sqrt{2}\alpha\mu h^{-\frac12}[\![v]\!]\Big)\|r\|.$$

Here we use the property (2.7) of $\partial_x\omega(x,t)$. Finally, we analyze the inequality (3.8). By the property of the piecewise constant L^2 projection,

$$(r - P_h^0 r, v_x)_{K_j(t)} = 0, \quad \forall v(\chi_j(\cdot,t)) \in P^1([-1,1]),$$

we have

$$\Big((\beta-\omega)v_x, r - P_h^0 r\Big)_{K_j(t)} = \Big((\omega_{j-\frac12}-\omega)v_x, r - P_h^0 r\Big)_{K_j(t)},$$

which yields

$$\begin{aligned}\Big((\beta-\omega)v_x, r - P_h^0 r\Big) &\le C_{wx}h\|v_x\|\|r - P_h^0 r\| \\ &\le \mu C_{wx}\|v\|\|r - P_h^0 r\|.\end{aligned}$$

Henceforth, replacing r with $r - P_h^0 r$ in (3.13) and by similar arguments, we obtain

$$|\mathcal{A}(v, r - P_h^0 r)(t)| \le \Big((\mu+1)C_{wx}\|v\| + \sqrt{2}\alpha\mu h^{-\frac12}[\![v]\!]\Big)\|r - P_h^0 r\|.$$

□

Lemma 3.3. *Suppose $\mathcal{A}$ is defined by (3.4) and $P_h v$ is the L^2 projection defined by (2.23). Denote $\eta = P_h v - v$, then for any $v \in H_h^1(t)$, $r \in V_h(t)$ and $t \in [t_n, t_{n+1}]$, we have*

$$|\mathcal{A}(\eta,r)(t)| \le \mu C_{wx}\|\eta\|\|r\| + \sqrt{2}\alpha\|\eta\|_{\Gamma_h(t)}[\![r]\!], \tag{3.15}$$

$$|\mathcal{A}(\eta,r)(t)| \le \mu C_{wx}\|\eta\|\|r\| + 2\alpha\mu h^{-\frac12}\|\eta\|_{\Gamma_h(t)}\|r\|. \tag{3.16}$$

Proof. From (3.9) and the definition of the L^2 projection (2.23), we have

$$\begin{aligned}\mathcal{A}(\eta,r)(t) &= -\Big((\beta-\omega)\eta, r_x\Big) - \sum_{j=1}^{N}\hat{g}(\omega,\eta)_{j+\frac12}[\![r]\!]_{j+\frac12}\\ &= \Big((\omega-\omega_{j-\frac12})\eta, r_x\Big) - \sum_{j=1}^{N}\hat{g}(\omega,\eta)_{j+\frac12}[\![r]\!]_{j+\frac12},\end{aligned}$$

which implies that

$$\begin{aligned}|\mathcal{A}(\eta,r)(t)| &\le C_{wx}h\|\eta\|\|r_x\| + \left(\sum_{j=1}^{N}\hat{g}(\omega,\eta)^2_{j+\frac12}\right)^{\frac12}[\![r]\!]\\ &\le \mu C_{wx}\|\eta\|\|r\| + \sqrt{2}\alpha\|\eta\|_{\Gamma_h(t)}[\![r]\!].\end{aligned}$$

Here we use the Cauchy–Schwarz inequality, the inverse inequality (2.26) as well as the estimate (3.10). It is easy to obtain (3.16) from (3.15) by using the estimate (3.11) and the inverse property (2.26). $\square$

Lemma 3.4. *For any v, $r \in H_h^1(t)$ and $t \in [t_n, t_{n+1}]$, we have*

$$\mathcal{A}(v,r)(t) + \mathcal{A}(r,v)(t) = \sum_{j=1}^{N}\alpha[\![v]\!]_{j+\frac12}[\![r]\!]_{j+\frac12} - \sum_{j=1}^{N}\omega_x(t)(v,r)_{K_j(t)}, \tag{3.17}$$

$$\mathcal{A}(v,v)(t) = \sum_{j=1}^{N}\frac{\alpha}{2}[\![v]\!]^2_{j+\frac12} - \sum_{j=1}^{N}\frac{\omega_x(t)}{2}\|v\|^2_{K_j(t)}. \tag{3.18}$$

Proof. With the representation of $\mathcal{A}(v,r)(t)$ in (3.9) and integration by parts, we can easily obtain

$$\begin{aligned}\mathcal{A}(v,r)(t)+\mathcal{A}(r,v)(t) &= -\Big((\beta-\omega)v, r_x\Big) - \sum_{j=1}^{N}\hat{g}(\omega,v)_{j+\frac12}[\![r]\!]_{j+\frac12}\\ &\quad - \Big((\beta-\omega)r, v_x\Big) - \sum_{j=1}^{N}\hat{g}(\omega,r)_{j+\frac12}[\![v]\!]_{j+\frac12}\\ &= -\sum_{j=1}^{N}\omega_x(t)(v,r)_{K_j(t)} + \sum_{j=1}^{N}\alpha[\![v]\!]_{j+\frac12}[\![r]\!]_{j+\frac12}\\ &\quad - \sum_{j=1}^{N}(\beta-\omega_{j+\frac12})v^-_{j+\frac12}r^-_{j+\frac12} + \sum_{j=1}^{N}(\beta-\omega_{j-\frac12})v^+_{j-\frac12}r^+_{j-\frac12}\\ &\quad - \sum_{j=1}^{N}(\beta-\omega_{j+\frac12})(\{\!\{v\}\!\}_{j+\frac12}[\![r]\!]_{j+\frac12} + \{\!\{r\}\!\}_{j+\frac12}[\![v]\!]_{j+\frac12})\\ &= -\sum_{j=1}^{N}\omega_x(t)(v,r)_{K_j(t)} + \sum_{j=1}^{N}\alpha[\![v]\!]_{j+\frac12}[\![r]\!]_{j+\frac12}.\end{aligned}$$

Here in the last step we use the periodic boundary condition and $[\![r]\!]\{\!\{v\}\!\} + [\![v]\!]\{\!\{r\}\!\} = [\![rv]\!]$. It is clear that (3.17) implies (3.18) if $r = v$. $\square$

It is worth pointing out that the properties of $\mathcal{A}$ in Lemmas 3.2–3.4 are similar to those in Zhang and Shu [25] developed for the static grids, which play very important roles in obtaining stability.

4. Stability analysis for linear conservation laws

In this section, we would like to analyze the stability of three fully discrete schemes, that is, the ALE-DG method coupled with Euler-forward, TVD-RK2 and TVD-RK3 time-marching schemes. In what follows, we denote the approximation of $u_h(t_n)$ by u_h^n.

4.1. First order scheme

The ALE-DG with the Euler-forward scheme is given in the following form: find $u_h^{n+1} \in V_h(t_{n+1})$, such that for any $v_h^n \equiv v_h(\cdot, t_n) \in V_h(t_n)$ and $1 \le j \le N$, there holds

$$(u_h^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} = (u_h^n, v_h^n)_{K_j^n} - \tau \mathcal{A}(u_h^n, v_h^n)_{K_j^n}. \tag{4.1}$$

Here $\widehat{v_h^n}$ is defined by (2.12).

Theorem 4.1. *Let u_h^{n+1} be the numerical solution of the fully discrete scheme (4.1), then we have for any n, that*

$$\|u_h^{n+1}\|^2 \le (1 + C\tau)\|u_h^n\|^2,$$

under the CFL condition

$$\alpha^2 \mu^2 \tau h^{-2} \le \frac{1}{9}\cdot \tag{4.2}$$

In particular, for the piecewise constant finite element space, $V_h(t) = \{v(\chi_j(\cdot, t)) \in P^0([-1, 1])\}$, we have the strong stability

$$\|u_h^{n+1}\| \le \|u_h^n\|,$$

with the usual CFL condition

$$\alpha \mu^2 \tau h^{-1} \le \frac{1}{4}\cdot \tag{4.3}$$

Here α is defined by (3.3), μ is the inverse constant (2.26), and C is a positive constant depending solely on C_{wx}.

Proof. To analyze the stability of the scheme (4.1), we need first to obtain the energy identity. Take $v_h^n = u_h^n$ in the scheme (4.1) to yield

$$(u_h^{n+1}, \widehat{u_h^n})_{K_j^{n+1}} = \|u_h^n\|^2_{K_j^n} - \tau \mathcal{A}(u_h^n, u_h^n)_{K_j^n}. \tag{4.4}$$

By the scaling argument with (2.15), we have

$$\|\widehat{u_h^n}\|^2_{K_j^{n+1}} = \frac{\Delta_j^{n+1}}{\Delta_j^n}\|u_h^n\|^2_{K_j^n} = (1 + s_1)\|u_h^n\|^2_{K_j^n}. \tag{4.5}$$

Noting that

$$(u_h^{n+1}, \widehat{u_h^n})_{K_j^{n+1}} = \frac{1}{2}\|u_h^{n+1}\|^2_{K_j^{n+1}} + \frac{1}{2}\|\widehat{u_h^n}\|^2_{K_j^{n+1}} - \frac{1}{2}\|u_h^{n+1} - \widehat{u_h^n}\|^2_{K_j^{n+1}},$$

we get the energy identity by summing up (4.4) over j,

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 &= \frac{1}{2}\|u_h^{n+1} - \widehat{u_h^n}\|^2 - \sum_{j=1}^{N} \frac{s_1}{2}\|u_h^n\|^2_{K_j^n} - \tau\mathcal{A}(u_h^n, u_h^n)(t_n)\\ &= \frac{1}{2}\|u_h^{n+1} - \widehat{u_h^n}\|^2 - \frac{\tau}{2}\alpha[\![u_h^n]\!]^2. \end{aligned}\tag{4.6}$$

Here in the last step we use the property (3.18) of $\mathcal{A}$ as well as the definition (2.18) of s_1. Next, we only need to analyze the first term of the right hand side in (4.6). Apply the scaling arguments and (2.15) again to get

$$(u_h^n, v_h^n)_{K_j^n} = \frac{\Delta_j^n}{\Delta_j^{n+1}}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} = (1 - s_2)(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},\tag{4.7}$$

and

$$\mathcal{A}(u_h^n, v_h^n)_{K_j^n} = \mathcal{A}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}}.\tag{4.8}$$

It implies the equivalent form of (4.1),

$$(u_h^{n+1} - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} = -s_2(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - \tau\mathcal{A}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}}.\tag{4.9}$$

$\mathbf{P^k}$ **case**. Take the test function $\widehat{v_h^n} = u_h^{n+1} - \widehat{u_h^n}$ in (4.9) and sum all over j to obtain

$$\|u_h^{n+1} - \widehat{u_h^n}\|^2 = -\sum_{j=1}^{N} s_2(\widehat{u_h^n}, u_h^{n+1} - \widehat{u_h^n})_{K_j^{n+1}} - \tau\mathcal{A}(\widehat{u_h^n}, u_h^{n+1} - \widehat{u_h^n})(t_{n+1}).\tag{4.10}$$

Using the Cauchy–Schwarz inequality and the boundedness (3.6) of the operator $\mathcal{A}$, we have

$$\|u_h^{n+1} - \widehat{u_h^n}\|^2 \le (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\widehat{u_h^n}\|\|u_h^{n+1} - \widehat{u_h^n}\|.$$

Here we use the fact that $|s_2| \le C_{wx}\tau$. Then divide both sides of the above inequality by $\|u_h^{n+1} - \widehat{u_h^n}\|$ to get,

$$\|u_h^{n+1} - \widehat{u_h^n}\| \le (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\widehat{u_h^n}\|,\tag{4.11}$$

which yields that

$$\frac{1}{2}\|u_h^{n+1} - \widehat{u_h^n}\|^2 \le (C_{wx}^2\tau^2 + 9\alpha^2\mu^2\tau^2 h^{-2})\|\widehat{u_h^n}\|^2.$$

Under the CFL condition (4.2), we obtain the following inequality,

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1} - \widehat{u_h^n}\|^2 &\le (C_{wx}^2\tau^2 + \tau)\|\widehat{u_h^n}\|^2\\ &\le C\tau\|u_h^n\|^2.\end{aligned}$$

Here the last step uses (2.21) and $\tau \le 1$. Consequently, the energy identity (4.6) implies that

$$\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 \le C\tau\|u_h^n\|^2.$$

$\mathbf{P^0}$ **case**. Apply the equivalent form (3.13) of the operator $\mathcal{A}$ to rewrite (4.9),

$$(u_h^{n+1} - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} = -\tau\mathcal{B}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},$$

due to the fact that $s_2 = \tau\omega_x(t_{n+1})$. Since the finite element space is piecewise constant, we have $\partial_x v_h^n = 0$, which is not available for $k \geq 1$. Take the test function $\widehat{v_h^n} = u_h^{n+1} - \widehat{u_h^n}$ in the above equality and use the boundedness (3.14) of the operator $\mathcal{B}$ to yield,

$$\|u_h^{n+1} - \widehat{u_h^n}\| \leq \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![u_h^n]\!], \tag{4.12}$$

which leads to

$$\frac{1}{2}\|u_h^{n+1} - \widehat{u_h^n}\|^2 \leq \alpha^2\mu^2\tau h^{-1}[\![u_h^n]\!]^2. \tag{4.13}$$

If

$$\alpha^2\mu^2\tau^2 h^{-1} - \frac{\tau}{4}\alpha \leq 0, \quad \text{that is,} \quad \alpha\mu^2\tau h^{-1} \leq \frac{1}{4},$$

we finish the proof by combining (4.6) and (4.13) together,

$$\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 \leq 0.$$

$\square$

In the following, we provide a remark to summarize the main difference between the case P^0 and P^k, $k \geq 1$ in the stability analysis of the first order scheme.

Remark 4.2. For the piecewise constant finite element space, $V_h(t) = \{v(\chi_j(\cdot,t)) \in P^0([-1,1])\}$, we have the property $\partial_x v_h = 0$ with $v_h \in V_h(t)$, which can not be extended to the finite element space with polynomial degree $k \geq 1$. Thus, the bound (4.12) is no longer available for the case P^k, $k \geq 1$. Instead, we use the inverse inequality to control $\partial_x v_h$ and get the bound (4.11). In the end, two different CFL conditions are obtained for the stability of the first order scheme.

4.2. Second order scheme

The ALE-DG with TVD-RK2 scheme is given in the following form: find $u_h^{n+1} \in V_h(t_{n+1})$, such that for any $v_h^n \equiv v_h(\cdot, t_n) \in V_h(t_n)$ and $1 \leq j \leq N$, there hold

$$\begin{aligned} (u_h^1, \widehat{v_h^n})_{K_j^{n+1}} &= (u_h^n, v_h^n)_{K_j^n} - \tau\mathcal{A}(u_h^n, v_h^n)_{K_j^n}, \\ (u_h^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} &= \frac{1}{2}(u_h^n, v_h^n)_{K_j^n} + \frac{1}{2}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{2}\mathcal{A}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}}. \end{aligned} \tag{4.14}$$

Here $\widehat{v_h^n}$ is defined by (2.12).

Theorem 4.3. *Let u_h^{n+1} be the numerical solution of the fully discrete scheme (4.14), then for any n, there holds*

$$\|u_h^{n+1}\|^2 \leq (1 + C\tau)\|u_h^n\|^2,$$

under the CFL condition

$$\tau h^{-4/3} \leq \sqrt[3]{\frac{4}{81(\alpha\mu)^4}}.$$

In particular, for the piecewise linear finite element space, $V_h(t) = \{v(\chi_j(\cdot,t)) \in P^1([-1,1])\}$, we just need the usual CFL condition,

$$\alpha\mu\tau h^{-1} \leq \min\left\{\frac{1}{32\mu}, \frac{1}{\sqrt[3]{16\mu}}\right\}. \tag{4.15}$$

Here α is defined by (3.3), μ is the inverse constant (2.26), and C is a positive constant depending solely on C_{wx} and μ.

Proof. Rewrite the scheme (4.14) such that all of the terms are in the same cell K_j^{n+1},

$$\begin{aligned}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} &= (1-s_2)(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - \tau\mathcal{A}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},\\ (u_h^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} &= \frac{1-s_2}{2}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} + \frac{1}{2}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{2}\mathcal{A}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}}.\end{aligned} \tag{4.16}$$

Here we use (4.7) and (4.8). By taking $\widehat{v_h^n} = \frac{1}{2}\widehat{u_h^n}$, u_h^1 in the above equalities, respectively, and adding them together, we have

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1}\|^2_{K_j^{n+1}} - \frac{1-s_2}{2}\|\widehat{u_h^n}\|^2_{K_j^{n+1}} = \frac{1}{2}\|u_h^{n+1} - u_h^1\|^2_{K_j^{n+1}} + \frac{s_2}{4}\|u_h^1 - \widehat{u_h^n}\|^2_{K_j^{n+1}} - \frac{s_2}{4}\|\widehat{u_h^n}\|^2_{K_j^{n+1}}\\ - \frac{s_2}{4}\|u_h^1\|^2_{K_j^{n+1}} - \frac{\tau}{2}\mathcal{A}(\widehat{u_h^n}, \widehat{u_h^n})_{K_j^{n+1}} - \frac{\tau}{2}\mathcal{A}(u_h^1, u_h^1)_{K_j^{n+1}}.\end{aligned} \tag{4.17}$$

Noticing that

$$\|u_h^n\|^2_{K_j^n} = (1-s_2)\|\widehat{u_h^n}\|^2_{K_j^{n+1}}, \quad s_2 = \tau\omega_x(t_{n+1}), \tag{4.18}$$

we obtain the energy identity by summing (4.17) over all j and using the property (3.18) of $\mathcal{A}$,

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 = \frac{1}{2}\|u_h^{n+1} - u_h^1\|^2 + \sum_{j=1}^{N}\frac{s_2}{4}\|u_h^1 - \widehat{u_h^n}\|^2_{K_j^{n+1}}\\ - \frac{\tau}{4}\alpha[\![u_h^n]\!]^2 - \frac{\tau}{4}\alpha[\![u_h^1]\!]^2.\end{aligned} \tag{4.19}$$

In order to obtain the stability, we just need to analyze the first two terms of the right hand side in the above equality. From (4.16), it is straightforward to get

$$(u_h^1 - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} = -\tau\mathcal{A}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - s_2(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}}, \tag{4.20}$$

$$(u_h^{n+1} - u_h^1, \widehat{v_h^n})_{K_j^{n+1}} = -\frac{\tau}{2}\mathcal{A}(u_h^1 - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}}. \tag{4.21}$$

$\mathbf{P^k}$ **case**. Take the test function $\widehat{v_h^n} = u_h^{n+1} - u_h^1$ in (4.21) and sum up all over j to yield,

$$\|u_h^{n+1} - u_h^1\|^2 = -\frac{\tau}{2}\mathcal{A}(u_h^1 - \widehat{u_h^n}, u_h^{n+1} - u_h^1)(t_{n+1}). \tag{4.22}$$

Using the boundedness (3.6) of $\mathcal{A}$, we have

$$\|u_h^{n+1} - u_h^1\| \le \frac{3}{2}\alpha\mu\tau h^{-1}\|u_h^1 - \widehat{u_h^n}\|. \tag{4.23}$$

Then by the similar arguments, taking the test function $\widehat{v_h^n} = u_h^1 - \widehat{u_h^n}$ in (4.20) and using the boundedness (3.6) lead to

$$\|u_h^1 - \widehat{u_h^n}\| \le (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\widehat{u_h^n}\|. \tag{4.24}$$

Denote $\lambda = \alpha\mu\tau h^{-1}$. Combine (4.23) and (4.24) to get that

$$\frac{1}{2}\|u_h^{n+1} - u_h^1\|^2 \le \frac{9}{4}\lambda^2(C_{wx}^2\tau^2 + 9\lambda^2)\|\widehat{u_h^n}\|^2.$$

If

$$\frac{81}{4}\lambda^4 \le \tau, \quad \text{that is,} \quad \tau h^{-\frac{4}{3}} \le \sqrt[3]{\frac{4}{81(\alpha\mu)^4}}, \tag{4.25}$$

then we have

$$\frac{1}{2}\|u_h^{n+1} - u_h^1\|^2 \le (\frac{C_{wx}^2}{2}\tau^{\frac{5}{2}} + \tau)\|\widehat{u_h^n}\|^2 \le C\tau\|u_h^n\|^2, \tag{4.26}$$

where we use the property (2.21) and $\tau \le 1$. Under the CFL condition (4.25), the estimate (4.24) turns out to be

$$\|u_h^1 - \widehat{u_h^n}\|^2 \le 2(C_{wx}^2\tau^2 + 2\sqrt{\tau})\|\widehat{u_h^n}\|^2,$$

which indicates that

$$\begin{aligned}\sum_{j=1}^{N}\frac{s_2}{4}\|u_h^1 - \widehat{u_h^n}\|^2_{K_j^{n+1}} &\le \frac{C_{wx}}{2}\tau(C_{wx}^2\tau^2 + 2\sqrt{\tau})\|\widehat{u_h^n}\|^2 \\ &\le C\tau\|u_h^n\|^2.\end{aligned} \tag{4.27}$$

Thus we combine (4.19), (4.26) and (4.27) to obtain

$$\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 \le C\tau\|u_h^n\|^2.$$

$\mathbf{P^1}$ case. Denote $z = u_h^1 - \widehat{u_h^n}$. Using the boundedness (3.7) of $\mathcal{A}$ in (4.22), we get

$$\|u_h^{n+1} - u_h^1\| \le \frac{\tau}{2}\alpha\|z_x\| + \frac{C_{wx}}{2}\tau\|z\| + \frac{\sqrt{2}}{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![z]\!]. \tag{4.28}$$

Now we will analyze $\|z_x\|$. Let $y = z - P_h^0 z$, where $P_h^0 z$ denotes the L^2 projection of z onto the piecewise constant finite element space. It follows from the property of the L^2 projection and the identity (4.20) as well as the boundedness (3.8) of $\mathcal{A}$,

$$\begin{aligned}\|y\|^2 &= \sum_{j=1}^{N}(z - P_h^0 z, y)_{K_j^{n+1}} = \sum_{j=1}^{N}(z, y)_{K_j^{n+1}} \\ &= -\tau\mathcal{A}(\widehat{u_h^n}, y)(t_{n+1}) - \sum_{j=1}^{N}s_2(\widehat{u_h^n}, y)_{K_j^{n+1}} \\ &\le \left((\mu+2)C_{wx}\tau\|\widehat{u_h^n}\| + \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![u_h^n]\!]\right)\|y\|.\end{aligned}$$

Here we also use the Cauchy–Schwarz inequality and $|s_2| \le C_{wx}\tau$ for the last step. Divide both sides of the above inequality by $\|y\|$ to yield,

$$\|y\| \le (\mu+2)C_{wx}\tau\|\widehat{u_h^n}\| + \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![u_h^n]\!].$$

Noting that $\partial_x(P_h^0 z) = 0$ and the inverse property (2.26), we get

$$\|z_x\| = \|y_x\| \le \mu h^{-1}\|y\|,$$

which implies that

$$\|z_x\| \le (\mu+2)C_{wx}\mu\tau h^{-1}\|\widehat{u_h^n}\| + \sqrt{2}\alpha\mu^2\tau h^{-\frac{3}{2}}[\![u_h^n]\!].$$

In addition, it is clear to observe that

$$[\![z]\!] \le \sqrt{2}([\![u_h^n]\!] + [\![u_h^1]\!]).$$

Thus the estimates (4.24) and (4.28) give that,

$$\begin{aligned}\|u_h^{n+1} - u_h^1\| \le \frac{\alpha}{2}\tau\left((\mu+2)C_{wx}\mu\tau h^{-1}\|\widehat{u_h^n}\| + \sqrt{2}\alpha\mu^2\tau h^{-\frac{3}{2}}[\![u_h^n]\!]\right)\\ + \frac{C_{wx}}{2}\tau\left((C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\widehat{u_h^n}\|\right) + \frac{\sqrt{2}}{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![z]\!]\\ \le C_1\tau\|\widehat{u_h^n}\| + C_2[\![u_h^n]\!] + C_3[\![u_h^1]\!],\end{aligned} \tag{4.29}$$

where

$$C_1 = \frac{(\mu+5)\lambda + C_{wx}\tau}{2}C_{wx}, \quad C_2 = \alpha\mu\tau h^{-\frac{1}{2}} + \frac{\sqrt{2}}{2}\alpha^2\mu^2\tau^2h^{-\frac{3}{2}}, \quad C_3 = \alpha\mu\tau h^{-\frac{1}{2}},$$

and $\lambda = \alpha\mu\tau h^{-1}$. Squaring the above inequality, we have

$$\frac{1}{2}\|u_h^{n+1} - u_h^1\|^2 \le C_2^2[\![u_h^n]\!]^2 + 2C_3^2[\![u_h^1]\!]^2 + 2C_1^2\tau^2\|\widehat{u_h^n}\|^2.$$

If we let

$$C_2^2 \le \frac{\alpha}{8}\tau, \quad 2C_3^2 \le \frac{\alpha}{8}\tau,$$

or furthermore,

$$2\alpha^2\mu^2\tau^2h^{-1} \le \frac{\alpha}{16}\tau, \quad \alpha^4\mu^4\tau^4h^{-3} \le \frac{\alpha}{16}\tau, \quad \textit{that is,} \quad \lambda \le \min\{\frac{1}{32\mu}, \frac{1}{\sqrt[3]{16\mu}}\}, \tag{4.30}$$

then

$$\frac{1}{2}\|u_h^{n+1} - u_h^1\|^2 \le \frac{\alpha}{8}\tau[\![u_h^n]\!]^2 + \frac{\alpha}{8}\tau[\![u_h^1]\!]^2 + C\tau\|u_h^n\|^2. \tag{4.31}$$

Here we use the property (2.21) and $\tau \le 1$. Owing to the choice of the CFL condition (4.30) and the estimate (4.24), we obtain

$$\begin{aligned}\sum_{j=1}^{N}\frac{s_2}{4}\|u_h^1 - \widehat{u_h^n}\|^2_{K_j^{n+1}} \le \frac{C_{wx}}{4}\tau(C_{wx}\tau + 3\lambda)^2\|\widehat{u_h^n}\|^2\\ \le C\tau\|u_h^n\|^2.\end{aligned} \tag{4.32}$$

Finally, we finish the proof by combining (4.19), (4.31) and (4.32),

$$\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 \le C\tau\|u_h^n\|^2.$$

$\square$

Similar to the first order scheme, we provide a remark to summarize the main difference between the case P^1 and P^k, $k \geq 2$ in the stability analysis for the second order scheme.

Remark 4.4. For the piecewise linear finite element space, $V_h(t) = \{v(\chi_j(\cdot,t)) \in P^1([-1,1])\}$, we have the orthogonal property

$$(r - P_h^0 r, v_x)_{K_j(t)} = 0, \quad \forall v(\chi_j(\cdot,t)) \in P^1([-1,1]),$$

where $P_h^0 r$ is the L^2 projection of r onto the piecewise constant finite element space. With the help of this property, we can obtain a sharp estimate for $\|u_h^{n+1} - u_h^1\|$, namely, the bound (4.29). Under the usual CFL condition, we derive the stability for piecewise linear polynomials. However, for higher order piecewise polynomials ($k \geq 2$), the above treatment breaks down since the orthogonal property is invalid. Thus we use the boundedness (3.6) of $\mathcal{A}$ directly and obtain the stability for the case P^k, $k \geq 2$ under a stronger time step restriction.

4.3. Third order scheme

The ALE-DG with TVD-RK3 scheme is given in the following form: find $u_h^{n+1} \in V_h(t_{n+1})$, such that for any $v_h^n \equiv v_h(\cdot, t_n) \in V_h(t_n)$ and $1 \leq j \leq N$, there hold

$$\begin{aligned}
(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} &= (u_h^n, v_h^n)_{K_j^n} - \tau \mathcal{A}(u_h^n, v_h^n)_{K_j^n},\\
(u_h^2, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} &= \frac{3}{4}(u_h^n, v_h^n)_{K_j^n} + \frac{1}{4}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} - \frac{1}{4}\tau \mathcal{A}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}},\\
(u_h^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} &= \frac{1}{3}(u_h^n, v_h^n)_{K_j^n} + \frac{2}{3}(u_h^2, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} - \frac{2}{3}\tau \mathcal{A}(u_h^2, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}},
\end{aligned} \tag{4.33}$$

where $\widehat{v_h^n}$, $\overline{v_h^n}$ are defined by (2.12). In this subsection, we are going to obtain the L^2-norm stability for the fully discrete scheme (4.33). Similar to the second order case, we first rewrite the scheme (4.33) such that all of the terms are in the same cell K_j^{n+1},

$$\begin{aligned}
(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} &= (\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - \tau \mathcal{A}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - s_2(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},\\
(\widehat{u_h^2}, \widehat{v_h^n})_{K_j^{n+1}} &= \frac{3}{4}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} + \frac{1}{4}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{4}\mathcal{A}(u_h^1, \widehat{v_h^n})_{K_j^{n+1}} + \frac{s_2}{2}(\widehat{u_h^2} - \frac{3}{2}\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},\\
(u_h^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} &= \frac{1}{3}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} + \frac{2}{3}(\widehat{u_h^2}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{2\tau}{3}\mathcal{A}(\widehat{u_h^2}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{s_2}{3}(\widehat{u_h^n} + \widehat{u_h^2}, \widehat{v_h^n})_{K_j^{n+1}}.
\end{aligned} \tag{4.34}$$

Here we use the scaling arguments and (2.15)–(2.17). Next, for the convenience of the analysis, we define

$$\mathbb{E}_1 = u_h^1 - \widehat{u_h^n}, \quad \mathbb{E}_2 = 2\widehat{u_h^2} - u_h^1 - \widehat{u_h^n}, \quad \mathbb{E}_3 = u_h^{n+1} - 2\widehat{u_h^2} + \widehat{u_h^n}. \tag{4.35}$$

Then we can achieve the following identities by a direct calculation,

$$\begin{aligned}
(\mathbb{E}_1, \widehat{v_h^n})_{K_j^{n+1}} &= -s_2(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - \tau \mathcal{A}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},\\
(\mathbb{E}_2, \widehat{v_h^n})_{K_j^{n+1}} &= -s_2(\widehat{u_h^n} - \widehat{u_h^2}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{2}\mathcal{A}(\mathbb{E}_1, \widehat{v_h^n})_{K_j^{n+1}},\\
(\mathbb{E}_3, \widehat{v_h^n})_{K_j^{n+1}} &= -s_2(\widehat{u_h^2} - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{3}\mathcal{A}(\mathbb{E}_2, \widehat{v_h^n})_{K_j^{n+1}}.
\end{aligned} \tag{4.36}$$

For the last identity, we rewrite $\mathbb{E}_3 = (u_h^{n+1} - \frac{1}{3}\widehat{u_h^n} - \frac{2}{3}\widehat{u_h^2}) - \frac{4}{3}(\widehat{u_h^2} - \widehat{u_h^n})$, and

$$(\widehat{u_h^2} - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} = -s_2(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} + \frac{s_2}{2}(\widehat{u_h^2}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{4}\mathcal{A}(u_h^1 + \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}}. \tag{4.37}$$

In the following, we will present the stability for the fully discrete scheme (4.33).

Theorem 4.5. *Let u_h^{n+1} be the numerical solution of the fully discrete scheme (4.33), then we have for any n, that*

$$\|u_h^{n+1}\|^2 \le (1 + C\tau)\|u_h^n\|^2,$$

under the CFL condition

$$\alpha\mu\tau h^{-1} \le \frac{1}{3}, \tag{4.38}$$

where α is defined by the Lax-Friedrichs numerical flux (3.3), μ is the inverse constant (2.26), and C is a positive constant depending solely on C_{wx}.

Proof. Similar to the second order scheme (4.14), we need first obtain the energy identity. Taking the test functions $\widehat{v_h^n} = \widehat{u_h^n}$, $4u_h^1$, and $6\widehat{u_h^2}$ in the identities (4.34), respectively, and adding them together, we have

$$\int_{K_j^{n+1}} \mathbb{F}\mathrm{d}x = -s_2\|\widehat{u_h^n}\|^2_{K_j^{n+1}} - 2s_2\|\widehat{u_h^2}\|^2_{K_j^{n+1}} + s_2\mathbb{R}_1 - \tau\mathbb{R}_2,$$

where

$$\begin{aligned}
\mathbb{F} &= -2u_h^1\widehat{u_h^n} - (\widehat{u_h^n})^2 + 4\widehat{u_h^2}u_h^1 - (u_h^1)^2 + 6u_h^{n+1}\widehat{u_h^2} - 2\widehat{u_h^2}\widehat{u_h^n} - 4(\widehat{u_h^2})^2,\\
\mathbb{R}_1 &= 2(\widehat{u_h^2}, u_h^1 - \widehat{u_h^n})_{K_j^{n+1}} - 3(\widehat{u_h^n}, u_h^1)_{K_j^{n+1}},\\
\mathbb{R}_2 &= \mathcal{A}(\widehat{u_h^n}, \widehat{u_h^n})_{K_j^{n+1}} + \mathcal{A}(u_h^1, u_h^1)_{K_j^{n+1}} + 4\mathcal{A}(\widehat{u_h^2}, \widehat{u_h^2})_{K_j^{n+1}}.
\end{aligned}$$

Noting that

$$\mathbb{F} = 3[(u_h^{n+1})^2 - (\widehat{u_h^n})^2] - \mathbb{E}_2^2 - 3(u_h^{n+1} - \widehat{u_h^n})\mathbb{E}_3,$$

we get the following identity by summing over all j,

$$\begin{aligned}
3\|u_h^{n+1}\|^2 - 3\|\widehat{u_h^n}\|^2 &= \|\mathbb{E}_2\|^2 + 3(u_h^{n+1} - \widehat{u_h^n}, \mathbb{E}_3) + \sum_{j=1}^N s_2\mathbb{R}_1\\
&\quad - \sum_{j=1}^N s_2\|\widehat{u_h^n}\|^2_{K_j^{n+1}} - 2\sum_{j=1}^N s_2\|\widehat{u_h^2}\|^2_{K_j^{n+1}} - \tau\sum_{j=1}^N \mathbb{R}_2.
\end{aligned} \tag{4.39}$$

Denote each line of the right hand side in the above equality by Θ_1, Θ_2, respectively. The definitions (4.35), the identities (4.36) and the properties (3.17)–(3.18) of $\mathcal{A}$ yield,

$$\begin{aligned}
\Theta_1 &= \|\mathbb{E}_2\|^2 + 3(\mathbb{E}_3, \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3) + \sum_{j=1}^N s_2\mathbb{R}_1\\
&= -\|\mathbb{E}_2\|^2 + 2(\mathbb{E}_2, \mathbb{E}_2) + 3(\mathbb{E}_3, \mathbb{E}_1) + 3(\mathbb{E}_3, \mathbb{E}_2) + 3\|\mathbb{E}_3\|^2 + \sum_{j=1}^N s_2\mathbb{R}_1\\
&= -\|\mathbb{E}_2\|^2 + 3\|\mathbb{E}_3\|^2 - \tau\mathcal{A}(\mathbb{E}_1, \mathbb{E}_2) - \tau\mathcal{A}(\mathbb{E}_2, \mathbb{E}_1) - \tau\mathcal{A}(\mathbb{E}_2, \mathbb{E}_2)\\
&\quad + \sum_{j=1}^N s_2(\widehat{u_h^n} - \widehat{u_h^2}, \mathbb{E}_2 + 3\mathbb{E}_1)_{K_j^{n+1}} + \sum_{j=1}^N s_2\mathbb{R}_1\\
&= -\|\mathbb{E}_2\|^2 + 3\|\mathbb{E}_3\|^2 - \tau\alpha\sum_{j=1}^N [\![\mathbb{E}_1]\!]_{j+\frac{1}{2}}[\![\mathbb{E}_2]\!]_{j+\frac{1}{2}} - \frac{\alpha}{2}\tau[\![\mathbb{E}_2]\!]^2 + \sum_{j=1}^N s_2\Theta_{11},
\end{aligned}$$

and

$$\begin{aligned}\Theta_{11} &= (\widehat{u_h^n} - \widehat{u_h^2}, \mathbb{E}_2 + 3\mathbb{E}_1)_{K_j^{n+1}} + \mathbb{R}_1 + (\mathbb{E}_1, \mathbb{E}_2)_{K_j^{n+1}} + \frac{1}{2}\|\mathbb{E}_2\|^2_{K_j^{n+1}} \\ &= -\frac{5}{2}\|\widehat{u_h^n}\|^2_{K_j^{n+1}} - \frac{1}{2}\|u_h^1\|^2_{K_j^{n+1}}.\end{aligned}$$

On the other hand, the property (3.18) of $\mathcal{A}$ implies that

$$\begin{aligned}\Theta_2 = &-\frac{\alpha}{2}\tau[\![u_h^n]\!]^2 - \frac{\alpha}{2}\tau[\![u_h^1]\!]^2 - 2\alpha\tau[\![u_h^2]\!]^2 \\ &- \sum_{j=1}^{N}\frac{s_2}{2}\|\widehat{u_h^n}\|^2_{K_j^{n+1}} + \sum_{j=1}^{N}\frac{s_2}{2}\|u_h^1\|^2_{K_j^{n+1}}.\end{aligned}$$

Recalling the relationship (4.18), we add Θ_1 and Θ_2 to the equality (4.39) and obtain the energy identity

$$3\|u_h^{n+1}\|^2 - 3\|u_h^n\|^2 = \mathbb{I}_1 + \mathbb{I}_2, \tag{4.40}$$

where

$$\mathbb{I}_1 = -\|\mathbb{E}_2\|^2 + 3\|\mathbb{E}_3\|^2 - \alpha\tau\sum_{j=1}^{N}[\![\mathbb{E}_1]\!]_{j+\frac{1}{2}}[\![\mathbb{E}_2]\!]_{j+\frac{1}{2}} - \frac{\alpha}{2}\tau[\![\mathbb{E}_2]\!]^2, \tag{4.41}$$

$$\mathbb{I}_2 = -\frac{\alpha}{2}\tau[\![u_h^n]\!]^2 - \frac{\alpha}{2}\tau[\![u_h^1]\!]^2 - 2\alpha\tau[\![u_h^2]\!]^2. \tag{4.42}$$

Denote the last two terms on the right hand side of (4.41) by $\mathbb{I}_{11}$ and we have the following estimate by applying the Young's inequality,

$$\begin{aligned}\mathbb{I}_{11} &\le \frac{\alpha}{4}\tau[\![\mathbb{E}_1]\!]^2 + \frac{\alpha}{2}\tau[\![\mathbb{E}_2]\!]^2 \\ &\le \frac{\alpha}{2}\tau[\![u_h^1]\!]^2 + \frac{\alpha}{2}\tau[\![u_h^n]\!]^2 + \alpha\tau\|\mathbb{E}_2\|^2_{\Gamma_h(t_{n+1})} \\ &\le \frac{\alpha}{2}\tau[\![u_h^1]\!]^2 + \frac{\alpha}{2}\tau[\![u_h^n]\!]^2 + \alpha\mu\tau h^{-1}\|\mathbb{E}_2\|^2,\end{aligned} \tag{4.43}$$

where we use the estimate (3.11) for the second inequality and the last inequality uses the inverse inequality (2.26). Next, we will analyze $3\|\mathbb{E}_3\|^2$ with the identity (4.36). Denote $\lambda = \alpha\mu\tau h^{-1}$ as before. Take the test function $\widehat{v_h^n} = \mathbb{E}_3$ in the third equality of (4.36), sum over all j and use the boundedness (3.6) of $\mathcal{A}$ to derive,

$$\|\mathbb{E}_3\|^2 \le \left(C_{wx}\tau\|\widehat{u_h^2} - \widehat{u_h^n}\| + \lambda\|\mathbb{E}_2\|\right)\|\mathbb{E}_3\|,$$

which implies that,

$$3\|\mathbb{E}_3\|^2 \le 6\lambda^2\|\mathbb{E}_2\|^2 + 6C_{wx}^2\tau^2\|\widehat{u_h^2} - \widehat{u_h^n}\|^2. \tag{4.44}$$

Here we use $|s_2| = |\omega_x(t_{n+1})\tau| \le C_{wx}\tau$. Then from (4.41)–(4.44) and under the CFL condition (4.38), we get,

$$\begin{aligned}\mathbb{I}_1 + \mathbb{I}_2 &\le 6C_{wx}^2\tau^2\|\widehat{u_h^2} - \widehat{u_h^n}\|^2 - (1 - 6\lambda^2 - \lambda)\|\mathbb{E}_2\|^2 \\ &\le 6C_{wx}^2\tau\|\widehat{u_h^2} - \widehat{u_h^n}\|^2,\end{aligned} \tag{4.45}$$

since $\tau \le 1$. In addition, it is clear to obtain the following equality from (4.37),

$$(1 - \frac{s_2}{2})(\widehat{u_h^2} - \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} = -\frac{s_2}{2}(\widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{4}\mathcal{A}(u_h^1 + \widehat{u_h^n}, \widehat{v_h^n})_{K_j^{n+1}},$$

which yields,

$$\begin{aligned}(\widehat{u_h^2}-\widehat{u_h^n},\widehat{v_h^n})_{K_j^{n+1}} &= -\frac{s_3}{2}(\widehat{u_h^n},\widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{4}(1+\frac{s_3}{2})\mathcal{A}(u_h^1+\widehat{u_h^n},\widehat{v_h^n})_{K_j^{n+1}}\\ &= -\frac{s_3}{2}(\widehat{u_h^n},\widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{4}(1+\frac{s_3}{2})\mathcal{A}(\mathbb{E}_1,\widehat{v_h^n})_{K_j^{n+1}}\\ &\quad - \frac{\tau}{2}(1+\frac{s_3}{2})\mathcal{A}(\widehat{u_h^n},\widehat{v_h^n})_{K_j^{n+1}}.\end{aligned}$$

Here we use the relationship (2.19) between s_2 and s_3 for the first step. Taking the test function $\widehat{v_h^n}=\widehat{u_h^2}-\widehat{u_h^n}$ in the above equality, summing it over all elements, and applying the boundedness (3.6) of $\mathcal{A}$, we have

$$\|\widehat{u_h^2}-\widehat{u_h^n}\| \le (\frac{1}{2}+\frac{3C_{wx}}{4}\tau)\|\widehat{u_h^n}\| + \frac{1}{4}(1+\frac{C_{wx}}{2}\tau)\|\mathbb{E}_1\|. \tag{4.46}$$

Here for the first step we use $|s_3|=|\omega_x(t_{n+\frac{1}{2}})\tau| \le C_{wx}\tau$ and the last step uses the CFL condition (4.38). For the estimate of $\|\mathbb{E}_1\|$, we take $\widehat{v_h^n}=\mathbb{E}_1$ in the first equality of (4.36) and use the similar arguments to derive,

$$\|\mathbb{E}_1\| \le (C_{wx}\tau+3\lambda)\|\widehat{u_h^n}\| \le (C_{wx}\tau+1)\|\widehat{u_h^n}\|. \tag{4.47}$$

As a result, we collect the estimates from (4.45) to (4.47) to achieve,

$$\mathbb{I}_1+\mathbb{I}_2 \le C_1\tau\|\widehat{u_h^n}\|^2,$$

where C_1 depends solely on C_{wx}. Recalling the energy identity (4.40), we will finish the proof,

$$3\|u_h^{n+1}\|^2 - 3\|u_h^n\|^2 \le C_1\tau\|\widehat{u_h^n}\|^2 \le C\tau\|u_h^n\|^2,$$

where we use the property (2.21) and $\tau \le 1$. □

5. Error estimates for linear conservation laws

In this section, we will present error estimates for the fully discrete schemes with the help of the stability analysis in the previous section. We begin with some preliminaries. To make it clear to construct the error equation, we first show the representation of equation (1.1) after a time-dependent coordinate transformation $x=x(\xi,t)$ defined by (2.8). For simplicity, we denote $\check{v}(\xi,t)=v(x(\xi,t),t)$ for any function $v(x,t)$. Then by the chain rule,

$$\check{u}_\xi = u_x x_\xi, \quad \check{u}_t = u_t + u_x x_t,$$

where $x_\xi=\frac{\Delta_j(t)}{2}$ and $x_t=\check{\omega}$. Thus in the reference coordinates (ξ,t), the equation (1.1) of $K_j(t)$, $t\in[t_n,t_{n+1}]$ becomes

$$\check{u}_t + \frac{2}{\Delta_j(t)}(\beta-\check{\omega})\check{u}_\xi = 0, \quad (\xi,t)\in[-1,1]\times[t_n,t_{n+1}]. \tag{5.1}$$

On the other hand, by (2.3), (2.8) and (2.5), we have

$$\check{\omega}_\xi = \frac{1}{2}(\omega_{j+\frac{1}{2}}-\omega_{j-\frac{1}{2}}) = \frac{\Delta_j'(t)}{2}. \tag{5.2}$$

Combine (5.1) and (5.2) to derive,

$$\partial_t\Big(\check{u}\Delta_j(t)\Big) + \partial_\xi\Big(2(\beta-\check{\omega})\check{u}\Big) = 0,$$

which is equivalent to the form,

$$\check{U}_t + (a\check{U})_\xi = 0, \quad \check{U}(\xi,t)=\check{u}\Delta_j(t), \quad a(\xi,t)=\frac{2(\beta-\check{\omega})}{\Delta_j(t)}. \tag{5.3}$$

5.1. First order scheme

In this subsection, we would like to show the error estimates for the ALE-DG spatial discretization coupled with the Euler-forward time marching scheme. Following the idea of analyzing the error estimates for static meshes [25], we first present the error equation.

5.1.1. *Error equation*

Denote $u^n = u(x, t_n)$ for any time level n. To proceed with the error equation, we need the following lemma, which describes the local truncation error in time.

Lemma 5.1. *Let u be the exact solution of equation (1.1). Suppose u is sufficiently smooth with bounded derivatives, then for any $v_h^n \in V_h(t_n)$ and $1 \le j \le N$, there holds,*

$$(u^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} = (u^n, v_h^n)_{K_j^n} - \tau\mathcal{A}(u^n, v_h^n)_{K_j^n} + (\varepsilon_1^n, v_h^n)_{K_j^n}, \tag{5.4}$$

where $\widehat{v_h^n}$ is defined by (2.12), ε_1^n is the local truncation error in time and $\|\varepsilon_1^n\|_{K_j^n} = \mathcal{O}(\tau^2)$ for any j and n.

Proof. By the Taylor expansion with Lagrange form of the remainder, we obviously have,

$$\check{U}(\xi, t+\tau) = \check{U}(\xi, t) - \tau(a\check{U})_\xi(\xi, t) + \frac{\tau^2}{2}\check{U}_{tt}(\xi, t_1), \quad t_1 \in (t, t+\tau),$$

where we use the definition of $\check{U}$ in (5.3). Let $t = t_n$ and we still use the notation t_1 to stand for a fixed value between t_n and t_{n+1}. Multiply the test function $\check{v}_h^n \in P^k([-1, 1])$ on both sides of the above equation, and integrate by parts to yield,

$$\int_{-1}^{1} \check{U}^{n+1}\check{v}_h^n \mathrm{d}\xi = \int_{-1}^{1} \check{U}^n\check{v}_h^n \mathrm{d}\xi - \tau\check{\mathcal{A}}(\check{U}^n, \check{v}_h^n) + \frac{\tau^2}{2}\int_{-1}^{1} \check{U}_{tt}(\xi, t_1)\check{v}_h^n \mathrm{d}\xi,$$

where

$$\check{\mathcal{A}}(\check{U}^n, \check{v}_h^n) = -\int_{-1}^{1} a^n\check{U}^n\partial_\xi(\check{v}_h^n)\mathrm{d}\xi + a^n\check{U}^n\check{v}_h^n|_{\xi=1}^{-} - a^n\check{U}^n\check{v}_h^n|_{\xi=-1}^{+}.$$

Noting that $x_\xi = \frac{\Delta_j(t)}{2}$, we can easily get, by the scaling arguments and (5.3),

$$\int_{-1}^{1} \check{U}^{n+1}\check{v}_h^n \mathrm{d}\xi = 2(u^{n+1}, \widehat{v_h^n})_{K_j^{n+1}}, \quad \int_{-1}^{1} \check{U}^n\check{v}_h^n \mathrm{d}\xi = 2(u^n, v_h^n)_{K_j^n}. \tag{5.5}$$

Owing to the smooth exact solution, we have $[\![u]\!]_{j-\frac{1}{2}} = 0$ at each element boundary point, which implies that,

$$\check{\mathcal{A}}(\check{U}^n, \check{v}_h^n) = 2\mathcal{A}(u^n, v_h^n)_{K_j^n}. \tag{5.6}$$

Moreover, from the definition (5.3) of $\check{U}$ and $\Delta_j'(t) = \omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}$, we obtain,

$$\check{U}_t = \Delta_j(t)\check{u}_t + \Delta_j'(t)\check{u}, \quad \check{U}_{tt} = \Delta_j(t)\check{u}_{tt} + 2\Delta_j'(t)\check{u}_t.$$

It is inferred that

$$\begin{aligned}
\frac{\tau^2}{2}\int_{-1}^{1} \check{U}_{tt}(\xi, t_1)\check{v}_h^n \mathrm{d}\xi &= \frac{\tau^2}{2}\int_{-1}^{1}\Big(\Delta_j\check{u}_{tt} + 2\Delta_j'\check{u}_t\Big)(\xi, t_1)\check{v}_h^n \mathrm{d}\xi \\
&= \frac{\Delta_j(t_1)}{\Delta_j^n}\tau^2\Big(\tilde{u}_{tt}(\chi(\cdot, t_n)), v_h^n\Big)_{K_j^n} + \frac{2\Delta_j'}{\Delta_j^n}\tau^2\Big(\tilde{u}_t(\chi(\cdot, t_n)), v_h^n\Big)_{K_j^n} \\
&= 2(\varepsilon_1^n, v_h^n)_{K_j^n},
\end{aligned} \tag{5.7}$$

where $\tilde{u}_{tt}(\chi(\cdot,t_n)) = u_{tt}(x(\cdot,t_1))$, $\tilde{u}_t(\chi(\cdot,t_n)) = u_t(x(\cdot,t_1))$ for $\chi(\cdot,t_n) \in K_j^n$, $x \in K_j(t_1)$, and

$$\varepsilon_1^n = \frac{\Delta_j(t_1)}{2\Delta_j^n}\tau^2\tilde{u}_{tt} + \frac{\Delta_j^{'}}{\Delta_j^n}\tau^2\tilde{u}_t.$$

By the quantity (2.4), (2.5) and (2.7), there hold,

$$\frac{\Delta_j(t_1)}{\Delta_j^n} = 1 + \omega_x(t_n)(t_1 - t_n) \le 1 + C_{wx}, \quad |\frac{\Delta_j^{'}}{\Delta_j^n}| = |\omega_x(t_n)| \le C_{wx},$$

which implies that

$$\|\varepsilon_1^n\|_{K_j^n} \le C\tau^2.$$

Here C is a positive constant independent of h and τ. Finally, we combine (5.5), (5.6) and (5.7) to finish the proof,

$$(u^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} = (u^n, v_h^n)_{K_j^n} - \tau\mathcal{A}(u^n, v_h^n)_{K_j^n} + (\varepsilon_1^n, v_h^n)_{K_j^n}.$$

□

For the convenience of analysis, we introduce some notations. The error between the exact solution and the numerical solution of the first order scheme (4.1) is denoted by $e^n = u(x,t_n) - u_h^n$ for any stage n. Let $\zeta^n = P_h u^n - u_h^n$ and $\eta^n = P_h u^n - u^n$, where $P_h u^n$ is the L^2 projection defined by (2.23). Subtracting (5.4) for the first order scheme (4.1), we obtain the error equation for any $v_h^n \in V_h(t_n)$ and $1 \le j \le N$,

$$(e^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} = (e^n, v_h^n)_{K_j^n} - \tau\mathcal{A}(e^n, v_h^n)_{K_j^n} + (\varepsilon_1^n, v_h^n)_{K_j^n}.$$

Moreover, $e^n = \zeta^n - \eta^n$ yields that

$$(\zeta^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} = (\zeta^n, v_h^n)_{K_j^n} - \tau\mathcal{A}(\zeta^n, v_h^n)_{K_j^n} + \mathcal{H}_j(\eta, v_h^n), \tag{5.8}$$

where

$$\mathcal{H}_j(\eta, v_h^n) = (\eta^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} - (\eta^n, v_h^n)_{K_j^n} + \tau\mathcal{A}(\eta^n, v_h^n)_{K_j^n} + (\varepsilon_1^n, v_h^n)_{K_j^n}. \tag{5.9}$$

In the end, we present the following estimates for the projection error.

Lemma 5.2. *Suppose u is sufficiently smooth with bounded derivatives, then there exists a positive constant C independent of h, τ and n, such that for any $v_h^n \in H_h^1(t_n)$ and $\forall n \le M$,*

$$\|\eta^n\| + h^{1/2}\|\eta^n\|_{\Gamma_h(t_n)} + h\|\partial_x\eta^n\| \le Ch^{k+1}, \tag{5.10}$$

$$(\eta^{n+1}, \widehat{v_h^n}) - (\eta^n, v_h^n) \le C\tau h^{k+1}\|v_h^n\|. \tag{5.11}$$

Proof. The estimate (5.10) can be obtained directly from (2.25). By the scaling arguments, the definition $\eta^n = P_h u^n - u^n$ and (5.3), we get,

$$\begin{aligned}
(\eta^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} - (\eta^n, v_h^n)_{K_j^n} &= \frac{\Delta_j^{n+1}}{2}\int_{-1}^{1}\check{\eta}^{n+1}\check{v}_h^n \mathrm{d}\xi - \frac{\Delta_j^n}{2}\int_{-1}^{1}\check{\eta}^n\check{v}_h^n \mathrm{d}\xi \\
&= \frac{1}{2}\int_{-1}^{1}\Big[(P_h\check{U}^{n+1} - \check{U}^{n+1}) - (P_h\check{U}^n - \check{U}^n)\Big]\check{v}_h^n \mathrm{d}\xi \\
&= \frac{1}{2}\int_{-1}^{1}\Big[P_h(\check{U}^{n+1} - \check{U}^n) - (\check{U}^{n+1} - \check{U}^n)\Big]\check{v}_h^n \mathrm{d}\xi \\
&\le \frac{C_1}{2}\tau\|\partial_\xi^{k+2}(a\check{U})(\xi,t_n)\|_{L^2([-1,1])}\|\check{v}_h^n\|_{L^2([-1,1])}.
\end{aligned}$$

Here we use the fact that the L^2 projection is linear in time on the static mesh as well as the exact solution is smooth enough. Noting that $a\check{U} = 2(\beta - \check{\omega})\check{u}$ from (5.3), we apply the scaling arguments again to derive,

$$\begin{aligned}\|\partial_\xi^{k+2}(a\check{U})(\xi,t_n)\|^2_{L^2([-1,1])} &= \int_{-1}^{1}\Big(\partial_\xi^{k+2}(a\check{U})(\xi,t_n)\Big)^2 \mathrm{d}\xi \\ &= 4\int_{K_j^n}\Big(\partial_x^{k+2}((\beta-\omega)u^n)\Big)^2 (x_\xi)^{2k+3}\mathrm{d}x \\ &= 4(\frac{\Delta_j^n}{2})^{2k+3}\int_{K_j^n}\Big(\partial_x^{k+2}((\beta-\omega)u^n)\Big)^2\mathrm{d}x,\end{aligned}$$

and

$$\|\check{v}_h^n\|^2_{L^2([-1,1])} = \int_{-1}^{1}(\check{v}_h^n)^2\mathrm{d}\xi = \frac{2}{\Delta_j^n}\int_{K_j^n}(v_h^n)^2\mathrm{d}x.$$

Finally, the above estimates yield,

$$(\eta^{n+1},\widehat{v_h^n}) - (\eta^n, v_h^n) \le C\tau h^{k+1}\|v_h^n\|.$$

The proof is completed. □

In the following, we use the notation C to stand for a generic positive constant independent of τ, h, n and u_h^n, but may depends on the exact solution u, the mesh speed function ω and the inverse constant μ in (2.26). It may have a different value in each occurrence.

5.1.2. Error estimate for the first order scheme

Theorem 5.3. *Let u_h^n be the numerical solution of the fully discrete scheme (4.1) with Euler-forward time-marching method, and u be the exact solution of equation (1.1). Suppose u is sufficiently smooth with bounded derivatives, then we have the following error estimate,*

$$\max_{n\tau\le T}\|u(x,t_n)-u_h^n\| \le C(h^{k+\frac12}+\tau),$$

under the CFL condition

$$\alpha^2\mu^2\tau h^{-2} \le \frac{1}{9}. \tag{5.12}$$

In particular, for the piecewise constant finite element space, $V_h(t) = \{v(\chi_j(\cdot,t)) \in P^0([-1,1])\}$, we just need the usual CFL condition

$$\alpha\mu^2\tau h^{-1} \le \frac{1}{8}. \tag{5.13}$$

Here α is defined by (3.3), μ is the inverse constant (2.26). The positive constant C is independent of h, τ, n and u_h.

Proof. Similar to the stability analysis, we take the test function $v_h^n = \zeta^n$ in the error equation (5.8) to derive the energy identity for ζ^n,

$$\frac12\|\zeta^{n+1}\|^2 - \frac12\|\zeta^n\|^2 = \frac12\|\zeta^{n+1}-\widehat{\zeta^n}\|^2 - \frac{\tau}{2}\alpha[\![\zeta^n]\!]^2 + \sum_{j=1}^{N}\mathcal{H}_j(\eta,\zeta^n). \tag{5.14}$$

In the following, we will estimate $\|\zeta^{n+1}-\widehat{\zeta^n}\|^2$ and $\sum_{j=1}^N \mathcal{H}_j(\eta,\zeta^n)$ separately. From the estimate (5.11) and Lemma 5.1, it is easy to conclude that,

$$(\eta^{n+1},\widehat{v_h^n})-(\eta^n,\widehat{v_h^n})+(\varepsilon_1^n,\widehat{v_h^n})\le C(\tau h^{k+1}+\tau^2)\|v_h^n\|. \tag{5.15}$$

By the scaling arguments with (2.15), we have,

$$(\zeta^n,v_h^n)_{K_j^n}=\frac{\Delta_j^n}{\Delta_j^{n+1}}(\widehat{\zeta^n},\widehat{v_h^n})_{K_j^{n+1}}=(1-s_2)(\widehat{\zeta^n},\widehat{v_h^n})_{K_j^{n+1}}.$$

Then the error equation (5.8) can be rewritten as

$$(\zeta^{n+1}-\widehat{\zeta^n},\widehat{v_h^n})_{K_j^{n+1}}=-s_2(\widehat{\zeta^n},\widehat{v_h^n})_{K_j^{n+1}}-\tau\mathcal{A}(\widehat{\zeta^n},\widehat{v_h^n})_{K_j^n}+\mathcal{H}_j(\eta,v_h^n).$$

Choose $\widehat{v_h^n}=\zeta^{n+1}-\widehat{\zeta^n}$ in the above equality and sum over all j to obtain

$$\begin{aligned}\|\zeta^{n+1}-\widehat{\zeta^n}\|^2=&-\sum_{j=1}^N s_2(\widehat{\zeta^n},\zeta^{n+1}-\widehat{\zeta^n})_{K_j^{n+1}}-\tau\mathcal{A}(\widehat{\zeta^n},\zeta^{n+1}-\widehat{\zeta^n})(t_{n+1})\\&+\sum_{j=1}^N\mathcal{H}_j(\eta,\widetilde{\zeta^{n+1}}-\zeta^n),\end{aligned}\tag{5.16}$$

where $\widetilde{\zeta^{n+1}}$ is defined by (2.14). Now we will divide the analysis into two cases as in the stability analysis.

P^k case. The boundedness (3.15) of $\mathcal{A}$ firstly gives that

$$\begin{aligned}\mathcal{A}(\eta^n,\zeta^n)&\le\mu C_{wx}\|\eta^n\|\|\zeta^n\|+\sqrt{2}\alpha\|\eta^n\|_{\Gamma_h(t_n)}[\![\zeta^n]\!]\\&\le Ch^{k+1}\|\zeta^n\|+C\alpha h^{k+\frac12}[\![\zeta^n]\!]\\&\le\|\zeta^n\|^2+\frac{\alpha}{2}[\![\zeta^n]\!]^2+Ch^{2k+1}.\end{aligned}$$

Here we use the estimate (5.10) for the second step and Young's inequality for the last step. Take $v_h^n=\zeta^n$ in (5.15) to infer that

$$\begin{aligned}\sum_{j=1}^N\mathcal{H}_j(\eta,\zeta^n)&=(\eta^{n+1},\widehat{\zeta^n})-(\eta^n,\zeta^n)+\tau\mathcal{A}(\eta^n,\zeta^n)+(\varepsilon_1^n,\zeta^n)\\&\le2\tau\|\zeta^n\|^2+\frac{\alpha}{2}\tau[\![\zeta^n]\!]^2+C\tau(h^{2k+1}+\tau^2).\end{aligned}\tag{5.17}$$

Here we use the Young's inequality again. As for the estimate of $\|\zeta^{n+1}-\widehat{\zeta^n}\|^2$, we will use the equality (5.16). By the boundedness (3.16) of $\mathcal{A}$ and (2.20), we have,

$$\begin{aligned}\mathcal{A}(\eta^n,\widetilde{\zeta^{n+1}}-\zeta^n)(t_n)&\le\Big(\mu C_{wx}\|\eta^n\|+2\alpha\mu h^{-\frac12}\|\eta^n\|_{\Gamma_h(t_n)}\Big)\|\widetilde{\zeta^{n+1}}-\zeta^n\|\\&\le C\Big(h^{k+1}+\alpha\mu h^k\Big)\|\zeta^{n+1}-\widehat{\zeta^n}\|.\end{aligned}$$

Owing to $\tau\le1$, it follows from taking $\widehat{v_h^n}=\zeta^{n+1}-\widehat{\zeta^n}$ in (5.15) and (2.20) that

$$\sum_{j=1}^N\mathcal{H}_j(\eta,\widetilde{\zeta^{n+1}}-\zeta^n)\le C\Big(\tau h^{k+1}+\alpha\mu\tau h^k+\tau^2\Big)\|\zeta^{n+1}-\widehat{\zeta^n}\|.$$

In addition, by the boundedness (3.6) of $\mathcal{A}$, we get,

$$\tau|\mathcal{A}(\widehat{\zeta^n}, \zeta^{n+1} - \widehat{\zeta^n})(t_{n+1})| \le 3\alpha\mu\tau h^{-1}\|\widehat{\zeta^n}\|\|\zeta^{n+1} - \widehat{\zeta^n}\|.$$

Recalling the equality (5.16), we obtain the following estimate by dividing both sides by $\|\zeta^{n+1} - \widehat{\zeta^n}\|$,

$$\|\zeta^{n+1} - \widehat{\zeta^n}\| \le C_{wx}\tau\|\widehat{\zeta^n}\| + 3\alpha\mu\tau h^{-1}\|\widehat{\zeta^n}\| + C(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^2),$$

which implies that

$$\begin{aligned}\frac{1}{2}\|\zeta^{n+1} - \widehat{\zeta^n}\|^2 &\le \Big(C\tau^2 + (3\alpha\mu\tau h^{-1})^2\Big)\|\widehat{\zeta^n}\|^2 + C\Big(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^2\Big)^2 \\ &\le C\tau\|\widehat{\zeta^n}\|^2 + C\tau(h^{2k+2} + \tau^2) \\ &\le C\tau\|\zeta^n\|^2 + C\tau(h^{2k+2} + \tau^2). \end{aligned} \tag{5.18}$$

Here we use the CFL condition (5.12), the relationship (2.21) and $\tau \le 1$. Hence, we combine (5.14), (5.17) and (5.18) to derive

$$\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2 \le C\tau\|\zeta^n\|^2 + C\tau(h^{2k+1} + \tau^2).$$

Summing over n and using Gronwall's inequality, we obtain,

$$\|\zeta^n\|^2 \le C(h^{2k+1} + \tau^2), \quad n \le M,$$

if we choose the initial condition $u_h(x,0) = P_h u(x,0)$. Finally, apply the estimate (5.10) to yield,

$$\|e^n\|^2 \le C(h^{2k+1} + \tau^2), \quad n \le M.$$

$\mathbf{P^0}$ **case**. Since the finite element space is piecewise constant, we have $\partial_x v_h^n = 0$, which indicates that

$$\begin{aligned}\mathcal{A}(\eta^n, v_h^n)(t_n) &= -\sum_{j=1}^{N} \hat{g}(\omega, \eta^n)_{j+\frac{1}{2}}[\![v_h^n]\!]_{j+\frac{1}{2}} \\ &\le \sqrt{2}\alpha\|\eta^n\|_{\Gamma_h(t_n)}[\![v_h^n]\!] \qquad (5.19)\\ &\le \frac{\alpha}{4}[\![v_h^n]\!]^2 + Ch. \qquad (5.20)\end{aligned}$$

Here we use the estimate (3.10) for the second step, Young's inequality and the estimate (5.10) are used for the third step. Then recalling the definition (5.9) of $\mathcal{H}_j$ and taking $v_h^n = \zeta^n$ in (5.15) and (5.20), we get

$$\sum_{j=1}^{N} \mathcal{H}_j(\eta, \zeta^n) \le \tau\|\zeta^n\|^2 + \frac{\alpha}{4}\tau[\![\zeta^n]\!]^2 + C\tau(h + \tau^2). \tag{5.21}$$

On the other hand, choose $v_h^n = \widetilde{\zeta^{n+1}} - \zeta^n$ in (5.19) to derive,

$$\begin{aligned}\mathcal{A}(\eta^n, \widetilde{\zeta^{n+1}} - \zeta^n)(t_n) &\le \sqrt{2}\alpha\|\eta^n\|_{\Gamma_h(t_n)}[\![\widetilde{\zeta^{n+1}} - \zeta^n]\!] \\ &\le 2\alpha\mu h^{-\frac{1}{2}}\|\eta^n\|_{\Gamma_h(t_n)}\|\widetilde{\zeta^{n+1}} - \zeta^n\| \\ &\le C\alpha\mu\|\zeta^{n+1} - \widehat{\zeta^n}\|,\end{aligned}$$

where the same reasons for obtaining (3.16) are used in the second step, and for the last step we use the estimates (5.10) and (2.20). It is inferred from taking $\widehat{v_h^n} = \zeta^{n+1} - \widehat{\zeta^n}$ in (5.15) and (2.20) that

$$\sum_{j=1}^{N} \mathcal{H}_j(\eta, \widetilde{\zeta^{n+1}} - \zeta^n) \le C\Big(\tau h + \alpha\mu\tau + \tau^2\Big)\|\zeta^{n+1} - \widehat{\zeta^n}\|.$$

Moreover, by the boundedness (3.7) of $\mathcal{A}$, we obtain,

$$\tau|\mathcal{A}(\widehat{\zeta^n}, \zeta^{n+1} - \widehat{\zeta^n})(t_{n+1})| \le \Big(C_{wx}\tau\|\widehat{\zeta^n}\| + \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![\zeta^n]\!]\Big)\|\zeta^{n+1} - \widehat{\zeta^n}\|.$$

In light of the equality (5.16), divide both sides by $\|\zeta^{n+1} - \widehat{\zeta^n}\|$ to yield,

$$\|\zeta^{n+1} - \widehat{\zeta^n}\| \le 2C_{wx}\tau\|\widehat{\zeta^n}\| + \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![\zeta^n]\!] + C(\tau h + \alpha\mu\tau + \tau^2).$$

Under the CFL condition (5.13), the above estimate leads to

$$\frac{1}{2}\|\zeta^{n+1} - \widehat{\zeta^n}\|^2 \le \frac{\alpha}{4}\tau[\![\zeta^n]\!]^2 + C\tau\|\zeta^n\|^2 + C\tau(h + \tau^2). \tag{5.22}$$

Here we also use (2.21). Consequently, the estimates (5.21)-(5.22) together with the energy identity (5.14) imply

$$\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2 \le C\tau\|\zeta^n\|^2 + C\tau(h + \tau^2).$$

Thus by the same arguments as in the P^k case, we can obtain the desired results for the case P^0,

$$\|e^n\|^2 \le C(h + \tau^2), \quad n \le M.$$

The proof is completed. □

5.2. Second order scheme

In this subsection, we will present the error estimate for the fully discrete scheme (4.14) with TVD-RK2 time-marching method. Similar to the first order case, we need first obtain the error equation.

5.2.1. Error equation

To obtain the error equation, we introduce the reference functions, which are in parallel to the TVD-RK2 time discretization stages. Similar to the first order case, we consider on the reference cell. To be more specific, let $\check{U}^{(0)}(\xi, t) = \Delta_j(t)\check{u}(\xi, t)$ be the exact solution of the equation (5.3) in the j-th cell, and

$$\check{U}^{(1)}(\xi, t) = \check{U}^{(0)}(\xi, t) + \tau\check{U}_t^{(0)}(\xi, t). \tag{5.23}$$

Then define

$$\check{u}^{(0)}(\xi, t) = \frac{1}{\Delta_j(t)}\check{U}^{(0)}(\xi, t), \quad \check{u}^{(1)}(\xi, t) = \frac{1}{\Delta_j(t + \tau)}\check{U}^{(1)}(\xi, t).$$

Denote $u^{n,l} = u^{(l)}(x, t_n) = \check{u}^{(l)}(\xi, t_n)$ for any time level n and $l = 0,\ 1$. Now we are ready to state the following lemma, which describes the local truncation error in time.

Lemma 5.4. *Let u be the exact solution of equation (1.1). Suppose u is sufficiently smooth with bounded derivatives, then for any $v_h^n \in V_h(t_n)$ and $1 \le j \le N$, there hold,*

$$(u^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} = (u^n, v_h^n)_{K_j^n} - \tau\mathcal{A}(u^n, v_h^n)_{K_j^n}, \tag{5.24}$$

$$\begin{aligned}(u^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} &= \frac{1}{2}(u^n, v_h^n)_{K_j^n} + \frac{1}{2}(u^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} \\ &\quad - \frac{\tau}{2}\mathcal{A}(u^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} + (\varepsilon_2^n, v_h^n)_{K_j^n},\end{aligned} \tag{5.25}$$

where $\widehat{v_h^n}$ is defined by (2.12), ε_2^n is the local truncation error in time and $\|\varepsilon_2^n\|_{K_j^n} = \mathcal{O}(\tau^3)$ for any j and n.

Proof. By the Taylor expansion with Lagrange form of the remainder and the definitions of the reference functions (5.23), it is not difficult to derive,

$$\begin{aligned}\check{U}^{(1)}(\xi,t) &= \check{U}^{(0)}(\xi,t) - \tau(a\check{U}^{(0)})_\xi(\xi,t), \\ \check{U}(\xi,t+\tau) &= \frac{1}{2}\check{U}^{(0)}(\xi,t) + \frac{1}{2}\check{U}^{(1)}(\xi,t) - \frac{\tau}{2}[a(\xi,t+\tau)\check{U}^{(1)}(\xi,t)]_\xi + \varepsilon(\xi,t),\end{aligned}$$

where

$$\varepsilon(\xi,t) = \frac{\tau^3}{6}\check{U}_{ttt}(t_{21}) + \frac{\tau^3}{2}[(a_t\check{U}_t)(t) + \frac{1}{2}a_{tt}(t_{22})\check{U}^{(1)}(t)]_\xi, \quad t_{21},\ t_{22} \in (t, t+\tau).$$

Recalling the definition of $\check{U}$ and a in (5.3) as well as $\Delta_j^{'}(t) = \omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}$, we have,

$$\begin{aligned}&\check{U}_t = \Delta_j(t)\check{u}_t + \Delta_j^{'}(t)\check{u}, \quad \check{U}_{ttt} = \Delta_j(t)\check{u}_{ttt} + 3\Delta_j^{'}(t)\check{u}_{tt}, \\ &a_t = -\frac{2\Delta_j^{'}}{\Delta_j^2(t)}(\beta - \check{\omega}), \quad a_{tt} = \frac{(2\Delta_j^{'})^2}{\Delta_j^3(t)}(\beta - \check{\omega}).\end{aligned}$$

Let $t = t_n$ and we still use the notations t_{21} and t_{22} to stand for fixed values between t_n and t_{n+1}. The scaling arguments imply that

$$\begin{aligned}\int_{-1}^{1}\check{U}_{ttt}(t_{21})\check{v}_h^n \mathrm{d}\xi &= \int_{-1}^{1}[\Delta_j\check{u}_{ttt} + 3\Delta_j^{'}(t)\check{u}_{tt}](t_{21})\check{v}_h^n \mathrm{d}\xi \\ &= \frac{2\Delta_j(t_{21})}{\Delta_j^n}(\tilde{u}_{ttt}, v_h^n)_{K_j^n} + \frac{6\Delta_j^{'}}{\Delta_j^n}(\tilde{u}_{tt}, v_h^n)_{K_j^n},\end{aligned}$$

where $\tilde{u}_{ttt}(\chi(\cdot,t_n)) = u_{ttt}(x(\cdot,t_{21}))$, $\tilde{u}_{tt}(\chi(\cdot,t_n)) = u_{tt}(x(\cdot,t_{21}))$ for $\chi(\cdot,t_n) \in K_j^n$, $x \in K_j(t_{21})$. Similarly,

$$\begin{aligned}\int_{-1}^{1}[a_t\check{U}_t]_\xi(t_n)\check{v}_h^n \mathrm{d}\xi &= -\frac{2\Delta_j^{'}}{\Delta_j^n}\int_{-1}^{1}[(\beta-\check{\omega})\check{u}_t]_\xi(t_n)\check{v}_h^n \mathrm{d}\xi - 2(\frac{\Delta_j^{'}}{\Delta_j^n})^2\int_{-1}^{1}[(\beta-\check{\omega})\check{u}]_\xi(t_n)\check{v}_h^n \mathrm{d}\xi \\ &= -\frac{2\Delta_j^{'}}{\Delta_j^n}\Big([(\beta-\omega)u_t]_x, v_h^n\Big)_{K_j^n} - 2(\frac{\Delta_j^{'}}{\Delta_j^n})^2\Big([(\beta-\omega)u]_x, v_h^n\Big)_{K_j^n},\end{aligned}$$

and

$$\begin{aligned}\int_{-1}^{1}[a_{tt}(t_{22})\check{U}^{(1)}(t_n)]_\xi \check{v}_h^n \mathrm{d}\xi &= \frac{(2\Delta_j^{'})^2}{\Delta_j^3(t_{22})}\int_{-1}^{1}[(\beta-\check{\omega})(\check{U}+\tau\check{U}_t)(t_n)]_\xi \check{v}_h^n \mathrm{d}\xi\\ &= \frac{(2\Delta_j^{'})^2\Delta_j^n}{\Delta_j^3(t_{22})}\int_{-1}^{1}[(\beta-\check{\omega})(\check{u}+\tau\check{u}_t)]_\xi(t_n)\check{v}_h^n\mathrm{d}\xi\\ &\quad+\frac{(2\Delta_j^{'})^3}{\Delta_j^3(t_{22})}\tau\int_{-1}^{1}[(\beta-\check{\omega})\check{u}]_\xi(t_n)\check{v}_h^n\mathrm{d}\xi\\ &= \frac{(2\Delta_j^{'})^2\Delta_j^n}{\Delta_j^3(t_{22})}\Big([(\beta-\omega)(u+\tau u_t)]_x(t_n),v_h^n\Big)_{K_j^n}\\ &\quad+\frac{(2\Delta_j^{'})^3}{\Delta_j^3(t_{22})}\tau\Big([(\beta-\omega)u]_x(t_n),v_h^n\Big)_{K_j^n}.\end{aligned}$$

As a result, we know that

$$\int_{-1}^{1}\varepsilon(\xi,t_n)\check{v}_h^n\mathrm{d}\xi = 2(\varepsilon_2^n,v_h^n)_{K_j^n},$$

where ε_2^n can be obtained by the above analysis and

$$\|\varepsilon_2^n\|_{K_j^n}\le C\tau^3.$$

Here we use the assumption that u is smooth enough, the quantity (2.4), (2.5)–(2.7) and the fact that $\tau\le 1$. The positive constant C is independent of τ, h and n. Finally, by the same arguments as Lemma 5.1, we obtain the desired results (5.24)–(5.25). □

As is customary in the error estimates, we introduce some notations. Denote the error at each stage by $e^{n,0}=u^n-u_h^n$ and $e^{n,1}=u^{n,1}-u_h^1$, where u_h^n and u_h^1 are the solutions of the fully discrete scheme (4.14). For simplicity, denote $u_h^{n,0}=u_h$ and $u_h^{n,1}=u_h^1$. Let

$$\zeta^{n,l}=P_hu^{n,l}-u_h^{n,l},\ \eta^{n,l}=P_hu^{n,l}-u^{n,l},\ l=0,1,$$

where $P_hu^{n,l}$ is the L^2 projection defined by (2.23). Noting that $e^{n,l}=\zeta^{n,l}-\eta^{n,l}$, we obtain the error equation for any $v_h^n\in V_h(t_n)$ and $1\le j\le N$ by Lemma 5.4 and the scheme (4.14),

$$(\zeta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}=(\zeta^n,v_h^n)_{K_j^n}-\tau\mathcal{A}(\zeta^n,v_h^n)_{K_j^n}+\mathcal{L}_j^1(\eta,v_h^n),\tag{5.26}$$

$$(\zeta^{n+1},\widehat{v_h^n})_{K_j^{n+1}}=\frac{1}{2}(\zeta^n,v_h^n)_{K_j^n}+\frac{1}{2}(\zeta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}-\frac{\tau}{2}\mathcal{A}(\zeta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}+\mathcal{L}_j^2(\eta,v_h^n),\tag{5.27}$$

where

$$\mathcal{L}_j^1(\eta,v_h^n)=(\eta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}-(\eta^n,v_h^n)_{K_j^n}+\tau\mathcal{A}(\eta^n,v_h^n)_{K_j^n},\tag{5.28}$$

$$\begin{aligned}\mathcal{L}_j^2(\eta,v_h^n)&=(\eta^{n+1},\widehat{v_h^n})_{K_j^{n+1}}-\frac{1}{2}(\eta^n,v_h^n)_{K_j^n}-\frac{1}{2}(\eta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}\\&\quad+\frac{\tau}{2}\mathcal{A}(\eta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}+(\varepsilon_2^n,v_h^n)_{K_j^n}.\end{aligned}\tag{5.29}$$

By the scaling arguments, we have

$$(\zeta^n,v_h^n)_{K_j^n}=(1-s_2)(\widehat{\zeta^n},\widehat{v_h^n})_{K_j^{n+1}},\ \mathcal{A}(\zeta^n,v_h^n)_{K_j^n}=\mathcal{A}(\widehat{\zeta^n},\widehat{v_h^n})_{K_j^{n+1}},$$

which indicates by a direct calculation,

$$(\zeta^{n,1} - \widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} = -s_2(\widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} - \tau\mathcal{A}(\widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} + \mathcal{L}_j^1(\eta, v_h^n), \tag{5.30}$$

$$(\zeta^{n+1} - \zeta^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} = -\frac{\tau}{2}\mathcal{A}(\zeta^{n,1} - \widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} + \mathcal{L}_j^2(\eta, v_h^n) - \frac{1}{2}\mathcal{L}_j^1(\eta, v_h^n). \tag{5.31}$$

Next, we will list some estimates for the projection error. The analysis is the same as that in Lemma 5.2, thus we only present the results without the detailed proof.

Lemma 5.5. *Suppose u is sufficiently smooth with bounded derivatives, then there exists a positive constant C independent of h, τ and n, such that for any $v_h^n \in H_h^1(t_n)$ and $\forall n \leq M$,*

$$\|\eta^{n,l}\| + h^{1/2}\|\eta^{n,l}\|_{\Gamma_h(t_{n+l})} + h\|\partial_x \eta^{n,l}\| \leq Ch^{k+1}, \ l = 0, 1, \tag{5.32}$$

$$d_1(\eta^{n+1}, \widehat{v_h^n}) + d_2(\eta^{n,1}, \widehat{v_h^n}) + d_3(\eta^n, v_h^n) \leq C\tau h^{k+1}\|v_h^n\|, \tag{5.33}$$

with any three constants restricted by $d_1 + d_2 + d_3 = 0$.

Based on the above estimates, we can easily get the following estimates, which is important for our analysis.

Lemma 5.6. *Suppose u is sufficiently smooth with bounded derivatives, then we have the following estimates, for $m = 1, 2$,*

$$\sum_{j=1}^{N} \mathcal{L}_j^m(\eta, v_h^n) \leq C(\tau h^{k+1} + \delta_{2m}\tau^3)\|v_h^n\| + C\alpha\tau h^{k+\frac{1}{2}}[\![v_h^n]\!], \tag{5.34}$$

$$\sum_{j=1}^{N} \mathcal{L}_j^m(\eta, v_h^n) \leq C(\tau h^{k+1} + \delta_{2m}\tau^3 + \alpha\mu\tau h^k)\|v_h^n\|, \tag{5.35}$$

where δ_{2m} is the Kronecker symbol.

Proof. It is straightforward to obtain the desired results by a combination of the estimates in Lemmas 3.3, 5.4 and 5.5. Here we also need use the property (2.21). □

5.2.2. Error estimate for the second order scheme

Theorem 5.7. *Let u_h^n be the numerical solution of the fully discrete scheme (4.14) with TVD-RK2 time-marching method, and u be the exact solution of equation (1.1). Suppose u is sufficiently smooth with bounded derivatives, then we have the following error estimate,*

$$\max_{n\tau \leq T} \|u(x, t_n) - u_h^n\| \leq C(h^{k+\frac{1}{2}} + \tau^2),$$

under the CFL condition

$$\tau \leq \rho h^{\frac{4}{3}}, \tag{5.36}$$

with any given positive constant ρ. In particular, for the piecewise linear finite element space, $V_h(t) = \{v(\chi_j(\cdot, t)) \in P^1([-1, 1])\}$, we just need the usual CFL condition,

$$\tau \leq \rho h. \tag{5.37}$$

Here ρ is a suitable positive constant depends solely on α and μ, where α is defined by (3.3) and μ is the inverse constant (2.26). The positive constant C is independent of h, τ, n and u_h.

Proof. To derive the energy identity for ζ^n, we take the test function $\widehat{v_h^n} = \frac{1}{2}\widehat{\zeta^n}$, $\zeta^{n,1}$ in the error equation (5.26) and (5.27), respectively, and add them together to yield,

$$\begin{aligned}\frac{1}{2}\|\zeta^{n+1}\|^2 - \frac{1}{2}\|\zeta^n\|^2 = \frac{1}{2}\|\zeta^{n+1} - \zeta^{n,1}\|^2 + \sum_{j=1}^{N}\frac{s_2}{4}\|\zeta^{n,1} - \widehat{\zeta^n}\|^2_{K_j^{n+1}} \\ - \frac{\alpha}{4}\tau[\![\zeta^n]\!]^2 - \frac{\alpha}{4}\tau[\![\zeta^{n,1}]\!]^2 + \sum_{j=1}^{N}\mathcal{L}_j^4(\eta,\zeta^n,\zeta^{n,1}),\end{aligned} \tag{5.38}$$

where

$$\mathcal{L}_j^4(\eta,\zeta^n,\zeta^{n,1}) = \frac{1}{2}\mathcal{L}_j^1(\eta,\zeta^n) + \mathcal{L}_j^2(\eta,\widetilde{\zeta^{n,1}}).$$

The following proof is decomposed into four steps.

Step 1. Bound on $\sum_{j=1}^N \mathcal{L}_j^4$. By the estimate (5.34) in Lemma 5.6, we get,

$$\begin{aligned}\sum_{j=1}^{N}\mathcal{L}_j^4(\eta,\zeta^n,\zeta^{n,1}) &\le C\tau h^{k+1}(\|\zeta^n\| + \|\widetilde{\zeta^{n,1}}\|) + C\tau^3\|\widetilde{\zeta^{n,1}}\| + C\alpha\tau h^{k+\frac{1}{2}}([\![\zeta^n]\!] + [\![\widetilde{\zeta^{n,1}}]\!]) \\ &\le \tau\|\zeta^n\|^2 + \tau\|\zeta^{n,1}\|^2 + \frac{\alpha}{8}\tau([\![\zeta^n]\!]^2 + [\![\zeta^{n,1}]\!]^2) + C\tau(h^{2k+1} + \tau^4),\end{aligned} \tag{5.39}$$

here we use the Young's inequality and (2.20).

Step 2. Bound on $\|\zeta^{n,1} - \widehat{\zeta^n}\|$. Take the test function $\widehat{v_h^n} = \zeta^{n,1} - \widehat{\zeta^n}$ in the equality (5.30) and sum over all j to yield,

$$\begin{aligned}\|\zeta^{n,1} - \widehat{\zeta^n}\|^2 \le (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\widehat{\zeta^n}\|\|\zeta^{n,1} - \widehat{\zeta^n}\| \\ + C(\tau h^{k+1} + \alpha\mu\tau h^k)\|\widetilde{\zeta^{n,1}} - \zeta^n\|.\end{aligned}$$

Here we use the Cauchy–Schwarz inequality, the boundedness (3.6) of $\mathcal{A}$ and the estimate (5.35). We also use the fact that $s_2 = \omega_x(t_{n+1})\tau$ and the assumption (2.7). By the property (2.20) and dividing both sides by $\|\zeta^{n,1} - \widehat{\zeta^n}\|$, we have,

$$\|\zeta^{n,1} - \widehat{\zeta^n}\| \le (C_{wx}\tau + 3\alpha\mu\tau h^{-1})\|\widehat{\zeta^n}\| + C(\tau h^{k+1} + \alpha\mu\tau h^k). \tag{5.40}$$

Step 3. Bound on $\|\zeta^{n,1}\|$. Taking the test function $\widehat{v_h^n} = \zeta^{n,1}$ in the error equation (5.26) and following the same lines as that in Step 2, we can easily get the boundedness of $\|\zeta^{n,1}\|$,

$$\|\zeta^{n,1}\| \le C(1 + 3\alpha\mu\tau h^{-1})\|\zeta^n\| + C(\tau h^{k+1} + \alpha\mu\tau h^k). \tag{5.41}$$

Step 4. Bound on $\|\zeta^{n+1} - \zeta^{n,1}\|$. This step will be divided into two cases, the general P^k case and the P^1 case, which is the same as in the stability analysis. Denote $\lambda = \alpha\mu\tau h^{-1}$ for simplicity.

$\mathbf{P^k}$ case. Take the test function $\widehat{v_h^n} = \zeta^{n+1} - \zeta^{n,1}$ in the equality (5.31) and apply the estimate (3.6) of $\mathcal{A}$ as well as (5.35) to derive,

$$\begin{aligned}\|\zeta^{n+1} - \zeta^{n,1}\| &\le \frac{3}{2}\alpha\mu\tau h^{-1}\|\zeta^{n,1} - \widehat{\zeta^n}\| + C(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^3) \\ &\le \frac{3}{2}\lambda(C_{wx}\tau + 3\lambda)\|\widehat{\zeta^n}\| + C(1 + \frac{3}{2}\lambda)(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^3).\end{aligned}$$

Here we use the estimate (5.40). If the time-step satisfies $\lambda^4 \le \rho\tau$ for any positive constant ρ, the above inequality indicates that

$$\frac{1}{2}\|\zeta^{n+1}-\zeta^{n,1}\|^2 \le C\tau\|\widehat{\zeta^n}\|^2 + C(\tau h^{2k+1}+\tau^5) \\ \le C\tau\|\zeta^n\|^2 + C(\tau h^{2k+1}+\tau^5), \tag{5.42}$$

where we use (2.21) and $\tau \le 1$. In addition, the estimates (5.40)–(5.41) turn out to be

$$\|\zeta^{n,1}-\widehat{\zeta^n}\| \le C\|\widehat{\zeta^n}\| + Ch^{k+1} \le C\|\zeta^n\| + Ch^{k+1}, \\ \|\zeta^{n,1}\| \le C\|\zeta^n\| + Ch^{k+1},$$

under the CFL condition (5.36). Then combine the energy identity (5.38), the estimates (5.39) and (5.42) to get,

$$\frac{1}{2}\|\zeta^{n+1}\|^2 - \frac{1}{2}\|\zeta^n\|^2 \le C\tau\|\zeta^n\|^2 + \tau\|\zeta^{n,1}\|^2 + C\tau(h^{2k+1}+\tau^4) + \frac{C_{wx}}{4}\tau\|\zeta^{n,1}-\widehat{\zeta^n}\|^2 \\ \le C\tau\|\zeta^n\|^2 + C\tau(h^{2k+1}+\tau^4).$$

Summing over n, using Gronwall's inequality and choosing the initial condition $u_h(x,0) = P_h u(x,0)$, we obtain,

$$\|\zeta^n\|^2 \le C(h^{2k+1}+\tau^4), \quad n \le M.$$

Finally, apply the estimate (5.32) to yield,

$$\|e^n\|^2 \le C(h^{2k+1}+\tau^4), \quad n \le M.$$

$\mathbf{P^1}$ **case**. Denote $z = \zeta^{n,1} - \widehat{\zeta^n}$. Choosing the test function $\widehat{v_h^n} = \zeta^{n+1} - \zeta^{n,1}$ in the equality (5.31) and using the estimate (3.7) of $\mathcal{A}$ as well as (5.35), we have,

$$\|\zeta^{n+1}-\zeta^{n,1}\| \le \frac{\alpha}{2}\tau\|z_x\| + \frac{C_{wx}}{2}\tau\|z\| + \frac{\sqrt{2}}{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![z]\!] + C(\tau h^2 + \alpha\mu\tau h + \tau^3). \tag{5.43}$$

Now we analyze $\|z_x\|$. Similar to the stability analysis, let $y = z - P_h^0 z$ and $P_h^0 z$ is the L^2 projection into the piecewise constant space. By the property of the L^2 projection and taking $\widehat{v_h^n} = y$ in the equality (5.30), we get,

$$\|y\|^2 = \sum_{j=1}^{N}(z,y)_{K_j^{n+1}} \le \left((\mu+2)C_{wx}\tau\|\widehat{\zeta^n}\| + \sqrt{2}\alpha\mu\tau h^{-\frac{1}{2}}[\![\zeta^n]\!]\right)\|y\| \\ + C(\tau h^2 + \alpha\mu\tau h)\|y\|. \tag{5.44}$$

Here we use the boundedness (3.8) of $\mathcal{A}$ and the estimate (5.35). Divide both sides of the above inequality by $\|y\|$ to obtain the estimate of $\|y\|$. In addition,

$$\|z_x\| = \|y_x\| \le \mu h^{-1}\|y\|. \tag{5.45}$$

Let $\tau \le \rho h$ with a positive constant ρ independent of τ and h, we will show the restriction of ρ in the following. Collect the estimates (5.43)–(5.45) to yield,

$$\|\zeta^{n+1}-\zeta^{n,1}\| \le \left(1+\frac{\sqrt{2}}{2}\alpha\mu\rho\right)\alpha\mu\tau h^{-\frac{1}{2}}[\![\zeta^n]\!] + \alpha\mu\tau h^{-\frac{1}{2}}[\![\zeta^{n,1}]\!] + \mathbb{L},$$

where

$$\mathbb{L} = C\rho\tau\|\widehat{\zeta^n}\| + \frac{C_{wx}}{2}\tau\|z\| + C\left(1 + \frac{\alpha\mu}{2}\rho\right)(\tau h^2 + \alpha\mu\tau h + \tau^3).$$

Hence,

$$\begin{aligned}\frac{1}{2}\|\zeta^{n+1} - \zeta^{n,1}\|^2 \leq (1 + \frac{\sqrt{2}}{2}\alpha\mu\rho)^2(\alpha\mu)^2\tau^2 h^{-1}[\![\zeta^n]\!]^2 \\ + 2(\alpha\mu)^2\tau^2 h^{-1}[\![\zeta^{n,1}]\!]^2 + 2\mathbb{L}^2.\end{aligned}$$

If

$$2(\alpha\mu)^2\tau^2 h^{-1} \leq \frac{\alpha}{16}\tau,\ (\alpha\mu)^4\rho^2\tau^2 h^{-1} \leq \frac{\alpha}{16}\tau,\ 2(\alpha\mu)^2\tau^2 h^{-1} \leq \frac{\alpha}{8}\tau,$$

that is,

$$\rho \leq \min\left\{\frac{1}{32\alpha\mu^2}, \frac{1}{\alpha\sqrt[3]{16\mu^4}}\right\},$$

then we have

$$\frac{1}{2}\|\zeta^{n+1} - \zeta^{n,1}\|^2 \leq \frac{\alpha}{8}\tau[\![\zeta^n]\!]^2 + \frac{\alpha}{8}\tau[\![\zeta^{n,1}]\!]^2 + 2\mathbb{L}^2. \tag{5.46}$$

With the CFL condition (5.37) and the property (2.21), the estimates (5.40)–(5.41) turn out to be,

$$\|z\| \leq C\|\zeta^n\| + Ch^2, \quad \|\zeta^{n,1}\| \leq C\|\zeta^n\| + Ch^2, \tag{5.47}$$

which imply that,

$$\mathbb{L}^2 \leq C\tau\|\zeta^n\|^2 + C\tau h^3 + \tau^5. \tag{5.48}$$

In light of the energy identity (5.38), we combine the estimates (5.39) and (5.46)–(5.48) to obtain,

$$\begin{aligned}\frac{1}{2}\|\zeta^{n+1}\|^2 - \frac{1}{2}\|\zeta^n\|^2 &\leq C\tau\|\zeta^n\|^2 + C\tau\|\zeta^{n,1}\|^2 + \frac{C_{wx}}{4}\tau\|z\|^2 + C\tau h^3 + \tau^5 \\ &\leq C\tau\|\zeta^n\|^2 + C\tau h^3 + \tau^5.\end{aligned}$$

In the end, by the same arguments as the general P^k case, we can obtain the desired results for the case P^1,

$$\|e^n\|^2 \leq C(h^3 + \tau^4), \quad n \leq M.$$

The proof is completed. □

5.3. Third order scheme

In this subsection, we will present the error estimate for the fully discrete scheme (4.33) with TVD-RK3 time-marching method. We begin with the error equation.

5.3.1. Error equation

Similar to second order case, the reference functions will be introduced to obtain the error equation. Considering the equation (5.3), define $\check{U}^{(0)}(\xi,t)=\Delta_j(t)\check{u}(\xi,t)$ as the exact equation, and

$$\begin{aligned}\check{U}^{(1)}(\xi,t)&=\check{U}^{(0)}(\xi,t)-\tau(a\check{U}^{(0)})_\xi(\xi,t),\\ \check{U}^{(2)}(\xi,t)&=\frac{3}{4}\check{U}^{(0)}(\xi,t)+\frac{1}{4}\check{U}^{(1)}(\xi,t)-\frac{\tau}{4}\Big(a(\xi,t+\tau)\check{U}^{(1)}(\xi,t)\Big)_\xi.\end{aligned}$$

Then let

$$\check{u}^{(0)}=\frac{1}{\Delta_j(t)}\check{U}^{(0)},\ \check{u}^{(1)}=\frac{1}{\Delta_j(t+\tau)}\check{U}^{(1)},\ \check{u}^{(2)}=\frac{1}{\Delta_j(t+\frac{\tau}{2})}\check{U}^{(2)},$$

here we omit the same symbol (ξ,t) on both sides of the above equalities. By the standard explicit TVD-RK3 scheme for the equation (5.3) and the same idea as that in Lemma 5.1 and Lemma 5.4, we can easily obtain the following lemma, which describes the local truncation error in time. Before doing that, denote $u^{n,l}=u^{(l)}(x,t_n)=\check{u}^{(l)}(\xi,t_n)$ for any time level n and $l=0,\,1,\,2$.

Lemma 5.8. *Let u be the exact solution of equation (1.1). Suppose u is sufficiently smooth with bounded derivatives, then for any $v_h^n\in V_h(t_n)$ and $1\le j\le N$, there hold,*

$$\begin{aligned}(u^{n,1},\widehat{v_h^n})_{K_j^{n+1}}&=(u^n,v_h^n)_{K_j^n}-\tau\mathcal{A}(u^n,v_h^n)_{K_j^n},\\ (u^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}&=\frac34(u^n,v_h^n)_{K_j^n}+\frac14(u^{n,1},\widehat{v_h^n})_{K_j^{n+1}}-\frac{\tau}{4}\mathcal{A}(u^{n,1},\widehat{v_h^n})_{K_j^{n+1}},\\ (u^{n+1},\widehat{v_h^n})_{K_j^{n+1}}&=\frac13(u^n,v_h^n)_{K_j^n}+\frac23(u^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}-\frac{2\tau}{3}\mathcal{A}(u^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}+(\varepsilon_3^n,v_h^n)_{K_j^n},\end{aligned}$$

where $\widehat{v_h^n}$ and $\overline{v_h^n}$ are defined by (2.12), ε_3^n is the local truncation error in time and $\|\varepsilon_3^n\|_{K_j^n}=\mathcal{O}(\tau^4)$ for any j and n.

Similar to the second order case, we denote the error at each stage by $e^{n,l}=u^{n,l}-u_h^{n,l}$ for any n and $l=0,\,1,\,2$, where $u_h^{n,l}=u_h^l$ with $l=1,\,2$ are the solutions of the fully discrete scheme (4.33) and $u_h^{n,0}=u_h^n$. In addition, the error can be rewritten as $e^{n,l}=\zeta^{n,l}-\eta^{n,l}$ with

$$\zeta^{n,l}=P_hu^{n,l}-u_h^{n,l},\ \eta^{n,l}=P_hu^{n,l}-u^{n,l},\ l=0,\ 1,\ 2.$$

Here $P_hu^{n,l}$ is the L^2 projection of $u^{n,l}$ defined by (2.23). We can obtain the error equation along with the scheme (4.33) and Lemma 5.8, for any $v_h^n\in V_h(t_n)$ and $1\le j\le N$,

$$\begin{aligned}(\zeta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}&=(\zeta^n,v_h^n)_{K_j^n}-\tau\mathcal{A}(\zeta^n,v_h^n)_{K_j^n}+\mathcal{T}_j^1(v_h^n),\\ (\zeta^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}&=\frac34(\zeta^n,v_h^n)_{K_j^n}+\frac14(\zeta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}-\frac{\tau}{4}\mathcal{A}(\zeta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}+\mathcal{T}_j^2(v_h^n),\\ (\zeta^{n+1},\widehat{v_h^n})_{K_j^{n+1}}&=\frac13(\zeta^n,v_h^n)_{K_j^n}+\frac23(\zeta^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}-\frac{2\tau}{3}\mathcal{A}(\zeta^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}+\mathcal{T}_j^3(v_h^n),\end{aligned}\tag{5.49}$$

where

$$\begin{aligned}\mathcal{T}_j^1(v_h^n)&=(\eta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}-(\eta^n,v_h^n)_{K_j^n}+\tau\mathcal{A}(\eta^n,v_h^n)_{K_j^n},\\ \mathcal{T}_j^2(v_h^n)&=(\eta^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}-\frac34(\eta^n,v_h^n)_{K_j^n}-\frac14(\eta^{n,1},\widehat{v_h^n})_{K_j^{n+1}}+\frac{\tau}{4}\mathcal{A}(\eta^{n,1},\widehat{v_h^n})_{K_j^{n+1}},\\ \mathcal{T}_j^3(v_h^n)&=(\eta^{n+1},\widehat{v_h^n})_{K_j^{n+1}}-\frac13(\eta^n,v_h^n)_{K_j^n}-\frac23(\eta^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}+\frac{2\tau}{3}\mathcal{A}(\eta^{n,2},\overline{v_h^n})_{K_j^{n+\frac12}}+(\varepsilon_3^n,v_h^n)_{K_j^n}.\end{aligned}$$

Similar to the stability analysis, we introduce some notations for simplicity,

$$\mathbb{D}_1 = \zeta^{n,1} - \widehat{\zeta^n},\ \mathbb{D}_2 = 2\widehat{\zeta^{n,2}} - \zeta^{n,1} - \widehat{\zeta^n},\ \mathbb{D}_3 = \zeta^{n+1} - 2\widehat{\zeta^{n,2}} + \widehat{\zeta^n}. \tag{5.50}$$

Recalling the fact that

$$(\zeta^n, v_h^n)_{K_j^n} = (1 - s_2)(\widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}},\ (\zeta^{n,2}, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} = (1 - \frac{s_2}{2})(\widehat{\zeta^{n,2}}, \widehat{v_h^n})_{K_j^{n+1}},$$

we follow the same lines as that in obtaining (4.36) to derive,

$$\begin{aligned}
(\mathbb{D}_1, \widehat{v_h^n})_{K_j^{n+1}} &= -s_2(\widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} - \tau\mathcal{A}(\widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} + \mathcal{T}_j^1(v_h^n),\\
(\mathbb{D}_2, \widehat{v_h^n})_{K_j^{n+1}} &= -s_2(\widehat{\zeta^n} - \widehat{\zeta^{n,2}}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{2}\mathcal{A}(\mathbb{D}_1, \widehat{v_h^n})_{K_j^{n+1}} + \mathcal{T}_j^4(v_h^n),\\
(\mathbb{D}_3, \widehat{v_h^n})_{K_j^{n+1}} &= -s_2(\widehat{\zeta^{n,2}} - \widehat{\zeta^n}, \widehat{v_h^n})_{K_j^{n+1}} - \frac{\tau}{3}\mathcal{A}(\mathbb{D}_2, \widehat{v_h^n})_{K_j^{n+1}} + \mathcal{T}_j^5(v_h^n),
\end{aligned} \tag{5.51}$$

where

$$\begin{aligned}
\mathcal{T}_j^4(v_h^n) &= 2(\eta^{n,2}, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} - (\eta^n, v_h^n)_{K_j^n} - (\eta^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} + \frac{\tau}{2}\mathcal{A}(\eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}},\\
\mathcal{T}_j^5(v_h^n) &= (\eta^{n+1}, \widehat{v_h^n})_{K_j^{n+1}} - 2(\eta^{n,2}, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} + (\eta^n, v_h^n)_{K_j^n} + (\varepsilon_3^n, v_h^n)_{K_j^n}\\
&\quad + \frac{\tau}{3}\mathcal{A}(2\widehat{\eta^{n,2}} - \eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}}.
\end{aligned} \tag{5.52}$$

Similar to the first and second order case, some estimates for the projection error will be shown. The proof follows the same lines as that in proving Lemma 5.2, therefore we just list the results without the detailed proof.

Lemma 5.9. *Suppose u is sufficiently smooth with bounded derivatives, then there exists a positive constant C independent of h, τ and n, such that for $\forall n \le M$,*

$$\|\eta^{n,l}\| + h^{1/2}\|\eta^{n,l}\|_{\Gamma_h(t_{n+l})} + h\|\partial_x\eta^{n,l}\| \le Ch^{k+1},\ l = 0, 1, \tag{5.53}$$

$$\|\eta^{n,2}\| + h^{1/2}\|\eta^{n,2}\|_{\Gamma_h(t_{n+\frac{1}{2}})} + h\|\partial_x\eta^{n,2}\| \le Ch^{k+1}. \tag{5.54}$$

Moreover, for any $v_h^n \in H_h^1(t_n)$,

$$(d_1\eta^{n+1} + d_2\eta^{n,1}, \widehat{v_h^n}) + d_3(\eta^{n,2}, \overline{v_h^n}) + d_4(\eta^n, v_h^n) \le C\tau h^{k+1}\|v_h^n\|,$$

with any four constants restricted by $d_1 + d_2 + d_3 + d_4 = 0$.

Denote $\mathcal{T}^m(v_h^n) = \sum_{j=1}^N \mathcal{T}_j^m(v_h^n)$ for $m = 1, \ldots, 5$. It is straightforward to derive the following results by a combination of the estimates in Lemmas 3.3, 5.8 and 5.9 as well as in (2.20)–(2.22).

Lemma 5.10. *Suppose u is sufficiently smooth with bounded derivatives, then we have the following estimates, for $m = 1, \ldots, 5$,*

$$\mathcal{T}^m(v_h^n) \le C(\tau h^{k+1} + \delta_{3m}\tau^4 + \delta_{5m}\tau^4)\|\widehat{v_h^n}\| + C\alpha\tau h^{k+\frac{1}{2}}[\![v_h^n]\!], \tag{5.55}$$

$$\mathcal{T}^m(v_h^n) \le C(\tau h^{k+1} + \delta_{3m}\tau^4 + \delta_{5m}\tau^4 + \alpha\mu\tau h^k)\|\widehat{v_h^n}\|, \tag{5.56}$$

where δ_{3m} and δ_{5m} are the Kronecker symbols.

In particular, we also get the following results of the estimates for $\mathcal{T}^4$ and $\mathcal{T}^5$.

Lemma 5.11. *Suppose u is sufficiently smooth with bounded derivatives, then we have the following estimates, for $m = 4, 5$,*

$$\mathcal{T}^m(v_h^n) \leq C(\tau h^{k+1} + \delta_{5m}\tau^4 + \alpha\mu\tau^2 h^k)\|\widehat{v_h^n}\|. \tag{5.57}$$

Proof. By the scaling arguments and the quantities (2.15)–(2.17), we have,

$$(\eta^n, v_h^n)_{K_j^n} = (1 - s_2)(\widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}}, \ (\eta^{n,2}, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} = (1 - \frac{s_2}{2})(\widehat{\eta^{n,2}}, \widehat{v_h^n})_{K_j^{n+1}},$$

which implies

$$\begin{aligned}(\eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}} &= (\eta^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} - (\eta^n, v_h^n)_{K_j^n} - s_2(\widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}},\\ (2\widehat{\eta^{n,2}} - \eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}} &= 2(\eta^{n,2}, \overline{v_h^n})_{K_j^{n+\frac{1}{2}}} - (\eta^{n,1}, \widehat{v_h^n})_{K_j^{n+1}} - (\eta^n, v_h^n)_{K_j^n}\\ &\quad + s_2(\widehat{\eta^{n,2}} - \widehat{\eta^n}, \widehat{v_h^n})_{K_j^{n+1}}.\end{aligned}$$

It follows from the estimates in Lemma 5.9 and $s_2 = \omega_x(t_{n+1})\tau$ that

$$\begin{aligned}(\eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n}) &\leq C\tau h^{k+1}\|\widehat{v_h^n}\|,\\ (2\widehat{\eta^{n,2}} - \eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n}) &\leq C\tau h^{k+1}\|\widehat{v_h^n}\|.\end{aligned} \tag{5.58}$$

Here we use the relationship (2.20)–(2.22) and $\tau \leq 1$. Take $\widehat{v_h^n} = \eta^{n,1} - \widehat{\eta^n}$ in (5.58) and divide both sides by $\|\eta^{n,1} - \widehat{\eta^n}\|$ to obtain,

$$\|\eta^{n,1} - \widehat{\eta^n}\| \leq C\tau h^{k+1}.$$

By the inverse equality (2.26), we derive,

$$\|\eta^{n,1} - \widehat{\eta^n}\|_{\Gamma_h(t_{n+1})} \leq C\tau h^{k+\frac{1}{2}}.$$

Similarly, we have

$$\|2\widehat{\eta^{n,2}} - \eta^{n,1} - \widehat{\eta^n}\| \leq C\tau h^{k+1}, \ \|2\widehat{\eta^{n,2}} - \eta^{n,1} - \widehat{\eta^n}\|_{\Gamma_h(t_{n+1})} \leq C\tau h^{k+\frac{1}{2}}.$$

Thus the boundedness (3.16) of $\mathcal{A}$ yields ,

$$\begin{aligned}\mathcal{A}(\eta^{n,1} - \widehat{\eta^n}, \widehat{v_h^n})(t_{n+1}) &\leq \left(\mu C_{wx}\|\eta^{n,1} - \widehat{\eta^n}\| + 2\alpha\mu h^{-\frac{1}{2}}\|\eta^{n,1} - \widehat{\eta^n}\|_{\Gamma_h(t_{n+1})}\right)\|\widehat{v_h^n}\|\\ &\leq \tau\left(Ch^{k+1} + 2C\alpha\mu h^k\right)\|\widehat{v_h^n}\|.\end{aligned}$$

Together with Lemma 5.9, we can easily get,

$$\begin{aligned}\mathcal{T}^4(v_h^n) &\leq C(\tau h^{k+1} + \tau^2 h^{k+1} + \alpha\mu\tau^2 h^k)\|\widehat{v_h^n}\|\\ &\leq C(\tau h^{k+1} + \alpha\mu\tau^2 h^k)\|\widehat{v_h^n}\|,\end{aligned}$$

Since $\tau \leq 1$. We follow the same lines and use Lemma 5.8 to obtain,

$$\mathcal{T}^5(v_h^n) \leq C(\tau h^{k+1} + \tau^4 + \alpha\mu\tau^2 h^k)\|\widehat{v_h^n}\|.$$

The proof is completed. □

5.3.2. Error estimate for the third order scheme

Theorem 5.12. *Let u_h^n be the numerical solution of the fully discrete scheme (4.33) with TVD-RK3 time-marching method, and u be the exact solution of equation (1.1). Suppose u is sufficiently smooth with bounded derivatives, then we have the following error estimate,*

$$\max_{n\tau\le T}\|u(x,t_n)-u_h^n\|\le C(h^{k+\frac{1}{2}}+\tau^3),$$

under the CFL condition $\tau h^{-1}\le\rho$ with a fixed constant $\rho>0$. Here the positive constant C is independent of h, τ, n and u_h.

Proof. Similar to the stability analysis, we take the test function $v_h^n=\zeta^n$, $4\zeta^{n,1}$ and $6\zeta^{n,2}$ in the error equation (5.49), respectively, and add them together to obtain the identity for ζ^n,

$$\begin{aligned}3\|\zeta^{n+1}\|^2-3\|\widehat{\zeta^n}\|^2=\|\mathbb{D}_2\|^2+3(\zeta^{n+1}-\widehat{\zeta^n},\mathbb{D}_3)+\sum_{j=1}^N s_2\mathbb{G}_1\\ -\sum_{j=1}^N s_2\|\widehat{\zeta^n}\|^2_{K_j^{n+1}}-2\sum_{j=1}^N s_2\|\widehat{\zeta^{n,2}}\|^2_{K_j^{n+1}}-\tau\sum_{j=1}^N\mathbb{G}_2+\mathcal{T}^6,\end{aligned}\tag{5.59}$$

where

$$\begin{aligned}\mathbb{G}_1&=2(\widehat{\zeta^{n,2}},\zeta^{n,1}-\widehat{\zeta^n})_{K_j^{n+1}}-3(\widehat{\zeta^n},\zeta^{n,1})_{K_j^{n+1}},\\ \mathbb{G}_2&=\mathcal{A}(\widehat{\zeta^n},\widehat{\zeta^n})_{K_j^{n+1}}+\mathcal{A}(\zeta^{n,1},\zeta^{n,1})_{K_j^{n+1}}+4\mathcal{A}(\widehat{\zeta^{n,2}},\widehat{\zeta^{n,2}})_{K_j^{n+1}},\\ \mathcal{T}^6&=\mathcal{T}^1(\zeta^n)+4\mathcal{T}^2(\widetilde{\zeta^{n,1}})+6\mathcal{T}^3(\widetilde{\zeta^{n,2}}).\end{aligned}\tag{5.60}$$

Denote each line on the right hand side of (5.59) by Φ_1 and Φ_2, respectively. In light of the definitions (5.50), the equalities (5.51) and the properties (3.17)–(3.18) of $\mathcal{A}$, we follow the same lines as that in the stability analysis to derive,

$$\begin{aligned}\Phi_1&=\|\mathbb{D}_2\|^2+3(\mathbb{D}_3,\mathbb{D}_1+\mathbb{D}_2+\mathbb{D}_3)+\sum_{j=1}^N s_2\mathbb{G}_1\\ &=-\|\mathbb{D}_2\|^2+2(\mathbb{D}_2,\mathbb{D}_2)+3(\mathbb{D}_3,\mathbb{D}_1)+3(\mathbb{D}_3,\mathbb{D}_2)+3(\mathbb{D}_3,\mathbb{D}_3)+\sum_{j=1}^N s_2\mathbb{G}_1\\ &=-\|\mathbb{D}_2\|^2+3\|\mathbb{D}_3\|^2-\tau\mathcal{A}(\mathbb{D}_1,\mathbb{D}_2)-\tau\mathcal{A}(\mathbb{D}_2,\mathbb{D}_1)-\tau\mathcal{A}(\mathbb{D}_2,\mathbb{D}_2)\\ &\quad+\sum_{j=1}^N s_2(\widehat{\zeta^n}-\widehat{\zeta}^2,\mathbb{D}_2+3\mathbb{D}_1)_{K_j^{n+1}}+\sum_{j=1}^N s_2\mathbb{G}_1+\mathcal{T}^7\\ &=-\|\mathbb{D}_2\|^2+3\|\mathbb{D}_3\|^2-\tau\alpha\sum_{j=1}^N[\![\mathbb{D}_1]\!]_{j+\frac{1}{2}}[\![\mathbb{D}_2]\!]_{j+\frac{1}{2}}-\frac{\alpha}{2}\tau[\![\mathbb{D}_2]\!]^2+\sum_{j=1}^N s_2\Phi_{11}+\mathcal{T}^7,\end{aligned}$$

and

$$\begin{aligned}\Phi_{11}&=(\widehat{\zeta^n}-\widehat{\zeta^{n,2}},\mathbb{D}_2+3\mathbb{D}_1)_{K_j^{n+1}}+\mathbb{G}_1+(\mathbb{D}_1,\mathbb{D}_2)_{K_j^{n+1}}+\frac{1}{2}\|\mathbb{D}_2\|^2_{K_j^{n+1}},\\ \mathcal{T}^7&=2\mathcal{T}^4(\widetilde{\mathbb{D}_2})+3\mathcal{T}^5(\widetilde{\mathbb{D}_1})+3\mathcal{T}^5(\widetilde{\mathbb{D}_2}).\end{aligned}\tag{5.61}$$

A direct calculation of Φ_{11} gives that,

$$\Phi_{11} = -\frac{5}{2}\|\widehat{\zeta^n}\|^2_{K_j^{n+1}} - \frac{1}{2}\|\zeta^{n,1}\|^2_{K_j^{n+1}}.$$

Moreover, by the property (3.18) of $\mathcal{A}$, we get,

$$\begin{aligned}\Phi_2 = &-\frac{\alpha}{2}\tau[\![\zeta^n]\!]^2 - \frac{\alpha}{2}\tau[\![\zeta^{n,1}]\!]^2 - 2\alpha\tau[\![\zeta^{n,2}]\!]^2 \\ &- \sum_{j=1}^{N}\frac{s_2}{2}\|\widehat{\zeta^n}\|^2_{K_j^{n+1}} + \sum_{j=1}^{N}\frac{s_2}{2}\|\zeta^{n,1}\|^2_{K_j^{n+1}} + \mathcal{T}^6.\end{aligned}$$

Plug Φ_1 and Φ_2 in the identity (5.59) to obtain the energy equality for ζ^n,

$$3\|\zeta^{n+1}\|^2 - 3\|\zeta^n\|^2 = \Lambda_1 + \Lambda_2 + \Lambda_3, \tag{5.62}$$

where

$$\begin{aligned}\Lambda_1 &= -\|\mathbb{D}_2\|^2 + 3\|\mathbb{D}_3\|^2 - \tau\alpha\sum_{j=1}^{N}[\![\mathbb{D}_1]\!]_{j+\frac{1}{2}}[\![\mathbb{D}_2]\!]_{j+\frac{1}{2}} - \frac{\alpha}{2}\tau[\![\mathbb{D}_2]\!]^2, \\ \Lambda_2 &= -\frac{\alpha}{2}\tau[\![\zeta^n]\!]^2 - \frac{\alpha}{2}\tau[\![\zeta^{n,1}]\!]^2 - 2\alpha\tau[\![\zeta^{n,2}]\!]^2, \\ \Lambda_3 &= \mathcal{T}^6 + \mathcal{T}^7.\end{aligned}$$

Here $\mathcal{T}^6$ and $\mathcal{T}^7$ are defined by (5.60) and (5.61), respectively. The following proof is decomposed into five steps.

Step 1. Bound on $\|\mathbb{D}_3\|$. Take the test function $\widehat{v_h^n} = \mathbb{D}_3$ in the third equality of (5.51), sum all over all j and apply the boundedness (3.6) of $\mathcal{A}$ to obtain,

$$\begin{aligned}\|\mathbb{D}_3\| \le\ & C_{wx}\tau\left(\|\widehat{\zeta^{n,2}}\| + \|\widehat{\zeta^n}\|\right) + \alpha\mu\tau h^{-1}\|\mathbb{D}_2\| \\ &+ C(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^4).\end{aligned}$$

Here we use the estimate (5.56). Denote $\lambda = \alpha\mu\tau h^{-1}$ as before. It is inferred that

$$\begin{aligned}3\|\mathbb{D}_3\|^2 \le\ & 6\lambda^2\|\mathbb{D}_2\|^2 + C\tau^2\left(\|\widehat{\zeta^{n,2}}\|^2 + \|\widehat{\zeta^n}\|^2\right) \\ &+ C(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^4)^2.\end{aligned}$$

Step 2. Bound on Λ_1. By the Young's inequality and the above estimate, we have,

$$\begin{aligned}\Lambda_1 &\le \frac{\alpha}{8}\tau[\![\mathbb{D}_1]\!]^2 + \frac{3\alpha}{2}\tau[\![\mathbb{D}_2]\!]^2 - \|\mathbb{D}_2\|^2 + 3\|\mathbb{D}_3\|^2 \\ &\le \frac{\alpha}{4}\tau[\![\zeta^{n,1}]\!]^2 + \frac{\alpha}{4}\tau[\![\zeta^n]\!]^2 - (1 - 3\lambda - 6\lambda^2)\|\mathbb{D}_2\|^2 \\ &\quad + C\tau^2\left(\|\widehat{\zeta^{n,2}}\|^2 + \|\widehat{\zeta^n}\|^2\right) + C(\tau h^{k+1} + \alpha\mu\tau h^k + \tau^4)^2.\end{aligned}$$

Thus we obtain the time step restriction

$$(1 - 3\lambda - 6\lambda^2) > 0.$$

It is sufficient to choose $\lambda \leq \frac{1}{5}$. Then

$$\Lambda_1 \leq \frac{\alpha}{4}\tau[\![\zeta^{n,1}]\!]^2 + \frac{\alpha}{4}\tau[\![\zeta^{n}]\!]^2 + C(\tau h^{2k+1} + \tau^7) + C\tau\left(\|\widehat{\zeta^{n,2}}\|^2 + \|\widehat{\zeta^{n}}\|^2\right). \tag{5.63}$$

Here we use $\tau \leq 1$.

Step 3. Bound on $\mathcal{T}^6$. Recalling the definition (5.60) of $\mathcal{T}^6$, we take $v_h^n = \zeta^n$, $\widetilde{\zeta^{n,1}}$ and $\widetilde{\zeta^{n,2}}$ for $m = 1$, 2 and 3, respectively, in the estimate (5.55) to yield,

$$\begin{aligned}\mathcal{T}^6 &\leq C(\tau h^{k+1} + \tau^4)\left(\|\widehat{\zeta^n}\| + \|\zeta^{n,1}\| + \|\widehat{\zeta^{n,2}}\|\right)\\ &\quad + C\alpha\tau h^{k+\frac{1}{2}}\left([\![\zeta^n]\!] + [\![\zeta^{n,1}]\!] + [\![\zeta^{n,2}]\!]\right)\\ &\leq \tau\left(\|\widehat{\zeta^n}\|^2 + \|\zeta^{n,1}\|^2 + \|\widehat{\zeta^{n,2}}\|^2\right)\\ &\quad + \frac{\alpha}{4}\tau\left([\![\zeta^n]\!]^2 + [\![\zeta^{n,1}]\!]^2 + [\![\zeta^{n,2}]\!]^2\right) + C\tau(h^{2k+1} + \tau^6).\end{aligned} \tag{5.64}$$

Step 4. Bound on $\mathcal{T}^7$. Taking $\widehat{v_h^n} = \mathbb{D}_2$ and $\mathbb{D}_1 + \mathbb{D}_2$ in the representations (5.52) of $\mathcal{T}^4$ and $\mathcal{T}^5$, respectively, we obtain the following estimate by Lemma 5.11,

$$\begin{aligned}\mathcal{T}^7 &\leq C\tau(h^{k+1} + \tau^3 + \alpha\mu\tau h^k)(\|\mathbb{D}_1\| + \|\mathbb{D}_2\|)\\ &\leq \tau\|\mathbb{D}_1\|^2 + \tau\|\mathbb{D}_2\|^2 + C\tau(h^{2k+2} + \tau^6 + \alpha^2\mu^2\tau^2 h^{2k})\\ &\leq C\tau\left(\|\widehat{\zeta^{n,2}}\|^2 + \|\widehat{\zeta^n}\|^2 + \|\zeta^{n,1}\|^2\right) + C\tau(h^{2k+2} + \tau^6).\end{aligned} \tag{5.65}$$

Here we use the CFL condition $\tau \leq \rho h$ and the fact that

$$\|\mathbb{D}_1\|^2 + \|\mathbb{D}_2\|^2 \leq C\left(\|\widehat{\zeta^{n,2}}\|^2 + \|\widehat{\zeta^n}\|^2 + \|\zeta^{n,1}\|^2\right).$$

Step 5. Bound on $\|\zeta^{n,1}\|^2$ and $\|\widehat{\zeta^{n,2}}\|^2$. Take the test function $\widehat{v_h^n} = \zeta^{n,1}$ in the first equality of the error equation (5.49) to derive,

$$\begin{aligned}\|\zeta^{n,1}\|^2 &\leq \|\zeta^n\|\|\widetilde{\zeta^{n,1}}\| + 3\alpha\mu\tau h^{-1}\|\zeta^n\|\|\widetilde{\zeta^{n,1}}\|\\ &\quad + C(\tau h^{k+1} + \alpha\mu\tau h^k)\|\zeta^{n,1}\|\\ &\leq \left(C\|\zeta^n\| + Ch^{k+1}\right)\|\zeta^{n,1}\|.\end{aligned}$$

Here we use the Cauchy–Schwarz inequality, boundedness (3.6) of $\mathcal{A}$ and the estimate (5.56) of $\mathcal{T}^1$ for the first step. The CFL condition and (2.20) are used for the second step. The above estimate indicates that

$$\|\zeta^{n,1}\| \leq C\|\zeta^n\| + Ch^{k+1}. \tag{5.66}$$

Taking the test function $\overline{v_h^n} = \zeta^{n,2}$ in the second equality of the error equation (5.49) and by the similar analysis, we have,

$$\|\zeta^{n,2}\| \leq C\|\zeta^n\| + C\|\zeta^{n,1}\| + Ch^{k+1},$$

which implies

$$\|\widehat{\zeta^{n,2}}\| \leq C\|\zeta^n\| + C\|\zeta^{n,1}\| + Ch^{k+1}. \tag{5.67}$$

Here the properties (2.20)–(2.22) are used frequently. Finally, we combine the estimates (5.63)–(5.67) and the energy equality (5.62) together to yield,

$$3\|\zeta^{n+1}\|^2 - 3\|\zeta^n\|^2 \leq C\tau\|\zeta^n\|^2 + C\tau(h^{2k+1} + \tau^6).$$

Sum over all n, use the Gronwall's inequality and choose the initial condition $u_h(x,0) = P_h u(x,0)$ to obtain,

$$\|\zeta^n\|^2 \leq C(h^{2k+1} + \tau^6), \quad n \leq M.$$

We finish the proof by applying the estimate (5.53),

$$\|e^n\|^2 \leq C(h^{2k+1} + \tau^6), \quad n \leq M.$$

□

Remark 5.13. We remark that it is not difficult to extend the error estimates to the upwind flux, which also starts from the energy identity and the analysis follows the same ways as the Lax-Friedrichs flux case, then we can obtain the optimal error estimates. The main differences lie in two places. One is the properties of the ALE-DG operator $\mathcal{A}$, which is also changed owing to the choice of the flux, and the other one is the Gauss-Radau projections (2.24) instead of L^2 projection.

6. Conclusion

In this paper, we have analyzed the stability and error estimates of the fully discrete ALE-DG schemes for linear conservation laws, when explicit TVD-RK time-marching methods up to third order are adopted. The coordinate transformations and scaling arguments are the main techniques in our work, which have been used to control the additional quantities owing to time-dependent cells, function spaces and velocity grid field. We have assumed that the velocity grid field is a piecewise linear polynomial with respect to the spatial variable. Moreover, the velocity grid field and its weak derivative in space are assumed to be bounded in our analysis. These assumptions are helpful to satisfy the discrete geometric conservation laws and are significant to our proof. We have proven that the three fully discrete schemes are stable under the appropriate CFL conditions. In the first order fully discrete scheme, we have considered two cases, P^0 and P^k, $k \geq 1$. The main difference between them is that $\partial_x v_h$ vanishes with v_h belonging to the piecewise constant finite element space, and a usual CFL condition is sufficient to preserve stability for the case P^0. However, a more restrictive condition is required for the finite element space with polynomial degree $k \geq 1$. In the second order fully discrete scheme, the cases P^1 and P^k, $k \geq 2$ have been analyzed separately. For the piecewise linear finite element space, we have used the orthogonality property of a L^2 projection defined onto the piecewise constant finite element space and derived the stability under the usual CFL condition, while for higher order piecewise polynomials ($k \geq 2$), the above treatment broke down and we required a stronger CFL condition for stability. For the third order fully discrete scheme, the combinations of the numerical solutions in different time stages have been used to derive the stability under the usual CFL condition. In addition, we have obtained the quasi-optimal error estimates in space and optimal convergence rates in time for sufficiently smooth solutions. The ALE-DG method itself can be extended to conservation laws on a simplex mesh in two dimensions. The analysis of the fully discrete ALE-DG scheme in the two dimensional case is more technical and will be considered in the future.

Acknowledgements. The work of the second author was supported by NSFC grants No. 11471306, 11871449, and the third author by ARO grant W911NF-15-1-0226 and NSF grant DMS-1719410.

References

[1] P.G. Ciarlet, The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978).

[2] B. Cockburn, S. Hou and C.-W. Shu, The Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case. *Math. Comput.* **54** (1990) 545–581.

[3] B. Cockburn, G.E. Karniadakis and C.-W. Shu, Discontinuous Galerkin Methods-Theory, Computation and Applications. In vol. 11 of *Lecture notes in Computational Science and Engineering.* Springer (2000).

[4] B. Cockburn, S.-Y. Lin and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one dimensional systems. *J. Comput. Phys.* **84** (1989) 90–113.

[5] B. Cockburn and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework. *Math. Comput.* **52** (1989) 411–435.

[6] B. Cockburn and C.-W. Shu, The Runge–Kutta discontinuous Galerkin method for conservation laws V: multidimensional systems. *J. Comput. Phys.* **141** (1998) 199–224.

[7] B. Cockburn and C.-W. Shu, The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.* **35** (1998) 2440–2463.

[8] B. Cockburn and C.-W. Shu, Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. *J. Sci. Comput.* **16** (2001) 173–261.

[9] J. Donea, A. Huerta, J.-P. Ponthot and A. Rodríguez-Ferran, Arbitrary Lagrangian–Eulerian methods, edited by E. Stein, R. De Borst and T.J.R. Hughes. In vol. 1 of Encyclopedia of Computational Mechanics. Fundamentals. Wiley (2004) 413–437.

[10] E. Burman, A. Ern and M. A. Fernández, Explicit Runge–Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems. *SIAM J. Numer. Anal.* **48** (2010) 2019–2042.

[11] H. Guillard and C. Farhat, On the significance of the geometric conservation law for flow computations on moving meshes. *Comput. Methods Appl. Mech. Eng.* **190** (2000) 1467–1482.

[12] J.S. Hesthaven and T. Warburton, Nodal Discontinuous Galerkin Methods, Algorithms, Analysis, and Applications. Springer, New York, NY (2008).

[13] C. Klingenberg, G. Schnücke and Y. Xia, Arbitrary Lagrangian–Eulerian discontinuous Galerkin method for conservation laws: analysis and application in one dimension. *Math. Comput.* **86** (2017) 1203–1232.

[14] C. Klingenberg, G. Schnücke and Y. Xia, An arbitrary Lagrangian–Eulerian local discontinuous Galerkin method for Hamilton–Jacobi equations. *J. Sci. Comput.* **73** (2017) 906–942.

[15] D. Levy and E. Tadmor, From semidiscrete to fully discrete: stability of Runge–Kutta schemes by the energy method. *SIAM Rev.* **40** (1998) 40–73.

[16] I. Lomtev, R.M. Kirby and G.E. Karniadakis, A discontinuous Galerkin ALE method for compressible viscous flows in moving domains. *J. Comp. Phys.* **155** (1999) 128–159.

[17] V.-T. Nguyen, An arbitrary Lagrangian–Eulerian discontinuous Galerkin method for simulations of flows over variable geometries. *J. Fluids Struct.* **26** (2010) 312–329.

[18] P.-O. Persson, J. Bonet and J. Peraire, Discontinuous Galerkin solution of the Navier–Stokes equations on deformable domains. *Comput. Methods Appl. Mech. Eng.* **198** (2009) 1585–1595.

[19] W.H. Reed and T.R. Hill, Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory (1973).

[20] C.-W. Shu, Discontinuous Galerkin methods general: approach and stability, Numerical Solutions of Partial Differential Equations, edited by S. Bertoluzza, S. Falletta, G. Russo, and C.-W. Shu. *Advanced Courses in Mathematics.* CRM Barcelona, Birkhäuser, Besel, Switzerland (2009) 149–201.

[21] E. Tadmor, From semidiscrete to fully discrete: stability of Runge–Kutta schemes by the energy method II. Collected lectures on the preservation of stability under discretization. In vol. 109 of Proceedings in Applied Mathematics. SIAM (2002) 25–49.

[22] Q. Zhang, Third order explicit Runge–Kutta discontinuous Galerkin method for linear conservation law with inflow boundary condition. *J. Sci. Comput.* **46** (2011) 294–313.

[23] Q. Zhang and C.-W. Shu, Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for scalar conservation laws. *SIAM J. Numer. Anal.* **42** (2004) 641–666.

[24] Q. Zhang and C.-W. Shu, Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for symmetrizable systems of conservation laws. *SIAM J. Numer. Anal.* **44** (2006) 1703–1720.

[25] Q. Zhang and C.-W. Shu, Stability analysis and a priori error estimates to the third order explicit Runge–Kutta discontinuous Galerkin method for scalar conservation laws. *SIAM J. Numer. Anal.* **48** (2010) 1038–1063.

[26] Q. Zhang and C.-W. Shu, Error estimates for the third order explicit Runge–Kutta discontinuous Galerkin method for a linear hyperbolic equation in one-dimension with discontinuous initial data. *Numer. Math.* **126** (2014) 703–740.