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CONVERGENCE OF DISCRETE AND CONTINUOUS UNILATERAL FLOWS FOR AMBROSIO–TORTORELLI ENERGIES AND APPLICATION TO MECHANICS

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Abstract. We study the convergence of an alternate minimization scheme for a Ginzburg–Landau phase-field model of fracture. This algorithm is characterized by the lack of irreversibility constraints in the minimization of the phase-field variable; the advantage of this choice, from a computational stand point, is in the efficiency of the numerical implementation. Irreversibility is then recovered *a posteriori* by a simple pointwise truncation. We exploit a time discretization procedure, with either a one-step or a multi (or infinite)-step alternate minimization algorithm. We prove that the time-discrete solutions converge to a unilateral L^2-gradient flow with respect to the phase-field variable, satisfying equilibrium of forces and energy identity. Convergence is proved in the continuous (Sobolev space) setting and in a discrete (finite element) setting, with any stopping criterion for the alternate minimization scheme. Numerical results show that the multi-step scheme is both more accurate and faster. It provides indeed good simulations for a large range of time increments, while the one-step scheme gives comparable results only for very small time increments.

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1. Introduction

In the last years, after [12], the use of phase-field models in computational fracture mechanics has been constantly increasing (see, *e.g.*, [2] for a review on different models). In the original formulation of [12], for quasi-static brittle fracture in linearly elastic bodies, the propagation of the crack, represented by a phase-field function v, is determined by means of equilibrium configurations of the energy

$$\mathcal{F}_\varepsilon(u,v) := \tfrac{1}{2}\int_\Omega (v^2+\eta_\epsilon)\,\boldsymbol{\sigma}(u):\boldsymbol{\epsilon}(u)\,\mathrm{d}x + G_c\int_\Omega \tfrac{1}{4\varepsilon}(v-1)^2+\varepsilon|\nabla v|^2\,\mathrm{d}x, \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, $v\in H^1(\Omega)$, $u\in H^1(\Omega;\mathbb{R}^n)$ is the displacement, $\boldsymbol{\epsilon}(u)$ denotes the symmetric part of the gradient of u, $\boldsymbol{\sigma}(u):=\mathbf{C}\boldsymbol{\epsilon}(u)$, $\mathbf{C}$ being the usual elasticity tensor, ε and η_ε are

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two small positive parameters, and $G_c > 0$ is the toughness, related to the physical properties of the elastic material under consideration. In particular, in (1.1), the function v takes values in the interval $[0,1]$, where $v(x) = 1$ means that the elastic body is safe at $x \in \Omega$, while $v(x) = 0$ means that the material is fractured at x.

From the computational stand point, the study of the functional (1.1) is very convenient in combination with the so-called alternate minimization (or staggered) algorithm [12]: equilibrium configurations of the energy are indeed computed iteratively, minimizing $\mathcal{F}_\varepsilon$ first w.r.t. u and then w.r.t. v. In this way, at each iteration we look for a minimum of a quadratic functional, which leads, in the numerical framework, to solve a linear system (actually with variable coefficients). Moreover, energies like $\mathcal{F}_\varepsilon$, defined in Sobolev spaces, can be easily discretized in finite element spaces or, alternatively, by finite differences.

This phase-field approach raises several questions, of interest both on the theoretical level and for the applications. First, it is important to understand the relationship between phase-field and sharp crack (or sharp interface) energies, obtained in the limit as $\varepsilon \to 0$. Results in this direction are usually framed within the theory of Γ-convergence [18] and BV-like spaces [3], in the spirit of the seminal work [4]. In our mechanical context, the Γ-limit of the energy $\mathcal{F}_\varepsilon$, as $\varepsilon \to 0$, takes the form

$$\tfrac{1}{2} \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u) \, \mathrm{d}x + G_c \mathcal{H}^{n-1}(J_u), \tag{1.2}$$

where J_u (the set of discontinuity points of u) represents the crack. Rigorous proofs have been provided by Chambolle [15], in the framework of SBD^2 spaces [11], and later by Iurlano [26] in the more general setting of $GSBD^2$ spaces [20].

Similar results hold also in the discrete setting, *i.e.*, for finite element discretizations, say $\mathcal{F}_{\varepsilon,h}$ (h being the mesh size), of the energy $\mathcal{F}_\varepsilon$ (1.1). In this case, the same Γ-limit (1.2) is recovered, as $\varepsilon \to 0$, under the condition $h = o(\varepsilon)$ (see, *e.g.*, [10]); the use of small meshes is indeed necessary for an accurate approximation of the transition layer, of order ε, of the phase-field function. On a static level, Γ-convergence provides a rigorous way to prove that phase-field energies are indeed "regularizations" of sharp crack energies. As a by-product of Γ-convergence, global minimizers of (1.1) converge to global minimizers of (1.2), under suitable compactness properties. At the present stage not much is known about the convergence of critical points and energy release (see, for instance, [23]). This is related to the fact that a "good" notion of energy release or slope in BV-like spaces is still missing (see, *e.g.*, [19]).

Let us turn our attention to the problem of evolution of the phase-field driven by the energy functional $\mathcal{F}_\varepsilon$ (1.1). First, we describe the scheme studied in [27]. In dimension $n = 2$, let $[0,T]$ be a time interval and consider, for instance, a time dependent boundary condition $u = g(t)$ on $\partial\Omega$ and initial conditions u_0 and v_0, with $0 \le v_0 \le 1$. As is by now typical in the study of many rate-independent processes (see, *e.g.*, [31, 32] and reference therein), we proceed with a time discretization. For every $k \in \mathbb{N} \setminus \{0\}$, let $\tau_k := T/k$ be a time increment and denote $t_i^k := i\tau_k$, for $i = 0, ..., k$. The discrete in time evolutions are defined recursively as follows. Known u_{i-1}^k and v_{i-1}^k (at time t_{i-1}^k), consider the auxiliary sequences $u_{i,j}^k$ and $v_{i,j}^k$, for $j \in \mathbb{N}$, defined by the following alternate minimization scheme: $u_{i,0}^k := u_{i-1}^k$, $v_{i,0}^k := v_{i-1}^k$, and, for $j \ge 1$,

$$u_{i,j}^k := \arg\min \{ \mathcal{F}_\varepsilon(u, v_{i,j-1}^k) : \, u \in H^1(\Omega;\mathbb{R}^2), \, u = g(t_i^k) \text{ on } \partial\Omega \}, \tag{1.3}$$

$$v_{i,j}^k := \arg\min \{ \mathcal{F}_\varepsilon(u_{i,j}^k, v) : \, v \in H^1(\Omega), \, v \le v_{i,j-1}^k \}. \tag{1.4}$$

Then set

$$u_i^k := \lim_{j \to +\infty} u_{i,j}^k \qquad \text{and} \qquad v_i^k := \lim_{j \to +\infty} v_{i,j}^k. \tag{1.5}$$

Once the discrete evolutions are known for every τ_k, it is natural to investigate their limit as the time increment $\tau_k \to 0$. A complete result in this direction has been obtained in [27] characterizing the limit in terms of BV-evolutions. Describing this result is out of scope here. We only mention that the limit evolution satisfies a phase-field version of Griffith's criterion. Therefore, the time discrete scheme defined by (1.3)–(1.5) provides an approximation of a quasi-static evolution for brittle fracture. Let us also mention that a discrete version of [27] in a finite element setting (*i.e.*, for an energy $\mathcal{F}_{\varepsilon,h}$) has been studied in [1] together with the limit of the evolutions as the mesh parameter h tends to 0.

We notice that in the minimization (1.4) w.r.t. the phase-field variable v, the irreversibility of the crack is enforced through the constraint $v \le v^k_{i,j-1}$. In the literature there are other alternatives to impose irreversibility, such as by sublevel sets [12] or by accumulated traction energy [30]. In the present work, following [35], we actually adopt a further way, computationally very convenient and still physically correct. In order to be more precise, we now briefly describe the alternate minimization scheme we are going to study in this paper. Let us start with the simplest possible time discrete scheme, as proposed in [35], based on a single-step alternate minimization. With the notation used above, known u^k_{i-1} and v^k_{i-1} (at time t^k_{i-1}), u^k_i and v^k_i are defined by

$$u^k_i := \arg\min\{\mathcal{F}_\varepsilon(u, v^k_{i,j-1}) : u \in H^1(\Omega;\mathbb{R}^2),\ u = g(t^k_i) \text{ on } \partial\Omega\}, \tag{1.6}$$

$$v^k_i := \min\{\tilde v^k_i, v^k_{i-1}\} \quad \text{where} \quad \tilde v^k_i := \arg\min\{\mathcal{F}_\varepsilon(u^k_i, v) + \tfrac{1}{2\tau_k}\|v - v^k_{i-1}\|^2_{L^2} : v \in H^1(\Omega)\}. \tag{1.7}$$

Some comments are due. First of all, irreversibility, in terms of monotonicity of v, is taken into account by a simple truncation after the minimization w.r.t. v, which is unconstrained. This is numerically very efficient since it does not require to handle a unilateral constraint and leads to solve a simple linear system in order to compute $\tilde v^k_i$ and v^k_i. Second, in the minimization with respect to v an L^2-penalization appears; this is indeed the choice in [35] and, as we will see, it will lead us to the construction of a unilateral L^2-gradient flow w.r.t. the phase-field variable. More precisely, we show that, as $\tau_k \to 0$, the time discrete evolutions converge to an evolution $t \mapsto (u(t), v(t))$ such that

$$u(t) \in \arg\min\{\mathcal{F}_\varepsilon(u, v(t)) : u \in H^1(\Omega;\mathbb{R}^2) \text{ with } u = g(t) \text{ on } \partial\Omega\},$$

v is monotone non-increasing (in time), $v(t)$ takes values in $[0,1]$, and the following energy balance identity holds:

$$\dot{\mathcal{F}}_\varepsilon(u(t), v(t)) = -\tfrac{1}{2}\|\dot v(t)\|^2_{L^2} - \tfrac{1}{2}|\partial^-_v \mathcal{F}_\varepsilon|^2(u(t), v(t)) + \mathcal{P}(u(t), v(t), \dot g(t)), \tag{1.8}$$

where $|\partial^-_v \mathcal{F}_\varepsilon|$ is the L^2-unilateral slope (see Def. 2.1 and Prop. 2.3), $\mathcal{P}$ is the power of external forces, and the dot denotes the time derivative. We refer to Definition 2.5 and to Section 4 for the precise statements. Evolutions $t \mapsto (u(t), v(t))$ satisfying (1.8) are, in a suitable weak sense [33], solutions of the system

$$\begin{cases} \dot v(t) = [a_\varepsilon \Delta v(t) + b_\varepsilon(1 - v(t)) + v(t)\boldsymbol{\sigma}(u(t)) : \boldsymbol{\epsilon}(u(t))]_+, \\ \operatorname{div}(\boldsymbol{\sigma}_{v(t)}(u(t))) = 0, \end{cases}$$

where $a_\varepsilon, b_\varepsilon > 0$, $[\cdot]_+$ denotes the positive part (which ensures irreversibility), and $\boldsymbol{\sigma}_{v(t)}(u(t)) := (v^2(t) + \eta_\varepsilon)\boldsymbol{\sigma}(u(t))$ is the phase-field stress. Unilateral L^2-evolutions of this type are frequently employed in computational fracture, in this form and under the name of Ginzburg–Landau models (see, *e.g.*, [2] and the references therein). A system of this type has been studied in [9] and employed, as a regularization, also in [28]. We recall that the vanishing viscosity limit of these rate independent evolutions are indeed quasi-static BV-evolutions [28, 33]. A different approach for a unilateral rate-independent model, coupled with elasto-dynamic, can be found in [29].

As we have already mentioned, the scheme (1.6) and (1.7) is characterized by a single alternate minimization. This choice is the simplest possible to provide in the limit a unilateral L^2-gradient flow. However, we realized that computationally it does not provide good enough solutions (at least for reasonable time increments). This is mainly due to the fact that the couples (u^k_i, v^k_i) are not equilibrium configurations for the energy $\mathcal{F}_\varepsilon$. For this reason, in Section 5 we study also the following infinite-step scheme. Known u^k_{i-1} and v^k_{i-1} (at time t^k_{i-1}), consider the sequences $u^k_{i,j}$ and $v^k_{i,j}$, for $j \in \mathbb{N}$, constructed using this alternate minimization procedure: we set $u^k_{i,0} := u^k_{i-1}$, $v^k_{i,0} := v^k_{i-1}$, and, for $j \geq 1$,

$$u^k_{i,j} := \arg\min \{\mathcal{F}_\varepsilon(u, v^k_{i,j-1}) : \, u \in H^1(\Omega; \mathbb{R}^2), \, u = g(t^k_i) \text{ on } \partial\Omega\}, \tag{1.9}$$

$$v^k_{i,j} := \min \{\tilde{v}^k_{i,j}, v^k_{i-1}\} \quad \text{where} \quad \tilde{v}^k_{i,j} := \arg\min \{\mathcal{F}_\varepsilon(u^k_{i,j}, v) + \tfrac{1}{2\tau_k}\|v - v^k_{i-1}\|^2_{L^2} : \, v \in H^1(\Omega)\}. \tag{1.10}$$

Then set

$$u^k_i := \lim_{j\to+\infty} u^k_{i,j} \qquad \text{and} \qquad v^k_i := \lim_{j\to+\infty} v^k_{i,j}.$$

Note that in the definition of $\tilde{v}^k_{i,j}$ it appears, in the L^2-penalization term, the function v^k_{i-1} (the configuration at time t^k_{i-1}) and not $v^k_{i,j-1}$ (the previous configuration in the alternate scheme). In a similar way, in the definition of $v^k_{i,j}$ we truncate $\tilde{v}^k_{i,j}$ with v^k_{i-1}, so that, possibly, the sequence $\{v^k_{i,j}\}_{j\in\mathbb{N}}$ is not monotone, but still satisfies the constraint $v^k_{i,j} \leq v^k_{i-1}$ for every j. This choice is again motivated by applications and simulations. Indeed, using $v^k_{i,j-1}$, as in [27], may lead in some cases to accumulation of numerical errors at each iteration. As for the one iteration scheme, in the limit as $\tau_k \to 0$ we obtain a unilateral L^2-gradient flow.

In Section 6 we deal with a space discrete approximation of a unilateral L^2-gradient flow. We consider a family of P_1 finite element spaces on acute angle triangulations $\mathcal{T}_h$, *i.e.*,

$$u \in \{z \in H^1(\Omega; \mathbb{R}^2) : \; z \text{ is piecewise affine on } \mathcal{T}_h\} \qquad v \in \{z \in H^1(\Omega) : \; z \text{ is piecewise affine on } \mathcal{T}_h\},$$

and a family of approximating energies of the form

$$\mathcal{F}_{\varepsilon,h}(u, v) := \tfrac{1}{2}\int_\Omega \big(\Pi_h(v^2) + \eta_\varepsilon\big)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u)\,\mathrm{d}x + G_c \int_\Omega \tfrac{1}{4\varepsilon}\Pi_h\big((1-v)^2\big) \; + \varepsilon|\nabla v|^2\,\mathrm{d}x,$$

where Π_h is the usual Lagrange interpolation operator [1]. We remark that $\mathcal{F}_{\varepsilon,h}$ is not, strictly speaking, the restriction of $\mathcal{F}_\varepsilon$ to the finite element spaces. Nevertheless, it is not too difficult to show that the Γ-limit as $\varepsilon \to 0$ and with $h = o(\varepsilon)$ is again of the form (1.2). Moreover, the operator Π_h and the acute angle triangulations allow to prove [1] that the phase-field variable in the space discrete setting takes values in the interval $[0, 1]$.

In this framework, we consider again a time discrete approach in which the incremental problem is obtained by an alternate minimization procedure, producing this time a finite number of iterations, according to some stopping criterion. In order to have a general result, including all possible criteria, we only assume that the number of iterations J^k_i, possibly depending on k and i, are bounded from above, uniformly w.r.t. k and i, by a certain arbitrarily large number J. Thus, known u^k_{i-1} and v^k_{i-1} (at time t^k_{i-1}), we consider the sequences $u^k_{i,j}$ and $v^k_{i,j}$, for $j \in \mathbb{N}$, defined by the following alternate minimization scheme: $u^k_{i,0} := u^k_{i-1}$, $v^k_{i,0} := v^k_{i-1}$, and, for $j = 1, ..., J^k_i$,

$$u^k_{i,j} := \arg\min \{\mathcal{F}_{\varepsilon,h}(u, v^k_{i,j-1}) : u = g(t^k_i) \text{ on } \partial\Omega\}, \tag{1.11}$$

$$v^k_{i,j} := \min \{\tilde{v}^k_{i,j}, v^k_{i-1}\} \quad \text{where} \quad \tilde{v}^k_{i,j} := \arg\min \{\mathcal{F}_{\varepsilon,h}(u^k_{i,j}, v) + \tfrac{1}{2\tau_k}\|v - v^k_{i-1}\|^2_{L^2}\}. \tag{1.12}$$

Then set

$$u^k_i := u^k_{i,J^k_i} \qquad \text{and} \qquad v^k_i := v^k_{i,J^k_i}.$$

We prove that in the limit as $\tau_k \to 0$ and $h \to 0$ we obtain again a unilateral L^2-gradient flow. We refer to Theorems 6.13 and 6.17 for the precise statements.

Finally, in Section 7 we provide a detailed set of numerical examples. Our aim is to show and compare the efficiency of the one-step and multi (or infinite) step schemes. As we have mentioned above, it turns out that the multi-step algorithm is more stable and computationally more convenient than the single-step scheme. In particular, we will see that comparable evolutions are obtained for time step sizes of the order 10^{-1}, using the former algorithm, and for time step sizes of the order 10^{-3}, using the latter. For this reason, the multi-step scheme is computationally faster. We remark again that, from a numerical viewpoint, the power of the alternate minimization scheme investigated in this work is in the lack of a priori constraints in the phase-field minimizations (1.7), (1.10), and (1.12). In this way, indeed, we are simply led to solve a linear system.

From the technical point of view it is important to stress that our result employs an argument based on a fine regularity estimate, proved in [25] and already employed in [28], together with Sobolev embeddings (see proof of Prop. 2.9) which holds only for $\Omega \subset \mathbb{R}^2$. Second, the structure of discrete scheme, with unconstained minimization and a posteriory truncation makes it very difficult, if not impossible, to obtain H^1 estimates and apply Gronwall type arguments for the speed of the phase-field variable. We are thus forced to work only with L^2 velocities and the energy identity cannot rely on the chain rule. We use instead, for the energy identity, the Riemann sum argument of [21].

2. Notation and setting of the problem

Let Ω be an open bounded subset of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Denote $\mathcal{U} := H^1(\Omega;\mathbb{R}^2)$ and $\mathcal{V} := H^1(\Omega) \cap L^\infty(\Omega)$. For every $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we define the *elastic energy*

$$\mathcal{E}(u,v) := \tfrac{1}{2} \int_\Omega (v^2 + \eta)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u)\, \mathrm{d}x, \tag{2.1}$$

where η is a positive parameter, $\boldsymbol{\epsilon}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the (linearized) strain, $\boldsymbol{\sigma}(u) := \mathbf{C}\boldsymbol{\epsilon}(u)$ stands for the (linearized) elastic stress, and $\mathbf{C}$ is the stiffness matrix. We assume that $\mathbf{C}$ is positive definite on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$. In few cases we will also employ the phase-field stress $\boldsymbol{\sigma}_v(u) := (v^2 + \eta)\, \boldsymbol{\sigma}(u)$.

We introduce the *dissipation potential* associated to the phase-field variable $v \in \mathcal{V}$ given by

$$\mathcal{D}(v) := \tfrac{1}{2} \int_\Omega |\nabla v|^2 + (1-v)^2 \, \mathrm{d}x. \tag{2.2}$$

Note that the *dissipation* (*i.e.*, rate of dissipated energy) turns out to be of the form $d\mathcal{D}(v)[\dot{v}]$ (under suitable time regularity of v), where the dot denotes the time derivative.

The *total energy* $\mathcal{F}\colon \mathcal{U} \times \mathcal{V} \to [0, +\infty)$ of the system is given by the sum of elastic energy (2.1) and dissipation potential (2.2), *i.e.*,

$$\mathcal{F}(u,v) := \mathcal{E}(u,v) + \mathcal{D}(v). \tag{2.3}$$

We notice that the functional $\mathcal{F}$ in (2.3) coincides with $\mathcal{F}_\varepsilon$ in (1.1) for $\varepsilon = \frac{1}{2}$ and $G_C = 1$. This choice is made for notational convenience and does not influence our analysis.

An important role in the definition of evolution we consider in this work is played by the following notion of *unilateral L^2-slope*.

Definition 2.1. For $u \in \mathcal{U}$ and $v \in \mathcal{V}$ we define the *unilateral L^2-slope* of $\mathcal{F}$ with respect to v at the point (u,v) as

$$|\partial_v^- \mathcal{F}|(u,v) := \limsup_{\substack{z \to v \\ z \in \mathcal{V},\, z \le v}} \frac{[\mathcal{F}(u,v) - \mathcal{F}(u,z)]_+}{\|v - z\|_{L^2}}, \tag{2.4}$$

where $[\cdot]_+$ denotes the positive part and the convergence is intended in the L^2-topology.

Remark 2.2. The minus sign appearing in the notation $|\partial_v^- \mathcal{F}|$ reminds that only negative variations are allowed; it should not be confused with a similar notation for the relaxed slope (see, *e.g.*, [5], Sect. 2.3).

For $u \in \mathcal{U}$ and $v, \varphi \in \mathcal{V}$ there exists finite the partial derivative of $\mathcal{F}$ with respect to v, *i.e.*,

$$\partial_v \mathcal{F}(u,v)[\varphi] = \int_\Omega v\varphi\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u)\, \mathrm{d}x + \int_\Omega \nabla v \cdot \nabla\varphi - (1-v)\varphi \, \mathrm{d}x. \tag{2.5}$$

The natural relationship between partial derivatives (2.5) and slope (2.4) is stated in the next lemma.

Lemma 2.3. *For $u \in \mathcal{U}$ and $v \in \mathcal{V}$ there holds*

$$|\partial_v^- \mathcal{F}|(u,v) = \sup\{-\partial_v \mathcal{F}(u,v)[\varphi] : \varphi \in \mathcal{V},\ \varphi \le 0,\ \|\varphi\|_{L^2} \le 1\}. \tag{2.6}$$

Proof. For all $\varphi \in \mathcal{V}$ with $\varphi \le 0$ and $\|\varphi\|_{L^2} \le 1$ there holds

$$\begin{aligned} -\partial_v \mathcal{F}(u,v)[\varphi] &= \lim_{s\to 0^+} \frac{\mathcal{F}(u,v) - \mathcal{F}(u, v+s\varphi)}{s} \\ &\le \limsup_{s\to 0^+} \frac{[\mathcal{F}(u,v) - \mathcal{F}(u,v+s\varphi)]_+}{\|v-(v+s\varphi)\|_{L^2}} \le \limsup_{\substack{z\to v \\ z\in\mathcal{V},\, z\le v}} \frac{[\mathcal{F}(u,v) - \mathcal{F}(u,z)]_+}{\|v-z\|_{L^2}}. \end{aligned}$$

Taking the supremum of all φ we get

$$\sup\{-\partial_v \mathcal{F}(u,v)[\varphi] : \varphi \in \mathcal{V},\ \varphi \le 0,\ \|\varphi\|_{L^2} \le 1\} \le |\partial_v^- \mathcal{F}|(u,v). \tag{2.7}$$

In order to show the opposite inequality, let (z_n) in $\mathcal{V}$ with $z_n \to v$ and $z_n \le v$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{[\mathcal{F}(u,v) - \mathcal{F}(u,z_n)]_+}{\|v-z_n\|_{L^2}} = |\partial_v^- \mathcal{F}|(u,v).$$

We can assume that $|\partial_v^- \mathcal{F}|(u,v) > 0$, otherwise the inequality is obvious since $\partial_v \mathcal{F}(u,v)[0] = 0$. Hence, for n sufficiently large we have $\mathcal{F}(u,v) \ge \mathcal{F}(u,z_n)$. Together with the convexity of $\mathcal{F}(u,\cdot)$ there holds

$$|\partial_v^- \mathcal{F}|(u,v) = \lim_{n\to\infty} \frac{\mathcal{F}(u,v) - \mathcal{F}(u,z_n)}{\|v-z_n\|_{L^2}} \le -\liminf_{n\to\infty} \partial_v \mathcal{F}(u,v)[z_n'], \tag{2.8}$$

where $z_n' = (z_n - v)/\|v - z_n\|_{L^2}$. Clearly $z_n' \in \mathcal{V}$, $z_n' \le 0$ and $\|z_n'\|_{L^2} \le 1$. This concludes the proof of the lemma. $\square$

Finally, let us define, for $u, z \in \mathcal{U}$ and $v \in \mathcal{V}$, the functional

$$\mathcal{P}(u,v,z) = \int_\Omega (v^2+\eta)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(z)\, \mathrm{d}x = \int_\Omega \boldsymbol{\sigma}_v(u) : \boldsymbol{\epsilon}(z)\, \mathrm{d}x. \tag{2.9}$$

We anticipate here a continuity property of $\mathcal{P}$ which will be useful in the forthcoming discussion.

Lemma 2.4. *If $u_m \rightharpoonup u$ in $\mathcal{U}$ and $v_m \to v$ in $L^2(\Omega;[0,1])$, then*

$$\lim_{m\to\infty} \mathcal{P}(u_m, v_m, z) = \mathcal{P}(u,v,z).$$

Proof. Remember that

$$\mathcal{P}(u,v,z) = \int_\Omega (v^2+\eta)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(z)\, \mathrm{d}x.$$

Consider a subsequence (not relabelled) such that $v_m \to v$ a.e. in Ω. By dominated convergence it is easy to see that $v_m^2\, \boldsymbol{\epsilon}(z) \to v^2\, \boldsymbol{\epsilon}(z)$ strongly in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Clearly $\boldsymbol{\sigma}(u_m) \rightharpoonup \boldsymbol{\sigma}(u)$ (weakly) in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Hence $\mathcal{P}(u_m, v_m, z) \to \mathcal{P}(u,v,z)$. Since the limit is independent of the subsequence, the convergence holds for the whole sequence. $\square$

We are now in a position to give the precise definition of gradient flow evolution we consider in this paper.

Definition 2.5. Let $T > 0$ and $g \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^2))$ for some $p > 2$. Let $u_0 \in \mathcal{U}$ with $u_0 = g(0)$ on $\partial\Omega$ and let $v_0 \in H^1(\Omega;[0,1])$ be such that

$$u_0 \in \arg\min\{\mathcal{E}(u, v_0) : u \in \mathcal{U} \text{ with } u = g(0) \text{ on } \partial\Omega\}. \tag{2.10}$$

We say that a pair $(u,v)\colon [0,T] \to \mathcal{U} \times \mathcal{V}$ is a *unilateral L^2-gradient flow* for the energy $\mathcal{F}$ with initial condition (u_0, v_0) and boundary condition g if the following properties are satisfied:

(a) *Time regularity*: $u \in C([0,T];\mathcal{U})$ and $v \in H^1([0,T];L^2(\Omega)) \cap L^\infty([0,T];H^1(\Omega))$ with $u(0) = u_0$ and $v(0) = v_0$;

(b) *Irreversibility*: $t \mapsto v(t)$ is non-increasing (*i.e.*, $v(s) \le v(t)$ a.e. in Ω for every $0 \le t \le s \le T$) and $0 \le v(t) \le 1$ for every $t \in [0,T]$;

(c) *Displacement equilibrium*: for every $t \in [0,T]$ we have $u(t) = g(t)$ on $\partial\Omega$ and

$$u(t) \in \arg\min \{\mathcal{E}(u, v(t)) : u \in \mathcal{U} \text{ with } u = g(t) \text{ on } \partial\Omega\};$$

(d) *Energy balance*: the map $t \mapsto \mathcal{F}(u(t), v(t))$ is absolutely continuous and for a.e. $t \in [0,T]$ it holds

$$\dot{\mathcal{F}}(u(t), v(t)) = -\tfrac{1}{2}\|\dot{v}(t)\|^2_{L^2} - \tfrac{1}{2}|\partial^-_v \mathcal{F}|^2(u(t), v(t)) + \mathcal{P}(u(t), v(t), \dot{g}(t)).$$

Remark 2.6. Note that $\mathcal{P}(u(t), v(t), \dot{g}(t))$ provides the power of external forces. Indeed, by equilibrium of $u(t)$,

$$\mathcal{P}(u(t), v(t), \dot{g}(t)) = -\int_\Omega \operatorname{div}(\boldsymbol{\sigma}_v(u))\, \dot{g}(t)\, \mathrm{d}x + (\boldsymbol{\sigma}_v(u)(t)\,\nu, \dot{g}(t)) = (\boldsymbol{\sigma}_v(u)(t)\,\nu, \dot{g}(t)),$$

where $(\cdot,\cdot)$ denotes the duality between $H^{-1/2}(\partial\Omega;\mathbb{R}^2)$ and $H^{1/2}(\partial\Omega;\mathbb{R}^2)$, and ν stands for the exterior unit normal vector to $\partial\Omega$. Hence $\mathcal{P}(u(t), v(t), \dot{g}(t))$ gives a weak formulation for the "classic power"

$$\int_{\partial\Omega} (\boldsymbol{\sigma}_v(u)(t)\,\nu) \cdot \dot{g}(t)\, \mathrm{d}\mathcal{H}^1.$$

Remark 2.7. If (u, v) is a unilateral L^2-gradient flow in the sense of Definition 2.5, then $v \in C([0,T]; H^1(\Omega))$. Indeed, if $t_n \to t$ then $u(t_n) \to u(t)$ (strongly) in $\mathcal{U}$ while $v(t_n) \rightharpoonup v(t)$ (weakly) in $H^1(\Omega)$. It is not difficult to check that, by the displacement equilibrium (c), $\mathcal{E}(u(t_n), v(t_n)) \to \mathcal{E}(u(t), v(t))$. Hence, the energy balance (d) implies $\mathcal{D}(v(t_n)) \to \mathcal{D}(v(t))$, and, since $v \in C([0,T]; L^2(\Omega))$, it follows that

$$\int_\Omega |\nabla v(t_n)|^2\, \mathrm{d}x \to \int_\Omega |\nabla v(t_n)|^2\, \mathrm{d}x.$$

From this we deduce, together with weak convergence, the continuity of $t \mapsto v(t)$ in $H^1(\Omega)$.

Our first goal is to prove the convergence to a unilateral L^2-gradient flow of the *time discrete* solutions obtained by a couple of iterative schemes (see Sects. 4 and 5) based on the "unconstrained" version [35] of the alternate minimization algorithm [12]. Our second aim is to show, in the spirit of [1], that the same convergence result holds true for the corresponding *space and time discrete* scheme, *i.e.*, when also a space discretization is considered, inspired by Finite Element approximation. We refer to Section 6 for the detailed presentation of this last topic.

Before starting any discussion about the construction and convergence of a unilateral L^2-gradient flow, let us comment on the energy equality (d). In particular, we show in Proposition 2.9 that only an energy inequality is sufficient. The proof is based on a combination of a quantitative regularity estimate proved in Theorem 1.1 from [25] and a Riemann sum argument inspired by Gianni Dal Maso [21].

Next lemma provides the regularity property needed in our setting. For a more general statement we refer to [25].

Lemma 2.8. *Let $g \in AC([0,T]; W^{1,p}(\Omega;\mathbb{R}^2))$ for $p > 2$. For $t \in [0,T]$ and $v \in H^1(\Omega;[0,1])$ denote*

$$u(t,v) := \arg\min \{\mathcal{E}(u,v) : u \in \mathcal{U} \text{ with } u = g(t) \text{ on } \partial\Omega\}.$$

Then there exist an exponent $2 < r < p$ and a constant $C > 0$ such that for every $t_1, t_2 \in [0,T]$ and every $v_1, v_2 \in H^1(\Omega;[0,1])$ it holds

$$\|u(t_2, v_2) - u(t_1, v_1)\|_{W^{1,r}} \le C(\|g(t_2) - g(t_1)\|_{W^{1,r}} + \|g\|_{L^\infty(0,T;W^{1,p})}\, \|v_2 - v_1\|_{L^q}),$$

where $1/q = 1/r - 1/p$.

Next proposition shows that the energy inequality (2.11) is actually equivalent to the energy identity (d) of Definition 2.5.

Proposition 2.9. *Let T, g, u_0 and v_0 be as in Definition 2.5. Assume that the pair $(u,v)\colon [0,T] \to \mathcal{U} \times \mathcal{V}$ satisfies properties (a)–(c) of Definition 2.5 and that for every $t \in [0,T]$*

$$\mathcal{F}(u(t),v(t)) \leq \mathcal{F}(u_0,v_0) - \tfrac{1}{2}\int_0^t |\partial_v^- \mathcal{F}|^2(u(s),v(s)) + \|\dot{v}(s)\|_{L^2}^2 \,\mathrm{d}s + \int_0^t \mathcal{P}(u(s),v(s),\dot{g}(s))\,\mathrm{d}s. \tag{2.11}$$

Then, (u,v) also fulfills the energy balance (d) of Definition 2.5.

Proof. In order to prove the proposition we need to show the opposite inequality of (2.11). We exploit here the Riemann sum argument proposed in Lemma 4.12 from [21]. Since by (2.11) the slope $|\partial_v^- \mathcal{F}|(u,v)$ is in $L^2(0,T)$, for every $t \in [0,T]$ there exists a sequence of subdivisions, denoted (by abuse of notation) by t_i^j, with

$$0 = t_0^j < t_1^j < \ldots < t_{I_j}^j = t, \qquad \lim_{j\to\infty} \max\{(t_{i+1}^j - t_i^j) : 0 \leq i \leq I_j - 1\} = 0,$$

and such that the piecewise constant functions

$$F_j(s) := \sum_{i=0}^{I_j} \mathbf{1}_{(t_i^j, t_{i+1}^j)}(s)|\partial_v^- \mathcal{F}|(u(t_i^j), v(t_i^j)) \tag{2.12}$$

converge to $|\partial_v^- \mathcal{F}|(u,v)$ strongly in $L^2(0,t)$.

By the quadratic structure of the functional $\mathcal{F}$, we can write

$$\begin{aligned}
\mathcal{F}(u(t_i^j), v(t_{i+1}^j)) &= \mathcal{F}(u(t_i^j), v(t_i^j) + (v(t_{i+1}^j) - v(t_i^j))) \\
&= \mathcal{F}(u(t_i^j), v(t_i^j)) + \partial_v \mathcal{F}(u(t_i^j), v(t_i^j))[v(t_{i+1}^j) - v(t_i^j)] \\
&\quad + \tfrac{1}{2}\int_\Omega (v(t_{i+1}^j) - v(t_i^j))^2 \boldsymbol{\sigma}(u(t_i^j)) : \boldsymbol{\epsilon}(u(t_i^j))\,\mathrm{d}x + \tfrac{1}{2}\int_\Omega |\nabla v(t_{i+1}^j) - \nabla v(t_i^j)|^2\,\mathrm{d}x \\
&\quad + \tfrac{1}{2}\int_\Omega (v(t_{i+1}^j) - v(t_i^j))^2\,\mathrm{d}x \\
&\geq \mathcal{F}(u(t_i^j), v(t_i^j)) + \partial_v \mathcal{F}(u(t_i^j), v(t_i^j))[v(t_{i+1}^j) - v(t_i^j)] + \tfrac{1}{2}\|v(t_{i+1}^j) - v(t_i^j)\|_{H^1}^2.
\end{aligned} \tag{2.13}$$

Reordering the terms in (2.13) and recalling Lemma 2.3, we get that

$$\begin{aligned}
\mathcal{F}(u(t_i^j), v(t_i^j)) &\leq \mathcal{F}(u(t_i^j), v(t_{i+1}^j)) - \partial_v \mathcal{F}(u(t_i^j), v(t_i^j))[v(t_{i+1}^j) - v(t_i^j)] - \tfrac{1}{2}\|v(t_{i+1}^j) - v(t_i^j)\|_{H^1}^2 \\
&\leq \mathcal{F}(u(t_i^j), v(t_{i+1}^j)) + \int_{t_i^j}^{t_{i+1}^j} |\partial_v^- \mathcal{F}|(u(t_i^j), v(t_i^j)) \frac{\|v(t_{i+1}^j) - v(t_i^j)\|_{L^2}}{(t_{i+1}^j - t_i^j)}\,\mathrm{d}s \\
&\quad - \tfrac{1}{2}\|v(t_{i+1}^j) - v(t_i^j)\|_{H^1}^2.
\end{aligned} \tag{2.14}$$

For every $j \in \mathbb{N}$ and every $i \in \{0, \ldots, I_j - 1\}$, we have that

$$\begin{aligned}
\mathcal{F}(u(t_i^j), v(t_{i+1}^j)) &= \mathcal{F}(u(t_i^j) + g(t_{i+1}^j) - g(t_i^j), v(t_{i+1}^j)) \\
&\quad - \int_{t_i^j}^{t_{i+1}^j} \partial_s \mathcal{E}(u(t_i^j) + g(s) - g(t_i^j), v(t_{i+1}^j))\,\mathrm{d}s,
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
\partial_s \mathcal{E}(u(t_i^j) + g(s) - g(t_i^j), v(t_{i+1}^j)) &= \partial_u \mathcal{E}(u(t_i^j) + g(s) - g(t_i^j), v(t_{i+1}^j))[\dot{g}(s)] \\
&= \int_\Omega (v^2(t_{i+1}^j) + \eta)\,\boldsymbol{\sigma}(u(t_i^j) + g(s) - g(t_i^j)) : \boldsymbol{\epsilon}(\dot{g}(s))\,\mathrm{d}x \\
&= \mathcal{P}(u(t_i^j) + g(s) - g(t_i^j), v(t_{i+1}^j), \dot{g}(s)).
\end{aligned}$$

Next, we know that $u(t) \in \arg\min\{\mathcal{E}(u, v(t)) : u \in \mathcal{U},\ u = g(t) \text{ on } \partial\Omega\}$ for every $t \in [0, T]$. Hence,

$$\int_\Omega (v^2(t) + \eta)\, \boldsymbol{\sigma}(u(t)) : \boldsymbol{\epsilon}(\phi)\, \mathrm{d}x = 0 \quad \text{for every } \phi \in H^1_0(\Omega, \mathbb{R}^2).$$

As a consequence, if $w = g(t)$ on $\partial\Omega$ we get

$$\begin{aligned}\mathcal{F}(w, v(t)) - \mathcal{F}(u(t), v(t)) &= \int_\Omega (v^2(t) + \eta)\, \boldsymbol{\sigma}(w + u(t)) : \boldsymbol{\epsilon}(w - u(t))\, \mathrm{d}x \\ &= \int_\Omega (v^2(t) + \eta)\, \boldsymbol{\sigma}(w - u(t)) : \boldsymbol{\epsilon}(w - u(t))\, \mathrm{d}x \\ &\le C\|w - u(t)\|^2_{H^1}.\end{aligned}$$

Choosing $t = t^j_{i+1}$ and $w = u(t^j_i) + g(t^j_{i+1}) - g(t^j_i)$ we can write

$$\begin{aligned}\mathcal{F}(u(t^j_i) + g(t^j_{i+1}) - g(t^j_i), v(t^j_{i+1})) &\le \mathcal{F}(u(t^j_{i+1}), v(t^j_{i+1})) \\ &\quad + C\|g(t^j_{i+1}) - g(t^j_i)\|^2_{H^1} + C\|u(t^j_{i+1}) - u(t^j_i)\|^2_{H^1}.\end{aligned} \tag{2.16}$$

Joining (2.14)–(2.16) we obtain

$$\begin{aligned}\mathcal{F}(u(t^j_i), v(t^j_i)) &\le \mathcal{F}(u(t^j_{i+1}), v(t^j_{i+1})) + \int_{t^j_i}^{t^j_{i+1}} |\partial^-_v \mathcal{F}|(u(t^j_i), v(t^j_i)) \frac{\|v(t^j_{i+1}) - v(t^j_i)\|_{L^2}}{(t^j_{i+1} - t^j_i)}\, \mathrm{d}s \\ &\quad - \int_{t^j_i}^{t^j_{i+1}} \mathcal{P}(u(t^j_i) + g(s) - g(t^j_i), v(t^j_{i+1}), \dot g(s))\, \mathrm{d}s \\ &\quad - \tfrac{1}{2}\|v(t^j_{i+1}) - v(t^j_i)\|^2_{H^1} + C\|g(t^j_{i+1}) - g(t^j_i)\|^2_{H^1} + C\|u(t^j_{i+1}) - u(t^j_i)\|^2_{H^1}.\end{aligned} \tag{2.17}$$

We now estimate the term $\|u(t^j_{i+1}) - u(t^j_i)\|^2_{H^1}$ in (2.17). By Lemma 2.8, we have that there exist $C > 0$ and $q \gg 2$ independent of i and j such that

$$\|u(t^j_{i+1}) - u(t^j_i)\|^2_{H^1} \le C\|g(t^j_{i+1}) - g(t^j_i)\|^2_{H^1} + C\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^q}.$$

By interpolation inequality, we can find $0 < \alpha < 1$ and $\bar q \gg 2$ such that

$$\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^q} \le \|v(t^j_{i+1}) - v(t^j_i)\|^{2\alpha}_{L^{\bar q}} \|v(t^j_{i+1}) - v(t^j_i)\|^{2(1-\alpha)}_{L^2}.$$

Applying a weighted Young inequality, we get that for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^q} \le \delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^{\bar q}} + C_\delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^2}.$$

In view of Sobolev embedding, we can continue with

$$\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^q} \le C\delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{H^1} + C_\delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^2}.$$

Combining all the previous inequalities we get

$$\|u(t^j_{i+1}) - u(t^j_i)\|^2_{H^1} \le C\|g(t^j_{i+1}) - g(t^j_i)\|^2_{H^1} + C\delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{H^1} + C_\delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^2}. \tag{2.18}$$

Substituting (2.18) in (2.17) and choosing $\delta > 0$ small enough so that $C\delta < \frac{1}{2}$, we obtain that

$$\begin{aligned}\mathcal{F}(u(t^j_i), v(t^j_i)) &\le \mathcal{F}(u(t^j_{i+1}), v(t^j_{i+1})) + \int_{t^j_i}^{t^j_{i+1}} |\partial^-_v \mathcal{F}|(u(t^j_i), v(t^j_i)) \frac{\|v(t^j_{i+1}) - v(t^j_i)\|_{L^2}}{(t^j_{i+1} - t^j_i)}\, \mathrm{d}s \\ &\quad - \int_{t^j_i}^{t^j_{i+1}} \mathcal{P}(u(t^j_i) + g(s) - g(t^j_i), v(t^j_{i+1}), \dot g(s))\, \mathrm{d}s \\ &\quad - \widetilde{C}\|v(t^j_{i+1}) - v(t^j_i)\|^2_{H^1} + C\|g(t^j_{i+1}) - g(t^j_i)\|^2_{H^1} + C_\delta\|v(t^j_{i+1}) - v(t^j_i)\|^2_{L^2},\end{aligned} \tag{2.19}$$

for some positive constants $C, \widetilde{C}$ independent of i and j.

Iterating inequality (2.19) for $i = 0, \ldots, I_j - 1$ and neglecting the terms with the H^1-norm of the phase-field variable (which are negative), we finally arrive at

$$\mathcal{F}(u_0, v_0) \leq \mathcal{F}(u(t), v(t)) + J_{1,j} + J_{2,j} + J_{3,j} + J_{4,j} \tag{2.20}$$

where

$$J_{1,j} := \sum_{i=0}^{I_j-1} \int_{t_i^j}^{t_{i+1}^j} |\partial_v^- \mathcal{F}|(u(t_i^j), v(t_i^j)) \frac{\|v(t_{i+1}^j) - v(t_i^j)\|_{L^2}}{(t_{i+1}^j - t_i^j)} \, \mathrm{d}s,$$

$$J_{2,j} := \sum_{i=0}^{I_j-1} \int_{t_i^j}^{t_{i+1}^j} \mathcal{P}(u(t_i^j) + g(s) - g(t_i^j), v(t_{i+1}^j), \dot{g}(s)) \, \mathrm{d}s,$$

$$J_{3,j} := C \sum_{i=0}^{I_j-1} \|g(t_{i+1}^j) - g(t_i^j)\|_{H^1}^2, \qquad J_{4,j} := C \sum_{i=0}^{I_j-1} \|v(t_{i+1}^j) - v(t_i^j)\|_{L^2}^2.$$

We now prove the following:

$$\lim_{j \to \infty} J_{1,j} = \int_0^t |\partial_v^- \mathcal{F}|(u(s), v(s)) \|\dot{v}(s)\|_{L^2} \, \mathrm{d}s, \tag{2.21}$$

$$\lim_{j \to \infty} J_{2,j} = \int_0^t \mathcal{P}(u(s), v(s), \dot{g}(s)) \, \mathrm{d}s, \tag{2.22}$$

$$\lim_{j \to \infty} J_{3,j} = \lim_{j \to \infty} J_{4,j} = 0. \tag{2.23}$$

As for (2.21), we first rewrite $J_{1,j}$ as

$$J_{1,j} = \int_0^t F_j(s) V_j(s) \, \mathrm{d}s,$$

where F_j has been introduced in (2.12) and V_j is defined by

$$V_j(s) := \sum_{i=0}^{I_j-1} \mathbf{1}_{(t_i^j, t_{i+1}^j)}(s) \frac{\|v(t_{i+1}^j) - v(t_i^j)\|_{L^2}}{(t_{i+1}^j - t_i^j)} \qquad \text{for } s \in [0,t].$$

We already know that, by the particular choice of the sequence of subdivisions of the interval $[0,t]$, the sequence F_j converges to $|\partial_v^- \mathcal{F}|(u,v)$ in $L^2(0,t)$. Hence, in order to get (2.22) it is enough to show that $V_j \rightharpoonup \|\dot{v}\|_{L^2}$ weakly in $L^2(0,t)$. Since $v \in H^1([0,T]; L^2(\Omega))$, we have that $V_j(s) \to \|\dot{v}(s)\|_{L^2}$ for a.e. $s \in [0,t]$. Moreover, V_j is bounded in $L^2([0,t])$. Indeed,

$$\|V_j\|_{L^2(0,t)}^2 = \sum_{i=1}^{I_j-1} \int_{t_i^j}^{t_{i+1}^j} \left\| \frac{v(t_{i+1}^j) - v(t_i^j)}{t_{i+1}^j - t_i^j} \right\|_{L^2}^2 \mathrm{d}s \leq \int_0^t \|\dot{v}(s)\|_{L^2}^2 \, \mathrm{d}s.$$

Therefore, we conclude that $V_j \rightharpoonup \|\dot{v}\|_{L^2}$ weakly in $L^2(0,t)$ and (2.21) holds true.

For the limit in (2.22) let us fix $s \in (0,t)$. For every $j \in \mathbb{N}$ let $s \in [t_{i_j}^j, t_{i_j+1}^j)$, so that $t_{i_j}^j \to s$ and $t_{i_j+1}^j \to s$. Since $u \in C([0,T]; \mathcal{U})$ and $v \in H^1([0,T]; L^2(\Omega))$, it is clear that $u(t_{i_j}^j) + g(s) - g(t_{i_j}^j) \to u(s)$ in $\mathcal{U}$ and $v(t_{i_j+1}^j) \to v(s)$ in $L^2(\Omega)$. By Lemma 2.4

$$\mathcal{P}(u(t_i^j) + g(s) - g(t_i^j), v(t_{i+1}^j), \dot{g}(s)) \to \mathcal{P}(u(s), v(s), \dot{g}(s)).$$

We get (2.22) by dominated convergence.

The limits (2.23) involving $J_{3,j}$ and $J_{4,j}$ follow, respectively, from the fact that the boundary datum $g \in AC([0,T]; \mathcal{U})$ and the phase-field $v \in H^1([0,T]; L^2(\Omega))$. This concludes the proof. □

3. Lemmata

We collect here some technical results that will be useful in the next sections.

Lemma 3.1. *Let $u_m, u \in \mathcal{U}$ and $v_m, v \in \mathcal{V}$. If $u_m \rightharpoonup u$ in $\mathcal{U}$ and $v_m \rightharpoonup v$ in $H^1(\Omega)$, then*

$$\mathcal{F}(u,v) \le \liminf_{m\to\infty} \mathcal{F}(u_m, v_m).$$

Proof. The lower semi-continuity of $\mathcal{D}$ is obvious, by convexity. The lower semi-continuity of $\mathcal{E}$ follows for instance from Theorem 7.5 of [22]. □

Now we prove a semicontinuity property of the slope $|\partial_v^- \mathcal{F}|$.

Lemma 3.2. *Let $u_m, u \in \mathcal{U}$ and $v_m, v \in \mathcal{V}$. If $u_m \to u$ in $\mathcal{U}$ and $v_m \rightharpoonup v$ in $H^1(\Omega)$, then*

$$|\partial_v^- \mathcal{F}|(u,v) \le \liminf_{m\to\infty} |\partial_v^- \mathcal{F}|(u_m, v_m). \tag{3.1}$$

Proof. Let us fix $\varphi \in \mathcal{V}$ with $\varphi \le 0$ and $\|\varphi\|_{L^2} \le 1$. Let us also assume, without loss of generality, that the $\liminf$ in (3.1) is a limit and that $\nabla u_m \to \nabla u$ and $v_m \to v$ pointwise a.e. in Ω. Then, by Lemma 2.3 we have that

$$\begin{aligned}\lim_{m\to\infty} |\partial_v^- \mathcal{F}|(u_m, v_m) &\ge \liminf_{m\to\infty} -\partial_v \mathcal{F}(u_m, v_m)[\varphi] \\ &= \liminf_{m\to\infty} \int_\Omega -v_m \varphi\, \boldsymbol{\sigma}(u_m) : \boldsymbol{\epsilon}(u_m)\, \mathrm{d}x - \int_\Omega \nabla v_m \cdot \nabla\varphi - (1-v_m)\varphi \, \mathrm{d}x.\end{aligned}$$

By (generalized) dominated convergence, for the first integral, and weak convergence, for the second integral, we get

$$\lim_{m\to\infty} |\partial_v^- \mathcal{F}|(u_m, v_m) \ge \int_\Omega -v\varphi\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u)\, \mathrm{d}x - \int_\Omega \nabla v \cdot \nabla \varphi - (1-v)\varphi\, \mathrm{d}x = -\partial_v \mathcal{F}(u,v)[\varphi].$$

Passing to the supremum with respect to φ in the previous inequality, we get (3.1). □

Finally, we will prove the following minimality properties.

Lemma 3.3. *Let $g_m, g_\infty, u_m, u_\infty \in \mathcal{U}$ and let $v_m, v_\infty \in H^1(\Omega; [0,1])$. Assume that $g_m \to g_\infty$ (strongly) and $u_m \rightharpoonup u_\infty$ (weakly) in $\mathcal{U}$, $v_m \rightharpoonup v$ (weakly) in $H^1(\Omega)$, and that*

$$u_m \in \arg\min \{\mathcal{E}(u, v_m) : \ u \in \mathcal{U} \text{ with } u = g_m \text{ on } \partial\Omega\}. \tag{3.2}$$

Then

$$u_\infty \in \arg\min \{\mathcal{E}(u, v_\infty) : \ u \in \mathcal{U} \text{ with } u = g_\infty \text{ on } \partial\Omega\} \tag{3.3}$$

and $u_m \to u_\infty$ strongly in $\mathcal{U}$.

Proof. Let $u \in \mathcal{U}$ be such that $u = g_\infty$ on $\partial\Omega$, then $u - g_\infty + g_m = g_m$ on $\partial\Omega$. By minimality, see (3.2), and by the lower-semicontinuity of $\mathcal{E}$, see Lemma 3.1, we get

$$\mathcal{E}(u_\infty, v_\infty) \le \liminf_{m\to\infty} \mathcal{E}(u_m, v_m) \le \liminf_{m\to\infty} \mathcal{E}(u + g_m - g_\infty, v_m). \tag{3.4}$$

Let us check that $\lim_{m\to\infty} \mathcal{E}(u + g_m - g_\infty, v_m) = \mathcal{E}(u, v_\infty)$ from which (3.3) follows. Write

$$\mathcal{E}(u + g_m - g_\infty, v_m) = \int_\Omega (v_m^2 + \eta)\, \boldsymbol{\sigma}(u + g_m - g_\infty) : \boldsymbol{\epsilon}(u + g_m - g_\infty)\, \mathrm{d}x.$$

Extract a subsequence (not relabelled) such that $v_m \to v_\infty$ and $\nabla g_m \to \nabla g_\infty$ a.e. in Ω. Then

$$(v_m^2 + \eta)\, \boldsymbol{\sigma}(u + g_m - g_\infty) : \boldsymbol{\epsilon}(u + g_m - g_\infty) \to (v_\infty + \eta)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u) \quad \text{a.e. in } \Omega.$$

Since $0 \le v_m \le 1$ and $g_m \to g_\infty$ (strongly) in $\mathcal{U}$ we can apply dominated convergence. We conclude because the limit is independent of the subsequence.

Rewriting the previous argument for $u = u_\infty$, we deduce that

$$\lim_{m\to\infty} \mathcal{E}(u_m, v_m) = \mathcal{E}(u_\infty, v_\infty).$$

For $v_m \in H^1(\Omega; [0,1])$ let us consider the space $L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})$ endowed with the weighted norm $\|\zeta\|_m^2 = \int_\Omega (v_m^2 + \eta)\zeta : \mathbf{C}\zeta \,\mathrm{d}x$ where $\mathbf{C}$ is the stiffness matrix. Since $\mathbf{C}$ is positive definite in $\mathbb{R}^{2\times 2}_{\text{sym}}$ and $\eta > 0$, it follows that there exists $c, C > 0$ (independent of k) such that $c\|\zeta\| \le \|\zeta\|_m \le C\|\zeta\|$, where $\|\cdot\|$ denotes the standard norm in $L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})$. We will prove that $\|\boldsymbol{\epsilon}(u_m - u_\infty)\|_m \to 0$, from which $\boldsymbol{\epsilon}(u_m) \to \boldsymbol{\epsilon}(u_\infty)$ strongly in $L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})$. Write

$$\begin{aligned} \|\boldsymbol{\epsilon}(u_m - u_\infty)\|_m^2 &= \int_\Omega (v_m^2 + \eta)\, \boldsymbol{\sigma}(u_m - u_\infty) : \boldsymbol{\epsilon}(u_m - u_\infty) \,\mathrm{d}x \\ &= \int_\Omega (v_m^2 + \eta)\, \boldsymbol{\sigma}(u_m) : \boldsymbol{\epsilon}(u_m) + (v_m^2 + \eta)\, \boldsymbol{\sigma}(u_\infty) : \boldsymbol{\epsilon}(u_\infty) - 2(v_m^2 + \eta)\, \boldsymbol{\sigma}(u_m) : \boldsymbol{\epsilon}(u_\infty) \,\mathrm{d}x. \end{aligned}$$

By convergence of the energies

$$\int_\Omega (v_m^2 + \eta)\, \boldsymbol{\sigma}(u_m) : \boldsymbol{\epsilon}(u_m) \,\mathrm{d}x \to \mathcal{E}(u_\infty, v_\infty).$$

By dominated convergence

$$\int_\Omega (v_m^2 + \eta)\, \boldsymbol{\sigma}(u_\infty) : \boldsymbol{\epsilon}(u_\infty) \,\mathrm{d}x \to \int_\Omega (v_\infty^2 + \eta)\, \boldsymbol{\sigma}(u_\infty) : \boldsymbol{\epsilon}(u_\infty) \,\mathrm{d}x = \mathcal{E}(u_\infty, v_\infty).$$

Finally, $\boldsymbol{\sigma}(u_m) \rightharpoonup \boldsymbol{\sigma}(u_\infty)$ in $L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})$ while $(v_m^2 + \eta)\boldsymbol{\epsilon}(u_\infty) \to (v_\infty^2 + \eta)\boldsymbol{\epsilon}(u_\infty)$ in $L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})$, again by dominated convergence. Hence

$$\int_\Omega (v_m^2 + \eta)\, \boldsymbol{\sigma}(u_m) : \boldsymbol{\epsilon}(u_\infty) \,\mathrm{d}x \to \int_\Omega (v_\infty^2 + \eta)\, \boldsymbol{\sigma}(u_\infty) : \boldsymbol{\epsilon}(u_\infty) \,\mathrm{d}x = \mathcal{E}(u_\infty, v_\infty).$$

Therefore $\|\boldsymbol{\epsilon}(u_m - u_\infty)\|_m^2 \to 0$. Since $g_m \to g_\infty$ in $\mathcal{U}$, we deduce that $\boldsymbol{\epsilon}(u_m - g_m) \to \boldsymbol{\epsilon}(u_\infty - g_\infty)$ in $L^2(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}})$. Since both $(u_m - g_m)$ and $(u_\infty - g_\infty)$ belong to $H^1_0(\Omega; \mathbb{R}^2)$, Korn's inequality implies that $\nabla u_m \to \nabla u_\infty$ in $L^2(\Omega; \mathbb{R}^2)$. As a consequence $u_m \to u_\infty$ strongly in $\mathcal{U}$. □

4. A one-step scheme

In this section we present a first time-discrete scheme, proposed in [35], converging to unilateral gradient flow, in the sense of Definition 2.5.

Given the time horizon $T > 0$, for every $k \in \mathbb{N}\backslash\{0\}$ we define the time step $\tau_k := \frac{T}{k}$. For every $i \in \{0, \dots, k\}$ we set the discrete time nodes $t_i^k := i\tau_k$ and we define recursively $u_i^k \in \mathcal{U}$ and $\tilde{v}_i^k, v_i^k \in H^1(\Omega)$ as follows: for $i = 0$ let $u_0^k := u_0$ and $\tilde{v}_0^k = v_0^k := v_0$, while, for $i \ge 1$, we define

$$u_i^k := \arg\min \{\mathcal{E}(u, v_{i-1}^k) : \, u \in \mathcal{U},\, u = g(t_i^k) \text{ on } \partial\Omega\}, \tag{4.1}$$

$$\tilde{v}_i^k := \arg\min \{\mathcal{F}(u_i^k, v) + \tfrac{1}{2\tau_k}\|v - v_{i-1}^k\|_{L^2}^2 : \, v \in H^1(\Omega)\}. \tag{4.2}$$

$$v_i^k := \min \{\tilde{v}_i^k, v_{i-1}^k\}. \tag{4.3}$$

We notice that the solutions of the minimum problems (4.1) and (4.2) exist and are unique by the strict convexity of the involved functionals. In particular, by the usual truncation argument, we have that $0 \le \tilde{v}_i^k \le 1$ in Ω whenever $0 \le v_{i-1}^k \le 1$ in Ω. By induction, this is guaranteed by the restriction $0 \le v_0 \le 1$ on the initial condition.

Remark 4.1. We stress that the minimum problem (4.2) for the phase-field variable is unconstrained, that is, we are not imposing any a priori irreversibility constraint of the form $v \le v_{i-1}^k$. The latter condition is instead imposed a posteriori by (4.3). Therefore, at the discrete level, the "non-increasing" phase-field variable v_i^k does not satisfy any equilibrium condition, while the "unconstrained" phase-field $\tilde{v}_i^k$ is not monotone, with respect to $i \in \mathbb{N}$. As discussed in [35], from a numerical viewpoint the approach described by (4.1)–(4.3) is computationally very convenient since, at every discrete time t_i^k, we have to solve a couple of unconstrained minimum problems for quadratic functionals.

We now show some consequences of (4.1)–(4.3).

Proposition 4.2. *For every $k \in \mathbb{N}\backslash\{0\}$ and every $i \in \{1, \dots, k\}$ let u_i^k, $\tilde{v}_i^k$ and v_i^k be defined as in (4.1)–(4.3). Then*

$$\frac{\|v_i^k - v_{i-1}^k\|_{L^2}}{\tau_k} = |\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k), \tag{4.4}$$

$$\partial_v \mathcal{F}(u_i^k, v_i^k)[v_i^k - v_{i-1}^k] = \partial_v \mathcal{F}(u_i^k, \tilde{v}_i^k)[v_i^k - v_{i-1}^k] = -|\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k)\|v_i^k - v_{i-1}^k\|_{L^2}. \tag{4.5}$$

Proof. By minimality of $\tilde{v}_i^k$, we have that

$$\partial_v \mathcal{F}(u_i^k, \tilde{v}_i^k)[\varphi] + \tfrac{1}{\tau_k} \int_\Omega (\tilde{v}_i^k - v_{i-1}^k)\varphi \,\mathrm{d}x = 0 \qquad \text{for every } \varphi \in \mathcal{V}. \tag{4.6}$$

Then, by Lemma 2.3, by the density of $\mathcal{V}$ in $L^2(\Omega)$, and since $v_i^k - v_{i-1}^k = -(\tilde{v}_i^k - v_{i-1}^k)_-$, we have that

$$\begin{aligned}
|\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k) &= \sup\{-\partial_v \mathcal{F}(u_i^k, \tilde{v}_i^k)[\varphi] : \varphi \in \mathcal{V}, \varphi \le 0, \|\varphi\|_{L^2} \le 1\} \\
&= \max\left\{ \tfrac{1}{\tau_k} \int_\Omega (\tilde{v}_i^k - v_{i-1}^k)\varphi \,\mathrm{d}x : \varphi \in L^2(\Omega), \varphi \le 0, \|\varphi\|_{L^2} \le 1 \right\} \\
&= \tfrac{1}{\tau_k} \int_\Omega (\tilde{v}_i^k - v_{i-1}^k)_- \frac{(\tilde{v}_i^k - v_{i-1}^k)_-}{\|(\tilde{v}_i^k - v_{i-1}^k)_-\|_{L^2}} \,\mathrm{d}x \\
&= \tfrac{1}{\tau_k} \|(\tilde{v}_i^k - v_{i-1}^k)_-\|_{L^2} = \tfrac{1}{\tau_k} \|v_i^k - v_{i-1}^k\|_{L^2},
\end{aligned}$$

which proves (4.4). In particular, since $(v_i^k - v_{i-1}^k) \in \mathcal{V}$, we also deduce the second part of (4.5), *i.e.*,

$$-\partial_v \mathcal{F}(u_i^k, \tilde{v}_i^k)[v_i^k - v_{i-1}^k] = |\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k)\|v_i^k - v_{i-1}^k\|_{L^2}. \tag{4.7}$$

Let us now define $\Omega_- := \{\tilde{v}_i^k \le v_{i-1}^k\}$ and $\Omega_+ := \{\tilde{v}_i^k > v_{i-1}^k\}$. Then, we claim that

$$\partial_v \mathcal{F}(u_i^k, v_i^k)[\varphi] + \tfrac{1}{\tau_k} \int_\Omega (v_i^k - v_{i-1}^k)\varphi \,\mathrm{d}x = 0 \tag{4.8}$$

for every $\varphi \in \mathcal{V}$ with $\varphi = 0$ on Ω_+. Note that the partial derivative of $\mathcal{F}$ is computed in v_i^k and not in $\tilde{v}_i^k$, as in (4.6). Being $\varphi = 0$ on Ω_+, we have

$$\int_{\Omega_+} v_i^k \varphi \, \boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k) \,\mathrm{d}x + \int_{\Omega_+} \nabla v_i^k \cdot \nabla\varphi \,\mathrm{d}x - \int_{\Omega_+} (1 - v_i^k)\varphi \,\mathrm{d}x + \tfrac{1}{\tau_k} \int_{\Omega_+} (v_i^k - v_{i-1}^k)\varphi \,\mathrm{d}x = 0.$$

On the other hand, by (4.3), on Ω_- we have $v_i^k = \tilde v_i^k$. Thus, in view of (4.6),

$$
\begin{aligned}
0 &= \partial_v \mathcal{F}(u_i^k, \tilde v_i^k)[\varphi] + \tfrac{1}{\tau_k}\int_\Omega (\tilde v_i^k - v_{i-1}^k)\varphi \,\mathrm{d}x \\
&= \int_{\Omega_-} \tilde v_i^k \varphi\, \boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k)\,\mathrm{d}x + \int_{\Omega_-} \nabla \tilde v_i^k \cdot \nabla\varphi \,\mathrm{d}x - \int_{\Omega_-} (1-\tilde v_i^k)\varphi\,\mathrm{d}x + \tfrac{1}{\tau_k}\int_{\Omega_-} (\tilde v_i^k - v_{i-1}^k)\varphi\,\mathrm{d}x \\
&= \int_{\Omega_-} v_i^k \varphi\, \boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k)\,\mathrm{d}x + \int_{\Omega_-} \nabla v_i^k \cdot \nabla\varphi \,\mathrm{d}x - \int_{\Omega_-} (1-v_i^k)\varphi\,\mathrm{d}x + \tfrac{1}{\tau_k}\int_{\Omega_-} (v_i^k - v_{i-1}^k)\varphi\,\mathrm{d}x.
\end{aligned}
$$

Hence (4.8) is proved.
Using now (4.6) and (4.8) with $\varphi = v_i^k - v_{i-1}^k$ we get

$$
\partial_v \mathcal{F}(u_i^k, \tilde v_i^k)[v_i^k - v_{i-1}^k] + \tfrac{1}{\tau_k}\int_\Omega (\tilde v_i^k - v_{i-1}^k)(v_i^k - v_{i-1}^k)\,\mathrm{d}x = 0, \tag{4.9}
$$

$$
\partial_v \mathcal{F}(u_i^k, v_i^k)[v_i^k - v_{i-1}^k] + \tfrac{1}{\tau_k}\int_\Omega (v_i^k - v_{i-1}^k)^2\,\mathrm{d}x = 0. \tag{4.10}
$$

It is easy to see that

$$
\int_\Omega (\tilde v_i^k - v_{i-1}^k)(v_i^k - v_{i-1}^k)\,\mathrm{d}x = \int_\Omega (v_i^k - v_{i-1}^k)^2\,\mathrm{d}x. \tag{4.11}
$$

Combining (4.9)–(4.11), we obtain

$$
\partial_v \mathcal{F}(u_i^k, \tilde v_i^k)[v_i^k - v_{i-1}^k] = \partial_v \mathcal{F}(u_i^k, v_i^k)[v_i^k - v_{i-1}^k],
$$

which, together with (4.7), concludes the proof of the proposition. □

Remark 4.3. In view of the equilibrium condition (4.6), we could define $\partial_v \mathcal{F}(u_i^k, \tilde v_i^k)$ as an element of $L^2(\Omega)$ by the relation

$$
\partial_v \mathcal{F}(u_i^k, \tilde v_i^k)[\varphi] = -\tfrac{1}{\tau_k}\int_\Omega (\tilde v_i^k - v_{i-1}^k)\varphi\,\mathrm{d}x \quad \text{for every } \varphi \in L^2(\Omega),
$$

that is, $\partial_v \mathcal{F}(u_i^k, \tilde v_i^k) = -\frac{1}{\tau_k}(\tilde v_i^k - v_{i-1}^k)$ in $L^2(\Omega)$.

We now define the following interpolation functions:

$$
v_k(t) := v_i^k + \frac{v_{i+1}^k - v_i^k}{\tau_k}(t - t_i^k) \qquad \text{for every } t \in [t_i^k, t_{i+1}^k), \tag{4.12}
$$

$$
\bar u_k(t) := u_i^k, \quad \bar v_k(t) := v_i^k, \quad \tilde v_k(t) := \tilde v_i^k, \quad t_k(t) := t_i^k \qquad \text{for every } t \in (t_{i-1}^k, t_i^k], \tag{4.13}
$$

$$
\underline{u}_k(t) := u_i^k, \quad \underline{v}_k(t) := v_i^k \qquad \text{for every } t \in [t_i^k, t_{i+1}^k). \tag{4.14}
$$

Next, we study compactness and energy balance for the sequences introduced just above.

Proposition 4.4. *The following facts hold:*

(a) *The sequence v_k is bounded in $L^\infty([0,T]; H^1(\Omega))$ and in $H^1([0,T]; L^2(\Omega))$;*
(b) *The sequences $\bar v_k$, $\tilde v_k$, and $\underline{v}_k$ are bounded in $L^\infty([0,T]; H^1(\Omega))$;*
(c) *The sequences $\bar u_k$ and $\underline{u}_k$ are bounded in $L^\infty([0,T]; \mathcal{U})$;*
(d) *For every $t \in [0,T]$ we have*

$$
\bar u_k(t) \in \arg\min\{\mathcal{E}(u, \underline{v}_k(t)) : u \in \mathcal{U},\ u = g(t_k(t)) \text{ on } \partial\Omega\};
$$

(e) *There exists a constant $C > 0$ (depending only on the stiffness tensor $\mathbf{C}$) such that for every $t \in [0,T]$*

$$\begin{aligned}\mathcal{F}(\bar{u}_k(t), \underline{v}_k(t)) \le \mathcal{F}(u_0, v_0) - \tfrac{1}{2}\int_0^{t_k(t)} \|\dot{v}_k(s)\|_2^2 + |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\,\mathrm{d}s \\ + \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s + C\sum_{i=1}^{I_t} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2,\end{aligned} \tag{4.15}$$

where $I_t = \min\{I \in \mathbb{N} \mid I \ge \frac{t}{\tau_k}\}$. In particular the energy $\mathcal{F}(\bar{u}_k(t), \underline{v}_k(t))$ is uniformly bounded, w.r.t. t and k.

Proof. We will start proving the energy estimate (e). Let us fix $k \in \mathbb{N}\backslash\{0\}$, $t \in (0,T]$, and $i \in \{1,\dots,k\}$ such that $t \in (t_{i-1}^k, t_i^k]$. By convexity of $v \mapsto \mathcal{F}(u_i^k, v)$, we have that

$$\begin{aligned}\mathcal{F}(u_i^k, v_{i-1}^k) &\ge \mathcal{F}(u_i^k, v_i^k) + \partial_v \mathcal{F}(u_i^k, v_i^k)[v_{i-1}^k - v_i^k] \\ &= \mathcal{F}(u_i^k, v_i^k) - \tau_k \partial_v \mathcal{F}(u_i^k, v_i^k)[\dot{v}_k(t)].\end{aligned} \tag{4.16}$$

Recalling Proposition 4.2, we can continue in (4.16) with

$$\begin{aligned}\mathcal{F}(u_i^k, v_{i-1}^k) &\ge \mathcal{F}(u_i^k, v_i^k) - \tau_k \partial_v \mathcal{F}(u_i^k, \tilde{v}_i^k)[\dot{v}_k(t)] \\ &= \mathcal{F}(u_i^k, v_i^k) + \tau_k |\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k)\|\dot{v}_k(t)\|_{L^2} \\ &= \mathcal{F}(u_i^k, v_i^k) + \tfrac{\tau_k}{2}\big(\|\dot{v}_k(t)\|_{L^2}^2 + |\partial_v^- \mathcal{F}|^2(u_i^k, \tilde{v}_i^k)\big) \\ &= \mathcal{F}(u_i^k, v_i^k) + \tfrac{1}{2}\int_{t_{i-1}^k}^{t_i^k} \|\dot{v}_k(s)\|_{L^2}^2 + |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\,\mathrm{d}s.\end{aligned} \tag{4.17}$$

Since $u_{i-1}^k + g(t_i^k) - g(t_{i-1}^k) = g(t_i^k)$ on $\partial\Omega$, in view of the minimality (4.1) of u_i^k and of the quadratic structure of the elastic energy $\mathcal{E}$, we have that

$$\begin{aligned}\mathcal{E}(u_i^k, v_{i-1}^k) &\le \mathcal{E}(u_{i-1}^k + g(t_i^k) - g(t_{i-1}^k), v_{i-1}^k) \\ &= \mathcal{E}(u_{i-1}^k, v_{i-1}^k) + \mathcal{E}(g(t_i^k) - g(t_{i-1}^k), v_{i-1}^k) \\ &\quad + \int_\Omega ((v_{i-1}^k)^2 + \eta)\,\boldsymbol{\sigma}(u_{i-1}^k) : \boldsymbol{\epsilon}(g(t_i^k) - g(t_{i-1}^k))\,\mathrm{d}x.\end{aligned} \tag{4.18}$$

The second term is estimated by

$$\begin{aligned}\mathcal{E}(g(t_i^k) - g(t_{i-1}^k), v_{i-1}^k) &= \tfrac{1}{2}\int_\Omega ((v_{i-1}^k)^2 + \eta)\,\boldsymbol{\sigma}(g(t_i^k) - g(t_{i-1}^k)) : \boldsymbol{\epsilon}(g(t_i^k) - g(t_{i-1}^k))\,\mathrm{d}x \\ &\le C\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2,\end{aligned}$$

while the last term is re-written as

$$\begin{aligned}\int_\Omega ((v_{i-1}^k)^2 + \eta)\,\boldsymbol{\sigma}(u_{i-1}^k) : \boldsymbol{\epsilon}(g(t_i^k) - g(t_{i-1}^k))\,\mathrm{d}x &= \int_{t_{i-1}^k}^{t_i^k} \left(\int_\Omega ((v_{i-1}^k)^2 + \eta)\,\boldsymbol{\sigma}(u_{i-1}^k) : \boldsymbol{\epsilon}(\dot{g}(s))\,\mathrm{d}x\right)\mathrm{d}s \\ &= \int_{t_{i-1}^k}^{t_i^k} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s.\end{aligned}$$

Hence, (4.18) gives

$$\mathcal{F}(u_i^k, v_{i-1}^k) \le \mathcal{F}(u_{i-1}^k, v_{i-1}^k) + \int_{t_{i-1}^k}^{t_i^k} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s + C\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2. \tag{4.19}$$

Combining inequalities (4.17) and (4.19) and iterating over i, we deduce (4.15).

Property (d) follows simply from the construction of u_i^k and the definition of $\bar{u}_k$ and $\underline{v}_k$. It remains to prove compactness and the uniform bound of the energy $\mathcal{F}(\bar{u}_k(t), \underline{v}_k(t))$. By minimality of u_i^k, for every $k \in \mathbb{N}\backslash\{0\}$ and every $i \in \{1, \dots, k\}$, we can estimate

$$\tfrac{1}{2}\int_\Omega \eta\, \boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k)\,\mathrm{d}x \le \mathcal{E}(u_i^k, v_{i-1}^k) \le \mathcal{E}(g(t_i^k), v_{i-1}^k) \le \tfrac{1}{2}\int_\Omega (1+\eta)\,\boldsymbol{\sigma}(g(t_i^k)) : \boldsymbol{\epsilon}(g(t_i^k))\,\mathrm{d}x \le C$$

for some positive constant C, independent of the indices k and i. Then, Korn's inequality implies that $\sup_{k,i} \|u_i^k\|_{H^1} < +\infty$, which proves (c). Using the fact that $\underline{v}_k(t)$ takes values in $[0,1]$ and that $\underline{u}_k$ is bounded in $L^\infty([0,T];\mathcal{U})$, we easily deduce that

$$\mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s)) = \int_\Omega (\underline{v}_k^2(s) + \eta)\,\boldsymbol{\sigma}(\underline{u}_k(s)) : \boldsymbol{\epsilon}(\dot{g}(s))\,\mathrm{d}x \le C\|\dot{g}(s)\|_{H^1}.$$

Since $g \in AC([0,T];\mathcal{U})$ we have

$$\int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s \le C,$$

$$\sum_{i=1}^{I_t} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 \le \Big(\sum_{i=1}^{k} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}\Big)^2 \le C.$$

These bounds, together with (4.15), imply that $\mathcal{F}(\bar{u}_k(t), \underline{v}_k(t))$ is uniformly bounded and that v_k is bounded in $H^1([0,T];L^2(\Omega))$. Since the energy is bounded, the phase-field sequences v_k, $\bar{v}_k$, and $\underline{v}_k$ (all taking value v_i^k in the points t_i^k) are bounded in $L^\infty([0,T];H^1(\Omega))$. By minimality of $\tilde{v}_i^k$ we have $\mathcal{F}(u_i^k, \tilde{v}_i^k) \le \mathcal{F}(u_i^k, v_i^k)$, hence the sequence $\tilde{v}_k$ is bounded in $L^\infty([0,T];H^1(\Omega))$ as well. □

We are now ready to prove the convergence of the one-step scheme (4.1)–(4.3) towards a unilateral L^2-gradient flow.

Theorem 4.5. *There exists a subsequence, not relabelled, of the pair $(\bar{u}_k, v_k)$ such that:*

(i) $v_k \rightharpoonup v$ *in* $H^1([0,T];L^2(\Omega))$;
(ii) $v_k(t) \rightharpoonup v(t)$ *in* $H^1(\Omega)$ *and* $\bar{u}_k(t) \to u(t)$ *in* $\mathcal{U}$ *for every* $t \in [0,T]$;
(iii) (u,v) *is a unilateral* L^2*-gradient flow for* $\mathcal{F}$*, in the sense of Definition 2.5.*

Proof. In view of Proposition 4.4 (a) and (c) we get (i). Upon identifying $v \in H^1([0,T];L^2(\Omega))$ with its continuous representative, we can write

$$v_k(t) = v_0 + \int_0^t \dot{v}_k(s)\,\mathrm{d}s \qquad \text{and} \qquad v(t) = v_0 + \int_0^t \dot{v}(s)\,\mathrm{d}s \qquad \text{for every } t \in [0,T].$$

Hence $v_k(t) \rightharpoonup v(t)$ in $L^2(\Omega)$ for every $t \in [0,T]$; as $v_k(t)$ is uniformly bounded in $H^1(\Omega)$ we actually have $v_k(t) \rightharpoonup v(t)$ in $H^1(\Omega)$. It is easy to check that v is non-increasing in time and takes values in the interval $[0,1]$. Remembering the definition of v_k, $\underline{v}_k$ and $\bar{v}_k$ we can write

$$\bar{v}_k(t) = v_0 + \int_0^{t_k(t)} \dot{v}_k(s)\,\mathrm{d}s.$$

Note that the integrand is still $\dot{v}_k$. Since $t_k(t) \to t$, for every $t \in [0,T]$, we have $\bar{v}_k(t) \rightharpoonup v(t)$ in $H^1(\Omega)$ for every $t \in [0,T]$. In a similar way we deduce that $\underline{v}_k(t) \rightharpoonup v(t)$ in $H^1(\Omega)$ for every $t \in [0,T]$. We also notice that, in view of the minimum problem (4.2),

$$\|\tilde{v}_k(t) - \underline{v}_k(t)\|_{L^2}^2 \le 2\tau_k \mathcal{F}(\bar{u}_k(t), \underline{v}_k(t)) \le C\tau_k$$

for some positive constant C independent of k. Therefore, $\tilde{v}_k(t) \rightharpoonup v(t)$ in $H^1(\Omega)$ for every $t \in [0,T]$.

We recall that for every $t \in [0,T]$ by Proposition 4.4 (d) it holds

$$\bar{u}_k(t) \in \arg\min \{\mathcal{E}(u, \underline{v}_k(t)) : u \in \mathcal{U},\ u = g(t_k(t)) \text{ on } \partial\Omega\}.$$

Taking into account that $\underline{v}_k(t) \rightharpoonup v(t)$ and $g(t_k(t)) \to g(t)$ everywhere in $[0,T]$, Lemma 3.3 implies that

$$u(t) \in \arg\min \{\mathcal{E}(u, v(t)) : u \in \mathcal{U},\ u = g(t) \text{ on } \partial\Omega\}$$

and that $\bar{u}_k(t) \to u(t)$ in $\mathcal{U}$ for every $t \in [0,T]$. At this point, using the time regularity of v and g, by Lemma 2.8 (or simply by Lem. 3.3) we deduce that $u \in C([0,T];\mathcal{U})$. Finally, let us see that $\underline{u}_k(t) \to u(t)$ in $\mathcal{U}$ for every $t \in [0,T]$. Indeed, it is enough to notice that $\underline{u}_k(t) = \bar{u}_k(t - \tau_k)$ satisfies

$$\underline{u}_k(t) \in \arg\min \{\mathcal{E}(u, \underline{v}_k(t - \tau_k)) : u \in \mathcal{U},\ u = g(t_k(t) - \tau_k) \text{ on } \partial\Omega\}.$$

Arguing as above, we conclude again by Lemma 3.3 and uniqueness of minimizers.

To complete the proof, it remains to show the energy balance (d) of Definition 2.5. To this end, we will pass to the limit in the energy estimate (4.15). Since $\bar{u}_k(t) \to u(t)$ in $\mathcal{U}$ and $\underline{v}_k(t) \rightharpoonup v(t)$ in $H^1(\Omega)$, by Lemma 3.1 and (4.15) we get

$$\begin{aligned}
\mathcal{F}(\bar{u}(t), v(t)) &\leq \liminf_{k\to\infty} \mathcal{F}(\bar{u}_k(t), \underline{v}_k(t)) \\
&\leq \limsup_{k\to\infty} \Big(\mathcal{F}(u_0, v_0) - \tfrac{1}{2} \int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2 + |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\, \mathrm{d}s \Big) \\
&\quad + \limsup_{k\to\infty} \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\, \mathrm{d}s + \limsup_{k\to\infty} \sum_{i=1}^{k} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 \\
&\leq \mathcal{F}(u_0, v_0) - \liminf_{k\to\infty} \tfrac{1}{2} \int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2\, \mathrm{d}s - \liminf_{k\to\infty} \tfrac{1}{2} \int_0^{t_k(t)} |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\, \mathrm{d}s \\
&\quad + \limsup_{k\to\infty} \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\, \mathrm{d}s + \limsup_{k\to\infty} \sum_{i=1}^{k} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 .
\end{aligned} \tag{4.20}$$

Since $v_k \rightharpoonup v$ in $H^1([0,T]; L^2(\Omega))$ and $t_k(t) \to t$ we get

$$-\liminf_{k\to\infty} \int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2\, \mathrm{d}s \leq -\int_0^t \|\dot{v}(s)\|_{L^2}^2\, \mathrm{d}s.$$

Since $\bar{u}_k \to u$ and $\tilde{v}_k \rightharpoonup v$ pointwise in $[0,T]$, applying Fatou's lemma and Lemma 3.2 we obtain

$$-\liminf_{k\to\infty} \int_0^{t_k(t)} |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\, \mathrm{d}s \leq -\int_0^t |\partial_v^- \mathcal{F}|^2(u(s), v(s))\, \mathrm{d}s\,.$$

Since $\underline{u}_k \to u$ and $\underline{v}_k \rightharpoonup v$ pointwise in $[0,T]$, by Lemma 2.4 we know that $\mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s)) \to \mathcal{P}(u(s), v(s), \dot{g}(s))$ pointwise in $[0,T]$. Since $0 \leq \underline{v}_k(s) \leq 1$ and $\underline{u}_k(s)$ is bounded in $\mathcal{U}$ we get (as in the proof of Prop. 4.4)

$$\mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s)) \leq C \|\dot{g}(s)\|_{H^1}.$$

Hence, by dominated convergence,

$$\limsup_{k\to\infty} \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\, \mathrm{d}s \leq \int_0^t \mathcal{P}(u(s), v(s), \dot{g}(s))\, \mathrm{d}s.$$

Finally, being $g \in AC([0,T];\mathcal{U})$, for every $\epsilon > 0$ we have $\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1} \le \epsilon$ for every $1 \le i \le k$ and every k sufficiently large. Hence

$$\sum_{i=1}^k \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 \le \epsilon \sum_{i=1}^k \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1} \le \epsilon \int_0^T \|\dot{g}(s)\|_{H^1}\,\mathrm{d}s \le C\epsilon.$$

Therefore

$$\limsup_{k\to\infty} \sum_{i=1}^k \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 = 0.$$

In conclusion

$$\mathcal{F}(u(t),v(t)) \le \mathcal{F}(u_0,v_0) - \tfrac{1}{2}\int_0^t \|\dot{v}(s)\|_{L^2}^2 + |\partial_v^- \mathcal{F}|^2(u(s),v(s))\,\mathrm{d}s + \int_0^t \mathcal{P}(u(s),v(s),\dot{g}(s))\,\mathrm{d}s.$$

The opposite inequality follows by Proposition 2.9. This concludes the proof. □

5. An infinite-step scheme

In this section we present another time-discrete scheme whose time-continuous limits provide unilateral L^2-gradient flows, in the sense of Definition 2.5. Here, in the spirit of previous works, such as [7,13,27], we consider an infinite-step scheme based, at each time increment, on an infinite alternate minimization process.

More precisely, consider again the time steps $\tau_k := T/k$, $k \in \mathbb{N}$, and the time nodes $t_i^k := i\tau_k$. We construct inductively the functions $u_i^k \in \mathcal{U}$ and $v_i^k \in \mathcal{V}$ providing the configuration of the system at time t_i^k. For $i = 0$ we set $u_0^k := u_0$ and $v_0^k = \tilde{v}_0^k := v_0$, being u_0, v_0 the initial conditions. For $i \in \{1,\dots,k\}$, *i.e.*, for each time increment, we introduce two auxiliary sequences $u_{i,j}^k$ and $v_{i,j}^k$, for $j \in \mathbb{N}$, defined as follows: for $j = 0$ we have $u_{i,0}^k := u_{i-1}^k$, $v_{i,0}^k := v_{i-1}^k$, and, for $j \ge 1$,

$$u_{i,j}^k := \arg\min\{\mathcal{E}(u, v_{i,j-1}^k) :\ u \in \mathcal{U},\ u = g(t_i^k) \text{ on } \partial\Omega\}, \tag{5.1}$$

$$\tilde{v}_{i,j}^k := \arg\min\{\mathcal{F}(u_{i,j}^k, v) + \tfrac{1}{2\tau_k}\|v - v_{i-1}^k\|_{L^2}^2 :\ v \in H^1(\Omega)\}, \tag{5.2}$$

$$v_{i,j}^k := \min\{\tilde{v}_{i,j}^k, v_{i-1}^k\}. \tag{5.3}$$

As usual, the minimum problems (5.1) and (5.2) admit unique solutions. It is not hard to check that the sequence $u_{i,j}^k$ is bounded in $\mathcal{U}$ and that the sequences $v_{i,j}^k$ and $\tilde{v}_{i,j}^k$ are bounded in $H^1(\Omega)$, with $0 \le v_{i,j}^k, \tilde{v}_{i,j} \le 1$. Hence, there exist $u_i^k \in \mathcal{U}$ and $v_i^k, \tilde{v}_i^k \in H^1(\Omega)$ such that, up to a subsequence, $u_{i,j}^k \rightharpoonup u_i^k$ weakly in $\mathcal{U}$, $v_{i,j}^k \rightharpoonup v_i^k$ and $\tilde{v}_{i,j}^k \rightharpoonup \tilde{v}_i^k$ weakly in $H^1(\Omega)$. Clearly, by strong convergence in $L^2(\Omega)$, $v_i^k = \min\{\tilde{v}_i^k, v_{i-1}^k\}$. Therefore, we also have $0 \le \tilde{v}_i^k \le 1$ and $0 \le v_i^k \le v_{i-1}^k \le 1$ for every $k \in \mathbb{N}$ and every $i \in \{0,\dots,k\}$.

Applying Lemma 3.3, we deduce that $u_{i,j}^k \to u_i^k$ strongly in $\mathcal{U}$ and that

$$u_i^k = \arg\min\{\mathcal{E}(u, v_i^k) :\ u \in \mathcal{U},\ u = g(t_i^k) \text{ on } \partial\Omega\}. \tag{5.4}$$

It is also easy to see (by Γ-convergence or using the Euler–Lagrange equations) that

$$\tilde{v}_i^k = \arg\min\{\mathcal{F}(u_i^k, v) + \tfrac{1}{2\tau_k}\|v - v_{i-1}^k\|_{L^2}^2 :\ v \in H^1(\Omega)\}. \tag{5.5}$$

In particular, the pair $(u_i^k, \tilde{v}_i^k)$ is a critical point of the time-discrete functional

$$\mathcal{F}(u,v) + \tfrac{1}{2\tau_k}\|v - v_{i-1}^k\|_{L^2}^2. \tag{5.6}$$

Note that this property is not satisfied by the configuration $(u_i^k, \tilde{v}_i^k)$ of the one-step algorithm. Finally, as in Proposition 4.2, in view of the stability condition (5.5) we obtain

$$\frac{\|v_i^k - v_{i-1}^k\|_{L^2}}{\tau_k} = |\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k), \tag{5.7}$$

$$\partial_v \mathcal{F}(u_i^k, v_i^k)[v_i^k - v_{i-1}^k] = \partial_v \mathcal{F}(u_i^k, \tilde{v}_i^k)[v_i^k - v_{i-1}^k] = -|\partial_v^- \mathcal{F}|(u_i^k, \tilde{v}_i^k)\|v_i^k - v_{i-1}^k\|_{L^2}. \tag{5.8}$$

Remark 5.1. Let us briefly comment on the algorithm (5.1)–(5.3). As in the one-step scheme, the minimization problem (5.2) involving the phase-field variable v is unconstrained, so that the computational cost of the single iteration is very low. Irreversibility of the phase-field function is taken into account a posteriori by (5.3). Note that in (5.3) the constraint is given by v_{i-1}^k (the configuration at the previous time node) and not by $v_{i,j-1}^k$ (the previous configuration of the alternate minimization scheme), as it is for instance in [27, 33]. The latter way of imposing the constraint, albeit theoretically correct, seems to be numerically more delicate as it may accumulate computational errors over the alternate iterations. On the other hand, the auxiliary sequence $v_{i,j}^k$ employed here is not monotone decreasing with respect to the index $j \in \mathbb{N}$.

As it will be pointed out later in the numerical simulations, the multi-step algorithm is significantly more stable than the single-step one. The reason is that the former allows us to produce at each time t_i^k a critical point $(u_i^k, \tilde{v}_i^k)$ of the functional (5.6), while the latter scheme generates pairs $(u_i^k, \tilde{v}_i^k)$ satisfying the weaker stability conditions (4.1) and (4.2), which are only a first rough approximation of the stability properties (5.4) and (5.5). For this reason, the one-step scheme often requires a time step adaptation procedure in the numerical simulations, while the multi-step scheme seems to be more robust.

As we did in (4.12)–(4.14), we define the following interpolation functions:

$$v_k(t) := v_i^k + \frac{v_{i+1}^k - v_i^k}{\tau_k}(t - t_i^k) \qquad \text{for every } t \in [t_i^k, t_{i+1}^k), \tag{5.9}$$

$$\bar{u}_k(t) := u_i^k, \quad \bar{v}_k(t) := v_i^k, \quad \tilde{v}_k(t) := \tilde{v}_i^k, \quad t_k(t) := t_i^k \qquad \text{for every } t \in (t_{i-1}^k, t_i^k], \tag{5.10}$$

$$\underline{u}_k(t) := u_i^k, \quad \underline{v}_k(t) := v_i^k \qquad \text{for every } t \in [t_i^k, t_{i+1}^k). \tag{5.11}$$

In what follows, we show that the interpolation functions (5.9)–(5.11) still converge to a unilateral L^2-gradient flow evolution. For sake of brevity, we only stress the main changes in the energy bounds proved in Proposition 4.4.

Proposition 5.2. *The following facts hold:*

(a) *The sequence v_k is bounded in $L^\infty([0,T]; H^1(\Omega))$ and in $H^1([0,T]; L^2(\Omega))$;*
(b) *The sequences $\bar{v}_k$, $\tilde{v}_k$, $\underline{v}_k$ are bounded in $L^\infty([0,T]; H^1(\Omega))$;*
(c) *The sequences $\bar{u}_k$ and $\underline{u}_k$ are bounded in $L^\infty([0,T]; \mathcal{U})$;*
(d) *For every $t \in [0,T]$ we have*

$$\bar{u}_k(t) \in \arg\min\{\mathcal{E}(u, \bar{v}_k(t)) : u \in \mathcal{U},\ u = g(t_k(t)) \text{ on } \partial\Omega\},$$
$$\tilde{v}_k(t) \in \arg\min\{\mathcal{F}(\bar{u}_k(t), v) + \tfrac{1}{\tau_k}\|v - \underline{v}_k(t)\|_{L^2}^2 : v \in H^1(\Omega)\};$$

(e) *There exists $R_k \to 0^+$ as $k \to +\infty$ such that for every $t \in [0,T]$ it holds*

$$\begin{aligned}\mathcal{F}(\bar{u}_k(t), \bar{v}_k(t)) \le \mathcal{F}(u_0, v_0) - \tfrac{1}{2}\int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2 + |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\,\mathrm{d}s \\ + \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s + R_k.\end{aligned} \tag{5.12}$$

In particular the energy $\mathcal{F}(\bar{u}_k(t), \underline{v}_k(t))$ is uniformly bounded, w.r.t. t and k.

Proof. We explain here how to adapt to the multi-step scheme the arguments used in the proof of Proposition 4.4.

Arguing as in (2.13), by the separate quadratic structure of the energy functional $\mathcal{F}$ and taking into account equalities (5.7) and (5.8), for every $k \in \mathbb{N}$ and every $i \in \{1, \dots, k\}$ we get

$$\begin{aligned}\mathcal{F}(u_i^k, v_{i-1}^k) &\geq \mathcal{F}(u_i^k, v_i^k) + \partial_v \mathcal{F}(u_i^k, v_i^k)[v_{i-1}^k - v_i^k] + \tfrac{1}{2}\|v_i^k - v_{i-1}^k\|_{H^1}^2 \\ &= \mathcal{F}(u_i^k, v_i^k) + \tau_k \Big(\tfrac{1}{2} |\partial_v^- \mathcal{F}|^2 (u_i^k, \tilde{v}_i^k) + \tfrac{1}{2} \frac{\|v_i^k - v_{i-1}^k\|_{L^2}^2}{\tau_k^2} \Big) + \tfrac{1}{2}\|v_i^k - v_{i-1}^k\|_{H^1}^2 .\end{aligned} \tag{5.13}$$

In order to pass from u_i^k to u_{i-1}^k in the left-hand side of (5.13), we first make an intermediate step exploiting the construction of $u_{i,1}^k$ in the multi-step algorithm. Exploiting again the separate quadratic structure of the functional $\mathcal{F}$, using the minimality of $u_{i,1}^k$, and recalling that $u_i^k = u_{i,1}^k = g(t_i^k)$ on $\partial\Omega$, we have that

$$\begin{aligned}\mathcal{F}(u_i^k, v_{i-1}^k) &= \mathcal{F}(u_{i,1}^k + (u_i^k - u_{i,1}^k), v_{i-1}^k) \\ &= \mathcal{F}(u_{i,1}^k, v_{i-1}^k) + \mathcal{E}(u_i^k - u_{i,1}^k, v_{i-1}^k) + \int_\Omega ((v_{i-1}^k)^2 + \eta) \boldsymbol{\sigma}(u_{i,1}^k) : \boldsymbol{\epsilon}(u_i^k - u_{i,1}^k) \, \mathrm{d}x \\ &= \mathcal{F}(u_{i,1}^k, v_{i-1}^k) + \mathcal{E}(u_i^k - u_{i,1}^k, v_{i-1}^k).\end{aligned} \tag{5.14}$$

Since v_{i-1}^k takes values in the interval $[0,1]$, in view of (5.14) there exists a positive constant C such that

$$\mathcal{F}(u_i^k, v_{i-1}^k) \leq \mathcal{F}(u_{i,1}^k, v_{i-1}^k) + C\|u_i^k - u_{i,1}^k\|_{H^1}^2 . \tag{5.15}$$

As we argued in (4.18) and (4.19), by the minimality of $u_{i,1}^k$ and the regularity of the boundary datum g, we can continue in (5.15) with

$$\begin{aligned}\mathcal{F}(u_i^k, v_{i-1}^k) \leq \mathcal{F}(u_{i-1}^k, v_{i-1}^k) &+ \int_{t_{i-1}^k}^{t_i^k} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s)) \, \mathrm{d}s \\ &+ C\|u_i^k - u_{i,1}^k\|_{H^1}^2 + C'\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 ,\end{aligned} \tag{5.16}$$

for some positive constant C' independent of k. Combining inequalities (5.13) and (5.16), we end up with

$$\begin{aligned}\mathcal{F}(u_i^k, v_i^k) &\leq \mathcal{F}(u_{i-1}^k, v_{i-1}^k) - \tfrac{1}{2} \int_{t_{i-1}^k}^{t_i^k} |\partial_v^- \mathcal{F}|^2 (\bar{u}_k(s), \tilde{v}_k(s)) + \frac{\|v_i^k - v_{i-1}^k\|_{L^2}^2}{\tau_k^2} \, \mathrm{d}s \\ &+ \int_{t_{i-1}^k}^{t_i^k} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s)) \, \mathrm{d}s + C\|u_i^k - u_{i,1}^k\|_{H^1}^2 + C'\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 - \tfrac{1}{2}\|v_i^k - v_{i-1}^k\|_{H^1}^2 .\end{aligned}$$

Recalling that u_i^k and $u_{i,1}^k$ are minimizers of the elastic energy $\mathcal{E}(\cdot, v_i^k)$ and $\mathcal{E}(\cdot, v_{i-1}^k)$, respectively, with the same boundary condition $g(t_i^k)$, applying Lemma 2.8 we get that

$$\|u_i^k - u_{i,1}^k\|_{H^1}^2 \leq C\|v_i^k - v_{i-1}^k\|_{L^q}^2 .$$

Following the same strategy leading to (2.18), we deduce that there exists $C_\delta > 0$ such that

$$\begin{aligned}\mathcal{F}(u_i^k, v_i^k) \leq \mathcal{F}(u_{i-1}^k, v_{i-1}^k) &- \tfrac{1}{2} \int_{t_{i-1}^k}^{t_i^k} \Big(|\partial_v^- \mathcal{F}|^2 (\bar{u}_k(s), \tilde{v}_k(s)) + \frac{\|v_i^k - v_{i-1}^k\|_{L^2}^2}{\tau_k^2} \Big) \mathrm{d}s \\ &+ \int_{t_{i-1}^k}^{t_i^k} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s)) \, \mathrm{d}s + C_\delta \|v_i^k - v_{i-1}^k\|_{L^2}^2 + C'\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 .\end{aligned} \tag{5.17}$$

Iterating inequality (5.17), for every $k \in \mathbb{N}$ and every $t \in [0, T]$ we get

$$\begin{aligned} \mathcal{F}(\bar{u}_k(t), \bar{v}_k(t)) \le \mathcal{F}(u_0, v_0) - \tfrac{1}{2} \int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2 + |\partial_v^- \mathcal{F}|^2(\bar{u}_k(s), \tilde{v}_k(s))\,\mathrm{d}s \\ + \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s \\ + C_\delta \sum_{i=1}^{K} \|v_i^k - v_{i-1}^k\|_{L^2}^2 + C' \sum_{i=1}^{K} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2. \end{aligned} \tag{5.18}$$

In order to proceed in the estimate (5.18), we notice that

$$\sum_{i=1}^{K} \|v_i^k - v_{i-1}^k\|_{L^2}^2 = \tau_k \int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2\,\mathrm{d}s. \tag{5.19}$$

Combining (5.18) and (5.19), we deduce that for k large enough it holds

$$\begin{aligned} \mathcal{F}(\bar{u}_k(t), \bar{v}_k(t)) + \tfrac{1}{4} \int_0^{t_k(t)} \|\dot{v}_k(s)\|_{L^2}^2\,\mathrm{d}s \\ \le \mathcal{F}(u_0, v_0) + \int_0^{t_k(t)} \mathcal{P}(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s + C' \sum_{i=1}^{K} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2. \end{aligned} \tag{5.20}$$

Following the argument of the proof of Proposition 4.4, we get that $\bar{u}_k$, $\underline{u}_k$ are bounded in $L^\infty([0,T];\mathcal{U})$, v_k, $\bar{v}_k$, $\underline{v}_k$, and $\tilde{v}_k$ are bounded $L^\infty([0,T]; H^1(\Omega))$, and v_k is bounded in $H^1([0,T]; L^2(\Omega))$.

In view of (5.18) and (5.19), we set

$$R_k := \tau_k \int_0^{T} \|\dot{v}_k(s)\|_{L^2}^2\,\mathrm{d}s + C' \sum_{i=1}^{K} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2.$$

Since v_k is bounded in $H^1([0,T]; L^2(\Omega))$ and $g \in AC([0,T]; \mathcal{U})$, we get that $R_k \to 0$ as $k \to +\infty$, and the proof of (5.12) is thus concluded. □

To conclude this section, we simply notice that, once we have proved the bounds of Proposition 5.2, the proof of the convergence to a unilateral L^2-gradient flow works as in the one-step algorithm. Therefore, we refer to Theorem 4.5 for the proof of the following result.

Theorem 5.3. *There exists a subsequence, not relabelled, of the pair* $(\bar{u}_k, v_k)$ *such that:*

(i) $v_k \rightharpoonup v$ *in* $H^1([0,T]; L^2(\Omega))$*;*
(ii) $v_k(t) \rightharpoonup v(t)$ *in* $H^1(\Omega)$ *and* $\bar{u}_k(t) \to u(t)$ *in* $\mathcal{U}$ *for every* $t \in [0,T]$*;*
(iii) (u, v) *is a unilateral* L^2*-gradient flow for* $\mathcal{F}$*, in the sense of Definition 2.5.*

6. Finite element approximation

In this section we present a finite element discretization for our unilateral L^2-gradient flow. Our aim is twofold: to provide a space-discrete (finite element) version of the unilateral L^2-gradient flow and then to show that its space-continuous limit is again a unilateral L^2-gradient flow, in the sense of Definition 2.5.

First, in Section 6.1, we will introduce a discrete energy $\mathcal{F}_h$ defined in discretized spaces $\mathcal{V}_h$ and $\mathcal{U}_h$ (h being the mesh size); the evolution will then be defined, in Section 6.2, using again a time discrete approach in which the time-incremental problem is provided by a finite-step algorithm. We stress here that this finite-step algorithm is flexible enough to cover every stopping criterion, including those employed in the numerical simulations of Section 7. This algorithm is, in some sense, intermediate between the simple one-step and the “theoretical” infinite-step schemes studied in Sections 4 and 5, respectively.

Finally, in Section 6.3 we will show that, as the mesh size vanishes, the finite element evolutions converge to a (space-continuous) unilateral L^2-gradient flow, in the sense of Definition 2.5.

6.1. Preliminaries

First, let us describe the space-discrete setting we are considering in this section. Let Ω be a polyhedral set in $\mathbb{R}^2$ and let $\{\mathcal{T}_h\}_{h>0}$ be a family of acute-angle triangulations of Ω. We will denote by K the (triangular) elements and assume that $\mathrm{diam}(K) \le h$. Furthermore, we denote by $\varDelta_h$ the set of all the vertices of $\mathcal{T}_h$ and we set $N_h := \#\varDelta_h$.

We denote by $\mathcal{U}_h$ and $\mathcal{V}_h$ the sets of continuous P_1 finite elements functions on Ω discretizing, respectively, the function spaces $\mathcal{U} = H^1(\Omega;\mathbb{R}^2)$ and $\mathcal{V} = H^1(\Omega) \cap L^\infty(\Omega)$.

In what follows, we will consider in $\mathcal{V}_h$ the basis of shape functions $\{\xi_l\}_{l=1}^{N_h}$, where

$$\xi_l(x_m) = \delta_{lm} \qquad \text{for every } x_m \in \varDelta_h, \tag{6.1}$$

being δ_{lm} the Kronecker delta. Accordingly, we introduce the Lagrangian interpolant $\Pi_h\colon C(\bar\Omega) \to \mathcal{V}_h$, *i.e.*, the linear operator such that

$$\Pi_h(\varphi)(x_l) = \varphi(x_l) \qquad \text{for every } \varphi \in C(\bar\Omega) \text{ and every } x_l \in \varDelta_h. \tag{6.2}$$

Note that, being $\mathcal{T}_h$ an acute-angle mesh, the basis $\{\xi_l\}_{l=1}^{N_h}$ satisfies the stiffness condition

$$\int_\Omega \nabla\xi_l \cdot \nabla\xi_m \,\mathrm{d}x \le 0 \qquad \text{for every } l, m \in \{1, \dots, N_h\},\ l \ne m, \tag{6.3}$$

which is the natural condition to have a discrete maximum principle in $\mathcal{V}_h$ (*e.g.*, [16,34]) and, in turn, to ensure that, in the evolution, phase-field functions will take values in $[0, 1]$ (see Prop. 6.14).

In general, $\mathcal{U}_h$ and $\mathcal{V}_h$ will be endowed with the usual H^1-norms. However, we will employ in $\mathcal{V}_h$ a further norm given by

$$\|v\|_{\mathcal{V}_h} := \left(\int_\Omega |\Pi_h(v^2)| \,\mathrm{d}x\right)^{1/2}. \tag{6.4}$$

Using the definition of the basis $\{\xi_l\}_{l=1}^{N_h}$, it is easy to check that $\|\cdot\|_{\mathcal{V}_h}$ is a norm in $\mathcal{V}_h$. Moreover, we have the following property.

Lemma 6.1. *For every $v \in \mathcal{V}_h$ we have $\|v\|_{L^2} \le \|v\|_{\mathcal{V}_h}$.*

Proof. Let $\{x_l\}_{l=1}^{N_h}$ be the vertices of the triangulation $\mathcal{T}_h$. By the convexity of the quadratic function and the fact that $\sum_{l=1}^{N_h} \xi_l = 1$ with $0 \le \xi_l \le 1$, for every $l \in \{1, \dots, N_h\}$, we have

$$v^2 = \left(\sum_{l=1}^{N_h} v(x_l)\xi_l\right)^2 \le \sum_{l=1}^{N_h} v^2(x_l)\xi_l = \Pi_h(v^2).$$

The assertion follows by intergration over Ω. □

Remark 6.2. Note that on each triangle K, denoting by $\{x_i\}$ for $i = 1, 2, 3$ the vertices of K, we have

$$\int_K \Pi_h(v^2) \,\mathrm{d}x = \sum_{i=1}^{3} v^2(x_i) \left(\int_K \xi_i \,\mathrm{d}x\right) = \sum_{i,j=1}^{3} v(x_i)v(x_j)D_{ij}$$

where D is the diagonal matrix with entries $D_{ij} = \delta_{ij}(\int_K \xi_i \, \mathrm{d}x)$. Without the interpolation operator Π_h we would have the L^2-norm

$$\int_K v^2 \, \mathrm{d}x = \int_K \Big(\sum_{i=1}^{3} v(x_i)\xi_i \Big)^2 \mathrm{d}x$$
$$= \sum_{i,j=1}^{3} v(x_i)v(x_j) \left(\int_K \xi_i \xi_j \, \mathrm{d}x \right) = \sum_{i,j=1}^{3} v(x_i)v(x_j) A_{ij}$$

where A is, in general, a full matrix. In practice, employing the operator Π_h results in a simpler numerical integration formula for the quadratic function v^2 and, in our case, for the elastic energy (see below).

In our finite element setting we introduce the discrete counterparts of the stored elastic energy (2.1) and of the dissipated energy (2.2): for every $u \in \mathcal{U}_h$ and every $v \in \mathcal{V}_h$ we set, respectively,

$$\mathcal{E}_h(u, v) := \tfrac{1}{2} \int_\Omega \big(\Pi_h(v^2) + \eta\big) \, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u) \, \mathrm{d}x, \tag{6.5}$$

$$\mathcal{D}_h(v) := \tfrac{1}{2} \int_\Omega |\nabla v|^2 \, \mathrm{d}x + \tfrac{1}{2} \int_\Omega \Pi_h\big((1-v)^2\big) \, \mathrm{d}x. \tag{6.6}$$

As in (2.3), the *discrete total energy* is the sum of $\mathcal{E}_h$ and $\mathcal{D}_h$. Hence, for $u \in \mathcal{U}_h$ and $v \in \mathcal{V}_h$, we define

$$\mathcal{F}_h(u, v) := \mathcal{E}_h(u, v) + \mathcal{D}_h(v). \tag{6.7}$$

Remark 6.3. In general the energy functional $\mathcal{F}$ is discretized simply by taking its restriction to the finite element spaces, *i.e.*, by setting $\mathcal{F}_h := \mathcal{F}|_{\mathcal{U}_h \times \mathcal{V}_h}$. Here, instead, following the ideas of [1,8], we redefine $\mathcal{F}_h$ using also the projection operator Π_h. In this way we ensure that during the evolution the phase-field function $v \in \mathcal{V}_h$ will take values in $[0, 1]$ (see Prop. 6.14).

We notice that, as in (2.5), for every $u \in \mathcal{U}_h$ and every $v, \varphi \in \mathcal{V}_h$ there exists the derivative $\partial_v \mathcal{F}_h$ of $\mathcal{F}_h$ with respect to v. By linearity of Π_h, it reads

$$\partial_v \mathcal{F}_h(u, v)[\varphi] = \int_\Omega \Pi_h(v\varphi) \, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u) \, \mathrm{d}x + \int_\Omega \nabla v \cdot \nabla \varphi \, \mathrm{d}x - \int_\Omega \Pi_h((1-v)\varphi) \, \mathrm{d}x. \tag{6.8}$$

Similarly to Definition 2.1, we introduce the *discrete unilateral* L^2*-slope* of $\mathcal{F}_h$.

Definition 6.4. For every $u \in \mathcal{U}_h$ and every $v \in \mathcal{V}_h$, we define the discrete unilateral L^2-slope of $\mathcal{F}_h$ as

$$|\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u, v) := \limsup_{\substack{z \to v \\ z \in \mathcal{V}_h, \, z \le v}} \frac{[\mathcal{F}_h(u, v) - \mathcal{F}_h(u, z)]_+}{\|z - v\|_{\mathcal{V}_h}}.$$

With the argument used in Lemma 2.3, we can show the following.

Lemma 6.5. *For every $h > 0$, every $u \in \mathcal{U}_h$, and every $v \in \mathcal{V}_h$,*

$$|\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u, v) = \sup \{ -\partial_v \mathcal{F}_h(u, v)[\varphi] : \ \varphi \in \mathcal{V}_h, \ \varphi \le 0, \ \|\varphi\|_{\mathcal{V}_h} \le 1 \}. \tag{6.9}$$

Remark 6.6. Note that here the normalization in (6.9) is with respect to the norm $\| \cdot \|_{\mathcal{V}_h}$.

We now prove a lower-semicontinuity property of the slope $|\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}$ similar to Lemma 3.2.

Lemma 6.7. *Fix $h > 0$. If $u_m \to u$ in $\mathcal{U}_h$ and $v_m \to v$ in $\mathcal{V}_h$, then*

$$|\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u,v) \le \liminf_{m\to\infty} |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_m,v_m).$$

Proof. The proof can be done as in Lemma 3.2. □

Following the steps of Section 2, we introduce the space-discrete counterpart of the power of external forces (2.9). For every $u, z \in \mathcal{U}_h$ and every $v \in \mathcal{V}_h$ we set

$$\mathcal{P}_h(u,v,z) := \int_\Omega (\Pi_h(v^2) + \eta)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(z)\, \mathrm{d}x. \tag{6.10}$$

We are now ready to give the definition of *finite-dimensional unilateral L^2-gradient flow.*

Definition 6.8. Let $h > 0$, $T > 0$, and let $g \in AC([0,T];\mathcal{U}_h)$. Let $u_0 \in \mathcal{U}_h$ with $u_0 = g(0)$ on $\partial\Omega$ and let $v_0 \in \mathcal{V}_h$ be such that $0 \le v_0 \le 1$ and

$$u_0 \in \arg\min \{\mathcal{E}_h(u,v_0) : u \in \mathcal{U}_h \text{ with } u = g(0) \text{ on } \partial\Omega\}. \tag{6.11}$$

We say that a pair $(u,v)\colon [0,T] \to \mathcal{U}_h \times \mathcal{V}_h$ is a *finite-dimensional unilateral L^2-gradient flow* for the energy $\mathcal{F}_h$ with initial condition (u_0,v_0) and boundary condition g if the following properties are satisfied:

(a) *Time regularity*: $u \in C([0,T];\mathcal{U}_h)$ and $v \in H^1([0,T];\mathcal{V}_h) \cap L^\infty([0,T];\mathcal{V}_h)$ with $u(0) = u_0$ and $v(0) = v_0$;
(b) *Irreversibility*: $t \mapsto v(t)$ is non-increasing (*i.e.*, $v(s) \le v(t)$ a.e. in Ω for every $0 \le t \le s \le T$) and $0 \le v(t) \le 1$ for every $t \in [0,T]$;
(c) *Displacement equilibrium*: for every $t \in [0,T]$ we have $u(t) = g(t)$ on $\partial\Omega$ and

$$u(t) \in \arg\min \{\mathcal{E}_h(u,v(t)) : u \in \mathcal{U}_h \text{ with } u = g(t) \text{ on } \partial\Omega\};$$

(d) *Energy balance*: for a.e. $t \in [0,T]$ it holds

$$\dot{\mathcal{F}}_h(u(t),v(t)) = -\tfrac{1}{2}\|\dot v(t)\|^2_{\mathcal{V}_h} - \tfrac{1}{2}|\partial_v^- \mathcal{F}_h|^2_h(u(t),v(t)) + \mathcal{P}_h(u(t),v(t),\dot g(t)).$$

As we have done in Section 4, we immediately show that in order to obtain the balance (d) of Definition 6.8, only an energy inequality is sufficient. This is the content of Proposition 6.9, whose proof is similar to the one of Proposition 2.9.

Proposition 6.9. *Let $T > 0$, $h > 0$, $g \in AC([0,T];\mathcal{U}_h)$, $u_0 \in \mathcal{U}_h$, and $v_0 \in \mathcal{V}_h$ be such that (6.11) holds. Assume that the pair $(u,v)\colon [0,T] \to \mathcal{U}_h \times \mathcal{V}_h$ satisfies properties (a)–(c) of Definition 6.8 and that for every $t \in [0,T]$*

$$\begin{aligned}\mathcal{F}_h(u(t),v(t)) \le \mathcal{F}_h(u_0,v_0) &- \tfrac{1}{2}\int_0^t |\partial_v^- \mathcal{F}_h|^2_{\mathcal{V}_h}(u(s),v(s)) + \|\dot v_h(s)\|^2_{\mathcal{V}_h}\, \mathrm{d}s \\ &+ \int_0^t \mathcal{P}_h(u(s),v(s),\dot g(s))\, \mathrm{d}s.\end{aligned} \tag{6.12}$$

Then, (u,v) also fulfills the energy balance (d) of Definition 6.8.

Finally, we conclude this subsection by providing a couple of general estimate regarding the discrete displacement field and the discrete phase-field function. These results will be useful in the upcoming discussion of the finite-step algorithm.

Lemma 6.10. *Let $h > 0$. For $i = 1, 2$, let $g_i \in \mathcal{U}_h$ with $\|g_i\|_{H^1} \le M$, $v_i \in \mathcal{V}_h$ with $0 \le v_i \le 1$, and let*

$$u_i := \arg\min \{\mathcal{E}_h(u, v_i) : \ u \in \mathcal{U}_h,\ u = g_i \text{ on } \partial\Omega\}. \tag{6.13}$$

Then, there exists $C_h > 0$, independent of g_i and v_i but depending on h, such that

$$\|u_1 - u_2\|_{H^1} \le C_h \|g_1 - g_2\|_{H^1} + C_h M \|v_1 - v_2\|_2. \tag{6.14}$$

Proof. We sketch the proof, which follows easily by Euler–Lagrange equations. Consider the auxiliary function $u_* := \arg\min \{\mathcal{E}_h(u, v_1) : \ u \in \mathcal{U}_h,\ u = g_2 \text{ on } \partial\Omega\}$. We estimate $\|u_1 - u_*\|_{H^1}$ and $\|u_* - u_2\|_{H^1}$. By continuous dependence with respect to the boundary data, it is easy to see that

$$\|u_1 - u_*\|_{H^1} \le C \|g_1 - g_2\|_{H^1},$$

where $C > 0$ is actually independent of $h > 0$. By continuous depence with respect to the coefficient it is also easy to see that

$$\|u_1 - u_*\|_{H^1} \le CM \|v_1 - v_2\|_{L^\infty} \le C_h M \|v_1 - v_2\|_{L^2},$$

where the last inequality follows from the equivalence of norms in the finite dimensional space $\mathcal{V}_h$. □

Lemma 6.11. *Let $h > 0$. For $i = 1, 2$, let $u_i \in \mathcal{U}_h$, $\bar{v} \in \mathcal{V}_h$ and let*

$$v_i \in \arg\min\{\mathcal{F}_h(u_i, v) + \tfrac{1}{2\tau_k} \|v - \bar{v}\|^2_{\mathcal{V}_h} : v \in \mathcal{V}_h\}. \tag{6.15}$$

Then, there exists a constant $C_h > 0$, independent of u_i and $\bar{v}$ but depending on $h > 0$, such that

$$\|v_1 - v_2\|_{\mathcal{V}_h} \le \tau_k C_h (\|u_1\|_{H^1} + \|u_2\|_{H^1}) \|u_1 - u_2\|_{H^1}. \tag{6.16}$$

Proof. In view of (6.15), for $i = 1, 2$ the following equality holds:

$$\partial_v \mathcal{F}_h(u_i, v_i)[v_2 - v_1] + \tfrac{1}{\tau_k} \int_\Omega \Pi_h((v_i - \bar{v})(v_2 - v_1)) \,\mathrm{d}x = 0. \tag{6.17}$$

Subtracting the equality (6.17) for $i = 1$ from the one for $i = 2$, we obtain that

$$(\partial_v \mathcal{F}_h(u_2, v_2) - \partial_v \mathcal{F}_h(u_1, v_1))[v_2 - v_1] + \tfrac{1}{\tau_k} \|v_2 - v_1\|^2_{\mathcal{V}_h} = 0.$$

Adding and subtracting the term $\partial_v \mathcal{F}_h(u_1, v_2)[v_2 - v_1]$ and rearranging the terms, we deduce that

$$\tfrac{1}{\tau_k} \|v_2 - v_1\|^2_{\mathcal{V}_h} + (\partial_v \mathcal{F}_h(u_1, v_2) - \partial_v \mathcal{F}_h(u_1, v_1))[v_2 - v_1] = (\partial_v \mathcal{F}_h(u_1, v_2) - \partial_v \mathcal{F}_h(u_2, v_2))[v_2 - v_1]. \tag{6.18}$$

The left-hand side of (6.18) can be simply estimated by

$$\tfrac{1}{\tau_k} \|v_2 - v_1\|^2_{\mathcal{V}_h} + (\partial_v \mathcal{F}_h(u_1, v_2) - \partial_v \mathcal{F}_h(u_1, v_1))[v_2 - v_1] \ge \tfrac{1}{\tau_k} \|v_2 - v_1\|^2_{\mathcal{V}_h}. \tag{6.19}$$

Indeed

$$\begin{aligned}\partial_v \mathcal{F}_h(u_1, v_2)[v_2 - v_1] = &\int_\Omega \Pi_h(v_2(v_2 - v_1))\, \boldsymbol{\sigma}(u_1) : \boldsymbol{\epsilon}(u_1) \\ &\int_\Omega \nabla v_2 \cdot \nabla(v_2 - v_1) \,\mathrm{d}x + \int_\Omega \Pi_h((v_2 - 1)(v_2 - v_1)) \,\mathrm{d}x.\end{aligned}$$

and, similarly,

$$\begin{aligned}\partial_v \mathcal{F}_h(u_1, v_1)[v_2 - v_1] = &\int_\Omega \Pi_h(v_1(v_2 - v_1))\, \boldsymbol{\sigma}(u_1) : \boldsymbol{\epsilon}(u_1) \\ &\int_\Omega \nabla v_1 \cdot \nabla(v_2 - v_1) \,\mathrm{d}x + \int_\Omega \Pi_h((v_1 - 1)(v_2 - v_1)) \,\mathrm{d}x.\end{aligned}$$

Using the linearity of Π_h we easily get

$$(\partial_v \mathcal{F}_h(u_1, v_2) - \partial_v \mathcal{F}_h(u_1, v_1))[v_2 - v_1] = \int_\Omega \Pi_h((v_2 - v_1)^2)\, \boldsymbol{\sigma}(u_1) : \boldsymbol{\epsilon}(u_1)\, \mathrm{d}x + \int_\Omega |\nabla(v_2 - v_1)|^2 + \Pi_h((v_2 - v_1)^2)\, \mathrm{d}x \geq 0.$$

As for the right-hand side of (6.18), we have that

$$\begin{aligned}(\partial_v \mathcal{F}_h(u_1, v_2) - \partial_v \mathcal{F}_h(u_2, v_2))[v_2 - v_1] &= \int_\Omega \Pi_h(v_2(v_2 - v_1))\, (\boldsymbol{\sigma}(u_1) : \boldsymbol{\epsilon}(u_1) - \boldsymbol{\sigma}(u_2) : \boldsymbol{\epsilon}(u_2))\, \mathrm{d}x \\ &= \int_\Omega \Pi_h(v_2(v_2 - v_1))\, \boldsymbol{\sigma}(u_1 + u_2) : \boldsymbol{\epsilon}(u_1 - u_2)\, \mathrm{d}x \\ &\leq \|\Pi_h(v_2(v_2 - v_1))\|_{L^2} \|\boldsymbol{\sigma}(u_1 + u_2)\|_{L^\infty} \|\boldsymbol{\epsilon}(u_1 - u_2)\|_{L^2}.\end{aligned} \tag{6.20}$$

Denote

$$w = \Pi_h(v_2(v_2 - v_1)) = \sum_{l=1}^{N_h} \big(v_2(x_l)(v_2(x_l) - v_1(x_l))\big)\, \xi_l = \sum_{l=1}^{N_h} w(x_l)\, \xi_l.$$

Since $w \in \mathcal{V}_h$ we can use Lemma 6.1, hence $\|w\|_{L^2} \leq \|w\|_{\mathcal{V}_h}$. Note that

$$\|w\|^2_{\mathcal{V}_h} = \sum_K \int_K |\Pi_h(w^2)|\, \mathrm{d}x, \qquad \int_K |\Pi_h(w^2)|\, \mathrm{d}x = \sum_{i=1}^{3} w^2(x_i) D_{ii},$$

where the points x_i are the vertices of the element K and the weights D_{ii} are non-negative. By assumption $0 \leq v_2 \leq 1$, hence

$$\int_K |\Pi_h(w^2)|\, \mathrm{d}x \leq \sum_{i=1}^{3} (v_2(x_l) - v_1(x_l))^2 D_{ii} = \int_K |\Pi_h((v_2 - v_1)^2)|\, \mathrm{d}x.$$

Taking the sum for $K \in \mathcal{K}_h$ we get

$$\|\Pi_h(v_2(v_2 - v_1))\|^2_{\mathcal{V}_h} = \|w\|^2_{\mathcal{V}_h} \leq \|v_2 - v_1\|^2_{\mathcal{V}_h}.$$

Hence (6.20) yields $(\partial_v \mathcal{F}_h(u_1, v_2) - \partial_v \mathcal{F}_h(u_2, v_2))[v_2 - v_1] \leq \|v_2 - v_1\|_{\mathcal{V}_h} \|\boldsymbol{\sigma}(u_1 + u_2)\|_{L^\infty} \|\boldsymbol{\epsilon}(u_1 - u_2)\|_{L^2}$. Combining the previous inequality with (6.18)-(6.19) yields

$$\tfrac{1}{\tau_k} \|v_1 - v_2\|_{\mathcal{V}_h} \leq C(\|u_1\|_{W^{1,\infty}} + \|u_2\|_{W^{1,\infty}}) \|u_1 - u_2\|_{H^1}.$$

from which we get (6.16) by equivalence of norms in the finite dimensional space $\mathcal{U}_h$; the proof of the lemma is thus concluded. □

6.2. Finite-step algorithm

Let $T > 0$, $h > 0$, $g \in AC([0, T]; \mathcal{U}_h)$, $u_0 \in \mathcal{U}_h$ and $v_0 \in \mathcal{V}_h$ with $u(0) = g(0)$ on $\partial\Omega$ and $0 \leq v_0 \leq 1$ in Ω. We now present the finite-step alternate minimization scheme, whose convergence is discussed in Theorems 6.13 and 6.17.

For every $k \in \mathbb{N}\backslash\{0\}$ we define the time step $\tau_k := \frac{T}{k}$, and, for every $i \in \{0, \ldots, k\}$, we set the discrete time nodes $t_i^k := i\tau_k$.

We construct recursively the displacement $u_i^k \in \mathcal{U}_h$ and the phase-field functions $\tilde{v}_i^k, v_i^k \in \mathcal{V}_h$ at time t_i^k as follows: For $i = 0$ we set $u_0^k := u_0$ and $\tilde{v}_0^k = v_0^k := v_0$, while, for $i \geq 1$, we set $u_{i,0}^k := u_{i-1}^k$, $v_{i,0}^k := v_{i-1}^k$, and, for $j \geq 1$,

$$u_{i,j}^k := \arg\min \{\mathcal{E}_h(u, v_{i,j-1}^k) : u \in \mathcal{U}_h,\ u = g(t_i^k) \text{ on } \partial\Omega\}, \tag{6.21}$$

$$\tilde{v}_{i,j}^k := \arg\min \{\mathcal{F}_h(u_{i,j}^k, v) + \tfrac{1}{2\tau_k}\|v - v_{i-1}^k\|^2_{\mathcal{V}_h} : v \in \mathcal{V}_h\}. \tag{6.22}$$

As for $v_{i,j}^k$, we define it as the unique element of $\mathcal{V}_h$ satisfying

$$v_{i,j}^k(x_l) = \min\{\tilde{v}_{i,j}^k(x_l), v_{i-1}^k(x_l)\} \quad \text{for each vertex } x_l \in \Delta_h \text{ of the triangulation } \mathcal{T}_h. \tag{6.23}$$

We notice that the minimum problems (6.21) and (6.22) admit unique solutions. We fix a priori an upper bound $J \geq 1$ on the number of steps of the algorithm. However, in order to take into account the cases in which the algorithm stops according to a certain criterion, as it is in the applications, we set

$$u_i^k := u_{i,J_i^k}^k, \qquad \tilde{v}_i^k := \tilde{v}_{i,J_i^k}^k, \qquad v_i^k := v_{i,J_i^k}^k, \tag{6.24}$$

where $1 \leq J_i^k \leq J$. Note that this setting includes any stopping criterion forcing an upper bound (arbitrarily large) on the number of iterations.

Remark 6.12. The algorithm described by (6.21)–(6.23) is a finite-dimensional adaptation of the infinite-step scheme discussed in Section 5. In particular, the phase-field minimum problem (6.22) is unconstrained, while the irreversibility is taken into account in (6.23), where the constraint is imposed only in the nodes of the triangulation $\mathcal{T}_h$; note indeed that the function $\min\{\tilde{v}_{i,j}^k, v_{i-1}^k\}$ (where the minimum is pointwise in Ω) in general does not belong to $\mathcal{V}_h$.

As in Sections 4 and 5, we define the interpolation functions

$$v_k(t) := v_i^k + \frac{v_{i+1}^k - v_i^k}{\tau_k}(t - t_i^k) \qquad \text{for every } t \in [t_i^k, t_{i+1}^k), \tag{6.25}$$

$$\bar{u}_k(t) := u_i^k, \quad \bar{v}_k(t) := v_i^k, \quad \tilde{v}_k(t) := \tilde{v}_i^k, \quad t_k(t) := t_i^k \qquad \text{for every } t \in (t_{i-1}^k, t_i^k], \tag{6.26}$$

$$\underline{u}_k(t) := u_i^k, \quad \underline{v}_k(t) := v_i^k \qquad \text{for every } t \in [t_i^k, t_{i+1}^k). \tag{6.27}$$

The convergence result obtained in this subsection is the subject of the following theorem.

Theorem 6.13. *There exists a subsequence, not relabelled, of the pair $(\bar{u}_k, v_k)$ such that:*

(i) $v_k \rightharpoonup v$ *in* $H^1([0,T]; \mathcal{V}_h)$*;*
(ii) $v_k(t) \to v(t)$ *in* $\mathcal{V}_h$ *and* $\bar{u}_k(t) \to u(t)$ *in* $\mathcal{U}_h$ *for every* $t \in [0,T]$*;*
(iii) (u, v) *is a finite-dimensional unilateral L^2-gradient flow for $\mathcal{F}$, in the sense of Definition 6.8.*

The rest of this subsection is devoted to the proof of Theorem 6.13. We start by showing some properties of the functions defined in (6.22) and (6.23).

Proposition 6.14. *Let $\tilde{v}_{i,j}^k$ and $v_{i,j}^k$ be as in (6.22) and (6.23), respectively. Then $0 \leq v_{i,j}^k, \tilde{v}_{i,j}^k \leq 1$ in Ω.*

Proof. In view of (6.23), it is enough to prove $0 \leq \tilde{v}_{i,j}^k \leq 1$ on Ω assuming that $0 \leq v_{i-1}^k \leq 1$ (remember that we assume $0 \leq v_0 \leq 1$ in Ω). By contradiction, let us first suppose that $\tilde{v}_{i,j}^k \not\geq 0$. Let $x_l \in \Delta_h$ be such that $\tilde{v}_{i,j}^k(x_l) \leq \tilde{v}_{i,j}^k(x_m)$ for every $m = 1, \dots, N_h$. In particular, we have $\tilde{v}_{i,j}^k(x_l) < 0$. Let $\xi_l \geq 0$ be the l-th element

of the basis of $\mathcal{V}_h$ defined by (6.1). By (6.22),
we deduce that

$$0 = \partial_v \mathcal{F}_h(u^k_{i,j}, \tilde{v}^k_{i,j})[\xi_l] = \int_\Omega \Pi_h(\tilde{v}^k_{i,j}\xi_l)\, \boldsymbol{\sigma}(u^k_{i,j}) : \boldsymbol{\epsilon}(u^k_{i,j})\,\mathrm{d}x + \int_\Omega \nabla \tilde{v}^k_{i,j}\cdot\nabla\xi_l\,\mathrm{d}x \\ - \int_\Omega \Pi_h\big((1-\tilde{v}^k_{i,j})\xi_l\big)\,\mathrm{d}x + \tfrac{1}{\tau_k}\int_\Omega \Pi_h\big((\tilde{v}^k_{i,j} - v^k_{i-1})\xi_l\big)\,\mathrm{d}x. \tag{6.28}$$

Since $\tilde{v}^k_{i,j}(x_l) < 0$ and $\xi_l(x_m) = \delta_{ml}$, we have

$$\Pi_h(\tilde{v}^k_{i,j}\xi_l) = \sum_{m=1}^{N_h}(\tilde{v}^k_{i,j}(x_m)\xi_l(x_m))\xi_m = \tilde{v}^k_{i,j}(x_l)\xi_l \le 0, \quad \Pi_h\big((\tilde{v}^k_{i,j} - v^k_{i-1})\xi_l\big) \le 0$$

$$\Pi_h\big((1-\tilde{v}^k_{i,j})\xi_l\big) = (1-\tilde{v}^k_{i,j}(x_l))\xi_l \ge 0, \quad \text{with } \int_\Omega \Pi_h\big((1-\tilde{v}^k_{i,j})\xi_l\big)\,\mathrm{d}x > 0.$$

Hence, from (6.28) we get

$$\int_\Omega \nabla\tilde{v}^k_{i,j}\cdot\nabla\xi_l\,\mathrm{d}x = -\int_\Omega \Pi_h(\tilde{v}^k_{i,j}\xi_l)\,\boldsymbol{\sigma}(u^k_{i,j}) : \boldsymbol{\epsilon}(u^k_{i,j})\,\mathrm{d}x \\ + \int_\Omega \Pi_h\big((1-\tilde{v}^k_{i,j})\xi_l\big)\,\mathrm{d}x - \tfrac{1}{\tau_k}\int_\Omega \Pi_h\big((\tilde{v}^k_{i,j} - v^k_{i-1})\xi_l\big)\,\mathrm{d}x > 0. \tag{6.29}$$

On the other hand, writing $\tilde{v}^k_{i,j} = \sum_{m=1}^{N_h}\tilde{v}^k_{i,j}(x_m)\xi_m$, by direct computation we get

$$\begin{aligned}\int_\Omega \nabla\tilde{v}^k_{i,j}\cdot\nabla\xi_l\,\mathrm{d}x &= \sum_{m=1}^{N_h}\tilde{v}^k_{i,j}(x_m)\int_\Omega\nabla\xi_m\cdot\nabla\xi_l\,\mathrm{d}x \\ &= \tilde{v}^k_{i,j}(x_l)\sum_{m=1}^{N_h}\int_\Omega\nabla\xi_m\cdot\nabla\xi_l\,\mathrm{d}x + \sum_{m=1}^{N_h}\big(\tilde{v}^k_{i,j}(x_m) - \tilde{v}^k_{i,j}(x_l)\big)\int_\Omega\nabla\xi_m\cdot\nabla\xi_l\,\mathrm{d}x \\ &= \sum_{m=1}^{N_h}\big(\tilde{v}^k_{i,j}(x_m) - \tilde{v}^k_{i,j}(x_l)\big)\int_\Omega\nabla\xi_m\cdot\nabla\xi_l\,\mathrm{d}x \le 0,\end{aligned} \tag{6.30}$$

where, in the last equality, we have used (6.3), the fact that $\tilde{v}^k_{i,j}(x_l) \le \tilde{v}^k_{i,j}(x_m)$ for every $m = 1, \ldots, N_h$, and

$$\sum_{m=1}^{N_h}\int_\Omega\nabla\xi_m\cdot\nabla\xi_l\,\mathrm{d}x = \int_\Omega\nabla 1\nabla\xi_l\,\mathrm{d}x = 0.$$

Therefore, combining (6.29) and (6.30) we get a contradiction, and thus $\tilde{v}^k_{i,j} \ge 0$.
With a similar argument, we can also show that $\tilde{v}^k_{i,j} \le 1$. □

The following proposition is the discrete counterpart of Proposition 4.2 for the discrete unilateral L^2-slope $|\partial^-_v\mathcal{F}_h|_{\mathcal{V}_h}$.

Proposition 6.15. *Let $h > 0$, $k \in \mathbb{N}\backslash\{0\}$, u^k_i, $\tilde{v}^k_i$, and v^k_i be defined as in (6.21)–(6.24), for every $i \in \{1,\ldots,k\}$. Then*

$$\frac{\|v^k_i - v^k_{i-1}\|_{\mathcal{V}_h}}{\tau_k} = |\partial^-_v\mathcal{F}_h|_{\mathcal{V}_h}(u^k_i, \tilde{v}^k_i), \tag{6.31}$$

$$\partial_v\mathcal{F}_h(u^k_i, \tilde{v}^k_i)[v^k_i - v^k_{i-1}] = -|\partial^-_v\mathcal{F}_h|_{\mathcal{V}_h}(u^k_i, \tilde{v}^k_i)\|v^k_i - v^k_{i-1}\|_{\mathcal{V}_h}, \tag{6.32}$$

$$\partial_v\mathcal{F}_h(u^k_i, v^k_i)[v^k_i - v^k_{i-1}] \le \partial_v\mathcal{F}_h(u^k_i, \tilde{v}^k_i)[v^k_i - v^k_{i-1}]. \tag{6.33}$$

Note that in the continuum setting the counterpart of (6.33) holds with an identity.

Proof. Let us start with (6.31). In view of the definition of $\tilde{v}_i^k$, for every $\varphi \in \mathcal{V}_h$ it holds

$$\partial_v \mathcal{F}_h(u_i^k, \tilde{v}_i^k)[\varphi] + \tfrac{1}{\tau_k} \int_\Omega \Pi_h((\tilde{v}_i^k - v_{i-1}^k)\varphi)\, \mathrm{d}x = 0. \tag{6.34}$$

Therefore,

$$\begin{aligned} |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_i^k, \tilde{v}_i^k) &= \sup \{-\partial_v \mathcal{F}_h(u_i^k, \tilde{v}_i^k)[\varphi] : \ \varphi \in \mathcal{V}_h,\, \varphi \le 0,\, \|\varphi\|_{\mathcal{V}_h} \le 1\} \\ &= \sup \Big\{ \tfrac{1}{\tau_k} \int_\Omega \Pi_h\big((\tilde{v}_i^k - v_{i-1}^k)\varphi\big)\, \mathrm{d}x : \ \varphi \in \mathcal{V}_h,\, \varphi \le 0,\, \|\varphi\|_{\mathcal{V}_h} \le 1 \Big\}. \end{aligned} \tag{6.35}$$

In order to obtain (6.31) and (6.32), we will show that the supremum in the right-hand side of (6.35) is attained in $\varphi = (v_i^k - v_{i-1}^k)/\|v_i^k - v_{i-1}^k\|_{\mathcal{V}_h}$. By definition of Π_h we can write

$$\int_\Omega \Pi_h\big((\tilde{v}_i^k - v_{i-1}^k)\varphi\big)\, \mathrm{d}x = \sum_{l=1}^{N_h} \big(\tilde{v}_i^k(x_l) - v_{i-1}^k(x_l)\big)\varphi(x_l) \int_\Omega \xi_l \, \mathrm{d}x.$$

Hence, being $\varphi \le 0$, we can rewrite (6.35) as

$$\begin{aligned} |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_i^k, \tilde{v}_i^k) = \sup \Big\{ \tfrac{1}{\tau_k} \int_\Omega \Pi_h\big((\tilde{v}_i^k - v_{i-1}^k)\varphi\big)\, \mathrm{d}x : \ &\varphi \in \mathcal{V}_h,\, \varphi \le 0,\, \|\varphi\|_{\mathcal{V}_h} \le 1, \\ &\varphi(x_l) = 0 \text{ if } x_l \in \mathit{\Delta}_h \text{ and } \tilde{v}_i^k(x_l) - v_{i-1}^k(x_l) > 0 \Big\}. \end{aligned} \tag{6.36}$$

Remember that $v_i^k(x_l) = \min\{\tilde{v}_i^k(x_l), v_{i-1}^k(x_l)\}$ in each vertex $x_l \in \mathit{\Delta}_h$. Hence, for every $\varphi \in \mathcal{V}_h$ satisfying the constraints in (6.36) we have

$$\Pi_h\big((\tilde{v}_i^k - v_{i-1}^k)\varphi\big) = \Pi_h\big((v_i^k - v_{i-1}^k)\varphi\big), \tag{6.37}$$

which implies, together with (6.36),

$$\begin{aligned} |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_i^k, \tilde{v}_i^k) = \sup \Big\{ \tfrac{1}{\tau_k} \int_\Omega \Pi_h\big((v_i^k - v_{i-1}^k)\varphi\big)\, \mathrm{d}x : \ &\varphi \in \mathcal{V}_h,\, \varphi \le 0,\, \|\varphi\|_{\mathcal{V}_h} \le 1, \\ &\varphi(x_l) = 0 \text{ if } x_l \in \mathit{\Delta}_h \text{ and } \tilde{v}_i^k(x_l) - v_{i-1}^k(x_l) > 0 \Big\}. \end{aligned} \tag{6.38}$$

By (6.23) and (6.24) we know that $v_i^k(x_l) = v_{i-1}^k(x_l)$ for every vertex $x_l \in \mathit{\Delta}_h$ such that $\tilde{v}_i^k(x_l) - v_{i-1}^k(x_l) > 0$. Thus, equality (6.38) can be rewritten in the simpler form

$$|\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_i^k, \tilde{v}_i^k) = \sup \Big\{ \tfrac{1}{\tau_k} \int_\Omega \Pi_h\big((v_i^k - v_{i-1}^k)\varphi\big)\, \mathrm{d}x : \ \varphi \in \mathcal{V}_h,\, \varphi \le 0,\, \|\varphi\|_{\mathcal{V}_h} \le 1 \Big\}. \tag{6.39}$$

It is then easy to see that the supremum in (6.39) is actually attained for $\varphi = (v_i^k - v_{i-1}^k)/\|v_i^k - v_{i-1}^k\|_{\mathcal{V}_h}$.
In order to prove (6.33), we need to estimate each term of

$$\begin{aligned} \partial_v \mathcal{F}_h(u_i^k, \tilde{v}_i^k)[v_i^k - v_{i-1}^k] = &\int_\Omega \Pi_h\big(\tilde{v}_i^k(v_i^k - v_{i-1}^k)\big)\, \boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k)\, \mathrm{d}x + \int_\Omega \nabla \tilde{v}_i^k \cdot \nabla(v_i^k - v_{i-1}^k)\, \mathrm{d}x \\ &- \int_\Omega \Pi_h\big((1 - \tilde{v}_i^k)(v_i^k - v_{i-1}^k)\big)\, \mathrm{d}x =: I_1 + I_2 + I_3. \end{aligned} \tag{6.40}$$

Let us start with I_1. By the same argument used in (6.37), we have that

$$\Pi_h\big(\tilde{v}_i^k(v_i^k - v_{i-1}^k)\big) = \Pi_h\big(v_i^k(v_i^k - v_{i-1}^k)\big),$$

so that

$$I_1 = \int_\Omega \Pi_h\big(v_i^k(v_i^k - v_{i-1}^k)\big)\,\boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k)\,\mathrm{d}x. \tag{6.41}$$

In a similar way, we can also show that

$$I_3 = -\int_\Omega \Pi_h\big((1 - v_i^k)(v_i^k - v_{i-1}^k)\big)\,\mathrm{d}x. \tag{6.42}$$

As for I_2, we write the scalar product in terms of the basis $\{\xi_l\}_{l=1}^{N_h}$ of $\mathcal{V}_h$, so that

$$\begin{aligned} I_2 &= \sum_{l,m=1}^{N_h} \tilde{v}_i^k(x_l)(v_i^k(x_m) - v_{i-1}^k(x_m)) \int_\Omega \nabla\xi_l \cdot \nabla\xi_m\,\mathrm{d}x \\ &= \sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m)) \sum_{l=1}^{N_h} \tilde{v}_i^k(x_l) \int_\Omega \nabla\xi_l \cdot \nabla\xi_m\,\mathrm{d}x \\ &= \sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m))\tilde{v}_i^k(x_m) \int_\Omega \nabla\xi_m \cdot \nabla\xi_m\,\mathrm{d}x \\ &\qquad + \sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m)) \sum_{\substack{l=1 \\ l\neq m}}^{N_h} \tilde{v}_i^k(x_l) \int_\Omega \nabla\xi_l \cdot \nabla\xi_m\,\mathrm{d}x. \end{aligned} \tag{6.43}$$

By construction we have that $v_i^k \leq \tilde{v}_i^k$ and $v_i^k \leq v_{i-1}^k$. Therefore, by (6.3) we easily get

$$\sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m)) \sum_{\substack{l=1 \\ l\neq m}}^{N_h} \tilde{v}_i^k(x_l) \int_\Omega \nabla\xi_l \cdot \nabla\xi_m \mathrm{d}x \geq \sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m)) \sum_{\substack{l=1 \\ l\neq m}}^{N_h} v_i^k(x_l) \int_\Omega \nabla\xi_l \cdot \nabla\xi_m \mathrm{d}x.$$

Moreover, arguing as in (6.37), we deduce that

$$\sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m))\tilde{v}_i^k(x_m) \int_\Omega \nabla\xi_m \cdot \nabla\xi_m\,\mathrm{d}x = \sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m))v_i^k(x_m) \int_\Omega \nabla\xi_m \cdot \nabla\xi_m\,\mathrm{d}x.$$

Hence, we obtain

$$I_2 \geq \sum_{m=1}^{N_h} (v_i^k(x_m) - v_{i-1}^k(x_m)) \sum_{l=1}^{N_h} v_i^k(x_l) \int_\Omega \nabla\xi_l \cdot \nabla\xi_m\,\mathrm{d}x = \int_\Omega \nabla v_i^k \cdot \nabla(v_i^k - v_{i-1}^k)\,\mathrm{d}x. \tag{6.44}$$

Finally, inequalities (6.40)–(6.42) and (6.44) imply that

$$\begin{aligned} \partial_v \mathcal{F}_h(u_i^k, \tilde{v}_i^k)[v_i^k - v_{i-1}^k] \geq &\int_\Omega \Pi_h\big(v_i^k(v_i^k - v_{i-1}^k)\big)\,\boldsymbol{\sigma}(u_i^k) : \boldsymbol{\epsilon}(u_i^k)\,\mathrm{d}x + \int_\Omega \nabla v_i^k \cdot \nabla(v_i^k - v_{i-1}^k)\,\mathrm{d}x \\ &- \int_\Omega \Pi_h\big((1 - v_i^k)(v_i^k - v_{i-1}^k)\big)\,\mathrm{d}x = \partial_v \mathcal{F}_h(u_i^k, v_i^k)[v_i^k - v_{i-1}^k], \end{aligned}$$

which is exactly (6.33). This concludes the proof of the proposition. □

In the following proposition, we obtain the finite-dimensional counterpart of the energy inequalities (4.15) and (5.12), as well as some uniform bounds for the sequences (6.25)–(6.27).

Proposition 6.16. *Let $h > 0$. Then, the following facts hold:*

(a) *The sequence v_k is bounded in $L^\infty([0,T];\mathcal{V}_h)$ and in $H^1([0,T];\mathcal{V}_h)$;*
(b) *The sequences $\bar{v}_k$, $\tilde{v}_k$, $\underline{v}_k$ are bounded in $L^\infty([0,T];H^1(\Omega))$;*
(c) *The sequences $\bar{u}_k$ and $\underline{u}_k$ are bounded in $L^\infty([0,T];\mathcal{U}_h)$;*
(d) *There exists $R_k \to 0^+$ as $k \to +\infty$ such that for every $t \in [0,T]$ it holds*

$$\begin{aligned}\mathcal{F}_h(\bar{u}_k(t),\bar{v}_k(t)) \leq \mathcal{F}_h(u_0,v_0) - \tfrac{1}{2}\int_0^{t_k(t)} \|\dot{v}_k(s)\|^2_{\mathcal{V}_h} + |\partial_v^- \mathcal{F}_h|^2_{\mathcal{V}_h}(\bar{u}_k(s),\tilde{v}_k(s))\,\mathrm{d}s \\ + \int_0^{t_k(t)} \mathcal{P}_h(\underline{u}_k(s),\underline{v}_k(s),\dot{g}(s))\,\mathrm{d}s + R_k.\end{aligned} \tag{6.45}$$

In particular the energy $\mathcal{F}_h(\bar{u}_k(t),\bar{v}_k(t))$ is uniformly bounded, w.r.t. t and k.

Proof. The argument used to prove this proposition is similar to the one presented in Propositions 4.4 and 5.2. We show here where to apply the estimates shown in Lemmas 6.10 and 6.11 and in Proposition 6.15.

Let us fix $k \in \mathbb{N}\setminus\{0\}$, $i \in \{1,\dots,k\}$, and $t \in (t^k_{i-1}, t^k_i]$. By convexity of $\mathcal{F}_h(u^k_i,\cdot)$, we have that

$$\mathcal{F}_h(u^k_i, v^k_{i-1}) \geq \mathcal{F}_h(u^k_i, v^k_i) + \partial_v \mathcal{F}_h(u^k_i, v^k_i)[v^k_{i-1} - v^k_i].$$

In view of (6.33), we can continue with

$$\mathcal{F}_h(u^k_i, v^k_{i-1}) \geq \mathcal{F}_h(u^k_i, v^k_i) + \partial_v \mathcal{F}_h(u^k_i, \tilde{v}^k_i)[v^k_{i-1} - v^k_i].$$

Taking into account (6.31) and (6.32), we deduce that

$$\mathcal{F}_h(u^k_i, v^k_{i-1}) \geq \mathcal{F}_h(u^k_i, v^k_i) + \tfrac{1}{2}\int_{t^k_{i-1}}^{t^k_i} \|\dot{v}_k(s)\|^2_{\mathcal{V}_h} + |\partial_v^- \mathcal{F}_h|^2_{\mathcal{V}_h}(\bar{u}_k(s),\tilde{v}_k(s))\,\mathrm{d}s. \tag{6.46}$$

With the same argument used in (5.14)–(5.16), we get that

$$\begin{aligned}\mathcal{F}_h(u^k_i, v^k_{i-1}) \leq \mathcal{F}_h(u^k_{i-1}, v^k_{i-1}) + \int_{t^k_{i-1}}^{t^k_i} \mathcal{P}_h(\underline{u}_k(s),\underline{v}_k(s),\dot{g}(s))\,\mathrm{d}s + C\|g(t^k_i) - g(t^k_{i-1})\|^2_{H^1} \\ + C\|u^k_i - u^k_{i,1}\|^2_{H^1},\end{aligned}$$

for some constant $C > 0$ depending only on the stiffness tensor $\mathbf{C}$. Thanks to Lemma 6.10, the previous inequality becomes

$$\begin{aligned}\mathcal{F}_h(u^k_i, v^k_{i-1}) \leq \mathcal{F}_h(u^k_{i-1}, v^k_{i-1}) + \int_{t^k_{i-1}}^{t^k_i} \mathcal{P}_h(\underline{u}_k(s),\underline{v}_k(s),\dot{g}(s))\,\mathrm{d}s + C\|g(t^k_i) - g(t^k_{i-1})\|^2_{H^1} \\ + C\|v^k_{i,J^k_i-1} - v^k_{i-1}\|^2_{\mathcal{V}_h}\end{aligned} \tag{6.47}$$

If $J^k_i \geq 2$ then we write

$$\begin{aligned}\mathcal{F}_h(u^k_i, v^k_{i-1}) \leq \mathcal{F}_h(u^k_{i-1}, v^k_{i-1}) + \int_{t^k_{i-1}}^{t^k_i} \mathcal{P}_h(\underline{u}_k(s),\underline{v}_k(s),\dot{g}(s))\,\mathrm{d}s + C\|g(t^k_i) - g(t^k_{i-1})\|^2_{H^1} \\ + C\|v^k_i - v^k_{i,J^k_i-1}\|^2_{\mathcal{V}_h} + C\|v^k_i - v^k_{i-1}\|^2_{\mathcal{V}_h}.\end{aligned} \tag{6.48}$$

Applying Lemma 6.11 to u^k_i, u^k_{i,J^k_i-1} and $\bar{v} = v^k_{i-1}$ we deduce that

$$\|v^k_i - v^k_{i,J^k_i-1}\|_{\mathcal{V}_h} \leq \|\tilde{v}^k_i - \tilde{v}^k_{i,J^k_i-1}\|_{\mathcal{V}_h} \leq C_h \tau_k (\|u^k_i\|_{H^1} + \|u^k_{i,J^k_i-1}\|_{H^1})\|u^k_i - u^k_{i,J^k_i-1}\|_{H^1}.$$

Note that by minimality

$$\mathcal{E}_h(u_i^k, v_{i,J_i^k-1}^k) \le C(1+\eta)\|g_i^k\|_{H^1} \qquad \text{and} \qquad \mathcal{E}_h(u_{i,J_i^k-1}^k, v_{i,J_i^k-2}^k) \le C(1+\eta)\|g_i^k\|_{H^1}.$$

Since g_i^k is bounded in H^1 uniformly with respect to i and k, by Korn-Poincaré inequality we get $\|u_i^k\|_{H^1} + \|u_{i,J_i^k-1}^k\|_{H^1}$ is bounded uniformly with respect to i and k; hence

$$\|v_i^k - v_{i,J_i^k-1}^k\|_{\mathcal{V}_h} \le C_h\tau_k, \tag{6.49}$$

for some positive constant C_h independent of i and k. Then, for every $J_i^k \ge 1$ we obtain

$$\begin{aligned}\mathcal{F}_h(u_i^k, v_{i-1}^k) \le \mathcal{F}_h(u_{i-1}^k, v_{i-1}^k) + \int_{t_{i-1}^k}^{t_i^k} \mathcal{P}_h(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s + C\|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 \\ + C_h\tau_k^2 + C\|v_i^k - v_{i-1}^k\|_{\mathcal{V}_h}^2.\end{aligned} \tag{6.50}$$

Combining inequalities (6.46) and (6.50) and iterating over i, we get the estimate

$$\begin{aligned}&\mathcal{F}_h(\bar{u}_k(t), \bar{v}_k(t)) + \tfrac{1}{2}\int_0^{t_k(t)} \|\dot{v}_k(s)\|_{\mathcal{V}_h}^2 + |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}^2(\bar{u}_k(s), \tilde{v}_k(s))\,\mathrm{d}s \\ &\qquad \le \mathcal{F}_h(u_0, v_0) + \int_0^{t_k(t)} \mathcal{P}_h(\underline{u}_k(s), \underline{v}_k(s), \dot{g}(s))\,\mathrm{d}s + C\sum_{i=1}^{I} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 \\ &\qquad\quad + C\tau_k T + C\sum_{i=1}^{I}\|v_i^k - v_{i-1}^k\|_{\mathcal{V}_h}^2,\end{aligned}$$

where $I \in \{1, \dots, k\}$ is such that $t_k(t) = t_I^k$. Arguing as in (5.20), we obtain that u_k, $\bar{u}_k$, and $\underline{u}_k$ are bounded in $L^\infty([0,T];\mathcal{U}_h)$, v_k is bounded in $H^1([0,T];\mathcal{V}_h)$, and v_k, $\tilde{v}_k$, $\bar{v}_k$, and $\underline{v}_k$ are bounded in $L^\infty([0,T];H^1(\Omega))$. To conclude, it is enough to define

$$R_k := C\sum_{i=1}^{k} \|g(t_i^k) - g(t_{i-1}^k)\|_{H^1}^2 + C\tau_k T + C\sum_{i=1}^{k}\|v_i^k - v_{i-1}^k\|_{\mathcal{V}_h}^2.$$

By the regularity of the boundary datum $g \in AC([0,T];\mathcal{U}_h)$ and the boundedness of v_k in $H^1([0,T];\mathcal{V}_h)$, we get that $R_k \to 0$ as $k \to +\infty$. □

We are now in a position prove Theorem 6.13 performing the passage to the time-continuous limit of the sequences of interpolation functions defined in (6.25)–(6.27).

Proof of Theorem 6.13. In view of the bounds (a) and (b) in Proposition 6.16, there exists a function $v \in H^1([0,T];\mathcal{V}_h)$ such that, up to a subsequence, $v_k \rightharpoonup v$ weakly in $H^1([0,T];\mathcal{V}_h)$. This implies that $v_k(t) \to v(t)$ in $\mathcal{V}_h$ for every $t \in [0,T]$ and that $v \in L^\infty([0,T];\mathcal{V}_h)$ (remember that in the finite-dimensional setting weak and strong topologies are equivalent). It is also easy to see that v satisfies the irreversibility condition (b) of Definition 6.8. Since, by construction,

$$\|\underline{v}_k(t) - v_k(t)\|_{\mathcal{V}_h} \le \tau_k^{1/2}\Big(\int_0^T \|\dot{v}_k(s)\|_{\mathcal{V}_h}^2\,\mathrm{d}s\Big)^{1/2} \qquad \text{for every } t \in [0,T],$$

we have that $\underline{v}_k(t) \to v(t)$ in $\mathcal{V}_h$ for $t \in [0,T]$. In a similar way, we also get that $\bar{v}_k(t) \to v(t)$ in $\mathcal{V}_h$ for every $t \in [0,T]$. Moreover, by (6.22) and by Proposition 6.16, we get

$$\|\tilde{v}_k(t) - \underline{v}_k(t)\|_{\mathcal{V}_h}^2 \le 2\tau_k\,\mathcal{F}_h(\bar{u}_k(t), \underline{v}_k(t)) \le C\tau_k,$$

for some positive constant C independent of k. Therefore, $\tilde{v}_k(t) \to v(t)$ in $\mathcal{V}_h$ for every $t \in [0,T]$.

As for the sequences $\bar{u}_k$ and $\underline{u}_k$, by (c) of Proposition 6.16 we have that for every $t \in [0,T]$ there exists $u(t) \in \mathcal{U}_h$ such that, up to a subsequence, $\bar{u}_k(t) \to u(t)$ in $\mathcal{U}_h$. Applying Lemma 3.2 from [1], we can prove that the converging subsequence does not depend on $t \in [0,T]$, that $\underline{u}_k(t) \to u(t)$ in $\mathcal{U}_h$ for every $t \in [0,T]$, and that the pair $(u(t), v(t))$ satisfies the displacement equilibrium condition (c) of Definition 6.8.

Since $v \in H^1([0,T]; \mathcal{V}_h)$, by continuous dependence for the displacement, see Lemma 6.10, we easily deduce the time regularity of u, that is, $u \in C([0,T]; \mathcal{U}_h)$.

It remains to prove the energy balance (d) of Definition 6.8. Applying (a) of Lemma 6.7 and Fatou Lemma, we can pass to the lim inf as $k \to +\infty$ in the energy estimate (6.45), obtaining the inequality

$$\begin{aligned}\mathcal{F}_h(u(t), v(t)) \le \mathcal{F}_h(u(0), v(0)) - \tfrac{1}{2} \int_0^t \|\dot{v}_h(s)\|^2_{\mathcal{V}_h} + |\partial_v^- \mathcal{F}_h|_h^2(u(s), v(s))\, \mathrm{d}s \\ + \int_0^t \mathcal{P}_h(u(s), v(s), \dot{g}(s))\, \mathrm{d}s.\end{aligned} \tag{6.51}$$

The opposite inequality follows from Proposition 6.9. □

6.3. Convergence to the continuum

We conclude this paper by showing that any limit of a sequence of finite-dimensional unilateral L^2-gradient flow taken as the mesh becomes finer and finer (*i.e.*, as $h \to 0$) is itself a unilateral L^2-gradient flow. This is the content of the following theorem.

Theorem 6.17. *Let $T > 0$, $g \in AC([0,T]; W^{1,p}(\Omega; \mathbb{R}^2))$, $v_0 \in H^1(\Omega)$ with $0 \le v_0 \le 1$, and*

$$u_0 \in \arg\min\{\mathcal{E}(u, v_0) : u \in \mathcal{U},\ u = g(0) \text{ on } \partial\Omega\}.$$

Assume that there exist the sequences $v_{0,h} \in \mathcal{V}_h$ and $g_h \in AC([0,T]; \mathcal{U}_h)$ such that $0 \le v_{0,h} \le 1$, $v_{0,h} \to v_0$ in $H^1(\Omega)$ and $g_h \to g$ in $W^{1,1}([0,T]; \mathcal{U})$, as $h \to 0$. Let

$$u_{0,h} \in \arg\min\{\mathcal{E}_h(u, v_0) : u \in \mathcal{U}_h,\ u = g_h(0) \text{ on } \partial\Omega\}.$$

For every $h > 0$, let $(u_h, v_h)\colon [0,T] \to \mathcal{U}_h \times \mathcal{V}_h$ be a finite-dimensional unilateral L^2-gradient flow for the energy $\mathcal{F}_h$ with initial conditions $(u_{0,h}, v_{0,h})$ and boundary condition g_h.

Then, there exists a unilateral L^2-gradient flow $(u,v)\colon [0,T] \to \mathcal{U} \times \mathcal{V}$ with initial conditions (u_0, v_0) and boundary conditions g such that, up to a subsequence independent of $t \in [0,T]$, $u_h(t) \to u(t)$ in $\mathcal{U}$ and $v_h(t) \rightharpoonup v(t)$ weakly in $H^1(\Omega)$.

In order to prove Theorem 6.17, we first need to show a convergence property for energy and unilateral slope, when passing from the space-discrete to the space-continuous setting.

Lemma 6.18. *Let $u_h \in \mathcal{U}_h$, $v_h \in \mathcal{V}_h$, $u \in \mathcal{U}$ and $v \in \mathcal{V}$ with $0 \le v_h, v \le 1$. If $u_h \to u$ in $\mathcal{U}$ and $v_h \rightharpoonup v$ weakly in $H^1(\Omega)$, then*

$$\mathcal{F}(u,v) \le \liminf_{h \to 0} \mathcal{F}_h(u_h, v_h) \qquad \text{and} \qquad |\partial_v^- \mathcal{F}|(u,v) \le \liminf_{h \to 0} |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_h, v_h).$$

Proof. As a preliminary step, let us show that $\Pi_h(v_h^2) \to v^2$ in $L^1(\Omega)$. By classical interpolation estimates, *e.g.*, [17], Theorem 3.1.6, for every element $K \in \mathcal{K}_h$ we have

$$\|\Pi_h(w) - w\|_{L^1(K)} \le Ch|w|_{W^{1,1}(K)}. \tag{6.52}$$

Hence,

$$\|\Pi_h(v_h^2) - v_h^2\|_{L^1(K)} \le Ch|v_h^2|_{W^{1,1}(K)} \le 2\,Ch\|v_h\|_{L^\infty}|v_h|_{W^{1,1}(K)}.$$

As $0 \leq v_h \leq 1$ and v_h is bounded in $H^1(\Omega)$, we have $\|\Pi_h(v_h^2) - v_h^2\|_{L^1(\Omega)} \leq Ch$. Since $v_h^2 \to v^2$ in $L^1(\Omega)$ we get that $\Pi_h(v_h^2) \to v^2$ in $L^1(\Omega)$, and actually in $L^q(\Omega)$ for every $1 \leq q < \infty$.

Knowing that $v_h \rightharpoonup v$ in $H^1(\Omega)$, $\Pi_h(v_h^2) \to v^2$ in L^1 and $u_h \to u$ in $\mathcal{U}$ it is easy to check that $\mathcal{F}(u,v) \leq \liminf_{h\to 0} \mathcal{F}_h(u_h, v_h)$.

Let us fix $\varphi \in C^\infty(\bar{\Omega})$ with $\varphi \leq 0$ and $\|\varphi\|_{L^2} \leq 1$. Denote $\varphi_h = \Pi_h\varphi$. First, let us check that $\|\varphi_h\|_{\mathcal{V}_h} \to \|\varphi\|_{L^2}$. By classical interpolation estimates, $\varphi_h \to \varphi$ in $H^1(\Omega)$ and thus $\|\varphi_h\|_{L^2} \to \|\varphi\|_{L^2}$. Remember that $\|\varphi_h\|_{\mathcal{V}_h} = \|\Pi_h(\varphi_h^2)\|_{L^1}^{1/2}$. Moreover, using the interpolation estimate (6.52), we get

$$\|\Pi_h(\varphi_h^2) - \varphi_h^2\|_{L^1} \leq Ch\|\varphi_h\|_{L^\infty} |\varphi_h|_{W^{1,1}} \leq C'h,$$

for some $C' > 0$ independent of h. Hence,

$$\left|\|\varphi_h\|_{L^2}^2 - \|\varphi_h\|_{\mathcal{V}_h}^2\right| = \left|\|\varphi_h^2\|_{L^1} - \|\Pi_h(\varphi_h^2)\|_{L^1}\right| \leq C'h,$$

which implies that $\|\varphi_h\|_{\mathcal{V}_h} \to \|\varphi\|_{L^2}$. We now define the sequence

$$\hat{\varphi}_h := \begin{cases} \dfrac{\varphi_h}{\|\varphi_h\|_{\mathcal{V}_h}} & \text{if } \|\varphi\|_{L^2} = 1, \\ \varphi_h & \text{if } \|\varphi\|_{L^2} < 1. \end{cases}$$

Clearly, $\hat{\varphi}_h \in \mathcal{V}_h$ and $\hat{\varphi}_h \leq 0$ in Ω. Since $\varphi_h \to \varphi$ in $H^1(\Omega)$ and $\|\varphi_h\|_{\mathcal{V}_h} \to \|\varphi\|_{L^2}$, we also have that $\hat{\varphi}_h \to \varphi$ in $H^1(\Omega)$ and, for h small enough, that $\|\hat{\varphi}_h\|_{\mathcal{V}_h} \leq 1$. Hence $\hat{\varphi}_h$ is an admissible test function in (6.9) and

$$\partial_v \mathcal{F}_h(u_h, v_h)[\hat{\varphi}_h] = \int_\Omega \Pi_h(v_h\hat{\varphi}_h)\, \boldsymbol{\sigma}(u_h) : \boldsymbol{\epsilon}(u_h)\, \mathrm{d}x + \int_\Omega \nabla v_h \cdot \nabla \hat{\varphi}_h\, \mathrm{d}x - \int_\Omega \Pi_h\big((1 - v_h)\hat{\varphi}_h\big)\, \mathrm{d}x.$$

Using again the interpolation estimate (6.52) we get

$$\|\Pi_h(v_h\hat{\varphi}_h) - v_h\hat{\varphi}_h\|_{L^1} \leq Ch\|v_h\hat{\varphi}_h\|_{L^\infty} |v_h\hat{\varphi}_h|_{W^{1,1}} \leq C'h.$$

Since $v_h\hat{\varphi}_h \to v\varphi$ in $L^1(\Omega)$ we get that $\Pi_h(v_h\hat{\varphi}_h) \to v\varphi$ in $L^1(\Omega)$. Remembering that $v_h \rightharpoonup v$ in $H^1(\Omega)$ and that $u_h \to u$ in $\mathcal{U}$ it is easy to check that

$$\liminf_{h\to 0} |\partial_v^- \mathcal{F}_h|_{\mathcal{V}_h}(u_h, v_h) \geq \liminf_{h\to 0} -\partial_v \mathcal{F}_h(u_h, v_h)[\hat{\varphi}_h] = -\partial_v \mathcal{F}(u,v)[\varphi].$$

Passing to the supremum over φ we conclude the proof. □

We are now ready to prove Theorem 6.17.

Proof of Theorem 6.17. First, let us see, briefly, that $u_{0,h} \to u_0$ in $\mathcal{U}$. By minimality

$$\mathcal{E}_h(u_{0,h}, v_{0,h}) \leq \mathcal{E}_h(g_{0,h}, v_{0,h}) \leq C(1+\eta)\|g_{0,h}\|_{H^1}.$$

By Korn-Poincaré inequality $u_{0,h}$ is then bounded in $\mathcal{U}$. Up to subsequences, not relabelled, $u_{0,h} \rightharpoonup w$ in $\mathcal{U}$. Since $\Pi_h(v_{0,h}^2) \to v_0^2$ in L^1, using the Euler–Lagrange equations and the arguments of Lemma 3.3 it is not difficult to check that $w = u_0$ and that $u_{0,h} \to u_0$ in $\mathcal{U}$.

Let $(u_h, v_h)\colon [0,T] \to \mathcal{U}_h \times \mathcal{V}_h$ be as in the statement of the theorem. In view of Definition 6.8, we have that the sequence u_h is bounded in $L^\infty([0,T];\mathcal{U})$, while the sequence v_h is bounded in $L^\infty([0,T];H^1(\Omega))$ and in $H^1([0,T];L^2(\Omega))$. Therefore, there exists $v \in H^1([0,T];L^2(\Omega))$ such that $v_h \rightharpoonup v$ weakly in $H^1([0,T];L^2(\Omega))$. With the same argument used in the proof of Theorem 4.5 in Section 4, we can also show that $v_h(t) \rightharpoonup v(t)$ weakly in $H^1(\Omega)$ for every $t \in [0,T]$.

Applying Lemma 5.2 from [1], we have that there exists $u \in L^\infty([0,T];\mathcal{U})$ such that $u_h(t) \to u(t)$ in $\mathcal{U}$ for every $t \in [0,T]$ and such that the pair $(u(t), v(t))$ satisfies the displacement equilibrium property (c) of Definition 2.5. The time regularity of u follows by Lemma 3.3 and by the regularity of v.

Passing to the lim inf in the energy inequality (d) of Definition 6.8, by the convergences shown above, by the hypotheses of the theorem, and by Lemma 6.18, we immediately get that

$$\mathcal{F}(u(t), v(t)) \leq \mathcal{F}(u_0, v_0) - \tfrac{1}{2} \int_0^t \|\dot{v}(s)\|_2^2 + |\partial_v^- \mathcal{F}|^2(u(s), v(s))\,\mathrm{d}s + \int_0^t \mathcal{P}(u(s), v(s), \dot{g}(s))\,\mathrm{d}s.$$

The opposite inequality follows by Proposition 2.9. □

7. Numerics

In this section we present some numerical experiments to show the applicability of the discrete schemes studied in Section 6. Our aim is to compare the efficiency of the one-step and multi-step schemes, validating the choices and the analysis made in the previous theoretical sections. We also refer to Remark 5.1 for further discussions on the stability of the algorithms.

In the first simulations, we compare the evolutions obtained by one-step and multi-step algorithm in a geometrically simple setting. For both schemes, we will apply the alternate minimization algorithm of Section 6 with $J = 1$ and $J \gg 1$, respectively (J being the upper bound on the number of iterations). We will see that, from a computational point of view, the multi-step scheme with an appropriate stopping criterion is the right choice. Indeed, it provides good solutions in a large range of time steps, while the one-step scheme seems to fail in some cases, for instance when the propagation is very fast (in our experiments when the crack reaches the boundary of the domain). Then, we briefly show some simulations, based only on the multi-step scheme, in which the crack path kinks and curves. All the simulations are computed using the partial differential solver `FreeFem++`.

Before showing examples we fix some details, describing the general numerical framework and how the alternate minimization schemes are precisely implemented. The finite dimensional energy functional is given by

$$\mathcal{F}_{\varepsilon,h}(u,v) := \tfrac{1}{2} \int_\Omega \big(\Pi_h(v^2) + \eta_\varepsilon\big)\, \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u)\,\mathrm{d}x + G_c \varepsilon \int_\Omega |\nabla v|^2\,\mathrm{d}x + \frac{G_c}{4\varepsilon} \int_\Omega \Pi_h\big((1-v)^2\big)\,\mathrm{d}x, \tag{7.1}$$

where $0 < \eta_\varepsilon \ll \varepsilon \ll 1$ are approximating parameters (related to the Γ-convergence of the Ambrosio–Tortorelli functional [6]) and $G_c > 0$ is the toughness. Note that, for notational convenience, in the previous sections we have set, without loss of generality, $G_c = 1$ and $\varepsilon = \frac{1}{2}$. For the following numerical experiments we keep $G_c = 1$ fixed and use $\varepsilon = 5 \times 10^{-3}$ and $\eta_\varepsilon = 10^{-5}$. Assuming a homogeneous and isotropic material the stress tensor is of the following form:

$$\boldsymbol{\sigma}(u) = \lambda \mathrm{tr}(\boldsymbol{\epsilon}(u))\mathbf{I} + 2\mu\boldsymbol{\epsilon}(u),$$

where λ and μ denote the first and second Lamè coefficient, respectively. In what follows, we fix $\lambda = 0$ and $\mu = 1$. Therefore, in the two dimensional framework we get

$$\tfrac{1}{2}\boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(u) = \boldsymbol{\epsilon}(u) : \boldsymbol{\epsilon}(u) = (\partial_1 u_1)^2 + (\partial_2 u_2)^2 + \tfrac{1}{2}(\partial_1 u_2 + \partial_2 u_1)^2.$$

Given a final time $T > 0$, the interval $[0,T]$ is discretized by a constant time step $\tau = (T/k) > 0$ (for some $k \gg 1$) so that we set $t_0 := 0$ and $t_i := i\tau$ for $1 \leq i \leq k$. In both the algorithms we are going to define u_i and v_i as in (6.21)–(6.23). Actually, the phase-field minimization in (6.22) is then performed with respect to the functional

$$\mathcal{F}_{\varepsilon,h}(u,v) + \tfrac{\alpha}{\tau}\|v - v_{i-1}\|^2_{\mathcal{V}_h}, \quad \text{for } \alpha > 0.$$

Note that, without loss of generality, in the previous sections we used $\alpha = \frac{1}{2}$. For our purposes we set $\alpha = 10^{-3}$, indeed here the L^2-gradient flow is intended as vanishing viscosity approximation for a quasi-static BV-evolution, *e.g.* [33].

The alternate minimizing iterations, with respect to the index j, are interrupted when $\|v_{i,j} - v_{i,j-1}\|_{L^\infty}$ is smaller than a certain threshold, which we call TOL_v and fix to the value 2×10^{-3}. In practice, the assumption of a uniform bound for the number of iterations, as required in Section 6, is not imposed; indeed, we will see that the stopping criterion is always reached and that the number of iterations, at each time step, is decreasing as τ becomes smaller. Therefore, we expect, a posteriori, that the number of iterations is again uniformly bounded with respect to τ.

On most parts of the domain the phase-field function will be nearly constant. Only close to the crack it is expected to be very steep. To get an appropriate interpolation error, the mesh has to be very fine in the neighborhood of the crack, while it can be coarse elsewhere. Thus, we use an adaptive triangulation refining the mesh where it is necessary. Such approaches have been investigated accurately in [8, 14]. For our purposes, we regularly adapt the mesh in the iteration procedure using the standard routine `adaptmesh` provided from `Freefem++`, which uses a standard anisotropic second order interpolation error estimate. We fix the error tolerance $\mathrm{TOL}_{\mathrm{interpol}} = 10^{-3}$.

The complete algorithms in the way how we implement them for the presented experiments are given in detail by Algorithms 1 and 2 below. All the appearing parameters and variables, which are fixed throughout the section, are summarized in Table 1.

Algorithm 1 Implementation of the one-step scheme with mesh adaptation.

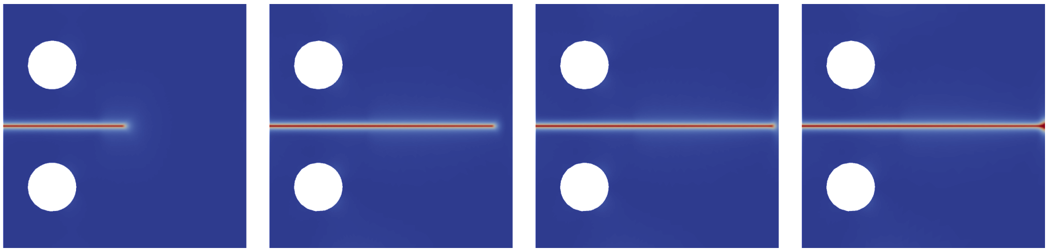
initialize v_0
for $i = 1$ to k **do**
  do
    $u_i \leftarrow \arg\min \{\mathcal{E}_h(u, v_{i-1}) : u \in \mathcal{U}_h, u = g(t_i) \text{ on } \partial\Omega\}$
    $v_i \leftarrow \arg\min \{\mathcal{F}_h(u_i, v) + \frac{\alpha}{\tau}\|v - v_{i-1}\|^2_{\mathcal{V}_h} : v \in \mathcal{V}_h\}$
    $v_i \leftarrow \min\{v_i, v_{i-1}\}$
    mesh adaption with error tolerance $\mathrm{TOL}_{\mathrm{interpol}}$
    $rel_{\mathrm{adapt}} \leftarrow$ "relative change of nodes"
  while $rel_{\mathrm{adapt}} > \mathrm{TOL}_{\mathrm{adapt}}$
end for

Algorithm 2 Implementation of the multi-step scheme with mesh adaptation.

initialize v_0
$\tilde{v}_0 \leftarrow v_0$
for $i = 1$ to k **do**
  do
    $j \leftarrow 0$
    do
      $j \leftarrow j + 1$
      $\tilde{u}_j \leftarrow \arg\min \{\mathcal{E}_h(u, \tilde{v}_{j-1}) : u \in \mathcal{U}_h, u = g(t_i) \text{ on } \partial\Omega\}$
      $\tilde{v}_j \leftarrow \arg\min \{\mathcal{F}_h(\tilde{u}_j, v) + \frac{\alpha}{\tau}\|v - v_{i-1}\|^2_{\mathcal{V}_h} : v \in \mathcal{V}_h\}$
      $\tilde{v}_j \leftarrow \min\{\tilde{v}_j, v_{i-1}\}$
    while $\|\tilde{v}_j - \tilde{v}_{j-1}\|_\infty > \mathrm{TOL}_v$ AND $j < 10$
    mesh adaption with error tolerance $\mathrm{TOL}_{\mathrm{interpol}}$
    $rel_{\mathrm{adapt}} \leftarrow$ "relative change of nodes"
    $\tilde{v}_0 \leftarrow \tilde{v}_j$
  while $rel_{\mathrm{adapt}} > \mathrm{TOL}_{\mathrm{adapt}}$ AND $\|\tilde{v}_j - \tilde{v}_{j-1}\|_\infty > \mathrm{TOL}_v$
  $v_i \leftarrow \tilde{v}_j$
  $u_i \leftarrow \tilde{u}_j$
end for

TABLE 1. Numerical parameters.

λ	μ	G_c	ε	η_ε	α	$\mathrm{TOL}_{\mathrm{interpol}}$	$\mathrm{TOL}_{\mathrm{adapt}}$	TOL_v
0	1	1	5×10^{-3}	10^{-5}	10^{-3}	10^{-3}	10^{-2}	2×10^{-3}



(A) $t = 1$ (B) $t = 2.73$ (C) $t = 2.896$ (D) $t = 2.901$

FIGURE 1. Phase-field at different times using the one step scheme with time step size $\tau = 10^{-3}$, the boundary condition g from (7.3) and the initial phase-field v_0 from (7.2).

Let us fix the domain, the boundary condition and the initial configuration. The domain Ω is given by $(0,1) \times (0,1) \setminus (B^+ \cup B^-)$, where B^+ is the (closed) ball with center $(0.2, 0.75)$ and radius 0.1, while B^- is the (symmetric) ball with center $(0.2, 0.25)$ and same radius. We will impose a boundary condition on $\partial B^+ \cup \partial B^-$. We consider a pre-existing crack given by the line segment with extrema $(0, 0.5)$ and $(0.4, 0.5)$. In the phase-field setting, the pre-crack is represented by the initial condition v_0. To this end we use the optimal profile functions rescaled by $\varepsilon > 0$. Precisely, we define

$$v_0(x,y) := \begin{cases} 1 - \exp\left(\frac{-|y-0.5|}{\varepsilon}\right) & \text{if } x < 0.4\,, \\ 1 - \exp\left(\frac{-\sqrt{(y-0.5)^2+(x-0.4)^2}}{\varepsilon}\right) & \text{if } x \geq 0.4\,. \end{cases} \tag{7.2}$$

7.1. One-step *vs.* multi-step

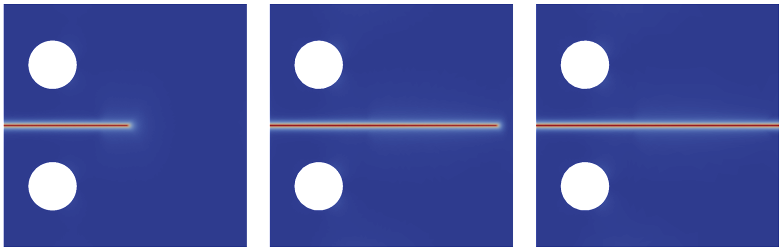
For the first example, we consider a symmetric setting, pulling the upper hole B^+ up and the lower hole B^- down monotonically in time. Concretely, we consider the Dirichlet condition

$$g(t) = \begin{cases} (0, t) & \text{on } \partial B^+, \\ (0, -t) & \text{on } \partial B^-. \end{cases} \tag{7.3}$$

Figures 1 and 2 show the phase field for the one-step scheme with $\tau = 10^{-3}$ and for the multi-step scheme with $\tau = 10^{-2}$, respectively.

As already mentioned in Remark 5.1, we expect the multi-step scheme to converge faster with respect to the time step τ, since in this algorithm we approximate a critical point of the energy functional for each time node. In order to investigate this phenomenon, we perform the simulation for several time step sizes and compare in Table 2 the time when the crack is completed, *i.e.*, when the domain is splitted in two subdomains and the elastic energy vanishes. Furthermore, in order to compare efficiency, in Table 2 we also show the number of iterations. Note that, due to the mesh adaptation, the number of iterations in the one-step scheme exceeds the number of time nodes.

We notice that, in the one-step scheme, for $\tau \geq 0.05$ we get qualitatively poor solution. Indeed, as shown in Figure 3, the crack spreads too much in the bulk.

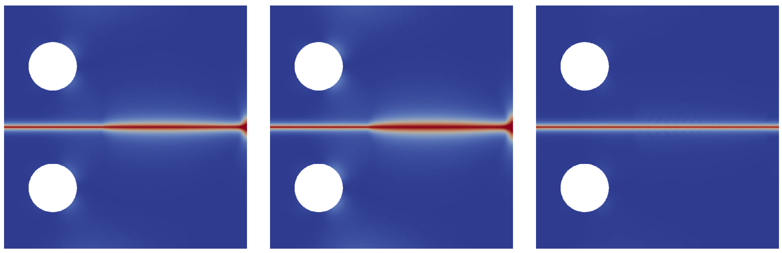


(A) $t = 1$ (B) $t = 2.73$ (C) $t = 2.74$

FIGURE 2. Phase-field at different times using the multi step scheme with time step size $\tau = 10^{-2}$, the boundary condition g from (7.3) and the initial phase-field v_0 from (7.2).

TABLE 2. Calculation time and the time t when the crack completes for different time step sizes τ.

Time step size τ		10^{-1}	5×10^{-2}	2×10^{-2}	10^{-2}	5×10^{-3}	2×10^{-3}	10^{-3}	5×10^{-4}
Time of crack completion	Single step	6	4.8	3.86	3.47	3.205	2.998	2.901	2.842
	Multi step	2.8	2.75	2.74	2.74	2.74	2.742	2.739	2.7415
Number of iterations	Single step	128	192	324	530	727	1555	2974	5934
	Multi step	3033	4735	7136	8932	10405	11851	12839	14003

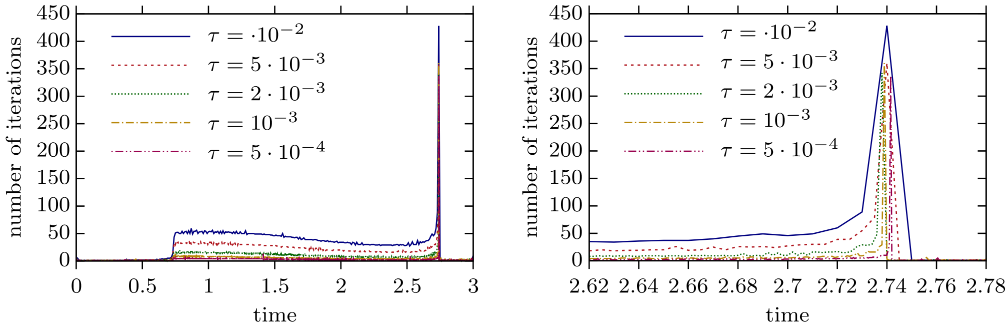


(A) One step scheme with $\tau = 0.05$. (B) One step scheme with $\tau = 0.1$. (C) Multi step scheme with $\tau = 0.1$.

FIGURE 3. Comparison of final phase-fields with big time step sizes.

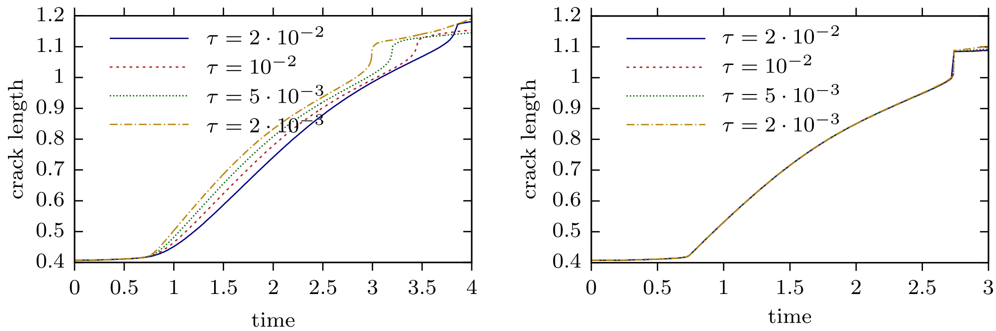
From Table 2 it is also clear that the time of crack completion decreases as the time step size decreases. On the contrary, with the multi-step scheme the crack always completes at around $t = 2.74$ and solutions are qualitatively very good even for $\tau = 0.1$. Moreover, even if the two algorithms give comparable results for τ of order 10^{-3} or 10^{-4}, a closer look (compare Fig. 1(B) with Fig. 2B) shows that the crucial difference between the two schemes comes up when the crack tip reaches the right boundary of the domain, *i.e.*, when the crack is expected to grow very fast. In particular (compare Fig. 1(D) with Fig. 2(C)), in this case the multi-step scheme produces a much sharper phase-field profile at the end of the crack. We notice that this fast behavior is close to a discontinuity in the quasi-static limit; indeed these results are consistent with those obtained for a toy model in Section 8 from [27]. The above observations indicate that we may in general expect that evolutions obtained with the one-step scheme converge, as $\tau \to 0$, much slower than evolutions obtained for the multi-step scheme.

In Figure 4 we plot, as a function of t_i, the number of iterations needed by the multi-step scheme to fulfill the stopping criterion. It is clear that the smaller the time step size the less iterations are needed. For τ small enough the multi-step scheme fulfills the stopping criterion more or less after one iteration until the time node t_i, where



(A) $t \in [0, 3]$. (B) Zoomed in for $t \in [2.62, 2.78]$.

FIGURE 4. Number of iterations, as a function of time, using the multi-step scheme for different time step sizes, for the boundary condition g from (7.3) and for the initial phase-field v_0 from (7.2) (color online).



(A) One Step Scheme. (B) Multi Step Scheme.

FIGURE 5. Crack length at each time step for different time step sizes, for the boundary condition g from (7.3) and for the initial phase-field v_0 from (7.2) (color online).

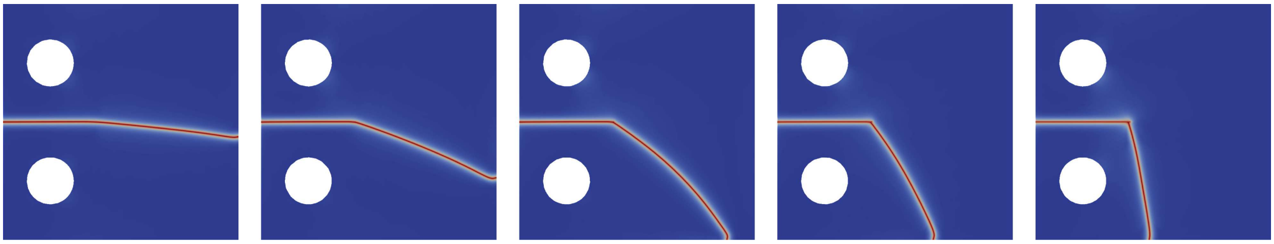
the last part of the crack appears almost instantaneously, is reached. At this node the number of iterations blows up. In Figure 5 we show the crack length as a function of time variable. The length of the fracture is estimated by the dissipative energy $\int_\Omega \frac{1}{4\varepsilon}(1-v)^2 + \varepsilon|\nabla v|^2 \, dx$. The physical maximum crack length of 1 is exceeded due to interpolation errors and diffusions of the phase field. We notice that, also in this plot, the last part of the crack is well visible as a jump in the evolution.

7.2. Asymmetric boundary condition

We extend our numerical experiments with a simulation of a brittle fracture evolution driven by an asymmetric boundary condition. The basic setting remains the same: we use the initial phase-field v_0 from (7.2) and force the boundary condition on the boundary of the two holes B^+ and B^-. The asymmetry appears by pulling the holes in a direction with a certain angle γ with respect to the vertical line. Precisely, we set

$$g(t) = \begin{cases} t\big(\sin(\gamma), \cos(\gamma)\big) & \text{on } \partial B^+, \\ -t\big(\sin(\gamma), \cos(\gamma)\big) & \text{on } \partial B^-. \end{cases} \tag{7.4}$$

We show the final phase-fields for different angles γ in Figure 6 computed with the multi-step scheme (see Algorithm 2) using the time step size $\tau = 0.01$. Note that for $\gamma = 0$ we are in the setting of the previous example.



(A) $\gamma = \frac{\pi}{18}$ (B) $\gamma = \frac{\pi}{6}$ (C) $\gamma = \frac{\pi}{4}$ (D) $\gamma = \frac{\pi}{3}$ (E) $\gamma = \frac{4\pi}{9}$

FIGURE 6. Phase-fields with asymmetric boundary condition $g(t)$ from (7.4) and different angles γ.

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