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AN OPTIMIZED SCHWARZ METHOD WITH RELAXATION FOR THE HELMHOLTZ EQUATION: THE NEGATIVE IMPACT OF OVERLAP

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Abstract. In this paper we study how the overlapping size influences the convergence rate of an optimized Schwarz domain decomposition (DD) method with relaxation in the two subdomain case for the Helmholtz equation. Through choosing suitable parameters, we find that the convergence rate is independent of the wave number k and mesh size h, but sensitively depends on the overlapping size. Furthermore, by careful analysis, we obtain that the convergence behavior deteriorates with the increase of the overlapping size. Numerical results which confirm our theory are given.

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1. Introduction

The Helmholtz problem is widely used in acoustics, elasticity, electromagnetics, quantum mechanics and geophysics. Efficient and accurate numerical approximation to the Helmholtz equation is significant to scientific computation.

In this paper, we consider the following Helmholtz problem with the Sommerfeld radiation condition, *i.e.*

$$\begin{aligned} -\triangle u - k^2 u = f \quad \text{in } \Omega, \\ \lim_{r\to\infty} r^{1/2}\left(\frac{\partial u}{\partial r} - \mathrm{i}ku\right) = 0, \end{aligned} \tag{1.1}$$

where the domain $\Omega = \mathbb{R}^2$, k is the wave number, $r = (x^2+y^2)^{1/2}$, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. The well known Sommerfeld radiation condition is imposed to exclude the incoming wave for this unbounded problem. However, for practical computation, a bounded problem is necessary. As a result, a truncated problem with some transparent condition added on the artificial boundary should be considered. However, the transparent condition, which is an exact condition, is usually non-local and related to the Dirichlet to Neumann operator.

Keywords and phrases. optimized Schwarz method, the Helmholtz equation, overlapping size.

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So the absorbing condition, which is a local approximation of the transparent condition, is used instead for numerical simulation.

To solve the bounded Helmholtz problem, usually at least eight points are demanded per wavelength for the discretization (see [27] for details). Even worse, if the pollution effect is also introduced, a more rigorous condition should be satisfied, *i.e.* $k^r h =$ constant, where $r > 1$, which implies that the larger wave number k, the more points per wavelength are needed. So for high frequency problem, there is a huge discrete system. On the other hand, large k may induce highly indefinite discrete algebraic system. Due to these two reasons, it is difficult for traditional methods to solve the Helmholtz problem. So there are many modifications based on some traditional methods in the literature, such as shifted Laplacian [11–13, 26], multigrid [3, 22] and DD [15, 16, 19] methods, but nowadays how to solve this problem fast and efficiently is still a challenging problem [14]. It is known that DD methods are important tools with their high parallel performance. In this paper, we shall study a type of DD iterative method with Robin-type transmission conditions on the subdomain interfaces for the Helmholtz equation. Furthermore, we shall analyze the influence from the overlapping size.

For the nonoverlapping case, a Robin-like DD method was first proposed for elliptic problems by Agoshkov and Lebedev in [1], where they introduced relaxation parameters and then include the Dirichlet–Neumann method as a particular case of it (see also [25] for details). However, based on the classical Schwarz alternating method, Lions first proposed a more simplified method without the relaxation steps in [21], so his method can be extended to the many subdomain case. He proved the convergence of his DD method by energy analysis. From then on, most relative works are based on Lions' Robin DD method. Japhet in [20] and Gander in [17] propose the well known optimized Schwarz method by optimizing the Robin transmission condition. Recently, Chen *et al.* reconsidered the optimized Schwarz method with a relaxation step for two subdomains in [7], where they proved that for suitable choice of the parameters, their DD method is better than the method in [17]. Moreover, we could find that the choice of parameters makes their method close to the Dirichlet–Neumann method in spectrum, which leads to the good behavior. Actually, for the complete symmetric partition of the domain, the Dirichlet–Neumann method may convergence in two iterations by choosing the relaxation parameter to be $\frac{1}{2}$ (see [25] for details). However, for the overlapping case, the performance of Dirichlet–Neuman method deteriorates for small overlap but returns well for large overlap.

The idea of using Robin-type transmission conditions for the Helmholtz equation without the relaxation step was first proposed by Després in [9]. He employed an transmission condition, which is the zeroth order approximation to the exact Dirichlet to Neumann operator, and used energy method to prove the convergence of his DD method for many subdomains. Then Gander, Magoules, and Nataf improved the convergence rate by using optimized Schwarz method for two subdomains in [19] with optimized zeroth order and second order transmission conditions. Afterwards, the optimized Schwarz method with two-sided Robin-type transmission condition for the Helmholtz equation was further introduced by Gander *et al.* in [18], where they gave the analysis for two subdomains that the convergence rate of their DD method is $1 - O(h^{\frac{1}{4}})$ for fixed k and $1 - O(k^{-\frac{1}{8}})$ for $kh =$ constant. Referencing the method in [1] and [7] for the elliptic problem, Chen *et al.* added a relaxation step in [6] for the Helmholtz problem, and by choosing suitable parameters they proved that the convergence rate for this improved method is independent of k for $kh =$ constant and also independent of h for fixed frequency k in the two subdomain case. Similar as in [7], the best choice of parameters makes their method close to the Dirichlet–Neumann method. However, because of the well-posedness of the Helmholtz problem, it is better to keep the Robin form transmission condition on the interface rather than use the Dirichlet–Neumann method instead. Besides the Robin transmission conditions, there are many DD methods based on some other types of transmission condition. In [2], Boubendir *et al.* used the high order Padé approximating to the Dirichlet to Neumann operator in the many subdomain case. Compared to low order approximation, they find that the convergence performance is better, but with more computational cost to pay for high order approximation. In [8] and [29], authors use the Perfect Match Layer (PML) method as an approximation, which also give good numerical behavior in the many subdomain case with the strip like domain decomposition.

For the overlapping case, the Schwarz method was introduced in [4, 23] for the Helmholtz problem, where the authors use some absorbing condition instead of the Dirichlet condition for each subdomain. Meanwhile, a restricted additive Schwarz method was proposed in [5] by Cai and Sarkis with better convergence behavior. Furthermore, St-Cyr, Gander and Thomas presented an optimized restrict additive Schwarz method with better numerical performance in [28]. However, these overlapping works lack theoretical analysis. For a complete review of DD method, see [10].

As a complement and an overlapping extension of the method proposed in [6], in this paper, we shall study an overlapping optimized Schwarz method with relaxation in the two subdomain case for the Helmholtz equation. By careful analysis, we shall show in this paper that for small overlap, similar to the result in [6], the convergence rate is independent of k for $kh =$ constant and also independent of h for fixed frequency k. More important, we shall obtain the result that the convergence rate deteriorates with the increase of the overlapping size, which is different from the traditional Schwarz method for the elliptic problem. Furthermore, compared with the overlap case of Dirichlet–Neumann method for the elliptic problem, we find that the convergence behavior of our method is also different.

The outline of this paper is as follows: In Section 2, we shall present our optimized Schwarz method. In Section 3, we shall analyze the convergence rate and the influence of overlapping size. Numerical implementation shall be given in Section 4. Finally in Section 5, we present the numerical results which confirm our theory.

2. DD algorithm

Based on model problem (1.1), we introduce and analyze the DD algorithm.

We assume that the domain Ω is decomposed into two subdomains $\Omega_1 = (-\infty, x_1)\times\mathbb{R}$ and $\Omega_2 = (x_2, +\infty)\times\mathbb{R}$, where $x_1 \geqslant x_2$. Obviously, the interface for subdomain Ω_1 is $x = x_1$, which is denoted by Γ_1. Similarly, the interface $x = x_2$ for subdomain Ω_2 is denoted by Γ_2. Let n_i to be the unit outward normal vector of Ω_i at each interface Γ_i ($i = 1$ or 2). Let $f_1 = f|_{\Omega_1}, f_2 = f|_{\Omega_2}$, γ_1 and γ_2 be the two constant parameters whose values shall be determined later. Then the DD iterative procedure can be defined as follows:

(1) Solve for u_1^n in Ω_1

$$\begin{cases} -\triangle u_1^n - k^2 u_1^n = f_1 \quad \text{in } \Omega_1, \\ \dfrac{\partial u_1^n}{\partial n_1} + \gamma_1 u_1^n = g_1^n \quad \text{on } \Gamma_1, \\ \lim\limits_{r\to\infty} r^{1/2}\left(\dfrac{\partial u_1^n}{\partial r} - iku_1^n\right) = 0. \end{cases} \tag{2.1}$$

(2) Update the transmission condition along Γ_2

$$g_2^n = \frac{\partial u_1^n}{\partial n_2} + \gamma_2 u_1^n.$$

(3) Solve the subproblem in Ω_2 to obtain u_2^n

$$\begin{cases} -\triangle u_2^n - k^2 u_2^n = f_2 \quad \text{in } \Omega_2, \\ \dfrac{\partial u_2^n}{\partial n_2} + \gamma_2 u_2^n = g_2^n \quad \text{on } \Gamma_2, \\ \lim\limits_{r\to\infty} r^{1/2}\left(\dfrac{\partial u_2^n}{\partial r} - iku_2^n\right) = 0. \end{cases} \tag{2.2}$$

(4) Update the transmission condition along Γ_1

$$g_1^* = \frac{\partial u_2^n}{\partial n_1} + \gamma_1 u_2^n.$$

(5) Relax the transmission condition

$$g_1^{n+1} = \theta g_1^* + (1-\theta) g_1^n.$$

Through suitably choosing parameters $\gamma_1, \gamma_2, \theta$, and the overlapping size $x_1 - x_2$, we shall optimize the convergence rate.

3. The convergence rate estimate

In this section, our theoretical analysis will be given for the algorithm introduced in the previous section.

According to our DD iteration procedure, there exists a fixed point of the iterative procedure, which is the exact solution u to the model problem (1.1). Based on this observation, it suffices to consider the transmission behavior of the error functions $u_1^n - u$ and $u_2^n - u$, as a result, we assume $f = 0$ and analyze the convergence to the zero solution. Let $f_1 = 0, f_2 = 0$ in (2.1) and (2.2). Performing a Fourier transform in the y direction for each iterative step, denoting the Fourier variable by η, we have functions $\widehat{u} = \widehat{u}(x,\eta)$ and $\widehat{g} = \widehat{g}(\eta)$ corresponding to $u(x,y)$ and $g(y)$ respectively. Similar as in the nonoverlapping case in [6], we obtain the following two ordinary differential equations in the variable x,

$$-\frac{\partial^2 \widehat{u}_1^n}{\partial x^2} - (k^2-\eta^2)\widehat{u}_1^n = 0 \quad x < x_1, \eta \in \mathbb{R}, \tag{3.1}$$

$$-\frac{\partial^2 \widehat{u}_2^n}{\partial x^2} - (k^2-\eta^2)\widehat{u}_2^n = 0 \quad x > x_2, \eta \in \mathbb{R}. \tag{3.2}$$

The transmission conditions for Γ_1 and Γ_2 are transformed into

$$\frac{\partial \widehat{u}_1^n(x_1,\eta)}{\partial n_1} + \gamma_1 \widehat{u}_1^n(x_1,\eta) = \widehat{g}_1^n(x_1,\eta), \tag{3.3}$$

$$\frac{\partial \widehat{u}_2^n(x_2,\eta)}{\partial n_2} + \gamma_2 \widehat{u}_2^n(x_2,\eta) = \widehat{g}_2^n(x_2,\eta). \tag{3.4}$$

According to step $2, 4, 5$ in the algorithm, which describe updates of the transmission conditions, we have the new forms

$$\widehat{g}_2^n(x_2,\eta) = \frac{\partial \widehat{u}_1^n(x_2,\eta)}{\partial n_2} + \gamma_2 \widehat{u}_1^n(x_2,\eta), \tag{3.5}$$

$$\widehat{g}_1^*(x_1,\eta) = \frac{\partial \widehat{u}_2^n(x_1,\eta)}{\partial n_1} + \gamma_1 \widehat{u}_2^n(x_1,\eta), \tag{3.6}$$

$$\widehat{g}_1^{n+1}(x_1,\eta) = \theta \widehat{g}_1^*(x_1,\eta) + (1-\theta)\widehat{g}_1^n(x_1,\eta). \tag{3.7}$$

Since the interfaces are $x = x_i, i = 1, 2$, we have the unit normal derivatives n_i in the x direction. Next, we fix the variable η for solving the ordinary differential equations above, which correspond to the partial differential equations in our algorithm. Let λ be the solution to the characteristic equation

$$\lambda^2 + (k^2 - \eta^2) = 0,$$

then,

$$\lambda(\eta) = \begin{cases} \sqrt{\eta^2 - k^2} & \text{if } |\eta| \geqslant k, \\ -\mathrm{i}\sqrt{k^2-\eta^2} & \text{if } |\eta| < k. \end{cases}$$

The general solutions to the ordinary differential equations (3.1) and (3.2) without boundary conditions can be written as follows:

$$\begin{aligned} \widehat{u}_1^n(x,\eta) &= A_1 \mathrm{e}^{\lambda(\eta)(x-x_1)} + B_1 \mathrm{e}^{-\lambda(\eta)(x-x_1)}, \quad x < x_1, \\ \widehat{u}_2^n(x,\eta) &= A_2 \mathrm{e}^{\lambda(\eta)(x-x_2)} + B_2 \mathrm{e}^{-\lambda(\eta)(x-x_2)}, \quad x > x_2. \end{aligned}$$

These expression of $\widehat{u}_1^n$ and $\widehat{u}_2^n$ can be further simplified due to the Sommerfeld radiation condition which excludes incoming waves and growing modes at infinity. Taking $\widehat{u}_1^n$ as an example, notice that the two basis functions of $\widehat{u}_1^n$ are $e_1(x) = \mathrm{e}^{\lambda(\eta)(x-x_1)}$ and $e_2(x) = \mathrm{e}^{-\lambda(\eta)(x-x_1)}$. First, for the case $|\eta| > k$, $\lambda(\eta)$ is real, so we have $\lim_{x\to-\infty} e_1(x) = 0$. However, for the other basis e_2, the fact that $\lim_{x\to-\infty} e_2(x) = \infty$ is a contradiction to the meaning of the radiation condition. Second, for the case $|\eta| < k$, $\lambda(\eta)$ is a complex number. $e_1(x)$ means the left going wave, and $e_2(x)$ means the right going wave, with the frequency $\sqrt{k^2-\eta^2}$. For the third case $|\eta| = k$, $\lambda(\eta) = 0$ and $e_1(x) = e_2(x)$, so we could delete $e_2(x)$ by setting $B_1 = 0$. Based on the above observation, in any case, the basis $e_2(x)$ should not appear in the formula of the solution $\widehat{u}_1^n$, which implies $B_1 = 0$. Similarly, we have $A_2 = 0$ in the solution $\widehat{u}_2^n$. So the simplified solutions can be expressed as

$$\widehat{u}_1^n(x,\eta) = \widehat{u}_1^n(x_1,\eta)\mathrm{e}^{\lambda(\eta)(x-x_1)}, \quad \text{and} \quad \widehat{u}_2^n(x,\eta) = \widehat{u}_2^n(x_2,\eta)\mathrm{e}^{-\lambda(\eta)(x-x_2)}.$$

Substituting these expressions into the transmission conditions (3.3), (3.5), (3.4) and (3.6), we have

$$\begin{aligned}
\widehat{g}_1^n(x_1,\eta) &= (\lambda(\eta)+\gamma_1)\widehat{u}_1^n(x_1,\eta),\\
\widehat{g}_2^n(x_2,\eta) &= (-\lambda(\eta)+\gamma_2)\widehat{u}_1^n(x_1,\eta)\mathrm{e}^{-\lambda(\eta)L}\\
&= (\lambda(\eta)+\gamma_2)\widehat{u}_2^n(x_2,\eta),\\
\widehat{g}_1^*(x_1,\eta) &= (-\lambda(\eta)+\gamma_1)\widehat{u}_2^n(x_2,\eta)\mathrm{e}^{-\lambda(\eta)L},
\end{aligned}$$

where $L = x_1 - x_2$ is the overlapping size. Combining the above equations with the relaxation step (3.7), we get the following error propagation equation

$$\widehat{g}_1^{n+1}(x_1,\eta) = \left[(1-\theta) + \theta\frac{-\lambda(\eta)+\gamma_1}{\lambda(\eta)+\gamma_2}\cdot\frac{-\lambda(\eta)+\gamma_2}{\lambda(\eta)+\gamma_1}\cdot\mathrm{e}^{-2\lambda(\eta)L}\right]\widehat{g}_1^n(x_1,\eta). \tag{3.8}$$

Since the solution $\widehat{u}_1^n$ and $\widehat{u}_2^n$ are complex functions, the parameters γ_1 and γ_2 should also be complex values, *i.e.* $\gamma_1 = p_1 - q_1 i, \gamma_2 = p_2 - q_2 i$ [18, 19]. To preserve the well-posedness for each subproblem, the parameters should satisfy $p_1, q_1, p_2, q_2 > 0$ [27]. Then the convergence rate ρ should be written as follows:

$$\rho = \begin{cases}
(1-\theta) + \theta\dfrac{p_1 - (q_1 - \sqrt{k^2-\eta^2})i}{p_2 - (q_2 + \sqrt{k^2-\eta^2})i}\cdot\dfrac{p_2 - (q_2 - \sqrt{k^2-\eta^2})i}{p_1 - (q_1 + \sqrt{k^2-\eta^2})i}\cdot\mathrm{e}^{2\sqrt{k^2-\eta^2}Li} & |\eta| < k,\\[2ex]
(1-\theta) + \theta\dfrac{(p_1 - \sqrt{\eta^2-k^2}) - q_1 i}{(p_2 + \sqrt{\eta^2-k^2}) - q_2 i}\cdot\dfrac{(p_2 - \sqrt{\eta^2-k^2}) - q_2 i}{(p_1 + \sqrt{\eta^2-k^2}) - q_1 i}\cdot\mathrm{e}^{-2\sqrt{\eta^2-k^2}L} & |\eta| \geqslant k.
\end{cases}$$

Actually, ρ is a function of the parameters $\eta, k, \theta, L, p_1, p_2, q_1, q_2$, *i.e.* $\rho(\eta, k, \theta, L, p_1, p_2, q_1, q_2)$. Note that for the case $|\eta| = k$, the convergence rate is $|\rho| = 1$ no matter what is chosen for $\theta, L, p_1, p_2, q_1, q_2$. In other words, the method does not converge for this case. The same phenomenon is also observed in [19], [18] and [6]. To deal with this problem, similar as in [6, 18, 19], we only consider the reduced case $|\eta| \leqslant k_-$ and $|\eta| \geqslant k_+$, where $k_- < k < k_+$ are close to k. This simplification is reasonable, because the computational domain is bounded in practical applications. For example, suppose the domain is $\widetilde{\Omega} = [a,b]\times[0,H]$, with homogeneous Dirichlet condition on its upper and bottom boundary, then the relevant frequencies are $\eta = \frac{j\pi}{H}, j \in \mathbb{N}$, so we may choose $k_- = k - \frac{\pi}{H}, k_+ = k + \frac{\pi}{H}$ and leave only one frequency $\eta = k$ which can be treated easily by the preconditioned Krylov method [18, 19]. If k satisfies $\frac{j\pi}{H} < k < \frac{(j+1)\pi}{H}$, then $\eta^2 - k^2 \neq 0$, the iterative method converges by choosing $k_- = \frac{j\pi}{H}$ and $k_+ = \frac{(j+1)\pi}{H}$.

Noticing that the lowest relevant frequency $|\eta|_{\min}$ usually described by the size of the domain, we always have $|\eta| \geqslant |\eta|_{\min} \geqslant 0$. In the continuous case, the value of $|\eta|$ can be arbitrarily large. However, for a discrete problem, we find that $|\eta| \leqslant |\eta|_{\max}$, where $|\eta|_{\max} = \frac{\pi}{h}$ is the highest frequency described by the mesh size. Then we obtain the range of $|\eta|$, which is $|\eta|_{\min} \leqslant |\eta| \leqslant |\eta|_{\max}$. Let $\underline{\eta} := |\eta|_{\min}$ and $\overline{\eta} := |\eta|_{\max}$. Let

$$\alpha = \begin{cases} \sqrt{k^2 - \eta^2} & \underline{\eta} \leqslant |\eta| \leqslant k_-, \\ \sqrt{\eta^2 - k^2} & k_+ \leqslant |\eta| \leqslant \overline{\eta}. \end{cases} \tag{3.9}$$

According to this definition, we know that α can be bounded by $0 < \underline{\alpha} \leqslant \alpha \leqslant \overline{\alpha}$, where $0 < \underline{\alpha} = \min\limits_{|\eta| \in [\underline{\eta}, k_-] \cup [k_+, \overline{\eta}]} \sqrt{|k^2 - \eta^2|}$ and $\overline{\alpha} = \max\limits_{|\eta| \in [\underline{\eta}, k_-] \cup [k_+, \overline{\eta}]} \sqrt{|k^2 - \eta^2|}$. Inserting (3.9) into the expression of ρ, by some elementary computation, the convergence factor should be

$$\rho = (1-\theta) + \theta \frac{A_1 + \mathrm{i}B_1}{D_1} \mathrm{e}^{2\alpha L i}, \quad \text{if} \quad \underline{\eta} \leqslant |\eta| \leqslant k_-,$$

where A_1, B_1 and D_1 are as follows:

$$\begin{aligned} A_1 &= [(p_1p_2 - q_1q_2) - \alpha^2]^2 + (p_1q_2 + p_2q_1)^2 - [(p_1+p_2)^2 + (q_1+q_2)^2]\alpha^2, \\ B_1 &= 2[(p_1q_2 + p_2q_1)(q_1+q_2) + ((p_1p_2 - q_1q_2) - \alpha^2)(p_1+p_2)]\alpha, \\ D_1 &= [(p_1p_2 - q_1q_2) - \alpha^2 - (q_1+q_2)\alpha]^2 + [(p_1q_2 + p_2q_1) + (p_1+p_2)\alpha]^2. \end{aligned}$$

For the other case of $|\eta|$, a simple calculation leads to

$$\rho = (1-\theta) + \theta \frac{A_2 + \mathrm{i}B_2}{D_2} \mathrm{e}^{-2\alpha L}, \quad \text{if} \quad k_+ \leqslant |\eta| \leqslant \overline{\eta},$$

where A_2, B_2 and D_2 are of the form

$$\begin{aligned} A_2 &= [(p_1p_2 - q_1q_2) + \alpha^2]^2 + (p_1q_2 + p_2q_1)^2 - [(p_1+p_2)^2 + (q_1+q_2)^2]\alpha^2, \\ B_2 &= 2[-(p_1q_2 + p_2q_1)(p_1+p_2) + ((p_1p_2 - q_1q_2) + \alpha^2)(q_1+q_2)]\alpha, \\ D_2 &= [(p_1p_2 - q_1q_2) + \alpha^2 + (p_1+p_2)\alpha]^2 + [(p_1q_2 + p_2q_1) + (q_1+q_2)\alpha]^2. \end{aligned}$$

Since it is too complicated to optimize the convergence factor ρ by freely choosing p_1, q_1, p_2, q_2, in the following, we set $p_1 = q_1, p_2 = q_2$ for simplicity and symmetry, then the expression of ρ is

$$\rho = \begin{cases} (1-\theta) + \theta \dfrac{A + \mathrm{i}B}{D} \mathrm{e}^{2\alpha L i}, & \text{if} \quad \underline{\eta} \leqslant |\eta| \leqslant k_-, \\ (1-\theta) + \theta \dfrac{A - \mathrm{i}B}{D} \mathrm{e}^{-2\alpha L}, & \text{if} \quad k_+ \leqslant |\eta| \leqslant \overline{\eta}, \end{cases} \tag{3.10}$$

where

$$\begin{aligned} A &= (2q_1q_2)^2 - 2(q_1+q_2)^2\alpha^2 + \alpha^4, \\ B &= 2(2q_1q_2 - \alpha^2)(q_1+q_2)\alpha, \\ D &= (2q_1q_2 + (q_1+q_2)\alpha)^2 + (q_1+q_2+\alpha)^2\alpha^2. \end{aligned}$$

By the above discussion, we have that ρ only depends on $\alpha, \theta, L, q_1, q_2$, *i.e.* $\rho(\alpha, \theta, L, q_1, q_2)$. Then after solving the problem

$$\min_{\theta, L, q_1, q_2} \max_{\alpha \in [\underline{\alpha}, \overline{\alpha}]} |\rho(\alpha, \theta, L, q_1, q_2)|,$$

we could obtain the optimal convergence rate. Unfortunately, it is still too difficult to directly solve it for the explicit parameters. Following the same way as we did in [6] for the nonoverlapping case, we constrain the choices of parameters q_1 and q_2, then we can obtain a convergence rate independent of the wave number k and mesh size h.

If we choose $\theta = 1$ in (3.10), which means no relaxation, then we obtain

$$\rho = \begin{cases} \rho_0 \mathrm{e}^{2\alpha L i}, & \text{if} \quad \underline{\eta} \leqslant |\eta| \leqslant k_-, \\ \rho_0 \mathrm{e}^{-2\alpha L}, & \text{if} \quad k_+ \leqslant |\eta| \leqslant \overline{\eta}, \end{cases}$$

where

$$\rho_0 = \frac{A + \mathrm{i}B}{D}$$

indicates the convergence factor for the nonoverlapping method. Compared to the nonoverlapping case, the overlapping improves the contraction factor for the evanescent modes, which is the case $k_+ \leqslant |\eta| \leqslant \overline{\eta}$. Meanwhile, it dose not impact the propagation mode, which is the case $\underline{\eta} \leqslant |\eta| \leqslant k_-$. By suitably choosing the parameters q_1, q_2, ρ_0 can be optimized. The nonoverlapping result is obtained in [19] for the optimized Schwarz method, and in [18] for the two-sided optimized Schwarz method.

For $\theta \neq 1$, by choosing $L = 0$ in (3.10), we come to the nonoverlapping optimized Schwarz method with relaxation, which was optimized in [6]. In this paper, we come to the general case, *i.e.* the analysis for (3.10) with $L \geqslant 0$.

For further discussion, we define $\phi := 2\alpha L$, and require the expression of $|\rho|^2$, which is the next lemma.

Lemma 3.1.

$$|\rho|^2 = \begin{cases} \left[1 + \left(\frac{A}{D}\right)^2 + \left(\frac{B}{D}\right)^2 - 2\left(\frac{A}{D}\cos\phi - \frac{B}{D}\sin\phi\right)\right]\theta^2 - 2\left[1 - \left(\frac{A}{D}\cos\phi - \frac{B}{D}\sin\phi\right)\right]\theta + 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if} \quad \underline{\eta} \leqslant |\eta| \leqslant k_-, \\ \left[\left(1 - \frac{A}{D}\mathrm{e}^{-\phi}\right)^2 + \left(\frac{B}{D}\mathrm{e}^{-\phi}\right)^2\right]\theta^2 - 2\left(1 - \frac{A}{D}\mathrm{e}^{-\phi}\right)\theta + 1, \quad \textit{if} \quad k_+ \leqslant |\eta| \leqslant \overline{\eta}. \end{cases} \tag{3.11}$$

Proof. we can obtain it directly from (3.10) by simple deducing. □

Lemma 3.2. *Let d_1, d_2 and d_3 be defined as:*

$$d_1 = \frac{2q_1q_2}{2q_1q_2 + (q_1+q_2)\alpha}, \quad d_2 = \frac{\alpha}{(q_1+q_2)+\alpha}, \quad d_3 = \frac{2q_1q_2 - \alpha^2}{2q_1q_2 + (q_1+q_2)\alpha} \tag{3.12}$$

then the bounds

$$2\min\{d_1, d_2\} \leqslant 1 + \frac{A}{D} \leqslant 2\max\{d_1, d_2\},$$

$$\frac{|B|}{|D|} \leqslant |d_3|,$$

hold.

Proof. By careful calculation,

$$\begin{aligned} 1 + \frac{A}{D} &= \frac{2}{D}\left(2q_1q_2(2q_1q_2 + (q_1+q_2)\alpha) + (q_1+q_2+\alpha)\alpha^3\right) \\ &= \frac{2}{D}\left(d_1(2q_1q_2 + (q_1+q_2)\alpha)^2 + d_2(q_1+q_2+\alpha)^2\alpha^2\right). \end{aligned}$$

From the definition of D, $1 + \frac{A}{D}$ can be bounded by

$$2\min\{d_1, d_2\} \leqslant 1 + \frac{A}{D} \leqslant 2\max\{d_1, d_2\}. \tag{3.13}$$

Note that $D \geqslant 2(2q_1q_2 + (q_1+q_2)\alpha)(q_1+q_2+\alpha)\alpha$, then

$$\frac{(2q_1q_2 + (q_1+q_2)\alpha)(q_1+q_2)\alpha}{D} \leqslant \frac{(2q_1q_2 + (q_1+q_2)\alpha)(q_1+q_2)\alpha}{2(2q_1q_2 + (q_1+q_2)\alpha)(q_1+q_2+\alpha)\alpha} \leqslant \frac{1}{2}.$$

So $\frac{B}{D}$ can be bounded by

$$\frac{|B|}{|D|} = \frac{|2(2q_1q_2 - \alpha^2)|}{2q_1q_2 + (q_1+q_2)\alpha} \frac{(2q_1q_2 + (q_1+q_2)\alpha)(q_1+q_2)\alpha}{D} \leqslant \left|\frac{2d_3}{2}\right| = |d_3|.$$

□

The next lemma we proved in [6] gives the bounds of d_1, d_2, d_3.

Lemma 3.3. *If $0 < \underline{\alpha} \leqslant \alpha \leqslant \overline{\alpha}$, and assume that*

$$q_1 \leqslant \frac{C_1}{2}\underline{\alpha}, \quad q_2 \geqslant \frac{1}{C_2}\overline{\alpha}, \tag{3.14}$$

where the constants C_j satisfy $0 < C_j < 1$ $(j = 1, 2)$, then d_1, d_2 and d_3 in (3.12) can be bounded by

$$\frac{q_1}{2\overline{\alpha}} < d_1 < \frac{C_1}{C_1+1}, \tag{3.15}$$

$$\frac{\underline{\alpha}}{3q_2} < d_2 \leqslant \frac{C_2}{1+C_2}, \tag{3.16}$$

$$|d_3| < \max\left\{\frac{C_1}{C_1+1}, C_2\right\}. \tag{3.17}$$

Combining Lemma 3.2 and Lemma 3.3, we have the next lemma.

Lemma 3.4. *If $0 < \underline{\alpha} \leqslant \alpha \leqslant \overline{\alpha}$, and*

$$q_1 \leqslant \frac{C_1}{2}\underline{\alpha}, \quad q_2 \geqslant \frac{1}{C_2}\overline{\alpha},$$

where $0 < C_j < 1 (j = 1, 2)$. Let $\overline{C} = \max\{\frac{C_1}{C_1+1}, C_2\}$, then d_1, d_2, d_3 can be bounded by

$$\begin{aligned} 0 < d_1 &< \overline{C}, \\ 0 < d_2 &\leqslant \overline{C}, \\ |d_3| &< \overline{C}. \end{aligned}$$

The bounds of $1 + \frac{A}{D}$ and $\frac{|B|}{|D|}$ are as follows:

$$0 < 1 + \frac{A}{D} \leqslant 2\overline{C}, \tag{3.18}$$

$$\frac{|B|}{|D|} \leqslant \overline{C}. \tag{3.19}$$

Because C_1, C_2 and thus $\overline{C}$ are independent of α, the bounds we obtain in Lemma 3.4 are valid for all the values of $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. Define

$$\rho_{\max}(\theta, L, q_1, q_2) := \max_{\alpha \in [\underline{\alpha}, \overline{\alpha}]} |\rho(\alpha, \theta, L, q_1, q_2)|. \tag{3.20}$$

For convenience of studying the bounds of $\rho_{\max}$, we need to restrict $\phi = 2\alpha L$ to be small enough, which leads to the next theorem.

Theorem 3.5. *Assume*

$$0 \leqslant \phi \leqslant \frac{\pi}{2}. \tag{3.21}$$

Choose C_1, C_2 *small enough, which make* $\overline{C} < \varepsilon$ *small enough, where* $\overline{C} = \max\{\frac{C_1}{C_1+1}, C_2\}$. *If* $0 < \underline{\alpha} \leqslant \alpha \leqslant \overline{\alpha}$, *and*

$$q_1 \leqslant \frac{C_1}{2}\underline{\alpha}, \quad q_2 \geqslant \frac{1}{C_2}\overline{\alpha}, \tag{3.22}$$

then the solution of

$$\min_{\theta \in \mathbb{R}} \rho_{\max} \tag{3.23}$$

is obtained at $\theta = \frac{1}{2}$, *which lead to* $\rho_{\max}(\theta = \frac{1}{2}, L, q_1, q_2) < 1$.

Proof. Using Lemma 3.4, noticing $\overline{C} < \varepsilon$, we know that the first equation in (3.11) for the case $\underline{\eta} \leqslant |\eta| \leqslant k_-$ can be written into

$$\begin{aligned}
&\left[1 + \left(\frac{A}{D}\right)^2 + \left(\frac{B}{D}\right)^2 - 2\left(\frac{A}{D}\cos\phi - \frac{B}{D}\sin\phi\right)\right]\theta^2 - 2\left[1 - \left(\frac{A}{D}\cos\phi - \frac{B}{D}\sin\phi\right)\right]\theta + 1 \\
&= \left(2 + \left(1 + \frac{A}{D}\right)^2 - 2\left(1 + \frac{A}{D}\right) + \left(\frac{B}{D}\right)^2 - 2\left[\left(1 + \frac{A}{D}\right)\cos\phi - \cos\phi - \frac{B}{D}\sin\phi\right]\right)\theta^2 \\
&- 2\left[1 + \cos\phi - \left(1 + \frac{A}{D}\right)\cos\phi + \frac{B}{D}\sin\phi\right]\theta + 1 \\
&= \left[2\left(1 + \cos\phi\right) + O(\varepsilon)\right]\theta^2 - \left[2\left(1 + \cos\phi\right) + O(\varepsilon)\right]\theta + 1.
\end{aligned} \tag{3.24}$$

We denote by $\rho_1 = [2(1 + \cos\phi) + O(\varepsilon)]\theta^2 - [2(1 + \cos\phi) + O(\varepsilon)]\theta + 1$. Similarly, for the case $k_+ \leqslant |\eta| \leqslant \overline{\eta}$, the second equation in (3.11) is

$$\begin{aligned}
&\left[\left(1 - \frac{A}{D}\mathrm{e}^{-\phi}\right)^2 + \left(\frac{B}{D}\mathrm{e}^{-\phi}\right)^2\right]\theta^2 - 2\left(1 - \frac{A}{D}\mathrm{e}^{-\phi}\right)\theta + 1 \\
&= \left[\left(1 - \left(1 + \frac{A}{D}\right)\mathrm{e}^{-\phi} + \mathrm{e}^{-\phi}\right)^2 + \left(\frac{B}{D}\mathrm{e}^{-\phi}\right)^2\right]\theta^2 - 2\left(1 + \mathrm{e}^{-\phi} - \left(1 + \frac{A}{D}\right)\mathrm{e}^{-\phi}\right)\theta + 1 \\
&= \left[\left(1 + \mathrm{e}^{-\phi}\right)^2 + O(\varepsilon)\right]\theta^2 - \left[2(1 + \mathrm{e}^{-\phi}) + O(\varepsilon)\right]\theta + 1.
\end{aligned} \tag{3.25}$$

We denote by $\rho_2 = [(1 + \mathrm{e}^{-\phi})^2 + O(\varepsilon)]\theta^2 - [2(1 + \mathrm{e}^{-\phi}) + O(\varepsilon)]\theta + 1$. Note $\phi = 2\alpha L$, then the convergence rate is

$$|\rho(\alpha, \theta, L, q_1, q_2)|^2 = \max\{\rho_1, \rho_2\}. \tag{3.26}$$

Here we only consider the case $\phi > 0$, *i.e.* $L > 0$. For the case $\phi = 0$, there is a discussion in [6]. Since ρ_1 and ρ_2 only depend on ϕ and θ by omitting the small value $O(\varepsilon)$, we could drop q_1 and q_2 here. Let

$$\rho_1'(\alpha, \theta, L) = [2(1 + \cos\phi)]\theta^2 - 2(1 + \cos\phi)\theta + 1, \tag{3.27}$$

$$\rho_2'(\alpha, \theta, L) = [(1 + \mathrm{e}^{-\phi})^2]\theta^2 - 2(1 + \mathrm{e}^{-\phi})\theta + 1, \tag{3.28}$$

then the convergence rate (3.26) can be approximately rewritten as

$$|\rho(\alpha, \theta, L)|^2 = \max\{\rho_1', \rho_2'\} + O(\varepsilon). \tag{3.29}$$

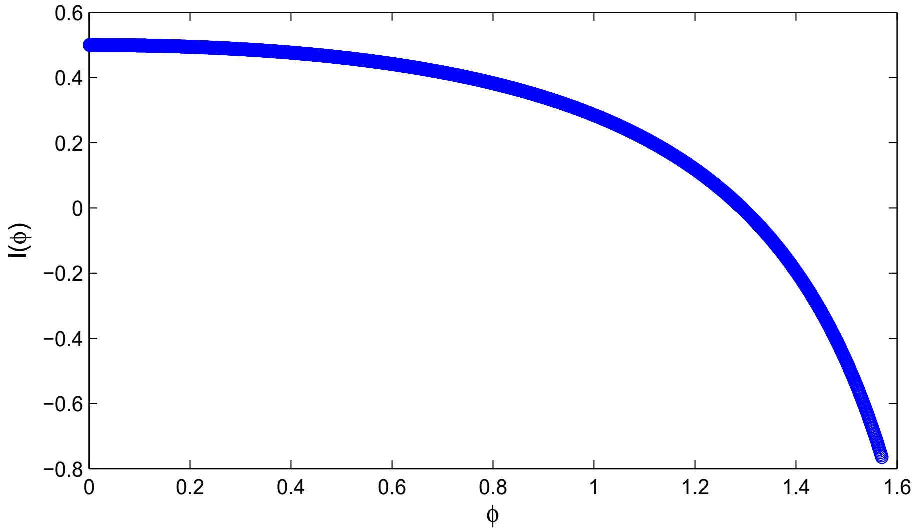

FIGURE 1. Function $l(\phi)$.

Noticing (3.21), if $\theta \leqslant 0$ or $\theta \geqslant 1$ in (3.27), then $\rho'_1 > 1$. Together with (3.29), we could have $|\rho(\alpha, \theta, L)| \geqslant 1 + O(\varepsilon)$ in this case for $\forall \alpha \in [\underline{\alpha}, \overline{\alpha}]$, which means divergence. Hence, to preserve the convergence, the necessary condition $0 < \theta < 1$ should be satisfied.

Based on the above discussion, it is sufficient to solve

$$\min_{\theta \in (0,1)} \rho_{\max}$$

rather than (3.23). According to (3.29) and the definition (3.20), we should compare the values ρ'_1, ρ'_2 first, *i.e.* study

$$\rho'_1 - \rho'_2 = [2(1 + \cos\phi) - (1 + \mathrm{e}^{-\phi})^2]\theta^2 - [2(1 + \cos\phi) - 2(1 + \mathrm{e}^{-\phi})]\theta.$$

Noticing (3.21), we have $2(1 + \cos\phi) - (1 + \mathrm{e}^{-\phi})^2 > 0$. Let function $l(\phi) := \frac{2(1+\cos\phi)-2(1+\mathrm{e}^{-\phi})}{2(1+\cos\phi)-(1+\mathrm{e}^{-\phi})^2}$. Since $\theta > 0$, it is easy to verify that when $\theta \geqslant l(\phi)$, then $\rho'_1 \geqslant \rho'_2$. Furthermore, by simple calculation, we have $l(\phi) \leqslant \frac{1}{2}$ (see Fig. 1), which leads to that if $\theta \geqslant \frac{1}{2} \geqslant l(\phi)$, then $\max\{\rho'_1, \rho'_2\} = \rho'_1$. From the expression (3.27), for L which satisfies (3.21), we conclude that

$$\begin{aligned}\min_{\theta \in [\frac{1}{2},1)} \rho^2_{\max} &= \min_{\theta \in [\frac{1}{2},1)} \max_{\alpha \in [\underline{\alpha},\overline{\alpha}]} \rho'_1(\alpha, \theta, L) + O(\varepsilon) \\ &= \max_{\alpha \in [\underline{\alpha},\overline{\alpha}]} \rho'_1(\alpha, \theta = \frac{1}{2}, L) + O(\varepsilon)\end{aligned}$$

On the other hand, if $0 < \theta < l(\phi) \leqslant \frac{1}{2}$, then $\rho'_1 < \rho'_2$. We have

$$\begin{aligned}\min_{\theta \in (0,\frac{1}{2})} \rho^2_{\max} &= \min_{\theta \in (0,\frac{1}{2})} \max_{\alpha \in [\underline{\alpha},\overline{\alpha}]} \rho'_2(\alpha, \theta, L) + O(\varepsilon) \\ &> \min_{\theta \in (0,\frac{1}{2})} \max_{\alpha \in [\underline{\alpha},\overline{\alpha}]} \rho'_1(\alpha, \theta, L) + O(\varepsilon) \\ &\geqslant \max_{\alpha \in [\underline{\alpha},\overline{\alpha}]} \rho'_1(\alpha, \theta = \frac{1}{2}, L) + O(\varepsilon).\end{aligned}$$

Combining both cases, we obtain

$$\begin{aligned}\min_{\theta\in\mathbb{R}}\rho_{\max}^2 &= \min_{\theta\in(0,1)}\rho_{\max}^2\\ &= \max_{\alpha\in[\underline{\alpha},\overline{\alpha}]}\rho_1'(\alpha,\theta=\frac{1}{2},L)+O(\varepsilon)\\ &= \max_{\alpha\in[\underline{\alpha},\overline{\alpha}]}\left\{1-\frac{1}{2}(1+\cos\phi)\right\}+O(\varepsilon)\\ &\leqslant \max_{\phi\in[0,\frac{\pi}{2}]}\left\{1-\frac{1}{2}(1+\cos\phi)\right\}+O(\varepsilon)\\ &\leqslant \frac{1}{2}+O(\varepsilon)<1,\end{aligned}\tag{3.30}$$

which finishes our proof. □

We remark that from the definition of α in (3.9),

$$\alpha\in[(2\tilde{C}_k k)^{\frac{1}{2}},C_0h^{-1}],\tag{3.31}$$

where $\tilde{C}_k=\min(k-k_-,k_+-k)$ is a sufficiently small value compared with k, and C_0 is a constant independent of h and k. Note that $\overline{\alpha}<\overline{\eta}=\frac{\pi}{h}$, we may choose q_2 dependent on the mesh size h. Similar to the result in [6], we obtain the next corollary.

Corollary 3.6. *If $0<\underline{\alpha}\leqslant\alpha\leqslant\overline{\alpha}$, and assume that*

$$q_1\leqslant\frac{C_1}{2}\underline{\alpha},\quad q_2\geqslant C_3h^{-m},\tag{3.32}$$

where $m\geqslant 1$ and $C_3=\frac{\pi}{C_2}$. Let $0\leqslant\phi\leqslant\frac{\pi}{2}$. The parameter is chosen to be $\theta=\frac{1}{2}$, then the convergence rate $|\rho|<1$ and:

- *is independent of k for $k^rh=constant$ where $r\geqslant 1$;*
- *is independent of h for fixed k with $kh\leqslant constant$.*

Remark 3.7. We claim that $|\rho|$ sensitively depends on the overlapping size L. Since $\phi=2\alpha L$, by equation (3.30), if $L=0$, we have $\min_{\theta\in\mathbb{R}}\rho_{\max}^2=O(\varepsilon)$, which indicates that the method converges fast.

Noticing that in (3.30), as $\phi=2\alpha L$ is increasing from 0 to π, which is equivalent to that L increases from 0 to $\frac{\pi}{2\overline{\alpha}}$, the function

$$\min_{\theta\in\mathbb{R}}\rho_{\max}(L)^2=\max_{\alpha\in[\underline{\alpha},\overline{\alpha}]}\left\{1-\frac{1}{2}(1+\cos\phi)\right\}+O(\varepsilon)=\max_{\alpha\in[\underline{\alpha},\overline{\alpha}]}\left\{1-\frac{1}{2}(1+\cos(2\alpha L))\right\}+O(\varepsilon)$$

increases. To satisfy the assumption (3.21), we need $L\leqslant\frac{\pi}{4\overline{\alpha}}$. Actually, this bound may be enlarged to $L<\frac{\pi}{2\overline{\alpha}}$. However, if $L\geqslant\frac{\pi}{2\overline{\alpha}}$, we could have $\min_{\theta\in\mathbb{R}}\rho_{\max}(L)^2=1+O(\varepsilon)$, which suggests that the method may not converge. By this observation, if $\overline{\alpha}\approx\frac{\pi}{h}$, the condition $L<\frac{h}{2}$ should be satisfied. It is not realizable in practical computing for the overlapping size to be less than one mesh size, unless $L=0$, which is the nonoverlapping method we introduced in [6].

In fact, the real bound is $\overline{\alpha}\approx k$ rather than $\overline{\alpha}\approx\frac{\pi}{h}$. We notice that $\overline{\alpha}\approx\frac{2\pi}{\lambda}=\frac{\pi}{h}$ represents the highest frequency described by the mesh size, which means $\lambda=2h$, *i.e.* there are only 2 discrete elements for each wavelength. However, the highest frequency $\frac{\pi}{h}$ usually could not be reached for this wave problem. Actually, according to our model problem (1.1), since the solution has uniform wave number k in the whole domain, the highest frequency approximates to be $2\pi k$, and the corresponding wavelength is $\lambda=\frac{2\pi}{2\pi k}=\frac{1}{k}$, which is independent of the mesh size h. So the value $\overline{\alpha}$ should satisfy $\overline{\alpha}\approx k$, which indicates that for $kh\approx 0.5$, the overlap should be relaxed to $L<4h$. The numerical tests confirm this conclusion.

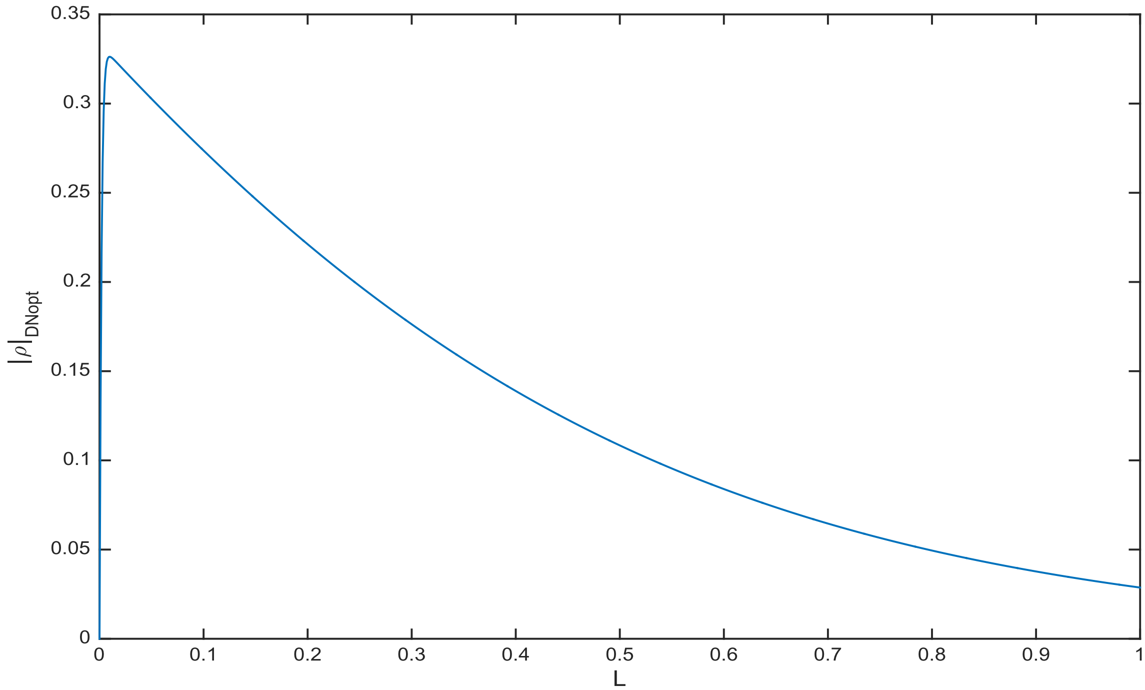

FIGURE 2. Function $|\rho|_{\mathrm{DNopt}}$, where $k=1$, $h=\frac{1}{100}$, $\underline{\eta}=1$ and $\overline{\eta}=\frac{\pi}{h}$.

Remark 3.8. For the elliptic problem

$$-\triangle u + k^2 u = f, \tag{3.33}$$

according to analysis in [17], we would have the contraction factor for the classical overlapping Schwarz method

$$|\rho|_{\mathrm{cla}} = \max_{|\eta|\in[\underline{\eta},\overline{\eta}]} \mathrm{e}^{-2\sqrt{k^2+\eta^2}L}.$$

It indicates that the convergence behavior is better for larger overlap. Similarly, following the same way of analyzing, we would find the contraction factor for the Dirichlet–Neumann method to be

$$|\rho|_{\mathrm{DN}} = \max_{|\eta|\in[\underline{\eta},\overline{\eta}]} |-\theta\mathrm{e}^{-2\sqrt{k^2+\eta^2}L} + (1-\theta)|.$$

The optimal θ should be

$$\theta_{\mathrm{opt}} = \frac{2}{2+\mathrm{e}^{-2\sqrt{k^2+\underline{\eta}^2}L}+\mathrm{e}^{-2\sqrt{k^2+\overline{\eta}^2}L}},$$

and then by direct calculation, the corresponding contraction factor is

$$|\rho|_{\mathrm{DNopt}} = \frac{\mathrm{e}^{-2\sqrt{k^2+\underline{\eta}^2}L}-\mathrm{e}^{-2\sqrt{k^2+\overline{\eta}^2}L}}{2+\mathrm{e}^{-2\sqrt{k^2+\underline{\eta}^2}L}+\mathrm{e}^{-2\sqrt{k^2+\overline{\eta}^2}L}}.$$

According to Figure 2, we could find that the optimal choice of overlap is $L=0$, and the worst choice is close to $L=h$. The convergence behavior is good for no overlap and the large overlap case.

However, the convergence behavior of our method for the Helmholtz problem is different. The optimal overlap is $L=0$ and it becomes worse when increasing L. It may not even converge for $L\geqslant 4h$. Also, according to the analysis in [6], if $L=0$, the optimal choice of parameters should satisfy (3.22), and it becomes worse for other choices of parameters.

4. Implementation of the DD algorithm

In this section, we shall derive the preconditioned system of our overlapping DD algorithm and discretize it by the finite element method.

As we mentioned in Section 1, in practical computations, the problem on a bounded region $\widetilde{\Omega}$ with some absorbing boundary condition, which actually approximates the Dirichlet to Neumann operator, is considered. A usual one is the zeroth order approximation on the boundary, *i.e.*

$$\frac{\partial u}{\partial n} - iku = 0, \tag{4.1}$$

where n is the unit outward normal vector at the boundary. We discuss this bounded domain problem in the Sobolev space $H^1(\widetilde{\Omega})$ and $L^2(\widetilde{\Omega})$. With this condition, the variational form is to find $u \in V = H^1(\widetilde{\Omega})$ such that

$$(\nabla u, \nabla \overline{v})_{\widetilde{\Omega}} - k^2(u, \overline{v})_{\widetilde{\Omega}} - \mathrm{i}k\langle u, \overline{v}\rangle_{\partial\widetilde{\Omega}} = (f, \overline{v})_{\widetilde{\Omega}} \quad \forall v \in V,$$

where $\overline{v}$ is the conjugated form of v and $(u, \overline{v})_{\widetilde{\Omega}} := \int_{\widetilde{\Omega}} u\overline{v}$, $\langle u, \overline{v}\rangle_{\partial\widetilde{\Omega}} := \int_{\partial\widetilde{\Omega}} u\overline{v}$. This variational form has a unique solution for $f \in L^2(\widetilde{\Omega})$ (see [27] for details). If a piece of the boundary condition is $\frac{\partial u}{\partial n} + \gamma u = g$ on $\widetilde{\Gamma}_r$ or Dirichlet on $\widetilde{\Gamma}_d$, the other part of the boundary condition is (4.1), the variational form is to find $u \in \widetilde{V} = H^1(\widetilde{\Omega}, \widetilde{\Gamma}_d)$ such that

$$(\nabla u, \nabla \overline{v})_{\widetilde{\Omega}} - k^2(u, \overline{v})_{\widetilde{\Omega}} + \gamma\langle u, \overline{v}\rangle_{\widetilde{\Gamma}_r} - \mathrm{i}k\langle u, \overline{v}\rangle_{\partial\widetilde{\Omega}\setminus(\widetilde{\Gamma}_r\cup\widetilde{\Gamma}_d)} = (f, \overline{v}) + \langle g, \overline{v}\rangle_{\widetilde{\Gamma}_r} \quad \forall v \in \widetilde{V},$$

where $H^1(\widetilde{\Omega}, \widetilde{\Gamma}_d) := \{w \big| \ \gamma_0(w)|_{\widetilde{\Gamma}_d} = 0, \forall w \in H^1(\widetilde{\Omega})\}$, here $\gamma_0(w)$ is the trace of w. It is proved in [27] that this variational problem has a solution if $\mathrm{Re}(\gamma) \geqslant 0, \mathrm{Im}(\gamma) \leqslant 0$. Approximating these variational forms by finite element method is popular in practical computation.

For simplicity we assume that $\widetilde{\Omega}$ is a bounded domain with homogeneous Dirichlet boundary condition on the upper and bottom boundary and the absorbing boundary condition (4.1) on the left and right boundary. Other absorbing boundary conditions can be discussed similarly. The Helmholtz equation is as follows:

$$\begin{aligned} -\triangle u - k^2 u = f \quad & \text{in } \widetilde{\Omega}, \\ u = 0 \quad & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} - iku = 0 \quad & \text{on } \Gamma_R, \end{aligned} \tag{4.2}$$

where Γ_D, Γ_R are the boundaries with Dirichlet and the absorbing boundary conditions respectively. The subproblems in our DD algorithm (Fig. 3) can be written down as:

$$\begin{aligned} -\triangle u_j^n - k^2 u_j^n = f_j \quad & \text{in } \widetilde{\Omega}_j, \\ \frac{\partial u_j^n}{\partial n_j} + \gamma_j u_j^n = g_j^n \quad & \text{on } \Gamma_j, \\ u_j^n = 0 \quad & \text{on } \Gamma_{Dj} \\ \frac{\partial u_j^n}{\partial n} - iku_j^n = 0 \quad & \text{on } \Gamma_{Rj}, \end{aligned}$$

where $j = 1, 2$, $\Gamma_{Dj} = \Gamma_D \cap \partial\widetilde{\Omega}_j, \Gamma_{Rj} = \Gamma_R \cap \partial\widetilde{\Omega}_j$ and Γ_j is the interface.

The discrete variational form of each subproblem is to find $u_{jh}^n \in V_{jh}$ such that

$$\begin{aligned} (\nabla_h u_{jh}^n, \nabla_h \overline{v}_h)_{\widetilde{\Omega}_j} - k^2(u_{jh}^n, \overline{v}_h)_{\widetilde{\Omega}_j} + \gamma_j\langle \Pi u_{jh}^n, \Pi\overline{v}_h\rangle_{\Gamma_j} \\ -\mathrm{i}k\langle \Pi u_{jh}^n, \Pi\overline{v}_h\rangle_{\Gamma_{Rj}} = (f_j, \overline{v}_h)_{\widetilde{\Omega}_j} + \langle g_j^n, \Pi\overline{v}_h\rangle_{\Gamma_j} \quad \forall v_h \in V_{jh}, \end{aligned} \tag{4.3}$$

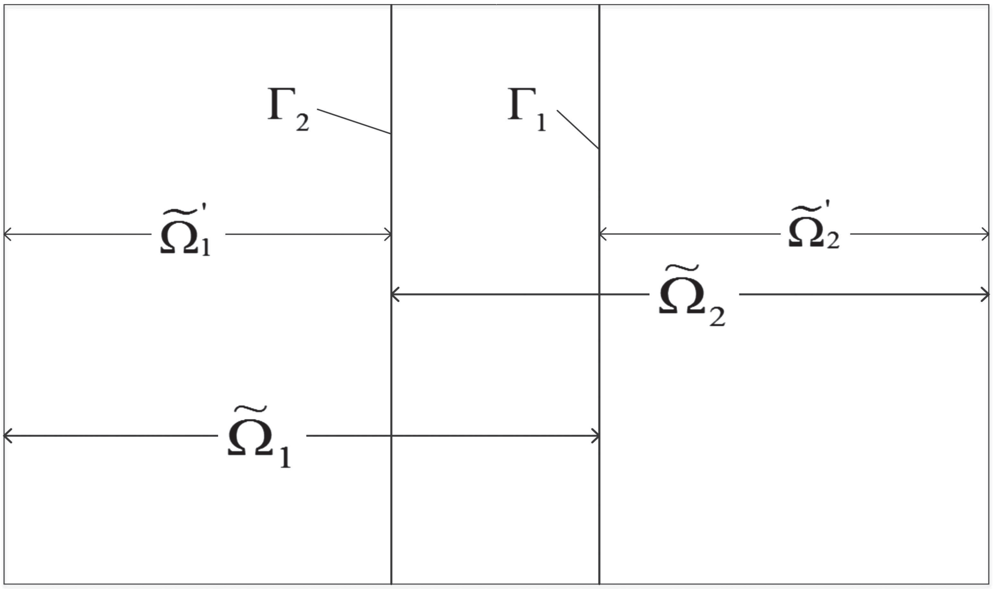

FIGURE 3. The decomposition of domain.

where $(\nabla_h u_{jh}^n, \nabla_h \overline{v}_h)_{\widetilde{\Omega}_j} = \sum_{e\in\tau(\widetilde{\Omega}_j)}(\nabla u_{jh}^n, \nabla \overline{v}_h)_e, \tau(\widetilde{\Omega}_j)$ is the triangulation of $\widetilde{\Omega}_j$, V_{jh} is the finite element space and the projection operator Π shall be defined later.

Let $\tilde{A}_j$ be the stiffness matrix of each problem on subdomain $\widetilde{\Omega}_j$, and M_{Γ_j} is the mass matrix on the interface corresponding to the term $\langle \Pi u_{jh}^n, \Pi \overline{v}_h\rangle_{\Gamma_j}$ in (4.3).

Define $\Pi : V_{jh}|_{\Gamma_j} \to M_{jh}$ to be the L^2 projection operator such that

$$\langle \Pi u_{jh}, \overline{v}_h\rangle_{\Gamma_j} = \langle u_{jh}, \overline{v}_h\rangle_{\Gamma_j}, \quad \forall v_h \in M_{jh},$$

where M_{jh} is the subspace of $V_{jh}|_{\Gamma_j}$. If V_{jh} is a P_1 conforming finite element space, $M_{jh} = V_{jh}|_{\Gamma_j}$ and Π is the identify operator in this case; if V_{jh} is a Crouzeix–Raviart non-conforming finite element space, M_{jh} is the piecewise constant space, *i.e.* [24],

$$M_{jh} := \{\psi_{jh} : \psi_{jh}|_{\Gamma_{je}} \text{ is constant for } \forall \Gamma_{je} \text{ the element of the triangulation on } \Gamma_j\}.$$

A direct observation from the definitions of the space M_{jh} leads to the assertion that M_{Γ_j} is the tridiagonal matrix or identity matrix I for the cases of P_1 conforming or Crouzeix–Raviart non-conforming finite element discretization respectively.

Define the matrix P_{Mj} to be the zero extension matrix from interface Γ_j to subdomain $\widetilde{\Omega}_j$. That is for any vector g_j, which is the degrees of freedom in Γ_j, we have $P_{Mj} g_j|_{\Gamma_j} = g_j$. Moreover, the other degrees of freedom of $P_{Mj} g_j$ corresponding to $\widetilde{\Omega}_j$ are 0. Let T_1, T_2 be the restriction matrices for domains $\widetilde{\Omega}_2, \widetilde{\Omega}_1$ respectively. For any degrees of freedom w_{2h} in $\widetilde{\Omega}_2$, we define $T_1 w_{2h} = w_{2h}|_{\Gamma_1}$. T_2 is defined similarly. Denote the subdomain $\widetilde{\Omega}_j' = \widetilde{\Omega}_j \backslash (\widetilde{\Omega}_1 \cap \widetilde{\Omega}_2)$, which means $\widetilde{\Omega}_j$ without its overlapping part.

The subproblem in $\widetilde{\Omega}'_j$ is

$$\begin{aligned}
-\triangle u_j^n - k^2 u_j^n &= f'_j \quad \text{in } \widetilde{\Omega}'_j,\\
\frac{\partial u_j^n}{\partial n_j} &= g_j^{n'} \quad \text{on } \Gamma'_j,\\
u_j^n &= 0 \quad \text{on } {\Gamma'}_{Dj},\\
\frac{\partial u_j^n}{\partial n} - iku_j^n &= 0 \quad \text{on } {\Gamma'}_{Rj},
\end{aligned}$$

where $j=1,2$, ${\Gamma'}_{Dj}=\Gamma_D\cap\partial\widetilde{\Omega}'_j, {\Gamma'}_{Rj}=\Gamma_R\cap\partial\widetilde{\Omega}'_j$, and $\Gamma'_1=\Gamma_2, \Gamma'_2=\Gamma_1$. Similar as (4.3), the variational form for this subproblem is to find $u_{jh}^n\in V'_{jh}$ such that

$$(\nabla_h u_{jh}^n, \nabla_h \overline{v}_h)_{\widetilde{\Omega}'_j} - k^2(u_{jh}^n,\overline{v}_h)_{\widetilde{\Omega}'_j} - \mathrm{i}k\langle \Pi u_{jh}^n, \Pi\overline{v}_h\rangle_{{\Gamma'}_{Rj}} = (f'_j,\overline{v}_h)_{\widetilde{\Omega}'_j} + \langle g_j^{n\prime}, \Pi\overline{v}_h\rangle_{\Gamma'_j} \quad \forall v_h\in V'_{jh}. \tag{4.4}$$

Denote $\tilde{A}'_j$ as the stiffness matrix for the subproblem in $\widetilde{\Omega}'_j$, and $M'_{\Gamma'_j}$ as the mass matrix on the interface. Let the matrix D'_j be the restriction from $\widetilde{\Omega}_j$ to $\widetilde{\Omega}'_j$, *i.e.* $D'_j w_{jh} = w_{jh}|_{\widetilde{\Omega}'_j}$ for any degrees of freedom w_{jh} in $\widetilde{\Omega}_j$. Similarly, we define the matrix T'_j to be the restriction from $\widetilde{\Omega}'_j$ to its interface Γ'_j, *i.e.* $T'_j w'_{jh} = w'_{jh}|_{\Gamma'_j}$ for any degrees of freedom w'_{jh} in $\widetilde{\Omega}'_j$.

According to (4.4), the normal derivative along the interface is

$$\langle g_j^{n\prime}, \Pi\overline{v}_h\rangle_{\Gamma'_j} = (\nabla_h u_{jh}^n, \nabla_h \overline{v}_h)_{\widetilde{\Omega}'_j} - k^2(u_{jh}^n,\overline{v}_h)_{\widetilde{\Omega}'_j} - \mathrm{i}k\langle \Pi u_{jh}^n, \Pi\overline{v}_h\rangle_{{\Gamma'}_{Rj}} - (f'_j,\overline{v}_h)_{\widetilde{\Omega}'_j} \quad \forall v_h\in V'_{jh}.$$

So if the vector w_{jh} for each subdomain $\widetilde{\Omega}_j$ is known, the vector form of the normal derivative $g_j^{n\prime}$ along the interface is

$$M'_{\Gamma'_j} g_j^{n\prime} = T'_j(\tilde{A}'_j D'_j w_{jh} - f'_j).$$

By the above definitions, the discrete form of the DD iteration procedure can be expressed explicitly as follows:

(1) Solve for w_{1h}^n in $\widetilde{\Omega}_1$

$$w_{1h}^n = \tilde{A}_1^{-1}(P_{M1}M_{\Gamma_1}g_1^n + f_1).$$

(2) Update the transmission condition along Γ_2

$$M_{\Gamma_2}g_2^n = \gamma_2 M_{\Gamma_2}T_2 w_{1h}^n - T'_1(\tilde{A}'_1 D'_1 w_{1h}^n - f'_1).$$

(3) Solve the subproblem in $\widetilde{\Omega}_2$ to get w_{2h}^n

$$w_{2h}^n = \tilde{A}_2^{-1}(P_{M2}M_{\Gamma_2}g_2^n + f_2).$$

(4) Update the transmission condition along Γ_1

$$M_{\Gamma_1}g_1^* = \gamma_1 M_{\Gamma_1}T_1 w_{2h}^n - T'_2(\tilde{A}'_2 D'_2 w_{2h}^n - f'_2).$$

(5) Relax the transmission condition

$$M_{\Gamma_1}g_1^{n+1} = \theta M_{\Gamma_1}g_1^* + (1-\theta)M_{\Gamma_1}g_1^n.$$

Eliminating the variables $w_{1h}^n, w_{2h}^n, g_2^n, g_1^*$, we get

$$\begin{aligned}
M_{\Gamma_1}g_1^{n+1} = M_{\Gamma_1}g_1^n + \theta\{&(\gamma_1 M_{\Gamma_1}T_1 - T'_2\tilde{A}'_2D'_2)\tilde{A}_2^{-1}P_{M2}(\gamma_2 M_{\Gamma_2}T_2 - T'_1\tilde{A}'_1D'_1)\tilde{A}_1^{-1}f_1\\
&+ (\gamma_1 M_{\Gamma_1}T_1 - T'_2\tilde{A}'_2D'_2)\tilde{A}_2^{-1}f_2\\
&+ (\gamma_1 M_{\Gamma_1}T_1 - T'_2\tilde{A}'_2D'_2)\tilde{A}_2^{-1}P_{M2}T'_1f'_1 + T'_2f'_2\\
&+ [(\gamma_1 M_{\Gamma_1}T_1 - T'_2\tilde{A}'_2D'_2)\tilde{A}_2^{-1}P_{M2}(\gamma_2 M_{\Gamma_2}T_2 - T'_1\tilde{A}'_1D'_1)\tilde{A}_1^{-1}P_{M1}M_{\Gamma_1} - M_{\Gamma_1}]g_1^n\}.
\end{aligned}$$

Then the preconditioned system corresponding to the above iteration method is

$$Pg_1 = f_{g1}, \tag{4.5}$$

where

$$P = M_{\Gamma_1} - (\gamma_1 M_{\Gamma_1} T_1 - T_2' \tilde{A}_2' D_2') \tilde{A}_2^{-1} P_{M2} (\gamma_2 M_{\Gamma_2} T_2 - T_1' \tilde{A}_1' D_1') \tilde{A}_1^{-1} P_{M1} M_{\Gamma_1}$$

and

$$\begin{aligned} f_{g1} &= (\gamma_1 M_{\Gamma_1} T_1 - T_2' \tilde{A}_2' D_2') \tilde{A}_2^{-1} P_{M2} (\gamma_2 M_{\Gamma_2} T_2 - T_1' \tilde{A}_1' D_1') \tilde{A}_1^{-1} f_1 \\ &\quad + (\gamma_1 M_{\Gamma_1} T_1 - T_2' \tilde{A}_2' D_2') \tilde{A}_2^{-1} f_2 \\ &\quad + (\gamma_1 M_{\Gamma_1} T_1 - T_2' \tilde{A}_2' D_2') \tilde{A}_2^{-1} P_{M2} T_1' f_1' + T_2' f_2'. \end{aligned}$$

The linear system (4.5) may be directly solved by GMRES and then w_{1h}^n, w_{2h}^n can be computed by further solving each subproblem.

Following the same way that we introduced in [6], we can then extend this two-subdomain algorithm to the many-subdomain case. However, we will not go to the details, since according to the above analysis and the following numerical results about the two-subdomain case, we find that the overlapping method is worse than the nonoverlapping method, which indicate using the nonoverlapping method in practice.

5. Numerical results

In this section, we shall give some numerical results to confirm our theory. Consider the following Helmholtz equation

$$\begin{aligned} -\triangle u - k^2 u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} - iku &= 0 \quad \text{on } \Gamma_R, \\ u &= 0 \quad \text{on } \Gamma_D, \end{aligned} \tag{5.1}$$

where Ω is the unit square $[0,1]\times[0,1]$, Γ_R denotes $x=0$ and $x=1$, which is the Robin boundary condition, and Γ_D is the boundary $y=0$ and $y=1$ with homogeneous Dirichlet boundary condition. The source term is chosen to be a Gaussian function $f=\exp\{-h^{-2}[(x-1/3)^2+(y-1/2)^2]\}$, which is an approximation to the point source $(\frac{1}{3},\frac{1}{2})$ with exponential decay off the center. Following the same way as in [6], we choose the Crouzeix–Raviart non-conforming finite element to discretize this model problem here.

The iteration of our method with different mesh sizes and different overlapping sizes is considered. According to our theory, the parameter is chosen to be $\theta=\frac{1}{2}$ in our iterative method. The algorithm stops when $\frac{\|u_{\text{iter}}-u_d\|_{l^2}}{\|u_d\|_{l^2}} \leqslant 10^{-10}$, where u_{iter} is the iteration solution, and u_d is the discrete mono-domain solution. The choices of the Robin transmission condition parameters are $\gamma_1=\frac{1}{100}k^{1/2}(1-i), \gamma_2=h^{-2}(1-i)$. The interfaces Γ_1 and Γ_2 are set close to the line $x=\frac{1}{2}$ for both nonoverlapping and overlapping case. We first fix the wave number k to test the iteration numbers corresponding to different mesh size h. It is shown in Table 1 that the iteration number does not change for each fixed overlapping size L and refined mesh. Furthermore, we observe that the iteration number increases quickly for increasing L. When $L=0$, which is the nonoverlapping case, the iteration number is the smallest. Similar phenomenon is seen in Table 2, where the DD method is used as a preconditioner in GMRES iterations, and the iteration numbers greatly decrease. The terminal precision of the PGMRES iteration is 10^{-10}.

Next, we fix $kh=$ constant to test how the iteration numbers depend on the wave number k. The result is shown in Table 3 for the iteration DD method, and Table 4 for PGMRES. When L is fixed to be $0, h$, or $2h$, the iteration number is stable, which means it does not change a lot, for increasing k. However, if $L=4h$, the iteration number is growing significantly when increasing k. This is mainly because, in this case, $kL\approx 2$, and

TABLE 1. The iteration numbers for different overlapping sizes L and different mesh sizes. The wave number $k = 19.5\pi$.

L	0	1/128	1/64	1/32
$h = \frac{1}{128}$	7	29	106	>500
$h = \frac{1}{256}$	6	29	106	>500
$h = \frac{1}{512}$	6	29	106	>500
$h = \frac{1}{1024}$	6	29	105	>500

TABLE 2. The iteration numbers of PGMRES for different overlapping sizes L and different mesh sizes. The wave number $k = 19.5\pi$.

L	0	1/128	1/64	1/32
$h = \frac{1}{128}$	4	11	15	27
$h = \frac{1}{256}$	4	11	14	27
$h = \frac{1}{512}$	4	11	14	26
$h = \frac{1}{1024}$	4	11	14	25

TABLE 3. The iteration numbers for different overlapping sizes L and different wave number k with $kh \approx 0.5$.

L	0	h	$2h$	$4h$
$k = 9.5\pi$	6	32	140	>500
$k = 19.5\pi$	7	30	125	>500
$k = 39.5\pi$	5	32	126	>500
$k = 79.5\pi$	7	31	127	>500
$k = 159.5\pi$	6	32	127	>500

TABLE 4. The iteration numbers of PGMRES with different wave number k and different overlapping sizes L for $kh \approx 0.5$.

L	0	h	$2h$	$4h$
$k = 9.5\pi$	5	11	12	14
$k = 19.5\pi$	4	12	15	27
$k = 39.5\pi$	4	12	17	47
$k = 79.5\pi$	4	13	18	84
$k = 159.5\pi$	4	15	19	164

by simple calculation, the range of $\phi = 2\alpha L$ may be $[0, 4]$. According to the discussion in Remark 3.7, this range contains the case $\phi = \pi$, which may lead to $\rho \geqslant 1$.

The influence of the relaxation parameter is considered in Table 5. It reveals that the convergence rate sensitively depends on the choice of θ and $\theta = \frac{1}{2}$ is optimal, which confirms our theoretical result.

The interfaces Γ_1 and Γ_2 are set close to the domain decomposition position x. For different choices of x, we obtain the results in Table 6. We find that for the nonoverlapping case, $x = \frac{1}{2}$ is the best. However, for the overlapping case, $x = \frac{3}{8}$, which is close to the point source $(\frac{1}{3}, \frac{1}{2})$, has the largest iteration numbers.

Finally, we test some other choice of parameters γ_1 and γ_2. From Table 7 we could find that if γ_1 and γ_2 are far away from each other, which satisfy Theorem 3.5, then the iteration numbers increase with the

TABLE 5. The iteration numbers for different choices of the relaxation parameter θ. Here "div"means divergence. $k = 19.5\pi$ and $kh \approx 0.5$.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$L = h$	135	65	43	33	30	41	87	div	div	div
$L = 2h$	390	210	154	132	125	131	151	>500	div	div

TABLE 6. The iteration numbers of PGMRES for different domain decomposition position x and different overlapping sizes L, where $k = 19.5\pi$, $h \approx \frac{1}{120}$ and $kh \approx 0.5$.

x	1/8	1/4	3/8	1/2	5/8	3/4	7/8
$L = 0$	11	10	10	4	11	12	11
$L = h$	13	14	17	12	12	13	12
$L = 2h$	16	18	20	15	15	15	15
$L = 4h$	27	28	29	27	26	25	24

TABLE 7. The iteration numbers of PGMRES with different choices of parameters γ_1, γ_2 and different overlapping sizes L, where $k = 19.5\pi$, $h \approx \frac{1}{120}$ and $kh \approx 0.5$.

γ_1	γ_2	$L = 0$	$L = h$	$L = 2h$	$L = 4h$
$\gamma_1 = \frac{1}{100}k^{1/2}(1-i)$	$\gamma_2 = h^{-2}(1-i)$	4	12	15	27
$\gamma_1 = k^{1/2}(1-i)$	$\gamma_2 = h^{-2}(1-i)$	12	13	15	23
$\gamma_1 = \frac{1}{1000}k^{1/2}(1-i)$	$\gamma_2 = h^{-2}(1-i)$	5	13	17	28
$\gamma_1 = \frac{1}{100}k^{1/2}(1-i)$	$\gamma_2 = h^{-1}(1-i)$	11	15	17	21
$\gamma_1 = \frac{1}{100}k^{1/2}(1-i)$	$\gamma_2 = 10h^{-2}(1-i)$	5	13	16	28
$\gamma_1 = k^{1/2}(1-i)$	$\gamma_2 = h^{-1}(1-i)$	11	13	16	17
$\gamma_1 = k^{1/2}(1-i)$	$\gamma_2 = 4h^{-1/2}(1-i)$	14	15	15	14
$\gamma_1 = k^{1/2}(1-i)$	$\gamma_2 = h^{-1/2}(1-i)$	19	18	15	14
$\gamma_1 = 1-i$	$\gamma_2 = 1-i$	24	20	17	16

increase of overlap, which indicates that the larger overlap leads to slower convergence. However, if γ_1 and γ_2 are close to each other, then the iteration numbers do not increase when we increase the overlap. The choice $\gamma_1 = \frac{1}{100}k^{1/2}(1-i)$ and $\gamma_2 = h^{-2}(1-i)$, which is used in our numerical test, is still the best for nonoverlap and small overlap. Also from this result, we suggest using the non-overlapping method in practical computing.

6. CONCLUSION

How the overlapping size influences the convergence rate of an optimized Schwarz method with relaxation for the Helmholtz equation is studied in this paper. We analyzed the convergence rate of this method for a model problem by Fourier transform. It is proved that the optimal choice of the relaxation parameter is $\theta = \frac{1}{2}$, and the overlapping size shall not be large for the efficiency of this DD method.

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