ESAIM: M2AN 53 (2019) 85–103
https://doi.org/10.1051/m2an/2018043

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

A VARIANT OF THE PLANE WAVE LEAST SQUARES METHOD FOR THE TIME-HARMONIC MAXWELL'S EQUATIONS

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Abstract. In this paper we are concerned with the plane wave method for the discretization of time-harmonic Maxwell's equations in three dimensions. As pointed out in Hiptmair *et al.* (*Math. Comput.* **82** (2013) 247–268), it is difficult to derive a satisfactory L^2 error estimate of the standard plane wave approximation of the time-harmonic Maxwell's equations. We propose a variant of the plane wave least squares (PWLS) method and show that the new plane wave approximations yield the desired L^2 error estimate. Moreover, the numerical results indicate that the new approximations have sightly smaller L^2 errors than the standard plane wave approximations. More importantly, the results are derived for more general models in inhomogeneous media.

Mathematics Subject Classification. 65N30, 65N55.

Received December 13, 2017. Accepted June 29, 2018.

1. Introduction

The plane wave method was first introduced to solve Helmholtz equations and was then extended to solve time-harmonic Maxwell's equations. This method is different from the traditional finite element method because of its special choice of basis functions. In the plane wave methods, the basis functions are chosen as the exact solutions of the differential equations without boundary conditions, so the resulting approximate solutions possess higher accuracies than that generated by the other methods. Over the last ten years, various plane wave methods were proposed in the literature, for example, the ultra weak variational formulation (UWVF) [2,4,15], the plane wave discontinuous Galerkin (PWDG) method [6,7] and the plane wave least squares (PWLS) method [11–13,17] (more references on the plane wave methods can be found in the survey article [5]).

Although there are many works to introduce plane wave methods and analyze the convergence of plane wave approximations for Helmholtz equation and time-harmonic Maxwell's equations, a satisfactory L^2 error estimate of the plane wave approximations for time-harmonic Maxwell's equations has not been obtained yet (see [6] for more detailed explanations on the reasons). Moreover, to our knowledge, there is no convergence analysis on the plane wave methods for Helmholtz equation or time-harmonic Maxwell's equations in inhomogeneous media.

In this paper, we consider the PWLS method since it results in a Hermitian and positive definite system. We propose a variant of the PWLS method to discretize time-harmonic homogeneous Maxwell's equations in

Keywords and phrases. Maxwell's equations, plane wave method, variational problem, L^2 error estimate.

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three dimensions. In the new PWLS method, we add the jump of the normal complement of the solutions into the objective functional defined in the existing PWLS methods. We prove that the error of the resulting plane wave approximation yields a desired L^2 estimate that is almost the same as the one in the case of Helmholtz equation. Here we need not to assume that the material coefficients are constants, namely, inhomogeneous media are allowed. The numerical results show that the plane wave approximations generated by the proposed method is slightly more accurate than that generated by the existing PWLS methods.

The paper is organized as follows: In Section 2, we present the proposed PWLS method for time-harmonic Maxwell's equations and the discretization of the variational problem. In Section 3, we give an important inequality about the L^2 norm of the errors. In Section 4, we give the L^2 error estimate for the approximate solutions generated by the new variational problem and simplify the variational formulation for the case of homogeneous material. Finally, in Section 5 we report some numerical results to confirm the effectiveness of the new method.

2. The PWLS method for time-harmonic Maxwell's equations

2.1. Description of time-harmonic Maxwell's equations

In this subsection we recall the first order system of Maxwell's equations and derive the corresponding second order system.

Let $\Omega \subset \mathbb{R}^3$ be a bounded, polyhedral domain. We denote by $\mathbf{n}$ the unit normal vector field on $\partial\Omega$ pointing outside Ω. We consider the following formulation of the three-dimensional time-harmonic Maxwell's equations in terms of electric filed $\mathbf{E}$ and magnetic field $\mathbf{H}$ written as the first-order system of equations in Ω

$$\begin{cases} \nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = 0, \\ \nabla \times \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0, \\ \nabla \cdot (\varepsilon\mathbf{E}) = 0, \\ \nabla \cdot (\mu\mathbf{H}) = 0, \end{cases} \tag{2.1}$$

with the lowest-order absorbing boundary condition

$$-\mathbf{E} \times \mathbf{n} + \sigma(\mathbf{H} \times \mathbf{n}) \times \mathbf{n} = \mathbf{g} \quad \text{on } \gamma = \partial\Omega. \tag{2.2}$$

Here $\omega > 0$ denotes the temporal frequency of the field, and $\mathbf{g} \in L^2(\partial\Omega)$. The material coefficients ε, μ and σ are understood as usual (refer to [15]). Notice that we often have $\sigma = \sqrt{\mu/|\varepsilon|}$ in applications. In particular, if ε is complex valued, then the material is known as an absorbing medium; otherwise the material is called a non-absorbing medium.

In applications, the parameter ε may have different values in different parts of the domain Ω, then the medium on Ω is called an inhomogeneous medium; but if ε is a constant on the whole domain, then the medium is called a homogeneous medium.

Based on the first equation of (2.1), we can write $\mathbf{H}$ in terms of $\mathbf{E}$ as $\mathbf{H} = \frac{1}{i\omega\mu}\nabla \times \mathbf{E}$. Substituting this expression into the second equation and into the boundary condition, we obtain the second-order Maxwell's equations

$$\begin{cases} \nabla \times (\dfrac{1}{i\omega\mu}\nabla \times \mathbf{E}) + i\omega\varepsilon\mathbf{E} = 0 & \text{in } \Omega, \\ -\mathbf{E} \times \mathbf{n} + \dfrac{\sigma}{i\omega\mu}((\nabla \times \mathbf{E}) \times \mathbf{n}) \times \mathbf{n} = \mathbf{g} & \text{on } \gamma. \end{cases} \tag{2.3}$$

2.2. Variational formulation

In this subsection we introduce details of the new variational problem of the Maxwell's equations based on triangulation.

Let Ω be partitioned into elements in the sense that

$$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k, \quad \Omega_l \bigcap \Omega_j = \emptyset, \quad \text{for } l \neq j.$$

For ease of presentation of the main ideas, we exclusively address quasi-uniform families of meshes and let $\mathcal{T}_h$ denote the triangulation comprised of elements $\{\Omega_k\}_{k=1}^{N}$, where h denotes the mesh width of the triangulation. Since the considered method belongs to the class of discontinuous Galerkin methods, the proposed method should still work without this assumption (refer to [8, 10, 14]).

Define

$$\mathit{\Gamma}_{lj} = \partial\Omega_l \bigcap \partial\Omega_j, \quad \text{for } l \neq j,$$

and

$$\gamma_k = \partial\Omega_k \bigcap \partial\Omega, \quad k = 1, \ldots, N, \quad \gamma = \bigcup_{k=1}^{N} \gamma_k.$$

For each element Ω_k, let $\mathbf{E}|_{\Omega_k} = \mathbf{E}_k$ and $\varepsilon|_{\Omega_k} = \varepsilon_k$. As usual, we can assume that μ and ε are constants on each element. In this paper we are interested in the case in inhomogeneous media, *i.e.*, $\varepsilon_j \neq \varepsilon_k$ for two different k and j. For simplicity of exposition, we assume that μ is a constant on Ω. If μ is not a constant, the proposed method still works with some obvious modifications, but there are more details in the discussions. Since ω denotes angle frequency in applications, ω is a constant on Ω.

Notice that we have the hidden relation $\mathrm{div}(\varepsilon\mathbf{E}) = 0$ from the first equation of (2.3). Then the reference problem (2.3) will be solved by finding the local electric field $\mathbf{E}_k$ such that

$$\nabla \times (\frac{1}{i\omega\mu}\nabla \times \mathbf{E}_k) + i\omega\varepsilon_k\mathbf{E}_k = 0 \quad \text{in } \Omega_k, \tag{2.4}$$

with the transmission conditions on each interface $\mathit{\Gamma}_{lj}$ (notice that $\mathbf{n}_l = -\mathbf{n}_j$ on $\mathit{\Gamma}_{lj}$)

$$\begin{cases} \mathbf{E}_l \times \mathbf{n}_l + \mathbf{E}_j \times \mathbf{n}_j = 0, \\ (\frac{1}{i\omega\mu}\nabla \times \mathbf{E}_l) \times \mathbf{n}_l + (\frac{1}{i\omega\mu}\nabla \times \mathbf{E}_j) \times \mathbf{n}_j = 0, \\ \varepsilon_l\mathbf{E}_l \cdot \mathbf{n}_l + \varepsilon_j\mathbf{E}_j \cdot \mathbf{n}_j = 0. \end{cases} \tag{2.5}$$

The boundary condition becomes

$$-\mathbf{E}_k \times \mathbf{n} + \frac{\sigma}{i\omega\mu}((\nabla \times \mathbf{E}_k) \times \mathbf{n}) \times \mathbf{n} = \mathbf{g}, \quad \text{on } \gamma_k. \tag{2.6}$$

For an element Ω_k, let $\mathbf{H}(\mathrm{curl}, \Omega_k)$ denote the standard Sobolev space. Set

$$\mathbf{V}(\Omega_k) = \{\mathbf{E}_k \in \mathbf{H}(\mathrm{curl}, \Omega_k) : \ \mathbf{E}_k \text{ satisfies the equation (2.4)}\} \tag{2.7}$$

and define

$$\mathbf{V}(\mathcal{T}_h) = \prod_{k=1}^{N} \mathbf{V}(\Omega_k).$$

Namely, $\mathbf{V}(\mathcal{T}_h)$ is a piecewise Trefftz space.

For $\mathbf{F} \in \mathbf{V}(\mathcal{T}_h)$, set $\mathbf{F}|_{\Omega_k} = \mathbf{F}_k$. For ease of notation, define

$$\mathit{\Phi}(\mathbf{F_k}) = \frac{\sigma}{i\omega\mu}((\nabla \times \mathbf{F_k}) \times \mathbf{n_k})$$

and

$$\mathit{\Psi}(\mathbf{F}_k) = \frac{1}{i\omega\mu}(\nabla \times \mathbf{F}_k).$$

Then the boundary condition (2.6) can be written as

$$-\mathbf{F}_k \times \mathbf{n}_k + \Phi(\mathbf{F}_k) \times \mathbf{n}_k = \mathbf{g} \quad \text{on } \gamma_k. \tag{2.8}$$

For each local interface Γ_{lj} $(l < j)$, we define the jumps on Γ_{lj} as follows (note that $\mathbf{n}_l = -\mathbf{n}_j$)

$$\begin{cases} [\![\mathbf{F} \times \mathbf{n}]\!] = \mathbf{F}_l \times \mathbf{n}_l + \mathbf{F}_j \times \mathbf{n}_j, \\ [\![\Psi(\mathbf{F}) \times \mathbf{n}]\!] = \Psi(\mathbf{F}_l) \times \mathbf{n}_l + \Psi(\mathbf{F}_j) \times \mathbf{n}_j, \\ [\![\varepsilon\mathbf{F} \cdot \mathbf{n}]\!] = \varepsilon_l \mathbf{F}_l \times \mathbf{n}_l + \varepsilon_j \mathbf{F}_j \cdot \mathbf{n}_j. \end{cases} \tag{2.9}$$

It is easy to see that the transmission conditions (2.5) are equivalent to the following conditions

$$[\![\mathbf{F} \times \mathbf{n}]\!] = 0, \quad [\![\Psi(\mathbf{F}) \times \mathbf{n}]\!] = 0 \quad \text{and} \quad [\![\varepsilon\mathbf{F} \cdot \mathbf{n}]\!] = 0 \quad \text{on } \Gamma_{lj}. \tag{2.10}$$

A vector field $\mathbf{E} \in \mathbf{V}(\mathcal{T}_h)$ is the solution of (2.4)–(2.6) if and only if the conditions (2.8) and (2.10) are satisfied. Set

$$\mathbf{V}^*(\mathcal{T}_h) = \{\mathbf{v} \in \mathbf{V}(\mathcal{T}_h) : \ \mathbf{v}_k, \nabla \times \mathbf{v}_k \in (H^{\delta_0}(\Omega_k))^3 \text{ with } \delta_0 > \frac{1}{2}, \ k = 1, \dots, N\}.$$

Based on this observation, we define the functional

$$\begin{aligned} J(\mathbf{F}) = &\sum_{k=1}^{N} \delta \int_{\gamma_k} |-\mathbf{F}_k \times \mathbf{n}_k + \Phi(\mathbf{F}_k) \times \mathbf{n}_k - \mathbf{g}|^2 \,\mathrm{d}s \\ &+ \sum_{l<j} \Big(\alpha \int_{\Gamma_{lj}} |[\![\mathbf{F} \times \mathbf{n}]\!]|^2 \,\mathrm{d}s + \beta \int_{\Gamma_{lj}} |[\![\Psi(\mathbf{F}) \times \mathbf{n}]\!]|^2 \,\mathrm{d}s \\ &+ \theta \int_{\Gamma_{lj}} |[\![\varepsilon\mathbf{F} \cdot \mathbf{n}]\!]|^2 \,\mathrm{d}s \Big), \quad \forall \mathbf{F} \in \mathbf{V}^*(\mathcal{T}_h) \end{aligned} \tag{2.11}$$

with δ, α, β and θ being positive numbers, which will be specified later. It is clear that the functional $J(\cdot)$ is well defined on $\mathbf{V}^*(\mathcal{T}_h)$ (by the trace theorem) and $J(\mathbf{F}) \geq 0$. Then we consider the following minimization problem: Find $\mathbf{E} \in \mathbf{V}^*(\mathcal{T}_h)$ such that

$$J(\mathbf{E}) = \min_{\mathbf{F} \in \mathbf{V}^*(\mathcal{T}_h)} J(\mathbf{F}). \tag{2.12}$$

If $\mathbf{E}$ is the solution of the equations (2.4)–(2.6), *i.e.* $\mathbf{E} \in \mathbf{V}^*(\mathcal{T}_h)$ satisfies the conditions (2.8) and (2.10), then we have $J(\mathbf{E}) = 0$, which implies that $\mathbf{E}$ is also the solution of the minimization problem (2.12). When $J(\mathbf{E})$ is very small, the vector field $\mathbf{E}$ should be an approximate solution of (2.4)–(2.6).

The variational problem associated with the minimization problem (2.12) can be expressed as follows: Find $\mathbf{E} \in \mathbf{V}^*(\mathcal{T}_h)$ such that

$$\begin{aligned} &\sum_{k=1}^{N} \delta \int_{\gamma_k} (-\mathbf{E}_k \times \mathbf{n}_k + \Phi(\mathbf{E}_k) \times \mathbf{n}_k - \mathbf{g}) \cdot \overline{-\mathbf{F}_k \times \mathbf{n}_k + \Phi(\mathbf{F}_k) \times \mathbf{n}_k} \,\mathrm{d}s \\ &\quad + \sum_{l<j} \Big(\alpha \int_{\Gamma_{lj}} [\![\mathbf{E} \times \mathbf{n}]\!] \cdot \overline{[\![\mathbf{F} \times \mathbf{n}]\!]} \,\mathrm{d}s + \beta \int_{\Gamma_{lj}} [\![\Psi(\mathbf{E}) \times \mathbf{n}]\!] \cdot \overline{[\![\Psi(\mathbf{F}) \times \mathbf{n}]\!]} \,\mathrm{d}s \\ &\quad + \theta \int_{\Gamma_{lj}} [\![\varepsilon\mathbf{E} \cdot \mathbf{n}]\!] \cdot \overline{[\![\varepsilon\mathbf{F} \cdot \mathbf{n}]\!]} \,\mathrm{d}s \Big) = 0, \qquad \forall \ \mathbf{F} \in \mathbf{V}^*(\mathcal{T}_h). \end{aligned} \tag{2.13}$$

The method described above can be viewed as a variant of the plane wave least-squares (PWLS) method proposed in [17] or [12], here we added the final term containing the jump of the normal complements of the considered vector-valued functions.

Define the sesquilinear form $A(\cdot,\cdot)$ by

$$\begin{aligned}A(\mathbf{E},\mathbf{F}) &= \sum_{k=1}^{N}\delta\int_{\gamma_k}(-\mathbf{E}_k\times\mathbf{n}_k+\Phi(\mathbf{E}_k)\times\mathbf{n}_k)\cdot\overline{-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k}\,\mathrm{d}s\\ &\quad+\sum_{l<j}\Big(\alpha\int_{\Gamma_{lj}}[\![\mathbf{E}\times\mathbf{n}]\!]\cdot\overline{[\![\mathbf{F}\times\mathbf{n}]\!]}\,\mathrm{d}s+\beta\int_{\Gamma_{lj}}[\![\Psi(\mathbf{E})\times\mathbf{n}]\!]\cdot\overline{[\![\Psi(\mathbf{F})\times\mathbf{n}]\!]}\,\mathrm{d}s\\ &\quad+\theta\int_{\Gamma_{lj}}[\![\varepsilon\mathbf{E}\cdot\mathbf{n}]\!]\cdot\overline{[\![\varepsilon\mathbf{F}\cdot\mathbf{n}]\!]}\,\mathrm{d}s\Big),\quad \mathbf{E},\ \mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h)\end{aligned}\tag{2.14}$$

and $\boldsymbol{\xi}\in\mathbf{V}(\mathcal{T}_h)$, via the Riesz representation theorem, by

$$(\boldsymbol{\xi},\ \mathbf{F})=\sum_{k=1}^{N}\delta\int_{\gamma_k}\mathbf{g}\cdot\overline{-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k}\,\mathrm{d}s,\qquad \mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).\tag{2.15}$$

Then (2.13) can be written as

$$\begin{cases}\text{Find }\mathbf{E}\in\mathbf{V}^*(\mathcal{T}_h),\ \text{s.t.}\\ A(\mathbf{E},\mathbf{F})=(\boldsymbol{\xi},\ \mathbf{F}),\quad\forall\,\mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).\end{cases}\tag{2.16}$$

Theorem 2.1. *A vector-valued function $\mathbf{E}\in\mathbf{V}^*(\mathcal{T}_h)$ is a solution of the reference problem (2.3) if and only if $\mathbf{E}$ is a solution of the variational problem (2.16).*

Proof. It is clear that the solution of the problem (2.3) is also the solution of the variational problem (2.16). We only need to verify the uniqueness of the solution of problem (2.16). The process of verification is standard.

Let us consider the two solutions $\mathbf{E}=(\mathbf{E}_1,\ldots,\mathbf{E}_N)$ and $\mathbf{E}'=(\mathbf{E}'_1,\ldots,\mathbf{E}'_N)$ of the variational problem, and let $\tilde{\mathbf{E}}=(\tilde{\mathbf{E}}_1,\ldots,\tilde{\mathbf{E}}_N)$ denote the difference between the two solutions. It follows by (2.16) that the difference satisfies

$$A(\tilde{\mathbf{E}},\mathbf{F})=0,\quad\forall\,\mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).\tag{2.17}$$

Taking $\mathbf{F}=\tilde{\mathbf{E}}$, the equation (2.17) becomes

$$A(\tilde{\mathbf{E}},\tilde{\mathbf{E}})=0,$$

which implies that

$$\int_{\gamma_k}|-\tilde{\mathbf{E}}_k\times\mathbf{n}_k+\Phi(\tilde{\mathbf{E}}_k)\times\mathbf{n}_k|^2\,\mathrm{d}s=0$$

and

$$\int_{\Gamma_{kj}}|[\![\tilde{\mathbf{E}}\times\mathbf{n}]\!]|^2\,\mathrm{d}s=\int_{\Gamma_{kj}}|[\![\Psi(\tilde{\mathbf{E}})\times\mathbf{n}]\!]|^2\,\mathrm{d}s=\int_{\Gamma_{kj}}|[\![\varepsilon\tilde{\mathbf{E}}\cdot\mathbf{n}]\!]|^2\,\mathrm{d}s=0.$$

These show that $\tilde{\mathbf{E}}$ satisfies the condition (2.10), *i.e.* the transmission continuity (2.5), and verify the initial Maxwell reference problem (2.3) (notice that $\tilde{\mathbf{E}}\in\mathbf{V}^*(\mathcal{T}_h)$) with the boundary condition. Thus the function $\tilde{\mathbf{E}}$ vanishes on Ω, which proves the uniqueness of the solution of (2.16). □

2.3. Discrete variational problem

Let $\mathbf{V}_p(\mathcal{T}_h)\subset\mathbf{V}^*(\mathcal{T}_h)$ be a plane wave space with finite dimension, where p denote the direction number of plane waves, which determine the number of basis functions on each element. In Section 4.1 we will give a concrete example of the space $\mathbf{V}_p(\mathcal{T}_h)$. Then the discrete variational problem associated with (2.16) can be described as follows

$$\begin{cases}\text{Find }\mathbf{E}_h\in\mathbf{V}_p(\mathcal{T}_h),\ \ \text{s.t.}\\ A(\mathbf{E}_h,\mathbf{F}_h)=(\boldsymbol{\xi},\ \mathbf{F}_h),\quad\forall\,\mathbf{F}_h\in\mathbf{V}_p(\mathcal{T}_h).\end{cases}\tag{2.18}$$

When the discrete space $\mathbf{V}_p(\mathcal{T}_h)$ is defined in suitable manner, the solution $\mathbf{E}_h$ should be good approximation of $\mathbf{E}$. In the rest of this paper, we will derive a L^2 error estimate of the approximate solution $\mathbf{E}_h$.

3. An important inequality for the L^2 norm of the errors

In this section, we derive an important inequality for the L^2 norm for the errors of the approximate solution $\mathbf{E}_h$ defined by (2.18). The analysis is based on the techniques developed in [6], as well as some new observations.

Set $\mathbf{e}_h = \mathbf{E} - \mathbf{E}_h$, where $\mathbf{E}$ and $\mathbf{E}_h$ are defined by (2.16) and (2.18) respectively. Clearly we have

$$\nabla \times (\nabla \times \mathbf{e}_h) - \kappa^2 \mathbf{e}_h = 0 \qquad \text{in } \Omega_k \tag{3.1}$$

with $\kappa = \omega\sqrt{\varepsilon\mu}$.

By the definition of $A(\cdot,\cdot)$ and the proof of Theorem 2.1, we have that $A(\mathbf{F},\mathbf{F}) \geq 0$ and that $A(\mathbf{F},\mathbf{F}) = 0$ for $\mathbf{F} \in \mathbf{V}^*(\mathcal{T}_h)$ if and only if $\mathbf{F} = 0$. Thus, $A(\cdot,\cdot)$ is a norm on $\mathbf{V}^*(\mathcal{T}_h)$. For ease of notation, this norm is denoted by $||\cdot||_{\mathbf{V}}$ in the rest of this section.

The key technique is to use a Helmholtz-type decomposition of $\mathbf{e}_h$

$$\mathbf{e}_h = \mathbf{w} + \nabla p, \tag{3.2}$$

where $p \in H_0^1(\Omega)$ is defined by

$$\int_\Omega (\bar{\varepsilon}\nabla p)\cdot\nabla\bar{q}\ \mathrm{d}x = \int_\Omega (\bar{\varepsilon}\mathbf{e}_h)\cdot\nabla\bar{q}\ \mathrm{d}x, \quad \forall\, q \in H_0^1(\Omega)$$

and $\mathbf{w} \in H(\mathrm{curl},\Omega)$ satisfying $\mathrm{div}(\bar{\varepsilon}\mathbf{w}) = 0$. We point out that, for the case of inhomogeneous media (*i.e.*, ε is not a constant), the definitions of the functions p and $\mathbf{w}$ in the decomposition (3.2) are different from that in the Helmholtz decomposition considered in [6].

Throughout this paper, we always use $|\varepsilon|_{\max}$ (rep. $|\varepsilon|_{\min}$) to denote the maximal (rep. minimal) value of the function $|\varepsilon(x)|$ on $\bar{\Omega}$. And we use $|\kappa|_{\max} = \omega\sqrt{|\varepsilon|_{\max}\mu}$ and $|\kappa|_{\min} = \omega\sqrt{|\varepsilon|_{\min}\mu}$.

Lemma 3.1. *For $\mathbf{w}$ and p defined above, the stabilities hold*

$$||\mathbf{w}||_{0,\Omega} \leq \frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}||\mathbf{e}_h||_{0,\Omega} \quad \textit{and} \quad |p|_{1,\Omega} \leq \frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}||\mathbf{e}_h||_{0,\Omega}. \tag{3.3}$$

Proof. From the equality $\mathrm{div}(\bar{\varepsilon}\mathbf{w}) = 0$, we have

$$\int_\Omega \bar{\varepsilon}\mathbf{w}\cdot\overline{\nabla p}\ \mathrm{d}x = -\int_\Omega \mathrm{div}(\bar{\varepsilon}\mathbf{w})\cdot\bar{p}\ \mathrm{d}x + \int_{\partial\Omega}(\bar{\varepsilon}\mathbf{w}\cdot\mathbf{n})\cdot\bar{p}\ \mathrm{d}s = 0. \tag{3.4}$$

Then, by (3.2) and (3.4) we can easily get

$$\begin{aligned} |\varepsilon|_{\min}||\mathbf{w}||_{0,\Omega}^2 &\leq |\int_\Omega \bar{\varepsilon}\mathbf{w}\cdot\bar{\mathbf{w}}\ \mathrm{d}x| = |\int_\Omega \bar{\varepsilon}\mathbf{w}\cdot\overline{(\mathbf{e}_h - \nabla p)}\ \mathrm{d}x| \\ &= |\int_\Omega \bar{\varepsilon}\mathbf{w}\cdot\bar{\mathbf{e}}_h\ \mathrm{d}x| \leq |\varepsilon|_{\max}||\mathbf{w}||_{0,\Omega}||\mathbf{e}_h||_{0,\Omega}, \end{aligned}$$

and

$$\begin{aligned} |\varepsilon|_{\min}||\nabla p||_{0,\Omega}^2 &\leq |\int_\Omega \varepsilon\nabla p\cdot\overline{\nabla p}\ \mathrm{d}x| = |\int_\Omega \varepsilon\nabla p\cdot\overline{(\mathbf{e}_h - \mathbf{w})}\ \mathrm{d}x| \\ &= |\int_\Omega \varepsilon\nabla p\cdot\bar{\mathbf{e}}_h\ \mathrm{d}x| \leq |\varepsilon|_{\max}||\nabla p||_{0,\Omega}||\mathbf{e}_h||_{0,\Omega}. \end{aligned}$$

Thus

$$||\mathbf{w}||_{0,\Omega} \leq \frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}||\mathbf{e}_h||_{0,\Omega} \quad \text{and} \quad |p|_{1,\Omega} \leq \frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}||\mathbf{e}_h||_{0,\Omega}.$$

□

In order to develop the argument needed for the error analysis below, we will consider the adjoint Maxwell's equations (the first equation is different from that in [6])

$$\begin{cases} \nabla \times (\nabla \times \mathbf{u}) - \overline{\kappa}^2 \mathbf{u} = \overline{\varepsilon} \mathbf{w} & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} + \dfrac{\sigma}{i\omega\mu} \big((\nabla \times \mathbf{u}) \times \mathbf{n} \big) \times \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

In the rest of this paper, we always use C to denote a generic positive constant independent of h, p, ω and ε, but its value might change at different occurrence. For ease of notation, we choose $\sigma = \sqrt{\mu/|\varepsilon|}$.

Lemma 3.2. *[9] Under the previous assumptions on Ω, the solution $\mathbf{u}$ of problem (3.5) belongs to $H^{1/2+s}(curl, \Omega)$, for all the real parameters $s > 0$ such that $s \leq \tilde{s}$, where $0 < \tilde{s} < 1/2$ is a parameter only depending on Ω. Moreover, there is positive constant C independent of ω and ε, such that*

$$||\nabla \times \mathbf{u}||_{0,\Omega} + \omega ||\overline{\varepsilon}^{1/2} \mathbf{u}||_{0,\Omega} \leq C ||\overline{\varepsilon} \mathbf{w}||_{0,\Omega} \tag{3.6}$$

and

$$||\nabla \times \mathbf{u}||_{1/2+s,\Omega} + \omega |\varepsilon|_{\min}^{1/2} ||\mathbf{u}||_{1/2+s,\Omega} \leq C\, \omega |\varepsilon|_{\max}^{1/2} ||\overline{\varepsilon} \mathbf{w}||_{0,\Omega}. \tag{3.7}$$

When Ω is convex, the inequality (3.7) holds for all $0 < s < 1/2$.

Remark 3.3. We would like to give simple explanations to above estimates. From the analysis in Theorem 3.1 of [9], we can get

$$||\nabla \times \mathbf{u}||_{0,\Omega} + \omega \mu^{\frac{1}{2}} ||\overline{\varepsilon}^{\frac{1}{2}} \mathbf{u}||_{0,\Omega} \leq \big(\frac{d^2}{\eta} (|\varepsilon|_{\max}^{\frac{1}{2}} |\sigma|_{\min} + 4\mu |\varepsilon|_{\max}^{-\frac{1}{2}} |\sigma|_{\min}^{-1}) \mu^{-\frac{1}{2}} + d^{\frac{1}{2}} \big) ||\overline{\varepsilon} \mathbf{w}||_{0,\Omega},$$

where $d = \operatorname{diam}(\Omega)$ and η is a real number measuring the nature of Ω as the coefficient γ defined in [9]. Then (3.6) can be derived directly by this inequality since $\sigma = \sqrt{\mu/|\varepsilon|}$. Furthermore, from the analysis in Theorem 4.4 of [9], we can obtain (3.7). It is not hard to see that the constant C in Lemma 3.2 is independent of the parameters ω and ε.

Theorem 3.4. *Let the parameters δ, α, β and θ in the functional (2.11) be chosen as*

$$\delta = \alpha = \frac{|\varepsilon|_{\max}^4}{|\varepsilon|_{\min}^4}, \quad \beta = \frac{|\varepsilon|_{\max}^4}{|\varepsilon|_{\min}^5} \quad \text{and} \quad \theta = \frac{|\varepsilon|_{\max}^2}{|\varepsilon|_{\min}^4}. \tag{3.8}$$

And we suppose there exists a constant $C_0 > 0$, such that $|\kappa|_{\max} h \leq C_0$. Then the error $\mathbf{e}_h = \mathbf{E} - \mathbf{E}_h$ satisfies

$$||\mathbf{e}_h||_{0,\Omega} \leq C |\kappa|_{\max}^{1/2} h^{-1/2} ||\mathbf{e}_h||_{\mathbf{V}}. \tag{3.9}$$

Proof. Using the decomposition (3.2), yields

$$|\varepsilon|_{\min} ||\mathbf{e}_h||_{0,\Omega}^2 \leq |\int_\Omega \varepsilon \mathbf{e}_h \cdot \overline{\mathbf{e}}_h \, dx| = |\int_\Omega \mathbf{e}_h \cdot \overline{\overline{\varepsilon} \mathbf{w}} \, dx + \int_\Omega \varepsilon \mathbf{e}_h \cdot \overline{\nabla p} \, dx|. \tag{3.10}$$

Integrating by parts, using equality (3.1) and taking into account the boundary conditions of (3.5), we deduce that

$$
\begin{aligned}
\int_\Omega \mathbf{e}_h \cdot \overline{\varepsilon \mathbf{w}}\ \mathrm{d}x &= \sum_{k=1}^N \int_{\Omega_k} \mathbf{e}_h \cdot \overline{\left(\nabla\times(\nabla\times\mathbf{u})-\overline{\kappa}^2\mathbf{u}\right)}\ \mathrm{d}x \\
&= \sum_{k=1}^N \Big(\int_{\Omega_k} \left(\nabla\times(\nabla\times\mathbf{e}_h)-\kappa^2\mathbf{e}_h\right)\cdot\overline{\mathbf{u}}\ \mathrm{d}x - \int_{\partial\Omega_k} \mathbf{e}_h\cdot\overline{(\nabla\times\mathbf{u})\times\mathbf{n_k}}\ \mathrm{d}s \\
&\quad - \int_{\partial\Omega_k} (\nabla\times\mathbf{e}_h)\cdot\overline{(\mathbf{u}\times\mathbf{n_k})}\ \mathrm{d}s\Big) \\
&= \sum_{k=1}^N \Big(\int_{\partial\Omega_k} (\mathbf{e}_h\times\mathbf{n_k})\cdot\overline{(\nabla\times\mathbf{u})}\ \mathrm{d}s + \int_{\partial\Omega_k} (\nabla\times\mathbf{e}_h)\times\mathbf{n_k}\cdot\overline{\mathbf{u}}\ \mathrm{d}s\Big) \\
&= \sum_{l<j} \Big(\int_{\Gamma_{lj}} [\![\mathbf{e}_h\times\mathbf{n}]\!]\cdot\overline{(\nabla\times\mathbf{u})}\ \mathrm{d}s + \int_{\Gamma_{lj}} [\![(\nabla\times\mathbf{e}_h)\times\mathbf{n}]\!]\cdot\overline{\mathbf{u}}\ \mathrm{d}s\Big) \\
&\quad + \sum_{k=1}^N \int_{\gamma_k} (-\mathbf{e}_h\times\mathbf{n_k}+\Phi(\mathbf{e}_h)\times\mathbf{n}_k)\cdot\overline{(-\nabla\times\mathbf{u})}\ \mathrm{d}s
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
\int_\Omega \varepsilon\mathbf{e}_h\cdot\overline{\nabla p}\ \mathrm{d}x &= \sum_{k=1}^N \int_{\Omega_k} \varepsilon_k\mathbf{e}_h\cdot\overline{\nabla p}\ \mathrm{d}x \\
&= \sum_{k=1}^N \Big(-\int_{\Omega_k} \varepsilon_k(\operatorname{div}\mathbf{e}_h)\cdot\overline{p}\ \mathrm{d}V + \int_{\partial\Omega_k} (\varepsilon_k\mathbf{e}_h\cdot\mathbf{n}_k)\cdot\overline{p}\ \mathrm{d}s\Big) \\
&= \sum_{k=1}^N \int_{\partial\Omega_k} (\varepsilon_k\mathbf{e}_h\cdot\mathbf{n}_k)\cdot\overline{p}\ \mathrm{d}s = \sum_{l<j}\int_{\Gamma_{lj}} [\![\varepsilon\mathbf{e}_h\cdot\mathbf{n}]\!]\cdot\overline{p}\ \mathrm{d}s.
\end{aligned}
\tag{3.12}
$$

Define

$$
G = \sum_{k=1}^N \delta^{-1}||\nabla\times\mathbf{u}||^2_{0,\gamma_k} + \sum_{l<j}\Big(\alpha^{-1}||\nabla\times\mathbf{u}||^2_{0,\Gamma_{lj}} + \beta^{-1}\omega^2||\mathbf{u}||^2_{0,\Gamma_{lj}} + \theta^{-1}||p||^2_{0,\Gamma_{lj}}\Big).
$$

Substituting (3.11) and (3.12) into (3.10), we get

$$
\begin{aligned}
|\varepsilon|_{\min}||\mathbf{e}_h||^2_{0,\Omega} &\le \Big|\sum_{k=1}^N \int_{\gamma_k} (-\mathbf{e}_h\times\mathbf{n_k}+\Phi(\mathbf{e}_h)\times\mathbf{n}_k)\cdot\overline{(-\nabla\times\mathbf{u})}\,\mathrm{d}s \\
&\quad + \sum_{l<j}\Big(\int_{\Gamma_{lj}} [\![\mathbf{e}_h\times\mathbf{n}]\!]\cdot\overline{(\nabla\times\mathbf{u})}\,\mathrm{d}s + \int_{\Gamma_{lj}} [\![(\nabla\times\mathbf{e}_h)\times\mathbf{n}]\!]\cdot\overline{\mathbf{u}}\mathrm{d}s + \int_{\Gamma_{lj}} [\![\varepsilon\mathbf{e}_h\cdot\mathbf{n}]\!]\cdot\overline{p}\,\mathrm{d}s\Big)\Big| \\
&\le C\Big(\sum_{k=1}^N ||-\mathbf{e}_h\times\mathbf{n_k}+\Phi(\mathbf{e}_h)\times\mathbf{n}_k||_{0,\gamma_k}\cdot||\nabla\times\mathbf{u}||_{0,\gamma_k} \\
&\quad + \sum_{l<j}\big(||[\![\mathbf{e}_h\times\mathbf{n}]\!]||_{0,\Gamma_{lj}}\cdot||\nabla\times\mathbf{u}||_{0,\Gamma_{lj}} + ||[\![\Psi(\mathbf{e}_h)\times\mathbf{n}]\!]||_{0,\Gamma_{lj}}\cdot||\omega\mathbf{u}||_{0,\Gamma_{lj}} \\
&\quad + ||[\![\varepsilon\mathbf{e}_h\cdot\mathbf{n}]\!]||_{0,\Gamma_{lj}}\cdot||p||_{0,\Gamma_{lj}}\big)\Big).
\end{aligned}
$$

Then, by Cauchy–Schwarz inequality, we further obtain

$$|\varepsilon|_{\min}||\mathbf{e}_h||^2_{0,\Omega} \leq C||\mathbf{e}_h||_{\mathbf{V}} \cdot G^{\frac{1}{2}}. \tag{3.13}$$

In the following we estimate the G. Taking $\delta = \alpha$ in G and using the trace inequality (see [16], Thm. A.2)

$$||u||^2_{0,\partial K} \leq C(h_K^{-1}||u||^2_{0,K} + h_K^{2\eta}|u|^2_{1/2+\eta,K}), \quad \forall\, u \in H^{1/2+\eta}(K) \quad (\eta > 0)$$

on each element $K \in \mathcal{T}_h$, leads to

$$\begin{aligned}
G &\leq \sum_{k=1}^{N} \Big(\alpha^{-1}||\nabla \times \mathbf{u}||^2_{0,\partial\Omega_k} + \beta^{-1}\omega^2||\mathbf{u}||^2_{0,\partial\Omega_k} + \theta^{-1}||p||^2_{0,\partial\Omega_k}\Big) \\
&\leq C\sum_{k=1}^{N} \Big(\alpha^{-1}h^{-1}||\nabla \times \mathbf{u}||^2_{0,\Omega_k} + \alpha^{-1}h^{2s}||\nabla \times \mathbf{u}||^2_{1/2+s,\Omega_k} + \beta^{-1}\omega^2 h^{-1}||\mathbf{u}||^2_{0,\Omega_k} \\
&\quad + \beta^{-1}\omega^2 h^{2s}||\mathbf{u}||^2_{1/2+s,\Omega_k} + \theta^{-1}h^{-1}||p||^2_{0,\Omega_k} + \theta^{-1}h|p|^2_{1,\Omega_k}\Big) \\
&\leq C\big(\alpha^{-1}h^{-1}||\nabla \times \mathbf{u}||^2_{0,\Omega} + \alpha^{-1}h^{2s}||\nabla \times \mathbf{u}||^2_{1/2+s,\Omega} + \beta^{-1}h^{-1}\omega^2||\mathbf{u}||^2_{0,\Omega} \\
&\quad + \beta^{-1}h^{2s}\omega^2||\mathbf{u}||^2_{1/2+s,\Omega} + \theta^{-1}h^{-1}||p||^2_{0,\Omega} + \theta^{-1}h|p|^2_{1,\Omega}\big).
\end{aligned}$$

Then from Lemma 3.1, we have $||p||_{0,\Omega} \leq C|p|_{1,\Omega} \leq C\frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}||\mathbf{e}_h||_{0,\Omega}$, because of $p \in H^1_0(\Omega)$. So using the stability estimates (3.6) and (3.7), Lemma 3.1 and the above inequality, we can bound G as follows

$$\begin{aligned}
G &\leq C\alpha^{-1}\big(h^{-1} + h^{2s}\omega^2|\varepsilon|_{\max}\big)\frac{|\varepsilon|^4_{\max}}{|\varepsilon|^2_{\min}}||\mathbf{e}_h||^2_{0,\Omega} \\
&\quad + C\beta^{-1}\big(h^{-1} + h^{2s}\omega^2|\varepsilon|_{\max}\big)\frac{|\varepsilon|^4_{\max}}{|\varepsilon|^3_{\min}}||\mathbf{e}_h||^2_{0,\Omega} + C\theta^{-1}h^{-1}\frac{|\varepsilon|^2_{\max}}{|\varepsilon|^2_{\min}}||\mathbf{e}_h||^2_{0,\Omega} \\
&\leq C|\varepsilon|^2_{\min}\big(h^{-1} + h^{2s}\omega^2|\varepsilon|_{\max}\big)||\mathbf{e}_h||^2_{0,\Omega}
\end{aligned}$$

with $\alpha = \frac{|\varepsilon|^4_{\max}}{|\varepsilon|^4_{\min}}$, $\quad \beta = \frac{|\varepsilon|^4_{\max}}{|\varepsilon|^5_{\min}}$ and $\theta = \frac{|\varepsilon|^2_{\max}}{|\varepsilon|^4_{\min}}$.

Thus we have

$$\begin{aligned}
G^{1/2} &\leq C|\varepsilon|_{\min}(h^{-1/2} + h^s\omega\sqrt{|\varepsilon|}_{\max})||\mathbf{e}_h||_{0,\Omega} \\
&\leq C|\varepsilon|_{\min}h^{-1/2}(1 + h^{s+1/2}|\kappa|_{\max})||\mathbf{e}_h||_{0,\Omega} \\
&\leq C|\varepsilon|_{\min}h^{-1/2}(1 + h^s|\kappa|^{1/2}_{\max})||\mathbf{e}_h||_{0,\Omega} \leq C|\varepsilon|_{\min}h^{-1/2}|\kappa|^{1/2}_{\max}||\mathbf{e}_h||_{0,\Omega}.
\end{aligned} \tag{3.14}$$

Taking (3.14) into (3.13), we obtain the estimate (3.9). □

4. L^2 ERROR ESTIMATE OF THE PLANE WAVE APPROXIMATIONS

In this section, we derive a L^2 estimate for the errors of the approximate solution $\mathbf{E}_h$ defined by (2.18) by using Theorem 3.4, as well as the approximation properties of plane wave space.

4.1. Basis functions of $\mathbf{V}_p(\mathcal{T}_h)$ and algebraic form of (2.16)

In this subsection we give a precise definition of the basis functions of the space $\mathbf{V}_p(\mathcal{T}_h)$.

In practice, following [3], a suitable family of plane waves, which are solutions of the constant-coefficient Maxwell equations, are generated on Ω_k by choosing p unit propagation directions $\mathbf{d}_l$ $(l = 1, \ldots, p)$ (we use the optimal spherical codes from [18]), and defining a real unit polarization vector $\mathbf{G}_l$ orthogonal to $\mathbf{d}_l$. Then the propagation directions and polarization vectors define the complex polarization vectors $\mathbf{F}_l$ and $\mathbf{F}_{l+p}$ by

$$\mathbf{F}_l = \mathbf{G}_l + \mathrm{i}\mathbf{G}_l \times \mathbf{d}_l, \quad \mathbf{F}_{l+p} = \mathbf{G}_l - \mathrm{i}\mathbf{G}_l \times \mathbf{d}_l \quad (l = 1, \dots, p).$$

It is clear that (2.4) can be written as

$$\nabla \times (\nabla \times \mathbf{E}_k) - \kappa^2 \mathbf{E}_k = 0 \quad \text{in } \Omega_k, \tag{4.1}$$

with $\kappa = \omega\sqrt{\mu\varepsilon}$. We then define the complex functions $\mathbf{E}_l$

$$\mathbf{E}_l = \sqrt{\mu}\ \mathbf{F}_l \exp(\mathrm{i}\kappa \mathbf{d}_l \cdot \mathbf{x}) \qquad l = 1, \dots, 2p. \tag{4.2}$$

It is easy to verify that every function $\mathbf{E}_l$ $(l = 1, \dots, 2p)$ satisfies Maxwell's equation (4.1). Notice that κ has different values on different elements for inhomogeneous media, so the plane wave basis functions $\mathbf{E}_l$ may have different form on different elements.

Let $\mathcal{Q}_{2p}$ denote the space spanned by the $2p$ plane wave functions $\mathbf{E}_l$ $(l = 1, \dots, 2p)$. Define the finite element space

$$\mathbf{V}_p(\mathcal{T}_h) = \left\{ \mathbf{v} \in L^2(\Omega) : \ \mathbf{v}|_K \in \mathcal{Q}_{2p} \quad \forall\, K \in \mathcal{T}_h \right\}, \tag{4.3}$$

which has $N \times 2p$ basis functions, which are defined by

$$\boldsymbol{\phi}_l^k(\mathbf{x}) = \begin{cases} \mathbf{E}_l(\mathbf{x}) & \mathbf{x} \in \Omega_k \\ 0 & \mathbf{x} \in \Omega_j \text{ when } j \neq k \end{cases} \quad (k = 1, \dots, N,\ l = 1, \dots, 2p). \tag{4.4}$$

Let $\mathbf{V}_p(\mathcal{T}_h)$ be defined as in (4.3). Now we define an approximation of $\mathbf{E}_k$ by

$$\mathbf{E}_h^k = \sum_{l=1}^{2p} x_l^k \boldsymbol{\phi}_l^k. \tag{4.5}$$

In this case, after solving problem (2.18), the approximate solution of Maxwell's equations (2.3) are obtained directly, because the unknown $\mathbf{E}_h$ is defined on the elements. Moreover, the structure of the sesquilinear form $A(\cdot,\cdot)$ is very simple, so the method seems easier to implement.

As in [12], let $\mathcal{A}$ be the stiffness matrix associated with the sesquilinear form $A(\cdot,\cdot)$, and let b denote the vector associated with the vector product $(\boldsymbol{\xi},\ \mathbf{F}_h)$. Then the discretized problem (2.18) leads to the algebraic system

$$\mathcal{A}X = b, \tag{4.6}$$

where $X = (x_{11}, x_{12}, \dots, x_{1\,2p}, x_{21}, \dots, x_{2\,2p}, \dots, x_{N1}, \dots, x_{N\,2p})^t \in \mathbb{C}^{2pN}$ is the vector of unknowns. Furthermore, we know that the matrix $\mathcal{A}$ is Hermitian positive definite from the definition of the bilinear form $A(\cdot,\cdot)$, so the system (4.6) is easier to be solved.

4.2. A L^2 Error estimate

For a domain $D \subset \Omega$, let $||\cdot||_{s,\kappa,D}$ be the weighted Sobolev norm defined by

$$||v||^2_{s,\kappa,D} = \sum_{j=0}^{s} |\kappa|^{2(s-j)} |v|^2_{j,D}.$$

Let the mesh triangulation $\mathcal{T}_h$ satisfy the definition stated in [6] and set $\lambda = \min_{K\in\mathcal{T}_h} \lambda_K$, where λ_K is the positive parameter that depends only on the shape of an element K of $\mathcal{T}_h$. Let r and q be given positive integers satisfying $q \geq 2r + 1$ and $q \geq 2(1 + 2^{1/\lambda})$. Let the number p of plane-wave propagation directions be chosen as $p = (q + 1)^2$.

The following approximation can be viewed as a version of Theorem 5.4 in [6].

Lemma 4.1. *([12], Lem. 4.2) Let* $\mathbf{E}$ *be the solution of the equations (2.3). Assume that* $\mathbf{E} \in H^{r+1}(curl, K)$*. Then there exists* $\boldsymbol{\xi}_{\mathbf{E}} \in \mathbf{V}_p(\mathcal{T}_h)|_K$ *such that*

$$\begin{aligned}||\mathbf{E}-\boldsymbol{\xi}_{\mathbf{E}}||_{j-1,\kappa,K} \leq C|\kappa|^{-2}(1+(|\kappa|h_K)^{q+j-r+8})\mathrm{e}^{(\frac{7}{4}-\frac{3}{4}\rho)|\kappa|h_K} h_K^{r+1-j} \\ \times [q^{-\lambda_K(r+1-j)}+(\rho q)^{-\frac{q-3}{2}}pq]\,||\nabla\times\mathbf{E}||_{r+1,\kappa,K}\end{aligned} \tag{4.7}$$

for every $1 \leq j \leq r$*. Here* ρ *is a real number measuring the nature of* Ω *as defined in [6].*

It is known that the plane wave methods are attractive only for the cases of middle and high frequency (refer to [5]), which are just the most challenging situations in applications. Thus, we can suppose there exits a constant $c_0 > 0$, such that $|\kappa|_{\min} \geq c_0$. Based on Theorem 3.4 and the above lemma, we can build a L^2 error estimate for the proposed method.

Theorem 4.2. *Assume that the analytical solution* $\mathbf{E}$ *of the Maxwell problem (2.3) belongs to* $H^{r+1}(curl, \Omega)$ *and there are constants* C_0 *and* c_0 *such that* $|\kappa|_{\max}h \leq C_0$ *and* $|\kappa|_{\min} \geq c_0$*. Let the parameters in (2.11) be defined by (3.8) and let* $\mathbf{E}_h \in \mathbf{V}_p(\mathcal{T}_h)$ *be the solution of (2.18). Then, for large* $p = (q+1)^2$ *and the integer* r *satisfying* $r \geq 3$ *and* $2r+1 \leq q$*, we have*

$$||\mathbf{E}-\mathbf{E}_h||_{0,\Omega} \leq C\Big(\frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}\Big)^{\frac{9}{4}}\frac{h^{r-2}}{q^{\lambda(r-\frac{3}{2})}}||\nabla\times\mathbf{E}||_{r+1,\kappa,\Omega}, \tag{4.8}$$

where C *is a constant independent of* p *but dependent on* κ *and* h *only through the product* $|\kappa|_{max}h$ *as an increasing function.*

Proof. Let $\mathbf{F}_h = \boldsymbol{\xi}_{\mathbf{E}} \in \mathbf{V}_p(\mathcal{T}_h)$ be defined by Lemma 4.1, then $\mathbf{E}_h - \mathbf{F}_h \in \mathbf{V}_p(\mathcal{T}_h)$. Notice that $\mathbf{V}_p(\mathcal{T}_h) \subset \mathbf{V}(\mathcal{T}_h)$, by the definition of $\mathbf{E}$ and $\mathbf{E}_h$, we have

$$A(\mathbf{E}-\mathbf{E}_h, \mathbf{E}_h-\mathbf{F}_h) = 0.$$

Then we get by the Cauchy–Schwarz inequality

$$||\mathbf{E}-\mathbf{E}_h||_{\mathbf{V}}^2 = A(\mathbf{E}-\mathbf{E}_h, \mathbf{E}-\mathbf{F}_h) \leq ||\mathbf{E}-\mathbf{E}_h||_{\mathbf{V}} \cdot ||\mathbf{E}-\mathbf{F}_h||_{\mathbf{V}},$$

which implies that

$$||\mathbf{E}-\mathbf{E}_h||_{\mathbf{V}} \leq ||\mathbf{E}-\mathbf{F}_h||_{\mathbf{V}}. \tag{4.9}$$

It suffices to estimate $||\mathbf{E}-\mathbf{F}_h||_{\mathbf{V}}$. For ease of notation, set $\boldsymbol{\epsilon}_h = \mathbf{E}-\mathbf{F}_h$. By the definition of the norm $||\cdot||_{\mathbf{V}}$, we get

$$||\boldsymbol{\epsilon}_h||_{\mathbf{V}}^2 \leq C\frac{|\varepsilon|_{\max}^4}{|\varepsilon|_{\min}^4}\sum_{k=1}^{N}\Big(\int_{\partial\Omega_k}|\boldsymbol{\epsilon}_{h,k}|^2\,\mathrm{d}s + |\kappa|_{\min}^{-2}\int_{\partial\Omega_k}|(\nabla\times\boldsymbol{\epsilon}_{h,k})\times\mathbf{n}_k|^2\,\mathrm{d}s\Big) \tag{4.10}$$

where $\boldsymbol{\epsilon}_{h,k} = \boldsymbol{\epsilon}_h|_{\Omega_k}$. In an analogous way with the proof of Corollary 5.5 in [6], we can prove that

$$\int_{\partial\Omega_k}|\boldsymbol{\epsilon}_{h,k}|^2\mathrm{d}s + |\kappa|_{\min}^{-2}\int_{\partial\Omega_k}|(\nabla\times\boldsymbol{\epsilon}_{h,k})\times\mathbf{n}_k|^2\mathrm{d}s \leq C|\kappa|_{\min}^{-4}\Big(\frac{h}{q^\lambda}\Big)^{2r-3}||\nabla\times\mathbf{E}||_{r+1,\kappa,\Omega_k}^2.$$

Substituting the above inequality into (4.10) and combing (4.9), yields

$$\begin{aligned}||\mathbf{E}-\mathbf{E}_h||_{\mathbf{V}} &\leq C\,|\kappa|_{\min}^{-2}\frac{|\varepsilon|_{\max}^2}{|\varepsilon|_{\min}^2}\Big(\frac{h}{q^\lambda}\Big)^{r-\frac{3}{2}}||\nabla\times\mathbf{E}||_{r+1,\kappa,\Omega} \\ &\leq C\,|\kappa|_{\min}^{-1/2}\frac{|\varepsilon|_{\max}^2}{|\varepsilon|_{\min}^2}\Big(\frac{h}{q^\lambda}\Big)^{r-\frac{3}{2}}||\nabla\times\mathbf{E}||_{r+1,\kappa,\Omega}.\end{aligned} \tag{4.11}$$

On the other hand, from the Theorem 3.4 we know that

$$||\mathbf{E}-\mathbf{E}_h||_{0,\Omega} \leq C|\kappa|_{\max}^{1/2}h^{-1/2}||\mathbf{E}-\mathbf{E}_h||_{\mathbf{V}}.$$

Then we get the desired estimate (4.8) by combining the above inequality with (4.11). $\square$

4.3. The case of homogeneous material

In this subsection we consider the special case that the coefficient ε in (2.1) is a constant on the whole domain Ω. For this special case, which was considered in the most existing papers, the proposed variational formula (2.13) can be simplified.

According to the choices of the parameters (see (3.8)), we have $\alpha=\delta=1$, $\beta=\frac{1}{|\varepsilon|}$ and $\theta=\frac{1}{|\varepsilon|^2}$. It is easy to see that the minimization functional (2.11) becomes

$$\begin{aligned}J(\mathbf{F})=&\sum_{k=1}^{N}\int_{\gamma_k}|-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k-\mathbf{g}|^2\ \mathrm{d}s\\&+\sum_{l<j}\Big(\int_{\Gamma_{lj}}|[\![\mathbf{F}\times\mathbf{n}]\!]|^2\ \mathrm{d}s+\frac{1}{|\varepsilon|}\int_{\Gamma_{lj}}|[\![\Psi(\mathbf{F})\times\mathbf{n}]\!]|^2\ \mathrm{d}s+\int_{\Gamma_{lj}}|[\![\mathbf{F}\cdot\mathbf{n}]\!]|^2\ \mathrm{d}s\Big)\\=&\sum_{k=1}^{N}\int_{\gamma_k}|-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k-\mathbf{g}|^2\ \mathrm{d}s\\&+\sum_{l<j}\Big(\int_{\Gamma_{lj}}|[\![\mathbf{F}]\!]|^2\ \mathrm{d}s+\frac{1}{|\varepsilon|}\int_{\Gamma_{lj}}|[\![\Psi(\mathbf{F})\times\mathbf{n}]\!]|^2\ \mathrm{d}s\Big),\quad\forall\mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).\end{aligned}$$

Here we have used the equality $|[\![\mathbf{F}\times\mathbf{n}]\!]|^2+|[\![\mathbf{F}\cdot\mathbf{n}]\!]|^2=|[\![\mathbf{F}]\!]|^2$ because of the relation $[\![\mathbf{F}]\!]=[\![\mathbf{n}\times(\mathbf{F}\times\mathbf{n})]\!]+[\![(\mathbf{F}\cdot\mathbf{n})\mathbf{n}]\!]$ and the orthogonality between $[\![\mathbf{n}\times(\mathbf{F}\times\mathbf{n})]\!]$ and $[\![(\mathbf{F}\cdot\mathbf{n})\mathbf{n}]\!]$.

Then the variational problem can be expressed as follows: Find $\mathbf{E}\in\mathbf{V}^*(\mathcal{T}_h)$ such that

$$\begin{aligned}&\sum_{k=1}^{N}\int_{\gamma_k}(-\mathbf{E}_k\times\mathbf{n}_k+\Phi(\mathbf{E}_k)\times\mathbf{n}_k-\mathbf{g})\cdot\overline{-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k}\ \mathrm{d}s\\&+\sum_{l<j}\big(\int_{\Gamma_{lj}}[\![\mathbf{E}]\!]\cdot[\![\overline{\mathbf{F}}]\!]\ \mathrm{d}s+\frac{1}{|\varepsilon|}\int_{\Gamma_{lj}}[\![\Psi(\mathbf{E})\times\mathbf{n}]\!]\cdot\overline{[\![\Psi(\mathbf{F})\times\mathbf{n}]\!]}\ \mathrm{d}s\big)=0,\quad\forall\ \mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).\end{aligned}$$

Thus the sesquilinear $A(\cdot,\cdot)$ and the functional $(\boldsymbol{\xi},\ \cdot)$ are written as

$$\begin{aligned}A(\mathbf{E},\mathbf{F})=&\sum_{k=1}^{N}\int_{\gamma_k}(-\mathbf{E}_k\times\mathbf{n}_k+\Phi(\mathbf{E}_k)\times\mathbf{n}_k)\cdot\overline{-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k}\ \mathrm{d}s\\&+\sum_{l<j}\big(\int_{\Gamma_{lj}}[\![\mathbf{E}]\!]\cdot[\![\overline{\mathbf{F}}]\!]\ \mathrm{d}s+\frac{1}{|\varepsilon|}\int_{\Gamma_{lj}}[\![\Psi(\mathbf{E})\times\mathbf{n}]\!]\cdot\overline{[\![\Psi(\mathbf{F})\times\mathbf{n}]\!]}\ \mathrm{d}s\big),\ \forall\ \mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).\end{aligned}$$

and

$$(\boldsymbol{\xi},\ \mathbf{F})=\sum_{k=1}^{N}\int_{\gamma_k}\mathbf{g}\cdot\overline{-\mathbf{F}_k\times\mathbf{n}_k+\Phi(\mathbf{F}_k)\times\mathbf{n}_k}\ \mathrm{d}s,\qquad\forall\ \mathbf{F}\in\mathbf{V}^*(\mathcal{T}_h).$$

It is clear that the expression of the above $A(\cdot,\cdot)$ is simpler than the one introduced in Section 2.2. With this expression of $A(\cdot,\cdot)$, the cost for the implementation of the proposed method can be reduced. Moreover, for this case the error estimate in (4.8) becomes

$$||\mathbf{E}-\mathbf{E}_h||_{0,\Omega}\le C\frac{h^{r-2}}{q^{\lambda(r-\frac{3}{2})}}||\nabla\times\mathbf{E}||_{r+1,\kappa,\Omega}.\tag{4.12}$$

Remark 4.3. We emphasize that the L^2 error estimates (4.8) and (4.12) are almost the same as the L^2 error estimates of the plane wave approximations for three-dimensional Helmholtz equations in homogeneous media (there is an extra factor $\big(\frac{|\varepsilon|_{\max}}{|\varepsilon|_{\min}}\big)^{\frac{9}{4}}$ in (4.8)).

5. Numerical experiments

For the examples tested in this section, we adopt a uniform triangulation $\mathcal{T}_h$ for the domain Ω as follows: Ω is divided into small cubes of equal meshwidth, where h is the length of the longest edge of the elements. As described in Section 4.3, we choose the number of basis functions p to be $p = (q+1)^2$ for all elements Ω_k, where q is a positive integer.

Meanwhile we choose a set of $\mathbf{d}_l$ using the optimal spherical codes from [18]. For a given $\mathbf{d}_l$, the vectors $\mathbf{G}_l$ that are orthogonal to $\mathbf{d}_l$ are not unique. In this section, when we write $\mathbf{d}_l = [a_l, b_l, c_l]^t$ (which satisfies $a_l^2 + b_l^2 + c_l^2 = 1$), we always choose G_l as follows

$$\mathbf{G}_l = \left[\frac{a_l b_l}{\sqrt{1-b_l^2}}, \ -\sqrt{1-b_l^2}, \ \frac{-a_l^2 b_l + y(1-b_l^2)}{c_l\sqrt{1-b_l^2}} \right]^t.$$

To measure the accuracy of the numerical solution, we introduce the following relative numerical error

$$\text{err.} = \frac{||\mathbf{E}_{\text{ex}} - \mathbf{E}_h||_{L^2(\Omega)}}{||\mathbf{E}_{\text{ex}}||_{L^2(\Omega)}}$$

for the exact solution $\mathbf{E}_{\text{ex}} \in L^2(\Omega)$.

We define the error order with respect to h in the standard manner. Meanwhile we introduce the error order with respect to p as

$$\text{order} = \frac{\ln(\text{err}_{p_2}/\text{err}_{p_1})}{\ln(p_2/p_1)},$$

where err_{p_1} and err_{p_2} denote the relative numerical errors with $p = p_1$ and $p = p_2$, respectively.

For the examples tested in this section, we assume that $\mu = 1$ and $\sigma = \sqrt{\mu/|\varepsilon|}$. Meanwhile we choose the parameters in (2.11) according to (3.8) in the numerical experiments. We perform all computations on a Dell Precision T7610 workstation using MATLAB implementations.

5.1. The case of homogeneous media

5.1.1. Electric dipole in free space for a smooth case

In this part we consider the example tested in [12, 15]. In Maxwell system (2.3) we choose $\Omega = [-0.5, 0.5]^3$ and set $\varepsilon = 1 + i$ (μ and σ are defined above). We compute the electric field due to an electric dipole source at the point $\mathbf{x}_0 = (0.6, 0.6, 0.6)$. The dipole point source can be defined as the solution of a homogeneous Maxwell system (2.3). The exact solution of the problems is

$$\mathbf{E}_{\text{ex}} = -\mathrm{i}\omega I \phi(\mathbf{x}, \mathbf{x}_0)\mathbf{a} + \frac{I}{\mathrm{i}\omega\varepsilon}\nabla(\nabla\phi \cdot \mathbf{a}) \tag{5.1}$$

with

$$\phi(\mathbf{x}, \mathbf{x}_0) = \frac{\exp(\mathrm{i}\omega\sqrt{\varepsilon}|\mathbf{x} - \mathbf{x}_0|)}{4\pi|\mathbf{x} - \mathbf{x}_0|}, \quad \mathbf{a} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^t$$

and I is the unit matrix.

Then the boundary data is computed by

$$\mathbf{g}_{\text{ex}} = -\mathbf{E}_{\text{ex}} \times \mathbf{n} + \frac{\sigma}{i\omega\mu}((\nabla \times \mathbf{E}_{\text{ex}}) \times \mathbf{n}) \times \mathbf{n}. \tag{5.2}$$

In Tables 1 and 2, we show the convergence orders of the resulting approximations with respect to h and p, respectively.

TABLE 1. Convergence orders of approximations with respect to h ($\omega = 4\pi$).

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{20}$
$p = 16$	err.	0.5164	0.1351	0.0483	0.0219	0.0116
	order		1.9345	2.5368	2.7494	2.8479
$p = 25$	err.	0.2630	0.0413	0.0108	0.0035	0.0014
	order		2.6708	3.3081	3.9168	4.1063

TABLE 2. Convergence orders of approximations with respect to p ($\omega = 6\pi$ and $h = \frac{1}{8}$).

p	16	25	36	49	64	81
err.	0.2627	0.0905	0.0293	0.0099	0.0026	0.0007
order		−2.3878	−3.0928	−3.5195	−5.0064	−5.4039

TABLE 3. Errors of approximations with increasing ω ($\omega h = \pi$ and $p = 25$).

ω	4π	8π	12π	16π
err.	0.2630	0.1711	0.1329	0.1116

TABLE 4. Errors of approximations with respect to h for the case of $\omega = 8\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$p = 16$	old PWLS	0.8214	0.4206	0.0918
	new PWLS	0.8207	0.4162	0.0851
$p = 25$	old PWLS	0.6372	0.1751	0.0214
	new PWLS	0.6359	0.1711	0.0180

The data in Tables 1 and 2 indicate that the proposed method indeed possess high accuracy and the convergence orders depend on the values of p. In Table 3, we report the errors of approximations with increasing ω when $\omega h = \pi$ is fixed.

It can be seen from Table 3 that the errors slightly become smaller when the values of ωh and p are fixed but the value of ω is increased, *i.e.*, there is no "wave number pollution". All the results in the above three tables show the validity of the theoretical results in Theorem 4.2 for this example.

Then we would like to compare the relative L^2 errors of the approximations generated by the proposed method and the PWLS method in [12]. For convenience, throughout this section we use "new PWLS" to represent the method proposed in this paper and "old PWLS" to represent the method introduced in [12] which doesn't have the jump of the normal component in the least-squares functional. The comparison results are shown in Table 4.

The results listed in Table 4 indicate that the approximations generated by the new PWLS method have smaller errors than that generated by the old PWLS method in [12].

5.1.2. Another example

The first example seems too special. In this part, we consider another example in the homogeneous media. We also choose $\Omega = [-0.5, 0.5]^3$ and assume that $\varepsilon = 1 + i$. The other quantities are kept the same. The analytical

Table 5. Convergence orders of approximations with respect to h ($\omega = 4\pi$).

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$	$\frac{1}{20}$
$p = 16$	err.	0.1549	0.0252	0.0076	0.0022	0.0008
	order		2.6198	2.9563	4.3092	4.5334
$p = 25$	err.	0.0568	0.0044	0.0010	0.0003	0.0001
	order		3.6903	3.6541	4.1851	4.9233

Table 6. Convergence orders of approximations with respect to p ($\omega = 6\pi,\ h = \frac{1}{8}$).

p	16	25	36	49	64	81
err.	6.83e-2	1.94e-2	3.30e-3	5.03e-4	9.31e-5	1.95e-5
order		−2.8202	−4.8578	−6.1015	−6.3166	−6.6362

Table 7. Errors of approximations with increasing ω ($\omega h = \pi,\ p = 25$).

ω	4π	8π	12π	16π
err.	0.0568	0.0519	0.0492	0.0474

Table 8. Errors of approximations with respect to h for the case of $\omega = 8\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$p = 16$	old PWLS	0.4792	0.1316	0.0218
	new PWLS	0.4784	0.1293	0.0171
$p = 25$	old PWLS	0.3481	0.0531	0.0050
	new PWLS	0.3476	0.0519	0.0042

solution is given by

$$\mathbf{E}_{\text{ex}} = (\omega\sqrt{\varepsilon}xz\,\cos(\omega\sqrt{\varepsilon}y), -z\,\sin(\omega\sqrt{\varepsilon}y), -\,\cos(\omega\sqrt{\varepsilon}x)^t. \tag{5.3}$$

In Tables 5 and 6, we show the orders of approximations with respect to h and p, respectively. In Table 7, we list the errors of approximations with increasing ω when $\omega h = \pi$ is fixed. These indicate the validity of the theoretical results in Theorem 4.2 for the homogeneous medium.

The above results also indicate the validity of the theoretical results in Theorem 4.2 for this example. Then in Table 8 we give some comparisons for the L^2 relative errors as in the last subsection.

Similarly to the first example, the results listed in Table 8 indicate that the approximations generated by the new PWLS method are slightly more accurate.

5.1.3. An example in non-convex polyhedron

In this part, we consider the case that the solution domain Ω is a non-convex polyhedron. It is known that the analytic solution $\mathbf{E}$ of (2.3) may be not in $(H^1(\Omega))^3$ if Ω is a non-convex polyhedron. Since such non-smooth solution cannot be directly given, as usual we replace the analytic solution by its "good" approximation generated by the standard finite element method with very fine grids in order to compute accuracies of the plane wave approximations generated by the proposed method.

Table 9. Convergence orders of approximations with respect to h ($\omega = 4\pi$).

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$
$p = 16$	err.	0.1868	0.0554	0.0271	0.0139
	order		1.7535	1.7635	2.3208
$p = 25$	err.	0.0894	0.0252	0.0118	0.0059
	order		1.8269	1.8713	2.4094

Table 10. Errors of approximations with respect to h ($\omega = 8\pi$).

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$p = 16$	old PWLS	0.4953	0.1682	0.0431
	new PWLS	0.4951	0.1659	0.0415
$p = 25$	old PWLS	0.3016	0.0943	0.0233
	new PWLS	0.3015	0.0918	0.0224

Let the boundary data $\mathbf{g}$ in the equations (2.3) be chosen as

$$\mathbf{g}_{\text{ex}} = (x,\, y,\, x + y + z)^t. \tag{5.4}$$

We choose $\Omega = [0,1] \times [0,1] \times [0,0.5] \cup [0,0.5] \times [0,0.5] \times [0.5,1]$ and $\varepsilon = 1 + i$. The other quantities are kept the same. Then the "good" approximation of the analytical solution can be generated by the standard finite element method with very fine grids.

In Tables 9 and 10, we show the convergence orders of approximations with respect to h and some comparisons between the proposed method and the original method.

Comparing Table 9 with Table 1 or Table 5, we find that the convergence orders for the case of non-convex polyhedron are indeed lower than that for the case of convex polyhedron since the analytic solution for the former has lower smoothness than the latter. The results listed in Table 10 indicate that the approximations generated by the new PWLS method also have sightly smaller errors than that generated by the old PWLS method in [12] for this example.

5.2. The case of inhomogeneous media

In this Subsection, we test examples in the inhomogeneous media. For the case of inhomogeneous media (*i.e.*, ε is not a constant), it is also difficult to give an analytic solution of the Maxwell system (2.3). So we also replace the analytic solution by a "good" approximation generated by the standard finite element method with very fine grids. For the examples tested in this section, we choose the size of the fine grids to be $\frac{1}{80}$.

5.2.1. Electric dipole in free space

Let the boundary data $\mathbf{g}$ in the equation (2.3) be chosen as the same function $\mathbf{g}_{\text{ex}}$ given in Section 5.1.1. But we choose $\varepsilon = 1 + i$ for the upper domain $z > 0$ and $\varepsilon = 2 + 2i$ for the region $z < 0$ (the other quantities are kept the same as in Sect. 5.1.1). Since the parameter ε is different from the one chosen in Section 5.1.1, the vector function $\mathbf{E}_{\text{ex}}$ in Section 5.1.1 is not the analytic solution of the current equations.

In Table 11, we show the convergence orders of approximations with respect to h. And then some comparisons for the L^2 relative errors are shown in Table 12.

The data in Table 11 show the high accuracy of the new method and the validity of the results in Theorem 4.2 for the inhomogeneous medium. The results in Table 12 indicate that the approximations generated by

TABLE 11. Orders of approximations with respect to h for the case of $\omega = 4\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$
$p = 16$	err.	0.5199	0.1416	0.0518	0.0253
	order		1.8764	2.4802	2.4909
$p = 25$	err.	0.2678	0.0455	0.0161	0.0065
	order		2.5572	2.5622	3.1528

TABLE 12. Errors of approximations with respect to h for the case of $\omega = 8\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$p = 16$	old PWLS	0.8210	0.4211	0.0921
	new PWLS	0.8207	0.4181	0.0889
$p = 25$	old PWLS	0.6376	0.1767	0.0308
	new PWLS	0.6369	0.1752	0.0290

TABLE 13. Orders of approximations with respect to h for the case of $\omega = 4\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$
$p = 16$	err.	0.1515	0.0441	0.0171	0.0084
	order		1.7805	2.3365	2.4709
$p = 25$	err.	0.0704	0.0189	0.0067	0.0028
	order		1.8972	2.5577	3.0328

this new PWLS method also have sightly smaller errors than that generated by the old PWLS method for a inhomogeneous medium.

5.2.2. Another example

In this part, we consider another example with more complicated structure of ε. We choose $\Omega = [0,\, 1]^3$ and define ε as

$$\varepsilon = \begin{cases} \frac{3}{2} + \frac{3}{2}i & y < 0.5, z < 0.5, \\ 1 + i & y > 0.5, z < 0.5, \\ 1 + i & y < 0.5, z > 0.5, \\ \frac{1}{2} + \frac{1}{2}i & y > 0.5, z > 0.5. \end{cases}$$

The other quantities are kept the same. The analytical boundary data $\mathbf{g}_{\mathrm{ex}}$ in the equation (2.3) be chosen as

$$\mathbf{g}_{\mathrm{ex}} = (x,\, y,\, x + y + z)^t. \tag{5.5}$$

In Tables 13 and 14, we show the convergence orders of approximations with respect to h and some comparisons between the new method and the original method.

The results listed in Tables 13 and 14 indicate the validity of the theoretical results in Theorem 4.2 and the advantage of the new PWLS method are kept even for the complicated inhomogeneous medium.

5.2.3. An example in non-convex polyhedron

In this part, we consider the case in non-convex polyhedron. We choose $\Omega = \big([0,1]\times[0,1]\times[0,0.5]\big)\cup\big([0,0.5]\times[0,0.5]\times[0.5,1]\big)$ and choose $\varepsilon = 1 + i$ for the domain $z > 0.5$ and $\varepsilon = 2 + 2i$ for the region $z < 0.5$. The other

TABLE 14. Errors of approximations with respect to h for the case of $\omega = 8\pi$

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$p = 16$	old PWLS	0.4148	0.1328	0.0275
	new PWLS	0.4132	0.1318	0.0272
$p = 25$	old PWLS	0.2572	0.0554	0.0093
	new PWLS	0.2570	0.0542	0.0088

TABLE 15. Orders of approximations with respect to h for the case of $\omega = 4\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{16}$
$p = 16$	err.	0.2959	0.1046	0.0494	0.0253
	order		1.5002	1.8502	2.3260
$p = 25$	err.	0.1439	0.0495	0.0232	0.0117
	order		1.5396	1.8690	2.3796

TABLE 16. Errors of approximations with respect to h for the case of $\omega = 8\pi$.

	h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
$p = 16$	old PWLS	0.6477	0.2914	0.0757
	new PWLS	0.6474	0.2862	0.0739
$p = 25$	old PWLS	0.4791	0.1602	0.0403
	new PWLS	0.4782	0.1563	0.0387

quantities are kept the same. The analytical boundary data $\mathbf{g}_{\rm ex}$ in the equation (2.3) is given by

$$\mathbf{g}_{\rm ex} = (x,\, y,\, x + y + z)^t. \tag{5.6}$$

In Table 15, we show the convergence orders of approximations with respect to h and in Table 16, we give some comparisons between the new method and the original method.

We find that the convergence orders for the case of non-convex polyhedron are also lower than that for the convex polyhedron in the case of inhomogeneous media by comparing Table 15 with Table 11 or Table 13. Meanwhile, the data in Table 16 indicate that the advantage of the new PWLS method is kept for this example.

Acknowledgements. This work was supported by Natural Science Foundation of China G11571352.

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