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CONVERGENCE OF A HOMOTOPY FINITE ELEMENT METHOD FOR COMPUTING STEADY STATES OF BURGERS' EQUATION

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Abstract. In this paper, the convergence of a homotopy method (1.1) for solving the steady state problem of Burgers' equation is considered. When ν is fixed, we prove that the solution of (1.1) converges to the unique steady state solution as $\epsilon \to 0$, which is independent of the initial conditions. Numerical examples are presented to confirm this conclusion by using the continuous finite element method. In contrast, when $\nu = \epsilon \to 0$, numerically we show that steady state solutions obtained by (1.1) indeed depend on initial conditions.

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1. Introduction

Studies of steady state problems for hyperbolic equations have been focused on time marching of the original problem. The argument behind time marching is that for a steady state problem, the transient components in the flow field will gradually decay as time tends towards infinity. Clearly, time marching can provide a physical path for propagating boundary information mildly into the whole flow field, especially at the initial stage of a simulation. However, the starting Courant-Friedrichs-Lewy condition (CFL) number and the strategy to advance it might vary in practical applications. Especially for stiff flow problems, *e.g.*, high Reynolds number turbulent flow, high-speed flow with shocks, etc., a desired convergence may not be achieved when time marching is used. This can dramatically hinder the application of these time marching numerical methods in industrial design processes.

Homotopy continuation, which has been widely adopted in Numerical Algebraic Geometry (NAG) (*e.g.*, Bates *et al.* [3], Sommese and Wampler [24]) and bifurcation analysis (Hao *et al.* [10]), provides an alternative way for solving nonlinear systems governing fluid flow. Recently the homotopy method has been developed to compute steady states of hyperbolic conservation laws which has been a major research and application area of computational mathematics in the last few decades. In Hao *et al.* [12], a homotopy continuation method coupled with some numerical techniques from NAG has demonstrated that the homotopy continuation approach is efficient to handle singular systems and can also be used to find multiple steady states. In Hicken and Zingg [16], Hicken *et al.* [15], a homotopy continuation algorithm, constructed based on a dissipation operator, has been designed as a general globalization method for a Newton–Krylov external aerodynamic flow solver. It is

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more efficient than time marching methods in some cases, especially inviscid flows, but is not competitive for turbulent flows.

Although homotopy continuation algorithms based on predictor-corrector methods can be easily implemented Hao *et al.* [10, 11, 13, 14], the convergence of numerical solution to the entropy solution by using the homotopy continuation setup is unclear. In this paper, we will use the homotopy continuation method to explore the steady state problem of the 1D Burgers' equation. In particular, we will study the following problem:

$$\left[\left(\frac{{u_\epsilon}^2}{2}\right)_{,x} - g - \nu u_{\epsilon,xx}\right](1-\epsilon) + \lambda\epsilon(u_\epsilon - u_0) = 0, \tag{1.1}$$

and pay more attention to the behavior of the weak solution $u_\epsilon(x)$ as $\epsilon \to 0$ and $\nu \to 0$. Since $u_\epsilon(x)$ depends on both ϵ and x, we use the subscript $(,x)$ to denote the derivative with respect to x. In (1.1), the vanishing viscosity method is introduced to obtain the unique entropy solution and ν is the artificial viscosity coefficient. The parameter λ in (1.1) is a constant quantity introduced to balance the physical dimension. Here the initial condition u_0 is related to the 1D Burgers' equation:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_{,x} = g(x), \quad x \in \Omega = [a,b] \\ u(0,x) = u_0(x), \\ u(t,a) = u(t,b) = 0. \end{cases} \tag{1.2}$$

The initial condition $u_0(x)$ should be consistent with two boundary conditions, $u_0(a) = u_0(b) = 0$, in the sense that they are both of inflow type at a and b for all $t \geq 0$. In other words, the data is entering the domain along the characteristic curves through two endpoints a and b; see *e.g.*, Bardos *et al.* [25], Strub and Bayen [2] and references therein. Furthermore, $g(x)$ is chosen so that the steady state problem is well-posed. In particular, this paper will consider

$$g(x) := \sin x \cos x, \quad u_0(x) := \beta \sin x, \quad \beta \in (-1,1) \text{ and } \Omega = [0,\pi], \tag{1.3}$$

which is studied in Salas *et al.* [23]. Here $\int_\Omega g = 0$ makes the problem conservative $\int_\Omega u(t,x) = \int_\Omega u_0(x)$ for all $t \geq 0$.

In this paper we will study $u_\epsilon(x)$ as a weak solution of (1.1), which is defined as $u \in H_0^1(\Omega)$ satisfying

$$-(1-\epsilon)(u^2/2, v_{,x}) - (1-\epsilon)(g,v) + (1-\epsilon)\nu(u_{,x}, v_{,x}) + \lambda\epsilon(u - u^0, v) = 0, \quad \forall v \in H_0^1(\Omega).$$

Here we use $(\cdot,\cdot)$ as the usual inner product of the Hilbert space $L^2(\Omega)$. For the steady state problem for the 1D Burgers' equation, there are two kinds of behavior of the steady state solution for the vanishing viscosity method $u_t + uu_{,x} = \nu u_{xx}$, as shown in Hopf [17]: the behavior as $t \to \infty$ while ν stays constant and the behavior while $\nu \to 0$. In the present paper, we will show that there are also two kinds of behavior of the limit solution of our homotopy continuation setup (1.1): the behavior as $\epsilon \to 0$ while ν stays constant and the behavior while $\nu = \epsilon \to 0$.

For the case of ν being kept constant, based on the energy estimate, first we will study the uniqueness of the solution $u_\epsilon(x)$ for a fixed ϵ and the initial condition u_0 in Section 2.1. The independence of $u_\epsilon(x)$ on the initial condition as $\epsilon \to 0$ is considered in Section 2.2, while the convergence for the given initial condition as $\epsilon \to 0$ is shown in Section 2.3. Based on the fixed point theorem, we will show the existence (and uniqueness) of the solution $u_\epsilon(x)$ for a fixed ϵ and the initial condition u_0 in Section 2.4.

For the case $\nu = \epsilon \to 0$, we will study it numerically. It is known that Burgers' equation can be solved by many numerical methods such as finite volume methods (*e.g.* Lax [20], Leer [21], Nessyahu and Tadmor [22]) and finite element methods (*e.g.* Guermond and Nazarov [8], Guermond and Popov [6], Guermond *et al.* [7, 9]). In the present paper, we choose the continuous finite element metho d to solve the homotopy continuation setup (1.1). The implementation can fully illustrate those two kinds of behavior of the steady state solutions and show the consistency between our homotopy continuation setup (1.1) and the vanishing viscosity setup in Hopf [17] when ν is fixed or $\nu = \epsilon \to 0$. When ν stays a constant, Kreiss and Kreiss [19] show the independence of the steady state solution on the initial condition for the vanishing viscosity setup. Similarly, we will show the independence for the homotopy continuation setup (1.1) as $\epsilon \to 0$.

The paper is organized as follows. In Section 2, we show some results of the behavior as $\epsilon \to 0$ while ν stays constant, including the existence and uniqueness of the solution $u_\epsilon(x, u_0)$, the existence of the limit as $\epsilon \to 0$ and the independence of the limit as $\epsilon \to 0$ on the initial condition. In Section 3, we use the continuous finite element and Newton's iteration methods to implement the homotopy continuation algorithm (1.1) and show that the numerical method is convergent and conservative. In Section 4, numerical tests are used to confirm two kinds of solution behaviors.

2. Convergence analysis of homotopy continuation method

To highlight the possible dependence of the weak solution of (1.1) on ϵ and u_0, we use the notation $u_\epsilon(x, u_0)$. In this section, we assume that ν is a positive constant, and will show that the solution $u_\epsilon(x, u_0)$ changes from u_0 to the steady state of the viscous Burgers' equation, as ϵ changes from 1 to 0 in (2.5). More precise requirements on ν will be presented in different statements. We will denote the $L^2(\Omega)$ norm by $\|\cdot\|$ and the $L^\infty(\Omega)$ norm by $\|\cdot\|_\infty$.

2.1. Uniqueness of $u_\epsilon(x, u_0)$

Lemma 2.1. *The solution $u_{\epsilon,x}(x, u_0)$ of (1.1) satisfies the following a priori estimates*

$$\|u_\epsilon(x,u_0)\| \le \frac{c_p^2}{\nu}\|g\| + \|u_0\| \text{ and } \|u_{\epsilon,x}(x,u_0)\| \le \sqrt{\frac{\lambda\epsilon}{(1-\epsilon)\nu}}\|u_0\| + \frac{c_p}{\nu}\|g\|,$$

where c_p is the constant in Poincaré inequality i.e., $\|v\| \le c_p\|v_{,x}\|$, $\forall v \in H_0^1(\Omega)$.

Proof. Testing (1.1) with $u_\epsilon(x, u_0)$, one gets

$$(1-\epsilon)\int_\Omega \left(\frac{u_\epsilon(x,u_0)^2}{2}\right)_{,x} u_\epsilon(x,u_0) - (1-\epsilon)\int_\Omega g(x)u_\epsilon(x,u_0)$$
$$+ \nu(1-\epsilon)\int_\Omega |u_{\epsilon,x}(x,u_0)|^2 + \lambda\epsilon\int_\Omega |u_\epsilon(x,u_0)|^2 = \lambda\epsilon\int_\Omega u_\epsilon(x,u_0)u_0.$$

The Dirichlet boundary condition implies that

$$\int_\Omega \left(\frac{u_\epsilon(x,u_0)^2}{2}\right)_{,x} u_\epsilon(x,u_0) = \int_\Omega \left(\frac{u_\epsilon(x,u_0)^3}{3}\right)_{,x} = 0,$$

which follows that

$$\nu(1-\epsilon)\int_\Omega |u_{\epsilon,x}(x,u_0)|^2 + \lambda\epsilon\int_\Omega |u_\epsilon(x,u_0)|^2 = \lambda\epsilon\int_\Omega u_\epsilon(x,u_0)u_0 + (1-\epsilon)\int_\Omega g(x)u_\epsilon(x,u_0). \tag{2.1}$$

Using Poincaré inequality $\|v\| \le c_p\|v_{,x}\|$, we have

$$(1-\epsilon)\int_\Omega g(x)u_\epsilon(x,u_0) \le (1-\epsilon)\|g\|\|u_\epsilon(x,u_0)\|$$
$$\le c_p(1-\epsilon)\|g\|\|u_{\epsilon,x}(x,u_0)\| \le \frac{(1-\epsilon)}{2\nu}c_p^2\|g\|^2 + \frac{(1-\epsilon)\nu}{2}\|u_{\epsilon,x}(x,u_0)\|^2.$$

From (2.1), we obtain

$$(1-\epsilon)\nu\|u_{\epsilon,x}(x,u_0)\|^2+\lambda\epsilon\|u_\epsilon(x,u_0)\|^2\le\lambda\epsilon\|u_0\|^2+(1-\epsilon)\frac{c_p^2}{\nu}\|g\|^2,$$

which implies that

$$\|u_{\epsilon,x}(x,u_0)\|^2\le\frac{\lambda\epsilon}{(1-\epsilon)\nu}\|u_0\|^2+\frac{c_p^2}{\nu^2}\|g\|^2.$$

Moreover, since $(1-\epsilon)\int_\Omega g(x)u_\epsilon(x,u_0)\le c_p(1-\epsilon)\|g\|\|u_{\epsilon,x}(x,u_0)\|$, from (2.1), we have

$$(1-\epsilon)\|u_{\epsilon,x}(x,u_0)\|\left(\nu\|u_{\epsilon,x}(x,u_0)\|-c_p\|g\|\right)+\frac{\lambda\epsilon}{2}\|u_\epsilon(x,u_0)\|^2\le\frac{\lambda\epsilon}{2}\|u_0\|^2.$$

Considering the following two cases

- if $\|u_{\epsilon,x}(x,u_0)\|\ge\frac{c_p}{\nu}\|g\|$, then $\|u_\epsilon(x,u_0)\|\le\|u_0\|$,
- if $\|u_{\epsilon,x}(x,u_0)\|\le\frac{c_p}{\nu}\|g\|$, then $\|u_\epsilon(x,u_0)\|\le\frac{c_p^2}{\nu}\|g\|$,

we conclude that$\|u_\epsilon(x,u_0)\|\le\frac{c_p^2}{\nu}\|g\|+\|u_0\|$. □

Lemma 2.2. *The solution $u_\epsilon(x,u_0)$ of (1.1) satisfies the following $L^\infty(\Omega)$ estimate*

$$\|u_\epsilon(x,u_0)\|_\infty\le|\Omega|^{-1/2}\|u_0\|+|\Omega|^{1/2}\sqrt{\frac{\lambda\epsilon}{(1-\epsilon)\nu}}\|u_0\|+(|\Omega|^{-1/2}c_p+|\Omega|^{1/2})\frac{c_p}{\nu}\|g\|. \tag{2.2}$$

Proof. Choose $p\in\Omega$ such that

$$|u_\epsilon(p,u_0)|\le|\Omega|^{-1/2}\|u_\epsilon(x,u_0)\|.$$

Since

$$u_\epsilon(x,u_0)-u_\epsilon(p,u_0)=\int_p^x u_{\epsilon,x}(x,u_0)\le|\Omega|^{1/2}\|u_{\epsilon,x}(x,u_0)\|,$$

we have

$$\|u_\epsilon(x,u_0)\|_\infty\le|\Omega|^{-1/2}\|u_\epsilon(x,u_0)\|+|\Omega|^{1/2}\|u_{\epsilon,x}(x,u_0)\|.$$

By Lemma 2.1, we get (2.2). □

Lemma 2.3. *Let $u_\epsilon(x,u_i)$ be solutions of (1.1) corresponding to two different initial conditions $u_i, i=0,1$. If ν satisfies*

$$2c_p\|u_\epsilon(x,u_0)\|_\infty\le\nu, \tag{2.3}$$

then

$$(1-\epsilon)\nu\|(u_\epsilon(x,u_1)-u_\epsilon(x,u_0))_{,x}\|^2+\lambda\epsilon\|u_\epsilon(x,u_1)-u_\epsilon(x,u_0)\|^2\le\lambda\epsilon\|u_1-u_0\|^2.$$

Proof. Since $u_\epsilon(x, u_i)$ are solutions of (1.1) under the initial condition u_i, we have

$$\left[\left(\frac{u_\epsilon(x,u_i)^2}{2}\right)_{,x} - g(x) - \nu u_{\epsilon,xx}(x,u_i)\right](1-\epsilon) + \lambda(u_\epsilon(x,u_i) - u_i)\epsilon = 0,$$

and then

$$\left[\left(\frac{u_\epsilon(x,u_1)^2 - u_\epsilon(x,u_0)^2}{2}\right)_{,x} - \nu\left(u_{\epsilon,xx}(x,u_1) - u_{\epsilon,xx}(x,u_0)\right)\right](1-\epsilon) \\ + \lambda\epsilon(u_\epsilon(x,u_1) - u_\epsilon(x,u_0)) = \lambda\epsilon(u_1 - u_0).$$

Denoting $w(x) := u_\epsilon(x,u_1) - u_\epsilon(x,u_0)$, we have

$$[u_\epsilon(x,u_1)w_{,x} + wu_{\epsilon,x}(x,u_0) - \nu w_{xx}](1-\epsilon) + \lambda\epsilon w = \lambda\epsilon(u_1 - u_0).$$

Taking w as a test function, we obtain that

$$(1-\epsilon)\int_\Omega [u_\epsilon(x,u_1)ww_{,x} + w^2 u_{\epsilon,x}(x,u_0)] + (1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \lambda\epsilon\int_\Omega |w|^2 = \lambda\epsilon\int_\Omega w(u_1 - u_0).$$

Since $u_\epsilon(x,u_0)|_{\partial\Omega} = 0$, it follows that

$$(1-\epsilon)\int_\Omega [w - u_\epsilon(x,u_0)]ww_{,x} + (1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \lambda\epsilon\int_\Omega |w|^2 = \lambda\epsilon\int_\Omega w(u_1 - u_0).$$

Since $\int_\Omega w^2 w_{,x} = 0$, we have

$$(1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \lambda\epsilon\int_\Omega |w|^2 = \lambda\epsilon\int_\Omega w(u_1 - u_0) + (1-\epsilon)\int_\Omega u_\epsilon(x,u_0)ww_{,x},$$

which implies that

$$\begin{aligned}(1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \frac{\lambda\epsilon}{2}\int_\Omega |w|^2 &\le \frac{\lambda\epsilon}{2}\|u_1 - u_0\|^2 + (1-\epsilon)\|u_\epsilon(x,u_0)\|_\infty \int_\Omega |ww_{,x}| \\ &\le \frac{\lambda\epsilon}{2}\|u_1 - u_0\|^2 + (1-\epsilon)\|u_\epsilon(x,u_0)\|_\infty \|w\|\|w_{,x}\| \\ &\le \frac{\lambda\epsilon}{2}\|u_1 - u_0\|^2 + (1-\epsilon)c_p\|u_\epsilon(x,u_0)\|_\infty \|w_{,x}\|^2.\end{aligned}$$

Therefore, we obtain

$$2(1-\epsilon)(\nu - c_p\|u_\epsilon(x,u_0)\|_\infty)\|w_{,x}\|^2 + \lambda\epsilon\|w\|^2 \le \lambda\epsilon\|u_1 - u_0\|^2.$$

Under the condition (2.3), we have

$$(1-\epsilon)\nu\|w_{,x}\|^2 + \lambda\epsilon\|w\|^2 \le \lambda\epsilon\|u_1 - u_0\|^2.$$

which completes the proof. ∎

Corollary 2.4. *For fixed $0 \le \epsilon < 1$ and given initial condition u_0, if ν satisfies the condition (2.3), then the solution $u_\epsilon(x,u_0)$ of the nonlinear problem (1.1) is unique in $H_0^1(\Omega)$.*

2.2. Uniqueness as $\epsilon \to 0$

In this subsection, we will show that the limit of $u_\epsilon(x,u_0)$ is independent of the initial condition u_0.

Theorem 2.5. *Let $u_\epsilon(x,u_i)$ be the solution of (1.1) corresponding to two different initial conditions $u_i, i=0,1$. Assuming $u_0 \equiv 0$, and if ν is large enough such that*

$$2(|\Omega|^{-1/2}c_p + |\Omega|^{1/2})c_p^2\|g\| \le \nu^2, \tag{2.4}$$

then

$$\|u_\epsilon(x,u_1) - u_\epsilon(x,u_0\equiv 0)\|_{H_0^1(\Omega)} \to_{\epsilon\to 0} 0.$$

Proof. By Lemma 2.2 we have

$$2c_p\|u_\epsilon(x,u_0\equiv 0)\|_\infty \le (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})\frac{2c_p^2}{\nu}\|g\| \le \nu,$$

under the condition (2.4). Therefore ν satisfies the condition (2.3). Applying Lemma 2.3, it follows that

$$(1-\epsilon)\nu\|(u_\epsilon(x,u_1) - u_{\epsilon,x}(x,u_0))\|^2 \le \lambda\epsilon\|u_1-u_0\|^2.$$

As $\epsilon\to 0$, since ν is independent of ϵ, it follows $\|(u_\epsilon(x,u_1) - u_{\epsilon,x}(x,u_0))\| \to 0$. Considering the homogeneous Dirichlet boundary condition and Poincaré inequality, it completes the proof. □

Remark 2.6. It seems that the condition (2.4) is not strong enough to guarantee the uniqueness of $u_\epsilon(x,u_1)$ by just using Lemma 2.3 or Corollary 2.4 by noticing that the condition (2.3) involves u_0 and not u_1. The sharpness of the condition (2.4) will be investigated in the future. However, one can choose a bigger ν to guarantee the uniqueness of $u_\epsilon(x,u_1)$ as stated in the following theorem.

Theorem 2.7. *Let $a_i := |\Omega|^{-1/2}\|u_i\|, b_i := |\Omega|^{1/2}\sqrt{\frac{\lambda\epsilon}{(1-\epsilon)}}\|u_i\|,\ i=0,1$ and $c := (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})c_p\|g\|$. Assuming that α_i is the largest positive solution of the equation $a_i x + b_i\sqrt{x} + c = \frac{x^2}{2c_p}$, if $\nu \ge \max\{\alpha_0,\alpha_1\}$, then $u_\epsilon(x,u_1)\to u_\epsilon(x,u_0)$ in $H_0^1(\Omega)$ as $\epsilon\to 0$.*

Proof. By Lemma 2.2, we have

$$2c_p\|u_\epsilon(x,u_i)\|_\infty \le 2c_p[a_i + b_i\frac{1}{\sqrt{\nu}} + c\frac{1}{\nu}] \le 2c_p[a_i + b_i\frac{1}{\sqrt{\alpha_i}} + c\frac{1}{\alpha_i}] = 2c_p\frac{a_i\alpha_i + b_i\sqrt{\alpha_i}+c}{\alpha_i} \le \alpha_i \le \nu,$$

which implies that the condition (2.3) is true for both u_0 and u_1. Therefore, the uniqueness of $u_\epsilon(x,u_0)$ and $u_\epsilon(x,u_1)$ is guaranteed.

Furthermore, by Lemma 2.3, we obtain that

$$(1-\epsilon)\nu\|(u_\epsilon(x,u_1) - u_{\epsilon,x}(x,u_0))\|^2 \le \lambda\epsilon\|u_1-u_0\|^2.$$

As $\epsilon\to 0$, since ν is independent of ϵ, it follows $\|(u_\epsilon(x,u_1) - u_{\epsilon,x}(x,u_0))\| \to 0$. Considering the homogeneous Dirichlet boundary condition and Poincaré inequality, it completes the proof. □

2.3. Convergence as $\epsilon \to 0$

Let $u^* \in H_0^1(\Omega)$ be the solution of the vanishing viscosity method

$$\begin{cases} \left(\frac{u^2}{2}\right)_{,x} - g(x) - \nu u_{,xx} = 0, \\ u(a) = u(b) = 0. \end{cases} \tag{2.5}$$

For any $\nu > 0$, the existence and uniqueness of u^* is followed by Theorem 2.2 in Kreiss and Kreiss [19].

In this section, we will show that the solution $u_\epsilon(x,u_0)$ of the homotopy problem (1.1) converges to u^* as $\epsilon\to 0$.

Lemma 2.8. *Let $u_\epsilon(x,u_0) \in H_0^1(\Omega)$ be the solution of (1.1) with the initial condition u_0. If ν is a fixed positive constant and satisfies (2.3), then $u_\epsilon(x,u_0) \to u^*$ in $H_0^1(\Omega)$ as $\epsilon \to 0$.*

Proof. Since $u_\epsilon(x,u_0)$ is the solution of (1.1) with the initial condition u_0, namely,

$$\left[\left(\frac{u_\epsilon(x,u_0)^2}{2}\right)_{,x} - g(x) - \nu u_{\epsilon,xx}(x,u_0)\right](1-\epsilon) + \lambda[u_\epsilon(x,u_0) - u_0]\epsilon = 0,$$

we have

$$\left[\left(\frac{u_\epsilon(x,u_0)^2 - u^*(x)^2}{2}\right)_{,x} - \nu\left(u_{\epsilon,xx}(x,u_0) - u^*_{xx}(x)\right)\right](1-\epsilon) + \lambda\epsilon u_\epsilon(x,u_0) = \lambda\epsilon u_0. \tag{2.6}$$

By denoting $w(x) := u_\epsilon(x,u_0) - u^*(x)$, (2.6) is rewritten as

$$[u_\epsilon(x,u_0)w_{,x} + wu^*(x)_{,x} - \nu w_{xx}](1-\epsilon) + \lambda\epsilon u_\epsilon(x,u_0) = \lambda\epsilon u_0.$$

Taking w as a test function, we deduce that

$$(1-\epsilon)\int_\Omega \left(u_\epsilon(x,u_0)ww_{,x} + w^2u^*(x)_{,x}\right) + (1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \lambda\epsilon\int_\Omega wu_\epsilon(x,u_0) = \lambda\epsilon\int_\Omega wu_0.$$

Considering the boundary condition $u_\epsilon(x,u_0)|_{\partial\Omega} = 0$, it follows that

$$(1-\epsilon)\int_\Omega \left(2w - u_\epsilon(x,u_0)\right)ww_{,x} + (1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \lambda\epsilon\int_\Omega wu_\epsilon(x,u_0) = \lambda\epsilon\int_\Omega wu_0.$$

Since $\int_\Omega w^2 w_{,x} = 0$, we have

$$(1-\epsilon)\nu\int_\Omega |w_{,x}|^2 + \lambda\epsilon\int_\Omega |w|^2 = \lambda\epsilon\int_\Omega wu_0 + (1-\epsilon)\int_\Omega u_\epsilon(x,u_0)ww_{,x} - \lambda\epsilon\int_\Omega wu^*.$$

Using Poincaré inequality we have

$$\begin{aligned}(1-\epsilon)\int_\Omega u_\epsilon(x,u_0)ww_{,x} &\le (1-\epsilon)\|u_\epsilon(x,u_0)\|_\infty\int_\Omega |ww_{,x}| \\ &\le (1-\epsilon)\|u_\epsilon(x,u_0)\|_\infty\|w\|\|w_{,x}\| \le c_p(1-\epsilon)\|u_\epsilon(x,u_0)\|_\infty\|w_{,x}\|^2\end{aligned}$$

and therefore

$$2(1-\epsilon)\|w_{,x}\|^2(\nu - c_p\|u_\epsilon(x,u_0)\|_\infty) + \lambda\epsilon\|w\| \le 2\lambda\epsilon\|u_0\| + 2\lambda\epsilon\|u^*\|.$$

Since ν satisfies the condition (2.3), it implies that

$$(1-\epsilon)\nu\|w_{,x}\|^2 + \lambda\epsilon\|w\| \le 2\lambda\epsilon\|u_0\| + 2\lambda\epsilon\|u^*\|.$$

Since ν is independent of ϵ, as $\epsilon \to 0$, it follows that $\|w_{,x}\| \to 0$. By Poincaré inequality, we conclude that$u_\epsilon(x,u_0) \to u^*$ in $H_0^1(\Omega)$ as $\epsilon \to 0$. ☐

Remark 2.9. To get a sufficient condition on ν to make it satisfy the condition (2.3), one can use Lemma 2.2 in the analogy of Theorem 2.7 and conclude that ν can be chosen as the largest positive root of the equation $a_0x + b_0\sqrt{x} + c = \frac{x^2}{2c_p}$, with $a_0 := |\Omega|^{-1/2}\|u_0\|, b_0 := |\Omega|^{1/2}\sqrt{\frac{\lambda\epsilon}{(1-\epsilon)}}\|u_0\|$ and $c := (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})c_p\|g\|$, which has at least one positive root for x by the Mean-Value Theorem.

2.4. Existence (and uniqueness) of $u_\epsilon(x, u_0)$

Lemma 2.10. *If $u \in H_0^1(\Omega)$ solves the equation $\left(\frac{u^2}{2}\right)_{,x} = g(x) + \nu u_{xx}(x)$, then $\|u_{,x}\| \le \frac{c_p}{\nu}\|g\|$.*

Proof. Since

$$\int_\Omega u\left(\frac{u^2}{2}\right)_{,x} + \nu\int_\Omega |u_{,x}|^2 = \int_\Omega gu$$

we have

$$\nu\int_\Omega |u_{,x}|^2 \le \|g\|\|u\| \le c_p\|g\|\|u_{,x}\|$$

which implies $\|u_{,x}\| \le \frac{c_p}{\nu}\|g\|$. □

Theorem 2.11. *If ν satisfies*

$$[|\Omega|^{-1/2}c_p + |\Omega|^{1/2}]\frac{c_p^2}{\nu}\left[\|g\| + \frac{2\lambda\epsilon}{1-\epsilon}\left(\frac{c_p^2}{\nu}\|g\| + \|u_0\|\right)\right] \le \frac{\nu}{2}, \tag{2.7}$$

and ϵ satisfies

$$\epsilon \le \frac{\nu}{2\lambda c_p^2 + \nu}, \tag{2.8}$$

then there exists a solution to the homotopy problem (1.1).

Proof. Define a nonlinear operator $A : u \mapsto v$. Here v is the unique solution of

$$\left(\frac{v^2}{2}\right)_{,x} = g(x) + \frac{\lambda\epsilon}{1-\epsilon}(u - u_0) + \nu v_{xx}(x)$$

guaranteed by Theorem 2.2 in Kreiss and Kreiss [19]. Define the set $D := \{u \in H_0^1(\Omega) : \|u\| \le \frac{2c_p^2}{\nu}\|g\| + \|u_0\|\}$. Since $\nu > \frac{2\lambda\epsilon c_p^2}{1-\epsilon}$ by (2.8), applying Lemma 2.10 for any $u \in D$ we have

$$\|v\| \le c_p\|v_{,x}\| \le \frac{c_p^2}{\nu}\|g\| + \frac{c_p^2}{\nu}\frac{\lambda\epsilon}{1-\epsilon}(\|u\| + \|u_0\|) \le \frac{c_p^2}{\nu}\|g\| + \frac{1}{2}(\|u\| + \|u_0\|) \le \frac{2c_p^2}{\nu}\|g\| + \|u_0\|$$

which implies that $AD \subset D$. Next we will show that the operator A is a strict contraction mapping on D when (2.7) and (2.8) hold. For $u_1, u_2 \in D$, define $v_1 = A(u_1)$ and $v_2 = A(u_2)$. It follows that

$$\left(\frac{v_1^2 - v_2^2}{2}\right)_{,x} = \frac{\lambda\epsilon}{1-\epsilon}(u_1 - u_2) + \nu(v_1 - v_2)_{xx}.$$

Introducing $w := v_1 - v_2$ and testing with w, we have

$$\int_\Omega (v_1 w w_{,x} + w^2 v_{2,x}) + \nu\int_\Omega |w_{,x}|^2 = \frac{\lambda\epsilon}{1-\epsilon}\int_\Omega (u_1 - u_2)w.$$

Since

$$\int_\Omega (v_1 w w_{,x} + w^2 v_{2,x}) = \int_\Omega (v_1 - 2v_2) w w_{,x} = \int_\Omega (w - v_2) w w_{,x} = -\int_\Omega v_2 w w_{,x},$$

we have

$$\nu \int_\Omega |w_{,x}|^2 = \frac{\lambda\epsilon}{1-\epsilon}\int_\Omega (u_1-u_2)w + \int_\Omega v_2 w w_{,x} \leq \frac{\lambda\epsilon}{1-\epsilon}\|u_1-u_2\|\|w\| + \|v_2\|_\infty \|w\|\|w_{,x}\|$$
$$\leq \frac{\lambda\epsilon c_p}{1-\epsilon}\|u_1-u_2\|\|w_{,x}\| + c_p\|v_2\|_\infty\|w_{,x}\|^2.$$

Similar to the inequality (2.2), using Lemma 2.10 and the definition of D, we have

$$\begin{aligned}\|v_2\|_\infty \leq |\Omega|^{-1/2}\|v_2\| + |\Omega|^{1/2}\|v_{2,x}\| &\leq (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})\|v_{2,x}\| \\ &\leq (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})\frac{c_p}{\nu}\|g + \frac{\lambda\epsilon}{1-\epsilon}(u_2-u_0)\| \\ &\leq (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})\frac{c_p}{\nu}\left(\|g\| + \frac{\lambda\epsilon}{1-\epsilon}\|u_2-u_0\|\right) \\ &\leq (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})\frac{c_p}{\nu}\left[\|g\| + \frac{2\lambda\epsilon}{1-\epsilon}\left(\frac{c_p^2}{\nu}\|g\| + \|u_0\|\right)\right].\end{aligned}$$

Under the condition (2.7), we obtain $c_p\|v_2\|_\infty \leq \frac{\nu}{2}$, and therefore

$$\nu\|w_{,x}\| \leq \frac{2\lambda\epsilon c_p}{1-\epsilon}\|u_1-u_2\| \leq \frac{2\lambda\epsilon c_p^2}{1-\epsilon}\|u_{1,x}-u_{2,x}\|.$$

It follows that the map A is a strict contraction mapping on D since $\nu > \frac{2\lambda\epsilon c_p^2}{1-\epsilon}$ by (2.8). By Lemma 2.1 and the Banach's fixed point theorem (see *e.g.* Evans [5]), there exists a unique solution of (1.1). ☐

Remark 2.12. Applying the condition (2.8) into (2.7), one can simplify the condition (2.7) as

$$[|\Omega|^{-1/2}c_p + |\Omega|^{1/2}]\left[\frac{2c_p^2}{\nu}\|g\| + \|u_0\|\right] \leq \frac{\nu}{2},$$

or equivalently

$$[|\Omega|^{-1/2}c_p + |\Omega|^{1/2}]\left[\|u_0\| + \sqrt{4c_p^2\|g\| + \|u_0\|^2}\right] \leq \nu.$$

Remark 2.13. For large enough ν satisfying (2.7) which depends on $\|g\|$ and $\|u_0\|$, the above theorem shows that the problem (1.1) has a unique solution for any smaller enough ϵ satisfying (2.8). For the case $\epsilon = 1$, the problem (1.1) is trivial and has the unique solution $u_\epsilon(x) = u_0(x)$. In contrast, for the case $\epsilon \in (\frac{\nu}{2\lambda c_p^2+\nu}, 1)$, we need other techniques to prove the existence. However, this theorem is strong enough for our study since we are interested in the case $\epsilon \to 0$.

3. Finite element implementation of homotopy continuation method

3.1. Finite element implementation

We will use the continuous finite element method, see *e.g.* Ern and Guermond [4], to solve (1.1): For $m \in \mathbb{N}$, let $\{x_i\}_{i\in\{0:m+1\}}$ be a sequence of equidistributed points in Ω. By denoting $I_i := [x_i, x_{i+1}]$ and $h := |I_i|$, we have $\overline{\Omega} = \cup_{i=0}^m I_i$. The mesh is defined as $\mathcal{T}_h := \{I_i\}_{i\in\{0:m\}}$, and $\widehat{I} := [0,1]$ is the reference element. The affine geometric transformation is denoted by $T_{I_i} : \widehat{I} \longrightarrow I_i$ such that $T_{I_i}(\widehat{x}) = x_i(1-\widehat{x}) + \widehat{x}x_{i+1}$. We will use

$$V_{h,\kappa} := \{\, v_h \in \mathcal{C}_0(\overline{\Omega}) \;\mid\; v_h(a) = v_h(b) = 0, v_h \circ T_I \in \mathbb{P}_\kappa,\ \forall I \in \mathcal{T}_h \,\},$$

where $\mathbb{P}_\kappa$ is the polynomial space with order κ. Note that the Dirichlet boundary condition is enforced explicitly in the definition of $V_{h,\kappa}$. The finite element method seeks a $u_h \in V_{h,\kappa}$ such that

$$-(1-\epsilon)(u_h^2/2, v_{,x}) - (1-\epsilon)(g, v) + (1-\epsilon)\nu(u_{h,x}, v_{,x}) + \lambda\epsilon(u_h - u^0, v) = 0 \tag{3.1}$$

for any $v \in V_{h,\kappa}$, where u^0 is assumed to be the L^2 projection of u_0 in $V_{h,\kappa}$. The existence of the solution u_h can be shown in the analogy with Theorem 2.11.

Assuming that $\{\, \phi_i : i = 0, \cdots, N \,\}$ is a basis of $V_{h,\kappa}$, and denoting $u_h = \sum_{i=0}^N U_i\phi_i$ and $u^0 = \sum_{i=0}^N U_i^0\phi_i$, it follows that $U := [U_0, \cdots, U_N] \in \mathbb{R}^{N+1}$ satisfies

$$\mathbf{\Phi}(U_0, \cdots, U_N) = \mathbf{0}, \tag{3.2}$$

where $\mathbf{\Phi} = [\Phi_0, \cdots, \Phi_N]^T$ and

$$\Phi_i(U_0, \cdots, U_N) := -(1-\epsilon)\frac{1}{2}\int_\Omega \left(\sum_{j=0}^N U_j\phi_j\right)^2 \phi_{i,x} - (1-\epsilon)\int_\Omega g\phi_i + (1-\epsilon)\nu\sum_{j=0}^N \int_\Omega U_j\phi_{j,x}\phi_{i,x} + \lambda\epsilon\sum_{j=0}^N \int_\Omega (U_j - U_j^0)\phi_j\phi_i.$$

We will use Newton's method to solve the nonlinear system (3.2) iteratively. It means that the sequence

$$U^{k+1} := U^k + \delta U^k \in \mathbb{R}^{N+1} \tag{3.3}$$

is used to approximate the solution of (3.2), where $U^k = [U_0^k, \cdots, U_N^k] \in \mathbb{R}^{N+1}$ and $\delta U^k \in \mathbb{R}^{N+1}$ satisfies

$$D\mathbf{\Phi}(U^k)\delta U^k = -\mathbf{\Phi}(U^k). \tag{3.4}$$

Since

$$\frac{\partial \Phi_i}{\partial U_j}(U^k) = -(1-\epsilon)\int_\Omega \left(\sum_{l=0}^N U_l^k\phi_l\right)\phi_j\phi_{i,x} + (1-\epsilon)\nu\int_\Omega \phi_{j,x}\phi_{i,x} + \lambda\epsilon\int_\Omega \phi_j\phi_i, \tag{3.5}$$

the problem (3.4) is equivalent to finding $\delta u^k := \sum_{i=0}^N \delta U_j^k\phi_j \in V_{h,\kappa}$ such that

$$\sum_j \frac{\partial \Phi_i}{\partial U_j}(U^k)\delta U_j^k = -\Phi_i(U^k)$$

or

$$\begin{aligned} &-(1-\epsilon)\int_\Omega u^k\delta u^k\phi_{i,x} + (1-\epsilon)\nu\int_\Omega \delta u_{,x}^k\phi_{i,x} + \lambda\epsilon\int_\Omega \delta u^k\phi_i \\ &\quad = (1-\epsilon)\frac{1}{2}\int_\Omega u^k(x)^2\phi_{i,x} + (1-\epsilon)\int_\Omega g\phi_i - (1-\epsilon)\nu\int_\Omega u_{,x}^k\phi_{i,x} - \lambda\epsilon\int_\Omega (u^k - u^0)\phi_i \end{aligned} \tag{3.6}$$

for any $i \in \{0, \cdots, N\}$, where $u^k := \sum_{i=0}^N U_j^k\phi_j \in V_{h,\kappa}$.

3.2. Convergence of Newton's iteration (3.3)

From (3.5), we have

$$D\mathbf{\Phi}(U^k) = -(1-\epsilon)A_k + (1-\epsilon)\nu S + \lambda\epsilon M,$$

where A_k, S and M are matrices in $\mathbb{R}^{(N+1)\times(N+1)}$ with ijth entry defined by $\int_\Omega u^k \phi_{i,x}\phi_j$, $\int_\Omega \phi_{i,x}\phi_{j,x}$, and $\int_\Omega \phi_i\phi_j$, respectively. The matrix M is called mass matrix in general.

Note that $D\mathbf{\Phi}(U)$ is continuous at any $U \in \mathbb{R}^{N+1}$ since A_k is linear with respect to U and other two terms do not depend on U. We will show that the matrix $D\mathbf{\Phi}(U)$ is non-singular at $U \in \mathbb{R}^{N+1}$ for large enough ν. For $V \in \mathbb{R}^{N+1}$ from (3.5), we have

$$\begin{aligned} V^T D\mathbf{\Phi}(U)V = &-(1-\epsilon)\int_\Omega \left(\sum_{l=0}^N U_l\phi_l\right)\left(\sum_{j=0}^N V_j\phi_j\right)\left(\sum_{i=0}^N V_i\phi_{i,x}\right) \\ &+ (1-\epsilon)\nu\int_\Omega \left(\sum_{j=0}^N V_j\phi_{j,x}\right)\left(\sum_{i=0}^N V_i\phi_{i,x}\right) + \lambda\epsilon\int_\Omega \left(\sum_{j=0}^N V_j\phi_j\right)\left(\sum_{i=0}^N V_i\phi_i\right). \end{aligned} \tag{3.7}$$

Introducing $v := \sum_{i=0} V_i\phi_i$ we can rewrite (3.7) as

$$V^T D\mathbf{\Phi}(U)V = -(1-\epsilon)\int_\Omega uvv_{,x} + (1-\epsilon)\nu\int_\Omega |v_{,x}|^2 + \lambda\epsilon\int_\Omega |v|^2. \tag{3.8}$$

Lemma 3.1. *For given $\epsilon \in [0,1], \nu > 0, \lambda > 0$ and $u = \sum_{i=0}^N U_i\phi_i$, if*

$$\nu \geq 2c_p\|u\|_\infty, \tag{3.9}$$

then

$$V^T D\mathbf{\Phi}(U)V \geq \frac{1}{c_p}\min\left\{\frac{\nu}{2c_p}, \lambda c_p\right\}\mu_{\min}|V|^2, \quad \forall V \in \mathbb{R}^{N+1}, \tag{3.10}$$

where $U = [U_0, \cdots, U_N] \in \mathbb{R}^{N+1}$ and $\mu_{\min}$ is the minimum eigenvalue of the mass matrix M. In particular, the matrix $D\mathbf{\Phi}(U)$ is invertible.

Proof. Since

$$\int_\Omega uvv_{,x} \leq \|u\|_\infty\|v\|\|v_{,x}\|,$$

from (3.8) we have

$$\begin{aligned} V^T D\mathbf{\Phi}(U)V &\geq (1-\epsilon)\|v_{,x}\|(\nu\|v_{,x}\| - \|u\|_\infty\|v\|) + \lambda\epsilon\|v\|^2 \\ &\geq \min\left\{\|v_{,x}\|(\nu\|v_{,x}\| - \|u\|_\infty\|v\|), \lambda\|v\|^2\right\} \geq \min\left\{\frac{\nu}{c_p} - \|u\|_\infty, \lambda c_p\right\}\frac{1}{c_p}\|v\|^2 \\ &\geq \frac{1}{c_p}\min\left\{\frac{\nu}{2c_p}, \lambda c_p\right\}V^T M V \geq \frac{1}{c_p}\min\left\{\frac{\nu}{2c_p}, \lambda c_p\right\}\mu_{\min}|V|^2, \end{aligned}$$

which completes the proof of (3.10) and implies the invertibility of $D\mathbf{\Phi}(U)$ since it holds for any $V \in \mathbb{R}^{N+1}$. $\square$

Combining (3.9) and Lemma 2.2, we get the convergence of Newton's method when ν is large enough. Specifically, we have the following result.

Theorem 3.2. *If ν satisfies the following inequality*

$$|\Omega|^{-1/2}\|u_0\| + |\Omega|^{1/2}\sqrt{\frac{\lambda\epsilon}{(1-\epsilon)\nu}}\|u_0\| + (|\Omega|^{-1/2}c_p + |\Omega|^{1/2})\frac{c_p}{\nu}\|g\| \le \frac{\nu}{2c_p}$$

then Theorem 2.2 in Izmailov and Solodov [18] holds. In particular, the Newton's method (3.3) starting from any point close enough to u^* converges to u^* and the rate of convergence is quadratic.

3.3. Weak conservation

For each $h := \frac{|\Omega|}{m+1}$, because of the convergence of Newton's method, there is a solution $u_m = \sum_{i=0}^N U_{m,i}\phi_i \in V_{h,\kappa}$ of the nonlinear equation (3.1). We will show that u_m satisfies the following weak conservation.

Theorem 3.3. *For fixed ϵ, ν, let u_m be the solution of (3.1) for given h. Assume that u_m converges to u, the classical solution of (1.1), in $L^2(\Omega)$. Then we have*

$$\lim_{m\to\infty}\int_\Omega u_m = \int_\Omega u_0.$$

Proof. Since u_m converges to u in $H_0^1(\Omega)$, we have

$$|\int_\Omega u_m - \int_\Omega u| \le \int_\Omega |u_m - u| \le \|u_m - u\|\|\Omega\|^{1/2} \to 0$$

as $m \to \infty$. Since u is the classical steady state solution of (1.1) by assumption, it satisfies $\int_\Omega u = \int_\Omega u_0$, which completes the proof. ☐

In the above theorem, the weak solution u of (1.1) has to be regular enough, at least a C^2 function, to become a classical solution. However, instead of this C^2 requirement, a weak local requirement at the two end points suffices to get the same result, as follows.

Theorem 3.4. *For fixed ϵ, ν, assume u_m converges to some function u in $H_0^1(\Omega)$, and u_m converges to u also in $C^1(\mathcal{O}_0)$ and $C^1(\mathcal{O}_\pi)$, where $\mathcal{O}_0$ and $\mathcal{O}_1$ are open neighborhoods of the two end points. Assume $u(0) = u(\pi) = 0$ and $u_{,x}(0) = u_{,x}(\pi)$. Then we have*

$$\lim_{m\to\infty}\int_\Omega u_m = \int_\Omega u_0.$$

Proof. Let us consider the case $\kappa = 1$, *i.e.*, $V_{h,1}$ is the function space of piecewise linear functions vanishing at two end points with $h := \frac{|\Omega|}{m+1}$. Since $u_m \in V_{h,1}$ is the solution of (3.1) by Newton's method, it satisfies

$$-(1-\epsilon)(u_m^2/2, v_{,x}) - (1-\epsilon)(g, v) + (1-\epsilon)\nu(u_{m,x}, v_{,x}) + \lambda\epsilon(u_m - u^0, v) = 0, \quad \forall v \in V_{h,1}.$$

Choosing $v = \phi_i$ and taking a sum over $i = 0, \cdots, N$, since $\sum_{i=0}^N \phi_i = 1 - \phi_{-1} - \phi_{N+1}$ with $\phi_{-1}(x) = (1-\frac{x}{h})\mathbb{1}_{I_0}$ and $\phi_{N+1}(x) = \frac{x-x_m}{h}\mathbb{1}_{I_m}$, we have

$$(1-\epsilon)\frac{1}{2}\int_\Omega u_m(x)^2\phi_{-1,x} + (1-\epsilon)\frac{1}{2}\int_\Omega u_m(x)^2\phi_{N+1,x} - (1-\epsilon)\int_\Omega g + (1-\epsilon)\int_\Omega g\phi_{-1} + (1-\epsilon)\int_\Omega g\phi_{N+1}$$
$$-(1-\epsilon)\nu\int_\Omega u_{m,x}\phi_{-1,x} - (1-\epsilon)\nu\int_\Omega u_{m,x}\phi_{N+1,x} + \lambda\epsilon\int_\Omega(u_m - u^0) - \lambda\epsilon\int_\Omega(u_m - u^0)\phi_{-1} - \lambda\epsilon\int_\Omega(u_m - u^0)\phi_{N+1} = 0$$

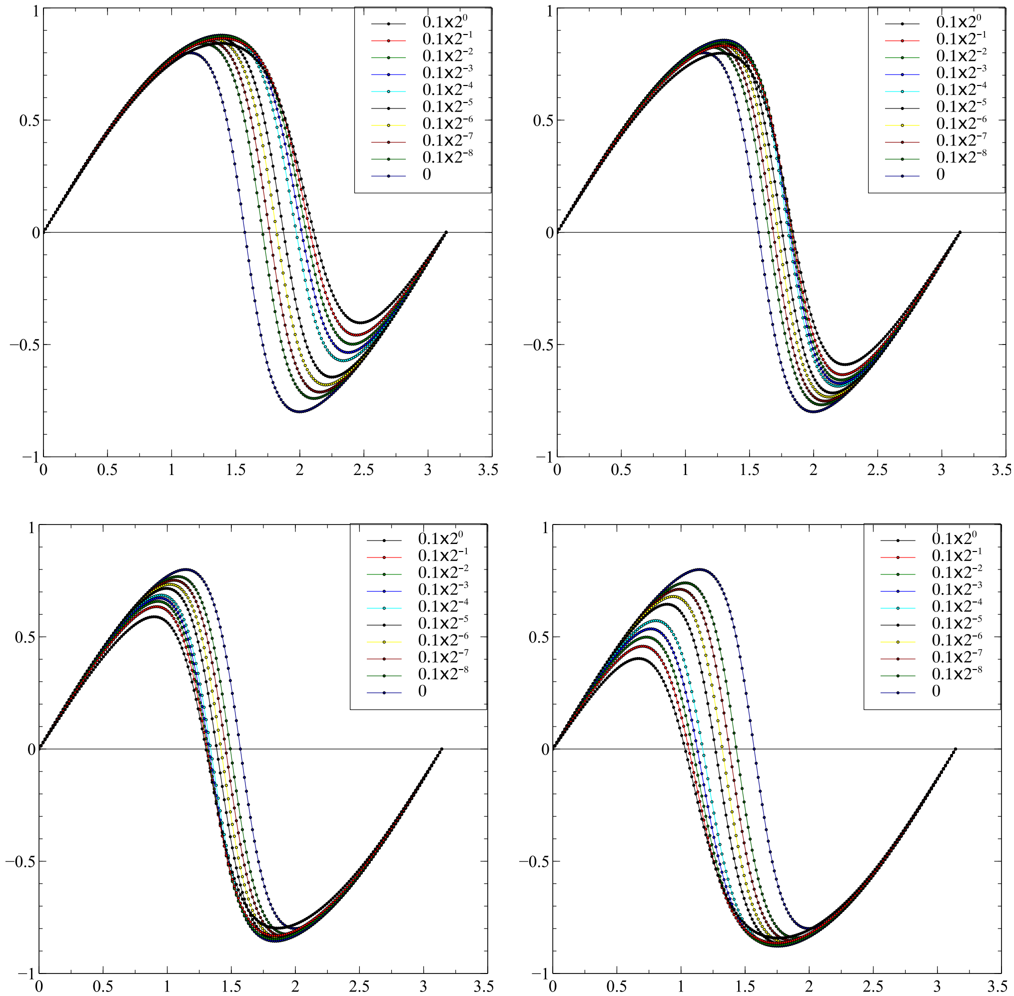

FIGURE 1. The numerical solutions of (1.1) for different β with fixed $\nu = 0.1$. $\beta = 0.5$ (*upper left*); $\beta = 0.25$ (*upper right*), $\beta = -0.25$ (*lower left*) and $\beta = -0.5$ (*lower right*).

As $m \to \infty$, since $\phi_{-1,x} = -\frac{1}{h}$, $I_0 = [0, h]$ and $u_m \to u$ in $C^1(\mathcal{O}_0)$, there exists $\xi_m \in (0, h)$ such that $\int_\Omega u_m(x)^2 \phi_{-1,x} = -u_m^2(\xi_m) \to -u^2(0) = 0$. Likewise, we have $\int_\Omega u_m(x)^2 \phi_{N+1,x} \to 0$, $\int_\Omega u_m \phi_{-1} = \frac{1}{6} u_m(h) \to 0$, $\int_\Omega u_m \phi_{N+1} \to 0$, $\int_\Omega u_{m,x} \phi_{-1,x} \to -u_{,x}(0)$ and $\int_\Omega u_{m,x} \phi_{N+1,x} \to u_{,x}(\pi)$. Since $u_{,x}(\pi) = u_{,x}(0)$ in our assumption, it follows that

$$-(1-\epsilon) \int_\Omega g + \lambda\epsilon \lim_{m \to \infty} \left(\int_\Omega u_m - \int_\Omega u^0 \right) = 0.$$

Since $\int_\Omega g = 0$, we conclude that $\lim_{m\to\infty} \int_\Omega u_m = \int_\Omega u_0$ for fixed ϵ and ν. □

4. NUMERICAL RESULTS

Two numerical tests for the problem (1.3) are presented in this section. In the first test, we confirm the results proved in Section 2, namely, the independence of the limit solution of $u_\epsilon(x, u_0)$ on the initial condition

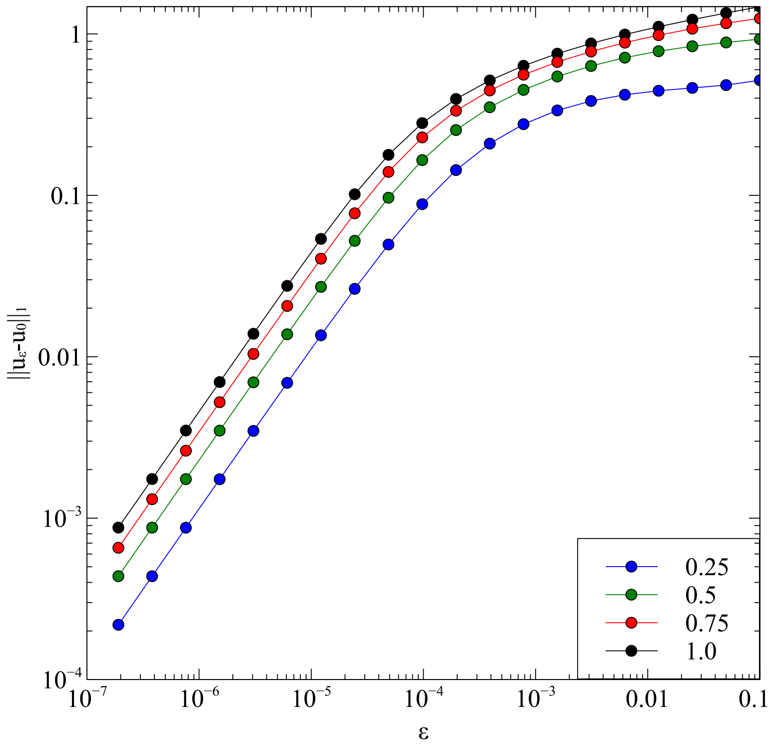

FIGURE 2. The L^1 error of solutions corresponding to different β *vs.* ϵ when $\nu = 0.1$. Note that u_0 is independent of β.

u_0 as $\epsilon \to 0$. In contrast, the second test shows that the limit solution of $u_\epsilon(x, u_0)$ indeed depends on the initial condition u_0 when $\nu = \epsilon \to 0$. The method (3.3) is implemented by using the deal.II library by Bangerth *et al.* [1]. All results shown here are obtained by using the piecewise linear finite element method, *i.e.*, $\kappa = 1$. Other high-order finite element methods, *e.g.*, $\kappa = 2, 3$, have also been tested and produce similar results.

4.1. Test 1

As shown in Theorem 2.2 of Kreiss and Kreiss [19], for any fixed $\nu > 0$, the steady state solution is unique with only one shock at $x_s = \frac{\pi}{2}$ and is independent on β. This is also true for the homotopy method (1.1) when ν is large enough (see Theorems 2.5 and 2.7). We verify this conclusion numerically by choosing different $\beta = 0.5, 0.25, -0.25, -0.5$. The convergence of numerical solutions of the homotopy problem (1.1) for different initial conditions is shown in Figure 1. We choose $\nu = 0.1$ and $\epsilon = 0.1 \times 2^{-i}, i = 0, \ldots, 8$. The stopping tolerance for Newton's method is 10^{-8} in the L^1 norm. The independence of the limit solution of the homotopy problem (1.1) on the initial condition is shown in Figure 2. The L^1 error of the solution corresponding to different β is shown to converge to zero as $\epsilon \to 0$.

4.2. Test 2

When $\nu = 0$, the steady state problem of (1.2) becomes

$$\begin{cases} \left(\frac{u^2}{2}\right)_{,x} = \sin x \cos x, \quad x \in \Omega = [0, \pi] \\ u(0, x) = \beta \sin x \\ u(t, 0) = u(t, \pi) = 0, \end{cases} \tag{4.1}$$

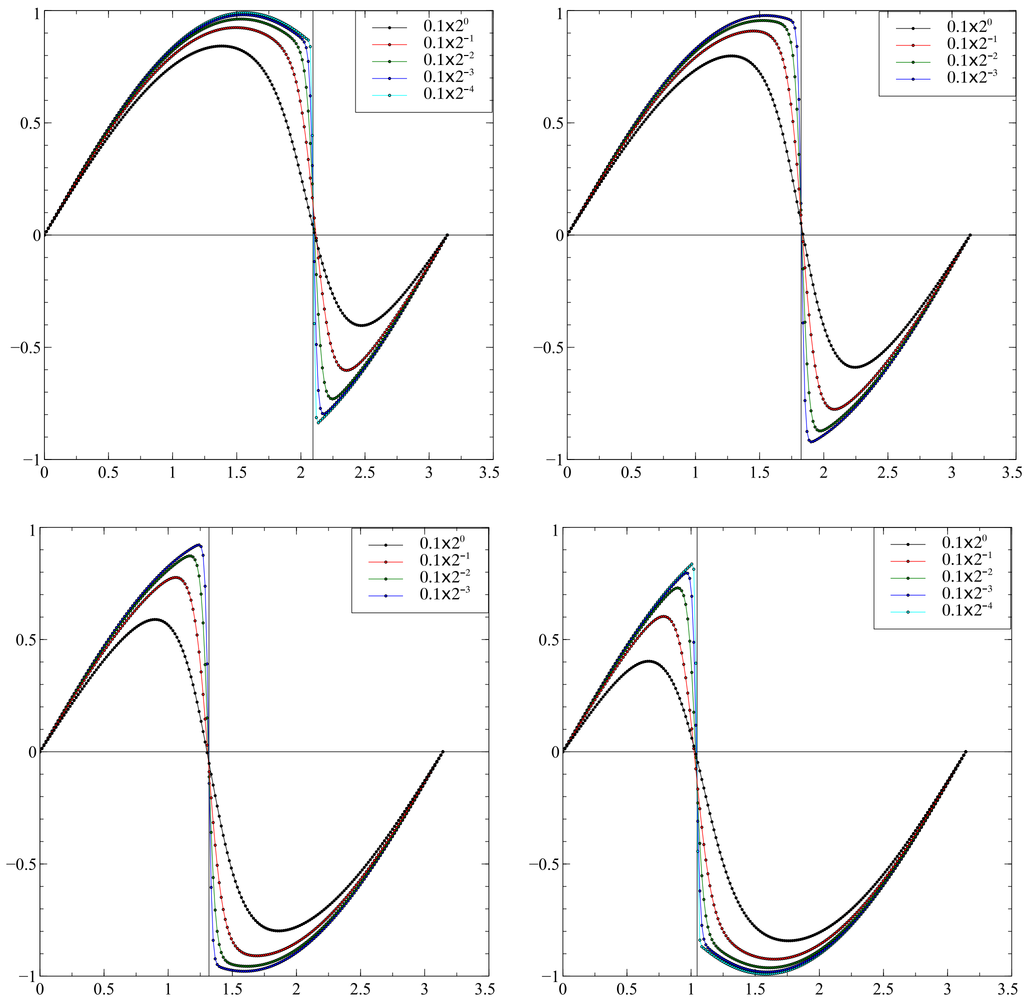

FIGURE 3. The numerical solutions for different β by setting $\nu = \epsilon$. *Upper left*: $\beta = 0.5$ the shock location is $x_s = 2\pi/3$. *Upper right*: $\beta = 0.25$ the shock location is $x_s = 1.82347$. *Lower left*: $\beta = -0.25$ the shock location is $x_s = 1.31812$. *Lower right*: $\beta = -0.5$ the shock location is $x_s = \pi/3$.

for $\beta \in (-1, 1)$. In Theorem 2 and its Corollary of Salas *et al.* [23], the only reachable steady state solution is $u(x) = \mathbb{1}_{x<x_s} \sin x - \mathbb{1}_{x>x_s} \sin x$ where the entropy condition is satisfied and

$$x_s = \begin{cases} \pi - \sin^{-1}\sqrt{1-\beta^2}, & \text{if } \beta > 0, \\ \sin^{-1}\sqrt{1-\beta^2}, & \text{if } \beta \le 0. \end{cases} \tag{4.2}$$

To confirm this conclusion, a single limit is used in the homotopy problem (1.1) by setting $\nu = \epsilon$. As shown in Figure 3 the solutions for different β have the shock locations exactly as (4.2). The dependence on the initial condition can be seen from the conservation property of the method (3.3) as shown in Theorem 3.3. Since (4.1) is a singular system, the homotopy problem becomes ill-conditioned when $\nu = \epsilon \to 0$. Numerically, we can only compute up to $\nu = \epsilon = 0.00625$ for $\kappa = 1$ and $\nu = \epsilon = 0.003125$ for $\kappa = 3$. In order to handle this singularity, one remedy is to use the Cauchy endgame algorithm in Hao *et al.* [12].

5. Conclusion

In this paper, we have studied the convergence of the homotopy continuation approach based on the continuous finite element method for computing steady state solutions of 1D Burgers' equation. We have proved the existence and uniqueness of the solution of the homotopy problem (1.1) and have also proved the independence of the limit solution as $\epsilon \to 0$ on the initial condition when ν is large enough. We have presented a finite element implementation of the homotopy problem (1.1) and shown the convergence of Newton's iteration method. These results for fixed ν are verified by numerical computations. As $\nu = \epsilon \to 0$, the numerical results show that the steady state solution indeed depends on the initial condition. In the future, we will explore further details of the convergence of the homotopy continuation method in these two cases and consider the two dimensional Burgers' equation as well as other hyperbolic systems.

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