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THE OUTGOING TIME-HARMONIC ELECTROMAGNETIC WAVE IN A HALF-SPACE WITH NON-ABSORBING IMPEDANCE BOUNDARY CONDITION

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Abstract. We show existence and uniqueness of the outgoing solution for the Maxwell problem with an impedance boundary condition of Leontovich type in a half-space. Due to the presence of surface waves guided by an infinite surface, the established radiation condition differs from the classical one when approaching the boundary of the half-space. This specific radiation pattern is derived from an accurate asymptotic analysis of the Green's dyad associated to this problem.

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1. Introduction

Multiple structures of interest in electrical and optical engineering require the analysis of electromagnetic (EM) fields scattered by objects composed of perfect conductors protected by dielectric coatings or layers. For instance, coatings are used to increase energy conversion efficiency of photovoltaic and thermo-photovoltaic cells by exciting surface plasmonic polaritons [25,28,30]. This amounts to the creation of electromagnetic waves whose energy concentrates strongly along the layer. A similar phenomenon occurs when using periodic gratings and dielectric or open waveguides used in applications ranging from spectrography and astronomy to long-haul optic communications [22]. As the complexity of such structures increases, so does the need to model and computationally simulate them more accurately.

For wavelengths relatively larger than the coating thickness, one can approximate the dielectric/perfect conductor layer by an impedance boundary condition [2,8,9,14]. Depending on the accuracy of the approximation, most relevant physical features will be qualitatively portrayed while also simplifying the numerical simulation. Specifically, meshing a coating layer can create highly ill-conditioned elements and/or increase the number of degrees of freedom beyond any practical use.

Still, when dealing with unbounded domains and infinite boundaries, canonically modeled by either half-spaces or half-planes, not only the computational effort is challenging [26], but also fundamental mathematical

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properties such as existence and uniqueness of solutions remain unclear. Specifically, when particular relations between material and geometric parameters exist *Surface Waves* (SWs) may appear. Contrary to the standard decaying $\mathcal{O}(1/r)$ behaviour of waves scattered by bounded domains, SWs portray an exponential decrease in amplitude perpendicularly to the surfaces while along them the decay is as $\mathcal{O}(1/\sqrt{r})$ (*cf.* [13]). This has also been proven for transverse electric (TE) and transverse magnetic (TM) electromagnetic modes occurring in dielectric layered systems [19–21].

In this work, we study the existence and uniqueness of time-harmonic electromagnetic scattered waves by an unbounded impedance half-space allowing the presence of SWs. First-order impedance boundary conditions can be expressed as a complex proportionality relationship between electric and magnetic fields, denoted by $\mathbf{E}$ and $\mathbf{H}$, respectively, as follows:

$$\mathbf{E}_T(\mathbf{x}) - i\kappa\beta\,\mathbf{H}(\mathbf{x}) \times \widehat{\mathbf{n}} = \mathbf{f}_T(\mathbf{x}), \qquad \text{for all } \mathbf{x} \text{ on surface},$$

where κ is the wavenumber, $\widehat{\mathbf{n}}$ is the outward unit vector normal to the surface, $\mathbf{E}_T := \widehat{\mathbf{n}} \times (\mathbf{E} \times \widehat{\mathbf{n}})$ is the tangential electric field component, $\beta \neq 0$ a complex constant, and $\mathbf{f}_T$ being the excitation over the boundary. This problem is known as the *Leontovich boundary condition problem* [4], for which SWs appear when $\mathrm{Im}\,(\beta) = 0$, as we will see later on. There exist several works in the literature concerning the impedance problem, most of them are exterior problems for a bounded scatterer (see [5, 6, 11, 17]). For these cases, SWs may appear but they are confined to the bounded boundary of the obstacle, without producing a radiation pattern contribution towards infinity. Unbounded scatters where considered by Ammari and Latiri-Grouz for the absorbing case, *i.e.* without the presence of SWs (see [3]), proving existence and uniqueness of solutions and developing also an integral equation method for the numerical resolution of this problem.

In this paper, we consider an unbounded scatterer (a plane) together with Leontovich-type boundary conditions allowing the propagation of SWs. We prove existence and uniqueness of Maxwell's solutions when the boundary data has compact support. Uniqueness is proved by first obtaining the adequate radiation pattern coming from an exhaustive asymptotic analysis of the associated electric field Green's dyad, leading to an equivalent Silver-Müller type radiation condition which allows presence of SWs. This condition is in agreement with those found in [20, 21] for open layered waveguides. Next, we prove that the only solutions for the homogeneous problem satisfying our radiation condition are null fields. Finally, existence is proved by obtaining an integral representation of the solutions in terms of the Green's dyad and boundary data.

This work can be seen as a natural extension of the acoustic case [13] due to the resemblance in the solutions' behaviors, as well as the similarity in the general procedure to obtain the desired results. However, the electromagnetic situation can not be obtained as a corollary of the acoustic one. Even though the SW behavior may be polarized, the general Maxwell problem can not be decoupled into Helmholtz-like problems without the assumption of additional conditions that affect the outgoing radiation pattern.

The outline of the paper is the following: in Section 2 we describe the model problem and geometry. In Section 3 we introduce the associated Green's dyad, together with an accurate asymptotic analysis of it. Section 4 defines the radiation condition and proves that the Green's dyad of the previous section satisfies this radiation pattern. Finally, Section 5 contains the uniqueness and existence theorems of outgoing solutions satisfying the previous radiation pattern, followed by some concluding remarks. Appendices are provided for more technical details.

2. Model problem

Let $\mathbb{R}^3_+ := \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_3 > 0\}$ be the upper half-space and denote by $\Gamma := \partial\mathbb{R}^3_+ = \{\mathbf{x} \in \mathbb{R}^3_+ \ : \ x_3 = 0\}$ its boundary with outward normal $\widehat{\mathbf{n}} = -\widehat{\mathbf{k}} = (0, 0, -1)$. We assume $\mathbb{R}^3_+$ to be filled with a homogeneous, linear, lossless and isotropic medium characterized by its permittivity $\epsilon > 0$ and permeability $\mu > 0$ that we set equal to one for the sake of simplicity. At Γ, we assume a homogeneous constant impedance plane with admittance $\beta \in \mathbb{R} \setminus \{0\}$ being excited by an external source $\mathbf{f}_T = (f_1, f_2, 0)$ with compact support over Γ. For a given wavenumber $\kappa \in \mathbb{R}_+$, we want to find the outgoing time-harmonic EM fields $(\mathbf{E}, \mathbf{H})$ satisfying the homogeneous Maxwell equations in $\mathbb{R}^3_+$:

$$\begin{cases} \mathbf{curl}\,\mathbf{E} - i\kappa\,\mathbf{H} = \mathbf{0}, \\ \mathbf{curl}\,\mathbf{H} + i\kappa\,\mathbf{E} = \mathbf{0}, \end{cases} \tag{2.1}$$

together with the impedance boundary condition over Γ:

$$\mathbf{E}_T - i\kappa\beta\,\mathbf{H} \times \widehat{\mathbf{n}} = \mathbf{f}_T, \tag{2.2}$$

where $\mathbf{E}_T := \widehat{\mathbf{n}} \times (\mathbf{E} \times \widehat{\mathbf{n}})$ denotes the tangential component of the electric field.

2.1. Surface wave solutions

Standard plane wave solutions of (2.1) and (2.2), with $f_T \equiv 0$, are described in the the book of Balanis [7]. We find it instructive to show here that the boundary condition (2.2) allows the existence of surface wave-type solutions. We emphasize that none of these solutions satisfy our upcoming outgoing radiation pattern.

According to the sign of β, we find two families of (SWs) satisfying the homogeneous equations (2.1) and (2.2) with $\mathbf{f}_T = \mathbf{0}$. These waves are guided along the plane Γ and propagate in a given direction $\psi \in [0, 2\pi)$. Setting $\mathbf{p} = (\sin\psi, -\cos\psi, 0)$, we distinguish two cases:

- If $\beta > 0$, we obtain the SW fields $\mathbf{E}_+ = \mathbf{p}\ \mathrm{e}^{i\sqrt{\kappa^2+\beta^{-2}}(\cos\psi x_1 + \sin\psi x_2)}\mathrm{e}^{-\beta^{-1}x_3}$ and $\mathbf{H}_+ = \dfrac{1}{i\kappa}\mathbf{curl}\,\mathbf{E}_+$.
- If $\beta < 0$, we obtain the SW fields $\mathbf{H}_- = \mathbf{p}\ \mathrm{e}^{i\kappa\sqrt{1+\beta^2\kappa^2}(\cos\psi x_1 + \sin\psi x_2)}\mathrm{e}^{-|\beta|\kappa^2 x_3}$ and $\mathbf{E}_- = -\dfrac{1}{i\kappa}\mathbf{curl}\,\mathbf{H}_-$.

As we can observe, the wavenumbers of these SWs are different from κ-the wavenumber related to volume waves. Hence, due to the presence of SWs in the far field and in order to characterize the proper outgoing wave behavior, it is required to set an adequate radiation condition that differs from the standard Silver-Mülller radiation condition [24] when approaching the horizontal surface Γ.

Another important observation is that these particular SWs are polarized. Indeed, the couple $(\mathbf{E}_+, \mathbf{H}_+)$ is TE polarized while the couple $(\mathbf{E}_-, \mathbf{H}_-)$ is TM (*cf.* [7]). Additionally, when $\beta \to 0$, the SW fields $(\mathbf{E}_-, \mathbf{H}_-)$ tend to a standard couple of planar wave fields guided by the plane, while the SW fields $(\mathbf{E}_+, \mathbf{H}_+)$ vanish. This is consistent with the theory, since SWs have not been reported for $\beta = 0$.

2.2. Preliminary notation

For $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$, we introduce cylindrical and spherical coordinates centered at the source point $\mathbf{x} = (x_1, x_2, x_3)$:

$$\begin{cases} y_1 - x_1 = \rho\cos\varphi, \\ y_2 - x_2 = \rho\sin\varphi, \\ y_3 - x_3 = y_3 - x_3, \end{cases} \quad \text{and} \quad \begin{cases} y_1 - x_1 = r\sin\theta\cos\varphi, \\ y_2 - x_2 = r\sin\theta\sin\varphi, \\ y_3 - x_3 = r\cos\theta. \end{cases} \tag{2.3}$$

Due to the cylindrical nature of the problem, during almost the entire text, we will consider the fields $\mathbf{E}, \mathbf{H}$ expanded in the coordinate system generated by the unitary basis vectors $\{\widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{\varphi}}, \widehat{\mathbf{k}}\}$. In Section 5.3, we also consider the vector fields $\mathbf{E}, \mathbf{H}$ in the Cartesian coordinate system $\{\widehat{\mathbf{i}}, \widehat{\mathbf{j}}, \widehat{\mathbf{k}}\}$. In any case, we use the general notation $\mathbf{U}$ when referring to either $\mathbf{E}$ or $\mathbf{H}$ with

$$\mathbf{U} = U_\rho\,\widehat{\boldsymbol{\rho}} + U_\varphi\,\widehat{\boldsymbol{\varphi}} + U_3\,\widehat{\mathbf{k}} \qquad \text{or} \qquad \mathbf{U} = U_1\,\widehat{\mathbf{i}} + U_2\,\widehat{\mathbf{j}} + U_3\,\widehat{\mathbf{k}}.$$

The tangential invariance of the problem suggests the use of a tangential Fourier transform, which corresponds to a double Fourier transform in the Cartesian coordinates x_1, x_2. The hat notation $\widehat{f}(\xi_1, \xi_2, y_3)$ denotes the transform of $f(y_1, y_2, y_3)$. Introducing the change of coordinates $\xi_1 = \xi\,\cos(\varphi_\xi)$, $\xi_2 = \xi\,\sin(\varphi_\xi)$, we will use the tilde notation $\widetilde{\mathbf{U}}$ to denote the Fourier transform of any vector function $\mathbf{U}$ in the cylindrical coordinate system (ξ, φ_ξ, y_3), and write:

$$\widetilde{\mathbf{U}} = \widetilde{U}_\xi \,\widehat{\boldsymbol{\xi}} + \widetilde{U}_{\varphi_\xi}\, \widehat{\boldsymbol{\varphi}}_\xi + \widetilde{U}_3 \,\widehat{\mathbf{k}}. \tag{2.4}$$

When referring to a 3×3 matrix of sparse nature, we will consider it as a linear combination of the set of matrices $\{\mathbf{e}_{ij}\}_{i,j=1}^3$, where $\mathbf{e}_{ij}$ denotes the matrix which has exactly one non-zero element equal to one at the (i, j) position. This notation is related to the matrix structure, regardless of the coordinate system that is being used. Further conventions will be properly introduced when required.

3. Asymptotic analysis of the Green's dyad

Following [3] for the scattering problem in an absorbing plane, we deduce the Green's dyad for the Leontovitch problem (2.1) and (2.2) in $\mathbb{R}^3_+$ and derive its asymptotics to obtain radiation conditions.

Let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ be a fixed excitation point and $\mathbf{y} = (y_1, y_2, y_3)$ be any observation point in $\mathbb{R}^3_+$. The Green's dyad $\mathsf{G}(\mathbf{x}, \mathbf{y})$ associated with the homogeneous Leontovitch problem in a half-space, satisfies:

$$\mathbf{curl}_\mathbf{y}\mathbf{curl}_\mathbf{y}\mathsf{G}(\mathbf{x}, \mathbf{y}) - \kappa^2\mathsf{G}(\mathbf{x}, \mathbf{y}) = \delta_\mathbf{x}(\mathbf{y})\mathsf{I}, \quad \text{in } \mathbb{R}^3_+, \tag{3.1a}$$

$$\mathsf{G}(\mathbf{x}, \mathbf{y})_T + \beta\, \mathbf{curl}_\mathbf{y}\mathsf{G}(\mathbf{x}, \mathbf{y}) \times \widehat{\mathbf{k}} = \mathbf{0}, \quad \text{on } \Gamma, \tag{3.1b}$$

where $\mathsf{G}(\mathbf{x}, \mathbf{y})_T := \widehat{\mathbf{k}} \times \left(\mathsf{G}(\mathbf{x}, \mathbf{y}) \times \widehat{\mathbf{k}}\right)$, $\delta_\mathbf{x}$ is the Dirac delta at $\mathbf{x}$, and I is the 3×3 identity matrix.

3.1. The spectral Green's dyad

Due to invariance along the horizontal axes, a tangential Fourier transform is applied to (3.1) obtaining a system of ordinary differential equations (ODEs) that has an analytic solution in the spectral cylindrical coordinate system (ξ, φ_ξ, y_3). The solution is given by (see Appendix A):

$$\begin{cases} \widetilde{\mathsf{G}}(\xi, \varphi_\xi, y_3) \;=\; \dfrac{1}{4\pi\kappa^2}\left(\widetilde{\mathsf{G}}^-(\xi, \varphi_\xi, y_3)\, \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}|x_3-y_3|} + \widetilde{\mathsf{G}}^+(\xi, \varphi_\xi, y_3)\, \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}(x_3+y_3)}\right), \\ \text{where} \\ \widetilde{\mathsf{G}}^-(\xi, \varphi_\xi, y_3) := -\sqrt{\xi^2-\kappa^2}\,\mathbf{e}_{11} + \dfrac{1}{\sqrt{\xi^2-\kappa^2}}\left(\kappa^2\,\mathbf{e}_{22} + \xi^2\,\mathbf{e}_{33}\right) \\ \qquad\qquad -i\xi \operatorname{sign}(y_3 - x_3)\left(\mathbf{e}_{13} + \mathbf{e}_{31}\right), \\ \widetilde{\mathsf{G}}^+(\xi, \varphi_\xi, y_3) := \sigma^- \sqrt{\xi^2-\kappa^2}\,\mathbf{e}_{11} + \dfrac{1}{\sqrt{\xi^2-\kappa^2}}\left(\sigma^+\kappa^2\,\mathbf{e}_{22} + \sigma^-\xi^2\,\mathbf{e}_{33}\right) \\ \qquad\qquad -i\xi\sigma^-\left(\mathbf{e}_{13} - \mathbf{e}_{31}\right), \\ \text{with} \\ \sigma^- \;:=\; \dfrac{\sqrt{\xi^2-\kappa^2} - \beta\kappa^2}{\sqrt{\xi^2-\kappa^2} + \beta\kappa^2} \quad \text{and} \quad \sigma^+ := \dfrac{\beta\sqrt{\xi^2-\kappa^2}+1}{\beta\sqrt{\xi^2-\kappa^2}-1}. \end{cases} \tag{3.2}$$

Remark 3.1. The spectral dyads $\widetilde{\mathsf{G}}^-$ and $\widetilde{\mathsf{G}}^+$ seem to be singular at $\xi = \kappa$. However, such singularity becomes removable when considering the total representation of $\widetilde{\mathsf{G}}$. Stronger singularities related to SWs will be analyzed next.

3.2. Singularities

It can be seen from expression (3.2) that, in addition to the singularity mentioned in Remark 3.1, some dyadic elements of $\widetilde{\mathsf{G}}^+$, related to σ^- and σ^+, present a singularity depending on the sign of β. Notice that the case $\beta = 0$ is non-singular since σ^- becomes $+1$ and σ^+ becomes -1. The singularity is given by:

$$\xi_p = \begin{cases} \kappa\sqrt{1+\beta^2\kappa^2}, & \text{if } \beta < 0, \\ \sqrt{\kappa^2+\beta^{-2}}, & \text{if } \beta > 0. \end{cases} \tag{3.3}$$

Observe that $\xi_p > \kappa$. We will see later that this singularity is related to the wavenumber of the SW. Since the derivation of proper radiation conditions is based on the *limiting absorption principle* [18], we will perturb κ by a purely imaginary small value ε (*cf.* [13, 21]), writing

$$\kappa = \lim_{\varepsilon \to 0^+} \kappa_\varepsilon := \lim_{\varepsilon \to 0^+} \kappa + i\varepsilon, \tag{3.4}$$

which requires a unique definition of the square root:

$$\xi \longmapsto \sqrt{\xi^2 - \kappa_\varepsilon^2}. \tag{3.5}$$

Assuming that ξ is complex, the square root in (3.5) can be seen as the product between $\sqrt{\xi - \kappa_\varepsilon}$ and $\sqrt{\xi + \kappa_\varepsilon}$, defined respectively as

$$\arg(\xi - \kappa_\varepsilon) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right) \quad \text{and} \quad \arg(\xi + \kappa_\varepsilon) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Remark 3.2. If $\varepsilon = 0$ and $\mathrm{Im}(\xi) = 0$, then $\arg(\xi - \kappa) \in \{-\pi, 0\}$ and $\arg(\xi + \kappa) \in \{0, \pi\}$, which implies that $\sqrt{\xi^2 - \kappa^2} = -i\sqrt{\kappa^2 - \xi^2}$ when $|\xi| < \kappa$.

Denote by $\xi_p(\varepsilon)$ the complex-valued function defined from the singularity associated with σ^- and σ^+ in equation (3.2) after perturbing $\kappa > 0$ by some $\varepsilon > 0$ (see Eq. (3.4)). The following lemma holds true:

Lemma 3.3. *For each* $\beta \in \mathbb{R} \setminus \{0\}$*, there exists* $\varepsilon_0 > 0$ *sufficiently small, such that* $\arg(\xi_p(\varepsilon)) \in (0, \pi/2)$ *and* $\mathrm{Re}(\xi_p(\varepsilon)) > \kappa$*, for all* $0 < \varepsilon < \varepsilon_0$*.*

Proof. Clearly, $\xi_p(0) > \kappa$. Then, by continuity on ε, it is enough to show that $\lim_{\varepsilon \to 0^+} \mathrm{Im}\left(\frac{\partial \xi_p}{\partial \varepsilon}\right) > 0$. To do this, we perform an implicit differentiation over the relation $f(\xi, \varepsilon) = 0$ that defines ξ_p, where

$$f(\xi, \varepsilon) = \begin{cases} \sqrt{\xi^2 - \kappa_\varepsilon^2} + \beta \kappa_\varepsilon^2, & \text{if } \beta < 0, \\ \beta \sqrt{\xi^2 - \kappa_\varepsilon^2} - 1, & \text{if } \beta > 0. \end{cases}$$

Denoting by $\partial_j f$ the partial derivative of f with respect to the jth variable, $j = 1, 2$, we get:

$$\begin{aligned} &\text{If } \beta < 0, \text{ then} \quad \partial_1 f(\xi_p(0), 0) = \frac{\xi_p}{|\beta|\, \kappa^2} \quad \text{and} \quad \partial_2 f(\xi_p(0), 0) = -i \frac{(1 + 2\beta^2 \kappa)}{|\beta|\, \kappa}. \\ &\text{If } \beta > 0, \text{ then} \quad \partial_1 f(\xi_p(0), 0) = \beta^2 \sqrt{\kappa^2 + \beta^{-2}} \quad \text{and} \quad \partial_2 f(\xi_p(0), 0) = -i\beta^2\, \kappa. \end{aligned}$$

The proof is obtained from the implicit definition $\frac{\partial \xi_p}{\partial \varepsilon} = -\frac{\partial_2 f}{\partial_1 f}$. □

3.3. Integral representation of the Green's dyad

The spatial Green's dyad can be written as $\mathsf{G} = \mathsf{G}^{(1)} + \mathsf{G}^{(2)}$, with

$$\mathsf{G}^{(1)} := \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\kappa} \widetilde{\mathsf{G}}\, \mathrm{e}^{i\xi r \sin\theta \cos(\varphi_\xi - \varphi)}\, \xi \,\mathrm{d}\xi \,\mathrm{d}\varphi_\xi, \tag{3.6a}$$

$$\mathsf{G}^{(2)} := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_0^{2\pi} \int_{\kappa}^{+\infty} \widetilde{\mathsf{G}}_\varepsilon\, \mathrm{e}^{i\xi r \sin\theta \cos(\varphi_\xi - \varphi)}\, \xi \,\mathrm{d}\xi \,\mathrm{d}\varphi_\xi, \tag{3.6b}$$

and where (r, φ, θ) represents the spherical coordinates system defined in (2.3). In (3.6b), $\widetilde{\mathsf{G}}_\varepsilon$ denotes the dyad $\widetilde{\mathsf{G}}$ (see Eq. (3.2)) after perturbation by ε (see Sect. 3.2). Notice that in $\mathsf{G}^{(1)}(\mathbf{x}, \mathbf{y})$, due to the absence of singularities, there is no need to consider a limiting case when $\varepsilon \to 0^+$. In order to obtain the correct radiation conditions for the problem (2.1) and (2.2), we need to study the asymptotic behavior of the integral representation of the Green's dyad G when $r \to +\infty$.

3.4. Asymptotic analysis

The following proposition holds true (*cf.* [21], Sect. 2.2.1):

Proposition 3.4. *Let* $\mathsf{G}^{(1)}$ *be the component of the Green's dyad defined in* (3.6a). *Then* $\mathsf{G}^{(1)}$ *satisfies the following asymptotic behavior:*

$$\mathsf{G}^{(1)} = \Pi(\theta, \kappa, \beta, x_3)\frac{\mathrm{e}^{i\kappa r}}{4\pi r} + \mathcal{O}(r^{-2}), \quad \text{when} \quad 0 \leq \theta < \frac{\pi}{2} \quad \text{and} \quad r \to +\infty, \tag{3.7}$$

where Π *is a bounded matrix function (see Rem.* 3.5*) satisfying*

$$|\Pi(\theta, \kappa, \beta, x_3)| = \mathcal{O}(\cos\theta), \text{when } \theta \to \frac{\pi}{2}^{-}. \tag{3.8}$$

Proof. Denote by $B_\kappa \subset \mathbb{R}^2$ the disk of radius $\kappa > 0$ centered at the origin. By means of Remark 3.2, $\mathsf{G}^{(1)}$ can be equivalently written as:

$$\mathsf{G}^{(1)} = \frac{1}{8\pi^2\kappa^2}\int_{B_\kappa}\left(\widetilde{G}^{-}(\xi_1, \xi_2) + \widetilde{G}^{+}(\xi_1, \xi_2)\mathrm{e}^{2i\sqrt{\kappa^2-\xi^2}\,x_3}\right)\mathrm{e}^{ir\phi(\xi_1,\xi_2)}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,$$

with

$$\phi(\xi_1, \xi_2) := \sqrt{\kappa^2 - \xi^2}\cos\theta + \sin\theta\cos\varphi\,\xi_1 + \sin\theta\sin\varphi\,\xi_2. \tag{3.9}$$

Relation (3.7) is obtained by applying the stationary phase method (see [15]), after noting that the phase function ϕ has only one stationary point in B_κ given by $(\xi_1^s, \xi_2^s) = \kappa\sin\theta(\cos\varphi, \sin\varphi)$. The verification of equation (3.8) is direct by taking the limit in the explicit representation of Π given in Remark 3.5 below. $\square$

Remark 3.5. The matrix function Π in Proposition 3.4 is:

$$\begin{cases} \Pi(\theta, \kappa, \beta, x_3) = \Pi^{-}(\theta) + \Pi^{+}(\theta, \kappa, \beta)\,\mathrm{e}^{-2i\kappa\cos\theta x_3}, \\ \text{with} \\ \Pi^{-}(\theta) \qquad := \cos^2\theta\,\mathbf{e}_{11} + \mathbf{e}_{22} + \sin^2\theta\,\mathbf{e}_{33} - \cos\theta\sin\theta\,(\mathbf{e}_{13} + \mathbf{e}_{31}), \\ \Pi^{+}(\theta, \kappa, \beta) \quad := \cos^2\theta\,\upsilon^{-}\,\mathbf{e}_{11} + \upsilon^{+}\,\mathbf{e}_{22} - \sin^2\theta\,\upsilon^{-}\,\mathbf{e}_{33} + \cos\theta\sin\theta\,\upsilon^{-}\,(\mathbf{e}_{13} - \mathbf{e}_{31}) \\ \text{and } \upsilon^{-} \qquad := \dfrac{\beta\kappa^2 + i\kappa\cos\theta}{\beta\kappa^2 - i\kappa\cos\theta}, \quad \text{and } \upsilon^{+} := \dfrac{i\kappa\beta\cos\theta - 1}{i\kappa\beta\cos\theta + 1}. \end{cases}$$

The following remarks are useful to establish the adequate radiation condition pattern of the solutions.

Remark 3.6. Using Remark 3.5, by direct evaluation one can show that each column of $\mathsf{G}^{(1)}(\mathbf{x}, \mathbf{y})$ satisfies the classical *Silver-Müller* radiation condition:

$$|\mathbf{curl}\mathbf{U} \times \widehat{\mathbf{r}} - i\kappa\mathbf{U}| = \mathcal{O}(r^{-2}), \qquad \text{when } r \to +\infty,$$

with $\widehat{\mathbf{r}}$ denoting the outward unit normal to the upper half-sphere of radius r. In fact, it is enough to observe that, in the cylindrical coordinate system, the dyadic $\mathbf{curl} \times \widehat{\mathbf{r}}$ operator is given by (*cf.* [3]):

$$\mathbf{curl} \times \widehat{\mathbf{r}} = \begin{pmatrix} \cos\theta\partial_{y_3} & 0 & -\cos\theta\partial_\rho \\ -\dfrac{\sin\theta}{\rho}\partial_\varphi & \cos\theta\partial_{y_3} + \dfrac{\sin\theta}{\rho}\partial_\rho(\rho\,\cdot) & -\dfrac{\cos\theta}{\rho}\partial_\varphi \\ -\sin\theta\partial_{y_3} & 0 & \sin\theta\partial_\rho \end{pmatrix}. \tag{3.10}$$

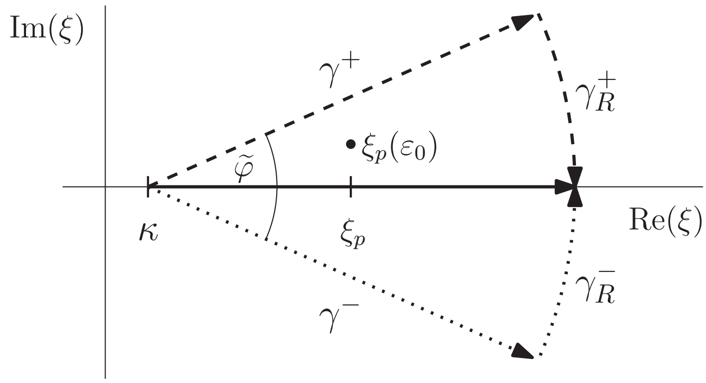

FIGURE 1. Complex paths.

Remark 3.7. The same procedure employed to prove Propostion 3.4 can be used to show that $\mathbf{curl}_\mathbf{y}\,\mathsf{G}^{(1)}$ also admits an asymptotic expression of the form:

$$\mathbf{curl}_\mathbf{y}\mathsf{G}^{(1)} = \Pi_{\mathbf{curl}}(\theta,\kappa,\beta,x_3)\frac{\mathrm{e}^{i\kappa r}}{4\pi r} + \mathcal{O}(r^{-2}), \quad \text{when } 0 \le \theta < \frac{\pi}{2} \quad \text{and } r \to +\infty,$$

where $\Pi_{\mathbf{curl}}$ is a bounded matrix function satisfying $|\Pi_{\mathbf{curl}}(\theta,\kappa,\beta,x_3)| = \mathcal{O}(\cos\theta)$, when $\theta \to \frac{\pi}{2}^-$. This implies that also each column of the dyad $\mathbf{curl}_\mathbf{y}\,\mathsf{G}^{(1)}$ satisfies the classical Silver-Müller radiation condition (see Rem. 3.6). Therefore, the couple $(\mathsf{G}^{(1)}, (i\kappa)^{-1}\mathbf{curl}_\mathbf{y}\,\mathsf{G}^{(1)})$ corresponds to the standard radiative contribution of the associated Green's dyad EM couple.

We next study the asymptotic behavior of $\mathsf{G}^{(2)}$ defined in equation (3.6b). Recall that, for a sufficiently small $\varepsilon > 0$, the singularity of $\widetilde{\mathsf{G}}_\varepsilon$ defines a pole $\xi_p(\varepsilon)$ satisfying $\mathrm{Re}\,(\xi_p(\varepsilon)) > \kappa$ and $\mathrm{Im}\,(\xi_p(\varepsilon)) > 0$ (see Lem. 3.3). Such singularity perches over the real axis when $\varepsilon = 0$. To avoid this, we apply *the Residue Theorem* of complex analysis (see [27]) and study the asymptotic contribution obtained from the residue plus the integral over an alternative integration path displayed in Figure 1. To choose an adequate contour, notice that, for any sufficient large ξ in the new integration path, we can write $\xi = \kappa + R\,\mathrm{e}^{i\omega}$, with $R > 0$ and $\omega \in (-\pi/2, \pi/2)$. It is easy to conclude that $\lim\limits_{R\to+\infty} \sqrt{(\kappa + R\,\mathrm{e}^{i\omega})^2 - \kappa_\varepsilon^2} = \lim\limits_{R\to+\infty} R\,\mathrm{e}^{i\omega}$ by looking at the square root definition (3.5). This implies that the exponential factors in the integrals defining $\mathsf{G}^{(2)}$, in the worst case, behave asymptotically like $\exp\left[-R\mathrm{e}^{i\omega} r\cos\theta + iR\mathrm{e}^{i\omega} r\sin\theta\cos(\varphi_\xi - \varphi)\right]$ (see Eq. (3.9)). Thus, to ensure exponential decay when $R \to +\infty$, we use integration paths over the upper complex half-plane when $\cos(\varphi_\xi - \varphi) \ge 0$, and when $\cos(\varphi_\xi - \varphi) < 0$, we choose paths in the lower part.

Summarizing, we decompose $\mathsf{G}^{(2)}$ as:

$$\mathsf{G}^{(2)} = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi}\int_0^{2\pi} \left(\mathsf{S}_\varepsilon^{\varphi_\xi} + \mathsf{R}_\varepsilon^{\varphi_\xi}\right) \mathrm{d}\varphi_\xi =: \lim_{\varepsilon\to 0^+} (\mathsf{S}_\varepsilon + \mathsf{R}_\varepsilon) =: \mathsf{S} + \mathsf{R}, \tag{3.11}$$

where, for a fixed small $\varepsilon > 0$ and for each angle $\varphi_\xi \in (0, 2\pi)$, $\mathsf{S}_\varepsilon^{\varphi_\xi}$ corresponds to the pole contribution and $\mathsf{R}_\varepsilon^{\varphi_\xi}$ to the line integral over the alternative integration contours. The next proposition provides asymptotic behaviors of the components S and R of $\mathsf{G}^{(2)}$. In particular, R is estimated differently in two complementary regions of the half-space depicted in Figure 2. This is necessary to avoid singularities in the estimation techniques.

Proposition 3.8. *Let (r,φ,θ) and (ρ,φ,x_3) represent the spherical and cylindrical coordinates systems (2.3) respectively. Regarding the decomposition $\mathsf{G}^{(2)} = \mathsf{S} + \mathsf{R}$ defined in equation (3.11), the following statements hold true:*

(i) $\mathsf{S} \to \mathbf{0}$ *exponentially when* $y_3 \to +\infty$.

(ii) *Horizontally,* $\mathsf{S} = \mathsf{F}(\beta,\kappa)\mathrm{e}^{-q(\beta,\kappa)(x_3+y_3)}\dfrac{\mathrm{e}^{i\rho\xi_p - i\pi/4}}{\sqrt{2\pi\rho\,\xi_p}} + \mathcal{O}(\rho^{-3/2})$, *when* $\rho \to +\infty$, *where* F *is a bounded matrix,* q *is a positive function, and* ξ_p *is the singularity characterized in equation* (3.3).

(iii) *For any* $\alpha > 0$ *fixed,* $|\mathsf{R}| = \mathcal{O}\left(r^{-2}\right) + \mathcal{O}\left(r^{-(2\alpha+1/2)}\right)$ *in the domain described by* $r\cos\theta > r^{\alpha}$.

(iv) $|\mathsf{R}| = \mathcal{O}(\rho^{-3/2})$ *in the complementary domain* $r\cos\theta \le r^{\alpha}$.

Proof. We start by studying the asymptotic behavior of S. For a fixed φ_ξ, we have the following cases for the pole contribution:

- If $\cos(\varphi_\xi - \varphi) < 0$, we use path integrals on the lower complex half-plane following the dotted contour $\gamma^- \cup \gamma_R^-$ in Figure 1, thus the pole is not included.

- If $\cos(\varphi_\xi - \varphi) \ge 0$, we integrate over $\mathrm{Im}\,(\xi) > 0$ enclosing the pole located at $\xi_p(\varepsilon)$ by following the dashed contour $\gamma^+ \cup \gamma_R^+$ in Figure 1.

For the last case, the residue contribution is

$$\mathsf{S}_\varepsilon^{\varphi_\xi} = \frac{i}{4\pi\kappa^2}\lim_{\xi\to\xi_p(\varepsilon)}\left(\xi - \xi_p(\varepsilon)\right)\xi\,\mathsf{G}_\varepsilon^{+}\,\mathrm{e}^{\rho\,\phi_\varepsilon(\xi,\varphi_\xi)},$$

with $\rho\,\phi_\varepsilon(\xi,\varphi_\xi)|_{\xi=\xi_p(\varepsilon)} = -\sqrt{\xi_p^2(\varepsilon) - \kappa_\varepsilon^2}\,(x_3+y_3) - i\,\xi_p(\varepsilon)\,r\sin\theta\,\cos(\varphi_\xi - \varphi)$. Thus, integrating in angle, taking the limit when $\varepsilon \to 0^+$, and noting that $\cos(\varphi_\xi - \varphi) \ge 0$ if and only if $-\pi/2 + \varphi \le \varphi_\xi \le \pi/2 + \varphi$, we obtain:

$$\mathsf{S} = \frac{\mathsf{F}(\beta,\kappa)}{2\pi}\mathrm{e}^{-q(\beta,\kappa)(x_3+y_3)}\int_{-\pi/2+\varphi}^{\pi/2+\varphi}\mathrm{e}^{i\rho\xi_p\cos(\varphi_\xi-\varphi)}\,\mathrm{d}\varphi_\xi, \tag{3.12}$$

where $\mathsf{F}(\beta,\kappa)$ is a bounded matrix function and $q(\beta,\kappa)$ is a scalar positive function (see Rem. 3.9). The proof of (i) is direct by observing that the integral in (3.12) is bounded. To prove (ii), we apply again the stationary phase method with phase function $\phi(\varphi_\xi) := \xi_p\cos(\varphi_\xi - \varphi)$. The only stationary point such that $\cos(\varphi_\xi - \varphi) \ge 0$ is $\varphi_\xi^s = \varphi$. We obtain:

$$\mathsf{S} = \mathsf{F}(\beta,\kappa)\mathrm{e}^{-q(\beta,\kappa)(x_3+y_3)}\frac{\mathrm{e}^{i\rho\xi_p - i\pi/4}}{\sqrt{2\pi\rho\,\xi_p}} + \mathcal{O}(\rho^{-3/2}), \text{ when } \rho \to +\infty. \tag{3.13}$$

It remains only to study the asymptotic behavior of R. We start by showing (iii). For a given $\widetilde{\varphi} \in (0,\pi/2)$, the straight complex line departing from κ has a parametrization:

$$\gamma : [0,\infty) \to \mathbb{C}, \quad \text{given by} \quad \gamma(\eta) = \begin{cases} \kappa + \eta\,\mathrm{e}^{i\widetilde{\varphi}}, & \text{if } \cos(\varphi_\xi - \varphi) \ge 0, \\ \kappa + \eta\,\mathrm{e}^{-i\widetilde{\varphi}}, & \text{if } \cos(\varphi_\xi - \varphi) < 0. \end{cases}$$

Note that for each component of R, their respective integral over γ admits a representation of form:

$$I_{ij}^{\gamma} := \int_0^{2\pi}\int_0^{+\infty} g_{ij}(\eta)\mathrm{e}^{r\phi(\eta)}\mathrm{d}\eta\,\mathrm{d}\varphi_\xi, \tag{3.14}$$

where $\phi(\eta) = -\sqrt{\eta^2\,\mathrm{e}^{\pm i2\widetilde{\varphi}} + 2\eta\kappa\,\mathrm{e}^{\pm i\widetilde{\varphi}}}\cos\theta + i(\kappa + \eta\,\mathrm{e}^{\pm i\widetilde{\varphi}})\sin\theta\cos(\varphi_\xi - \varphi)$ and the sign of the exponential terms depends on the sign of $\cos(\varphi_\xi - \varphi)$. Moreover, g_{ij} is an analytic function in $(0,+\infty)$ having a removable singularity at $\eta = 0$ (see Rem. 3.1). It can be easily checked that $\mathrm{Re}\,(\phi) < 0$, for all η and $\widetilde{\varphi}$. Thus, the integrand in equation (3.14) vanishes at infinity implying a null contribution over the curve $\gamma_R^{\pm}$. On the other hand, a double integration by parts in equation (3.14) gives:

$$I_{ij}^{\gamma} = \frac{1}{r^2}\int_0^{2\pi}\int_0^{+\infty} \partial_\eta\left(\frac{\phi' g_{ij}' - \phi'' g_{ij}}{(\phi')^3}\right) \mathrm{e}^{r\phi}\,\mathrm{d}\eta\,\mathrm{d}\varphi_\xi - \frac{c_{ij}}{r^2\cos^2\theta}\int_0^{2\pi} \mathrm{e}^{ir\kappa\sin\theta\cos(\varphi_\xi-\varphi)}\mathrm{d}\varphi_\xi,$$

with c_{ij} being a constant. The first term in the right-hand side of the previous equality behaves as $\mathcal{O}(r^{-2})$, while the second term will behave as $\mathcal{O}(r^{-(2\alpha+1/2)})$. In fact, the stationary phase technique applied to the integral of the second term gives a $\mathcal{O}(r^{-1/2})$ behavior, while the additional $\mathcal{O}(r^{-2\alpha})$ contribution comes from the assumption $r\cos\theta > r^\alpha$.

To prove (iv), we can compute the value of R as the difference:

$$\mathsf{R} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} \xi\left(\widetilde{\mathsf{G}} - \widetilde{\mathsf{S}}\right)\mathrm{e}^{i\rho\xi\cos(\varphi_\xi-\varphi)}\,\mathrm{d}\varphi_\xi\,\mathrm{d}\xi,$$

with $\widetilde{\mathsf{S}}$ denoting the Fourier transform of the SW dyad S (see Eq. (3.12)). It can be easily checked that the difference $\widetilde{\mathsf{G}} - \widetilde{\mathsf{S}}$ is continuous, since $\widetilde{\mathsf{S}}$ have the same singularity that $\widetilde{\mathsf{G}}$ has. Thus, applying the stationary phase technique to the angle integral, we obtain:

$$\begin{aligned}\mathsf{R} &= \frac{\mathrm{e}^{-i\pi/4}}{\sqrt{2\pi\rho}}\int_0^{+\infty}\sqrt{\xi}\left(\widetilde{\mathsf{G}} - \widetilde{\mathsf{S}}\right)\mathrm{e}^{i\rho\xi}\mathrm{d}\xi \\ &\quad + \frac{\mathrm{e}^{i\pi/4}}{\sqrt{2\pi\rho}}\int_0^{+\infty}\sqrt{\xi}\left(\widetilde{\mathsf{G}} - \widetilde{\mathsf{S}}\right)\mathrm{e}^{-i\rho\xi}\mathrm{d}\xi + \mathcal{O}(\rho^{-3/2}) \\ &= \mathcal{O}(\rho^{-3/2}),\ \text{when } \rho \to +\infty,\end{aligned}$$

where the last equality comes from an integration by parts argument in ξ to obtain an extra power of ρ. □

Remark 3.9. We have the following cases for the functions $\mathsf{F}(\beta,\kappa)$ and $q(\beta,\kappa)$ used in equations (3.12) and (3.13):

$$\text{If } \beta > 0 : q = \beta^{-1} \quad\text{and}\quad \mathsf{F} = i\beta^2\kappa^2\,\mathbf{e}_{22}.$$

$$\text{If } \beta < 0 : q = |\beta|\kappa^2 \quad\text{and}\quad \mathsf{F} = i\beta^2\kappa^2\left(\kappa^2|\beta|\,\mathbf{e}_{11} + \frac{\xi_p^2}{\kappa^2|\beta|}\,\mathbf{e}_{33} - i\xi_p(\mathbf{e}_{13} - \mathbf{e}_{31})\right).$$

In any case, the limiting situation $\beta \to 0$ corresponds to $\mathsf{F}(\beta,\kappa)\mathrm{e}^{-q(\beta,\kappa)} \equiv 0$, which implies $\mathsf{S} \equiv 0$ as expected.

Remark 3.10. Note that S represents a SW and its horizontal behavior is the characteristic one for an outgoing 2D wave (see Eq. (3.13)).

Remark 3.11. Using an analog argument to the one in Proposition 3.8, we can prove that the expression $\mathbf{curl_y}\,\mathsf{G}^{(2)} := \mathbf{curl_y}\,\mathsf{S} + \mathbf{curl_y}\,\mathsf{R}$ also satisfies the following asymptotic behaviors:

(i) $\mathbf{curl_y}\,\mathsf{S} \to \mathbf{0}$ exponentially when $y_3 \to +\infty$.
(ii) $\mathbf{curl_y}\,\mathsf{S} = \mathsf{F}_{\mathbf{curl}}(\beta,\kappa)\mathrm{e}^{-q(\beta,\kappa)(x_3+y_3)}\dfrac{\mathrm{e}^{i\rho\xi_p - i\pi/4}}{\sqrt{2\pi\rho\,\xi_p}} + \mathcal{O}(\rho^{-3/2})$, when $\rho \to +\infty$, where $\mathsf{F}_{\mathbf{curl}}$ a bounded matrix, q is positive (see Rem. 3.9), and ξ_p is the singularity characterized in (3.3).
(iii) For any $\alpha > 0$ fixed, $|\mathbf{curl_y}\,\mathsf{R}| = \mathcal{O}(r^{-2}) + \mathcal{O}(r^{-(2\alpha+1/2)})$ in the domain described by $r\cos\theta > r^\alpha$.
(iv) $|\mathbf{curl_y}\,\mathsf{R}| = \mathcal{O}(\rho^{-3/2})$ in the complementary domain $r\cos\theta \le r^\alpha$.

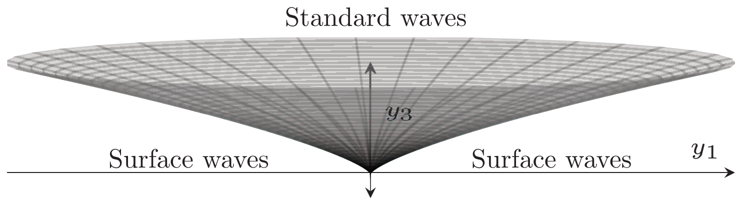

FIGURE 2. Half-space sub-division by the surface $y_3 = r^\alpha$ for $0 < \alpha < 1/2$.

4. The radiation condition

Recall that (r, φ, θ) and (ρ, φ, x_3) represent the spherical and cylindrical coordinates systems (2.3) respectively. The next notion of a radiative couple of electromagnetic fields follows from the asymptotic analysis of the previous section (*cf.* [21], Prop. 1.2 for the open transverse waveguide). Indeed, Theorem 4.6 in this section establishes that the Green's dyad is radiative in the sense of Definition 4.1. Moreover, radiative couples are unique (see Sect. 5.3). In particular, an analog of the Rellich Lemma for acoustic radiative solutions (*cf.* [11]) can be stated (see Prop. 5.10), which justifies the choice of $\alpha \in (\frac{1}{4}, \frac{1}{2})$ in the following definition.

Definition 4.1. We say that a couple of EM fields $(\mathbf{E}, \mathbf{H})$ satisfying (2.1) and the boundary condition (2.2) is radiative, if and only if, there exists a constant $\alpha \in (\frac{1}{4}, \frac{1}{2})$, such that the following Silver-Müller-type radiation condition holds true:

$$|\mathbf{H} \times \widehat{\mathbf{r}} - \mathbf{E}| = \mathcal{O}\left(r^{-(2\alpha+1/2)}\right), \text{ for } y_3 > r^\alpha, \tag{4.1a}$$

$$|\mathbf{H} \times \widehat{\mathbf{r}} - \mathsf{M}\,\mathbf{E}| = \mathcal{O}\left(r^{-(3/2-\alpha)}\right), \text{ for } y_3 \le r^\alpha, \tag{4.1b}$$

with $\widehat{\mathbf{r}}$ being the outer unit normal to the upper half-sphere of radius r, and M is the matrix defined by:

$$\mathsf{M} = \begin{cases} \dfrac{\xi_p}{\kappa}\mathsf{I}, & \text{if } \beta > 0, \\ \dfrac{\kappa}{\xi_p}\mathsf{I} + \dfrac{i\kappa^3|\beta|}{\xi_p^2}\mathbf{e}_{13}, & \text{if } \beta < 0, \end{cases} \tag{4.2}$$

where I is the identity matrix.

Remark 4.2. The matrix M is invertible, with inverse:

$$\mathsf{M}^{-1} = \begin{cases} \dfrac{\kappa}{\xi_p}\mathsf{I}, & \text{if } \beta > 0, \\ \dfrac{\xi_p}{\kappa}\mathsf{I} - i\kappa|\beta|\mathbf{e}_{13}, & \text{if } \beta < 0, \end{cases} \tag{4.3}$$

For the matrices M and M^{-1}, the following proposition holds true:

Proposition 4.3. *Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^3$. The matrices M and M^{-1} are positive-definite in the following sense:*

(i) $\operatorname{Re}(\overline{\mathbf{u}} \cdot \mathsf{M}\mathbf{u}) \ge \gamma_{\mathsf{M}} \|\mathbf{u}\|^2$, *with* $\gamma_{\mathsf{M}} = \begin{cases} \dfrac{\xi_p}{\kappa}, & \text{if } \beta > 0, \\ \dfrac{\kappa}{\xi_p^2}\left(\xi_p - \kappa^2|\beta|\right), & \text{if } \beta < 0. \end{cases}$

(ii) $\mathrm{Re}\left(\mathbf{u}\cdot\overline{\mathsf{M}^{-1}}\,\overline{\mathbf{u}}\right) \geq \gamma_{\mathsf{M}^{-1}}\,\|\mathbf{u}\|^2$, *with* $\gamma_{\mathsf{M}^{-1}} = \begin{cases} \dfrac{\kappa}{\xi_p}, & \text{if } \beta > 0, \\ \kappa^{-1}\left(\xi_p - \kappa^2|\beta|\right), & \text{if } \beta < 0. \end{cases}$

Proof. The proof of 4.3 for the case $\beta > 0$ is direct. For the case $\beta < 0$ observe that:

$$\mathrm{Re}\left(\overline{\mathbf{u}}\cdot\mathsf{M}\mathbf{u}\right) = \frac{\kappa}{\xi_p^2}\Big(\xi_p\|\mathbf{u}\|^2 - \kappa^2|\beta|\mathrm{Im}\left(\overline{u}_1\,u_3\right)\Big) \geq \frac{\kappa}{\xi_p^2}\left(\xi_p - \kappa^2|\beta|\right)\|\mathbf{u}\|^2.$$

The same procedure can be used to prove 4.3. □

Remark 4.4. It is clear that γ_{M} and $\gamma_{\mathsf{M}^{-1}}$ are always positive. Indeed, when $\beta < 0$, we have

$$\xi_p = \kappa\sqrt{1+\kappa^2\beta^2} > \kappa^2|\beta|.$$

Remark 4.5. Note that (4.1b) can be equivalently written in terms of M^{-1} (see Eq. 4.3) as

$$\left|\mathsf{M}^{-1}\mathbf{H}\times\widehat{\mathbf{r}} - \mathbf{E}\right| = \mathcal{O}\left(r^{-(3/2-\alpha)}\right), \quad \text{for} \quad y_3 \leq r^{\alpha}.$$

Theorem 4.6. *Denote by* G_j *the* j*th column of the Green's dyad* G*. If* $x_3 = 0$*, for* $j = 1,2,3$*, the electromagnetic couple* $\left(\mathsf{G}_j, \dfrac{1}{i\kappa}\mathbf{curl}\mathsf{G}_j\right)$ *satisfies the radiation condition* (4.1) *for any* $\alpha \in (\frac{1}{4}, \frac{1}{2})$*.*

Proof. Recall that, asymptotically, the Green's dyad admits the decomposition (see Eqs. (3.6) and (3.11)) $\mathsf{G} = \mathsf{G}^{(1)} + \mathsf{G}^{(2)} = \mathsf{G}^{(1)} + \mathsf{S} + \mathsf{R}$.

When in the region $y_3 > r^{\alpha}$, the proof of (4.1a) for $\mathsf{G}^{(1)}$ comes directly as consequence of Remark 3.6, while the proof for $\mathsf{G}^{(2)}$ is deduced from Proposition 3.8 and Remark 3.11. Opposingly, when considering the sub-domain $y_3 \leq r^{\alpha}$, first it can be demonstrated that each column of S satisfies the following asymptotic behavior (see Prop. 3.8 and Rem. 3.9):

$$|\mathbf{curl}\mathbf{U}\times\widehat{\boldsymbol{\rho}} - i\kappa\,\mathsf{M}\mathbf{U}| = \mathcal{O}(\rho^{-3/2}), \text{ when } \rho\to+\infty. \tag{4.4}$$

In fact, observe that the differential operator $\mathbf{curl}(\cdot)\times\widehat{\boldsymbol{\rho}}$ is nothing but the differential operator $\mathbf{curl}(\cdot)\times\widehat{\mathbf{r}}$ in equation (3.10) with $\theta = \frac{\pi}{2}$. Next, since $\rho = r\sin\theta$ and $\sin\theta$ is bounded away from zero in the domain $y_3 \leq r^{\alpha}$, we have that $\left|\mathbf{curl}\mathsf{G}_j^{(2)}\right| = \mathcal{O}(\rho^{-1/2}) + \mathcal{O}(\rho^{-3/2}) = \mathcal{O}(r^{-1/2})$ (see Rem. 3.11), with $\mathsf{G}_j^{(2)}$ denoting the jth column of $\mathsf{G}^{(2)}$, for $j = 1,2,3$. On the other hand, since $1-\sin\theta \leq 1-\sin^2\theta = \cos^2\theta \leq \cos\theta \leq r^{-(1-\alpha)}$, we get $|\widehat{\mathbf{r}} - \widehat{\boldsymbol{\rho}}| \leq r^{-(1-\alpha)}$. Thus, gathering all this information together with equation (4.4), Proposition 3.8 and Remark 3.11, we deduce

$$\begin{aligned}\left|\mathbf{curl}\mathsf{G}_j^{(2)}\times\widehat{\mathbf{r}} - i\kappa\,\mathsf{M}\mathsf{G}_j^{(2)}\right| &\leq \left|\mathbf{curl}\mathsf{G}_j^{(2)}\times(\widehat{\mathbf{r}} - \widehat{\boldsymbol{\rho}})\right| + \left|\mathbf{curl}\mathsf{G}_j^{(2)}\times\widehat{\boldsymbol{\rho}} - i\kappa\,\mathsf{M}\mathsf{G}_j^{(2)}\right| \\ &= \mathcal{O}(r^{-(3/2-\alpha)}) + \mathcal{O}(\rho^{-3/2}) = \mathcal{O}(r^{-(3/2-\alpha)}).\end{aligned}$$

The proof is then completed by noting that $\mathsf{G}^{(1)} = \dfrac{\mathcal{O}(\cos\theta)}{r} + \mathcal{O}(r^{-2})$, when θ is close to $\pi/2$ and $r\to+\infty$ (see Prop. 3.4). □

5. The uniqueness and existence results

Before stating the main results of this paper, we set the functional space framework needed (Sect. 5.1), together with some properties of *spherical Maxwell solutions* (Sect. 5.2).

5.1. Functional spaces

Let Ω be an open subset of $\mathbb{R}^3_+$, and denote by $\partial\Omega$ its boundary. Let $\mathscr{D}(\Omega)$ be the space of $C^\infty(\Omega)$-functions with compact support in Ω, and let $\mathscr{D}'(\Omega)$ be the dual space of distributions (see [27]). The space $\mathscr{D}\left(\overline{\Omega}\right)$ will denote the restriction to Ω of functions in the space $\mathscr{D}\left(\mathbb{R}^3\right)$. As usual, let us denote by $L^2(\Omega)$ the Hilbert space of square-integrable functions over Ω and $\boldsymbol{L}^2(\Omega) := \left[L^2(\Omega)\right]^3$. Without loss of generality, denote by $(\cdot\,,\,\cdot)$ the inner product of $L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$ (when it corresponds), and by $\|\cdot\|_0$ their respective norms. Consider the following Sobolev spaces:

$$
\begin{aligned}
H^1(\Omega) &:= \left\{V \in \mathscr{D}'(\Omega)\,:\, V \in L^2(\Omega) \text{ and } \nabla V \in \boldsymbol{L}^2(\Omega)\right\}, \quad \boldsymbol{H}^1(\Omega) := [H^1(\Omega)]^3,\\
\boldsymbol{H}(\mathbf{curl},\Omega) &:= \left\{\mathbf{V} \in [\mathscr{D}'(\Omega)]^3\,:\, \mathbf{V} \in \boldsymbol{L}^2(\Omega) \text{ and } \mathbf{curl}\,\mathbf{V} \in \boldsymbol{L}^2(\Omega)\right\},
\end{aligned}
$$

endowed with the natural norms $\|V\|_1^2 := \|V\|_0^2 + \|\nabla V\|_0^2$, $\|\mathbf{V}\|_1^2 := \sum_{i=1}^3 \|V_i\|_1^2$, and $\|\mathbf{V}\|^2_{\mathbf{curl}} := \|\mathbf{V}\|_0^2 + \|\mathbf{curl}\mathbf{V}\|_0^2$.

For $\mathbf{V} \in \left[\mathscr{D}\left(\overline{\Omega}\right)\right]^3$, define the following trace operators:

$$
\gamma_0(\mathbf{V}) := \mathbf{V}|_{\partial\Omega}, \qquad \gamma_t(\mathbf{V}) := \gamma_0(\mathbf{V}) \times \widehat{\mathbf{n}} \quad \text{and} \quad \gamma_T(\mathbf{V}) := \widehat{\mathbf{n}} \times \gamma_t(\mathbf{V}). \tag{5.1}
$$

It is well known that $[\mathscr{D}\left(\overline{\Omega}\right)]^3$ is dense in $\boldsymbol{H}(\mathbf{curl},\Omega)$, and that the trace γ_0 extends continuously from $\boldsymbol{H}^1(\Omega)$ onto the trace space $\boldsymbol{H}^{1/2}(\partial\Omega) := [H^{1/2}(\partial\Omega)]^3$ (see [10]). The dual space of $\boldsymbol{H}^{1/2}(\partial\Omega)$ will be denoted by $\boldsymbol{H}^{-1/2}(\partial\Omega)$. These trace spaces are endowed with the norms:

$$
\|\mathbf{v}\|_{1/2} = \inf_{\substack{\mathbf{V}\in\boldsymbol{H}^1(\Omega)\\ \gamma_0(\mathbf{V})=\mathbf{v}}} \|\mathbf{V}\|_1 \qquad \text{and} \qquad \|\mathbf{w}\|_{-1/2} = \sup_{0\neq\mathbf{v}\in\boldsymbol{H}^{1/2}(\partial\Omega)} \frac{\langle \mathbf{w}\,,\,\mathbf{v}\rangle}{\|\mathbf{v}\|_{1/2}}\,, \tag{5.2}
$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the duality pairing between $\boldsymbol{H}^{-1/2}(\partial\Omega)$ and $\boldsymbol{H}^{1/2}(\partial\Omega)$. For the traces γ_t and γ_T, we have the next result (see [10] for a general version of it):

Proposition 5.1. *For $\Omega = \mathbb{R}^3_+$, the traces γ_t and γ_T extend continuously from $\boldsymbol{H}(\mathbf{curl},\Omega)$ into $\boldsymbol{H}^{-1/2}(\partial\Omega)$, and the following inequality holds:*

$$
\|\gamma_T(\mathbf{U})\|_{-1/2} = \|\gamma_t(\mathbf{U})\|_{-1/2} \leq \|\mathbf{U}\|_{\mathbf{curl}} \qquad \forall\ \mathbf{U} \in \boldsymbol{H}(\mathbf{curl},\Omega)..
$$

Proof. The trace γ_t is defined by the action:

$$
\langle \gamma_t(\mathbf{U}),\, \gamma_0(\mathbf{V})\rangle := (\mathbf{U}\,,\,\mathbf{curl}\,\mathbf{V}) - (\mathbf{curl}\,\mathbf{U}\,,\,\mathbf{V}). \tag{5.3}
$$

It follows that $\left|\int_{\partial\Omega}\gamma_t(\mathbf{U})\cdot\overline{\gamma_0(\mathbf{V})}\right| = |\langle\gamma_t(\mathbf{U}),\,\gamma_0(\mathbf{V})\rangle| \leq \|\mathbf{U}\|_{\mathbf{curl}}\|\mathbf{V}\|_1$, for all $\mathbf{U} \in [\mathscr{D}\left(\overline{\Omega}\right)]^3$ and for all $\mathbf{V} \in \boldsymbol{H}^1(\Omega)$. Using the definition of the dual norm in equation (5.2), we obtain that $\|\gamma_t(\mathbf{U})\|_{-1/2} \leq \|\mathbf{U}\|_{\mathbf{curl}}$ for all $\mathbf{U} \in [\mathscr{D}\left(\overline{\Omega}\right)]^3$. The result follows from a standard density argument and the fact that, over the plane $x_3 = 0$, it holds:

$$
\langle\gamma_T(\mathbf{U}),\,\gamma_0(\mathbf{V})\rangle = -\langle\gamma_t(\mathbf{U}),\,\gamma_0(\mathbf{V}^*)\rangle\,,
$$

with $\mathbf{V}^* = (-V_2, V_1, V_3) \in \boldsymbol{H}^1(\Omega)$, which implies that $\|\gamma_T(\mathbf{U})\|_{-1/2} = \|\gamma_t(\mathbf{U})\|_{-1/2}$. $\square$

Remark 5.2. We recall that the traces γ_t and γ_T are not surjective in $\boldsymbol{H}^{-1/2}(\partial\Omega)$. However, continuity is enough for the purposes of this work.

The asymptotic behavior of the Green's dyad (see Propositions 3.4 and 3.8) suggests the use of weighted Sobolev spaces in order to allow $\boldsymbol{L}^2$-integrability in infinite domains. Thus, we consider the weight $\varrho(\rho, x_3) = \sqrt{1+\rho^2+x_3^2}$ where $\rho = \sqrt{x_1^2+x_2^2}$, and for any $\sigma > 1/2$, we define the weighted spaces:

$$\begin{aligned} H^1_\sigma(\Omega) &:= \left\{ V \in \mathscr{D}'(\Omega) \,:\, \varrho^\sigma\, V \in L^2(\Omega) \text{ and } \varrho^\sigma\, \nabla V \in \boldsymbol{L}^2(\Omega) \right\} \subset H^1(\Omega), \\ \boldsymbol{H}^1_\sigma(\Omega) &:= \left[H^1_\sigma(\Omega) \right]^3 \subset \boldsymbol{H}^1(\Omega), \\ \boldsymbol{H}_{-\sigma}(\mathbf{curl}, \Omega) &:= \left\{ \mathbf{V} \in [\mathscr{D}'(\Omega)]^3 \,:\, \varrho^{-\sigma}\, \mathbf{V} \in \boldsymbol{L}^2(\Omega) \text{ and } \varrho^{-\sigma} \mathbf{curl}\, \mathbf{V} \in \boldsymbol{L}^2(\Omega) \right\}, \end{aligned}$$

endowed with the norms $\|V\|^2_{1,\sigma} := \|\varrho^\sigma\, V\|^2_0 + \|\varrho^\sigma\, \nabla V\|^2_0$, $\|\mathbf{V}\|^2_{1,\sigma} := \sum_{i=1}^3 \|V_i\|^2_{1,\sigma}$, and $\|\mathbf{V}\|^2_{\mathbf{curl},-\sigma} := \|\varrho^{-\sigma}\, \mathbf{V}\|^2_0 + \|\varrho^{-\sigma}\, \mathbf{curl}\mathbf{V}\|^2_0$. Observe that $\boldsymbol{H}_{-\sigma}(\mathbf{curl}, \Omega) \supset \boldsymbol{H}(\mathbf{curl}, \Omega)$.

We define the trace space $\boldsymbol{H}^{1/2}_\sigma(\partial\Omega) := \gamma_0(\boldsymbol{H}_\sigma(\Omega)) \subset \boldsymbol{H}^{1/2}(\partial\Omega)$, and denote by $\boldsymbol{H}^{-1/2}_{-\sigma}(\partial\Omega)$ its dual. These trace spaces are equipped with the norms:

$$\|\mathbf{v}\|_{1/2,\sigma} = \inf_{\substack{\mathbf{V}\in \boldsymbol{H}^1_\sigma(\Omega) \\ \gamma_0(\mathbf{V}) = \mathbf{v}}} \|\boldsymbol{\phi}\|_{1,\sigma} \qquad \text{and} \qquad \|\mathbf{w}\|_{-1/2,-\sigma} = \sup_{0\neq \boldsymbol{\phi} \in \boldsymbol{H}^{1/2}_\sigma(\partial\Omega)} \frac{\langle \mathbf{w}\,,\, \boldsymbol{\phi}\rangle_\sigma}{\|\boldsymbol{\phi}\|_{1/2,\sigma}}\,,$$

where $\langle\cdot\,,\,\cdot\rangle_\sigma$ denotes the duality pairing between $\boldsymbol{H}^{-1/2}_{-\sigma}(\partial\Omega)$ and $\boldsymbol{H}^{1/2}_\sigma(\partial\Omega)$. Analogously to Proposition 5.1, the following lemma holds true:

Lemma 5.3. *For $\sigma > 1/2$ and $\Omega = \mathbb{R}^3_+$, the traces γ_t and γ_T defined in (5.1) extend continuously from $\boldsymbol{H}_{-\sigma}(\mathbf{curl}, \Omega)$ into $\boldsymbol{H}^{-1/2}_{-\sigma}(\partial\Omega)$. Moreover, the following inequality holds true:*

$$\|\gamma_T(\mathbf{U})\|_{-1/2,-\sigma} = \|\gamma_t(\mathbf{U})\|_{-1/2,-\sigma} \le \|\mathbf{U}\|_{\mathbf{curl},-\sigma} \qquad \forall\ \mathbf{U} \in \boldsymbol{H}_{-\sigma}(\mathbf{curl}, \Omega).$$

Proof. The proof is analogous to the one of Proposition 5.1 by considering the duality pairing:

$$\begin{aligned} \langle \gamma_t(\mathbf{U}),\, \gamma_0(\mathbf{V})\rangle_\sigma &:= (\varrho^{-\sigma}\, \mathbf{U}\,, \varrho^\sigma\, \mathbf{curl}\, \mathbf{V}) - (\varrho^{-\sigma}\, \mathbf{curl}\, \mathbf{U}\,,\, \varrho^\sigma\, \mathbf{V}), \\ \langle \gamma_T(\mathbf{U}),\, \gamma_0(\mathbf{V})\rangle_\sigma &:= -\, \langle \gamma_t(\mathbf{U}),\, \gamma_0(\mathbf{V}^*)\rangle_\sigma\,, \end{aligned}$$

with $\mathbf{V}^* = (-v_2, v_1, v_3) \in \boldsymbol{H}^1_\sigma(\Omega)$. ☐

In what follows, it will be useful to work with functions at a given tangential level $x_3 = c$, with $c \ge 0$. Thus, for $\mathbf{V} \in \boldsymbol{H}_{-\sigma}(\mathbf{curl}, \mathbb{R}^3_+)$, we define $\mathbf{V}^c : \mathbb{R}^2 \to \mathbb{C}^3$ as $\mathbf{V}^c(\rho, \varphi) = \mathbf{V}(\rho, \varphi, c)$. We conclude with the next lemma:

Lemma 5.4. *For any $c \ge 0$ and $\mathbf{V} \in \boldsymbol{H}_{-\sigma}\left(\mathbf{curl}, \mathbb{R}^3_+\right)$, the traces $\mathbf{V}^c_t := \widehat{\mathbf{k}} \times \mathbf{V}^c$ and $\mathbf{V}^c_T := \widehat{\mathbf{k}} \times \mathbf{V}^c \times \widehat{\mathbf{k}}$ belong to the space $\boldsymbol{H}^{-1/2}_{-\sigma}\left(\mathbb{R}^2\right)$, their respective norms are equal, and the following estimate holds:*

$$\|\mathbf{V}^c_t\|_{-1/2,-\sigma} = \|\mathbf{V}^c_T\|_{-1/2,-\sigma} \le \left(1 + \frac{c^2 + c\sqrt{c^2+4}}{2} \right)^\sigma \|\mathbf{V}\|_{\mathbf{curl},-\sigma}. \tag{5.4}$$

Proof. Define $\mathbf{U} \in \boldsymbol{H}_{-\sigma}(\mathbf{curl}, \mathbb{R}^3_+)$ as $\mathbf{U}(\rho, \varphi, x_3) := \mathbf{V}(\rho, \varphi, x_3 + c)$. Then, by Lemma 5.3, $\mathbf{V}^c_t = \gamma_t(\mathbf{U}) \in \boldsymbol{H}^{-1/2}_{-\sigma}\left(\mathbb{R}^2\right)$ and the following inequality holds:

$$\|\mathbf{V}^c_t\|_{-1/2,-\sigma} = \|\gamma_t(\mathbf{U})\|_{-1/2,-\sigma} \le \|\mathbf{U}\|_{\mathbf{curl},-\sigma}.$$

On the other hand,

$$\|(\varrho(\rho, x_3))^{-\sigma} \mathbf{U}\|^2_0 \le \|(\varrho(\rho, x_3 - c))^{-\sigma} \mathbf{v}\|^2_0 \le \left\| \frac{\varrho(\rho, x_3 - c)}{\varrho(\rho, x_3)} \right\|^{-2\sigma}_\infty \|\varrho^{-\sigma}(\rho, x_3)\, \mathbf{V}\|^2_0.$$

The same estimate holds when considering $\mathbf{curl}\,\mathbf{U}$ and $\mathbf{curl}\,\mathbf{V}$ instead of $\mathbf{U}$ and $\mathbf{V}$, respectively. Thus,

$$\|\mathbf{U}\|_{\mathbf{curl},-\sigma} \leq \left\| \frac{\varrho(\rho,x_3)}{\varrho(\rho,x_3-c)} \right\|_{\infty}^{2\sigma} \|\mathbf{V}\|_{\mathbf{curl},-\sigma}.$$

Finally, a straightforward computation of the maximum for the two-variable function $\dfrac{\varrho(\rho,x_3)}{\varrho(\rho,x_3-c)}$ gives the bound in the expression (5.4), thereby completing the proof. □

Inequality (5.4) is sub-optimal for $\mathbf{V}_t^c = \widehat{\mathbf{k}} \times \mathbf{V}^c$, in fact, more regularity can be considered for the space of solutions by noting that, for any $\varepsilon > 0$, the Green's dyad components satisfy the following condition:

$$\varrho^{-(1/2+\varepsilon)} \left(\widehat{\mathbf{k}} \times \mathbf{V}^c \right) \in \boldsymbol{L}^2\left(\mathbb{R}^2\right), \qquad \forall\, c \geq 0. \tag{5.5}$$

Thus, a more adequate space to seek for solutions of the Leontovich problem is

$$\boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right) = \left\{ \mathbf{V} \in \boldsymbol{H}_{-\sigma}\left(\mathbf{curl}, \mathbb{R}^3_+\right) \,:\, \widehat{\mathbf{k}} \times \mathbf{V}^c \text{ satisfies } (5.5),\ \forall\, c \geq 0 \right\}. \tag{5.6}$$

We have the following result:

Lemma 5.5. *If $\mathbf{V} \in \boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right)$, then for all $c \geq 0$ and $\varepsilon > 0$, we have that $\mathbf{V}_t^c = \widehat{\mathbf{k}} \times \mathbf{V}^c \in \boldsymbol{H}^{-(1/2+\varepsilon)}\left(\mathbb{R}^2\right)$.*

Proof. This follows immediately from the Fourier definition of the fractional Sobolev space (see [29]):

$$H^s\left(\mathbb{R}^2\right) := \left\{ \phi \in \mathscr{D}'\left(\mathbb{R}^2\right) \,:\, (1+\xi^2)^{s/2}\widehat{\phi} \in L^2\left(\mathbb{R}^2\right) \right\}$$

and its extension to vector fields. □

5.2. Spherical Maxwell's solutions

For $l \in \mathbb{N}_0$, let j_l denote the spherical Bessel function of the first kind and order l (see [1]). Using the spherical coordinates system (r,φ,θ), let us consider the following family of functions $\{v_l^m(r,\varphi,\theta) := j_l(\kappa r)\,\mathrm{e}^{im\varphi}\,\mathbb{P}_l^m(\cos\theta)\}$, where $\mathbb{P}_l^m$ denotes the Legendre functions of order $l \in \mathbb{N}_0$ and degree $m \in \mathbb{Z}$ such that $|m| \leq l$ (see [1]). For $\mathbf{x} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$, it is well known (*cf.* [23]) that the following vectorial spherical functions:

$$\mathbf{M}_l^m := \mathbf{curl}\,\left(v_l^m\,\mathbf{x}\right) \qquad \text{and} \qquad \mathbf{N}_l^m := \frac{1}{\kappa}\,\mathbf{curl}\,\mathbf{M}_l^m, \tag{5.7}$$

are solutions of the equation $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2\,\mathbf{u} = \mathbf{0}$.

Lemma 5.6. *For $\alpha_l^m, \beta_l^m \in \mathbb{R}$, the electromagnetic couple $(\mathbf{E}_l^m, \mathbf{H}_l^m)$, where*

$$\mathbf{E}_l^m := \alpha_l^m \mathbf{M}_l^m + \beta_l^m \mathbf{N}_l^m \quad \textit{and} \quad i\,\mathbf{H}_l^m := \alpha_l^m \mathbf{N}_l^m + \beta_l^m \mathbf{M}_l^m,$$

is a solution of the Maxwell system (2.1).

Proof. See Theorem 9.16 and Remark 9.18 from [23]. □

Remark 5.7. Notice that

$$\mathbf{curl}\,\mathbf{M}_l^m = \kappa\,\mathbf{N}_l^m \quad \text{and} \quad \mathbf{curl}\,\mathbf{N}_l^m = \kappa\,\mathbf{M}_l^m. \tag{5.8}$$

The next property concerns traces of functions $\mathbf{M}_l^m$ and $\mathbf{N}_l^m$ and will be used in Step 2 of the proof of the uniqueness theorem (see Sect. 5.3).

Lemma 5.8. *If $l+m$ is an odd integer, then*

$$\gamma_T\left(\mathbf{M}_l^m\right) = j_l(\kappa\rho)\,\mathrm{e}^{im\varphi}\,\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_l^m(0)\ \widehat{\boldsymbol{\varphi}} \text{ over } x_3 = 0 \ (i.e.\ \theta = \pi/2),$$
$$\gamma_T\left(\mathbf{N}_l^m\right) = \gamma_t\left(\mathbf{N}_l^m\right) = \mathbf{0} \qquad \text{over } x_3 = 0,$$

where $\widehat{\boldsymbol{\varphi}}$ is the unitary basis vector associated to the azimuthal angle.

Proof. The results are a direct consequence of the tangential representations (B.1), (B.2) in Appendix B.1, and the fact that $\mathbb{P}_l^m(0) = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\mathbb{P}_l^m(0) = 0$ when $l+m$ is odd, where the latter can be deduced from Formulas 8.5.3, 8.5.4 and 8.6.1 of [1]. ☐

5.3. The uniqueness theorem

Before establishing the main theorem, we provide some properties of solutions satisfying the homogenous problem (2.1) and (2.2), *i.e.* with $\mathbf{f}_T(\mathbf{x}) = \mathbf{0}$.

Lemma 5.9. *Consider the space $\boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right)$ defined in (5.6). Let $(\mathbf{E},\mathbf{H}) \in \boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right) \times \boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right)$ satisfy the Maxwell system (2.1). Then, for any Lipschitz open bounded set $\Omega \subset \mathbb{R}^3_+$, we have that $(\mathbf{E},\mathbf{H})$ is in $\boldsymbol{H}^1_{\mathrm{loc}}(\Omega) \times \boldsymbol{H}^1_{\mathrm{loc}}(\Omega)$. In particular, all their traces are well defined as functions in $\boldsymbol{L}^2(\partial\Omega)$ when $\partial\Omega \cap \Gamma = \emptyset$.*

Proof. Since $\mathbf{E},\mathbf{H} \in \boldsymbol{W}_{-\sigma}(\mathbb{R}^3_+)$, then in particular $\mathbf{E},\mathbf{H} \in \boldsymbol{H}(\mathbf{curl},\Omega)$ by the boundedness of Ω. Moreover, since the couple $(\mathbf{E},\mathbf{H})$ satisfies the homogeneous Maxwell equations (2.1), then $\operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0$. Hence, both $\mathbf{E},\mathbf{H} \in \boldsymbol{H}^1_{\mathrm{loc}}(\Omega)$ (see [16], Cor. 2.10). ☐

Proposition 5.10. *Consider an outgoing radiative EM couple $(\mathbf{E},\mathbf{H})$ (see Def. 4.1), with $\mathbf{f}_T = 0$ as boundary data in (2.2), satisfying the same hypothesis of Lemma 5.9. Let $R > 0$ and denote by S_R the surface of the half-ball of radius R contained in $\mathbb{R}^3_+$. We have:*

$$\lim_{R\to+\infty} \|\mathbf{E}\|_{L^2(S_R)} = 0 \ \ \textit{and} \ \ \lim_{R\to+\infty} \|\mathbf{H}\times\widehat{\mathbf{r}}\|_{L^2(S_R)} = 0. \tag{5.9}$$

Proof. Denote by Ω_R the half-ball of radius R contained in $\mathbb{R}^3_+$, and denote by D_R the disk of radius R within the surface Γ. For any $\varepsilon > 0$, let $\Omega_R^\varepsilon \subset \mathbb{R}^3_+$ be the vertical ε-translation of Ω_R towards the direction $\widehat{\mathbf{k}}$, *i.e.* $\Omega_R^\varepsilon := \Omega_R + \varepsilon\widehat{\mathbf{k}}$.

Finally, let S_R^ε and D_R^ε be the upper and lower boundaries of Ω_R^ε, respectively. Lemma 5.9 implies that, for any $\varepsilon > 0$, the following equality holds true:

$$\begin{aligned}(\mathbf{H},\mathbf{curl}\,\mathbf{E})_{\Omega_R^\varepsilon} - (\mathbf{curl}\,\mathbf{H},\mathbf{E})_{\Omega_R^\varepsilon} &= \int_{S_R^\varepsilon}(\mathbf{H}\times\widehat{\mathbf{n}})\cdot\overline{\mathbf{E}} + \int_{D_R^\varepsilon}(\mathbf{H}\times\widehat{\mathbf{n}})\cdot\overline{\mathbf{E}}\\ &= \int_{S_R^\varepsilon}(\mathbf{H}\times\widehat{\mathbf{r}})\cdot\overline{\mathbf{E}} - \int_{D_R^\varepsilon}(\mathbf{H}\times\widehat{\mathbf{k}})\cdot\overline{\mathbf{E}_T}.\end{aligned}$$

Taking $\varepsilon \to 0^+$, and using the curl relations (2.1) together with the homogeneous boundary condition (2.2), we obtain:

$$\int_{S_R}(\mathbf{H}\times\widehat{\mathbf{r}})\cdot\overline{\mathbf{E}} = i\kappa\int_{\Omega_R}\left(|\mathbf{E}|^2 - |\mathbf{H}|^2\right) - (i\kappa\beta)^{-1}\int_{D_R}|\mathbf{E}_T|^2.$$

Thus,

$$\operatorname{Re}\left(\int_{S_R}(\mathbf{H}\times\widehat{\mathbf{r}})\cdot\overline{\mathbf{E}}\right) = 0. \tag{5.10}$$

In order to use the radiation condition equation (4.1), take $\alpha \in (\frac{1}{4}, \frac{1}{2})$ guaranteed by Definition 4.1 and define the piece-wise constant matrix:

$$\widetilde{\mathsf{M}}(\mathbf{x}) = \begin{cases} \mathsf{I}, & \text{if } x_3 > r^\alpha, \\ \mathsf{M}, & \text{if } x_3 \le r^\alpha, \end{cases}$$

with matrices I and M defined in (4.2). By adding an adequate zero term to $\mathbf{H}\times\widehat{\mathbf{r}}$ in equation (5.10), we conclude

$$-\mathrm{Re}\left(\int_{S_R} \overline{\mathbf{E}} \cdot \widetilde{\mathsf{M}}\,\mathbf{E}\right) = \mathrm{Re}\left(\int_{S_R} \left(\mathbf{H} \times \widehat{\mathbf{r}} - \widetilde{\mathsf{M}}\,\mathbf{E}\right) \cdot \overline{\mathbf{E}}\right).$$

Next, using Proposition 4.3 and the Silver-Müller-type radiation condition (4.1), we obtain for large R:

$$\|\mathbf{E}\|^2_{\boldsymbol{L}^2(S_R)} \le C \max\left\{\frac{1}{R^{2\alpha+\frac{1}{2}}}, \frac{1}{R^{\frac{3}{2}-\alpha}}\right\} \int_{S_R} |\mathbf{E}|.$$

By Hölder's inequality, it holds

$$\|\mathbf{E}\|_{\boldsymbol{L}^2(S_R)} \le C \max\left\{\frac{1}{R^{2\alpha-\frac{1}{2}}}, \frac{1}{R^{\frac{1}{2}-\alpha}}\right\}.$$

In a similar way, adding an adequate zero term to $\overline{\mathbf{E}}$ in relation (5.10), using Proposition 4.3 4.3, and the radiation condition (4.1) written in terms of $\widetilde{\mathsf{M}}^{-1}$ (see Rem. 4.5), it is also easy to prove the asymptotic:

$$\|\mathbf{H} \times \widehat{\mathbf{r}}\|_{\boldsymbol{L}^2(S_R)} \le C \max\left\{\frac{1}{R^{2\alpha-\frac{1}{2}}}, \frac{1}{R^{\frac{1}{2}-\alpha}}\right\}.$$

The result follows by taking the limit when $R \to +\infty$, using the fact that $\frac{1}{4} < \alpha < \frac{1}{2}$. □

The next Lemma is an orthogonality result verified by EM couples satisfying the hypothesis of Proposition 5.10. This result is critical in the proof of Theorem 5.12 to imply that the Fourier transform of the traces of such a couple vanish over a disk of radius $\kappa > 0$.

Lemma 5.11. *Let D_R be the disk of radius R within the surface $\Gamma = \partial\mathbb{R}^3_+$. Consider an outgoing radiative electromagnetic couple $(\mathbf{E}, \mathbf{H})$ satisfying the same hypothesis of Proposition 5.10. Let $\mathbf{E}^0_T := \widehat{\mathbf{k}} \times \mathbf{E} \times \widehat{\mathbf{k}}$ and $\mathbf{H}^0_t := \widehat{\mathbf{k}} \times \mathbf{H}$ be tangential traces of $(\mathbf{E}, \mathbf{H})$ over Γ. If $(l+m)$ is an odd integer, then:*

$$\lim_{R\to\infty} \int_{D_R} j_l(\kappa\rho)\, \mathrm{e}^{-im\varphi}\, \mathbf{E}^0_T = \mathbf{0} \qquad \textit{and} \qquad \lim_{R\to\infty} \int_{D_R} j_l(\kappa\rho)\, \mathrm{e}^{-im\varphi}\, \mathbf{H}^0_t = \mathbf{0}. \tag{5.11}$$

Proof. Let Ω_R be the half-ball of radius R contained in $\mathbb{R}^3_+$ having D_R as its bottom and let $S_R := \partial\Omega_R \setminus D_R$. Let the couple $(\mathbf{V}^m_l, \mathbf{W}^m_l)$ be any $(\mathbf{M}^m_l, \mathbf{N}^m_l)$ or $(\mathbf{N}^m_l, \mathbf{M}^m_l)$ (see (5.7) and (5.8)). First notice that, for any open and bounded set $\Omega \subset \mathbb{R}^3_+$, the curl relations (2.1) and (5.8), together with Lemma 5.9, imply that:

$$\int_{\partial\Omega} (\mathbf{H} \times \widehat{\mathbf{n}}) \cdot \overline{\mathbf{V}^m_l} = \kappa\, (\mathbf{H}, \mathbf{W}^m_l)_\Omega + i\kappa (\mathbf{E}, \mathbf{W}^m_l)_\Omega,$$

and

$$\int_{\partial\Omega} (\mathbf{W}^m_l \times \widehat{\mathbf{n}}) \cdot \overline{\mathbf{E}} = -i\kappa\, (\mathbf{V}^m_l, \mathbf{H})_\Omega - \kappa (\mathbf{V}^m_l, \mathbf{E})_\Omega.$$

Therefore, using the same limit argument as in the proof of Proposition 5.10, we obtain:

$$0 = i \int_{\partial\Omega_R} (\mathbf{H} \times \widehat{\mathbf{n}}) \cdot \overline{\mathbf{V}^m_l} - \int_{\partial\Omega_R} \mathbf{E} \cdot (\overline{\mathbf{W}^m_l} \times \widehat{\mathbf{n}}).$$

The asymptotic decay of Bessel functions (see [1]) and the limit properties (5.9) of Proposition 5.10 imply that the integral contribution over S_R vanishes when $R \to +\infty$. Thus,

$$\lim_{R\to\infty} i \int_{D_R} \mathbf{H}_t^0 \cdot \overline{\mathbf{V}_l^m} = \lim_{R\to\infty} \int_{D_R} \mathbf{E}_T^0 \cdot (\widehat{\mathbf{k}} \times \overline{\mathbf{W}_l^m}).$$

The boundary condition (2.2) implies that, in the previous equality, we have:

$$\lim_{R\to\infty} \frac{1}{\kappa\,\beta} \int_{D_R} \mathbf{E}_T^0 \cdot \overline{\mathbf{V}_l^m} = \lim_{R\to\infty} \int_{D_R} \mathbf{E}_T^0 \cdot (\widehat{\mathbf{k}} \times \overline{\mathbf{W}_l^m}).$$

Since the couple $(\mathbf{V}_l^m, \mathbf{W}_l^m)$ can be any $(\mathbf{M}_l^m, \mathbf{N}_l^m)$ or $(\mathbf{N}_l^m, \mathbf{M}_l^m)$, if $l+m$ is an odd integer, Lemma 5.8 implies that the right-hand side of previous equality vanishes when $\mathbf{V}_l^m = \mathbf{M}_l^m$ whereas the left-hand side is equal to zero when $\mathbf{V}_l^m = \mathbf{N}_l^m$. This proves the orthogonality property in (5.11) for $\mathbf{E}_T^0$, and also the orthogonality property for $\mathbf{H}_t^0$ as a consequence of the homogeneous impedance boundary condition. □

We now present the main theorem of this section.

Theorem 5.12 (Uniqueness)**.** *Consider the space $\boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right)$ defined in (5.6). There exists at most one radiative couple $(\mathbf{E}, \mathbf{H})$ in $\boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right) \times \boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right)$ satisfying Maxwell equations (2.1) and the Leontovich boundary condition (2.2).*

Proof. As usual, we consider the case when $\mathbf{f}_T = \mathbf{0}$ and conclude that the only solution of this problem is $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$. The absence of forcing terms in (2.1) implies that, in a distributional sense, each EM field must satisfy:

$$-\Delta\,\mathbf{E} - \kappa^2\,\mathbf{E} = \mathbf{0}. \tag{5.12}$$

Since we are looking for solutions in $\boldsymbol{W}_{-\sigma}\left(\mathbb{R}^3_+\right)$ (see (5.6)), the field $\mathbf{E}$ has at most polynomial growth. Hence, for any fixed $x_3 > 0$, $\mathbf{E}^{x_3}(\rho, \varphi) := \mathbf{E}(\rho, \varphi, x_3)$ induces a tempered distribution in the space $\left[\mathscr{S}'\left(\mathbb{R}^2\right)\right]^3$ (see [27] for the definition and properties of $\mathscr{S}'\left(\mathbb{R}^2\right)$). In particular, the two-dimensional Fourier transform application is well defined on $\mathbf{E}^{x_3}$ and we thus denote it by $\widetilde{\mathbf{E}}^{x_3} \in \left[\mathscr{S}'\left(\mathbb{R}^2\right)\right]^3$. That is, the Fourier transform in the coordinate system (ξ, φ_ξ, x_3) (see Eq. (2.4)). Using equation (5.12), it can be easily deduced that $\widetilde{\mathbf{E}}(\xi, \varphi_\xi, x_3) := \widetilde{\mathbf{E}}^{x_3}$ must satisfy:

$$\partial_3^2 \widetilde{\mathbf{E}} + (\kappa^2 - \xi^2)\widetilde{\mathbf{E}} = \mathbf{0}.$$

General solutions of the previous equation are of the form:

$$\widetilde{\mathbf{E}}^{x_3} = \mathbf{T}^-(\xi) \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3} + \mathbf{T}^+(\xi) \mathrm{e}^{\sqrt{\xi^2-\kappa^2}x_3}, \tag{5.13}$$

with $\mathbf{T}^\pm$ being distributions that are independent from φ_ξ and x_3. Note that the same reasoning applies to $\mathbf{H}$ instead of $\mathbf{E}$ in the previous analysis.

On the other hand, from equation (2.1) we derive the following additional relations for the two-dimensional Fourier transform of the EM fields in the spectral cylindrical coordinate system (ξ, φ_ξ, x_3):

$$\begin{aligned} i\kappa \left(\widetilde{H}_\xi,\, \widetilde{H}_{\varphi_\xi},\, \widetilde{H}_3\right) &= \left(-\partial_3 \widetilde{E}_{\varphi_\xi},\, \partial_3 \widetilde{E}_\xi - i\xi \widetilde{E}_3,\, i\xi \widetilde{E}_{\varphi_\xi}\right), \\ -i\kappa \left(\widetilde{E}_\xi,\, \widetilde{E}_{\varphi_\xi},\, \widetilde{E}_3\right) &= \left(-\partial_3 \widetilde{H}_{\varphi_\xi},\, \partial_3 \widetilde{H}_\xi - i\xi \widetilde{H}_3,\, i\xi \widetilde{H}_{\varphi_\xi}\right). \end{aligned} \tag{5.14}$$

The next steps of the proof require the analysis of the horizontal Fourier traces:

$$\widetilde{\mathbf{E}}_T^{x_3} := \widehat{\mathbf{k}} \times \widetilde{\mathbf{E}}^{x_3} \times \widehat{\mathbf{k}} \qquad \text{and} \qquad \widetilde{\mathbf{H}}_t^{x_3} := \widehat{\mathbf{k}} \times \widetilde{\mathbf{H}}^{x_3},$$

which are nothing but the Fourier counterparts of the horizontal traces:

$$\mathbf{E}_T^{x_3} := \widehat{\mathbf{k}} \times \mathbf{E}^{x_3} \times \widehat{\mathbf{k}} \qquad \text{and} \qquad \mathbf{H}_t^{x_3} := \widehat{\mathbf{k}} \times \mathbf{H}^{x_3}.$$

Observe that the homogeneous impedance boundary condition (2.2) implies that:

$$\mathbf{E}_T^{0} = i\kappa\beta \mathbf{H}_t^{0} \qquad \text{and} \qquad \widetilde{\mathbf{E}}_T^{0} = i\kappa\beta \widetilde{\mathbf{H}}_t^{0}. \tag{5.15}$$

The analysis will be divided into four steps:

(1) For fixed $x_3 \geq 0$, we prove that $\widetilde{\mathbf{E}}_T^{x_3}$ and $\widetilde{\mathbf{H}}_t^{x_3}$ have compact support in the horizontal plane $\mathbb{R}^2$.
(2) We show that $\widetilde{\mathbf{E}}_T^{0}$ and $\widetilde{\mathbf{H}}_t^{0}$ vanish over the disk of radius $\kappa > 0$.
(3) We use the previous information to conclude that $\widetilde{\mathbf{E}}^{x_3} = \mathbf{0}$ and $\widetilde{\mathbf{H}}^{x_3} = \mathbf{0}$, which implies $\mathbf{E} = \mathbf{H} = \mathbf{0}$.

5.3.1. Step 1

Let us denote by $D_\kappa \subset \mathbb{R}^2$ the open disk of radius κ located at the origin, and by $\overline{D}_\kappa^c = \mathbb{R}^2 \backslash \overline{D}_\kappa$ its open complement on $\mathbb{R}^2$. By representation (5.13), we observe that the distribution $\widetilde{\mathbf{E}}_T^{x_3}$ over $\overline{D}_\kappa^c$ is the sum of exponentially increasing and decreasing terms. Using Lemma 5.4, we show that $\widetilde{\mathbf{E}}_T^{x_3}$ can not be exponentially increasing in $\overline{D}_\kappa^c$. In fact, for $\boldsymbol{\phi}_t = (\phi_\xi, \phi_{\varphi_\xi}, 0) \in \left[\mathscr{D}\left(\overline{D}_\kappa^c\right)\right]^3$, it holds that

$$\begin{aligned}\left|\left\langle \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}\, \widetilde{\mathbf{E}}_T^{x_3}\,,\, \boldsymbol{\phi}_t \right\rangle\right| &= \left|\langle \mathbf{E}_T^{x_3}\,,\, \mathcal{F}(\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}\, \boldsymbol{\phi}_t)\rangle\right| \\ &\leq C(x_3,\sigma)\, \|\mathbf{E}\|_{\mathbf{curl},-\sigma} \|\mathcal{F}(\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3} \boldsymbol{\phi}_t)\|_{1/2,\sigma},\end{aligned}$$

with $\mathcal{F}$ denoting the two-dimensional Fourier transform, and $C(x_3, \sigma)$ being the constant in equation (5.4), which has an asymptotic polynomial growth in x_3. The product $C(x_3,\sigma)\, \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3} \boldsymbol{\phi}_t$ belongs to the space $\left[\mathscr{S}\left(\mathbb{R}^2\right)\right]^3$, as well as its Fourier transform, implying that the right-hand side of the previous inequality goes to zero when $x_3 \to +\infty$. The last statement allows us to conclude that $\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}\, \widetilde{\mathbf{E}}_T^{x_3}$ goes to zero as a distribution in $\left[\mathscr{D}'\left(\overline{D}_\kappa^c\right)\right]^3$. The same applies to $\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}\, \widetilde{\mathbf{H}}_t^{x_3}$ as well. Having in mind the representation (5.13) for $\widetilde{\mathbf{E}}^{x_3}$ and $\widetilde{\mathbf{H}}^{x_3}$, we arrive at the following characterization of $\widetilde{\mathbf{E}}_T^{x_3}$ and $\widetilde{\mathbf{H}}_t^{x_3}$ in $\left[\mathscr{D}'\left(\overline{D}_\kappa^c\right)\right]^3$:

$$\widetilde{\mathbf{E}}_T^{x_3} = \mathbf{T}^{(1)}\, \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3} \qquad \text{and} \qquad \widetilde{\mathbf{H}}_t^{x_3} = \mathbf{T}^{(2)}\, \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}, \tag{5.16}$$

for some x_3-independent distributions $\mathbf{T}^{(j)} = (T_\xi^{(j)}, T_{\varphi_\xi}^{(j)}, 0)$, with $j = 1, 2$.

On the other hand, from equation (5.14) we obtain that $\widetilde{E}_3 = -\dfrac{i\xi}{\xi^2-\kappa^2}\partial_3 \widetilde{E}_\xi$ and thus $\widehat{\mathbf{k}} \times \widetilde{\mathbf{H}}$ can be characterized as

$$\widehat{\mathbf{k}} \times \widetilde{\mathbf{H}} = \left(-\widetilde{H}_{\varphi_\xi}, \widetilde{H}_\xi, 0\right) = -\left(\frac{i\kappa}{\xi^2-\kappa^2}\partial_3 \widetilde{E}_\xi,\; \frac{1}{i\kappa}\partial_3 \widetilde{E}_{\varphi_\xi},\; 0\right). \tag{5.17}$$

For any $\boldsymbol{\psi}_t = (\psi_\xi, \psi_{\varphi_\xi}, 0) \in \left[\mathscr{D}\left(\overline{D}_\kappa^c\right)\right]^3$, the previous relation implies:

$$\left\langle \widetilde{\mathbf{H}}_t^{x_3}, \boldsymbol{\psi}_t \right\rangle = \left\langle \widetilde{H}_\xi^{x_3}, \psi_{\varphi_\xi} \right\rangle - \left\langle \widetilde{H}_{\varphi_\xi}^{x_3}, \psi_\xi \right\rangle, \tag{5.18}$$

where, by using (5.16) and (5.17), we obtain

$$\begin{aligned}\langle \widetilde{H}_\xi^{x_3}, \psi_{\varphi_\xi}\rangle &= \frac{1}{i\kappa}\left\langle \sqrt{\xi^2-\kappa^2}\, T_{\varphi_\xi}^{(1)}\, \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}, \psi_{\varphi_\xi} \right\rangle, \\ \langle \widetilde{H}_{\varphi_\xi}^{x_3}, \psi_\xi\rangle &= -i\kappa \left\langle \frac{T_\xi^{(1)}}{\sqrt{\xi^2-\kappa^2}} \mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}, \psi_\xi \right\rangle.\end{aligned}$$

Moreover, the impedance boundary condition (2.2) implies that

$$0 = \lim_{x_3\to 0^+} \left\langle \widetilde{\mathbf{E}}_T^{x_3} - i\kappa\beta\,\widetilde{\mathbf{H}}_t^{x_3}\,,\,\boldsymbol{\psi}_t \right\rangle = \lim_{x_3\to 0^+} \left\langle \widetilde{E}_\xi^{x_3} + i\kappa\beta\,\widetilde{H}_{\varphi_\xi}^{x_3}, \psi_\xi \right\rangle + \left\langle \widetilde{E}_{\varphi_\xi}^{x_3} - i\kappa\beta\,\widetilde{H}_\xi^{x_3}, \psi_{\varphi_\xi} \right\rangle, \tag{5.19}$$

for all $\boldsymbol{\psi}_t = (\psi_\xi, \psi_{\varphi_\xi}, 0)$. In particular, for $\boldsymbol{\psi}_t = (\psi, 0, 0)$ and $\boldsymbol{\psi}_t = (0, \psi, 0)$, with $\psi \in \mathscr{D}(\overline{D}_\kappa^c)$. Therefore, combining equations (5.16), (5.18), (5.19) and using the fact that $\left[\mathscr{D}(\overline{D}_\kappa^c)\right]^3$ is not affected under multiplication by $\sqrt{\xi^2-\kappa^2}$, we finally conclude that

$$\left\langle \left(\sqrt{\xi^2-\kappa^2} + \kappa^2\beta\right) T_\xi^{(1)}, \psi \right\rangle = 0 \qquad \text{and} \qquad \left\langle \left(1 - \beta\sqrt{\xi^2-\kappa^2}\right) T_{\varphi_\xi}^{(1)}, \psi \right\rangle = 0\,,$$

for any $\psi \in \mathscr{D}(\overline{D}_\kappa^c)$, which implies that the support of $\mathbf{T}^{(1)}$ in $\overline{D}_\kappa^c$ is contained in the curve $\xi^2 = \xi_p^2$ (see Sect. 3.2). A similar argument shows that the support of $\mathbf{T}^{(2)}$ is also contained on the same curve.

Finally, from the representations (5.16) we conclude that, for any $x_3 \geq 0$, $\widetilde{\mathbf{E}}_T^{x_3}$ and $\widetilde{\mathbf{H}}_t^{x_3}$ belong to $\left[\mathscr{E}'\left(\mathbb{R}^2\right)\right]^3$, the space of distributions with compact support. In particular, their inverse Fourier transforms $\mathbf{E}_T^{x_3}$ and $\mathbf{H}_t^{x_3}$ belong to $C^\infty\left(\mathbb{R}^2\right)$ (see [27]).

5.3.2. Step 2

In the following, we will consider the trace $\mathbf{E}_T^0$ given in terms of its tangential Cartesian components $(E_1^0, E_2^0, 0)$. The cylindrical counterpart $(E_\rho^0, E_\varphi^0, 0)$ can be obtained using the following identities:

$$E_\rho^0 = \cos\varphi E_1^0 + \sin\varphi E_2^0 = \frac{\mathrm{e}^{-i\varphi}}{2}(E_1^0 + iE_2^0) + \frac{\mathrm{e}^{i\varphi}}{2}(E_1^0 - iE_2^0), \tag{5.20}$$

$$E_\varphi^0 = -\sin\varphi E_1^0 + \cos\varphi E_2^0 = -i\frac{\mathrm{e}^{-i\varphi}}{2}(E_1^0 + iE_2^0) + i\frac{\mathrm{e}^{i\varphi}}{2}(E_1^0 - iE_2^0). \tag{5.21}$$

Combining equations (5.20) and (5.21), the conclusion of Lemma 5.11 can be equivalently written as:

$$\begin{cases} \text{If } (l+m) \text{ is an odd integer:} \\ \displaystyle\lim_{R\to\infty} \int_0^R \int_0^{2\pi} \rho\, j_l(\kappa\rho)\, \mathrm{e}^{-i(m+1)\varphi} \left(E_1^0 + i\,E_2^0\right)\, \mathrm{d}\varphi\, \mathrm{d}\rho = 0, \\ \displaystyle\lim_{R\to\infty} \int_0^R \int_0^{2\pi} \rho\, j_l(\kappa\rho)\, \mathrm{e}^{-i(m-1)\varphi} \left(E_1^0 - i\,E_2^0\right)\, \mathrm{d}\varphi\, \mathrm{d}\rho = 0. \end{cases} \tag{5.22}$$

Let us define the following Fourier coefficients:

$$c_j^m(\mathbf{E}, \rho) = \frac{1}{2\pi} \int_0^{2\pi} E_j^0\, \mathrm{e}^{-im\varphi}\, d\varphi, \qquad \text{for } j = 1, 2.$$

Thus, from equation (5.22) we deduce that:

$$\begin{cases} \text{If } (l+m) \text{ is an even integer:} \\ \displaystyle\lim_{R\to\infty} \int_0^R \rho\, j_l(\kappa\rho)\, c_j^m(\mathbf{E}, \rho)\, d\rho = 0, \qquad \text{for } j = 1, 2. \end{cases} \tag{5.23}$$

It is possible to prove (*cf.* [13], Eq. (79)) that the Fourier transform of E_j^0 can be written as the limit:

$$\widehat{E}_j^0(\xi, \varphi_\xi) = \lim_{R\to\infty} \sum_{\substack{|m|\leq l \\ (l+m)\ \text{even}}} \wp_l^m(\xi, \varphi_\xi) \int_0^R \rho\, j_l(\kappa\rho)\, c_j^m(\mathbf{E}, \rho)\, d\rho, \qquad \forall \xi \leq \kappa,$$

where $\wp_l^m(\xi,\varphi_\xi) := (i)^l(2l+1)\mathrm{e}^{im\varphi_\xi}\int_0^{2\pi}\mathbb{P}_l\left(\frac{\xi}{\kappa}\cos(\varphi)\right)\mathrm{e}^{im\varphi}d\varphi$ and $\mathbb{P}_l$ corresponds to the lth Legendre polynomial. Using the orthogonality relation (5.23), we conclude that $\widehat{E}_1^0 = \widehat{E}_2^0 = 0$ (and thus $\widetilde{\mathbf{E}}_T^0 = \mathbf{0}$), for all $\xi \leq \kappa$. Using the Fourier representation for the impedance boundary condition (see (5.15)), we conclude also that $\widetilde{\mathbf{H}}_t^0 = \mathbf{0}$, for all $\xi \leq \kappa$.

5.3.3. Step 3

Collecting all the previous information, we have proved that the tangential traces $\widetilde{\mathbf{E}}_T^0$ and $\widetilde{\mathbf{H}}_t^0$ are supported in the circumference $\{\xi = \xi_p\}$. Using equation (5.14) we conclude that $\widetilde{\mathbf{E}}^0$ and $\widetilde{\mathbf{H}}^0$ are also supported in the circumference $\{\xi = \xi_p\}$. On another hand, Lemma 5.5 states that $\widetilde{\mathbf{E}}^0, \widetilde{\mathbf{H}}^0 \in \boldsymbol{H}^{-(1/2+\varepsilon)}(\mathbb{R}^2)$. Since the derivatives of the delta distribution do not belong to $H^{-(1/2+\varepsilon)}(\mathbb{R}^2)$, the Fourier traces of the EM fields must be of the form

$$\widetilde{\mathbf{E}}^0 = \Phi(\varphi_\xi)\,\delta(\xi-\xi_p) \quad \text{and} \quad \widetilde{\mathbf{H}}^0 = \Psi(\varphi_\xi)\,\delta(\xi-\xi_p), \tag{5.24}$$

for some vectorial functions $\Phi = (\Phi_\rho, \Phi_\varphi, \Phi_3)$ and $\Psi = (\Psi_\rho, \Psi_\varphi, \Psi_3)$. The last conclusion implies that, up to a constant, the trace couple $(\mathbf{E}^0, \mathbf{H}^0)$ is of the form:

$$\left(\mathbf{E}^0(\rho,\varphi), \mathbf{H}^0(\rho,\varphi)\right) = \int_0^{2\pi} \left(\Phi(\varphi_\xi), \Psi(\varphi_\xi)\right)\mathrm{e}^{i\rho\xi_p\cos(\varphi_\xi-\varphi)}\,\mathrm{d}\varphi_\xi.$$

Applying the stationary phase technique (see [15]) to the previous expressions, we obtain

$$\left(\mathbf{E}^0, \mathbf{H}^0\right) = (\Phi(\varphi), \Psi(\varphi))\frac{\mathrm{e}^{i(\rho\xi_p-\pi/4)}}{\sqrt{\rho}} + (\Phi(\varphi+\pi), \Psi(\varphi+\pi))\frac{\mathrm{e}^{-i(\rho\xi_p-\pi/4)}}{\sqrt{\rho}} + \mathcal{O}(\rho^{-3/2}).$$

Let us assume that the impedance parameter β is positive, then the radiation condition (4.1) over the plane $x_3 = 0$ reads $|\mathbf{H}^0 \times \widehat{\boldsymbol{\rho}} - \xi_p\mathbf{E}^0| = \mathcal{O}(\rho^{-3/2})$. Note that the radiation condition is satisfied if and only if each component of the vectorial expression:

$$\mathbf{H}^0 \times \widehat{\boldsymbol{\rho}} - \xi_p\mathbf{E}^0 = (-\xi_p E_\rho^0, H_3^0 - \xi_p E_\varphi^0, -H_\varphi^0 - \xi_p E_3^0) \tag{5.25}$$

is of order $\rho^{-3/2}$. The first component of (5.25) immediately implies that $E_\rho = \mathcal{O}(\rho^{-3/2})$ and, this is satisfied only if $\Phi_\rho(\varphi) = \Phi_\rho(\varphi+\pi) = 0$. Since φ is arbitrary, we deduce that $\Phi_\rho \equiv 0$. The second component of (5.25) gives that $H_3^0 - \xi_p E_\varphi^0 = \mathcal{O}(\rho^{-3/2})$. We can obtain additional information by using the **curl** relations (2.1). In particular, $H_3^0 = (i\kappa\rho)^{-1}(E_\varphi^0 + \rho\partial_\rho E_\varphi^0 - \partial_\varphi E_\rho^0)$. Since $\rho^{-1}E_\varphi^0$ and $\rho^{-1}\partial_\varphi E_\rho^0$ are already of order $\rho^{-3/2}$, we conclude that $\partial_\rho E_\varphi^0 + i\kappa\xi_p E_\varphi^0 = \mathcal{O}(\rho^{-3/2})$. A straightforward computation gives that this is satisfied only if $\Phi_\varphi \equiv 0$. The latter implies that $H_3^0 = \mathcal{O}(\rho^{-3/2})$ and thus $\Psi_3 \equiv 0$. Also, the impedance boundary condition (2.2) gives the following additional identities: $E_\rho^0 = -i\kappa\beta H_\varphi^0$ and $E_\varphi^0 = i\kappa\beta H_\rho^0$. From here, we can also deduce that $\Psi_\rho \equiv \Phi_\varphi \equiv 0$. Finally, the third component of (5.25) and the fact that $H_\varphi^0 = \mathcal{O}(\rho^{-3/2})$ imply that $\Phi_3 \equiv 0$. Summarizing, we have proved that $\Phi \equiv \Psi \equiv \mathbf{0}$, implying that $\widetilde{\mathbf{E}}^0 = \widetilde{\mathbf{H}}^0 = \mathbf{0}$ by equation (5.24). A similar procedure shows the same result for the case $\beta < 0$.

We show now that $\widetilde{\mathbf{E}}^{x_3} = \widetilde{\mathbf{H}}^{x_3} = \mathbf{0}$, for all $x_3 > 0$. First note that, the representation for $\widetilde{\mathbf{E}}^{x_3}$ given in equation (5.13) evaluated at $x_3 = 0$, implies that $\mathbf{T}^+(\xi,\varphi_\xi) = -\mathbf{T}^-(\xi,\varphi_\xi)$. Thus, for any $x_3 \geq 0$, the electric field can be written as:

$$\widetilde{\mathbf{E}}^{x_3} = \mathbf{T}^-(\xi,\varphi_\xi)\left(\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3} - \mathrm{e}^{\sqrt{\xi^2-\kappa^2}x_3}\right). \tag{5.26}$$

On another hand, since $\widetilde{\mathbf{H}}_t^0 = \mathbf{0}$, the relation (5.17) together with (5.26) imply that

$$\frac{i\kappa}{\sqrt{\xi^2-\kappa^2}}T_\xi^-(\xi,\varphi_\xi) = \frac{1}{i\kappa}\sqrt{\xi^2-\kappa^2}\,T_{\varphi_\xi}^-(\xi,\varphi_\xi) = 0,$$

concluding then that the support of $\widetilde{\mathbf{E}}_T^{x_3} = (T_\xi^-, T_{\varphi_\xi}^-, 0)\left(\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3} - \mathrm{e}^{\sqrt{\xi^2-\kappa^2}x_3}\right)$ is over the circumference $\{\xi = \kappa\}$, as well as the support of $\widetilde{\mathbf{H}}_t^{x_3}$ by relation (5.17). Finally, as a consequence of relations (5.14), we conclude that the support of the whole fields $\widetilde{\mathbf{E}}^{x_3}$ and $\widetilde{\mathbf{H}}^{x_3}$ is over the circumference $\{\xi = \kappa\}$. As we did with $\widetilde{\mathbf{E}}^0$ and $\widetilde{\mathbf{H}}^0$ (see Eq. (5.24)) to conclude that $\mathbf{E}^0 = \mathbf{H}^0 \equiv \mathbf{0}$ –using the stationary phase technique and the radiation condition–, we can also prove that $\mathbf{E}^{x_3} = \mathbf{H}^{x_3} \equiv \mathbf{0}$, for all $x_3 \geq 0$, from the fact that $\widetilde{\mathbf{E}}^{x_3}$ and $\widetilde{\mathbf{H}}^{x_3}$ are compactly supported in $\{\xi = \xi_p\}$. □

5.4. Existence

Theorem 5.13 (Existence)**.** *Let $\mathbf{f}_T = (f_1, f_2, 0) \in \boldsymbol{L}^2(\Gamma)$ be a compactly supported function. Let $(\mathbf{E}, \mathbf{H})$ be the EM couple defined as*

$$\mathbf{E}(\mathbf{x}) = -\frac{1}{\beta}\int_\Gamma \mathsf{G}^T(\mathbf{x},\mathbf{y})\,\mathbf{f}_T(\mathbf{y})\,\mathrm{d}S(\mathbf{y}),\ \ \mathbf{H}(\mathbf{x}) = \frac{1}{i\kappa}\mathbf{curl}\,\mathbf{E}(\mathbf{x}), \tag{5.27}$$

with $\mathsf{G}^T(\mathbf{x},\mathbf{y})$ denoting the transpose of the Green's dyad (3.2). Then, $(\mathbf{E}, \mathbf{H})$ belongs to the space $\boldsymbol{W}_{-\sigma}(\mathbb{R}^3_+) \times \boldsymbol{W}_{-\sigma}(\mathbb{R}^3_+)$, it is a radiative solution of Maxwell equations (2.1) and satisfies the impedance boundary condition (2.2).

Remark 5.14. In equation (5.27), $\mathsf{G}^T(\mathbf{x},\mathbf{y})\,\mathbf{f}_T(\mathbf{y})$ must be understood as a matrix-vector multiplication.

Proof. Notice that $\mathsf{G}^T(\mathbf{x},\mathbf{y}) = \mathsf{G}(\mathbf{y},\mathbf{x})$ (*cf.* [3]). Hence, $\mathsf{G}^T(\mathbf{x},\mathbf{y})$ satisfies the homogeneous electric field problem (3.1a), for any $\mathbf{x} \in \mathbb{R}^3_+$ and $\mathbf{y} \in \mathbb{R}^2 \times \{0\}$ by switching the roles of $\mathbf{x}$ and $\mathbf{y}$. Thus, it is clear that the EM couple $(\mathbf{E}, \mathbf{H})$ satisfies the homogeneous Maxwell equations (2.1). On the other hand, as a consequence of Theorem 4.6, it can be easily proved that the EM couple $(\mathbf{E}, \mathbf{H})$ is also radiative.

It only remains to prove that the couple $(\mathbf{E}, \mathbf{H})$ satisfies the impedance boundary condition (2.2). Without loss of generality, we assume that the couple $(\mathbf{E}, \mathbf{H})$ and the boundary data $\mathbf{f}_T$ are given in the cylindrical coordinate system. For $y_3 = 0$ and any $x_3 > 0$, one performs the computation of the boundary operator $(\mathsf{G}_T(\mathbf{y},\mathbf{x})\,f_T(\mathbf{y}) - \beta\,\widehat{\mathbf{k}} \times \mathbf{curl}_{\mathbf{x}}\,\mathsf{G}(\mathbf{y},\mathbf{x})\,f_T(\mathbf{y}))$ in spectral domain and cylindrical coordinates. Then, taking the limit as $x_3 \to 0^+$ and taking an inverse Fourier transform, it is easy to check that:

$$\left(\mathsf{G}_T\,f_T - \beta\,\widehat{\mathbf{k}} \times \mathbf{curl}_{\mathbf{x}}\,\mathsf{G}\,f_T\right)\Big|_{x_3=0} = -\beta\delta_{(x_1,x_2)}(y_1,y_2)\,(\mathbf{e}_{11} + \mathbf{e}_{22})\,f_T(\mathbf{y}),$$

completing the proof. □

6. Conclusions

We have proven the existence and uniqueness of solutions for the Maxwell problem over a half-space with an impedance non-absorbing boundary condition. For this, we established adequate Silver-Müller-type radiation conditions to be satisfied by the solutions through a detailed study of the Green's dyad associated to this problem. Our existence result is achieved with the help of an integral representation of the solution in terms of the Green's dyad. As immediate future work, it remains to prove existence and uniqueness for a locally perturbed half-space. More general problems such as the impedance problem for the full Engquist-Nédélec boundary condition must be considered. Finally, this work paves the way for obtaining similar results for scattering problems defined over layered media.

Appendix A. Derivation of the Green's dyad

Due to tangential symmetry, we can set the coordinate axes so that $\mathbf{x} = (0, 0, x_3)$. Following [3], we split the Green's dyad (3.1) as $\mathsf{G}(\mathbf{x},\mathbf{y}) = \mathsf{G}^0(\mathbf{x},\mathbf{y}) + \mathsf{P}^\beta(\mathbf{x},\mathbf{y})$, where $\mathsf{G}^0(\mathbf{x},\mathbf{y})$ is the Green's dyad associated with the perfect conductor problem:

$$\begin{cases} \mathbf{curl_y curl_y} \mathsf{G}^0 - \kappa^2 \, \mathsf{G}^0 = \delta_{\mathbf{x}}(\mathbf{y})\mathsf{I} & \text{in } \mathbb{R}^3_+, \\ \mathsf{G}^0_T = \mathbf{0} & \text{on } \Gamma, \end{cases} \tag{A.1}$$

and $\mathsf{P}^\beta(\mathbf{x},\mathbf{y})$ is the radiated solution of the problem:

$$\begin{cases} \mathbf{curl_y curl_y} \mathsf{P}^\beta - \kappa^2 \mathsf{P}^\beta = \mathbf{0} & \text{in } \mathbb{R}^3_+, \\ \mathsf{P}^\beta_T - \beta \, \mathbf{curl_y} \mathsf{P}^\beta \times \mathbf{n} = \beta \, \mathbf{curl_y} \mathsf{G}^0 \times \mathbf{n} & \text{on } \Gamma. \end{cases} \tag{A.2}$$

Denoting by $\mathbf{x}^* = (x_1, x_2, -x_3)$ the transversal reflection point of $\mathbf{x}$, by g the fundamental solution of the scalar Helmholtz equation in $\mathbb{R}^3$:

$$g(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{i\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|},$$

and defining $g^*(\mathbf{x},\mathbf{y}) := g(\mathbf{x}^*,\mathbf{y})$, the solution of problem (A.1) can be written as (*cf.* [12,23]):

$$\mathsf{G}^0(\mathbf{x},\mathbf{y}) := \left(\mathsf{I} + \frac{\mathbf{grad}\,\mathrm{div}}{\kappa^2}\right) \begin{pmatrix} g - g^* & 0 & 0 \\ 0 & g - g^* & 0 \\ 0 & 0 & g + g^* \end{pmatrix}. \tag{A.3}$$

To obtain the solution of problem (A.2), the next steps are followed:

- A Fourier transform is applied along the horizontal axes, reducing the spatial problem into an ODE problem in the vertical variable.
- The resulting ODEs are solved analytically and the spatial solution is represented via an integral form by applying the inverse Fourier transform.

A.1. The spectral problem

We define the tangential Fourier transform as:

$$\mathcal{F}_{y_1,y_2}(f)(\xi_1,\xi_2) =: \widehat{f} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y_1,y_2)\mathrm{e}^{-i(y_1\xi_1+y_2\xi_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Making the change of variables $(\xi_1,\xi_2) = \xi(\cos\varphi_\xi, \sin\varphi_\xi)$, the Fourier transform of the expression $(\mathbf{curlcurl} - \kappa^2)\mathsf{P}^\beta(\mathbf{x},\mathbf{y})$ in cylindrical spectral coordinates (ξ,φ_ξ,y_3) becomes:

$$\begin{pmatrix} -\partial^2_{y_3} - \kappa^2 & 0 & i\xi\partial_{y_3} \\ 0 & -\partial^2_{y_3} + \xi^2 - \kappa^2 & 0 \\ i\xi\partial_{y_3} & 0 & \xi^2 - \kappa^2 \end{pmatrix} \widetilde{\mathsf{P}}^\beta = \mathbf{0}, \tag{A.4}$$

and the Fourier transform of the Leontovich boundary condition reads:

$$\begin{pmatrix} 1 + \beta\partial_{y_3} & 0 & -i\beta\xi \\ 0 & 1 + \beta\partial_{y_3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \widetilde{\mathsf{P}}^\beta = -\beta \begin{pmatrix} \partial_{y_3} & 0 & -i\xi \\ 0 & \partial_{y_3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \widetilde{\mathsf{G}}^0, \tag{A.5}$$

with $\widetilde{\mathsf{P}}^\beta$ and $\widetilde{\mathsf{G}}^0$ denoting the Fourier transforms of P^β and G^0, respectively. The right-hand expression in the previous equality depends on $\widetilde{\mathsf{G}}^0$. We note that

$$\mathcal{F}_{y_1,y_2}(g)(\xi_1,\xi_2) =: \widetilde{g}(\xi) = \frac{\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}|x_3-y_3|}}{4\pi\sqrt{\xi^2-\kappa^2}},$$

$$\mathcal{F}_{y_1,y_2}(g^*)(\xi_1,\xi_2) =: \widetilde{g^*}(\xi) = \frac{\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}(x_3+y_3)}}{4\pi\sqrt{\xi^2-\kappa^2}}, \text{ for } y_3 \geq 0.$$

Thus, by (A.3) we get:

$$\widetilde{\mathsf{G}}^0 = \frac{1}{\kappa^2}\begin{pmatrix} \kappa^2-\xi^2 & 0 & i\xi\partial_{y_3} \\ 0 & \kappa^2 & 0 \\ i\xi\partial_{y_3} & 0 & \kappa^2+\partial_{y_3}^2 \end{pmatrix}\begin{pmatrix} \widetilde{g}-\widetilde{g}^* & 0 & 0 \\ 0 & \widetilde{g}-\widetilde{g}^* & 0 \\ 0 & 0 & \widetilde{g}+\widetilde{g}^* \end{pmatrix}$$

$$= \frac{1}{4\pi\kappa^2}\left(\widetilde{\mathsf{G}}^{0-}\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}|x_3-y_3|} + \widetilde{\mathsf{G}}^{0+}\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}(x_3+y_3)}\right),$$

with

$$\begin{cases} \widetilde{\mathsf{G}}^{0-} := \begin{pmatrix} -\sqrt{\xi^2-\kappa^2} & 0 & -i\xi\,\mathrm{sign}(y_3-x_3) \\ 0 & \dfrac{\kappa^2}{\sqrt{\xi^2-\kappa^2}} & 0 \\ -i\xi\,\mathrm{sign}(y_3-x_3) & 0 & \dfrac{\xi^2}{\sqrt{\xi^2-\kappa^2}} \end{pmatrix} \\ \text{and} \\ \widetilde{\mathsf{G}}^{0+} := \begin{pmatrix} \sqrt{\xi^2-\kappa^2} & 0 & -i\xi \\ 0 & -\dfrac{\kappa^2}{\sqrt{\xi^2-\kappa^2}} & 0 \\ i\xi & 0 & \dfrac{\xi^2}{\sqrt{\xi^2-\kappa^2}} \end{pmatrix}. \end{cases}$$

In conclusion, the right-hand side of equation (A.5) is given by:

$$-\beta\begin{pmatrix} \partial_{y_3} & 0 & -i\xi \\ 0 & \partial_{y_3} & 0 \\ 0 & 0 & 0 \end{pmatrix}\widetilde{\mathsf{G}}^0 = -\frac{\beta}{2\pi}\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}x_3}\begin{pmatrix} 1 & 0 & -\dfrac{i\xi}{\sqrt{\xi^2-\kappa^2}} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By direct computation, one can see that $\widetilde{\mathsf{P}}^\beta$ has the following structure:

$$\widetilde{\mathsf{P}}^\beta = \frac{\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}(x_3+y_3)}}{4\pi\kappa^2}\widetilde{\mathsf{P}}^{\beta+}, \qquad \text{for some } \widetilde{\mathsf{P}}^{\beta+} = \begin{pmatrix} \widetilde{\mathsf{P}}^\beta_{\xi\xi} & 0 & \widetilde{\mathsf{P}}^\beta_{\xi 3} \\ 0 & \widetilde{\mathsf{P}}^\beta_{\varphi_\xi\varphi_\xi} & 0 \\ \widetilde{\mathsf{P}}^\beta_{3\xi} & 0 & \widetilde{\mathsf{P}}^\beta_{33} \end{pmatrix}.$$

To obtain the unknown terms, first note that the third row in equation (A.4) gives the following relations:

$$\widetilde{\mathsf{P}}^\beta_{3\xi} = -\frac{i\xi}{\sqrt{\kappa^2-\xi^2}}\widetilde{\mathsf{P}}^\beta_{\xi\xi} \qquad \text{and} \qquad \widetilde{\mathsf{P}}^\beta_{33} = -\frac{i\xi}{\sqrt{\kappa^2-\xi^2}}\widetilde{\mathsf{P}}^\beta_{\xi 3}.$$

Using the relations (A.5) and the fact that $\widetilde{\mathsf{P}}^\beta$ is divergence-free, one can deduce the following expressions for $\widetilde{\mathsf{P}}^\beta_{\xi\xi}, \widetilde{\mathsf{P}}^\beta_{\varphi_\xi\varphi_\xi}$ and $\widetilde{\mathsf{P}}^\beta_{\xi 3}$ respectively:

$$\widetilde{\mathsf{P}}^\beta_{\xi\xi} = -\frac{2\beta\kappa^2\sqrt{\xi^2-\kappa^2}}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2}, \qquad \widetilde{\mathsf{P}}^\beta_{\varphi_\xi\varphi_\xi} = \frac{2\beta\kappa^2}{\beta\sqrt{\xi^2-\kappa^2}-1} \qquad \text{and} \qquad \widetilde{\mathsf{P}}^\beta_{\xi 3} = \frac{i2\beta\kappa^2\xi}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2}.$$

Consequently, we can also derive

$$\widetilde{\mathsf{P}}^\beta_{3\xi} = -\frac{2i\beta\kappa^2\xi}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2} \qquad \text{and} \qquad \widetilde{\mathsf{P}}^\beta_{33} = -\frac{2\beta\kappa^2\xi^2}{\sqrt{\xi^2-\kappa^2}\left(\sqrt{\xi^2-\kappa^2}+\beta\kappa^2\right)}.$$

Summarizing, the spectral Electric Green's dyad $\widetilde{\mathsf{G}}$ is given by:

$$\widetilde{\mathsf{G}} = \frac{1}{4\pi\kappa^2}\left(\widetilde{\mathsf{G}}^{-}\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}|x_3-y_3|} + \widetilde{\mathsf{G}}^{+}\mathrm{e}^{-\sqrt{\xi^2-\kappa^2}(x_3+y_3)}\right),$$

with $\widetilde{\mathsf{G}}^{-} := \widetilde{\mathsf{G}}^{0-}$ and

$$\widetilde{\mathsf{G}}^{+} := \widetilde{\mathsf{G}}^{0+} + \widetilde{\mathsf{P}}^{\beta+} = \begin{pmatrix} -\sqrt{\xi^2-\kappa^2}\dfrac{\sqrt{\xi^2-\kappa^2}-\beta\kappa^2}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2} & 0 & -i\xi\dfrac{\sqrt{\xi^2-\kappa^2}-\beta\kappa^2}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2} \\ 0 & \dfrac{\beta\sqrt{\xi^2-\kappa^2}+1}{\beta\sqrt{\xi^2-\kappa^2}-1} & 0 \\ i\xi\dfrac{\sqrt{\xi^2-\kappa^2}-\beta\kappa^2}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2} & 0 & \dfrac{\xi^2}{\sqrt{\xi^2-\kappa^2}}\dfrac{\sqrt{\xi^2-\kappa^2}-\beta\kappa^2}{\sqrt{\xi^2-\kappa^2}+\beta\kappa^2} \end{pmatrix}.$$

Appendix B. Spherical representation of $\mathbf{M}_l^m[f_l]$ and $\mathbf{N}_l^m[f_l]$

Consider the local orthogonal unit vectors in the directions of increasing r, θ and φ respectively,

$$\widehat{\mathbf{r}} = \sin\theta\,\widehat{\boldsymbol{\rho}} + \cos\theta\,\widehat{\mathbf{k}}\,,\ \widehat{\boldsymbol{\theta}} = \cos\theta\,\widehat{\boldsymbol{\rho}} - \sin\theta\,\widehat{\mathbf{k}}\,,\ \widehat{\boldsymbol{\varphi}} = \widehat{\boldsymbol{\varphi}}.$$

For a field $A = A_r\,\widehat{\mathbf{r}} + A_\theta\,\widehat{\boldsymbol{\theta}} + A_\varphi\,\widehat{\boldsymbol{\varphi}}$, its **curl** has the form:

$$\begin{aligned}\mathbf{curl}\,\mathbf{A} = {}& \frac{1}{r\sin\theta}\left(\partial_\theta(A_\varphi\sin\theta) - \partial_\varphi A_\theta\right)\widehat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\partial_\varphi A_r - \partial_r(rA_\varphi)\right)\widehat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left(\partial_r(rA_\theta) - \partial_\theta A_r\right)\widehat{\boldsymbol{\varphi}}.\end{aligned}$$

The associated vectorial functions in spherical coordinate system are given by

$$\begin{aligned}\mathbf{M}_l^m[f](r,\theta,\varphi) &= f_l(\kappa r)\mathrm{e}^{im\varphi}\left(\frac{im}{\sin\theta}\mathbb{P}_l^m(\cos\theta)\,\widehat{\boldsymbol{\theta}} + \sin\theta\,\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_l^m(\cos\theta)\,\widehat{\boldsymbol{\varphi}}\right), \\ \mathbf{N}_l^m[f](r,\theta,\varphi) &= \frac{f_l(\kappa r)\mathrm{e}^{im\varphi}}{\kappa r}h_r\,\widehat{\mathbf{r}} - \frac{\sin\theta}{\kappa r}\,h_\theta\,\widehat{\boldsymbol{\theta}} + \frac{im}{\sin\theta\kappa r}\,h_\varphi\,\widehat{\boldsymbol{\varphi}},\end{aligned}$$

with coefficients

$$\begin{aligned}h_r &:= 2\cos\theta\,\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_l^m(\cos\theta) - \sin^2\theta\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathbb{P}_l^m(\cos\theta) + \frac{m^2}{\sin^2\theta}\mathbb{P}_l^m(\cos\theta), \\ h_\theta &:= \left(f_l(\kappa r) + \kappa r\frac{\mathrm{d}}{\mathrm{d}x}f_l(\kappa r)\right)\mathrm{e}^{im\varphi}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_l^m(\cos\theta), \\ h_\varphi &:= \left(f_l(\kappa r) + \kappa r\frac{\mathrm{d}}{\mathrm{d}x}f_l(\kappa r)\right)\mathrm{e}^{im\varphi}\mathbb{P}_l^m(\cos\theta).\end{aligned}$$

B.1. Tangential traces of $\mathbf{M}_l^m[f], \mathbf{N}_l^m[f]$ over the plane $\theta = \pi/2$

Since we need to calculate integrals over the plane $x_3 = 0$, we are interested on the behavior of the vectorial functions $\mathbf{M}_l^m[f]$ and $\mathbf{N}_l^m[f]$ when $\theta = \pi/2$. Therefore, the coordinate system becomes

$$\widehat{\mathbf{r}}_{(\theta=\pi/2)} = \widehat{\boldsymbol{\rho}},\ \widehat{\boldsymbol{\theta}}_{(\theta=\pi/2)} = -\widehat{\mathbf{k}},\ \widehat{\boldsymbol{\varphi}}_{(\theta=\pi/2)} = \widehat{\boldsymbol{\varphi}},$$

and the spherical representations are of the form:

$$\gamma_T\left(\mathbf{M}_l^m[f_l]\right) = f_l(\kappa r)\mathrm{e}^{im\varphi}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_l^m(0)\,\widehat{\boldsymbol{\varphi}} = (l+m)f_l(\kappa r)\mathrm{e}^{im\varphi}\mathbb{P}_{(l-1)}^m(0)\,\widehat{\boldsymbol{\varphi}}, \tag{B.1}$$

and

$$\gamma_T\left(\mathbf{N}_l^m[f_l]\right) = \frac{f_l(\kappa r)}{\kappa r}e^{im\varphi}\left(m^2\mathbb{P}_l^m(0) - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathbb{P}_l^m(0)\right)\widehat{\boldsymbol{\rho}} + im\left(f_l(\kappa r) + \kappa r\frac{\mathrm{d}}{\mathrm{d}x}f_l(\kappa r)\right)e^{im\varphi}\mathbb{P}_l^m(0)\,\widehat{\boldsymbol{\varphi}}. \tag{B.2}$$

(B.3)

Remark B.1. Last equality in equation (B.1) comes from the recursive relation (*cf.* [1]):

$$(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}_l^m(x) = l\,x\,\mathbb{P}_l^m(x) - (l+m)\mathbb{P}_{l-1}^m(x).$$

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