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SUBSTRUCTURING PRECONDITIONERS FOR $h-p$ MORTAR FEM *

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Abstract. We build and analyze a substructuring preconditioner for the Mortar method, applied to elliptic problems, in the *h-p* finite element framework. Particular attention is given to the construction of the coarse component of the preconditioner in this framework, in which continuity at the cross points is not required. Two variants are proposed: the first one is an improved version of a coarse preconditioner already presented in [S. Bertoluzza and M. Pennacchio, *Appl. Numer. Anal. Comput. Math.* **1** (2004) 434–454]. The second is new and is built by using a Discontinuous Galerkin interior penalty method as coarse problem. A bound of the condition number is proven for both variants and their efficiency and scalability is illustrated by numerical experiments.

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1. Introduction

Introduced in the early nineties by Bernardi *et al.* [10] as a tool to couple spectral and finite element method for the solution of second order elliptic PDE's, the *mortar method* has been quickly extended to treat many different application fields [1, 7, 9, 27–29], turning out to be well suited for parallel implementation and to the coupling of many different approximation spaces. The method has gained a wide popularity, since it offers the possibility to use different, non matching, possibly heterogeneous discretizations in different regions of the domain of definition of the problem at hand. However, in order to make such technique more competitive for real life applications, one has to deal with the problem of the efficient solution of the associated linear system of equations. The design of efficient preconditioners for such linear system is then a fundamental task. Different approaches were considered in the literature: iterative substructuring [2], additive Schwarz with overlap [24],

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FETI-DP [18,22,23] and BDDC [25]. To the best of our knowledge, all these methods deal the h-version of the Mortar FEM, and the explicit dependence on the polynomial degree relative to the FEM space considered has never been analysed before.

Here we deal with the construction of preconditioners for the h-p Mortar Finite Element method. We start by considering the approach proposed in the framework of conforming domain decomposition by Bramble *et al.* [15], which has already been extended to the h version of the Mortar method by Achdou *et al.* [2]. In doing this we will extend to the h-p version some tools that are common to the analysis of a wide range of substructuring preconditioner. This approach consists in considering a suitable splitting of the nonconforming discretization space in terms of "interior", "edge" and "vertex" degrees of freedom and then using the related block-Jacobi type preconditioners. While the "interior" and the "edge" blocks can be treated essentially as in the conforming case, the treatment of the vertex block deserves some additional considerations.

Indeed, a problem that, in our opinion, has not until now been tackled in a satifactory way for the Mortar method is the design of the coarse vertex block of the preconditioner (which is responsible for the good scaling properties of the preconditioners considered). In fact, when building preconditioners for the Mortar method, we have to deal with the fact that the coarse space depends on the fine discretization, *via* the the action of the "Mortar projection operator". Moreover, the design of such block is further complicated by the the presence of multiple degrees of freedom at each cross point (we recall, in fact, that in the definition of the Mortar method, continuity at cross points is not required). The solution considered in [2] is to use as a coarse preconditioner the vertex block of the Schur complement. This is clearly not efficient, since it implies actually assembling at least a block of the Schur complement (which is a task that we would like to avoid) and, for a high number of subdomains, it is definitely not practically feasible. Here, we propose two different coarse preconditioners. The first one is the vertex block of the Schur complement for a fixed auxiliary order one mesh with a small number of degrees of freedom per subdomain. This idea was presented in Reference [13] for the case of linear finite elements. We combine it, here, with a suitable balancing between vertex and edge component, yielding a better estimate for the condition number of the preconditioned matrix. This alternative allows to avoid the need of recomputing the coarse block of the preconditioner when refining the mesh. It still demands assembling a Schur complement matrix (though starting from a coarse mesh) and it is therefore quite expensive, at least when considering a large number of subdomains. In order to be able to tackle this kind of configuration, and obtain a feasible, scalable method even in a massively parallel environment we propose here, as a further alternative, to build the coarse preconditioner by giving up weak continuity and use, as a coarse preconditioner, a (non consistent) Discontinuous Galerkin type interior penalty method defined on the coarse mesh whose elements are the (quadrangular) subdomains. This approach turns out to be quite efficient even for a very a large number of subdomains (as we show in the numerical tests section).

We applied the theoretical approach first presented in Reference [11], that allows to provide a much more general analysis than [2, 15], to a model elliptic problem with continuity and coercivity constants of order one (the generalisation to coefficients strongly varying across the interface being considered in Ref. [32]). For this model problem, we were able to prove, for both choices of the coarse preconditioner, that the condition number of the preconditioned matrix is bounded by a constant times

$$p^{3/2}(1 + \log\left(Hp^2/h\right)^2$$

where H, h and p are the subdomain mesh-size, the fine mesh-size and the polynomial order respectively, (see Cor. 4.2 and Thm. 4.5). Numerical experiments seem, however, to indicate that this bound is not optimal: the condition number appears to behave in a polylogarithmic way, and there is no numerical evidence of the presence of the factor $p^{3/2}$. The same kind of behavior (loss of a power of p in the theoretical estimate that does not appear in the numerical tests) was observed also for the first error estimates for the h-p Mortar method [35]. Such estimate was then improved by applying an interpolation argument [8] that, unfortunately, cannot be applied for the type of bound that we are considering. The factor $p^{3/2}$ in the theoretical estimate derives from the boundedness estimates for the Mortar projector (2.33) and (2.34), which were shown to be sharp in [34]. We observe that the norm of such projection operator also comes into play in the analysis of other preconditioners (like, for instance, the FETI method) so that a generalization of the related theoretical estimates to the h-p version would also suffer of the loss of a factor $p^{3/2}$.

The paper is organized as follows. The basic notation, functional setting and the description of the Mortar method are given in Section 2. Some technical tools required in the construction and analysis of the proposed preconditioners are revised in the same Section. The substructuring preconditioner is introduced and analyzed in Section 3 whereas two different choices for the vertex block of the preconditioner are presented in Section 4. Numerical experiments are presented in Section 5.

We are interested here in explicitly studying the dependence of the estimates that we are going to prove on the number and size of the subdomains and on the degree of the polynomial used. To this end, in the following we will employ the notation $A \lesssim B$ (resp. $A \gtrsim B$) to say that the quantity A is bounded from above (resp. from below) by cB, with a constant c independent of ℓ, of the H_ℓ's, as well as of any mesh size parameter and of the polynomial degree p_ℓ. The expression $A \simeq B$ will stand for $A \lesssim B \lesssim A$.

2. The Mortar method.

Let us at first recall the definition of the Mortar method, see *e.g.* [36] and the literature therein. For simplicity we will consider the following simple model problem (though the results that we present here will very easily extend to a more general situation): letting $\Omega \subset \mathbb{R}^2$ be a polygonal domain and given $f \in L^2(\Omega)$, find u satisfying

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{2.1}$$

We assume that for almost all $x \in \Omega$ the matrix $a(x) = (a_{ij}(x))_{i,j=1,2}$ is symmetric positive definite, with smallest eigenvalue $\check{\alpha}(x) \geq \alpha > 0$ and largest eigenvalue $\hat{\alpha}(x) \leq \alpha'$, where $\check{\alpha}$ and $\hat{\alpha}$ are positive constants independent of x, which we assume to be of order one.

In order to discretize the above problem, we start by considering a decomposition of Ω as the union of L non overlapping subdomains Ω_ℓ,

$$\Omega = \bigcup_{\ell=1,\dots,L} \overline{\Omega}_\ell. \tag{2.2}$$

We assume that each subdomain Ω_ℓ satisfies the following assumption: there exist orientation preserving affine mappings $F_\ell : x \in [0,1]^2 \to F_\ell(x) = B_\ell x + b_\ell$ with B_ℓ and invertible 2×2 matrix and b_ℓ a vector of $\mathbb{R}^2$ such that

$$H_\ell^{-1}\|B_\ell\| \lesssim 1, \qquad H_\ell\|B^{-1}\| \lesssim 1, \tag{2.3}$$

where $\|\cdot\|$ stands for the matrix norm associated to the Euclidean norm in $\mathbb{R}^2$, and where H_ℓ is the diameter of the subdomain Ω_ℓ.

We set

$$\overline{\Gamma}_{\ell n} = \partial\overline{\Omega}_n \cap \partial\overline{\Omega}_\ell, \qquad \overline{S} = \cup \overline{\Gamma}_{\ell n} \tag{2.4}$$

and we denote by γ_ℓ^i $(i = 1, \dots, 4)$ the ith side of the ℓth domain:

$$\partial\Omega_\ell = \bigcup_{i=1}^{4} \bar{\gamma}_\ell^i.$$

Here we deal with the case of a geometrically conforming decomposition: each edge γ_ℓ^i coincides with $\Gamma_{\ell n}$ for some n. The extension to the case of a geometrically non-conforming decomposition will be considered in future works.

Remark 2.1. The assumption that the subdomains Ω_ℓ are quadrangles obtained from the unit square *via* an affine mapping satisfying (2.3) can be relaxed into asking that they are the union of a small number of shape regular triangles or quadrangles of diameter H_ℓ. All the arguments in this paper can be adapted quite easily to cover such a case, with the notable exception of the ones related to the construction and implementation of the DG coarse preconditioner, and in particular to the evaluation of the bilinear form (4.10), which will have to be modified. In such a case, a VEM [6] based approach can be considered.

Remark 2.2. In this paper we assume the coercivity and continuity constants $\check{\alpha}$ and $\hat{\alpha}$ to be of order one. Clearly, if this is not the case, the constants in the estimates that we present here will explode when α'/α goes to infinity. By suitably weighting the different components that add up to form the preconditioner that we proposed, it is possible (see [32]) to extend it to the case where we only ask that, for each ℓ, $\min_{x\in\Omega_\ell}\check{\alpha}(x) \simeq \max_{x\in\Omega_\ell}\hat{\alpha}(x)$. This will for instance allow for large jumps in the coefficients of (2.1), provided such jumps are aligned with the interface.

Functional spaces.

Let us at first introduce the necessary functional setting. For $\hat{\Omega}$ any domain in $\mathbb{R}^d$, $d = 1, 2$ we introduce the following unscaled norms and seminorms (with $0 < s < 1$):

$$\|\hat{u}\|^2_{0,\hat{\Omega}} = \int_{\hat{\Omega}} |\hat{u}|^2, \qquad |\hat{u}|^2_{1,\hat{\Omega}} = \int_{\hat{\Omega}} |\nabla u|^2, \qquad |\hat{u}|_{s,\hat{\Omega}} = \int_{\hat{\Omega}} \mathrm{d}x \int_{\hat{\Omega}} \mathrm{d}y \frac{|\hat{u}(x) - \hat{u}(y)|^2}{|x-y|^{d+2s}}\cdot$$

We then introduce the following suitably scaled norms and seminorms: for two dimensional entities

$$\|u\|^2_{H^1(\Omega_\ell)} = H_\ell^{-2} \int_{\Omega_\ell} |u|^2 \,\mathrm{d}x + \int_{\Omega_\ell} |\nabla u|^2 \,\mathrm{d}x, \qquad |u|^2_{H^1(\Omega_\ell)} = \int_{\Omega_\ell} |\nabla u|^2 \,\mathrm{d}x, \tag{2.5}$$

and for one dimensional entities (γ being either γ_ℓ^i or $\partial\Omega_\ell$)

$$|\eta|^2_{H^s(\gamma)} = H_\ell^{2s-1} \int_\gamma \int_\gamma \frac{|\eta(x) - \eta(y)|^2}{|x-y|^{2s+1}} \,\mathrm{d}x \,\mathrm{d}y, \quad s \in (0,1) \tag{2.6}$$

$$\|\eta\|^2_{L^2(\gamma)} = H_\ell^{-1} \int_\gamma |\eta|^2 \,\mathrm{d}s, \qquad \|\eta\|^2_{H^s(\gamma)} = |\eta|^2_{H^s(\gamma)} + \|\eta\|^2_{L^2(\gamma)}, \quad s \in (0,1). \tag{2.7}$$

Remark that the above norms are defined in such a way that they are scaling invariant, that is they are preserved when Ω_ℓ is rescaled to the reference domain $]0,1[^2$.

In the following for γ_ℓ^i edge of Ω_ℓ we will also make explicit use of the spaces $H^s_0(\gamma_\ell^i)$ and $H^{1/2}_{00}(\gamma_\ell^i)$, which are defined as the subspaces of those functions η of $H^s(\gamma_\ell^i)$ (resp. $H^{1/2}(\gamma_\ell^i)$) such that the function $\hat{\eta}$ defined as $\hat{\eta} = \eta$ on γ_ℓ^i and $\hat{\eta} = 0$ on $\partial\Omega_\ell \setminus \gamma_\ell^i$ belongs to $H^s(\partial\Omega_\ell)$ (resp. to $H^{1/2}(\partial\Omega_\ell)$). The spaces $H^s_0(\gamma_\ell^i)$ and $H^{1/2}_{00}(\gamma_\ell^i)$ are endowed with the norms

$$\|\eta\|_{H^s_0(\gamma_\ell^i)} = \|\hat{\eta}\|_{H^s(\partial\Omega^\ell)} \qquad\qquad \|\eta\|_{H^{1/2}_{00}(\gamma_\ell^i)} = \|\hat{\eta}\|_{H^{1/2}(\partial\Omega^\ell)}.$$

Let the spaces X and T be defined as

$$X = \prod_{\ell=1}^{L} \{u_\ell \in H^1(\Omega_\ell) | \;\; u_\ell = 0 \text{ on } \partial\Omega \cap \partial\Omega_\ell\}, \qquad T = \prod_{\ell=1}^{L} H^{1/2}_*(\partial\Omega_\ell), \tag{2.8}$$

where $H^{1/2}_*(\Omega_\ell)$ is defined by

$$H^{1/2}_*(\partial\Omega_\ell) = H^{1/2}(\partial\Omega_\ell) \qquad \text{if } |\partial\Omega_\ell \cap \partial\Omega| = 0$$

and

$$H_*^{1/2}(\partial\Omega_\ell) = H_{00}^{1/2}(\partial\Omega_\ell \setminus \partial\Omega)$$

otherwise.

Discretizations.

We consider for each ℓ a family $\mathcal{K}_h^\ell$ of compatible shape regular decompositions of Ω_ℓ, each made of open elements K, which, to fix the ideas, we assume to be triangular (the extension to quadrilateral elements being trivial), and we let h_K denote the diameter of K. We choose, for each element K, a polynomial degree p_K, and we let $X_h^\ell \subset H^1(\Omega_\ell)$ be the finite element space of order p_K in each K, defined on the decomposition $\mathcal{K}_h^\ell$ and satisfying an homogeneous boundary condition on $\partial\Omega \cap \partial\Omega_\ell$:

$$X_h^\ell = \{v \in C^0(\overline{\Omega}_\ell) \text{ s.t. } v|_K \in P_{p_K}(K),\ K \in \mathcal{K}_h^\ell\} \cap H_0^1(\Omega),$$

where $P_q(K)$ stands for the space of polynomials of degree at most q in K. We do not assume that the meshes are quasi uniform but that they are regular in shape and graded in the following sense: there exists two constants ρ and θ independent of the h_K's and of the p_K's, such that for any neighbouring elements K, K' it holds:

$$\rho^{-1} \leq \frac{h_K}{h_{K'}} \leq \rho, \qquad |p_K - p_{K'}| \leq \theta. \tag{2.9}$$

The analysis of the preconditioners that we will consider will mainly rely on inverse inequalities, therefore it will be convenient to introduce the following notation:

$$h_\ell = \min_{K \in \mathcal{K}_h^\ell} h_K \qquad p_\ell = \max_{K \in \mathcal{K}_h^\ell} p_K \tag{2.10}$$

and

$$h = \min_\ell h_\ell \qquad p = \max_\ell p_\ell \qquad H = \max_\ell H_\ell.$$

We set

$$T_h^\ell = X_h^\ell|_{\partial\Omega_\ell}, \tag{2.11}$$

and, for each edge γ_ℓ^i of the subdomain Ω_ℓ, we define

$$T^{\ell,i} = \{\eta : \quad \eta \text{ is the trace on } \gamma_\ell^i \text{ of some } u_\ell \in X_h^\ell\ \} \tag{2.12}$$

$$T_0^{\ell,i} = \{\eta \in T^{\ell,i} : \quad \eta = 0 \text{ at the extrema of } \gamma_\ell^i\}. \tag{2.13}$$

Finally, we set

$$X_h = \prod_{\ell=1}^L X_h^\ell\ \subset X, \qquad T_h = \prod_{\ell=1}^L T_h^\ell\ \subset T. \tag{2.14}$$

On X and T we introduce the following broken norm and semi-norm:

$$\|u\|_X = \left(\sum_{\ell=1}^L \|u\|_{H^1(\Omega_\ell)}^2\right)^{\frac{1}{2}}, \qquad |u|_X = \left(\sum_{\ell=1}^L |u|_{H^1(\Omega_\ell)}^2\right)^{\frac{1}{2}}, \tag{2.15}$$

$$\|\eta\|_T = \left(\sum_{\ell=1}^L \|\eta_\ell\|_{H^{1/2}(\partial\Omega_\ell)}^2\right)^{1/2} \qquad |\eta|_T = \left(\sum_{\ell=1}^L |\eta_\ell|_{H^{1/2}(\partial\Omega_\ell)}^2\right)^{1/2}. \tag{2.16}$$

The spaces considered satisfy classical inverse inequalities (see *e.g.* [5,17,33]), which, in view of (4.10) and of the scaling (2.5), take the form: for all $\eta \in T^0_{\ell,i}$ and for all $s, r \neq 1/2$ such that $0 \leq s < r \leq 1$

$$\|\eta\|_{H^r_0(\gamma^i_\ell)} \lesssim p_\ell^{2(r-s)} \left(\frac{h_\ell}{H_\ell}\right)^{s-r} \|\eta\|_{H^s_0(\gamma^i_\ell)}, \qquad |\eta|_{H^r_0(\gamma^i_\ell)} \lesssim p_\ell^{2(r-s)} \left(\frac{h_\ell}{H_\ell}\right)^{s-r} |\eta|_{H^s_0(\gamma^i_\ell)}, \tag{2.17}$$

with constants independent of r, s. For $s = 1/2$ or $r = 1/2$ (2.17) holds with H^s_0 (resp. H^r_0) replaced by $H^{1/2}_{00}$.

Mortar Problem

Let now a composite bilinear form $a_X : X \times X \longrightarrow \mathbb{R}$ be defined as follows:

$$a_X(u, v) = \sum_\ell a_\ell(u_\ell, v_\ell) \quad \text{with} \quad a_\ell(u_\ell, v_\ell) = \int_{\Omega_\ell} \sum_{i,j} a_{ij}(x) \frac{\partial u_\ell}{\partial x_i} \frac{\partial v_\ell}{\partial\, x_j} \,\mathrm{d}x. \tag{2.18}$$

The bilinear form a_X is clearly not coercive on X. In order to obtain a well posed problem we will then consider proper subspaces of X, consisting of functions satisfying a suitable weak continuity constraint. For defining such constraint, according to the Mortar method, we start by choosing for each segment $\Gamma_{\ell n} = \gamma^i_\ell = \gamma^j_n$, one side (let us say ℓ) to be the slave side, while the other side will be the master side. The skeleton $\mathcal{S}$ can be decomposed as

$$\mathcal{S} = \bigcup_{\gamma^i_\ell \text{ slave side}} \overline{\gamma}^i_\ell = \bigcup_{\gamma^j_n \text{ master side}} \overline{\gamma}^j_n.$$

For each slave side γ^i_ℓ, a multiplier space $M^{\ell,i}$ is defined by suitably modifying the corresponding trace space $T^{\ell,i}$. Letting e_k, $k = 0, \dots, N$ denote the elements of the one dimensional mesh induced on γ^i_ℓ by $\mathcal{K}^\ell_h$ (with e_0 and e_N extremal elements), we observe that $T^{\ell,i}$ will take the form

$$T^{\ell,i} = \{\eta \in C^0(\gamma^i_\ell) : \ \eta|_{e_k} \in P_{p_k}(e_k), \ k = 0, \dots, N\}.$$

The multiplier space will be obtained by reducing by one the polynomial degree at the extremal elements:

$$M^{\ell,i} = \{\eta \in C^0(\gamma^i_\ell) : \eta|_{e_k} \in P_{p_k}(e_k), \ 0 < k < N, \ \eta|_{e_0} \in P_{p_0-1}(e_0), \ \eta|_{e_N} \in P_{p_N-1}(e_N)\}.$$

Remark that $\dim(M^{\ell,i}) = \dim(T^{\ell,i}_0)$. We set:

$$M_h = \{\eta \in L^2(\mathcal{S}), \ \forall \gamma^i_\ell \text{ slave side}, \ \eta_{|_{\gamma^i_\ell}} \in M^{\ell,i}\}. \tag{2.19}$$

The constrained approximation and trace spaces $\mathcal{X}_h$ and $\mathcal{T}_h$ are then defined as follows:

$$\mathcal{X}_h = \left\{ v_h \in X_h, \int_\mathcal{S} [v_h] \lambda \,\mathrm{d}s = 0, \ \forall \lambda \in M_h \right\}, \tag{2.20}$$

$$\mathcal{T}_h = \left\{ \eta \in T_h, \int_\mathcal{S} [\eta] \lambda \,\mathrm{d}s = 0, \ \forall \lambda \in M_h \right\}, \tag{2.21}$$

where, on $\gamma^i_\ell = \gamma^j_n$, γ^i_ℓ slave side, we set $[\eta] = \eta_\ell - \eta_n$.

We can now introduce the following discrete problem:

Problem 2.3. *Find $u_h \in \mathcal{X}_h$ such that for all $v_h \in \mathcal{X}_h$*

$$a_X(u_h, v_h) = \int_\Omega f v_h \,\mathrm{d}x. \tag{2.22}$$

It is known that Problem 2.3 admits a unique solution u_h. For an error estimate, see [8].

Classical bounds and technical tools

With the chosen scaling, several classical bounds hold with constants independent of H_ℓ. In particular we have:

Poincaré type inequalities.
For all η with $\int_\gamma \eta \, \mathrm{d}s = 0$, γ being either γ_ℓ^i or $\partial\Omega_\ell$, it holds that

$$\|\eta\|_{H^s(\gamma)} \lesssim |\eta|_{H^s(\gamma)}. \tag{2.23}$$

Injection of H^s in H_0^s for $s < 1/2$.
We recall that for $s < 1/2$ the spaces $H^s(\gamma_\ell^i)$ and $H_0^s(\gamma_\ell^i)$ coincide as sets and have equivalent norms. However, the constants in the norm equivalence goes to infinity as s tends to $1/2$. For all $\varphi \in H^{1/2}(\gamma_\ell^i)$ the following bound can be shown (see [12]): for $\beta \in \mathbb{R}$ arbitrary it holds that

$$|\varphi|_{H_0^s(\gamma_\ell^i)} \lesssim \frac{1}{1/2-s}\|\varphi-\beta\|_{H^{1/2}(\gamma_\ell^i)} + \frac{1}{\sqrt{1/2-s}}|\beta|. \tag{2.24}$$

If φ is linear, bound (2.24) can be improved to

$$|\varphi|_{H_0^s(\gamma_\ell^i)} \lesssim \frac{1}{\sqrt{1/2-s}}(\|\varphi-\beta\|_{H^{1/2}(\gamma_\ell^i)} + |\beta|). \tag{2.25}$$

We now revise some technical tools that will be required in the construction and analysis of our preconditioner. For the reader's convenience we report the following two results, which are a generalisation to the hp-version of (Lems. 3.1 and 3.4 in [11,15] respectively) and of (Lem. 3.5 in [15]), and they can be easily proven by adapting the proof in [11] which essentially relies on inverse inequalities analogous to (2.17) and on properties (2.24) and (2.25) (see also [21]).

Lemma 2.4. *The following bound holds: for all $\xi \in T_h^\ell$ such that $\xi(P) = 0$ for some $P \in \bar\gamma$, γ being either γ_ℓ^i or $\partial\Omega_\ell$, it holds*

$$\|\xi\|^2_{L^\infty(\gamma)} \lesssim \left(1 + \log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right)|\xi|^2_{1/2,\gamma}. \tag{2.26}$$

Lemma 2.5. *Let $\xi \in T_h^\ell$ vanishing at all vertices of Ω_ℓ, and let $\zeta_L \in H^{1/2}(\partial\Omega_\ell)$, ζ_L linear on each edge of Ω_ℓ. Then it holds*

$$\sum_{i=1}^4 \|\xi\|^2_{H_{00}^{1/2}(\gamma_\ell^i)} \lesssim \left(1 + \log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right)^2 |\xi + \zeta_L|^2_{H^{1/2}(\partial\Omega_\ell)}. \tag{2.27}$$

Finally, the following lemma holds.

Lemma 2.6. *Let $\sigma : \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}$ be defined as*

$$\sigma(\alpha,\beta) = \sum_{\ell,n:|\Gamma_{\ell n}|>0} (\alpha_\ell - \alpha_n)(\beta_\ell - \beta_n). \tag{2.28}$$

For $\eta \in T$ let $\bar\eta \in \mathbb{R}^L$ be defined by

$$\bar\eta = (\bar\eta_\ell)_{\ell=1,\dots,L}, \qquad \bar\eta_\ell = |\partial\Omega_\ell|^{-1}\int_{\partial\Omega_\ell} \eta_\ell \, \mathrm{d}s. \tag{2.29}$$

Then, if $\eta \in T$ verifies

$$\int_{\gamma_\ell^i} [\eta] \, \mathrm{d}s = 0, \quad \forall \gamma_\ell^i \text{ slave side}, \tag{2.30}$$

we have

$$\sigma(\bar\eta,\bar\eta) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta|_T^2. \tag{2.31}$$

Proof. For $\ell = 1,\dots,L$, $i=1,\dots,4$ and for $\eta \in T$ let

$$\bar\eta_\ell^i = \frac{1}{|\gamma_\ell^i|}\int_{\gamma_\ell^i} \eta_\ell \,\mathrm{d}s.$$

For η satisfying (2.30) and $\Gamma_{\ell n} = \gamma_\ell^i = \gamma_n^j$, we have $\bar\eta_\ell^i = \bar\eta_n^j$. We can then introduce a single constant $\bar\eta_{\ell n}$, depending only on ℓ and n, defined as

$$\bar\eta_{\ell n} = \bar\eta_\ell^i = \bar\eta_n^j.$$

Then we have

$$\begin{aligned}\sigma(\bar\eta,\bar\eta) &= \sum_{\ell,n:|\Gamma_{\ell n}|>0} |\bar\eta_\ell - \bar\eta_{\ell n} - (\bar\eta_n - \bar\eta_{\ell n})|^2 \lesssim \sum_\ell \sum_{n:|\Gamma_{\ell n}|>0} |\bar\eta_\ell - \bar\eta_{\ell n}|^2 \\ &= \sum_\ell\sum_{i\in\mathcal{E}_\ell} |\bar\eta_\ell - \bar\eta_\ell^i + \eta_\ell(x_i^\ell) - \eta_\ell(x_i^\ell)|^2 \\ &\lesssim \sum_\ell\sum_{i\in\mathcal{E}_\ell} |\eta_\ell(x_i^\ell) - \bar\eta_\ell|^2 + \sum_\ell\sum_{i\in\mathcal{E}_\ell} |\eta_\ell(x_i^\ell) - \bar\eta_\ell^i|^2,\end{aligned}$$

where, for each ℓ, we let $\mathcal{E}_\ell = \{i : \ \gamma_\ell^i \text{ is an interior edge}\}$ and where x_i^ℓ, $i=1,\dots,4$ are the vertices of the subdomain, which are ordered in such a way that x_i^ℓ is a vertex of γ_ℓ^i.

We have

$$|\bar\eta_\ell - \eta_\ell(x_i^\ell)|^2 \lesssim \|\eta_\ell - \bar\eta_\ell\|^2_{L^\infty(\partial\Omega_\ell)}.$$

We observe that $\int_{\partial\Omega_\ell} \eta_\ell - \bar\eta_\ell = 0$, which, since $\eta_\ell - \bar\eta_\ell \in C^0(\partial\Omega_\ell)$, implies that $\eta_\ell - \bar\eta_\ell$ vanishes at some point of $\partial\Omega_\ell$. We can then apply bound (2.26), which yields

$$|\bar\eta_\ell - \eta_\ell(x_i^\ell)|^2 \lesssim \left(1+\log\left(\frac{H_\ell p_\ell^2}{h_\ell}\right)\right)|\eta_\ell|^2_{1/2,\partial\Omega_\ell}.$$

The term $|\eta_\ell(x_i^\ell) - \bar\eta_\ell^i|^2$ is bounded analogously. The thesis is obtained since the cardinality of the set $\mathcal{E}_\ell$ is bounded. ☐

The Mortar correction operator

For all slave side γ_ℓ^i, we let $\pi_\ell^i : L^2(\gamma_\ell^i) \longrightarrow T_0^{\ell,i}$ be the bounded projector defined as

$$\int_{\gamma_\ell^i} (\eta - \pi_\ell^i\eta)\lambda = 0, \quad \forall\lambda \in M^{\ell,i}. \tag{2.32}$$

The projection π_ℓ^i is well defined and satisfies (see [34, 35]):

Theorem 2.7. *There exists $\rho_0 \geq 4$ such that, if the grading parameter ρ verifies $\rho < \rho_0$, then for γ_ℓ^i slave side, it holds:*

$$\|\pi_\ell^i\eta\|_{L^2(\gamma_\ell^i)} \lesssim p_\ell^{\frac12}\|\eta\|_{L^2(\gamma_\ell^i)} \ \forall\eta\in L^2(\gamma_\ell^i), \tag{2.33}$$

$$|\pi_\ell^i\eta|_{H^1(\gamma_\ell^i)} \lesssim p_\ell\,|\eta|_{H^1(\gamma_\ell^i)} \ \ \forall\eta\in H_0^1(\gamma_\ell^i). \tag{2.34}$$

Remark 2.8. The results are proven in [34] for a graded mesh with fixed polynomial degree. It is not particularly difficult to verify that the proof holds essentially unchanged also for variable polynomial degree provided condition (2.9) holds for $\rho < \rho_0 = \min_{K\in\mathcal{K}_h^\ell}(p_K+1)^2$ and for any $\theta > 0$.

Remark 2.9. The problem of whether (2.33) and (2.34) are optimal was studied in [34], where, through an eigenvalue analysis the dependence on p to the power 1/2 and 1 of the norm of the projector appearing respectively in (2.33) and (2.34) was confirmed. This dependence does not seem to affect the asymptotic rate of the error, which, as observed in [34], seems to be only slightly suboptimal (loss of a factor $C(\varepsilon)p^\varepsilon$ for ε arbitrarily small). In [8] this good behavior of the error was proven, for sufficiently smooth solutions, thanks to an interpolation argument.

By space interpolation and using the Poincaré inequality we immediately get the following corollary

Corollary 2.10. *For all s, $0 < s < 1$, $s \neq 1/2$, for all $\eta \in H_0^s(\gamma_\ell^i)$ we have*

$$|\pi_\ell^i \eta|_{H_0^s(\gamma_\ell^i)} \lesssim p_\ell^{(1+s)/2}\,|\eta|_{H_0^s(\gamma_\ell^i)}, \tag{2.35}$$

uniformly in s. For all $\eta \in H_{00}^{1/2}(\gamma_\ell^i)$ we have

$$|\pi_\ell^i \eta|_{H_{00}^{1/2}(\gamma_\ell^i)} \lesssim p_\ell^{3/4}\,|\eta|_{H_{00}^{1/2}(\gamma_\ell^i)}. \tag{2.36}$$

We now define a global linear operator

$$\pi_h : \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell) \longrightarrow \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell)$$

as follows: for $\eta = (\eta_\ell)_{\ell=1,\dots,L} \in \Pi_\ell L^2(\partial\Omega_\ell)$, we set $\pi_h(\eta) = (\eta_\ell^*)_{\ell=1,\dots,L}$, where $\eta_\ell^* \in T_h^\ell$ is defined on slave sides as π_ℓ^i applied to the jump of η, while it is set identically zero on master sides and on the external boundary $\partial\Omega$: on $\gamma_\ell^i = \gamma_n^j$, ($\ell$ slave side, n master side)

$$\eta_\ell^*|_{\gamma_\ell^i} = \pi_\ell^i([\eta]|_{\gamma_\ell^i}), \qquad \eta_n^*|_{\gamma_\ell^i} = 0,$$

and for all ℓ

$$\eta_\ell^* = 0 \text{ on } \partial\Omega_\ell \cap \partial\Omega.$$

The following bound holds.

Lemma 2.11. *For all $\eta = (\eta_\ell)_{\ell=1,\dots,L} \in T$ and for all $\alpha = (\alpha_\ell)_{\ell=1,\dots,L} \in \mathbb{R}^L$, it holds*

$$|\pi_h(\eta)|_T^2 \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 \|\eta-\alpha\|_T^2 + p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\sigma(\alpha,\alpha) \tag{2.37}$$

with σ defined by (2.28). If, in addition, each η_ℓ is linear on each γ_ℓ^i, then the bound can be improved to

$$|\pi_h(\eta)|_T^2 \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\left(\|\eta-\alpha\|_T^2 + \sigma(\alpha,\alpha)\right). \tag{2.38}$$

Proof. We have

$$|\pi_h(\eta)|^2_T \lesssim \sum_{\gamma^i_\ell \text{ slave side}} |\pi^i_\ell([\eta])|^2_{H^{1/2}_{00}(\gamma^i_\ell)} \lesssim \sum_{\gamma^i_\ell \text{ slave side}} H^{2\varepsilon}_\ell p^{4\varepsilon}_\ell h^{-2\varepsilon}_\ell |\pi^i_\ell([\eta])|^2_{H^{1/2-\varepsilon}_0(\gamma^i_\ell)} \lesssim$$
$$\lesssim p^{3/2} \sum_{\gamma^i_\ell \text{ slave side}} h^{-2\varepsilon}_\ell H^{2\varepsilon}_\ell p^{4\varepsilon}_\ell |[\eta]|^2_{H^{1/2-\varepsilon}_0(\gamma^i_\ell)}.$$

We now observe that, for γ^i_ℓ slave side and γ^j_n corresponding master side, (2.24) with $\beta = \alpha_\ell - \alpha_n$ yields

$$|[\eta]|^2_{H^{1/2-\varepsilon}_0(\gamma^i_\ell)} \lesssim \frac{1}{\varepsilon^2}\|[\eta - \alpha]\|^2_{H^{1/2}(\gamma^i_\ell)} + \frac{1}{\varepsilon}|\alpha_\ell - \alpha_n|^2. \tag{2.39}$$

Then we have

$$|\pi_h(\eta)|^2_T \lesssim p^{3/2}\frac{H^{2\varepsilon}p^{4\varepsilon}}{h^{2\varepsilon}}\left(\frac{1}{\varepsilon^2}\sum_{\gamma^i_\ell \text{ slave side}} \|[\eta-\alpha]\|^2_{H^{1/2}(\gamma^i_\ell)} + \frac{1}{\varepsilon}\sigma(\alpha,\alpha)\right).$$

Observing that,

$$\|[\eta-\alpha]\|^2_{H^{1/2}(\gamma^i_\ell)} \le \|\eta_\ell - \alpha_\ell\|^2_{H^{1/2}(\gamma^i_\ell)} + \|\eta_n - \alpha_n\|^2_{H^{1/2}(\gamma^i_\ell)},$$

and by choosing $\varepsilon = 1/\log(Hp^2/h)$, we get (2.37). The bound (2.38) is obtained by noting that, if each η_ℓ is linear on each γ^i_ℓ, thanks to (2.25), the bound (2.39) can be improved to

$$|[\eta]|^2_{H^{1/2-\varepsilon}_0(\gamma^i_\ell)} \lesssim \frac{1}{\varepsilon}(\|[\eta-\alpha]\|^2_{H^{1/2}(\gamma^i_\ell)} + |\alpha_\ell - \alpha_n|^2). \qquad \square$$

Finally, by observing that

$$|(\mathbf{1} - \pi_h)(\eta)|^2_T \lesssim |\eta|^2_T + |\pi_h(\eta)|^2_T = |\eta - \alpha|^2_T + |\pi_h(\eta)|^2_T \lesssim \|\eta - \alpha\|^2_T + |\pi_h(\eta)|^2_T$$

with $\mathbf{1}$ denoting the identity operator, we easily obtain the following corollary.

Corollary 2.12. *For all $\eta = (\eta_\ell)_{\ell=1,\ldots,L} \in T$ and for all $\alpha = (\alpha_\ell)_{\ell=1,\ldots,L} \in \mathbb{R}^L$, it holds*

$$|(\mathbf{1} - \pi_h)(\eta)|^2_T \lesssim p^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 \|\eta - \alpha\|^2_T + p^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)\sigma(\alpha,\alpha) \tag{2.40}$$

with σ defined by (2.28). If, in addition, each η_ℓ is linear on each γ^i_ℓ, then the bound can be improved to

$$|(\mathbf{1} - \pi_h)(\eta)|^2_T \lesssim p^{3/2}\left(1 + \log\left(\frac{Hp^2}{h}\right)\right)\left(\|\eta - \alpha\|^2_T + \sigma(\alpha,\alpha)\right). \tag{2.41}$$

3. Substructuring preconditioners for the Mortar method

The main idea of substructuring preconditioners consists in splitting the functions $u \in \mathcal{X}_h$ as the sum of three suitably defined components: $u = u^0 + u^E + u^V$ identified respectively by *interior* degrees of freedom (corresponding to basis functions vanishing on the skeleton and supported on one subdomain), *edge* degrees of freedom, and *vertex* degrees of freedom, and consider preconditioners that, when expressed in a basis related to such a splitting, are block diagonal.

More precisely, we start as usual by introducing the discrete lifting operator $R_h : T_h \to X_h$ defined as follows. For $\eta = (\eta_\ell)_{\ell=1,\ldots,L} \in T_h$ we let $R_h\eta = (R^\ell_h\eta_\ell)_{\ell=1,\ldots,K} \in X_h$ with $R^\ell_h\eta_\ell \in X^\ell_h$ solution of

$$R^\ell_h\eta_\ell = \eta_\ell \text{ on } \partial\Omega_\ell, \qquad a_\ell(R^\ell_h\eta_\ell, v^\ell_h) = 0, \quad \forall v_h \in X^\ell_h \cap H^1_0(\Omega_\ell).$$

It is immediate to check that the spaces X_h of unconstrained functions and $\mathcal{X}_h$ of constrained functions can be split as direct sums of an interior and of a (respectively unconstrained or constrained) trace component:

$$X_h = \mathcal{X}_h^0 \oplus R_h(T_h), \qquad \mathcal{X}_h = \mathcal{X}_h^0 \oplus R_h(\mathcal{T}_h), \tag{3.1}$$

with

$$\mathcal{X}_h^0 = \prod_{\ell=1}^{L} X_h^\ell \cap H_0^1(\Omega_\ell).$$

We can easily verify that for $w = w^0 + R_h\eta$, $v = v^0 + R_h\zeta$ (with $w^0, v^0 \in \mathcal{X}_h^0$) $a_X : X_h \times X_h \to \mathbb{R}$ satisfies

$$a_X(w,v) = a_X(w^0,v^0) + a_X(R_h\eta, R_h\zeta) := a_X(w^0,v^0) + s(\eta,\zeta), \tag{3.2}$$

where the *discrete Steklov–Poincaré* operator $s : T_h \times T_h \to \mathbb{R}$ is defined by

$$s(\xi,\eta) := \sum_\ell a_\ell(R_h^\ell \xi_\ell, R_h^\ell \eta_\ell). \tag{3.3}$$

Finally, it is well known that

$$\|R_h^\ell \eta_\ell\|_{H^1(\Omega_\ell)} \simeq \|\eta_\ell\|_{1/2,\partial\Omega_\ell}, \qquad |R_h^\ell \eta_\ell|_{H^1(\Omega_\ell)} \simeq |\eta_\ell|_{1/2,\partial\Omega_\ell}. \tag{3.4}$$

see [5, 35], whence

$$\|R_h\eta\|_X \simeq \|\eta\|_T, \qquad |R_h\eta|_X \simeq |\eta|_T. \tag{3.5}$$

The following result for the *Steklov–Poincaré* operator follows easily from the definition of $s(\cdot,\cdot)$, the continuity and coercivity of $a_X(\cdot,\cdot)$ and (3.5).

Corollary 3.1. *For all $\xi \in T_h$, it holds*

$$s(\xi,\xi) \simeq |\xi|_T^2. \tag{3.6}$$

The problem of preconditioning the matrix $\mathbf{A}$ associated to the discretization of a_X, reduces to finding good preconditioners for the matrices $\mathbf{A}_0$ and $\mathbf{S}$ corresponding respectively to the bilinear forms a_X restricted to $\mathcal{X}_h^0$ and to s. Here we assume that we have good preconditioners for the stiffness matrix $\mathbf{A}_0$ and we concentrate therefore only on the discrete Steklov–Poincaré operator s.

We start by observing that the space of constrained trace functions $\mathcal{T}_h$ defined in (2.21) can be further split as the direct sum of *vertex* and *edge* spaces. More specifically, if we denote by $\mathfrak{L}$ the space

$$\mathfrak{L} = \{(\eta_\ell)_{\ell=1,\dots,L},\ \eta_\ell \in C^0(\partial\Omega_\ell) \text{ is linear on each edge of } \Omega_\ell\}, \tag{3.7}$$

then we can define the space of constrained *vertex* functions as

$$\mathcal{T}_h^V = (\mathbf{1} - \pi_h)\mathfrak{L}. \tag{3.8}$$

We observe that $\mathfrak{L} \subset T_h$, which yields $\mathcal{T}_h^V \subset \mathcal{T}_h$. We then introduce the space of constrained *edge* functions $\mathcal{T}_h^E \subset \mathcal{T}_h$ defined by

$$\mathcal{T}_h^E = \{\eta = (\eta_\ell)_{\ell=1,\dots,L} \in \mathcal{T}_h,\ \eta_\ell(x_i^\ell) = 0,\ i = 1,\dots,4\} \tag{3.9}$$

and we can easily verify that

$$\mathcal{T}_h = \mathcal{T}_h^V \oplus \mathcal{T}_h^E. \tag{3.10}$$

Moreover it is quite simple to check that a function in $\mathcal{T}_h^E$ is uniquely defined by its value on master edges, the value on slave edges being forced by the constraint.

It will be useful in the following to introduce the linear interpolation operator $\Lambda : T_h \to \mathfrak{L}$ defined as

$$\Lambda\eta = (\Lambda_\ell \eta_\ell)_{\ell=1,\dots,L}, \qquad \Lambda_\ell \eta_\ell(x_i^\ell) = \eta_\ell(x_i^\ell),\ i = 1, \dots, 4,$$

(where, we recall, x_i^ℓ, $i = 1, \dots, 4$ are the four vertices of the subdomain Ω_ℓ). Observe that for $\eta \in \mathcal{T}_h$ we have

$$\eta^V = (\eta_\ell^V)_{\ell=1,\dots,L} = (\mathbf{1} - \pi_h)\Lambda\eta \in \mathcal{T}_h^V \text{ and } \eta^E = (\eta_\ell^E)_{\ell=1,\dots,L} = \eta - (\mathbf{1} - \pi_h)\Lambda\eta \in \mathcal{T}_h^E.$$

The following Lemma can be proven exactely as the analogous result in [11] (see also [21]).

Lemma 3.2. *For all $\eta = (\eta_\ell)_{\ell=1,\dots,L} \in T_h$, it holds*

$$|\Lambda\eta|_T^2 \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) |\eta|_T^2, \qquad \|\Lambda\eta\|_T^2 \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right) \|\eta\|_T^2 \tag{3.11}$$

The preconditioner that we consider is built by introducing two bilinear forms:

$$b^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \to \mathbb{R} \qquad \text{and} \qquad b^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \to \mathbb{R}.$$

Let us start by introducing the bilinear form relative to the edges: for any master side γ_n^j, let $b_{n,j} : T_0^{n,j} \times T_0^{n,j} \longrightarrow \mathbb{R}$ be a symmetric bilinear form satisfying, for all $\eta \in T_0^{n,j}$

$$b_{n,j}(\eta, \eta) \simeq \|\eta\|^2_{H_{00}^{1/2}(\gamma_n^j)}. \tag{3.12}$$

Then, the block diagonal bilinear form $b^E : \mathcal{T}_h^E \times \mathcal{T}_h^E \longrightarrow \mathbb{R}$ is defined by

$$b^E(\eta, \xi) = \sum_{\gamma_n^j \text{ master side}} b_{n,j}(\eta_n, \xi_n). \tag{3.13}$$

Applying Lemma 2.5 we easily get

$$b^E(\eta^E, \eta^E) \lesssim \left(1 + \log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta, \eta). \tag{3.14}$$

Moreover, using the fact that η^E verifies the weak continuity constraint and thar η_ℓ^E vanishes at the cross points we immediately get that for γ_ℓ^i slave side and γ_n^j corresponding master side we have $\eta_\ell^E|_{\gamma_\ell^i} = \pi_\ell^i(\eta_n^E|_{\gamma_n^j})$ and, by (2.36),

$$|\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_\ell^i)} \lesssim p^{3/2} |\eta_n^E|^2_{H_{00}^{1/2}(\gamma_n^j)},$$

which allows us to write

$$|\eta^E|_T^2 \lesssim \sum_{\gamma_n^j \text{ master side}} |\eta_n^E|^2_{H_{00}^{1/2}(\gamma_n^j)} + \sum_{\gamma_\ell^i \text{ slave side}} |\eta_\ell^E|^2_{H_{00}^{1/2}(\gamma_\ell^i)} \tag{3.15}$$

$$\lesssim p^{3/2} \sum_{\gamma_n^j \text{ master side}} |\eta_n^E|^2_{H_{00}^{1/2}(\gamma_n^j)} \lesssim p^{3/2} b^E(\eta^E, \eta^E). \tag{3.16}$$

The construction of the vertex block of the preconditioner in the Mortar method framework is not standard, since we need to take into account the weak continuity constraint. In the P_1 framework, Achdou, Maday, Widlund in [2], propose to use

$$b_0^V(\eta^V, \zeta^V) = s(\eta^V, \zeta^V). \tag{3.17}$$

This choice immediately yields the bound

$$s(\eta,\eta) \lesssim b_0^V(\eta^V,\eta^V) + p^{3/2} b^E(\eta^E,\eta^E).$$

Let us bound $b_0^V(\eta^V,\eta^V)$ in terms of $s(\eta,\eta)$. Let $\bar\eta = (\bar\eta_\ell)_{\ell=1,\dots,L}$ be defined as in (2.29). Using (2.41), we can write

$$b_0^V(\eta^V,\eta^V) \lesssim |(\mathbf{1}-\pi_h)\Lambda\eta|_T^2 \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\left(\|\Lambda(\eta-\bar\eta)\|_T^2 + \sigma(\bar\eta,\bar\eta)\right).$$

(where we used that $\Lambda\bar\eta = \bar\eta$). Now, thanks to a Poincaré inequality, Lemmas 3.2 and 2.6, we obtain

$$b_0^V(\eta^V,\eta^V) \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 |\eta|_T^2.$$

Then we have

$$b_0^V(\eta^V,\eta^V) + p^{3/2} b^E(\eta^E,\eta^E) \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta).$$

This bound would suggest to choose, as a preconditioner for the matrix $\mathbf{S}$, the matrix $\mathbf{P}_0$ corresponding to the bilinear form

$$s_0(\eta,\zeta) = b_0^V(\eta^V,\zeta^V) + p^{3/2} b^E(\eta^E,\zeta^E).$$

With this choice we would have the bound

$$\mathrm{Cond}(\mathbf{P}_0^{-1}\mathbf{S}) \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2.$$

4. The vertex block of the preconditioner

Building the vertex block of the preconditioner according to (3.17) implies assembling at least a portion of the Schur complement matrix $\mathbf{S}$. This turns out to be too expensive, in particular when fine meshes, high order approximations and large number of subdomains are involved. First, the number of local Schur subdomain solves is proportional to the number of interior subdomain vertices and their respective computational cost depend on the local grid size and polynomial order. Second, the communication cost scales with the local number of Schur degrees of freedom. In the present section we therefore propose two more efficient alternatives.

4.1. A "coarse" vertex block preconditioner

The first option that we considered is to build the vertex block of the preconditioner using a fixed auxiliary coarse mesh, independent of the space discretization and of the polynomial degree. This idea was presented in [13] for the case of P_1 finite elements. We combine it here with a suitable balancing between vertex and edge component, yielding a better estimate for the condition number of the preconditioned matrix.

Let $\mathtt{n}_c$ be a fixed small integer (to fix the ideas, in all our tests we have $\mathtt{n}_c = 3$). We build coarse auxiliary quasi-uniform triangular meshes $\mathcal{K}_\delta^\ell$ with mesh size $\delta_\ell = \frac{H_\ell}{\mathtt{n}_c}$. We do not assume that $\mathcal{K}_\delta^\ell$ and $\mathcal{K}_h^\ell$ are nested. We define a coarse auxiliary P_1 discretization spaces $X_\delta^\ell \subset H^1(\Omega_\ell)\cap C^0(\bar\Omega_\ell)$ defined by

$$X_\delta^\ell = \{v \in C^0(\bar\Omega_\ell) \text{ s.t. } v|_K \in P_1(K),\ K \in \mathcal{T}_\delta^\ell\ \} \cap H_0^1(\Omega).$$

For each γ_ℓ^i slave side we also consider the corresponding auxiliary multiplier space $M_\delta^{\ell,i} \subset L^2(\gamma_\ell^i)$, defined analogously to (2.19).

The spaces X_δ, M_δ, $\mathcal{X}_\delta$, and T_δ^ℓ, T_δ, $\mathcal{T}_\delta$ are built starting from the X_δ^ℓ's and the $M_\delta^{\ell,i}$'s in the same way as the spaces X_h, M_h, $\mathcal{X}_h$ and T_h^ℓ, T_h, $\mathcal{T}_h$ by using definitions similar to (2.12), (2.13), (2.14), (2.19) and (2.20).

Analogously to π_h we can define the operator $\pi_\delta : \prod_{\ell=1}^{L} L^2(\partial\Omega_\ell) \longrightarrow T_\delta$. Using Lemma 2.11 we obtain for all $\eta \in T$ and $\alpha = (\alpha_\ell)_{\ell=1,\dots,L} \in \mathbb{R}^L$

$$|(\mathbf{1}-\pi_\delta)\eta|_T^2 \lesssim \|\eta-\alpha\|_T^2 + \sigma(\alpha,\alpha), \tag{4.1}$$

and for $\eta \in \mathfrak{L}$

$$|(\mathbf{1}-\pi_\delta)\eta|_T^2 \lesssim \|\eta-\alpha\|_T^2 + \sigma(\alpha,\alpha). \tag{4.2}$$

Moreover, Lemma 3.2 yields that for all $\eta \in T_\delta$

$$|\Lambda\eta|_T^2 \lesssim |\eta|_T^2. \tag{4.3}$$

Clearly, the constants in the inequalities (4.1), (4.2) and (3.2) depend on $\mathtt{n}_c$, which is however a fixed small number, independent of the h_ℓ's, the H_ℓ's and the p_ℓ's, and that we can, therefore, consider as a constant.

Analogously to R_h^ℓ we can define a local coarse lifting operator R_δ^ℓ. By standard arguments this verifies, for all $\eta \in T_\delta$,

$$\|R_\delta\eta\|_X \simeq \|\eta\|_T, \qquad |R_\delta\eta|_X \simeq |\eta|_T. \tag{4.4}$$

We define the vertex block of the preconditioner as $b_1^V : \mathcal{T}_h^V \times \mathcal{T}_h^V \to \mathbb{R}$ as

$$b_1^V(\eta^V,\xi^V) := \sum_\ell \int_{\Omega_\ell} a(x)\nabla(R_\delta^\ell(\mathbf{1}-\pi_\delta)\Lambda\eta^V)\cdot\nabla(R_\delta^\ell(1-\pi_\delta)\Lambda\xi^V)\,\mathrm{d}x. \tag{4.5}$$

The second preconditioner we propose is then:

$$\begin{aligned} &s_1 : \mathcal{T}_h\times\mathcal{T}_h\longrightarrow\mathbb{R}\\ &s_1(\eta,\xi) = b^E(\eta^E,\xi^E) + \left(1+\log\left(\frac{Hp^2}{h}\right)\right) b_1^V(\eta^V,\xi^V). \end{aligned}\tag{4.6}$$

Remark that $(1-\pi_\delta)\Lambda\mathcal{T}_h^V = \mathcal{T}_\delta^V$. In view of this identity it is not difficult to realize that computing the vertex block of this preconditioner only implies assembling the Schur complement matrix for the auxiliary Mortar problem corresponding to the fixed coarse discretization. This operation is then totally independent of the mesh size h. More details will be given in the next section.

The following theorem holds:

Theorem 4.1. *For all $\eta \in \mathcal{T}_h$ we have:*

$$p^{-3/2}s(\eta,\eta) \lesssim s_1(\eta,\eta) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta). \tag{4.7}$$

Proof. By using (3.15) we get

$$s(\eta,\eta) \lesssim |\eta^E|_T^2 + |\eta^V|_T^2 \lesssim p^{3/2} b^E(\eta^E,\eta^E) + |\eta^V|_T^2. \tag{4.8}$$

Let now $\eta^{V,\delta} = (\eta_\ell^{V,\delta})_{\ell=1,\dots,L} = (1-\pi_\delta)\Lambda\eta$. We observe that $\eta^V = (1-\pi_h)\Lambda\eta^{V,\delta}$. We also observe that since both M_h and M_δ contain the functions constant on each γ_ℓ^i, we have that on each edge $\gamma_\ell^i = \gamma_n^j$ (with γ_ℓ^i slave side) the integral of $\eta_\ell^{V,\delta}$, $\eta_n^{V,\delta}$, η_ℓ^V and η_n^V are all equals, since they coincide with the integral of $\Lambda_n\eta_n^V = \Lambda_n\eta_n^{V,\delta}$. This implies that the integral on $\partial\Omega_\ell$ of $\eta_\ell^{V,\delta}$ and of η_ℓ^V coincide. We can then introduce $\bar\eta = (\bar\eta_\ell)_{\ell=1,\dots,L} \in \mathbb{R}^L$ with

$$\bar\eta_\ell = |\partial\Omega_\ell|^{-1}\int_{\partial\Omega_\ell}\eta_\ell^{V,\delta}\,\mathrm{d}s = |\partial\Omega_\ell|^{-1}\int_{\partial\Omega_\ell}\eta_\ell^V\,\mathrm{d}s.$$

Using Corollary 2.12 and (4.3), as well as (2.23), we have ($\Lambda\bar{\eta}=\bar{\eta}$)

$$
\begin{aligned}
|\eta^V|_T^2 = |(\mathbf{1}-\pi_h)\Lambda\eta^{V,\delta}|_T^2 &\lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\left(\|\Lambda(\eta^{V,\delta}-\bar{\eta})\|_T^2+\sigma(\bar{\eta},\bar{\eta})\right)\\
&\lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\left(\|\eta^{V,\delta}-\bar{\eta}\|_T^2+|\eta^{V,\delta}|_T^2\right)\\
&\lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta^{V,\delta}|_T^2\\
&\lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)b_1^V(\eta^V,\eta^V).
\end{aligned}
$$

Then we have

$$
s(\eta,\eta)\lesssim|\eta^E|_T^2+|\eta^V|_T^2\lesssim p^{3/2}b^E(\eta^E,\eta^E)+p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)b_1^V(\eta^V,\eta^V)=p^{3/2}s_1(\eta,\eta),
$$

that is the first part of the theorem.

Let us now bound $s_1(\eta,\eta)$ in terms of $s(\eta,\eta)$. Using (4.2), Lemmas 3.2 and 2.6, and (2.23) we obtain

$$
\begin{aligned}
b_1^V(\eta^V,\eta^V)&\lesssim|(\mathbf{1}-\pi_\delta)\Lambda\eta|_T^2\lesssim\left(\|\Lambda(\eta-\bar{\eta})\|_T^2+\sigma(\bar{\eta},\bar{\eta})\right)\\
&\lesssim\left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta|_T^2\lesssim\left(1+\log\left(\frac{Hp^2}{h}\right)\right)s(\eta,\eta).
\end{aligned}
$$

Thanks to (3.14) and the definition (4.6) we get that

$$
s_1(\eta,\eta)=b^E(\eta^E,\eta^E)+\left(1+\log\left(\frac{Hp^2}{h}\right)\right)b_1^V(\eta^V,\eta^V)\lesssim\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta),
$$

that concludes the proof of the Theorem 4.1. □

Let $\mathbf{S}$ and $\mathbf{P}_1$ be the matrices obtained by discretizing respectively s and s_1 then, by using the lower and upper bounds for the eigenvalues of $\mathbf{P}_1^{-1}\mathbf{S}$ which are a direct consequence of Theorem 4.1, we obtain:

Corollary 4.2. *The condition number of the preconditioned matrix* $\mathbf{P}_1^{-1}\mathbf{S}$ *satisfies:*

$$
\mathrm{Cond}(\mathbf{P}_1^{-1}\mathbf{S})\lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2. \tag{4.9}
$$

4.2. A discontinuous Galerkin vertex block preconditioner

As a further alternative, we propose to construct the vertex block of the preconditioner, by completely giving up weak continuity and by using, instead, a Discontinuous Galerkin interior penalty method as coarse problem.

More precisely, letting $\mathcal{H}_\ell : H^{1/2}(\partial\Omega_\ell)\to H^1(\Omega_\ell)$ denote the continuous harmonic lifting, we set

$$
b_\#^V(\eta_\ell^V,\zeta_\ell^V)=\sum_\ell a_\ell(\mathcal{H}_\ell\Lambda_\ell\eta^V,\mathcal{H}_\ell\Lambda_\ell\zeta^V), \tag{4.10}
$$

$$
b_{[]}^V(\eta^V,\eta^V)=\sum_{\gamma_\ell^i\ \text{slave side}}|\gamma_\ell^i|^{-1}\int_{\gamma_\ell^i}|[\Lambda\eta]|^2\,\mathrm{d}s. \tag{4.11}
$$

Then, as vertex block of the preconditioner, we consider:

$$
b_2^V(\eta,\eta)=\tau_1\,b_\#^V(\eta_\ell^V,\eta_\ell^V)+\tau_2\,b_{[]}^V(\eta_\ell^V,\eta_\ell^V) \tag{4.12}
$$

with $\tau_1,\tau_2>0$ constants.

The global preconditioner is then assembled as

$$s_2(\eta,\eta) = b^E(\eta^E,\eta^E) + \left(1+\log\left(\frac{Hp^2}{h}\right)\right) b_2^V(\eta^V,\eta^V). \tag{4.13}$$

We have the following theorem.

Theorem 4.3. *For all $\eta \in \mathcal{T}_h$ we have:*

$$p^{-3/2}s(\eta,\eta) \lesssim s_2(\eta,\eta) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta). \tag{4.14}$$

Proof. Thanks to Lemma 3.2 we have

$$b_{\#}^V(\eta^V,\eta^V) \lesssim |\Lambda\eta|_T^2 \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta|_T^2.$$

Let us then bound $b_{[]}^V(\eta^V,\eta^V)$. For each slave side γ_ℓ^i and corresponding master side γ_n^j we introduce once again the constants

$$\bar\eta_\ell^i = \frac{1}{|\gamma_\ell^i|}\int_{\gamma_\ell^i} \eta_\ell^V \ \mathrm{d}s \qquad \bar\eta_n^j = \frac{1}{|\gamma_n^j|}\int_{\gamma_n^j} \eta_n^V \ \mathrm{d}s$$

Thanks to the weak continuity constraint, we have that $\bar\eta_\ell^i = \bar\eta_n^j$.

Letting a_ℓ^i and b_ℓ^i denote the two extrema of γ_ℓ^i, and observing that $[\Lambda\eta]$ coincides with $[\eta]$ when evaluated at a_ℓ^i and b_ℓ^i, then we can write

$$b_{[]}^V(\eta^V,\eta^V) = \sum_{\gamma_\ell^i \text{ slave side}} |\gamma_\ell^i|^{-1}\int_{\gamma_\ell^i} |[\Lambda\eta]|^2 \ \mathrm{d}s \simeq \sum_{\gamma_\ell^i \text{ slave side}} (|[\eta](a_\ell^i)|^2 + |[\eta](b_\ell^i)|^2).$$

Observing that for (n,j) such that $\gamma_\ell^i = \gamma_n^j$ and for $x \in \bar\gamma_\ell^i$ we have that

$$|[\eta](x)|^2 = |\eta_\ell(x)-\eta_n(x)|^2 = |\eta_\ell(x) - \bar\eta_\ell^i - (\eta_n(x)-\bar\eta_n^j)|^2 \lesssim |\eta_\ell(x)-\bar\eta_\ell^i|^2 + |\eta_n(x)-\bar\eta_n^j|^2,$$

we immediately obtain that

$$b_{[]}^V(\eta^V,\eta^V) \lesssim \sum_\ell \sum_{i=1}^4 |\eta_\ell(x_i^\ell) - \bar\eta_\ell^i|^2,$$

where, once again, x_ℓ^i, $\ell = 1,\dots,4$ denote the vertices of Ω_ℓ. Now, reasoning as in the proof of Lemma 2.6 we obtain

$$|\eta_\ell(x_i^\ell) - \bar\eta_\ell^i|^2 \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)|\eta|^2_{H^{1/2}(\partial\Omega_\ell)}.$$

Putting all together we obtain

$$b_2^V(\eta^V,\eta^V) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right) s(\eta,\eta). \tag{4.15}$$

Combining (4.15), (3.14) with (4.13), we obtain

$$s_2(\eta,\eta) \lesssim \left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2 s(\eta,\eta).$$

Let us now prove the reverse bound. We have

$$s(\eta,\eta) \lesssim |\eta^V|_T^2 + |\eta^E|_T^2$$

and

$$|\eta^V|_T^2 = |(\mathbf{1}-\pi_h)\Lambda\eta|_T^2 \lesssim |\Lambda\eta|_T^2 + |\pi_h\Lambda\eta|_T^2.$$

We bound the two terms on the right hand side separately. We have (see [26])

$$|\Lambda\eta|_T^2 \lesssim \sum_\ell |\mathcal{H}_\ell\Lambda_\ell\eta_\ell|^2_{H^1(\Omega_\ell)} \lesssim b^V_\#(\eta^V,\eta^V).$$

As far as the second term is concerned, proceeding as in the proof of Lemma 2.11 and using (2.25) with $\varepsilon = 1/\log(Hp^2/h)$ we obtain

$$|\pi_h(\Lambda\eta)|_T^2 \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)\sum_{\gamma_\ell^i \text{ slave side}} \|[\Lambda\eta]\|^2_{H^{1/2-\varepsilon}(\gamma_\ell^i)}.$$

Now we have (recall that $\|\cdot\|_{L^2(\Gamma_\ell)}$ is the scaled L^2 norm defined in (2.7))

$$\|[\Lambda\eta]\|^2_{H^{1/2-\varepsilon}(\gamma_\ell^i)} \lesssim \|[\Lambda\eta]\|^2_{L^2(\gamma_\ell^i)} + |[\Lambda\eta]|^2_{H^{1/2}(\gamma_\ell^i)} \lesssim |\gamma_\ell^i|^{-1}\int_{\gamma_\ell^i}|[\Lambda\eta]|^2\,\mathrm{d}s,$$

where the last inverse type inequality is obtained by a scaling argument, using the linearity of $\Lambda\eta$ on γ_ℓ^i. Combining the bounds on the two contributions we obtain

$$s(\eta^V,\eta^V) \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right) b_2^V(\eta^V,\eta^V).$$

which finally yields

$$s(\eta,\eta) \lesssim p^{3/2} s_2(\eta,\eta). \qquad \square$$

Remark 4.4. We observe that if the Ω_ℓ's are rectangles, for $\eta \in \mathfrak{L}$ we have that $\mathcal{H}_\ell\eta_\ell$ is the Q_1 function (polynomial of degree ≤ 1 in each of the two unknowns) coinciding with η_ℓ at the four vertices of Ω_ℓ. The local matrix corresponding to the block b_1^V can then be replaced by the elementary Q1 stifness matrix for the problem considered.

Let $\mathbf{S}$ and $\mathbf{P}_2$ be the matrices obtained by discretizing respectively s and s_2 then, by using the lower and upper bounds for the eigenvalues of $\mathbf{P}_2^{-1}\mathbf{S}$ implicitly given by Theorem 4.3, we obtain:

Corollary 4.5. *The condition number of the preconditioned matrix $\mathbf{P}_2^{-1}\mathbf{S}$ satisfies:*

$$\mathrm{Cond}(\mathbf{P}_2^{-1}\mathbf{S}) \lesssim p^{3/2}\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2. \tag{4.16}$$

5. Numerical results

In this section, we test the properties of the preconditioners previously proposed, by performing a p-, H- and h-convergence study. We consider the model problem

$$-\Delta u = f \quad \text{in } \Omega =]0,1[^2, \qquad u = 0 \quad \text{on } \partial\Omega$$

and, for all tests, we set $f = 1$. A geometrically conforming, domain decomposition of Ω in $N = 2^m \times 2^m$ subdomains, $m = 2, 3, 4, \ldots$, with a quasiuniform mesh of order $n \times n$ in each subdomain, is considered.

Let $\mathbf{S}$ be the matrix associated to the discrete Steklov–Poincaré operator $s(\cdot,\cdot)$ defined in (3.3) and let $\widehat{\mathbf{S}}$ be the matrix obtained after the change of basis corresponding to switching from the standard nodal basis to the basis related to the splitting (3.10). From now on, we focus on testing the efficiency of the preconditioners for the transformed Schur complement system

$$\widehat{\mathbf{S}}\,\widehat{\mathbf{u}} = \widehat{\mathbf{g}} \tag{5.1}$$

where the matrix $\widehat{\mathbf{S}}$, after ordering of the indices as nodes lying on the edges and on the vertices, can be written as:

$$\widehat{\mathbf{S}} = \begin{pmatrix} \widehat{\mathbf{S}}_{\mathsf{ee}} & \widehat{\mathbf{S}}_{\mathsf{ev}} \\ \widehat{\mathbf{S}}_{\mathsf{ev}}^T & \widehat{\mathbf{S}}_{\mathsf{vv}} \end{pmatrix}.$$

The Preconditioned Conjugate Gradient (PCG) method, with a relative tolerance set equal to 10^{-6}, was used to solve the transformed Schur complement system (5.1). The condition number of the (preconditioned) Schur complement matrix has been estimated as a byproduct of the Conjugate-Gradient method (see [20], Sects. 9.3, 10.2).

Remark 5.1. Of course the matrix $\widehat{\mathbf{S}}$ is never assembled within the solution procedure as only the computation of its action on a given vector is needed.

It is beyond the scope of this paper to go into details on how the preconditionrd mortar method considered in this paper can be efficiently implemented in a parallel framework. This issue will be thoroughly examined in [32], where extensive tests will be presented aimed at evaluating the performance of the method. To give an idea of its potential let us just mention that, in the biggest tests presented in this paper (corresponding to the result presented on 6, last line, second to last column) we solved, using $\mathbf{P}_2$ as preconditioner, a discrete problem on 40 000 cores, for $p = 4$ and $h = 0.625e^{-4}$ in 23 iterations in under 3 min.

The preconditioner for $\widehat{\mathbf{S}}$ will be of block-Jacobi type: one block for each one of the master edges and an additional block for the vertices.

For the edge block of the preconditioner, we need the matrix counterpart of (3.13). In the literature it is possible to find different ways to build bilinear forms $b^E(\cdot,\cdot)$ that satisfy (3.12) and (3.13). The choice we followed here for defining $b^E(\cdot,\cdot)$ is the one proposed in [15] and it is based on an equivalence result for the $H^{1/2}_{00}$ norm, see [3, 14] for a detailed description of its construction. We denote by $\boldsymbol{\eta}^E$ the vector representation of $\eta^E \in T_0^{\ell,i}$. Then it can be verified that, for each edgee γ_n^j, we have (see [14] p. 1110 and [19])

$$|\eta^E|^2_{H^{1/2}(\gamma_n^j)} \simeq (l_0^{1/2}\eta^E, \eta^E)_{\gamma_n^j} = \boldsymbol{\eta}^{E^T}\widehat{\mathbf{K}}_E\boldsymbol{\eta}^E,$$

with $\widehat{\mathbf{K}}_E = \mathbf{M}_E^{1/2}(\mathbf{M}_E^{-1/2}\mathbf{R}_E\mathbf{M}_E^{-1/2})^{1/2}\mathbf{M}_E^{1/2}$, where $\mathbf{M}_E$ and $\mathbf{R}_E$ are the mass and stiffness matrices associated to the discretization of the operator $-\mathrm{d}^2/\mathrm{d}s^2$ (in $T_0^{n,j}$) with homogeneous Dirichlet boundary conditions at the extrema a and b of γ_ℓ^i. Thus, the edge block of the preconditioner can be written as:

$$\widehat{\mathbf{K}}_{\mathsf{ee}} = \begin{pmatrix} \widehat{\mathbf{K}}_{E_1} & 0 & 0 & 0 \\ 0 & \widehat{\mathbf{K}}_{E_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \widehat{\mathbf{K}}_{E_M} \end{pmatrix} \tag{5.2}$$

with one block for each master edge where M is the number of masters.

The preconditioner $\mathbf{P}_1$. Concerning the vertex block of our preconditioner, following Section 4.1, we introduce a coarse auxiliary mesh in each subdomain made of 3×3 elements and we fix the polynomial order $p=1$. Let $\widehat{\mathbf{S}}^c$ be the matrix obtained after applying the change of basis to the associated Schur complement system. $\widehat{\mathbf{S}}^c$ takes the form

$$\widehat{\mathbf{S}}^c = \begin{pmatrix} \widehat{\mathbf{S}}^c_{\mathsf{ee}} & \widehat{\mathbf{S}}^c_{\mathsf{ve}} \\ \widehat{\mathbf{S}}^{c\,T}_{\mathsf{ve}} & \widehat{\mathbf{S}}^c_{\mathsf{vv}} \end{pmatrix}. \tag{5.3}$$

The preconditioner $\mathbf{P}_1$, described in Section 4.1, can then be written as:

$$\mathbf{P}_1 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}^c_{\mathsf{v}} \end{pmatrix}, \qquad \text{with } \mathbf{P}^c_{\mathsf{v}} = \left(1+\log\left(\frac{Hp^2}{h}\right)\right)\widehat{\mathbf{S}}^c_{\mathsf{vv}}. \tag{5.4}$$

The Preconditioner $\mathbf{P}_2$. Let $\mathbf{P}_{\#}$ and $\mathbf{P}_{[]}$ be the matrix counterparts of (4.10) and of (4.11) respectively and let

$$\mathbf{P}^{DG}_v = \left(1+\log\left(\frac{Hp^2}{h}\right)\right)(\tau_1\mathbf{P}_{\#}+\tau_2\mathbf{P}_{[]}).$$

Then the new preconditioner we propose is:

$$\mathbf{P}_2 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}^{DG}_v \end{pmatrix}. \tag{5.5}$$

All the tests presented relate to $\tau_1 = 1/10$ and $\tau_2 = 2$. These values of τ_1 and τ_2 were obtained by trial-and-error on small tests problems. Remark that the ratio between τ_1 and τ_2 is consistent with the choice that is usually done in the framework of interior penalty DG methods.

In summary, the numerical tests relate the following two preconditioners for the transformed Schur complement system:

$$\mathbf{P}_1 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}^c_{\mathsf{v}} \end{pmatrix} \qquad \text{and} \qquad \mathbf{P}_2 = \begin{pmatrix} \widehat{\mathbf{K}}_{\mathsf{ee}} & 0 \\ 0 & \mathbf{P}^{DG}_v \end{pmatrix}. \tag{5.6}$$

We report the condition number estimates of the preconditioned Schur complement matrix $\kappa(\widehat{\mathbf{P}}^{-1}\widehat{\mathbf{S}})$ where $\widehat{\mathbf{P}}$ is either one of the preconditioners defined in (5.6), the number of iterations and the following two ratios:

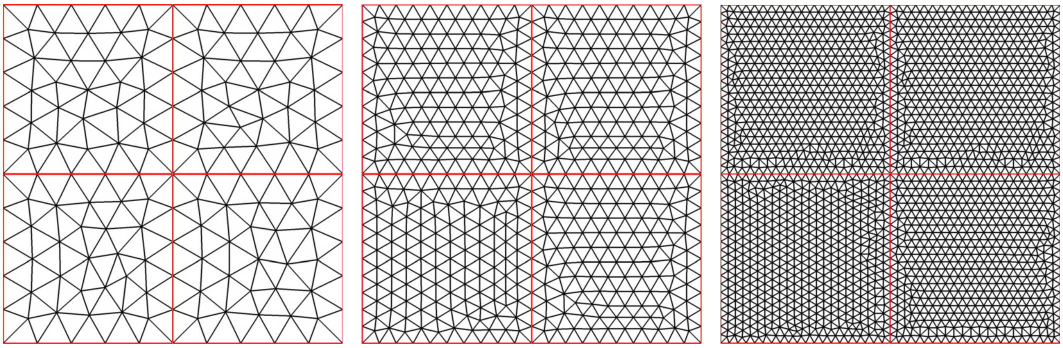
$$R_2 = \frac{\kappa(\widehat{\mathbf{P}}^{-1}\widehat{\mathbf{S}})}{\left(1+\log\left(\frac{Hp^2}{h}\right)\right)^2} \qquad\qquad R_{2p} = \frac{R_2}{p^{3/2}} \tag{5.7}$$

where H is the coarse mesh-size, h the fine mesh-size and p the polynomial order.

5.1. Computation platforms

For the implementation of the methodology described in this paper, we developed the code in C++11 using the library Feel++ [30, 31], which allows for a wide variety of numerical methods including continuous and discontinuous Galerkin methods from 1D to 3D and, of course, the Mortar method we are dealing with. Feel++ uses MPI for parallel computing and its data structures can be customized with respect to MPI communicators, which allows to implement the various preconditioners presented in this paper. Finally linear algebra is handled by PETSc both in sequential in the subdomains, and in parallel for the coarse preconditioner. The implementation details as well as more extensive results with respect to strong and weak scalability are presented in [32].

The simulations presented in the next sections were partly performed on `hpc-login` at MesoCentre@Strasbourg. MesoCentre is a supercomputer with 288 compute nodes interconnected by an infiniband QDR



1.1. Level 1 1.2. Level 2 1.3. Level 3

FIGURE 1. First three levels of refinements for unstructured triangular grids on a subdomain partition made of 4 squares.

TABLE 1. Unpreconditioned system. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	44	59	84	105	155	240	354
64	59	77	109	150	213	298	468
256	81	99	127	178	250	327	484

TABLE 2. Preconditioner $\mathbf{P}_1$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	26	27	28	31	33	34	36
64	24	27	29	31	33	35	36
256	21	23	25	28	30	33	35

network. The system is Scientific Linux based on Intel Xeon Ivy Bridge processors with 16 cores and 64GO of RAM running at 2.6 Ghz. MesoCentre has a theoretical peak performance of 70 TFLOP/s. The simulations on a large number of cores, 1024, 4096, 16 384, 22 500 and 40 000, were done on Curie at the TGCC, a TIER-0 system which is part of PRACE. Curie has 5040 B510 bullx nodes and for each node a 2 eight-core Intel processors Sandy Bridge cadenced at 2.7 GHz with 64 GB.

Linear elements. In the first set of experiments, we consider piecewise linear elements ($p = 1$), and compute the estimated condition number when varying the number of subdomains and the mesh size. We split the domain Ω in $N = 4^m$ subdomains, $m = 2, 3, 4$ with a quasiuniform mesh of order $n \times n$ in each subdomain. These results were obtained on a sequence of triangular grids like the ones shown in Figure 1

In the conforming case, the condition number of $\mathbf{S}$ is known to behave (for $p = 1$) like $(Hh)^{-1}$ [16]. In Table 1 we show the number of iterations required by the solution of the transformed Schur complement system without preconditioning as a function of N and of n. We observe that this number roughly behaves like $(Hh)^{1/2}$.

Table 2 shows the number of iterations to solve the system (5.1), preconditioned with $\mathbf{P}_1$, when increasing N and n. Analogous results obtained with preconditioned $\mathbf{P}_2$ are reported in Table 3. As expected, a logarithmic growth is clearly observed for both preconditioner $\mathbf{P}_1$ and $\mathbf{P}_2$, whereas, the number of iterations without preconditioning shown in Table 1 has a strong increase as the mesh size h goes to zero, as expected from the theoretical estimation of the condition number of $\hat{\mathbf{S}}$.

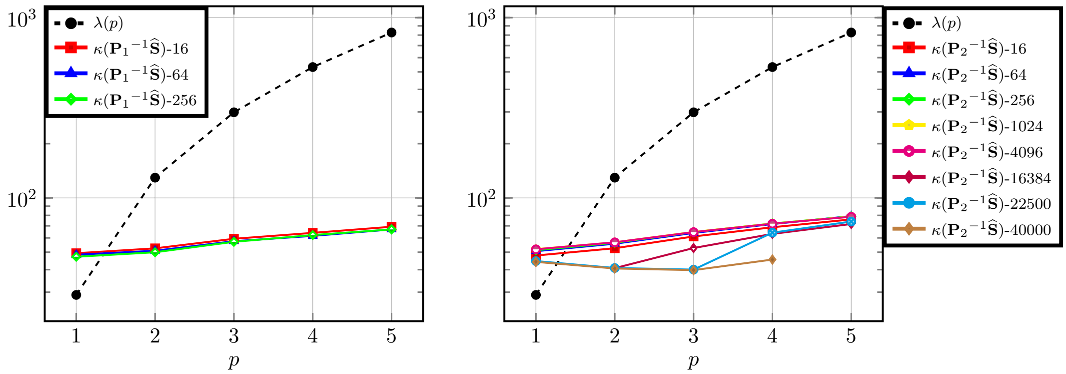
High-order elements. We now present some computations obtained with high-order elements. We run the same set of experiments carried out for linear FEM, but now we increase the polynomial order p up to 5. To study

TABLE 3. Preconditioner $\mathbf{P}_2$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	23	24	26	28	31	33	35
64	22	23	26	29	31	33	35
256	20	21	23	26	28	30	33

TABLE 4. Condition number estimate $\kappa(\hat{\mathbf{S}})$ and number of iterations (between parenthesis) for $n/N = 80$.

$N\backslash p$	1	2	3	4	5
16	155	256	344	444	533
64	213	330	468	613	765
256	250	356	495	631	787



2.1. Preconditioner $\mathbf{P}_1$ 2.2. Preconditioner $\mathbf{P}_2$

FIGURE 2. Condition number of the preconditioned system as function of p with 16, 64, 256, 1024, 4096 subdomains and $n/N = 80$

the dependence on p of our preconditioners, we report the condition number estimate for the preconditioned system, as function of p with H/h constant. For the sake of comparison, before showing the performance of our preconditioners, we report in Table 4, for $H/h \simeq n/N = 80$ constant and for increasing values of the polynomial order p, the number of iterations required to solve system (5.1) without preconditioning.

Let the function λ be defined as $\lambda(p) = p^{3/2}(1 + \log(\dfrac{Hp^2}{h}))^2$. In Figure 2, we plot the condition number of the transformed Schur system, preconditioned with $\mathbf{P}_1$ and $\mathbf{P}_2$, and $\lambda(p)$ as function of p.

To highlight the dependence on p of our preconditioners, in Tables 5 and 6 we report, for $n/N = 80$ fixed and for increasing values of the polynomial order p, the ratio R_2 defined in (5.7). Higher values of p are considered in Table 7 where the ratio R_2 and the number of iterations are reported for $n/N = 10$ fixed and polynomial order $p \in [6, 14]$.

We remark that the dependence on the factor $p^{3/2}$ of the condition number does not show up in these numerical tests. Indeed, the numerical results seem to show an even better behaviour than the polylogarithmic dependence on Hp^2/h. In particular, in Table 5, for fixed H/h and p the ratio R_2 seems to be slightly decreasing rather than constant. This is not in contrast with the theory. In fact, it is well known that, within the conjugate gradient iterations, the condition number estimate might be underestimated [4, 20].

TABLE 5. Ratio R2 for $n/N = 80$, preconditioner P_1 and increasing values of the polynomial order p. Between parenthesis the number of iterations.

$N\backslash p$	1	2	3	4	5
16	170(33)	114(31)	103(32)	0.96(33)	0.93(34)
64	167(33)	111(32)	1.0(34)	0.92(34)	0.90(34)
256	162(30)	109(31)	0.99(32)	0.93(33)	0.90(33)

TABLE 6. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_2$ and increasing values of the polynomial order p. Between parenthesis the number of iterations. The results at 40 000 cores and $p = 5$ are not available.

$N\backslash p$	1	2	3	4	5
16	165(31)	114(32)	106(33)	103(38)	102(39)
64	174(31)	121(33)	111(35)	107(40)	107(42)
256	176(28)	123(32)	112(34)	108(36)	106(40)
1024	178(27)	123(29)	112(31)	108(32)	106(34)
4096	179(25)	123(28)	112(29)	108(31)	106(31)
16 384	152(20)	0.88(22)	0.91(26)	0.94(27)	0.96(28)
22 500	152(19)	0.88(20)	0.69(22)	0.95(26)	0.99(27)
40 000	152(17)	0.88(20)	0.69(22)	0.68(23)	0.00(0)

TABLE 7. Ratio R_2 for $n/N = 10$, preconditioner $\mathbf{P}_2$ and increasing values of the polynomial order $p \in [6, 14]$. Between parenthesis the number of iterations.

$N\backslash p$	6	8	10	12	14
16	1.03(30)	1.02(35)	1.04(36)	1.06(37)	1.07(38)
64	1.08(32)	1.07(37)	1.09(39)	1.10(40)	1.11(41)
256	1.09(29)	1.07(33)	1.07(35)	1.07(37)	1.08(38)
1,024	1.09(28)	1.07(28)	1.06(28)	1.06(29)	1.06(30)
4,096	1.09(26)	1.07(27)	1.06(28)	1.06(28)	1.06(29)

Nonmatching grids. The tests performed until now deal with decomposition with matching grid (though the solution is non conforming, due to the lack of continuity at the cross points). We now turn to the case of non-matching triangulations. As before, we split the domain Ω in $N = 2^m \times 2^m$ subdomains, $m = 2, 3, 4$ but now we take quasiuniform meshes with two different mesh sizes: $h_{\text{fine}} = 1/(2n)$ and $h_{\text{coarse}} = 1/n$. We deliberately choose embedded grids in order to ensure exact numerical integration for the constraints. On the interface the master subdomains are chosen to be the ones corresponding to the coarser mesh.

We start as before with the linear case, $p = 1$, and we report the number of iterations when increasing the number of subdomains N and the number of elements n of the fine mesh. Then, for $n/N = 80$ constant and increasing values of p we report, for the two preconditioners $\mathbf{P}_1$ and $\mathbf{P}_2$, the ratio R_2 introduced in (5.7).

Similar behavior is obtained with nonconforming grids that are not embedded, these results will be presented in [32].

L shaped domain with graded meshes. Finally we present results for an L-shaped domain. Figure 5 displays two different decompositions of the domain made up of respectively 12 and 48 subdomains, and the two associated mesh. The polynomial order is fixed subdomain-wise and it goes from 1 to 5 as the distance of

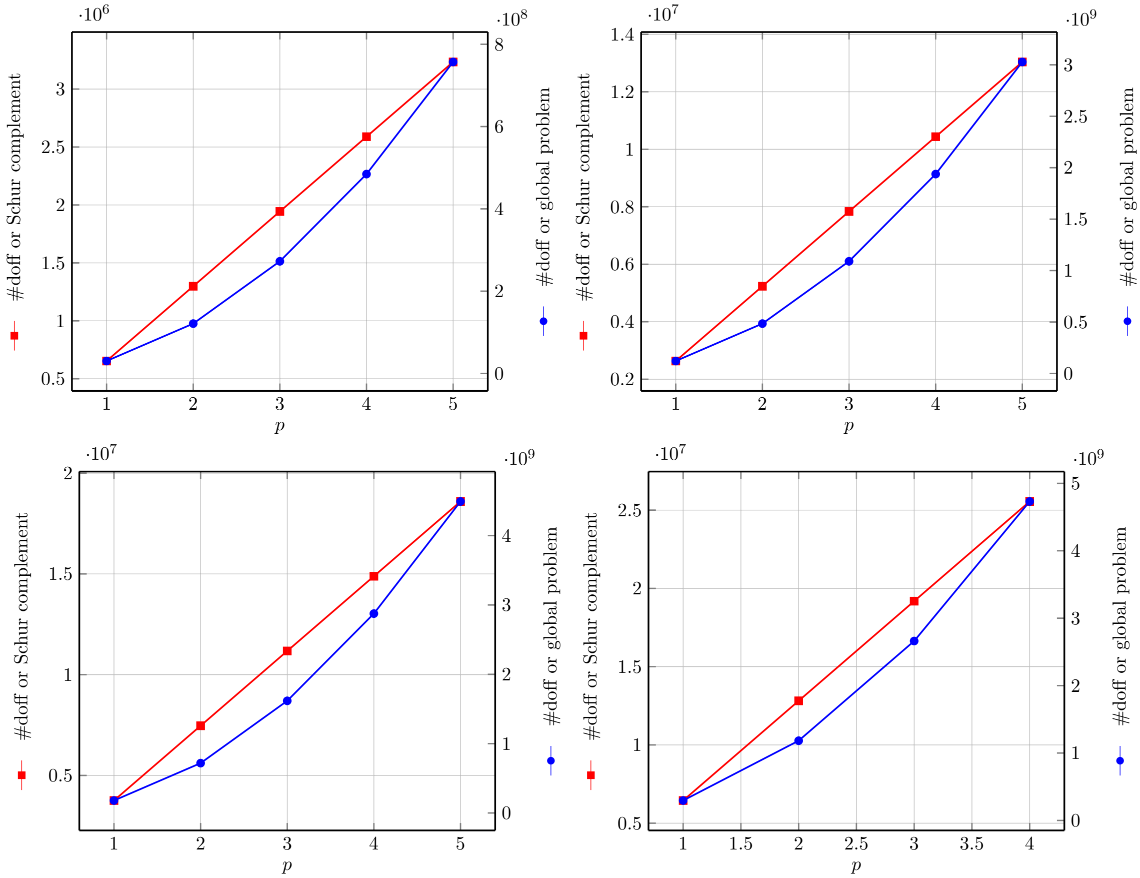


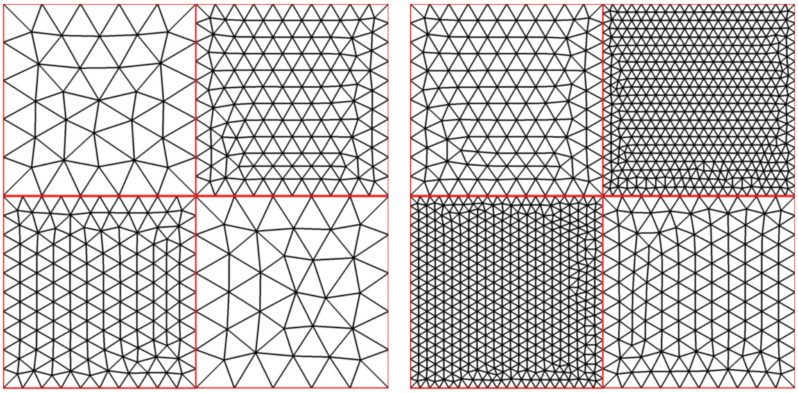
FIGURE 3. Number of degrees of freedom as function of p with 4096, 16 384, 22 500 and 40 000 subdomains. Note that two different scalings are used for the global problem (scaling marked on the right side of each figure) and for the Schur complements (on the left of each figure). For $p = 4$ and 40 000 subdomains the global problem has about 5 billion of unknowns, and it is solved in less than 3 min.

TABLE 8. Polynomial order $p = 1$. Preconditioner $\mathbf{P}_1$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	12	17	19	20	20	21	21
64	14	18	20	22	24	27	30
256	12	15	17	19	21	21	23

TABLE 9. Polynomial order $p = 1$. Preconditioner $\mathbf{P}_2$. Number of iterations required by PCG.

$N\backslash n$	5	10	20	40	80	180	320
16	14	17	18	18	20	22	23
64	18	18	19	20	23	26	28
256	16	16	17	19	22	24	27



4.1. Level 2   4.2. Level 3

FIGURE 4. Nonconforming decompositions with unstructured meshes in the case of 4 subdomains.

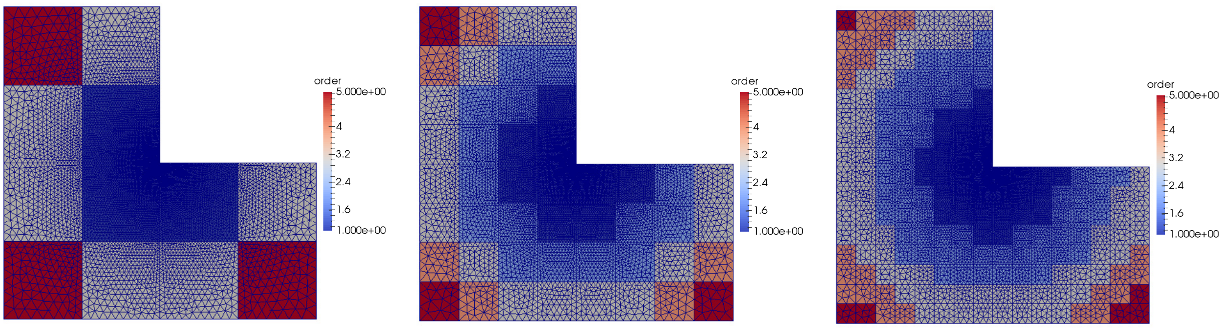


FIGURE 5. L-shaped domain decomposed in 12 (*left*), 48 (*middle*) and 192 (*right*) subdomains and the associated meshes; the minimum mesh size is $h_{\min} = 0.0125$. The polynomial order p goes from 1 to 5 as the distance from the singularities increases and it is fixed subdomain-wise.

TABLE 10. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_1$ and increasing values of the polynomial order p. Between parentheses the number of iterations.

$N\backslash p$	1	2	3	4	5
16	0.82(22)	0.65(24)	0.63(24)	0.62(24)	0.62(26)
64	0.82(25)	0.65(26)	0.63(27)	0.62(29)	0.61(29)
256	0.77(21)	0.63(21)	0.60(21)	0.60(22)	0.59(23)

TABLE 11. Ratio R_2 for $n/N = 80$, preconditioner $\mathbf{P}_2$ and increasing values of the polynomial order p. Between parentheses the number of iterations.

$N\backslash p$	1	2	3	4	5
16	0.74(22)	0.70(27)	0.71(28)	0.73(28)	0.74(28)
64	0.76(22)	0.73(28)	0.75(30)	0.76(31)	0.77(32)
256	0.77(21)	0.72(25)	0.74(30)	0.76(31)	0.78(31)
1024	0.77(19)	0.72(23)	0.71(25)	0.72(25)	0.73(29)
4096	0.71(17)	0.72(21)	0.71(22)	0.72(23)	0.72(24)

TABLE 12. Ratio R_2, number of iterations (between parenthesis) and number of degrees of freedom for an L-shaped domain decomposed in 12, 48 and 192 subdomains. Uniform mesh with $p = 5$ in each subdomain and a graded mesh with p from 1 to 5. Results for the graded mesh with 192 subdomains and $h_{\min} = .05$ and .025 are not available, since the subdomain size would, in this case, be to small with respect to the largest elements of the mesh.

	12 subdomains	
$h_{\min}$	uniform $p = 5$	graded mesh $p = 1:5$
0.05	2.74(36) 10 250	3.70(48) 5090
0.025	2.63(41) 37 175	3.44(50) 13 943
0.0125	2.58(40) 142 250	3.33(47) 34 437
0.00625	2.74(45) 558 675	3.23(50) 94 910
	48 subdomains	
$h_{\min}$	uniform $p = 5$	graded mesh $p = 1:5$
0.05	3.01(37) 15 839	4.50(48) 5215
0.025	2.75(39) 40 349	3.57(50) 11 069
0.0125	2.63(43) 147 699	3.17(53) 27 739
0.00625	2.64(48) 567 149	2.93(53) 71 939
	192 subdomains	
$h_{\min}$	uniform $p = 5$	graded mesh $p = 1:5$
0.05	3.43(34) 33 917	–
0.025	2.89(36) 62 877	–
0.0125	2.71(40) 160 122	2.73(46) 33 188
0.00625	2.61(44) 588 222	2.75(50) 81 162

the subdomain from the singularity increases, see Figure 5 The mesh is graded and becomes increasingly finer towards the singularity.

In Table 12 we compare the performance of preconditioner $\mathbf{P}_2$ for such meshes, with the performance of the same preconditioner on a uniform mesh with polynomial order $p = 5$ in all subdomains and uniform mesh size equal to the diameter of the smallest triangle in the graded mesh (which we denote by $h_{\min}$). For both uniform and graded mesh, the table reports, for four different values of $h_{\min}$, the ratio R_2, the number of iterations (between parentheses) and the number of degrees of freedom. We observe that the performance of our preconditioner on the graded mesh is slightly worse than it is on the corresponding uniform mesh. However we believe that this is more than compensated by the difference in the number of degrees of freedom. Moreover, the difference between the two cases (uniform *vs.* graded) becomes smaller and smaller as $h_{\min}$ goes to 0.

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