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ERROR ESTIMATES OF RUNGE–KUTTA DISCONTINUOUS GALERKIN METHODS FOR THE VLASOV–MAXWELL SYSTEM *

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Abstract. In this paper, error analysis is established for Runge–Kutta discontinuous Galerkin (RKDG) methods to solve the Vlasov–Maxwell system. This nonlinear hyperbolic system describes the time evolution of collisionless plasma particles of a single species under the self-consistent electromagnetic field, and it models many phenomena in both laboratory and astrophysical plasmas. The methods involve a third order TVD Runge–Kutta discretization in time and upwind discontinuous Galerkin discretizations of arbitrary order in phase domain. With the assumption that the exact solutions have sufficient regularity, the L^2 errors of the particle number density function as well as electric and magnetic fields at any given time T are bounded by $Ch^{k+1/2}+C\tau^3$ under a CFL condition $\tau/h \leq \gamma$. Here k is the polynomial degree used in phase space discretization, satisfying $k > \frac{d_x+1}{2}$ (with d_x being the dimension of spatial domain), τ is the time step, and h is the maximum mesh size in phase space. Both C and γ are positive constants independent of h and τ, and they may depend on the polynomial degree k, time T, the size of the phase domain, certain mesh parameters, and some Sobolev norms of the exact solution. The analysis can be extended to RKDG methods with other numerical fluxes and to RKDG methods solving relativistic Vlasov–Maxwell equations.

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1. Introduction

In this paper, we will establish error estimates of the Runge–Kutta discontinuous Galerkin (RKDG) methods for solving the dimensionless Vlasov–Maxwell (VM) equations

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + (E + v\times B)\cdot\nabla_v f = 0, \\ \partial_t E = \nabla\times B - J, \quad \partial_t B = -\nabla\times E, \\ \nabla\cdot E = \rho - \rho_i, \quad \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

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with

$$\rho(x,t)=\int_{\Omega_v} f(x,v,t)\mathrm{d}v, \quad J(x,t)=\int_{\Omega_v} f(x,v,t)v\mathrm{d}v. \tag{1.2}$$

This system describes the time evolution of collisionless plasma particles of a single species, such as electrons or ions, under the self-consistent electromagnetic field. Here $f(x,v,t)\geq 0$ is the particle number density function in the phase space with (x,v) at time t, $E(x,t)$ is the electric field, $B(x,t)$ is the magnetic field, $J(x,t)$ is the current density, $\rho(x,t)$ is the charge density, and ρ_i is the charge density of the background particles. The system (1.1) is defined on the phase domain $\Omega=\Omega_x\times\Omega_v$, where $\Omega_x=[L_{x,1},L_{x,2}]^{d_x}$ is the spatial domain and $\Omega_v=[L_{v,1},L_{v,2}]^{d_v}$ is the velocity domain ($d_x,d_v=1,2$ or 3), with periodic boundary conditions in x. In v direction, f is assumed to have compact support. We further assume the VM system is globally neutral, *i.e.* $\int_{\Omega_x}(\rho-\rho_i)\mathrm{d}x=0$. Note that this is compatible with the periodic boundary conditions in x.

The VM equations model many phenomena in both laboratory and astrophysical plasmas, and accurate and reliable numerical simulation of this system has fundamental importance. Particle methods [3,10,15] have been widely used since 60's because of their low computational cost especially when the dimension of the phase space is high, yet with their numerical noise, it is hard for the methods to produce very accurate approximations. In recent years, many high order Eulerian methods have been developed in the context of Vlasov–Poisson equations or the VM equations. Some examples include semi-Lagrangian methods [6,17,18,20], continuous finite element methods [24,25], and methods based on Fourier transform [12,13,16].

In reference [5], one of the authors and her collaborators proposed and analyzed semi-discrete discontinuous Galerkin (DG) methods for the VM system. The methods were further combined with Runge–Kutta time discretizations, resulting in Runge–Kutta discontinuous Galerkin (RKDG) methods, and their performance in accuracy, stability, and conservation was demonstrated numerically. Note that DG methods were previously proposed and studied in references [1,2,4,14] for the Vlasov–Poisson system. DG discretizations are chosen for the phase domain in reference [5] due to their high accuracy, compactness, high efficiency in parallel implementations, flexibility with complicated geometry as well as boundary conditions and adaptive simulations. All aforementioned properties make the methods a competitive candidate to simulate the VM system accurately with reasonable computational cost especially for lower dimensional cases (*e.g.* the 1D2V or 2D2V system). This is even so if one further makes good use of the modern computer architectures. More importantly, with properly designed numerical fluxes and up to some boundary effect, semi-discrete DG methods have *provable* conservation property of both mass and total energy, and this is shared by very few high order methods. On the other hand, though RKDG methods have been widely used in many applications since they were introduced [8,9], theoretical analysis for such fully discrete methods is relatively little. Error estimations based on Fourier analysis and some symbolic computations were carried out in references [23,29] when RKDG methods are applied to linear problems on uniform meshes. Important developments were made by Zhang and Shu in references [26–28], where error estimates to the smooth solutions on uniform or non-uniform meshes were developed for RKDG methods with the second order RK time discretization for scaler [26] and symmetrizable conservation laws [27], and with the third order RK method for scaler conservations laws in reference [28]. In reference [28], L^2-norm stability was also obtained for linear conservation laws, and such analysis so far is unavailable for nonlinear cases. In this paper, we use the idea in reference [28] to obtain the error estimates of the fully discrete RKDG methods for the VM system when the exact solutions have sufficient regularity. In particular, a third order TVD Runge–Kutta method [19], a commonly used Runge–Kutta method in RKDG schemes, is considered as time discretization together with DG discretizations of arbitrary accuracy in phase domain. The second order Runge–Kutta time discretization analyzed in reference [26] is not examined here partially due to their restrictive condition on timestep when the spatial accuracy is higher than second order [26,27]. Our analysis is based on Taylor expansion, energy analysis, some techniques for analyzing the semi-discrete DG methods of the VM system [5] and for analyzing the third order Runge–Kutta time discretization within the method of lines framework. To treat the nonlinearity due to the nonlinear coupling of the Vlasov and Maxwell parts, the polynomial degree k is required to satisfy $k>\frac{d_x+1}{2}$. In addition, *a priori* assumption is made for the L^∞ error of the electric and magnetic fields. This assumption will be proved later by mathematical induction. Even though the Gauss's laws are not considered in formulating the numerical methods, they are shown to be approximated accurately when the exact solutions have sufficient regularty.

The remaining of this paper is organized as follows. In Section 2, the formulations of RKDG methods are presented for the VM system. We also introduce notations and review some standard approximation results and inverse inequalities in finite element methods. In Section 3, error estimates are established for the RKDG methods. Here we start with the error equations and energy equations. Based on these equations, the errors from the Vlasov and Maxwell parts are estimated and then combined. To better present the analysis, the proofs of some lemmas are given in Section 4. Finally, we summarize and generalize our work in Section 5.

2. Formulation of Runge–Kutta discontinuous Galerkin methods

In this section, we will introduce notations, review some standard approximation results and inverse inequalities, and present the formulation of Runge–Kutta discontinuous Galerkin methods for the Vlasov–Maxwell system (1.1).

2.1. Some preliminaries

Throughout this paper, standard notations are used for Sobolev spaces and norms: for a bounded domain D and any nonnegative integer m, we denote the L^2-Sobolev space of order m by $H^m(D)$ equipped with Sobolev norm $||\cdot||_{m,D}$, and the L^∞-Sobolev space of order m by $W^{m,\infty}(D)$ with the Sobolev norm $||\cdot||_{m,\infty,D}$. When $m=0$, $L^2(D)$ is used instead of $H^0(D)$, so is $L^\infty(D)$ instead of $W^{0,\infty}(D)$. For the brevity of notation, we use $\star = x$ or v in this subsection. For the computational domain $\Omega=\Omega_x\times\Omega_v$, assume $T_h^\star=\{K_\star\}$ is a partition of $\Omega_\star$, with $K_\star$ being a (rotated) Cartesian element or a simplex, then $T_h=\{K: K=K_x\times K_v, \forall K_x\in T_h^x, \forall K_v\in T_h^v\}$ defines a partition of Ω. Let $\mathcal{E}_\star$ be the set of the edges of $T_h^\star$, then the edges of T_h will be $\mathcal{E}=\{T_h^x\times\mathcal{E}_v\}\cup\{\mathcal{E}_x\times T_h^v\}$. In addition, let $\mathcal{E}_v=\mathcal{E}_v^i\cup\mathcal{E}_v^b$ with $\mathcal{E}_v^i$ (resp. $\mathcal{E}_v^b$) consisting of all interior (resp. boundary) edges of T_h^v. The mesh size of T_h is denoted as $h=\max(h_x,h_v)=\max_{K\in T_h} h_K$, where $h_\star=\max_{K_\star\in T_h^\star} h_{K_\star}$ with $h_{K_\star}=\mathrm{diam}(K_\star)$, and $h_K=\max(h_{K_x},h_{K_v})$ with $K=K_x\times K_v$. When the mesh is refined, we assume both $\frac{h_x}{h_{v,\min}}:=\frac{h_x}{\min_{K_v\in T_h^v} h_{K_v}}$ and $\frac{h_v}{h_{x,\min}}:=\frac{h_v}{\min_{K_x\in T_h^x} h_{K_x}}$ are uniformly bounded above by a positive constant σ_0. Therefore in our analysis we do not always distinguish h, h_x, h_v, h_{K_x} and h_{K_v}. It is further assumed that $\{T_h^\star\}_h$ is shape-regular. That is, if $\rho_{K_\star}$ denotes the diameter of the largest sphere included in $K_\star$, there is $\frac{h_{K_\star}}{\rho_{K_\star}}\le\sigma_\star$, $\forall K_\star\in\mathcal{T}_h^\star$, for a positive constant $\sigma_\star$ independent of $h_\star$.

Next we introduce two finite dimensional discrete spaces

$$\mathcal{G}_h^k=\left\{g\in L^2(\Omega): g|_K\in P^k(K), \forall K\in T_h\right\},$$
$$\mathcal{U}_h^k=\left\{U\in\left[L^2(\Omega_x)\right]^3: U|_{K_x}\in\left[P^k(K_x)\right]^3, \forall K_x\in T_h^x\right\},$$

where $P^k(D)$ is the set of polynomials of the *total* degree at most k on D, with k being any nonnegative integer. Note that functions in $\mathcal{G}_h^k$ (resp. $\mathcal{U}_h^k$) are piecewise defined with respect to T_h (resp. T_h^x). For such function, we would need the notations of jumps and averages. Given an edge $e=(K_x^+\cap K_x^-)\in\mathcal{E}_x$ with $n_x^\pm$ as the outward unit normal vector of $K_x^\pm$, for any $g\in\mathcal{G}_h^k$ and $U\in\mathcal{U}_h^k$ with $g^\pm=g|_{K_x^\pm}$ and $U^\pm=U|_{K_x^\pm}$, the averages of g and U across e are:

$$\{g\}_x=\frac{1}{2}\left(g^++g^-\right),\quad \{U\}_x=\frac{1}{2}\left(U^++U^-\right),$$

and the jumps are:

$$[g]_x=g^+n_x^++g^-n_x^-,\quad [U]_x=U^+\cdot n_x^++U^-\cdot n_x^-,\quad [U]_{\tan}=U^+\times n_x^++U^-\times n_x^-.$$

The averages and jumps across any interior edge $e=(K_v^+\cap K_v^-)\in\mathcal{E}_v^i$ can be defined similarly. For a boundary edge in $\mathcal{E}_v^b$ with n_v being the outward unit normal vector, we set $[g]_v=gn_v$ and $\{g\}_v=\frac{1}{2}g$. This is consistent with the exact solution f being compactly supported in Ω_v. Below are some equalities which will be frequently used in the analysis and can be easily verified,

$$\frac{1}{2}[g^2]_\star = \{g\}_\star [g]_\star, \tag{2.3a}$$

$$[g_1 g_2]_x - \{g_1\}_x [g_2]_x - \{g_2\}_x [g_1]_x = 0, \tag{2.3b}$$

$$[U \times V]_x + \{V\}_x \cdot [U]_{\tan} - \{U\}_x \cdot [V]_{\tan} = 0, \tag{2.3c}$$

where $g, g_1, g_2 \in \mathcal{G}_h^k$, and $U, V \in \mathcal{U}_h^k$. We also introduce some shorthand notations, $\int_\Omega = \int_{T_h} = \sum_{K\in T_h} \int_K$, $\int_{\Omega_\star} = \int_{T_h^\star} = \sum_{K_\star \in T_h^\star} \int_{K_\star}$, $\int_{\mathcal{E}_\star} = \sum_{e\in\mathcal{E}_\star} \int_e$. Additionally, $||g||_{0,\mathcal{E}} = (||g||^2_{0,\mathcal{E}_x\times T_h^v} + ||g||^2_{0,T_h^x\times\mathcal{E}_v})^{\frac{1}{2}}$ with $||g||_{0,\mathcal{E}_x\times T_h^v} = \left(\int_{\mathcal{E}_x}\int_{T_h^v} g^2 \mathrm{d}v\mathrm{d}s_x\right)^{\frac{1}{2}}$, $||g||_{0,T_h^x\times\mathcal{E}_v} = \left(\int_{T_h^x}\int_{\mathcal{E}_v} g^2 \mathrm{d}s_v\mathrm{d}x\right)^{\frac{1}{2}}$, and $||g||_{0,\mathcal{E}_x} = (\int_{\mathcal{E}_x} g^2 \mathrm{d}s_x)^{\frac{1}{2}}$.

Let Π^k denote the L^2 projection onto $\mathcal{G}_h^k$, and Π_x^k be the L^2 projection onto $\mathcal{U}_h^k$. In this paper, the following approximation properties in (2.4) and inverse inequalities in (2.5) will be used: there exists a constant $C > 0$, such that $\forall g \in H^{k+1}(\Omega)$, $\forall U \in [H^{k+1}(\Omega_x)]^3$,

$$\begin{cases} ||g-\Pi^k g||_{0,K} + h_K^{\frac{1}{2}}||g-\Pi^k g||_{0,\partial K} \le C h_K^{k+1}||g||_{k+1,K}, \quad \forall K\in T_h, \\ ||U-\Pi_x^k U||_{0,K_x} + h_{K_x}||U-\Pi_x^k U||_{1,K_x} + h_{K_x}^{\frac{1}{2}}||U-\Pi_x^k U||_{0,\partial K_x} \le C h_{K_x}^{k+1}||U||_{k+1,K_x}, \quad \forall K_x \in T_h^x, \\ ||U-\Pi_x^k U||_{0,\infty,K_x} \le C h_{K_x}^{k+1}||U||_{k+1,\infty,K_x}, \forall K_x\in T_h^x. \end{cases} \tag{2.4}$$

In addition, there exists a constant $C > 0$, such that $\forall g \in P^k(K)$, $\forall U \in [P^k(K_x)]^3$,

$$\begin{cases} ||\nabla_x g||_{0,K} \le C h_{K_x}^{-1}||g||_{0,K}, \quad ||\nabla_v g||_{0,K} \le C h_{K_v}^{-1}||g||_{0,K}, \\ ||U||_{0,\infty,K_x} \le C h_{K_x}^{-\frac{d_x}{2}}||U||_{0,K_x}, \quad ||U||_{0,\partial K_x} \le C h_{K_x}^{-\frac{1}{2}}||U||_{0,K_x}. \end{cases} \tag{2.5}$$

Each positive constant C in (2.4) and (2.5) is independent of the mesh sizes h_{K_x} and h_{K_v}, and it depends on k and the shape regularity parameters σ_x and (or) σ_v of the mesh. One can refer to [7] for more details of such standard results.

Throughout the paper, τ is used to denote the time step and $t^n = n\tau$. Without loss of generality, we assume the time steps are uniform and τ, $h \le 1$. The analysis in this paper also holds for non-uniform time steps. Even though the numerical methods and error analysis will be presented when both the (exact and approximated) electric and magnetic fields have three components, they can be easily adapted to reduced VM equations, such as the one to study Weibel instability in [5] when $d_x = 1$ and $d_v = 2$, where some components of E and B vanish and need not be approximated numerically.

2.2. Runge–Kutta discontinuous Galerkin methods

Now we are ready to present the RKDG methods for the VM system, where upwind DG methods of arbitrary accuracy are used as the spatial and phase discretization and a third order TVD Runge–Kutta method [19] is used as the time discretization. Note that on the PDE level, the two Gauss's laws in (1.1) involving the divergence of the magnetic and electric fields can be derived from the remaining equations of the VM system as long as they are satisfied by the initial data, these equations will not be discretized numerically just as in reference [5]. For sufficiently smooth solutions as considered in this work, the divergence equations can be approximated accurately by the proposed methods (see Thm. 3.1). One should be aware that for general cases, imposing divergence equations in numerical simulations can be important. To initialize the simulation, let $f_h^0 = \Pi^k f_0$, $E_h^0 = \Pi_x^k E_0$ and $B_h^0 = \Pi_x^k B_0$, where f_0, E_0 and B_0 are the initial data of the VM system. Then for $n \ge 0$, the approximate solutions at time $t^{n+1} = (n+1)\tau$ are defined as follows. We look for $f_h^{n,1}, f_h^{n,2}, f_h^{n+1} \in \mathcal{G}_h^k$, and

$E_h^{n,1}, E_h^{n,2}, E_h^{n+1}, B_h^{n,1}, B_h^{n,2}, B_h^{n+1} \in \mathcal{U}_h^k$ satisfying

$$\begin{aligned}
\left(f_h^{n,1}, g\right)_\Omega &= (f_h^n, g)_\Omega + \tau a_h\left(f_h^n, E_h^n, B_h^n; g\right), \qquad (2.6a)\\
\left(E_h^{n,1}, U\right)_{\Omega_x} + \left(B_h^{n,1}, V\right)_{\Omega_x} &= (E_h^n, U)_{\Omega_x} + (B_h^n, V)_{\Omega_x} + \tau b_h\left(E_h^n, B_h^n, f_h^n; U, V\right),\\
\left(f_h^{n,2}, g\right)_\Omega &= \left(\frac{3}{4}f_h^n + \frac{1}{4}f_h^{n,1}, g\right)_\Omega + \frac{\tau}{4}a_h\left(f_h^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right), \qquad (2.6b)\\
\left(E_h^{n,2}, U\right)_{\Omega_x} + \left(B_h^{n,2}, V\right)_{\Omega_x} &= \left(\frac{3}{4}E_h^n + \frac{1}{4}E_h^{n,1}, U\right)_{\Omega_x} + \left(\frac{3}{4}B_h^n + \frac{1}{4}B_h^{n,1}, V\right)_{\Omega_x} + \frac{\tau}{4}b_h\left(E_h^{n,1}, B_h^{n,1}, f_h^{n,1}; U, V\right),\\
\left(f_h^{n+1}, g\right)_\Omega &= \left(\frac{1}{3}f_h^n + \frac{2}{3}f_h^{n,2}, g\right)_\Omega + \frac{2\tau}{3}a_h\left(f_h^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right), \qquad (2.6c)\\
\left(E_h^{n+1}, U\right)_{\Omega_x} + \left(B_h^{n+1}, V\right)_{\Omega_x} &= \left(\frac{1}{3}E_h^n + \frac{2}{3}E_h^{n,2}, U\right)_{\Omega_x} + \left(\frac{1}{3}B_h^n + \frac{2}{3}B_h^{n,2}, V\right)_{\Omega_x} + \frac{2\tau}{3}b_h\left(E_h^{n,2}, B_h^{n,2}, f_h^{n,2}; U, V\right),
\end{aligned}$$

for any $g \in \mathcal{G}_h^k$ and $U, V \in \mathcal{U}_h^k$, where

$$\begin{aligned}
a_h(f_h, E_h, B_h; g) &= \int_\Omega f_h v \cdot \nabla_x g + f_h(E_h + v \times B_h) \cdot \nabla_v g \mathrm{d}x\mathrm{d}v\\
&\quad - \sum_{K = K_x \times K_v \in T_h}\left(\int_{K_v}\int_{\partial K_x} \widehat{f_h v \cdot n_x} g \mathrm{d}s_x \mathrm{d}v + \int_{K_x}\int_{\partial K_v} \widehat{f_h(E_h + v \times B_h) \cdot n_v} g \mathrm{d}s_v \mathrm{d}x\right),\\
b_h(E_h, B_h, f_h; U, V) &= \int_{\Omega_x} B_h \cdot \nabla \times U - E_h \cdot \nabla \times V \mathrm{d}x + \sum_{K_x \in T_h^x}\int_{\partial K_x}\left(\widehat{n_x \times B_h} \cdot U - \widehat{n_x \times E_h} \cdot V\right)\mathrm{d}s_x\\
&\quad - \int_{\Omega_x} J_h \cdot U \mathrm{d}x, \quad J_h(x,t) = \int_{\Omega_v} f_h(x,v,t) v \mathrm{d}v.
\end{aligned}$$

Here n_x and n_v are outward unit normal vectors of ∂K_x and ∂K_v, respectively. All the hat functions are upwinding numerical fluxes defined as

$$\begin{aligned}
\widehat{f_h v \cdot n_x} &= \left(\{f_h v\}_x + \frac{|v \cdot n_x|}{2}[f_h]_x\right) \cdot n_x,\\
f_h\left(E_h \widehat{+ v \times} B_h\right) \cdot n_v &= \left(\{f_h(E_h + v \times B_h)\}_v + \frac{|(E_h + v \times B_h) \cdot n_v|}{2}[f_h]_v\right) \cdot n_v,\\
\widehat{n_x \times E_h} &= n_x \times \left(\{E_h\}_x + \frac{1}{2}[B_h]_{\tan}\right), \quad \widehat{n_x \times B_h} = n_x \times \left(\{B_h\}_x - \frac{1}{2}[E_h]_{\tan}\right),
\end{aligned}$$

and they further specify $a_h(f_h, E_h, B_h; g) = a_{h,1}(f_h; g) + a_{h,2}(f_h, E_h, B_h; g)$ with

$$\begin{aligned}
a_{h,1}(f_h; g) &= \int_\Omega f_h v \cdot \nabla_x g \mathrm{d}x\mathrm{d}v - \int_{T_h^v}\int_{\mathcal{E}_x}\left(\{f_h v\}_x + \frac{|v \cdot n_x|}{2}[f_h]_x\right) \cdot [g]_x \mathrm{d}s_x \mathrm{d}v,\\
a_{h,2}(f_h, E_h, B_h; g) &= \int_\Omega f_h(E_h + v \times B_h) \cdot \nabla_v g \mathrm{d}x\mathrm{d}v\\
&\quad - \int_{T_h^x}\int_{\mathcal{E}_v}\left(\{f_h(E_h + v \times B_h)\}_v + \frac{|(E_h + v \times B_h) \cdot n_v|}{2}[f_h]_v\right) \cdot [g]_v \mathrm{d}s_v \mathrm{d}x,
\end{aligned}$$

and

$$b_h(E_h, B_h, f_h; U, V) = \int_{\Omega_x} (B_h \cdot \nabla \times U - E_h \cdot \nabla \times V)\,\mathrm{d}x - \int_{\Omega_x} J_h U \mathrm{d}x$$
$$+ \int_{\mathcal{E}_x} \left(\{B_h\}_x - \frac{1}{2}[E_h]_{\tan}\right) \cdot [U]_{\tan} - \left(\{E_h\}_x + \frac{1}{2}[B_h]_{\tan}\right) \cdot [V]_{\tan} \mathrm{d}s_x.$$

Note that both $a_{h,1}$ and b_h are linear with respect to each argument, yet $a_{h,2}(f_h, E_h, B_h; g)$ is linear with respect to f_h and g only. The overall RKDG methods are consistent, and this will be used to derive the error equations in next section.

3. Error estimates

This section is devoted to the main result of the paper, which is given in Theorem 3.1. More specifically, we will establish error estimates at any given time $T > 0$ for the fully discrete RKDG methods in Section 2.2 when they are used to solve sufficiently smooth solutions. The analysis of the present work is based on the following assumptions on the regularity of the exact solutions.

Regularity Assumption

(a.1) $f(\cdot,\cdot,t), E(\cdot,t), B(\cdot,t) \in C^4([0,T])$;

(a.2) $\partial_t^4 f(\cdot,\cdot,t) \in L^2(\Omega)$, $\partial_t^4 E(\cdot,t), \partial_t^4 B(\cdot,t) \in L^2(\Omega_x)$ uniformly in $t \in [0,T]$;

(a.3) $(\partial_t^s E + v\times\partial_t^s B)\nabla_v((\partial_t E + v\times\partial_t B)\cdot\nabla_v(\partial_t f))$, $\nabla_x((\partial_t E + v\times\partial_t B)\cdot\nabla_v(\partial_t f))$, $\partial_t((\partial_t E + v\times\partial_t B)\partial_t(\nabla_v f))$, $(\partial_t^2 E + v\times\partial_t^2 B)\partial_t^2(\nabla_v f) \in L^2(\Omega)$, $\partial_t^s E, \partial_t^s B \in W^{1,\infty}(\Omega_x)$, $\nabla_v(\partial_t^s f), \nabla_v((\partial_t E + v\times\partial_t B)\cdot\nabla_v(\partial_t f)) \in L^\infty(\Omega)$ uniformly in t for $s = 0,1,2$;

(a.4) $\partial_t^s E(\cdot,t), \partial_t^s B(\cdot,t) \in H^{k+1}(\Omega_x), \partial_t^s f(\cdot,\cdot,t) \in H^{k+1}(\Omega)$, $(\partial_t E + v\times\partial_t B)\cdot\nabla_v(\partial_t f) \in H^{k+1}(\Omega)$ uniformly in $t \in [0,T]$, $s = 0,1,2$. These, together with equations (3.8), will ensure $f^{n,s} \in H^{k+1}(\Omega)$, $E^{n,s}$, $B^{n,s} \in H^{k+1}(\Omega_x)$ uniformly in n for $s = 0,1,2$.

The assumption (a.3) can be replaced by the following,

(a.3′) All the mixed partial derivatives (in t, x, and v) of E and B up to the second order, in addition to $\partial_t^3 E$ and $\partial_t^3 B$, are in $L^\infty(\Omega_x)$ uniformly in $t \in [0,T]$; $\nabla_v(\partial_t^s f), s = 0,1,2$ and $\nabla_v(\nabla_v \partial_t f)$ are in $L^\infty(\Omega)$ uniformly in $t \in [0,T]$; All the mixed partial derivatives (in t, x, and v) of f up to the third order are in $L^2(\Omega)$ uniformly in $t \in [0,T]$.

The assumption on L^2 norms of functions together with (a.1) is for the third order accuracy in time discretization, while (a.4) is for the accuracy in the spatial and/or phase domain. The assumption on L^∞ norms of functions is used to deal with the nonlinear coupling in the Vlasov part.

Unless otherwise specified, C is used to denote a generic positive constant, and it can take different values at different occurrences. This constant is independent of n, h, τ, and may depend on polynomial degree k, mesh parameter $\sigma_0, \sigma_x, \sigma_v$, domain parameters $L_{x,i}$, $L_{v,i}$, $i = 1,2$, and the time T. It may also depend on the exact solution in the form of its certain Sobolev norms or semi-norms. The constant γ_1 in Theorem 3.10, γ_2 in Theorem 3.16, and γ in Theorem 3.1 have similar dependence as the generic constant C. For convenience, we do not distinguish the upper indices $n,0$ and n. For instance, we regard $g^{n,0} = g^n$ for any function g. With the analysis being very technical, to make it easier to follow, the proofs of some lemmas are given later in Section 4.

Theorem 3.1. *Let (f, E, B) be a sufficiently smooth exact solution to the VM system (1.1). Let $(f_h^n, E_h^n, B_h^n) \in \mathcal{G}_h^k \times \mathcal{U}_h^k \times \mathcal{U}_h^k$ be the solution to the scheme (2.6) at time t^n with $k > \frac{d_x+1}{2}$. Under a CFL condition $\tau \leq \gamma h$, with $h < h_0$ for some $h_0 > 0$, we have*

$$||f(\cdot,\cdot,t^n) - f_h^n||_{0,\Omega}^2 + ||E(\cdot,t^n) - E_h^n||_{0,\Omega_x}^2 + ||B(\cdot,t^n) - B_h^n||_{0,\Omega_x}^2 \leq Ch^{2k+1} + C\tau^6$$

for any $n \le T/\tau$. *In addition,*

$$||E(\cdot,t^m) - E_h^m||_{0,\infty,\Omega_x} \le Ch, \quad ||B(\cdot,t^m) - B_h^m||_{0,\infty,\Omega_x} \le Ch, \qquad \forall m+1 \le T/\tau,$$

and the Gauss's laws in (1.1) *are approximated as below for any* $n \le T/\tau$,

$$||\nabla\cdot E_h^n - \rho_h^n + \rho_i^n||_{0,\Omega_x} \le C\frac{1}{h_x}\left(h^{k+\frac12}+\tau^3\right), \quad ||\nabla\cdot B_h^n||_{0,\Omega_x} \le C\frac{1}{h_x}\left(h^{k+\frac12}+\tau^3\right).$$

Here $\rho_h^n(\cdot) = \int_{\Omega_v} f_h^n(\cdot,v)\mathrm{d}v$, *and* $\rho_i^n(\cdot) = \rho_i(\cdot,t^n)$.

3.1. Error equations

Let $(f(x,v,t^n), E(x,t^n), B(x,t^n))$ be the exact solution to the system (1.1) and (1.2) at time $t^n = n\tau$. We denote $f^n(x,v) = f(x,v,t^n)$, $E^n(x) = E(x,t^n)$, $B^n(x) = B(x,t^n)$, $J^n(x) = \int_{\Omega_v} f^n(x,v)v\mathrm{d}v$, and define

$$\begin{cases} f^{n,1} = f^n - \tau\left(v\cdot\nabla_x f^n + (E^n + v\times B^n)\cdot\nabla_v f^n\right), \\ E^{n,1} = E^n + \tau\left(\nabla\times B^n - J^n\right), \quad B^{n,1} = B^n - \tau\left(\nabla\times E^n\right), \\ f^{n,2} = \dfrac{3}{4}f^n + \dfrac{1}{4}f^{n,1} - \dfrac{\tau}{4}\left(v\cdot\nabla_x f^{n,1} + \left(E^{n,1}+v\times B^{n,1}\right)\cdot\nabla_v f^{n,1}\right), \\ J^{n,1} = \displaystyle\int_{\Omega_v} f^{n,1}v\mathrm{d}v, \quad J^{n,2} = \int_{\Omega_v} f^{n,2}v\mathrm{d}v, \\ E^{n,2} = \dfrac{3}{4}E^n + \dfrac{1}{4}E^{n,1} + \dfrac{\tau}{4}\left(\nabla\times B^{n,1} - J^{n,1}\right), \quad B^{n,2} = \dfrac{3}{4}B^n + \dfrac{1}{4}B^{n,1} - \dfrac{\tau}{4}\left(\nabla\times E^{n,1}\right). \end{cases} \tag{3.7}$$

Equations in (3.7) are obtained by applying one step of the third order Runge–Kutta time discretization in reference [19] to the VM system (without the divergence conditions) from $t = t^n$ with the exact solution as the initial data at t_n. From (1.1) and (3.7), one can further represent $f^{n,\sharp}$, $E^{n,\sharp}$ and $B^{n,\sharp}$ ($\sharp = 1,2$) in terms of f^n, E^n, B^n and their derivatives as below,

$$\begin{cases} f^{n,1} = f^n + \tau\partial_t f^n, \quad E^{n,1} = E^n + \tau\partial_t E^n, \quad B^{n,1} = B^n + \tau\partial_t B^n, \\ f^{n,2} = f^n + \dfrac{\tau}{2}\partial_t f^n + \dfrac{\tau^2}{4}\partial_t^2 f^n - \dfrac{\tau^3}{4}\left(\partial_t E^n + v\times\partial_t B^n\right)\cdot\nabla_v\left(\partial_t f^n\right), \\ E^{n,2} = E^n + \dfrac{\tau}{2}\partial_t E^n + \dfrac{\tau^2}{4}\partial_t^2 E^n, \quad B^{n,2} = B^n + \dfrac{\tau}{2}\partial_t B^n + \dfrac{\tau^2}{4}\partial_t^2 B^n. \end{cases} \tag{3.8}$$

In the next lemma, local truncation errors from each step of the Runge–Kutta time discretization are given. The results can be verified straightforwardly based on (3.8) and Taylor expansion, and the proof is omitted.

Lemma 3.2. *If we define*

$$\begin{aligned} f\left(x,v,t^{n+1}\right) &= \frac{1}{3}f^n + \frac{2}{3}f^{n,2} - \frac{2\tau}{3}\left(v\cdot\nabla_x f^{n,2} + \left(E^{n,2}+v\times B^{n,2}\right)\cdot\nabla_v f^{n,2}\right) + T_f^n(x,v), \\ E\left(x,t^{n+1}\right) &= \frac{1}{3}E^n + \frac{2}{3}E^{n,2} + \frac{2\tau}{3}\left(\nabla\times B^{n,2} - J^{n,2}\right) + T_E^n(x), \\ B\left(x,t^{n+1}\right) &= \frac{1}{3}B^n + \frac{2}{3}B^{n,2} - \frac{2\tau}{3}\left(\nabla\times E^{n,2}\right) + T_B^n(x), \end{aligned} \tag{3.9}$$

where $T_f^n(x,v), T_B^n(x), T_E^n(x)$ *are the local truncation errors in the n-th time step* $\forall n : (n+1)\tau \le T$, *then*

$$\left\|T_f^n\right\|_{0,\Omega}, \|T_E^n\|_{0,\Omega_x}, \|T_B^n\|_{0,\Omega_x} \le C\tau^4.$$

For each stage of the Runge–Kutta method, we denote the error by $e_f^{n,\sharp} = f^{n,\sharp} - f_h^{n,\sharp} = \xi_f^{n,\sharp} - \eta_f^{n,\sharp}$, where $\xi_f^{n,\sharp} = \Pi^k f^{n,\sharp} - f_h^{n,\sharp}$ and $\eta_f^{n,\sharp} = \Pi^k f^{n,\sharp} - f^{n,\sharp}$, for $\sharp = 0,1,2$. Given that $\eta_f^{n,\sharp}$ can be estimated in a standard way from the approximation results in (2.4) of the discrete spaces, our error analysis will focus on the term $\xi_f^{n,\sharp}$, which is also called the projected error due to $\xi_f^{n,\sharp} = \Pi^k e_f^{n,\sharp}$. Similar comments and conventions on notation go to $e_E^{n,\sharp}, e_B^{n,\sharp}, \xi_E^{n,\sharp}, \xi_B^{n,\sharp}, \eta_E^{n,\sharp}$ and $\eta_B^{n,\sharp}$. Next we multiply an arbitrary test function $g \in \mathcal{G}_h^k$ (resp. $U, V \in \mathcal{U}_h^k$) on both sides of the equations corresponding to the Vlasov equation (resp. Maxwell equations) in (3.7) and (3.9), integrate over each mesh element K (resp. K_x), take integration by parts, and sum up with respect to all $K \in T_h$ (or all $K_x \in T_h^x$). We then subtract (2.6a)–(2.6c) from the resulting equations, and reach the error equations,

$$\begin{cases} \left(\xi_f^{n,1}, g\right)_\Omega = \left(\xi_f^n, g\right)_\Omega + \tau \mathcal{J}(g), \\ \left(\xi_f^{n,2}, g\right)_\Omega = \left(\frac{3}{4}\xi_f^n + \frac{1}{4}\xi_f^{n,1}, g\right)_\Omega + \frac{\tau}{4}\mathcal{K}(g), \\ \left(\xi_f^{n+1}, g\right)_\Omega = \left(\frac{1}{3}\xi_f^n + \frac{2}{3}\xi_f^{n,2}, g\right)_\Omega + \frac{2\tau}{3}\mathcal{L}(g), \end{cases} \tag{3.10}$$

$$\begin{cases} \left(\xi_E^{n,1}, U\right)_{\Omega_x} + \left(\xi_B^{n,1}, V\right)_{\Omega_x} = \left(\xi_E^n, U\right)_{\Omega_x} + \left(\xi_B^n, V\right)_{\Omega_x} + \tau \mathcal{Q}(U,V), \\ \left(\xi_E^{n,2}, U\right)_{\Omega_x} + \left(\xi_B^{n,2}, V\right)_{\Omega_x} = \left(\frac{3}{4}\xi_E^n + \frac{1}{4}\xi_E^{n,1}, U\right)_{\Omega_x} + \left(\frac{3}{4}\xi_B^n + \frac{1}{4}\xi_B^{n,1}, V\right)_{\Omega_x} + \frac{\tau}{4}\mathcal{R}(U,V), \\ \left(\xi_E^{n+1}, U\right)_{\Omega_x} + \left(\xi_B^{n+1}, V\right)_{\Omega_x} = \left(\frac{1}{3}\xi_E^n + \frac{2}{3}\xi_E^{n,2}, U\right)_{\Omega_x} + \left(\frac{1}{3}\xi_B^n + \frac{2}{3}\xi_B^{n,2}, V\right)_{\Omega_x} + \frac{2\tau}{3}\mathcal{S}(U,V). \end{cases} \tag{3.11}$$

Here

$$\begin{aligned} \mathcal{J}(g) &= \left(\frac{\eta_f^{n,1} - \eta_f^n}{\tau}, g\right)_\Omega + a_h\left(f^n, E^n, B^n; g\right) - a_h\left(f_h^n, E_h^n, B_h^n; g\right), \\ \mathcal{K}(g) &= \left(\frac{4\eta_f^{n,2} - 3\eta_f^n - \eta_f^{n,1}}{\tau}, g\right)_\Omega + a_h\left(f^{n,1}, E^{n,1}, B^{n,1}; g\right) - a_h\left(f_h^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right), \\ \mathcal{L}(g) &= \left(\frac{3\eta_f^{n+1} - \eta_f^n - 2\eta_f^{n,2} + 3T_f^n(x,v)}{2\tau}, g\right)_\Omega + a_h\left(f^{n,2}, E^{n,2}, B^{n,2}; g\right) - a_h\left(f_h^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right), \end{aligned}$$

$$\begin{aligned} \mathcal{Q}(U,V) &= \left(\frac{\eta_E^{n,1} - \eta_E^n}{\tau}, U\right)_{\Omega_x} + \left(\frac{\eta_B^{n,1} - \eta_B^n}{\tau}, V\right)_{\Omega_x} + b_h\left(e_E^n, e_B^n, e_f^n; U, V\right), \\ \mathcal{R}(U,V) &= \left(\frac{4\eta_E^{n,2} - 3\eta_E^n - \eta_E^{n,1}}{\tau}, U\right)_{\Omega_x} + \left(\frac{4\eta_B^{n,2} - 3\eta_B^n - \eta_B^{n,1}}{\tau}, V\right)_{\Omega_x} + b_h\left(e_E^{n,1}, e_B^{n,1}, e_f^{n,1}; U, V\right), \\ \mathcal{S}(U,V) &= \left(\frac{3\eta_E^{n+1} - \eta_E^n - 2\eta_E^{n,2} + 3T_E^n(x)}{2\tau}, U\right)_{\Omega_x} + \left(\frac{3\eta_B^{n+1} - \eta_B^n - 2\eta_B^{n,2} + 3T_B^n(x)}{2\tau}, V\right)_{\Omega_x} \\ &\quad + b_h\left(e_E^{n,2}, e_B^{n,2}, e_f^{n,2}; U, V\right), \end{aligned}$$

with any test functions $g \in \mathcal{G}_h^k$ and $U, V \in \mathcal{U}_h^k$. For the functional $\mathcal{J}(\cdot)$, we denote $\mathcal{J}_1(g) = \left(\frac{\eta_f^{n,1} - \eta_f^n}{\tau}, g\right)_\Omega$ and $\mathcal{J}_2(g) = a_h(f^n, E^n, B^n; g) - a_h(f_h^n, E_h^n, B_h^n; g)$. Similarly, one can define $\mathcal{K}_\sharp$, $\mathcal{L}_\sharp$, $\mathcal{Q}_\sharp$, $\mathcal{R}_\sharp$, $\mathcal{S}_\sharp$, $\sharp = 1,2$.

We now take the test function $g = \xi_f^n, 4\xi_f^{n,1}$ and $6\xi_f^{n,2}$ in each equation of (3.10), respectively, sum them up and obtain the energy equations

$$3\left\|\xi_f^{n+1}\right\|_{0,\Omega}^2 - 3\left\|\xi_f^n\right\|_{0,\Omega}^2 = \tau\left[\mathcal{J}\left(\xi_f^n\right) + \mathcal{K}\left(\xi_f^{n,1}\right) + 4\mathcal{L}\left(\xi_f^{n,2}\right)\right] + \left\|2\xi_f^{n,2} - \xi_f^{n,1} - \xi_f^n\right\|_{0,\Omega}^2 + 3\left(\xi_f^{n+1} - \xi_f^n, \xi_f^{n+1} - 2\xi_f^{n,2} + \xi_f^n\right)_\Omega. \tag{3.12}$$

Similarly, the following equation holds

$$\begin{aligned}
&3\left(\|\xi_E^{n+1}\|_{0,\Omega_x}^2 + \|\xi_B^{n+1}\|_{0,\Omega_x}^2\right) - 3\left(\|\xi_E^n\|_{0,\Omega_x}^2 + \|\xi_B^n\|_{0,\Omega_x}^2\right)\\
&= \tau\left[\mathcal{Q}\left(\xi_E^n, \xi_B^n\right) + \mathcal{R}\left(\xi_E^{n,1}, \xi_B^{n,1}\right) + 4\mathcal{S}\left(\xi_E^{n,2}, \xi_B^{n,2}\right)\right] + \left\|2\xi_E^{n,2} - \xi_E^{n,1} - \xi_E^n\right\|_{0,\Omega_x}^2 + \left\|2\xi_B^{n,2} - \xi_B^{n,1} - \xi_B^n\right\|_{0,\Omega_x}^2\\
&+ 3\left(\xi_E^{n+1} - \xi_E^n, \xi_E^{n+1} - 2\xi_E^{n,2} + \xi_E^n\right)_{\Omega_x} + 3\left(\xi_B^{n+1} - \xi_B^n, \xi_B^{n+1} - 2\xi_B^{n,2} + \xi_B^n\right)_{\Omega_x}.
\end{aligned} \tag{3.13}$$

The main error estimate will be established based on equations (3.12)–(3.13) which describe how the L^2 norms of the projected errors are accumulated in one time step. In particular, in the next two subsections, we will estimate the errors from the Vlasov and Maxwell solvers, respectively, and the results will be combined in Section 3.4 to get the main result of this paper.

Before continuing, we will make *a priori* assumption for the L^∞ error of the magnetic and electric fields,

$\mathbf{L^\infty}$-Assumption: For any integer $n+1 \le T/\tau$, $||e_E^{n,\sharp}||_{0,\infty,\Omega_x}$, $||e_B^{n,\sharp}||_{0,\infty,\Omega_x} \le \mathcal{C}h$, with $\sharp = 0, 1, 2$, hold for small enough h, where $\mathcal{C}$ is a fixed positive constant depending only on the initial conditions.

This assumption will be used in Section 3.2 and Lemma 3.3-(2) to estimate terms with $a_{h,2}$ as this is where the nonlinear coupling of the Vlasov and Maxwell parts lies, and this assumption will eventually be established rigorously by mathematical induction in Section 3.4.

Our presentation will also benefit from the following shorthand notations,

$$\mathbf{STAB}_f^\star = \int_{\mathcal{T}_h^v}\int_{\mathcal{E}_x} |v\cdot n_x||[\xi_f^\star]_x|^2 \mathrm{d}s_x \mathrm{d}v + \int_{\mathcal{T}_h^x}\int_{\mathcal{E}_v} |(E_h^\star + v\times B_h^\star)\cdot n_v||[\xi_f^\star]_v|^2 \mathrm{d}s_v \mathrm{d}x, \tag{3.14a}$$

$$\mathbf{STAB}_{EB}^\star = \int_{\mathcal{E}_x} (|[\xi_E^\star]_{\tan}|^2 + |[\xi_B^\star]_{\tan}|^2)\mathrm{d}s_x, \quad \mathbf{N}^\star = ||\xi_f^\star||_{0,\Omega}^2 + ||\xi_E^\star||_{0,\Omega_x}^2 + ||\xi_B^\star||_{0,\Omega_x}^2. \tag{3.14b}$$

where $\star = n$ or $n,\sharp$. The terms **STAB** with different subscripts or superscripts provide stability mechanism due to the upwind phase and spatial discretizations. Later on another type of stability mechanism will emerge which is due to the temporal discretization.

In our analysis, we will frequently encounter certain linear combinations of $\eta_\diamond^n, \eta_\diamond^{n,1}, \eta_\diamond^{n,2}$ and $\eta_\diamond^{n+1}$, $\diamond = f, E, B$. In the next lemma, the estimates for such terms are summarized, with their proofs given in Section 4.

Lemma 3.3. *Let $d_\diamond^n = d_0\eta_\diamond^n + d_1\eta_\diamond^{n,1} + d_2\eta_\diamond^{n,2} + d_3\eta_\diamond^{n+1}$, $\diamond = f, E, B$, where d_0, d_1, d_2, d_3 are four constants satisfying $d_0 + d_1 + d_2 + d_3 = 0$ and independent of n, h, τ. Then for any $g \in \mathcal{G}_h^k$ and $U, V \in \mathcal{U}_h^k$, we have*

$$(1)\quad \left\|d_f^n\right\|_{0,\Omega} + h^{\frac{1}{2}}\left\|d_f^n\right\|_{0,\mathcal{E}} \le C\tau h^{k+1}, \quad \|d_\star^n\|_{0,\Omega_x} + h_x^{\frac{1}{2}}\|d_\star^n\|_{0,\mathcal{E}_x} \le C\tau h_x^{k+1},\ \star = E, B; \tag{3.15a}$$

$$(2)\quad \left|a_h\left(d_f^n, E_h^{n,s}, B_h^{n,s}; g\right)\right| \le C\frac{\tau}{h}h^{k+1}||g||_{0,\Omega} \le C\frac{\tau}{h}\left(h^{2k+2} + ||g||_{0,\Omega}^2\right),\ s = 0, 1, 2; \tag{3.15b}$$

$$(3)\quad \left|b_h\left(d_E^n, d_B^n, d_f^n; U, V\right)\right| \le C\tau h^k\left(||U||_{0,\Omega_x}^2 + ||V||_{0,\Omega_x}^2\right). \tag{3.15c}$$

3.2. The Vlasov equation part

We start with a key decomposition of the error change in one time step [28], namely, $\xi_f^{n+1}-\xi_f^n = G_1^n+G_2^n+G_3^n$, where $G_1^n = \xi_f^{n,1} - \xi_f^n, G_2^n = 2\xi_f^{n,2} - \xi_f^{n,1} - \xi_f^n$ and $G_3^n = \xi_f^{n+1} - 2\xi_f^{n,2} + \xi_f^n$. It is obvious that

$$||G_i^n||_{0,\Omega}^2 \le C \sum_{\sharp=0}^{2} ||\xi_f^{n,\sharp}||_{0,\Omega}^2, \quad i = 1,2,3. \tag{3.16}$$

From equations (3.10), one gets

$$(G_2^n, g)_\Omega = \frac{\tau}{2}(\mathcal{K}(g) - \mathcal{J}(g)) \equiv \frac{\tau}{2}\mathcal{K}_{RK}(g), \tag{3.17a}$$

$$(G_3^n, g)_\Omega = \frac{\tau}{3}(2\mathcal{L}(g) - \mathcal{K}(g) - \mathcal{J}(g)) \equiv \frac{\tau}{3}\mathcal{L}_{RK}(g), \tag{3.17b}$$

for any $g \in \mathcal{G}_h^k$. In addition, one can verify based on (3.12) that

$$3||\xi_f^{n+1}||_{0,\Omega}^2 - 3||\xi_f^n||_{0,\Omega}^2 = \Xi_1 + \Xi_2 + \Xi_3, \tag{3.18}$$

where $\Xi_i = \tau[\mathcal{J}_i\left(\xi_f^n\right) + \mathcal{K}_i\left(\xi_f^{n,1}\right) + 4\mathcal{L}_i(\xi_f^{n,2})]$, $i = 1,2$ and

$$\Xi_3 = (G_2^n, G_2^n)_\Omega + 3\left(G_1^n, G_3^n\right)_\Omega + 3\left(G_2^n, G_3^n\right)_\Omega + 3\left(G_3^n, G_3^n\right)_\Omega.$$

In particular, Ξ_2 relies on the phase space discretizations, for which some results were essentially established in the analysis of the semi-discrete DG methods for the VM system in [5]. On the other hand, Ξ_1 and Ξ_3 characterize more the contribution of the time discretization. One will see that there are two mechanisms contributing to numerical stability, one is $\mathbf{STAB}_f^\star$ ($\star$ can be n or $n,\sharp$) which comes from the phase space discretization and is also used in analyzing the semi-discrete method in [5], the other one is $||G_2^n||^2$ which comes from the third order Runge–Kutta time discretization.

We first summarize in Lemma 3.4 some estimates, which are based on the phase space discretization and are essentially available in the analysis of the semi-discrete upwind DG method in reference [5]. For completeness, the proofs are given in Section 4.

Lemma 3.4. *For $\sharp = 0,1,2$, we have*

$$(1) \quad a_h\left(\xi_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; \xi_f^{n,\sharp}\right) = -\frac{1}{2}\mathbf{STAB}_f^{n,\sharp}, \tag{3.19a}$$

$$(2) \quad a_h\left(\eta_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; \xi_f^{n,\sharp}\right) \le Ch^{2k+1} + C\mathbf{N}^{n,\sharp} + \frac{1}{16}\mathbf{STAB}_f^{n,\sharp}, \quad \text{for } k \ge \frac{d_x}{2}, \tag{3.19b}$$

$$(3) \quad \left|a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_h\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right)\right| \le C\left(||e_E^{n,\sharp}||_{0,\Omega} + C||e_B^{n,\sharp}||_{0,\Omega}\right)||g||_{0,\Omega}, \quad \forall g \in \mathcal{G}_h^k, \tag{3.19c}$$

$$\text{moreover} \quad \left|a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; \xi_f^{n,\sharp}\right) - a_h\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; \xi_f^{n,\sharp}\right)\right| \le Ch^{2k+2} + C\mathbf{N}^{n,\sharp}. \tag{3.19d}$$

Proposition 3.5. *The following estimates hold for Ξ_1 and Ξ_2,*

$$(1) \quad \Xi_1 \le C\tau\left(h^{2k+2} + \tau^6\right) + C\tau \sum_{\sharp=0}^{2} ||\xi_f^{n,\sharp}||_{0,\Omega}^2; \tag{3.20a}$$

$$(2) \quad \Xi_2 \le C\tau h^{2k+1} + C\tau \sum_{\sharp=0}^{2} \mathbf{N}^{n,\sharp} - \frac{7}{16}\tau\left(\mathbf{STAB}_f^n + \mathbf{STAB}_f^{n,1} + 4\,\mathbf{STAB}_f^{n,2}\right), \forall k \ge \frac{d_x}{2}\cdot \tag{3.20b}$$

Proof. Recall $\Xi_1 = \tau[\mathcal{J}_1(\xi_f^n) + \mathcal{K}_1(\xi_f^{n,1}) + 4\mathcal{L}_1(\xi_f^{n,2})]$, with similarity, we only estimate $\mathcal{L}_1(\xi_f^{n,2})$. Applying Cauchy–Schwarz inequality, (3.15a) in Lemma 3.3, and truncation error estimate in Lemma 3.2, we get

$$\begin{aligned}
\mathcal{L}_1\left(\xi_f^{n,2}\right) &= \left(\frac{3\eta_f^{n+1} - \eta_f^n - 2\eta_f^{n,2} + 3T_f^n(x,v)}{2\tau}, \xi_f^{n,2}\right)_\Omega \\
&\le \frac{1}{2\tau}\left(\left\|3\eta_f^{n+1} - \eta_f^n - 2\eta_f^{n,2}\right\|_{0,\Omega} + 3\left\|T_f^n\right\|_{0,\Omega}\right)\left\|\xi_f^{n,2}\right\|_{0,\Omega} \\
&\le C\left(h^{k+1} + \tau^3\right)\left\|\xi_f^{n,2}\right\|_{0,\Omega} \le C\left(h^{2k+2} + \tau^6\right) + C\left\|\xi_f^{n,2}\right\|_{0,\Omega}^2.
\end{aligned} \tag{3.21}$$

To estimate Ξ_2, due to similarity, we will only estimate $\mathcal{J}_2(\xi_f^n)$. Using the results in Lemma 3.4, one has

$$\begin{aligned}
\mathcal{J}_2\left(\xi_f^n\right) &= a_h\left(f^n, E^n, B^n; \xi_f^n\right) - a_h\left(f_h^n, E_h^n, B_h^n; \xi_f^n\right) \\
&= a_h\left(\xi_f^n, E_h^n, B_h^n; \xi_f^n\right) + \left(a_h\left(f^n, E^n, B^n; \xi_f^n\right) - a_h\left(f^n, E_h^n, B_h^n; \xi_f^n\right)\right) - a_h\left(\eta_f^n, E_h^n, B_h^n; \xi_f^n\right) \\
&\le Ch^{2k+1} + C\mathbf{N}^n - \frac{7}{16}\mathbf{STAB}_f^n.
\end{aligned}$$

□

Next we will estimate Ξ_3. One key is to use $-||G_2^n||_{0,\Omega}^2$ to control $(C\frac{\tau}{h} + C\frac{\tau^2}{h^2})||G_2^n||_{0,\Omega}^2$ under some condition on the time step τ. To make the details tractable, we first give some preparatory results.

Lemma 3.6. *For $r = 0, 1, 2$, $s = 0, 1$, and any $g \in \mathcal{G}_h^k$*

$$(1)\ \ |a_h\left(\eta_f^{n,r}, E_h^{n,s+1}, B_h^{n,s+1}; g\right) - a_h\left(\eta_f^{n,r}, E_h^{n,s}, B_h^{n,s}; g\right)| \le C\left(1 + \frac{\tau}{h}\right)h^{k+1}||g||_{0,\Omega}, \tag{3.22a}$$

$$(2)\ \ |a_h\left(\xi_f^{n,r}, E_h^{n,s+1}, B_h^{n,s+1}; g\right) - a_h\left(\xi_f^{n,r}, E_h^{n,s}, B_h^{n,s}; g\right)| \le C\left(1 + \frac{\tau}{h}\right)\left\|\xi_f^{n,r}\right\|_{0,\Omega}||g||_{0,\Omega}. \tag{3.22b}$$

Lemma 3.7. *For any $g \in \mathcal{G}_h^k$, we have*

$$\mathcal{L}_{RK}(g) \le C\left(\left(1+\frac{\tau}{h}\right)\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} + h^{k+1}\right) + \tau^3\right)||g||_{0,\Omega} + a_h\left(G_2^n, E_h^{n,1}, B_h^{n,1}; g\right) \tag{3.23a}$$

$$\le C\left(\left(1+\frac{\tau}{h}\right)\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} + h^{k+1}\right) + \tau^3 + \frac{\|G_2^n\|_{0,\Omega}}{h}\right)||g||_{0,\Omega}. \tag{3.23b}$$

$$\mathcal{K}_{RK}(g) \le C\left(1+\frac{\tau}{h}\right)\left(\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} + h^{k+1}\right)\right)||g||_{0,\Omega} + a_h\left(G_1^n, E_h^{n,1}, B_h^{n,1}; g\right). \tag{3.23c}$$

Lemma 3.8.

$$\begin{aligned}
&\left|a_h\left(G_1^n, E_h^{n,1}, B_h^{n,1}; G_2^n\right) + a_h\left(G_2^n, E_h^{n,1}, B_h^{n,1}; G_1^n\right)\right| \\
&\qquad \le C\left(1+\frac{\tau}{h}\right)\left\|\xi_f^n\right\|_{0,\Omega}^2 + \frac{C}{h}\left\|G_2^n\right\|_{0,\Omega}^2 + \frac{1}{16}\left(\mathbf{STAB}_f^n + \mathbf{STAB}_f^{n,1}\right).
\end{aligned} \tag{3.24}$$

With all the preparation in Lemmas 3.6–3.8, we are now ready to estimate Ξ_3.

Proposition 3.9.

$$\Xi_3 \leq C\left(\tau\left(1+\frac{\tau}{h}\right)+\tau^2\left(1+\frac{\tau}{h}\right)^2\right)\left(\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}+h^{2k+2}\right)+C\tau^7$$
$$+\left(-1+C\frac{\tau}{h}+C\frac{\tau^2}{h^2}\right)||G_2^n||_{0,\Omega}^2+\frac{\tau}{16}\left(\mathbf{STAB}_f^n+\mathbf{STAB}_f^{n,1}\right). \tag{3.25}$$

Proof. First note that

$$\Xi_3 = -\left\|G_2^n\right\|_{0,\Omega}^2+2\left(G_2^n,G_2^n\right)_\Omega+3\left(G_1^n,G_3^n\right)_\Omega+3\left(G_2^n,G_3^n\right)_\Omega+3\left(G_3^n,G_3^n\right)_\Omega$$
$$= -\left\|G_2^n\right\|_{0,\Omega}^2+\tau\left(\mathcal{K}_{RK}\left(G_2^n\right)+\mathcal{L}_{RK}\left(G_1^n\right)+\mathcal{L}_{RK}\left(G_2^n\right)\right)+3\left(G_3^n,G_3^n\right)_\Omega. \tag{3.26}$$

Based on (3.23a) and (3.23c),

$$\mathcal{K}_{RK}\left(G_2^n\right)+\mathcal{L}_{RK}\left(G_1^n\right) \leq a_h\left(G_2^n,E_h^{n,1},B_h^{n,1};G_1^n\right)+a_h\left(G_1^n,E_h^{n,1},B_h^{n,1};G_2^n\right)$$
$$+C\left(\left(1+\frac{\tau}{h}\right)\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}+h^{k+1}\right)+\tau^3\right)\left(\left\|G_1^n\right\|_{0,\Omega}+\left\|G_2^n\right\|_{0,\Omega}\right).$$

We now apply Lemma 3.8 to estimate the first two terms on the right, and apply Cauchy–Schwartz inequality and (3.16) to estimate the last term, and get

$$\mathcal{K}_{RK}\left(G_2^n\right)+\mathcal{L}_{RK}\left(G_1^n\right) \leq C\left(1+\frac{\tau}{h}\right)\left(\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}+h^{2k+2}\right)+C\tau^6+\frac{C}{h}\left\|G_2^n\right\|_{0,\Omega}^2+\frac{1}{16}\left(\mathbf{STAB}_f^n+\mathbf{STAB}_f^{n,1}\right). \tag{3.27}$$

In order to estimate $\mathcal{L}_{RK}(G_2^n)$ in (3.26), we apply (3.23b) of Lemma 3.7. In particular, take $g=G_2^n$ in (3.23b) and use (3.16), we have

$$\mathcal{L}_{RK}\left(G_2^n\right) \leq C\left(\left(1+\frac{\tau}{h}\right)\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}+h^{k+1}\right)+\tau^3+\frac{\left\|G_2^n\right\|_{0,\Omega}}{h}\right)\left\|G_2^n\right\|_{0,\Omega}$$
$$\leq C\left(1+\frac{\tau}{h}\right)\left(\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}+h^{2k+2}\right)+C\tau^6+\frac{C}{h}\left\|G_2^n\right\|_{0,\Omega}^2. \tag{3.28}$$

For $3(G_3^n,G_3^n)_\Omega=\tau\mathcal{L}_{RK}(G_3^n)$, we take $g=G_3^n$ in (3.23b) and obtain

$$3\left\|G_3^n\right\|_{0,\Omega}^2 \leq C\tau\left(\left(1+\frac{\tau}{h}\right)\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}+h^{k+1}\right)+\tau^3+\frac{\left\|G_2^n\right\|_{0,\Omega}}{h}\right)\left\|G_3^n\right\|_{0,\Omega}$$
$$\leq C\tau^2\left(1+\frac{\tau}{h}\right)^2\left(\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}+h^{2k+2}\right)+C\tau^8+C\frac{\tau^2}{h^2}\left\|G_2^n\right\|_{0,\Omega}^2+\left\|G_3^n\right\|_{0,\Omega}^2.$$

Therefore, with a different constant C we have,

$$3\left(G_3^n,G_3^n\right)_\Omega \leq C\tau^2\left(1+\frac{\tau}{h}\right)^2\left(\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}+h^{2k+2}\right)+C\tau^8+C\frac{\tau^2}{h^2}\left\|G_2^n\right\|_{0,\Omega}^2. \tag{3.29}$$

Finally, we complete the proof by combing (3.26)–(3.29). □

Now we are ready to establish the main error estimate result for the Vlasov solver.

Theorem 3.10. *Let (f, E, B) be a sufficiently smooth exact solution to equations (1.1). Let $(f_h^n, E_h^n, B_h^n) \in \mathcal{G}_h^k \times \mathcal{U}_h^k \times \mathcal{U}_h^k$ be the solution to the scheme (2.6) at time t^n with $k \geq \frac{d_x}{2}$. Under the L^∞-Assumption, there exists a positive constant γ_1, such that for any $\frac{\tau}{h} \leq \gamma_1$, the following estimate holds for $\forall n : n+1 \leq T/\tau$*

$$3\left\|\xi_f^{n+1}\right\|_{0,\Omega}^2 - 3\left\|\xi_f^n\right\|_{0,\Omega}^2 \leq C\tau h^{2k+1} + C\tau^7 + C\tau \sum_{\sharp=0}^{2} \mathbf{N}^{n,\sharp}. \tag{3.30}$$

Proof. Since the constant C in the result (3.25) is independent of n, h, τ, there exists a positive constant γ_1 independent of n, h, τ, such that $-1+C\frac{\tau}{h}+C\frac{\tau^2}{h^2} \leq -\frac{1}{2}$ as long as $\frac{\tau}{h} \leq \gamma_1$. Under such condition, we combine (3.18) and the estimates in Propositions 3.5 and 3.9 and get

$$\begin{aligned}
&3\left\|\xi_f^{n+1}\right\|_{0,\Omega}^2 - 3\left\|\xi_f^n\right\|_{0,\Omega}^2 = \Xi_1 + \Xi_2 + \Xi_3 \\
&\leq C\tau h^{2k+1} + C\tau^7 + C\tau \sum_{\sharp=0}^{2} \mathbf{N}^{n,\sharp} - \frac{1}{2}\left\|G_2^n\right\|_{0,\Omega}^2 - \frac{\tau}{8}\left(3\mathbf{STAB}_f^n + 3\mathbf{STAB}_f^{n,1} + 14\mathbf{STAB}_f^{n,2}\right) \\
&\leq C\tau h^{2k+1} + C\tau^7 + C\tau \sum_{\sharp=0}^{2} \mathbf{N}^{n,\sharp}. \qquad \square
\end{aligned}$$

3.3. The Maxwell equations part

In this section, we estimate how error accumulates in the Maxwell solver. The procedure is parallel to the Vlasov part in Section 3.2. We start with a decomposition of the error change in one time step

$$\xi_E^{n+1} - \xi_E^n = X_1^n + X_2^n + X_3^n, \quad \xi_B^{n+1} - \xi_B^n = Z_1^n + Z_2^n + Z_3^n,$$

where $X_1^n = \xi_E^{n,1} - \xi_E^n$, $X_2^n = 2\xi_E^{n,2} - \xi_E^{n,1} - \xi_E^n$, $X_3^n = \xi_E^{n+1} - 2\xi_E^{n,2} + \xi_E^n$, $Z_1^n = \xi_B^{n,1} - \xi_B^n$, $Z_2^n = 2\xi_B^{n,2} - \xi_B^{n,1} - \xi_B^n$, and $Z_3^n = \xi_B^{n+1} - 2\xi_B^{n,2} + \xi_B^n$. It is obvious that

$$\left\|X_i^n\right\|_{0,\Omega}^2 \leq C\sum_{\sharp=0}^{2}\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}^2, \left\|Z_i^n\right\|_{0,\Omega}^2 \leq C\sum_{\sharp=0}^{2}\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}^2, \quad i = 1,2,3. \tag{3.31}$$

Based on equations (3.11), the following hold for any $U, V \in \mathcal{U}_h^k$.

$$\begin{aligned}
(X_2^n, U)_{\Omega_x} + (Z_2^n, V)_{\Omega_x} &= \frac{\tau}{2}\left(\mathcal{R}(U,V) - \mathcal{Q}(U,V)\right) \equiv \frac{\tau}{2}\mathcal{R}_{RK}(U,V), \\
(X_3^n, U)_{\Omega_x} + (Z_3^n, V)_{\Omega_x} &= \frac{\tau}{3}\left(2\mathcal{S}(U,V) - \mathcal{R}(U,V) - \mathcal{Q}(U,V)\right) \equiv \frac{\tau}{3}\mathcal{S}_{RK}(U,V).
\end{aligned}$$

In addition, one can verify based on (3.13) that

$$3\left(\left\|\xi_E^{n+1}\right\|_{0,\Omega_x}^2 + \left\|\xi_B^{n+1}\right\|_{0,\Omega_x}^2\right) - 3\left(\left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2\right) = \Theta_1 + \Theta_2 + \Theta_3, \tag{3.32}$$

where

$$\Theta_i = \tau\left(\mathcal{Q}_i\left(\xi_E^n, \xi_B^n\right) + \mathcal{R}_i\left(\xi_E^{n,1}, \xi_B^{n,1}\right) + 4\mathcal{S}_i\left(\xi_E^{n,2}, \xi_B^{n,2}\right)\right), \quad i = 1,2 \tag{3.33}$$

$$\begin{aligned}
\Theta_3 = \; &(X_2^n, X_2^n)_{\Omega_x} + 3(X_1^n, X_3^n)_{\Omega_x} + 3(X_2^n, X_3^n)_{\Omega_x} + 3(X_3^n, X_3^n)_{\Omega_x} \\
&+ (Z_2^n, Z_2^n)_{\Omega_x} + 3(Z_1^n, Z_3^n)_{\Omega_x} + 3(Z_2^n, Z_3^n)_{\Omega_x} + 3(Z_3^n, Z_3^n)_{\Omega_x}.
\end{aligned} \tag{3.34}$$

In particular, Θ_2 depends on the spatial discretization, while Θ_1 and Θ_3 characterizes the contribution from the time discretization. Similarly as in the Vlasov part, there are two stability mechanisms, with one being $\mathbf{STAB}_{EB}^{\star}$ ($\star$ can be n or $n,\sharp$) from the spatial discretization, and the other is related to $||X_2^n||^2$, $||Z_2^n||^2$ arising from the third order Runge–Kutta time discretization.

Next we will estimate Θ_1 and Θ_2. Some estimates for the spatial discretizations of the Maxwell part are summarized in Lemma 3.11.

Lemma 3.11. *For $\sharp = 0, 1, 2$, we have*

$$\text{(i)} \quad b_h\left(\xi_E^{n,\sharp}, \xi_B^{n,\sharp}, \xi_f^{n,\sharp}; \xi_E^{n,\sharp}, \xi_B^{n,\sharp}\right) \le C\left(\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}^2 + \left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}^2\right) - \frac{1}{2}\mathbf{STAB}_{EB}^{n,\sharp}, \tag{3.35a}$$

$$\text{(ii)} \quad \left|b_h\left(\eta_E^{n,\sharp}, \eta_B^{n,\sharp}, \eta_f^{n,\sharp}; \xi_E^{n,\sharp}, \xi_B^{n,\sharp}\right)\right| \le Ch^{2k+1} + C\left\|\xi_E^{n,\sharp}\right\|_{0,\mathcal{E}_x}^2 + \frac{1}{16}\mathbf{STAB}_{EB}^{n,\sharp}. \tag{3.35b}$$

Proposition 3.12. *The following estimates hold for Θ_1 and Θ_2.*

$$\Theta_1 \le C\tau\left(h^{2k+2} + \tau^6\right) + C\tau\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp} \tag{3.36a}$$

$$\Theta_2 \le C\tau h^{2k+1} + C\tau\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp} - \frac{7}{16}\tau\left(\mathbf{STAB}_{EB}^{n} + \mathbf{STAB}_{EB}^{n,1} + 4\mathbf{STAB}_{EB}^{n,2}\right). \tag{3.36b}$$

Proof. Recall that $\Theta_1 = \tau\left(\mathcal{Q}_1(\xi_E^n, \xi_B^n) + \mathcal{R}_1(\xi_E^{n,1}, \xi_B^{n,1}) + 4\mathcal{S}_1(\xi_E^{n,2}, \xi_B^{n,2})\right)$, with similarity, we only estimate $\mathcal{S}_1(\xi_E^{n,2}, \xi_B^{n,2})$. Applying Cauchy–Schwarz inequality, Lemma 3.2, Lemma 3.3(1), one gets

$$\begin{aligned}\mathcal{S}_1\left(\xi_E^{n,2}, \xi_B^{n,2}\right) &= \left(\frac{3\eta_E^{n+1} - \eta_E^n - 2\eta_E^{n,2} + 3T_E^n(x)}{2\tau}, \xi_E^{n,2}\right)_{\Omega_x} + \left(\frac{3\eta_B^{n+1} - \eta_B^n - 2\eta_B^{n,2} + 3T_B^n(x)}{2\tau}, \xi_B^{n,2}\right)_{\Omega_x} \\ &\le C\left(\left\|\xi_E^{n,2}\right\|_{0,\Omega_x}^2 + \left\|\xi_B^{n,2}\right\|_{0,\Omega_x}^2 + h^{2k+2} + \tau^6\right) \le C\left(\mathbf{N}^{n,2} + h^{2k+2} + \tau^6\right).\end{aligned}$$

To bound $\Theta_2 = \tau\left(\mathcal{Q}_2(\xi_E^n, \xi_B^n) + \mathcal{R}_2(\xi_E^{n,1}, \xi_B^{n,1}) + 4\mathcal{S}_2(\xi_E^{n,2}, \xi_B^{n,2})\right)$, with similarity, we only need to look at $\mathcal{Q}_2(\xi_E^n, \xi_B^n)$. By definition, and Lemma 3.11,

$$\begin{aligned}\mathcal{Q}_2\left(\xi_E^n, \xi_B^n\right) &= b_h\left(e_E^n, e_B^n, e_f^n; \xi_E^n, \xi_B^n\right) = b_h\left(\xi_E^n, \xi_B^n, \xi_f^n; \xi_E^n, \xi_B^n\right) - b_h\left(\eta_E^n, \eta_B^n, \eta_f^n; \xi_E^n, \xi_B^n\right) \\ &\le Ch^{2k+1} + C\left(\left\|\xi_f^n\right\|_{0,\Omega}^2 + \|\xi_E^n\|_{0,\Omega_x}^2\right) - \frac{7}{16}\mathbf{STAB}_{EB}^n \\ &\le Ch^{2k+1} + C\mathbf{N}^n - \frac{7}{16}\mathbf{STAB}_{EB}^n.\end{aligned}$$

□

Next we will estimate Θ_3. One key is to use $-||X_2^n||_{0,\Omega_x}^2 - ||Z_2^n||_{0,\Omega_x}^2$ to control $C\frac{\tau}{h}(||X_2^n||_{0,\Omega_x}^2 + ||Z_2^n||_{0,\Omega_x}^2)$. To make the analysis easy to follow, we first present some preparatory results in two lemmas.

Lemma 3.13. *For any $U, V \in \mathcal{U}_h^k$, we have*

$$\mathcal{S}_{RK}(U, V) \le C\left(h^{k+1} + \tau h^k + \tau^3\right)\left(||U||_{0,\Omega_x} + ||V||_{0,\Omega_x}\right) + b_h\left(X_2^n, Z_2^n, G_2^n; U, V\right) \tag{3.37a}$$

$$\begin{aligned}&\le C\left(h^{k+1} + \tau h^k + \tau^3 + \frac{1}{h}\left(\|Z_2^n\|_{0,\Omega_x} + \|X_2^n\|_{0,\Omega_x}\right)\right)\left(||U||_{0,\Omega_x} + ||V||_{0,\Omega_x}\right) \\ &\quad + C\left\|G_2^n\right\|_{0,\Omega}||U||_{0,\Omega_x}.\end{aligned} \tag{3.37b}$$

$$\mathcal{R}_{RK}(U, V) \le C\left(h^{k+1} + \tau h^k\right)\left(||U||_{0,\Omega_x} + ||V||_{0,\Omega_x}\right) + b_h\left(X_1^n, Z_1^n, G_1^n; U, V\right). \tag{3.37c}$$

Lemma 3.14.

$$
\begin{aligned}
& b_h\left(X_1^n, Z_1^n, G_1^n; X_2^n, Z_2^n\right) + b_h\left(X_2^n, Z_2^n, G_2^n; X_1^n, Z_1^n\right) \\
& \le \frac{C}{h}\left(\|X_2^n\|_{0,\Omega_x}^2 + \|Z_2^n\|_{0,\Omega_x}^2\right) + C\sum_{j=1}^{2}\left(\left\|G_j^n\right\|_{0,\Omega}^2 + \left\|X_j^n\right\|_{0,\Omega_x}^2\right) + \frac{1}{16}\left(\mathbf{STAB}_{EB}^n + \mathbf{STAB}_{EB}^{n,1}\right).
\end{aligned}
$$

Now we get ready to estimate Θ_3.

Proposition 3.15.

$$
\begin{aligned}
\Theta_3 \le\ & C\tau\left(h^{2k+2} + \tau^2 h^{2k} + \tau^6\right) + \left(-1 + C\frac{\tau}{h} + C\frac{\tau^2}{h^2}\right)\left(\|X_2^n\|_{0,\Omega_x}^2 + \|Z_2^n\|_{0,\Omega_x}^2\right) \\
& + \frac{\tau}{16}\left(\mathbf{STAB}_{EB}^n + \mathbf{STAB}_{EB}^{n,1}\right) + C\tau\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}.
\end{aligned} \tag{3.38}
$$

Proof. First note that

$$
\begin{aligned}
\Theta_3 =\ & -\|X_2^n\|_{0,\Omega_x}^2 - \|Z_2^n\|_{0,\Omega_x}^2 \\
& + \tau\left(\mathcal{R}_{RK}\left(X_2^n, Z_2^n\right) + \mathcal{S}_{RK}\left(X_1^n, Z_1^n\right) + \mathcal{S}_{RK}\left(X_2^n, Z_2^n\right)\right) + 3\|X_3^n\|_{0,\Omega_x}^2 + 3\|Z_3^n\|_{0,\Omega_x}^2 .
\end{aligned} \tag{3.39}
$$

Based on (3.37a), (3.37c), and Lemma 3.14, in addition to (3.16) and (3.31) we have

$$
\begin{aligned}
& \mathcal{R}_{RK}\left(X_2^n, Z_2^n\right) + \mathcal{S}_{RK}\left(X_1^n, Z_1^n\right) \\
& \le b_h\left(X_2^n, Z_2^n, G_2^n; X_1^n, Z_1^n\right) + b_h\left(X_1^n, Z_1^n, G_1^n; X_2^n, Z_2^n\right) + C\left(h^{k+1} + \tau h^k + \tau^3\right)\left(\|X_1^n\|_{0,\Omega_x} + \|Z_1^n\|_{0,\Omega_x}\right) \\
& \quad + C\left(h^{k+1} + \tau h^k\right)\left(\|X_2^n\|_{0,\Omega_x} + \|Z_2^n\|_{0,\Omega_x}\right) \\
& \le \frac{C}{h}\left(\|X_2^n\|_{0,\Omega_x}^2 + \|Z_2^n\|_{0,\Omega_x}^2\right) + \frac{1}{16}\left(\mathbf{STAB}_{EB}^n + \mathbf{STAB}_{EB}^{n,1}\right) + C\left(h^{2k+2} + \tau^2 h^{2k} + \tau^6\right) \\
& \quad + C\sum_{j=1}^{2}\left(\left\|G_j^n\right\|_{0,\Omega}^2 + \left\|X_j^n\right\|_{0,\Omega_x}^2 + \left\|Z_j^n\right\|_{0,\Omega_x}^2\right) \\
& \le \frac{C}{h}\left(\|X_2^n\|_{0,\Omega_x}^2 + \|Z_2^n\|_{0,\Omega_x}^2\right) + \frac{1}{16}\left(\mathbf{STAB}_{EB}^n + \mathbf{STAB}_{EB}^{n,1}\right) + C\left(h^{2k+2} + \tau^2 h^{2k} + \tau^6\right) + C\sum_{j=1}^{2}\mathbf{N}^{n,\sharp}.
\end{aligned} \tag{3.40}
$$

Next we estimate $\mathcal{S}_{RK}(X_2^n, Z_2^n)$ by using (3.37b) in Lemma 3.13.

$$
\begin{aligned}
\mathcal{S}_{RK}\left(X_2^n, Z_2^n\right) & \le C\left(h^{k+1} + \tau h^k + \tau^3 + \frac{1}{h}\left(\|Z_2^n\|_{0,\Omega_x} + \|X_2^n\|_{0,\Omega_x}\right)\right)\left(\|X_2^n\|_{0,\Omega_x} + \|Z_2^n\|_{0,\Omega_x}\right) + C\|G_2^n\|_{0,\Omega}\|X_2^n\|_{0,\Omega_x} \\
& \le C\left(h^{2k+2} + \tau^2 h^{2k} + \tau^6\right) + C\sum_{j=1}^{2}\mathbf{N}^{n,\sharp} + \frac{C}{h}\left(\|Z_2^n\|_{0,\Omega_x}^2 + \|X_2^n\|_{0,\Omega_x}^2\right).
\end{aligned} \tag{3.41}
$$

Finally we turn to $3||X_3^n||^2_{0,\Omega_x} + 3||Z_3^n||^2_{0,\Omega_x}$ in (3.39). By applying (3.37b) in Lemma 3.13,

$$\begin{aligned}
&3\left\|X_3^n\right\|_{0,\Omega_x}^2 + 3\left\|Z_3^n\right\|_{0,\Omega_x}^2 = \tau\mathcal{S}_{RK}\left(X_3^n, Z_3^n\right)\\
&\le C\tau\left(h^{k+1} + \tau h^k + \tau^3 + \frac{1}{h}\left(\left\|Z_2^n\right\|_{0,\Omega_x} + \left\|X_2^n\right\|_{0,\Omega_x}\right)\right)\left(\left\|X_3^n\right\|_{0,\Omega_x} + \left\|Z_3^n\right\|_{0,\Omega_x}\right) + C\tau\left\|G_2^n\right\|_{0,\Omega}\left\|X_3^n\right\|_{0,\Omega_x}\\
&\le C\left(\tau^2h^{2k+2} + \tau^4h^{2k} + \tau^8\right) + C\frac{\tau^2}{h^2}\left(\left\|Z_2^n\right\|_{0,\Omega_x}^2 + \left\|X_2^n\right\|_{0,\Omega_x}^2\right) + C\tau^2\left\|G_2^n\right\|_{0,\Omega}^2 + \left\|X_3^n\right\|_{0,\Omega_x}^2 + \left\|Z_3^n\right\|_{0,\Omega_x}^2,
\end{aligned}$$

therefore, with a different value of C, we have

$$3\left\|X_3^n\right\|_{0,\Omega_x}^2 + 3\left\|Z_3^n\right\|_{0,\Omega_x}^2 \le C\left(\tau^2h^{2k+2} + \tau^4h^{2k} + \tau^8\right) + C\frac{\tau^2}{h^2}\left(\left\|Z_2^n\right\|_{0,\Omega_x}^2 + \left\|X_2^n\right\|_{0,\Omega_x}^2\right) + C\tau^2\left\|G_2^n\right\|_{0,\Omega}^2. \quad (3.42)$$

Now by combining the results in (3.39)–(3.42) and (3.16), we can conclude the estimate for Θ_3 in (3.38). □

The main error estimate result for the Maxwell solver is now established as following.

Theorem 3.16. *Let (f, E, B) be a sufficiently smooth exact solution to equations (1.1). Let $(f_h^n, E_h^n, B_h^n) \in \mathcal{G}_h^k \times \mathcal{U}_h^k \times \mathcal{U}_h^k$ be the solution to the scheme (2.6) at time t^n. There exists a positive constant γ_2, such that for $\frac{\tau}{h} \le \gamma_2$, the following estimate holds for $\forall n : n+1 \le T/\tau$*

$$3\left(\left\|\xi_E^{n+1}\right\|_{0,\Omega_x}^2 + \left\|\xi_B^{n+1}\right\|_{0,\Omega_x}^2\right) - 3\left(\left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2\right) \le C\tau h^{2k+1} + C\tau^7 + C\tau\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}. \quad (3.43)$$

Proof. Since the constant C in the estimate (3.38) is independent of n, h, τ, there exists a positive constant γ_2 independent of n, h, τ, such that $-1 + C\frac{\tau}{h} + C\frac{\tau^2}{h^2} \le -\frac{1}{2}$ as long as $\frac{\tau}{h} \le \gamma_2$. Under this condition on the time step τ, we also have $\tau^2h^{2k} \le \gamma_2^2h^{2k+2}$. Now we combine the estimates for Θ_i, $i = 1, 2, 3$ in Propositions 3.12 and 3.15, and get

$$\begin{aligned}
&3\left(\left\|\xi_E^{n+1}\right\|_{0,\Omega_x}^2 + \left\|\xi_B^{n+1}\right\|_{0,\Omega_x}^2\right) - 3\left(\left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2\right) = \Theta_1 + \Theta_2 + \Theta_3\\
&\le C\tau h^{2k+1} + C\tau^7 + C\tau\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp} - \frac{1}{2}\left(\left\|X_2^n\right\|_{0,\Omega_x}^2 + \left\|Z_2^n\right\|_{0,\Omega_x}^2\right) - \frac{\tau}{8}\left(3\mathbf{STAB}_{EB}^n + 3\mathbf{STAB}_{EB}^{n,1} + 14\mathbf{STAB}_{EB}^{n,2}\right)\\
&\le C\tau h^{2k+1} + C\tau^7 + C\tau\sum_{\sharp=0}^{2}\mathbf{N}^{n,\sharp}.
\end{aligned}$$

□

Remark 3.17. Unlike in Theorem 3.10, the *a priori* assumption about the L^∞ norm of the error in the magnetic and electric fields, together with the condition of $k \ge \frac{d_x}{2}$, are not needed in Theorem 3.16. This difference is due to the nonlinear coupling terms in the Vlasov equation.

3.4. Proof of the main result: Theorem 3.1

The following lemma is the final preparation.

Lemma 3.18. *Suppose $\frac{\tau}{h} \le \alpha$, where α is a positive constant independent of n, h, τ. Then under the L^∞-Assumption, the following inequalities are satisfied*

$$\begin{aligned}
\left\|\xi_f^{n,1}\right\|_{0,\Omega}^2 &\le Ch^{2k+2} + C\left(\left\|\xi_f^n\right\|_{0,\Omega}^2 + \tau^2\left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \tau^2\left\|\xi_B^n\right\|_{0,\Omega_x}^2\right)\\
\left\|\xi_f^{n,2}\right\|_{0,\Omega}^2 &\le Ch^{2k+2} + C\left(\left\|\xi_f^n\right\|_{0,\Omega}^2 + \left\|\xi_f^{n,1}\right\|_{0,\Omega}^2 + \tau^2\left\|\xi_E^{n,1}\right\|_{0,\Omega_x}^2 + \tau^2\left\|\xi_B^{n,1}\right\|_{0,\Omega_x}^2\right)\\
\left\|\xi_E^{n,1}\right\|_{0,\Omega_x}^2 &\le Ch^{2k+2} + C\left(\tau^2\left\|\xi_f^n\right\|_{0,\Omega}^2 + \left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2\right)\\
\left\|\xi_E^{n,2}\right\|_{0,\Omega_x}^2 &\le Ch^{2k+2} + C\left(\tau^2\left\|\xi_f^{n,1}\right\|_{0,\Omega_x}^2 + \left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_E^{n,1}\right\|_{0,\Omega_x}^2 + \left\|\xi_B^{n,1}\right\|_{0,\Omega_x}^2\right)\\
\left\|\xi_B^{n,1}\right\|_{0,\Omega_x}^2 &\le Ch^{2k+2} + C\left(\left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2\right)\\
\left\|\xi_B^{n,2}\right\|_{0,\Omega_x}^2 &\le Ch^{2k+2} + C\left(\left\|\xi_E^{n,1}\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^{n,1}\right\|_{0,\Omega_x}^2\right).
\end{aligned}$$

A direct consequence of these inequalities is

$$\sum_{\sharp=0}^{2} \mathbf{N}^{n,\sharp} \le C\mathbf{N}^n + Ch^{2k+2}. \tag{3.44}$$

Now we are ready for the proof of Theorem 3.1.

Proof. First we make the *a priori* assumption for the L^∞ error of the magnetic and electric fields as in L^∞-Assumption. Based on Theorem 3.10 and Theorem 3.16, for any $\frac{\tau}{h} \le \gamma := \min(\gamma_1, \gamma_2)$, we get

$$3\mathbf{N}^{n+1} - 3\mathbf{N}^n \le C\tau^7 + C\tau h^{2k+1} + C\tau \sum_{\sharp=0}^{2} \mathbf{N}^{n,\sharp} \le \hat{C}\left(\tau^7 + \tau h^{2k+1} + \tau \mathbf{N}^n\right). \tag{3.45}$$

Here (3.44) is used to get the last inequality. Let $\Upsilon_n = \mathbf{N}^n/(1+\frac{\hat{C}\tau}{3})^n$, then (3.45) leads to $\Upsilon_n - \Upsilon_{n-1} \le \frac{\hat{C}(\tau^7+\tau h^{2k+1})}{3(1+\frac{\hat{C}}{3}\tau)^n}$. We now sum up $\Upsilon_\sharp - \Upsilon_{\sharp-1}$ and use $\Upsilon_0 = 0$ to obtain

$$\Upsilon_n \le \sum_{\sharp=1}^{n} \frac{\hat{C}\left(\tau^7 + \tau h^{2k+1}\right)}{3\left(1+\frac{\hat{C}}{3}\tau\right)^\sharp} \le \tau^6 + h^{2k+1}. \tag{3.46}$$

Therefore

$$\mathbf{N}^n = \left(1 + \frac{\hat{C}\tau}{3}\right)^n \Upsilon_n \le \mathrm{e}^{\frac{\hat{C}n\tau}{3}}\left(\tau^6 + h^{2k+1}\right) \le \mathrm{e}^{\frac{\hat{C}T}{3}}\left(\tau^6 + h^{2k+1}\right), \tag{3.47}$$

for any $n \le T/\tau$. This can also be written as

$$\left\|\xi_f^n\right\|_{0,\Omega}^2 + \left\|\xi_E^n\right\|_{0,\Omega_x}^2 + \left\|\xi_B^n\right\|_{0,\Omega_x}^2 \le C\left(\tau^6 + h^{2k+1}\right). \tag{3.48}$$

Next we will establish the L^∞-Assumption using mathematical induction. For $n = 0$, $\xi_f^0 = \xi_E^0 = \xi_B^0 = 0$ and the approximation property (2.4) implies that $||e_E^0||_{0,\infty,\Omega_x}$, $||e_B^0||_{0,\infty,\Omega_x} \le \mathcal{C}h$, where $\mathcal{C}$ depends only on the initial conditions and is independent of n, h, τ. Furthermore, Lemma 3.18 shows $||\xi_f^{0,\sharp}||_{0,\Omega}$, $||\xi_E^{0,\sharp}||_{0,\Omega_x}$, $||\xi_B^{0,\sharp}||_{0,\Omega_x} \le Ch^{k+1}$ for $\sharp = 1, 2$. Thus

$$||e_E^{0,\sharp}||_{0,\infty,\Omega_x} \le ||\xi_E^{0,\sharp}||_{0,\infty,\Omega_x} + ||\eta_E^{0,\sharp}||_{0,\infty,\Omega_x} \le Ch^{-\frac{d_x}{2}}||\xi_E^{0,\sharp}||_{0,\Omega_x} + Ch^{k+1} \le Ch^{k+1-\frac{d_x}{2}} \le \mathcal{C}h,$$

for small enough h. Here the inverse inequality (2.5) and the condition $k > \frac{d_x+1}{2} > \frac{d_x}{2}$ are used. Similarly $||e_B^{0,\sharp}||_{0,\infty,\Omega_x} \leq \mathcal{C}h$. Suppose $||e_B^n||_{0,\infty,\Omega_x}, ||e_E^n||_{0,\infty,\Omega_x} \leq \mathcal{C}h$ for all $n \leq m$, therefore inequality (3.48) with $n = m+1$ is satisfied. Combining this with Lemma 3.18 implies $||\xi_f^{m+1,\sharp}||_{0,\Omega}, ||\xi_E^{m+1,\sharp}||_{0,\Omega_x}, ||\xi_B^{m+1,\sharp}||_{0,\Omega_x} \leq C(\tau^3 + h^{k+\frac{1}{2}})$ for $\sharp = 0, 1, 2$. Thus

$$\begin{aligned}\left\|e_E^{m+1,\sharp}\right\|_{0,\infty,\Omega_x} &\leq Ch^{-\frac{d_x}{2}}\left\|\xi_E^{m+1,\sharp}\right\|_{0,\Omega_x} + Ch^{k+1} \leq Ch^{-\frac{d_x}{2}}\left(\tau^3 + h^{k+\frac{1}{2}}\right) + Ch^{k+1}\\ &\leq Ch^{-\frac{d_x}{2}}\tau^3 + Ch^{k+\frac{1}{2}-\frac{d_x}{2}} \leq C\gamma^3 h^{3-\frac{d_x}{2}} + Ch^{k+\frac{1}{2}-\frac{d_x}{2}} \leq C_0 h^{1+\epsilon},\end{aligned} \tag{3.49}$$

for $k > \frac{d_x+1}{2}$ (recall $d_x \leq 3$), under the condition $\frac{\tau}{h} \leq \gamma$ and with $\epsilon = \min(k - \frac{d_x+1}{2}, 2 - \frac{d_x}{2}) > 0$. The positive constant C_0 depends on T in general and is independent of m, h, τ. One can see that there exists $h_0 > 0$, such that $C_0 h^\epsilon < \mathcal{C}$, hence $||e_E^{m+1,\sharp}||_{0,\infty,\Omega_x} \leq \mathcal{C}h$ for $h < h_0$. Up to now the *a priori* assumption is established, and this also completes the L^2 error estimates of f, E and B.

Finally we will show how the Gauss's laws are approximated by the proposed methods. Based on the fact that the Gauss's laws are satisfied by the exact solutions, using inverse inequalities (2.5) and approximating properties of the discrete spaces in (2.4), and applying the L^2 error estimate of f and E, we get

$$\begin{aligned}\|\nabla\cdot E_h^n - \rho_h^n + \rho_i^n\|_{0,\Omega_x} &= \|\nabla\cdot E_h^n - \rho_h^n + \rho_i^n - (\nabla\cdot E^n - \rho^n + \rho_i^n)\|_{0,\Omega_x}\\ &= \|\nabla\cdot(E_h^n - E^n) - (\rho_h^n - \rho^n)\|_{0,\Omega_x}\\ &\leq \|\nabla\cdot\eta_E^n\|_{0,\Omega_x} + \|\nabla\cdot\xi_E^n\|_{0,\Omega_x} + C\left\|e_f^n\right\|_{0,\Omega}\\ &\leq Ch_x^k\|E^n\|_{k+1,\Omega_x} + \frac{C}{h_x}\|\xi_E^n\|_{0,\Omega_x} + C\left(h^{k+\frac{1}{2}} + \tau^3\right) \leq \frac{C}{h_x}\left(h^{k+\frac{1}{2}} + \tau^3\right).\end{aligned}$$

Similarly one can get $||\nabla\cdot B_h^n||_{0,\Omega_x} \leq \frac{C}{h_x}(h^{k+\frac{1}{2}} + \tau^3)$. This completes all the estimates in Theorem 3.1. $\square$

Remark 3.19. The condition $k > \frac{d_x+1}{2}$ is needed only to establish the L^∞-Assumption, while for all the other results, the requirement $k \geq \frac{d_x}{2}$ is enough.

4. Proofs of some lemmas

In this section, we will provide the proofs of some lemmas in Section 3.

4.1. Proof of Lemma 3.3

To get (3.15a), we start with $f^{n,1}$ and $f^{n,2}$ given in (3.8), and get

$$\begin{aligned}&d_0 f^n + d_1 f^{n,1} + d_2 f^{n,2} + d_3 f^{n+1}\\ &\quad= d_3\left(f^{n+1} - f^n\right) + \tau\left(d_1 + \frac{d_2}{2}\right)\partial_t f^n + \frac{\tau^2}{4}d_2\partial_t^2 f^n - \frac{\tau^3}{4}d_2\left(\partial_t E^n + v\times\partial_t B^n\right)\cdot\nabla_v\left(\partial_t f^n\right)\\ &\quad= \tau d_3\partial_t f\left(x,v,t^\star\right) + \tau\left(d_1 + \frac{d_2}{2}\right)\partial_t f^n + \frac{\tau^2}{4}d_2\partial_t^2 f^n - \frac{\tau^3}{4}d_2\left(\partial_t E^n + v\times\partial_t B^n\right)\cdot\nabla_v\left(\partial_t f^n\right),\end{aligned} \tag{4.50}$$

for some $t^\star \in [t^n, t^{n+1}]$. Here we have used $d_0 + d_1 + d_2 + d_3 = 0$. Therefore, with I as the identity operator,

$$d_f^n = \left(\Pi^k - I\right)\left(\tau d_3\partial_t f\left(x,v,t^\star\right) + \tau\left(d_1 + \frac{d_2}{2}\right)\partial_t f^n + \frac{\tau^2}{4}d_2\partial_t^2 f^n - \frac{\tau^3}{4}d_2\left(\partial_t E^n + v\times\partial_t B^n\right)\cdot\nabla_v\left(\partial_t f^n\right)\right).$$

For sufficiently smooth solution, there is

$$\left\|d_f^n\right\|_{0,\Omega} \leq C\tau h^{k+1} \max_{\forall t\in[0,T]} \left(||\partial_t f||_{k+1,\Omega} + \tau \left\|\partial_t^2 f\right\|_{k+1,\Omega} + \tau^2 \left\|(\partial_t E + v\times\partial_t B)\cdot\nabla_v(\partial_t f)\right\|_{k+1,\Omega}\right).$$

Recall that $\tau \leq 1$, we further have $||d_f^n||_{0,\Omega} \leq C\tau h^{k+1}$. Similarly one can show $||d_f^n||_{0,\mathcal{E}} \leq C\tau h^{k+\frac{1}{2}}$. These two estimates will lead to the upper bound in (3.15a). The proof of the results for E and B can be proceeded similarly.

To get (3.15b), based on definition

$$\begin{aligned}
&\left|a_h\left(d_f^n, E_h^{n,s}, B_h^{n,s}; g\right)\right| \\
&\leq \int_\Omega \left|d_f^n v\cdot\nabla_x g\right| \mathrm{d}x\mathrm{d}v + \int_{T_h^v}\int_{\mathcal{E}_x} \left(\left|\left\{d_f^n v\right\}_x + \frac{|v\cdot n_x|}{2}[d_f^n]_x\right|\right)|[g]_x|\mathrm{d}s_x\mathrm{d}v + \int_\Omega \left|d_f^n\left(E_h^{n,s}+v\times B_h^{n,s}\right)\nabla_v g\right|\mathrm{d}x\mathrm{d}v \\
&\quad + \int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(\left\{d_f^n\left(E_h^{n,s}+v\times B_h^{n,s}\right)\right\}_v + \frac{|(E_h^{n,s}+v\times B_h^{n,s})\cdot n_v|}{2}\left[d_f^n\right]_v\right)\cdot[g]_v\right|\mathrm{d}s_v\mathrm{d}x \\
&\leq C\left(\left\|d_f^n\right\|_{0,\Omega}||\nabla_x g||_{0,\Omega} + \left\|d_f^n\right\|_{0,T_h^v\times\mathcal{E}_x}||g||_{0,T_h^v\times\mathcal{E}_x}\right) \\
&\quad + C\left(\|E_h^{n,s}\|_{0,\infty,\Omega_x} + \|B_h^{n,s}\|_{0,\infty,\Omega_x}\right)\left(\left\|d_f^n\right\|_{0,\Omega}||\nabla_v g||_{0,\Omega} + \left\|d_f^n\right\|_{0,T_h^x\times\mathcal{E}_v}||g||_{0,T_h^x\times\mathcal{E}_v}\right) \\
&\leq \frac{C}{h}||g||_{0,\Omega}\left(\left\|d_f^n\right\|_{0,\Omega} + h^{\frac{1}{2}}\left\|d_f^n\right\|_{0,T_h^v\times\mathcal{E}_x}\right) \\
&\quad + \frac{C}{h}||g||_{0,\Omega}\left(\|E_h^{n,s}\|_{0,\infty,\Omega_x} + \|B_h^{n,s}\|_{0,\infty,\Omega_x}\right)\left(\left\|d_f^n\right\|_{0,\Omega} + h^{\frac{1}{2}}\left\|d_f^n\right\|_{0,T_h^x\times\mathcal{E}_v}\right).
\end{aligned}$$

By the *a priori* L^∞-Assumption, there is

$$\|E_h^{n,s}\|_{0,\infty,\Omega_x} \leq \|e_E^{n,s}\|_{0,\infty,\Omega_x} + \|E^{n,s}\|_{0,\infty,\Omega_x} \leq Ch + C \leq C, \tag{4.51}$$

and similarly,

$$\|B_h^{n,s}\|_{0,\infty,\Omega_x} \leq C. \tag{4.52}$$

Finally we apply (3.15a) and conclude

$$\left|a_h\left(d_f^n, E_h^{n,s}, B_h^{n,s}; g\right)\right| \leq C\frac{\tau}{h}||g||_{0,\Omega}h^{k+1} \leq C\frac{\tau}{h}\left(h^{2k+2} + ||g||_{0,\Omega}^2\right). \tag{4.53}$$

To get (3.15c),

$$\begin{aligned}
b_h\left(d_E^n, d_B^n, d_f^n; U, V\right) &= \int_{\Omega_x} d_B^n\cdot(\nabla\times U)\mathrm{d}x - \int_{\Omega_x} d_E^n\cdot(\nabla\times V)\mathrm{d}x - \int_{\Omega_x}\left(\int_{\Omega_v} d_f^n v\mathrm{d}v\right)\cdot U\mathrm{d}x \\
&\quad + \int_{\mathcal{E}_x}\left(\{d_B^n\}_x - \frac{1}{2}\left[d_E^n\right]_{\tan}\right)\cdot[U]_{\tan}\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{d_E^n\}_x + \frac{1}{2}\left[d_B^n\right]_{\tan}\right)\cdot[V]_{\tan}\mathrm{d}s_x \\
&= -\int_{\Omega_x}\left(\int_{\Omega_v} d_f^n v\mathrm{d}v\right)\cdot U\mathrm{d}x + \int_{\mathcal{E}_x}\left(\{d_B^n\}_x - \frac{1}{2}\left[d_E^n\right]_{\tan}\right)\cdot[U]_{\tan}\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{d_E^n\}_x + \frac{1}{2}\left[d_B^n\right]_{\tan}\right)\cdot[V]_{\tan}\mathrm{d}s_x,
\end{aligned}$$

where the second equality is due to $\nabla\times U, \nabla\times V \in \mathcal{U}_h^k$, which leads to $\int_{\Omega_x} d_B^n\cdot(\nabla\times U)\mathrm{d}x = \int_{\Omega_x} d_E^n\cdot(\nabla\times V)\mathrm{d}x = 0$. We now can apply inverse inequality (2.5) and (3.15a) to obtain (3.15c).

4.2. Proof of Lemma 3.4

To get (3.19a), based on definition, for any $g \in \mathcal{G}_h^k$, $U, V \in \mathcal{U}_h^k$

$$
\begin{aligned}
a_{h,1}(g;g) &= \int_\Omega g v \cdot \nabla_x g \mathrm{d}x\mathrm{d}v - \int_{T_h^v}\int_{\mathcal{E}_x}\left(\{gv\}_x + \frac{|v\cdot n_x|}{2}[g]_x\right)\cdot [g]_x \mathrm{d}s_x\mathrm{d}v \\
&= \frac{1}{2}\int_{T_h^v}\sum_{K_x\in T_h^x}\int_{\partial K_x} v\cdot n_x g^2 \mathrm{d}s_x\mathrm{d}v - \int_{T_h^v}\int_{\mathcal{E}_x}\left(\{gv\}_x + \frac{|v\cdot n_x|}{2}[g]_x\right)\cdot [g]_x \mathrm{d}s_x\mathrm{d}v \\
&= \int_{T_h^v}\int_{\mathcal{E}_x}\left(\frac{1}{2}v\cdot[g^2]_x - \left(\{gv\}_x + \frac{|v\cdot n_x|}{2}[g]_x\right)\cdot[g]_x\right)\mathrm{d}s_x\mathrm{d}v \\
&= \int_{T_h^v}\int_{\mathcal{E}_x}\left(\left(\frac{1}{2}[g^2]_x - \{g\}_x[g]_x\right)\cdot v - \frac{|v\cdot n_x|}{2}|[g]_x|^2\right)\mathrm{d}s_x\mathrm{d}v \\
&= -\frac{1}{2}\int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x||[g]_x|^2\mathrm{d}s_x\mathrm{d}v.
\end{aligned} \tag{4.54}
$$

Here the property of jump and average in (2.3a) is used to get the last equality in (4.54). Similarly, there is $a_{h,2}(g,U,V;g) = -\frac{1}{2}\int_{T_h^x}\int_{\mathcal{E}_v}|(U+v\times V)\cdot n_v||[g]_v|^2\mathrm{d}s_v\mathrm{d}x$. Finally, taking $g=\xi_f^{n,\sharp}$, $U=E_h^{n,\sharp}$, $V=B_h^{n,\sharp}$ leads to (3.19a).

To get (3.19b), we will proceed the proof in two steps.

Step 1: To estimate $a_{h,1}(\eta_f^{n,\sharp};\xi_f^{n,\sharp})$. By definition,

$$
a_{h,1}\left(\eta_f^{n,\sharp};\xi_f^{n,\sharp}\right) = \int_\Omega \eta_f^{n,\sharp} v\cdot\nabla_x\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v - \int_{T_h^v}\int_{\mathcal{E}_x}\left(\left\{\eta_f^{n,\sharp}v\right\}_x + \frac{|v\cdot n_x|}{2}\left[\eta_f^{n,\sharp}\right]_x\right)\cdot\left[\xi_f^{n,\sharp}\right]_x\mathrm{d}s_x\mathrm{d}v. \tag{4.55}
$$

Let v_0 be the L^2 projection of the function v onto the piecewise constant space with respect to T_h^v, then

$$
\begin{aligned}
\int_\Omega \eta_f^{n,\sharp} v\cdot\nabla_x\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v &= \int_\Omega \eta_f^{n,\sharp}(v-v_0)\cdot\nabla_x\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v + \int_\Omega \eta_f^{n,\sharp}v_0\cdot\nabla_x\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v \\
&= \int_\Omega \eta_f^{n,\sharp}(v-v_0)\cdot\nabla_x\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v.
\end{aligned} \tag{4.56}
$$

The last equality is satisfied because $v_0\cdot\nabla_x\xi_f^{n,\sharp}\in\mathcal{G}_h^k$ and the L^2 projection of $\eta_f^{n,\sharp}$ onto $\mathcal{G}_h^k$ vanishes. We further have

$$
\begin{aligned}
\left|\int_\Omega \eta_f^{n,\sharp} v\cdot\nabla_x\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v\right| &\le \|v-v_0\|_{0,\infty,\Omega_v}\sum_{K\in T_h}\left(h_{K_x}^{-1}\left\|\eta_f^{n,\sharp}\right\|_{0,K}\right)\left(h_{K_x}\left\|\nabla_x\xi_f^{n,\sharp}\right\|_{0,K}\right) \\
&\le Ch_v||v||_{1,\infty,\Omega_v}\sum_{K\in T_h}h_K^{k+1}h_{K_x}^{-1}\left\|f^{n,\sharp}\right\|_{k+1,K}\left\|\xi_f^{n,\sharp}\right\|_{0,K} \le Ch^{k+1}\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}.
\end{aligned} \tag{4.57}
$$

Here Cauchy–Schwarz inequality and approximate properties (2.4) are used for the first and second inequality above, respectively. Applying the similar technique to the second term of $a_{h,1}(\eta_f^{n,\sharp};\xi_f^{n,\sharp})$, we get

$$\begin{aligned}
&\left|\int_{T_h^v}\int_{\mathcal{E}_x}\left(\left\{\eta_f^{n,\sharp}v\right\}_x+\frac{|v\cdot n_x|}{2}\left[\eta_f^{n,\sharp}\right]_x\right)\cdot\left[\xi_f^{n,\sharp}\right]_x \mathrm{d}s_x\mathrm{d}v\right| \\
&\le \int_{T_h^v}\int_{\mathcal{E}_x}\left(|v\cdot n_x|\left(\left|\left\{\eta_f^{n,\sharp}\right\}_x\right|+\frac{\left|\left[\eta_f^{n,\sharp}\right]_x\right|}{2}\right)\right)\left|\left[\xi_f^{n,\sharp}\right]_x\right|\mathrm{d}s_x\mathrm{d}v \\
&\le \left(\int_{T_h^v}\int_{\mathcal{E}_x}2\left(\left|\left\{\eta_f^{n,\sharp}\right\}_x\right|^2+\left(\frac{\left|\left[\eta_f^{n,\sharp}\right]_x\right|}{2}\right)^2\right)|v\cdot n_x|\mathrm{d}s_x\mathrm{d}v\right)^{\frac12}\left(\int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left|\left[\xi_f^{n,\sharp}\right]_x\right|^2\mathrm{d}s_x\mathrm{d}v\right)^{\frac12} \\
&\le C\left\|\eta_f^{n,\sharp}\right\|_{0,T_h^v\times\mathcal{E}_x}\left(\int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left|\left[\xi_f^{n,\sharp}\right]_x\right|^2\mathrm{d}s_x\mathrm{d}v\right)^{\frac12}\le Ch^{k+\frac12}\left(\int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left|\left[\xi_f^{n,\sharp}\right]_x\right|^2\mathrm{d}s_x\mathrm{d}v\right)^{\frac12}. \quad (4.58)
\end{aligned}$$

By (4.57), (4.58) and Young's inequality, there is

$$a_{h,1}\left(\eta_f^{n,\sharp};\xi_f^{n,\sharp}\right)\le C\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}^2+Ch^{2k+1}+\frac{1}{16}\int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left|\left[\xi_f^{n,\sharp}\right]_x\right|^2\mathrm{d}s_x\mathrm{d}v. \tag{4.59}$$

Step 2: To estimate $a_{h,2}(\eta_f^{n,\sharp},E_h^{n,\sharp},B_h^{n,\sharp};\xi_f^{n,\sharp})$. Let $E_0=\Pi_x^0E^{n,\sharp}$, $B_0=\Pi_x^0B^{n,\sharp}$ be the L^2 projection of $E^{n,\sharp},B^{n,\sharp}$ onto piecewise constant vector space with respect to T_h^x, then

$$\begin{aligned}
\left|\int_\Omega\eta_f^{n,\sharp}\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot\nabla_v\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v\right| &= \left|\int_\Omega\eta_f^{n,\sharp}\left(E_h^{n,\sharp}-E_0+v\times\left(B_h^{n,\sharp}-B_0\right)\right)\cdot\nabla_v\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v\right| \\
&\le\left(\left\|E_h^{n,\sharp}-E_0\right\|_{0,\infty,\Omega_x}+C\left\|B_h^{n,\sharp}-B_0\right\|_{0,\infty,\Omega_x}\right)\left\|\eta_f^{n,\sharp}\right\|_{0,\Omega}\left\|\nabla_v\xi_f^{n,\sharp}\right\|_{0,\Omega}.
\end{aligned}\tag{4.60}$$

The first equality of (4.60) is satisfied because $(E_0+v\times B_0)\cdot\nabla_v\xi_f^{n,\sharp}\in\mathcal{G}_h^k$, combining with the fact that the L^2 projection of $\eta_f^{n,\sharp}$ onto $\mathcal{G}_h^k$ is equal to zero. Since the operator Π_x^k is bounded in any L^p $(1\le p\le\infty)$ norm [11], we can further estimate $||E_h^{n,\sharp}-E_0||_{0,\infty,\Omega_x}\le||E_h^{n,\sharp}-\Pi_x^kE^{n,\sharp}||_{0,\infty,\Omega_x}+||\Pi_x^kE^{n,\sharp}-E_0||_{0,\infty,\Omega_x}$, and $||\Pi_x^kE^{n,\sharp}-E_0||_{0,\infty,\Omega_x}=||\Pi_x^k(E^{n,\sharp}-E_0)||_{0,\infty,\Omega_x}\le C||E^{n,\sharp}-E_0||_{0,\infty,\Omega_x}\le Ch_x||E^{n,\sharp}||_{1,\infty,\Omega_x}$. Thus, $||E_h^{n,\sharp}-E_0||_{0,\infty,\Omega_x}\le||\xi_E^{n,\sharp}||_{0,\infty,\Omega_x}+Ch_x$. Similar treatment can be applied to $||B_h^{n,\sharp}-B_0||_{0,\infty,\Omega_x}$. Therefore,

$$\begin{aligned}
\left|\int_\Omega\eta_f^{n,\sharp}\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot\nabla_v\xi_f^{n,\sharp}\mathrm{d}x\mathrm{d}v\right| &\le\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\infty,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\infty,\Omega_x}+Ch_x\right)\left\|\eta_f^{n,\sharp}\right\|_{0,\Omega}\left\|\nabla_v\xi_f^{n,\sharp}\right\|_{0,\Omega} \\
&\le\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\infty,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\infty,\Omega_x}+Ch_x\right)Ch^k\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} \\
&\le Ch^{k-\frac{d_x}{2}}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}\right)\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}+Ch^{k+1}\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}.
\end{aligned}\tag{4.61}$$

Here we have applied the inverse inequality in (2.5) to the last inequality above. The second term in $a_{h,2}(\eta_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; \xi_f^{n,\sharp})$ can be estimated as

$$
\begin{aligned}
&\left|\int_{T_h^x}\int_{\mathcal{E}_v}\left(\left\{\eta_f^{n,\sharp}\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\right\}_v+\frac{\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|}{2}\left[\eta_f^{n,\sharp}\right]_v\right)\cdot\left[\xi_f^{n,\sharp}\right]_v \mathrm{d}s_v\mathrm{d}x\right| \qquad (4.62)\\
&\le \int_{T_h^x}\int_{\mathcal{E}_v}\left(\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left(\left|\left\{\eta_f^{n,\sharp}\right\}_v\right|+\frac{\left|\left[\eta_f^{n,\sharp}\right]_v\right|}{2}\right)\right)\left|\left[\xi_f^{n,\sharp}\right]_v\right|\mathrm{d}s_v\mathrm{d}x\\
&\le \left(\int_{T_h^x}\int_{\mathcal{E}_v}2\left|\left\{\left(\eta_f^{n,\sharp}\right)^2\right\}_v\right|\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\mathrm{d}s_v\mathrm{d}x\right)^{\frac12}\left(\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left|\left[\xi_f^{n,\sharp}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x\right)^{\frac12}\\
&\le C\left\|\eta_f^{n,\sharp}\right\|_{0,T_h^x\times\mathcal{E}_v}\left(\left\|E_h^{n,\sharp}\right\|^{\frac12}_{0,\infty,\Omega_x}+\left\|B_h^{n,\sharp}\right\|^{\frac12}_{0,\infty,\Omega_x}\right)\left(\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left|\left[\xi_f^{n,\sharp}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x\right)^{\frac12}\\
&\le Ch^{k+\frac12}\left(\left\|E_h^{n,\sharp}\right\|^{\frac12}_{0,\infty,\Omega_x}+\left\|B_h^{n,\sharp}\right\|^{\frac12}_{0,\infty,\Omega_x}\right)\left(\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left|\left[\xi_f^{n,\sharp}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x\right)^{\frac12}.
\end{aligned}
$$

Note $||E_h^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x} \le ||\xi_E^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x} + ||\Pi_x^k E^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x} \le ||\xi_E^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x} + C||E^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x}$. From the inverse inequality in (2.5), there is $||\xi_E^{n,\sharp}||_{0,\infty,\Omega_x} \le Ch^{-\frac{d_x}{2}}||\xi_E^{n,\sharp}||_{0,\Omega_x}$. Since $||E^{n,\sharp}||_{0,\infty,\Omega_x}$ is bounded, $||E_h^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x} \le C(1+h^{-\frac{d_x}{4}}||\xi_E^{n,\sharp}||^{\frac12}_{0,\Omega_x})$. A similar estimate also holds for $||B_h^{n,\sharp}||^{\frac12}_{0,\infty,\Omega_x}$. Combing the results, we have

$$
\left\|E_h^{n,\sharp}\right\|^{\frac12}_{0,\infty,\Omega_x}+\left\|B_h^{n,\sharp}\right\|^{\frac12}_{0,\infty,\Omega_x}\le C\left(1+h^{-\frac{d_x}{4}}\left(\left\|\xi_E^{n,\sharp}\right\|^{\frac12}_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|^{\frac12}_{0,\Omega_x}\right)\right). \qquad (4.63)
$$

Equations (4.61)–(4.63) lead to

$$
\begin{aligned}
a_{h,2}\left(\eta_f^{n,\sharp},E_h^{n,\sharp},B_h^{n,\sharp};\xi_f^{n,\sharp}\right)&\le Ch^{k-\frac{d_x}{2}}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}\right)\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}+Ch^{k+1}\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega}\\
&\quad+Ch^{k+\frac12}\left(1+h^{-\frac{d_x}{4}}\left(\left\|\xi_E^{n,\sharp}\right\|^{\frac12}_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|^{\frac12}_{0,\Omega_x}\right)\right)\left(\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left|\left[\xi_f^{n,\sharp}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x\right)^{\frac12}\\
&\le Ch^{2k-d_x}\left(\left\|\xi_E^{n,\sharp}\right\|^2_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|^2_{0,\Omega_x}\right)+C\left\|\xi_f^{n,\sharp}\right\|^2_{0,\Omega}+Ch^{2k+2}\\
&\quad+Ch^{2k+1}\left(1+h^{-\frac{d_x}{2}}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x}+\left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x}\right)\right)+\frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left|\left[\xi_f^{n,\sharp}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x.
\end{aligned}
$$

Since $h^{2k+1}h^{-\frac{d_x}{2}}(||\xi_E^{n,\sharp}||_{0,\Omega_x}+||\xi_B^{n,\sharp}||_{0,\Omega_x}) \le C(h^{4k+2-d_x}+||\xi_E^{n,\sharp}||^2_{0,\Omega_x}+||\xi_B^{n,\sharp}||^2_{0,\Omega_x}) \le C(h^{2k+2}+||\xi_E^{n,\sharp}||^2_{0,\Omega_x}+||\xi_B^{n,\sharp}||^2_{0,\Omega_x})$ and $h^{2k-d_x}\le 1$ for $k\ge\frac{d_x}{2}$, we further have

$$
a_{h,2}\left(\eta_f^{n,\sharp},E_h^{n,\sharp},B_h^{n,\sharp};\xi_f^{n,\sharp}\right)\le Ch^{2k+1}+C\mathbf{N}^{n,\sharp}+\frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,\sharp}+v\times B_h^{n,\sharp}\right)\cdot n_v\right|\left|\left[\xi_f^{n,\sharp}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x. \qquad (4.64)
$$

Now with the results in (4.59) and (4.64), we can conclude (3.19b).

To get (3.19c) and (3.19d), note that with f being smooth with compact support in Ω_v, one has $[f^{n,\sharp}]_v = 0$, $\{f^{n,\sharp}\}_v = f^{n,\sharp}$ for any $e \in \mathcal{E}_v$. Thus, applying the divergence theorem gives

$$\begin{aligned} a_{h,2}\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right) &= \int_\Omega f^{n,\sharp}\left(E_h^{n,\sharp} + v \times B_h^{n,\sharp}\right) \cdot \nabla_v g \mathrm{d}x\mathrm{d}v - \int_{\Omega_x}\int_{\mathcal{E}_v} f^{n,\sharp}\left(E_h^{n,\sharp} + v \times B_h^{n,\sharp}\right) \cdot [g]_v \mathrm{d}s_v \mathrm{d}x \\ &= -\int_\Omega \nabla_v f^{n,\sharp} \cdot \left(E_h^{n,\sharp} + v \times B_h^{n,\sharp}\right) g \mathrm{d}x\mathrm{d}v. \end{aligned}$$

Similarly, $a_{h,2}(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g) = -\int_\Omega \nabla_v f^{n,\sharp} \cdot (E^{n,\sharp} + v \times B^{n,\sharp}) g \mathrm{d}x\mathrm{d}v$. Therefore,

$$\begin{aligned} \left|a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_h\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right)\right| &= \left|a_{h,2}\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_{h,2}\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right)\right| \\ &= \left|\int_\Omega \nabla_v f^{n,\sharp} \cdot \left(E_h^{n,\sharp} - E^{n,\sharp} + v \times \left(B_h^{n,\sharp} - B^{n,\sharp}\right)\right) g \mathrm{d}x\mathrm{d}v\right| \\ &\le \left\|\nabla_v f^{n,\sharp}\right\|_{0,\infty,\Omega}\left(\left\|e_E^{n,\sharp}\right\|_{0,\Omega} + C\left\|e_B^{n,\sharp}\right\|_{0,\Omega}\right) ||g||_{0,\Omega} \\ &\le C\left(\left\|e_E^{n,\sharp}\right\|_{0,\Omega} + C\left\|e_B^{n,\sharp}\right\|_{0,\Omega}\right) ||g||_{0,\Omega}. \end{aligned} \tag{4.65}$$

This gives (3.19c). Taking $g = \xi_f^{n,\sharp}$, and with the approximation property in (2.4), we further get (3.19d),

$$\begin{aligned} &\left|a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; \xi_f^{n,\sharp}\right) - a_h\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; \xi_f^{n,\sharp}\right)\right| \\ &\quad \le C\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\eta_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\eta_B^{n,\sharp}\right\|_{0,\Omega_x}\right)\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} \le Ch^{2k+2} + C\mathbf{N}^{n,\sharp}. \end{aligned}$$

4.3. Proof of Lemma 3.6

To get (3.22a), based on the definition of a_h, approximation property in (2.4) and inverse inequality (2.5),

$$\begin{aligned} &\left|a_h\left(\eta_f^{n,r}, E_h^{n,s+1}, B_h^{n,s+1}; g\right) - a_h\left(\eta_f^{n,r}, E_h^{n,s}, B_h^{n,s}; g\right)\right| \\ &= \left|a_{h,2}\left(\eta_f^{n,r}, E_h^{n,s+1}, B_h^{n,s+1}; g\right) - a_{h,2}\left(\eta_f^{n,r}, E_h^{n,s}, B_h^{n,s}; g\right)\right| \\ &\le \left|\int_\Omega \eta_f^{n,r}\left(E_h^{n,s+1} - E_h^{n,s} + v \times \left(B_h^{n,s+1} - B_h^{n,s}\right)\right) \nabla_v g \mathrm{d}x\mathrm{d}v\right| \\ &\quad + \left|\int_{\mathcal{T}_h^x}\int_{\mathcal{E}_v} \left\{\eta_f^{n,r}\right\}_v\left(E_h^{n,s+1} - E_h^{n,s} + v \times \left(B_h^{n,s+1} - B_h^{n,s}\right)\right) \cdot [g]_v \mathrm{d}s_v \mathrm{d}x\right| \\ &\quad + \frac{1}{2}\int_{\mathcal{T}_h^x}\int_{\mathcal{E}_v} \left|\left[\eta_f^{n,r}\right]_v\right| \left|\left|\left(E_h^{n,s+1} + v \times B_h^{n,s+1}\right) \cdot n_v\right| - \left|(E_h^{n,s} + v \times B_h^{n,s}) \cdot n_v\right|\right| |[g]_v| \mathrm{d}s_v \mathrm{d}x \\ &\le C\left(\left\|E_h^{n,s+1} - E_h^{n,s}\right\|_{0,\infty,\Omega_x} + \left\|B_h^{n,s+1} - B_h^{n,s}\right\|_{0,\infty,\Omega_x}\right)\left(\left\|\eta_f^{n,r}\right\|_{0,\Omega} \|\nabla_v g\|_{0,\Omega} + \left\|\eta_f^{n,r}\right\|_{0,\mathcal{T}_h^x \times \mathcal{E}_v} ||g||_{0,\mathcal{T}_h^x \times \mathcal{E}_v}\right) \\ &\le Ch^k ||g||_{0,\Omega}\left(\left\|E_h^{n,s+1} - E_h^{n,s}\right\|_{0,\infty,\Omega_x} + \left\|B_h^{n,s+1} - B_h^{n,s}\right\|_{0,\infty,\Omega_x}\right). \end{aligned}$$

By the *a priori* assumption and equation (3.8), we have

$$\begin{aligned} \left\|E_h^{n,s+1} - E_h^{n,s}\right\|_{0,\infty,\Omega_x} &= \left\|E_h^{n,s+1} - E^{n,s+1} + E^{n,s} - E_h^{n,s} + E^{n,s+1} - E^{n,s}\right\|_{0,\infty,\Omega_x} \\ &\le \left\|e_E^{n,s+1}\right\|_{0,\infty,\Omega_x} + \|e_E^{n,s}\|_{0,\infty,\Omega_x} + \left\|E^{n,s+1} - E^{n,s}\right\|_{0,\infty,\Omega_x} \le C(h + \tau). \end{aligned} \tag{4.66}$$

Similarly, $||B_h^{n,s+1} - B_h^{n,s}||_{0,\infty,\Omega_x} \le C(h+\tau)$. Therefore,

$$\left|a_h\left(\eta_f^{n,r}, E_h^{n,s+1}, B_h^{n,s+1}; g\right) - a_h\left(\eta_f^{n,r}, E_h^{n,s}, B_h^{n,s}; g\right)\right| \le C\left(1+\frac{\tau}{h}\right) h^{k+1}||g||_{0,\Omega}.$$

To get (3.22b), we follow the similar lines as to prove (3.22a) and obtain

$$\begin{aligned}
&\left|a_h\left(\xi_f^{n,r}, E_h^{n,s+1}, B_h^{n,s+1}; g\right) - a_h\left(\xi_f^{n,r}, E_h^{n,s}, B_h^{n,s}; g\right)\right| \\
&\le C\left(\left\|E_h^{n,s+1} - E_h^{n,s}\right\|_{0,\infty,\Omega_x} + \left\|B_h^{n,s+1} - B_h^{n,s}\right\|_{0,\infty,\Omega_x}\right)\left(\left\|\xi_f^{n,r}\right\|_{0,\Omega}||\nabla_v g||_{0,\Omega} + \left\|\xi_f^{n,r}\right\|_{0,T_h^x\times\mathcal{E}_v}||g||_{0,T_h^x\times\mathcal{E}_v}\right) \\
&\le C(h+\tau)\left(\left\|\xi_f^{n,r}\right\|_{0,\Omega}||\nabla_v g||_{0,\Omega} + \left\|\xi_f^{n,r}\right\|_{0,T_h^x\times\mathcal{E}_v}||g||_{0,T_h^x\times\mathcal{E}_v}\right) \le C\left(1+\frac{\tau}{h}\right)\left\|\xi_f^{n,r}\right\|_{0,\Omega}||g||_{0,\Omega}.
\end{aligned}$$

Here (4.66) is used for the second inequality, and the inverse inequality (2.5) is used for the third one.

4.4. Proof of Lemma 3.7

By definition, we have

$$\begin{aligned}
\mathcal{L}_{RK}(g) &= 2\mathcal{L}(g) - \mathcal{K}(g) - \mathcal{J}(g) \\
&= \int_{T_h} \frac{3\eta_f^{n+1} + 3\eta_f^{n} - 6\eta_f^{n,2} + 3T_f^n(x,v)}{\tau} g\,\mathrm{d}x\mathrm{d}v + 2a_h\left(f^{n,2}, E^{n,2}, B^{n,2}; g\right) - 2a_h\left(f_h^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right) \\
&\quad - \left(a_h\left(f^{n,1}, E^{n,1}, B^{n,1}; g\right) - a_h\left(f_h^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right)\right) - \left(a_h\left(f^{n}, E^{n}, B^{n}; g\right) - a_h\left(f_h^{n}, E_h^{n}, B_h^{n}; g\right)\right) \\
&= \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &= \frac{1}{\tau}\int_{T_h}\left(3\eta_f^{n+1} + 3\eta_f^{n} - 6\eta_f^{n,2} + 3T_f^n(x,v)\right) g\,\mathrm{d}x\mathrm{d}v, \\
\Lambda_2 &= 2a_h\left(f^{n,2}, E^{n,2}, B^{n,2}; g\right) - 2a_h\left(f^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right) \\
&\quad - \left(a_h\left(f^{n,1}, E^{n,1}, B^{n,1}; g\right) - a_h\left(f^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right)\right) - \left(a_h\left(f^{n}, E^{n}, B^{n}; g\right) - a\left(f^{n}, E_h^{n}, B_h^{n}; g\right)\right), \\
\Lambda_3 &= -2a_h\left(\eta_f^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right) + a_h\left(\eta_f^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right) + a_h\left(\eta_f^{n}, E_h^{n}, B_h^{n}; g\right), \\
\Lambda_4 &= 2a_h\left(\xi_f^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right) - a_h\left(\xi_f^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right) - a_h\left(\xi_f^{n}, E_h^{n}, B_h^{n}; g\right).
\end{aligned}$$

The term Λ_1 can be estimated by applying Lemmas 3.2 and 3.3,

$$|\Lambda_1| \le \frac{1}{\tau}\left(\left\|3\eta_f^{n+1} + 3\eta_f^{n} - 6\eta_f^{n,2}\right\|_{0,\Omega} + 3\left\|T_f^n(x,v)\right\|_{0,\Omega}\right)||g||_{0,\Omega} \le C\left(h^{k+1}+\tau^3\right)||g||_{0,\Omega}.$$

To estimate Λ_2, we apply (3.19c) in Lemma 3.4, the approximation property (2.4), and obtain

$$|\Lambda_2| \le C\sum_{\sharp=0}^{2}\left(\left\|e_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|e_B^{n,\sharp}\right\|_{0,\Omega_x}\right)||g||_{0,\Omega} \le C\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + h^{k+1}\right)||g||_{0,\Omega}.$$

For Λ_3, we first rewrite it as below.

$$\begin{aligned}
\Lambda_3 = &-2a_h\left(\eta_f^{n,2} - \eta_f^{n,1}, E_h^{n,2}, B_h^{n,2}; g\right) - a_h\left(\eta_f^{n,1} - \eta_f^{n}, E_h^{n}, B_h^{n}; g\right) \\
&-2a_h\left(\eta_f^{n,1}, E_h^{n,2}, B_h^{n,2}; g\right) + 2a_h\left(\eta_f^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right) \\
&-a_h\left(\eta_f^{n,1}, E_h^{n,1}, B_h^{n,1}; g\right) + a_h\left(\eta_f^{n,1}, E_h^{n}, B_h^{n}; g\right).
\end{aligned}$$

Applying Lemma 3.3 to the first line, and Lemma 3.6–(1) to the second and the third lines, one has

$$\Lambda_3 \le C\left(1+\frac{\tau}{h}\right) h^{k+1}||g||_{0,\Omega}.$$

For Λ_4, recall $G_2^n = 2\xi_f^{n,2} - \xi_f^{n,1} - \xi_f^n$, and using Lemma 3.6–(2), we have

$$\begin{aligned}
\Lambda_4 &= \left(2a_h\left(\xi_f^{n,2}, E_h^{n,2}, B_h^{n,2}; g\right) - 2a_h\left(\xi_f^{n,2}, E_h^{n,1}, B_h^{n,1}; g\right)\right) + a_h\left(G_2^n, E_h^{n,1}, B_h^{n,1}; g\right)\\
&\quad + \left(a_h\left(\xi_f^n, E_h^{n,1}, B_h^{n,1}; g\right) - a_h\left(\xi_f^n, E_h^n, B_h^n; g\right)\right)\\
&\le C\left(1+\frac{\tau}{h}\right)\left(\left\|\xi_f^{n,2}\right\|_{0,\Omega} + \left\|\xi_f^n\right\|_{0,\Omega}\right)||g||_{0,\Omega} + a_h\left(G_2^n, E_h^{n,1}, B_h^{n,1}; g\right).
\end{aligned}$$

Now we combine the estimates for $\Lambda_i, i = 1, \ldots 4$, and can get (3.23a),

$$\mathcal{L}_{RK}(g) \le C\left(\left(1+\frac{\tau}{h}\right)\sum_{\sharp=0}^{2}\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} + h^{k+1}\right) + \tau^3\right)||g||_{0,\Omega} + a_h\left(G_2^n, E_h^{n,1}, B_h^{n,1}; g\right).$$

To further bound the last term and hence obtain (3.23b), we apply the inverse inequality (2.5) and $||E_h^{n,1}||_{0,\infty,\Omega_x} + ||B_h^{n,1}||_{0,\infty,\Omega_x}$ being bounded by (4.51)–(4.52), and get

$$\begin{aligned}
&\left|a_h\left(G_2^n, E_h^{n,1}, B_h^{n,1}; g\right)\right| \le C\left(\|G_2^n\|_{0,\Omega}\,||\nabla_x g||_{0,\Omega} + \|G_2^n\|_{0,\mathcal{T}_h^v\times\mathcal{E}_x}\,||g||_{0,\mathcal{T}_h^v\times\mathcal{E}_x}\right)\\
&\quad + C\left(\left\|E_h^{n,1}\right\|_{0,\infty,\Omega_x} + \left\|B_h^{n,1}\right\|_{0,\infty,\Omega_x}\right)\left(\|G_2^n\|_{0,\Omega}\,||\nabla_v g||_{0,\Omega} + \|G_2^n\|_{0,\mathcal{T}_h^x\times\mathcal{E}_v}\,||g||_{0,\mathcal{T}_h^x\times\mathcal{E}_v}\right)\\
&\le \frac{C}{h}\,\|G_2^n\|_{0,\Omega}\,||g||_{0,\Omega}.
\end{aligned}$$

The estimate for $\mathcal{K}_{RK}(g)$ in (3.23c) can be proved very similarly.

4.5. Proof of Lemma 3.8

First we consider $a_{h,1}(G_1^n; G_2^n) + a_{h,1}(G_2^n; G_1^n)$.

$$\begin{aligned}
&a_{h,1}(G_1^n; G_2^n) + a_{h,1}(G_2^n; G_1^n)\\
&= \int_\Omega \left(G_1^n v\cdot\nabla_x G_2^n + G_2^n v\cdot\nabla_x G_1^n\right)\mathrm{d}x\mathrm{d}v - \int_{\mathcal{T}_h^v}\int_{\mathcal{E}_x}\left(v\left\{G_1^n\right\}_x + \frac{|v\cdot n_x|}{2}\left[G_1^n\right]_x\right)\cdot\left[G_2^n\right]_x\,\mathrm{d}s_x\mathrm{d}v\\
&\quad - \int_{\mathcal{T}_h^v}\int_{\mathcal{E}_x}\left(v\left\{G_2^n\right\}_x + \frac{|v\cdot n_x|}{2}\left[G_2^n\right]_x\right)\cdot\left[G_1^n\right]_x\,\mathrm{d}s_x\mathrm{d}v\\
&= \int_{\mathcal{T}_h^v}\int_{\mathcal{E}_x} v\cdot\left(\left[G_1^n G_2^n\right]_x - \left\{G_1^n\right\}_x\left[G_2^n\right]_x - \left\{G_2^n\right\}_x\left[G_1^n\right]_x\right)\mathrm{d}s_x\mathrm{d}v - \int_{\mathcal{T}_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left[G_1^n\right]_x\cdot\left[G_2^n\right]_x\,\mathrm{d}s_x\mathrm{d}v\\
&= -\int_{\mathcal{T}_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left[G_1^n\right]_x\cdot\left[G_2^n\right]_x\,\mathrm{d}s_x\mathrm{d}v. \qquad (4.67)
\end{aligned}$$

The third equality above is due to (2.3b). Hence,

$$
\begin{aligned}
|a_{h,1}(G_1^n;G_2^n)+a_{h,1}(G_2^n;G_1^n)| &\le \int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left(\frac{1}{32}|[G_1^n]_x|^2+8|[G_2^n]_x|^2\right)\mathrm{d}s_x\mathrm{d}v\\
&\le \frac{1}{16}\int_{T_h^v}\int_{\mathcal{E}_x}|v\cdot n_x|\left(\left|\left[\xi_f^n\right]_x\right|^2+\left|\left[\xi_f^{n,1}\right]_x\right|^2\right)\mathrm{d}s_x\mathrm{d}v+\frac{C}{h}\|G_2^n\|_{0,\Omega}^2\,. \qquad (4.68)
\end{aligned}
$$

The last inequality in (4.68) is satisfied because of the inequality $|[G_1^n]_x|^2=|[\xi_f^{n,1}-\xi_f^n]_x|^2\le 2(|[\xi_f^n]_x|^2+|[\xi_f^{n,1}]_x|^2)$ and the inverse inequality in (2.5). Similarly, we can also show

$$
\begin{aligned}
&\left|a_{h,2}\left(G_1^n,E_h^{n,1},B_h^{n,1};G_2^n\right)+a_{h,2}\left(G_2^n,E_h^{n,1},B_h^{n,1};G_1^n\right)\right|\\
&=\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,1}+v\times B_h^{n,1}\right)\cdot n_v\right|[G_1^n]_v\cdot[G_2^n]_v\,\mathrm{d}s_v\mathrm{d}x\le\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,1}+v\times B_h^{n,1}\right)\cdot n_v\right|\left|\left[\xi_f^{n,1}\right]_v\right||[G_2^n]_v|\,\mathrm{d}s_v\mathrm{d}x\\
&\quad+\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,1}+v\times B_h^{n,1}\right)\cdot n_v\right|\left|\left[\xi_f^n\right]_v\right||[G_2^n]_v|\,\mathrm{d}s_v\mathrm{d}x. \qquad (4.69)
\end{aligned}
$$

Denote each term on the right side of (4.69) as Λ_1 and Λ_2, respectively, then

$$
\begin{aligned}
\Lambda_1 &\le \int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,1}+v\times B_h^{n,1}\right)\cdot n_v\right|\left(\frac{1}{16}\left|\left[\xi_f^{n,1}\right]_v\right|^2+4|[G_2^n]_v|^2\right)\mathrm{d}s_v\mathrm{d}x\\
&\le \frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,1}+v\times B_h^{n,1}\right)\cdot n_v\right|\left|\left[\xi_f^{n,1}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x+C\left\|E_h^{n,1}+v\times B_h^{n,1}\right\|_{0,\infty,\Omega_x}\frac{\|G_2^n\|_{0,\Omega}^2}{h}\\
&\le \frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}\left|\left(E_h^{n,1}+v\times B_h^{n,1}\right)\cdot n_v\right|\left|\left[\xi_f^{n,1}\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x+\frac{C}{h}\|G_2^n\|_{0,\Omega}^2\,, \qquad (4.70)
\end{aligned}
$$

here we have used the fact that $||E_h^{n,1}+v\times B_h^{n,1}||_{0,\infty,\Omega_x}\le C$ due to (4.51)–(4.52). For the estimate of Λ_2,

$$
\begin{aligned}
\Lambda_2 &\le \int_{T_h^x}\int_{\mathcal{E}_v}\left|E_h^{n,1}-E_h^n+v\times\left(B_h^{n,1}-B_h^n\right)\right|\left|\left[\xi_f^n\right]_v\right||[G_2^n]_v|\,\mathrm{d}s_v\mathrm{d}x\\
&\quad+\int_{T_h^x}\int_{\mathcal{E}_v}|(E_h^n+v\times B_h^n)\cdot n_v|\left|\left[\xi_f^n\right]_v\right||[G_2^n]_v|\,\mathrm{d}s_v\mathrm{d}x\\
&\le C\left(\left\|E_h^{n,1}-E_h^n\right\|_{0,\infty,\Omega_x}+\left\|B_h^{n,1}-B_h^n\right\|_{0,\infty,\Omega_x}\right)\|\xi_f^n\|_{0,T_h^x\times\mathcal{E}_v}\|G_2^n\|_{0,T_h^x\times\mathcal{E}_v}\\
&\quad+\frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}|(E_h^n+v\times B_h^n)\cdot n_v|\left|\left[\xi_f^n\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x+\frac{C}{h}\|G_2^n\|_{0,\Omega}^2\,. \qquad (4.71)
\end{aligned}
$$

As implied in the Proof of Lemma 3.6, $||E_h^{n,1}-E_h^n||_{0,\infty,\Omega_x}\le C(h+\tau)$. So we further have

$$
\begin{aligned}
\Lambda_2 &\le C\left(1+\frac{\tau}{h}\right)\|\xi_f^n\|_{0,\Omega}\|G_2^n\|_{0,\Omega}+\frac{C}{h}\|G_2^n\|_{0,\Omega}^2+\frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}|(E_h^n+v\times B_h^n)\cdot n_v|\left|\left[\xi_f^n\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x\\
&\le C\left(1+\frac{\tau}{h}\right)\|\xi_f^n\|_{0,\Omega}^2+C\left(1+\frac{\tau}{h}+\frac{1}{h}\right)\|G_2^n\|_{0,\Omega}^2+\frac{1}{16}\int_{T_h^x}\int_{\mathcal{E}_v}|(E_h^n+v\times B_h^n)\cdot n_v|\left|\left[\xi_f^n\right]_v\right|^2\mathrm{d}s_v\mathrm{d}x. \qquad (4.72)
\end{aligned}
$$

Note that $1,\frac{\tau}{h}\le\frac{1}{h}$, hence $1+\frac{\tau}{h}+\frac{1}{h}\le\frac{C}{h}$. We now combine (4.68), (4.70) and (4.72) and conclude this lemma.

4.6. Proof of Lemma 3.11

We only consider the case when $\sharp = 0$. The proof for other cases follows the same line. For part (i), with divergence theorem and equality (2.3c),

$$\begin{aligned} b_h\left(\xi_E^n, \xi_B^n, \xi_f^n; \xi_E^n, \xi_B^n\right) &= \int_{\Omega_x} \xi_B^n \cdot (\nabla \times \xi_E^n)\,\mathrm{d}x - \int_{\Omega_x} \xi_E^n \cdot (\nabla \times \xi_B^n)\,\mathrm{d}x - \int_{\Omega_x}\left(\int_{\Omega_v} v\xi_f^n \mathrm{d}v\right)\cdot \xi_E^n \mathrm{d}x \\ &\quad + \int_{\mathcal{E}_x}\left(\{\xi_B^n\}_x - \frac{1}{2}[\xi_E^n]_{\tan}\right)\cdot [\xi_E^n]_{\tan}\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{\xi_E^n\}_x + \frac{1}{2}[\xi_B^n]_{\tan}\right)\cdot[\xi_B^n]_{\tan}\,\mathrm{d}s_x \\ &= \int_{\mathcal{E}_x}\left([\xi_E^n \times \xi_B^n]_x + \{\xi_B^n\}_x [\xi_E^n]_{\tan} - \{\xi_E^n\}_x [\xi_B^n]_{\tan}\right)\mathrm{d}s_x - \int_{\Omega_x}\left(\int_{\Omega_v} v\xi_f^n\mathrm{d}v\right)\cdot\xi_E^n\mathrm{d}x - \frac{1}{2}\mathbf{STAB}_{EB}^n \\ &= -\int_{\Omega_x}\left(\int_{\Omega_v} v\xi_f^n\mathrm{d}v\right)\cdot\xi_E^n\mathrm{d}x - \frac{1}{2}\mathbf{STAB}_{EB}^n \le C\left(\left\|\xi_f^n\right\|_{0,\Omega}^2 + \|\xi_E^n\|_{0,\Omega_x}^2\right) - \frac{1}{2}\mathbf{STAB}_{EB}^n. \end{aligned}$$

For part (ii), since $\nabla \times \xi_E^n \in \mathcal{U}_h^k$, there is $\int_{\Omega_x} \eta_B^n \cdot (\nabla \times \xi_E^n)\mathrm{d}x = 0$. Similarly, $\int_{\Omega_x} \eta_E^n \cdot (\nabla \times \xi_B^n)\mathrm{d}x = 0$. Then

$$\begin{aligned} &\left|b_h\left(\eta_E^n, \eta_B^n, \eta_f^n; \xi_E^n, \xi_B^n\right)\right| \\ &= \left|\int_{\Omega_x}\left(\int_{\Omega_v} v\eta_f^n\mathrm{d}v\right)\cdot\xi_E^n\mathrm{d}x + \int_{\mathcal{E}_x}\left(\{\eta_B^n\}_x - \frac{1}{2}[\eta_E^n]_{\tan}\right)\cdot[\xi_E^n]_{\tan}\,\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{\eta_E^n\}_x + \frac{1}{2}[\eta_B^n]_{\tan}\right)\cdot[\xi_B^n]_{\tan}\,\mathrm{d}s_x\right| \\ &\le C\left\|\eta_f^n\right\|_{0,\Omega}\|\xi_E^n\|_{0,\Omega_x} + C\left(\|\eta_E^n\|_{0,\mathcal{E}_x} + \|\eta_B^n\|_{0,\mathcal{E}_x}\right)\left(\int_{\mathcal{E}_x}|[\xi_B^n]_{\tan}|^2 + |[\xi_E^n]_{\tan}|^2\,\mathrm{d}s_x\right)^{\frac{1}{2}} \\ &\le Ch^{2k+1} + C\|\xi_E^n\|_{0,\mathcal{E}_x}^2 + \frac{1}{16}\mathbf{STAB}_{EB}^n. \end{aligned}$$

4.7. Proof of Lemma 3.13

Based on definitions of $\mathcal{S}_{RK}(U,V)$, X_2^n, Z_2^n, and G_2^n, in addition to Lemmas 3.2 and 3.3–(1), we have

$$\begin{aligned} \mathcal{S}_{RK}(U,V) &= 2\mathcal{S}(U,V) - \mathcal{R}(U,V) - \mathcal{Q}(U,V) \\ &\le \left(\frac{3\eta_E^{n+1} - 6\eta_E^{n,2} + 3\eta_E^n + 3T_E^n(x)}{\tau}, U\right)_{\Omega_x} + \left(\frac{3\eta_B^{n+1} - 6\eta_B^{n,2} + 3\eta_B^n + 3T_B^n(x)}{\tau}, V\right)_{\Omega_x} \\ &\quad + b_h\left(2e_E^{n,2} - e_E^{n,1} - e_E^n, 2e_B^{n,2} - e_B^{n,1} - e_B^n, 2e_f^{n,2} - e_f^{n,1} - e_f^n; U, V\right) \\ &\le C\left(h^{k+1} + \tau^3\right)(||U||_{0,\Omega_x} + ||V||_{0,\Omega_x}) + b_h\left(X_2^n, Z_2^n, G_2^n; U, V\right) - b_h\left(d_E^n, d_B^n, d_f^n; U, V\right), \end{aligned} \tag{4.73}$$

where $d_\star^n = 2\eta_\star^{n,2} - \eta_\star^{n,1} - \eta_\star^n$ with $\star = E, B, f$. We further apply (3.15c) in Lemma 3.3, and this gives (3.37a),

$$\mathcal{S}_{RK}(U,V) \le C\left(h^{k+1} + \tau h^k + \tau^3\right)(||U||_{0,\Omega_x} + ||V||_{0,\Omega_x}) + b_h\left(X_2^n, Z_2^n, G_2^n; U, V\right). \tag{4.74}$$

To further obtain (3.37b), we need to estimate $b_h(X_2^n, Z_2^n, G_2^n; U, V)$.

$$\begin{aligned} b_h\left(X_2^n, Z_2^n, G_2^n; U, V\right) &= \int_{\Omega_x} Z_2^n \cdot (\nabla \times U)\mathrm{d}x - \int_{\Omega_x} X_2^n \cdot (\nabla \times V)\mathrm{d}x - \int_{\Omega_x}\left(\int_{\Omega_v} G_2^n v\mathrm{d}v\right)\cdot U\mathrm{d}x \\ &\quad + \int_{\mathcal{E}_x}\left(\{Z_2^n\}_x - \frac{1}{2}[X_2^n]_{\tan}\right)\cdot[U]_{\tan}\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{X_2^n\}_x + \frac{1}{2}[Z_2^n]_{\tan}\right)\cdot[V]_{\tan}\mathrm{d}s_x \\ &\le \|Z_2^n\|_{0,\Omega_x}||\nabla \times U||_{0,\Omega_x} + \|X_2^n\|_{0,\Omega_x}||\nabla \times V||_{0,\Omega_x} + C\|G_2^n\|_{0,\Omega}||U||_{0,\Omega_x} \\ &\quad + C\left(\|Z_2^n\|_{0,\mathcal{E}_x} + \|X_2^n\|_{0,\mathcal{E}_x}\right)(||U||_{0,\mathcal{E}_x} + ||V||_{0,\mathcal{E}_x}). \end{aligned} \tag{4.75}$$

Now we can apply inverse equalities in (2.5), get

$$b_h\left(X_2^n, Z_2^n, G_2^n; U, V\right) \leq \frac{C}{h}\left(\|Z_2^n\|_{0,\Omega_x} + \|X_2^n\|_{0,\Omega_x}\right)\left(||U||_{0,\Omega_x} + ||V||_{0,\Omega_x}\right) + C\left\|G_2^n\right\|_{0,\Omega}||U||_{0,\Omega_x},$$

hence (3.37b).

Similarly, one can get the estimate (3.37c) for $\mathcal{R}_{RK}(U,V)$.

4.8. Proof of Lemma 3.14

From the definition of b_h,

$$\begin{aligned}
&b_h\left(X_1^n, Z_1^n, G_1^n; X_2^n, Z_2^n\right) + b_h\left(X_2^n, Z_2^n, G_2^n; X_1^n, Z_1^n\right)\\
&= \int_{\Omega_x}\left(Z_1^n\cdot\left(\nabla\times X_2^n\right)\mathrm{d}x - X_1^n\cdot\left(\nabla\times Z_2^n\right) + Z_2^n\cdot\left(\nabla\times X_1^n\right)\mathrm{d}x - X_2^n\cdot\left(\nabla\times Z_1^n\right)\right)\mathrm{d}x\\
&\quad + \int_{\mathcal{E}_x}\left(\{Z_1^n\}_x - \frac{1}{2}\left[X_1^n\right]_{\tan}\right)\cdot\left[X_2^n\right]_{\tan}\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{X_1^n\}_x + \frac{1}{2}\left[Z_1^n\right]_{\tan}\right)\cdot\left[Z_2^n\right]_{\tan}\mathrm{d}s_x\\
&\quad + \int_{\mathcal{E}_x}\left(\{Z_2^n\}_x - \frac{1}{2}\left[X_2^n\right]_{\tan}\right)\cdot\left[X_1^n\right]_{\tan}\mathrm{d}s_x - \int_{\mathcal{E}_x}\left(\{X_2^n\}_x + \frac{1}{2}\left[Z_2^n\right]_{\tan}\right)\cdot\left[Z_1^n\right]_{\tan}\mathrm{d}s_x\\
&\quad - \int_{\Omega_x}\left(\int_{\Omega_v} G_1^n v\mathrm{d}v\right)\cdot X_2^n\mathrm{d}x - \int_{\Omega_x}\left(\int_{\Omega_v} G_2^n v\mathrm{d}v\right)\cdot X_1^n\mathrm{d}x.
\end{aligned}$$

Using divergence theorem and equality (2.3c), in addition to inverse inequality (2.5) and Young's inequality, we can simplify the inequality above as

$$\begin{aligned}
&b_h\left(X_1^n, Z_1^n, G_1^n; X_2^n, Z_2^n\right) + b_h\left(X_2^n, Z_2^n, G_2^n; X_1^n, Z_1^n\right)\\
&= -\int_{\mathcal{E}_x}\left(\left[X_1^n\right]_{\tan}\cdot\left[X_2^n\right]_{\tan} + \left[Z_1^n\right]_{\tan}\cdot\left[Z_2^n\right]_{\tan}\right)\mathrm{d}s_x - \int_{\Omega}\left(G_1^n v\cdot X_2^n + G_2^n v\cdot X_1^n\right)\mathrm{d}x\mathrm{d}v\\
&\leq \int_{\mathcal{E}_x}\left(\frac{1}{32}\left|\left[X_1^n\right]_{\tan}\right|^2 + 8\left|\left[X_2^n\right]_{\tan}\right|^2 + \frac{1}{32}\left|\left[Z_1^n\right]_{\tan}\right|^2 + 8\left|\left[Z_2^n\right]_{\tan}\right|^2\right)\mathrm{d}s_x + C\sum_{j=1}^{2}\left(\left\|G_j^n\right\|_{0,\Omega}^2 + \left\|X_j^n\right\|_{0,\Omega_x}^2\right)\\
&\leq \frac{C}{h}\left(\|X_2^n\|_{0,\Omega_x}^2 + \|Z_2^n\|_{0,\Omega_x}^2\right) + C\sum_{j=1}^{2}\left(\left\|G_j^n\right\|_{0,\Omega}^2 + \left\|X_j^n\right\|_{0,\Omega_x}^2\right) + \frac{1}{16}\left(\mathbf{STAB}_{EB}^n + \mathbf{STAB}_{EB}^{n,1}\right).
\end{aligned} \tag{4.76}$$

The last inequality of (4.76) is due to that

$$\left|\left[Z_1^n\right]_{\tan}\right|^2 = \left|\left[\xi_B^{n,1} - \xi_B^n\right]_{\tan}\right|^2 \leq 2\left|\left[\xi_B^{n,1}\right]_{\tan}\right|^2 + 2\left|\left[\xi_B^n\right]_{\tan}\right|^2,$$

and a similar bound for $\left|\left[X_1^n\right]_{\tan}\right|^2$.

4.9. Proof of Lemma 3.18

For any $g \in \mathcal{G}_h^k$, we first consider $a_h(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g) - a_h(f_h^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g)$, $\sharp = 0, 1, 2$,

$$\begin{aligned}
&a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_h\left(f_h^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right) \\
&= -a_h\left(\eta_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right) + \left(a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_h\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right)\right) + a_h\left(\xi_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right).
\end{aligned}$$

With Cauchy–Schwarz inequality, approximation property in (2.4), inverse inequality (2.5), boundedness of $||E_h^{n,1}||_{0,\infty,\Omega_x} + ||B_h^{n,1}||_{0,\infty,\Omega_x}$ in (4.51) and (4.52), we have

$$\begin{aligned}
\left|a_h\left(\eta_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right)\right| &\le C\left(\left\|\eta_f^{n,\sharp}\right\|_{0,\Omega} \|\nabla_x g\|_{0,\Omega} + \left\|\eta_f^{n,\sharp}\right\|_{0,T_h^v\times\mathcal{E}_x} ||g||_{0,T_h^v\times\mathcal{E}_x}\right) \\
&\quad + C\left(\left\|E_h^{n,\sharp}\right\|_{0,\infty,\Omega_x} + \left\|B_h^{n,\sharp}\right\|_{0,\infty,\Omega_x}\right)\left(\left\|\eta_f^{n,\sharp}\right\|_{0,\Omega} \|\nabla_v g\|_{0,\Omega} + \left\|\eta_f^{n,\sharp}\right\|_{0,T_h^v\times\mathcal{E}_x} ||g||_{0,T_h^v\times\mathcal{E}_x}\right) \\
&\le Ch^k ||g||_{0,\Omega}, \qquad (4.77)
\end{aligned}$$

$$\begin{aligned}
\left|a_h\left(\xi_f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right)\right| &\le C\left(\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} \|\nabla_x g\|_{0,\Omega} + \left\|\xi_f^{n,\sharp}\right\|_{0,T_h^v\times\mathcal{E}_x} ||g||_{0,T_h^v\times\mathcal{E}_x}\right) \\
&\quad + C\left(\left\|E_h^{n,\sharp}\right\|_{0,\infty,\Omega_x} + \left\|B_h^{n,\sharp}\right\|_{0,\infty,\Omega_x}\right)\left(\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} \|\nabla_v g\|_{0,\Omega} + \left\|\xi_f^{n,\sharp}\right\|_{0,T_h^v\times\mathcal{E}_x} ||g||_{0,T_h^v\times\mathcal{E}_x}\right) \\
&\le \frac{C}{h}\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} ||g||_{0,\Omega}. \qquad (4.78)
\end{aligned}$$

Following the derivation to get (4.65) in Lemma 3.4, we have

$$\begin{aligned}
&a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_h\left(f^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right) \\
&\quad = \int_\Omega \nabla_v f^{n,\sharp}\cdot\left(E_h^{n,\sharp} - E^{n,\sharp} + v\times\left(B_h^{n,\sharp} - B^{n,\sharp}\right)\right) g\mathrm{d}x\mathrm{d}v \\
&\quad \le C\left(\left\|e_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|e_B^{n,\sharp}\right\|_{0,\Omega_x}\right)||g||_{0,\Omega} \le C\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + h^{k+1}\right)||g||_{0,\Omega}. \qquad (4.79)
\end{aligned}$$

Equations (4.77)–(4.79) lead to

$$\begin{aligned}
a_h\left(f^{n,\sharp}, E^{n,\sharp}, B^{n,\sharp}; g\right) - a_h\left(f_h^{n,\sharp}, E_h^{n,\sharp}, B_h^{n,\sharp}; g\right) &\le C\left(\left\|\xi_E^{n,\sharp}\right\|_{0,\Omega_x} + \left\|\xi_B^{n,\sharp}\right\|_{0,\Omega_x} + h^k\right)||g||_{0,\Omega} \\
&\quad + \frac{C}{h}\left\|\xi_f^{n,\sharp}\right\|_{0,\Omega} ||g||_{0,\Omega}. \qquad (4.80)
\end{aligned}$$

Now based on (4.80) and Lemma 3.3, we can bound $\mathcal{J}(\xi_f^{n,1})$.

$$\begin{aligned}
\mathcal{J}\left(\xi_f^{n,1}\right) &= \left(\frac{\eta_f^{n,1} - \eta_f^n}{\tau}, \xi_f^{n,1}\right)_\Omega + a_h\left(f^n, E^n, B^n; \xi_f^{n,1}\right) - a_h\left(f_h^n, E_h^n, B_h^n; \xi_f^{n,1}\right) \\
&\le C\left(\|\xi_E^n\|_{0,\Omega_x} + \|\xi_B^n\|_{0,\Omega_x} + h^k\right)\left\|\xi_f^{n,1}\right\|_{0,\Omega} + \frac{C}{h}\|\xi_f^n\|_{0,\Omega}\left\|\xi_f^{n,1}\right\|_{0,\Omega}. \qquad (4.81)
\end{aligned}$$

Based on equations (4.81) and (3.10), we get

$$\left\|\xi_f^{n,1}\right\|_{0,\Omega}^2 = \left(\xi_f^{n,1}, \xi_f^n\right)_\Omega + \tau \mathcal{J}\left(\xi_f^{n,1}\right)$$
$$\leq C\tau\left(\|\xi_E^n\|_{0,\Omega_x} + \|\xi_B^n\|_{0,\Omega_x} + h^k\right)\left\|\xi_f^{n,1}\right\|_{0,\Omega} + \left(1 + C\frac{\tau}{h}\right)\|\xi_f^n\|_{0,\Omega}\left\|\xi_f^{n,1}\right\|_{0,\Omega}.$$

Cancel $||\xi_f^{n,1}||_{0,\Omega}$ from both sides of the inequality, and take the square of both sides, we have

$$\left\|\xi_f^{n,1}\right\|_{0,\Omega}^2 \leq C\tau^2\left(\|\xi_E^n\|_{0,\Omega_x}^2 + \|\xi_B^n\|_{0,\Omega_x}^2 + h^{2k}\right) + \left(1 + C\frac{\tau}{h}\right)^2\|\xi_f^n\|_{0,\Omega}^2. \tag{4.82}$$

By further using $\tau \leq \alpha h$, we obtain the upper bound of $||\xi_f^{n,1}||_{0,\Omega}^2$. Similarly, $||\xi_f^{n,2}||_{0,\Omega}^2$ can be estimated based on (4.80) and (3.10). Likewise, we can establish the estimates for $||\xi_B^{n,\sharp}||_{0,\Omega_x}, ||\xi_E^{n,\sharp}||_{0,\Omega_x}$ for $\sharp = 1, 2$.

5. Extension and conclusion

In this paper we prove the error estimates of fully discrete methods, which involve a third order Runge–Kutta time discretization and upwind DG discretizations of arbitrary order of accuracy in phase domain, for solving the Vlasov–Maxwell system. When the exact solutions have enough regularity, we show that the L^2 errors of the numerical solutions by such methods are of $O(h^{k+\frac{1}{2}} + \tau^3)$ for $k > \frac{d_x+1}{2}$. The third order Runge–Kutta time integration contributes to the error $O(\tau^3)$, while the error from the DG approximation is $O(h^{k+\frac{1}{2}})$, which is expected for hyperbolic systems with upwind numerical fluxes on general meshes.

The techniques used in this paper can be applied to the RKDG methods which involve other numerical fluxes, such as central or alternating fluxes,

$$(\widehat{n_x \times E_h}, \widehat{n_x \times B_h}) = (n_x \times \{E_h\}, n_x \times \{B_h\}), \qquad \text{(central)}$$
$$(\widehat{n_x \times E_h}, \widehat{n_x \times B_h}) = n_x \times (E_h^-, B_h^+), \text{or } n_x \times (E_h^+, B_h^-), \qquad \text{(alternating)},$$

in the Maxwell solver. It was shown that these fluxes will result better energy conservation in semi-discrete DG methods [5]. On the other hand, for RKDG methods with such fluxes, the stabilization mechanism $\mathbf{STAB}_{EB}^{n,\sharp}$, $\sharp = 0, 1, 2$, in the form of the tangential jump

$$\int_{\mathcal{E}_x} |[\xi_E^{n,\sharp}]_{\tan}|_{0,\Omega_x}^2 + |[\xi_B^{n,\sharp}]_{\tan}|_{0,\Omega_x}^2 \mathrm{d}s_x$$

is no longer available from the Maxwell solver (see part (i) of Lem. 3.11), and this in general will lead to a sub-optimal L^2-norm error estimate: $Ch^k + C\tau^3$. (For alternating fluxes, better estimates can be obtained on Cartesian meshes and with the tensor structured polynomial discrete spaces.) With some insignificant modification to the details, almost the same error estimates can be established for the RKDG methods solving the smooth solutions of the relativistic Vlasov–Maxwell system of one species [21, 22],

$$\begin{cases} \partial_t f + \dfrac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f + (E + \dfrac{v}{\sqrt{1+|v|^2}} \times B) \cdot \nabla_v f = 0, \\ \partial_t E = \nabla \times B - J, \quad \partial_t B = -\nabla \times E, \\ \nabla \cdot E = \rho - \rho_i, \quad \nabla \cdot B = 0, \end{cases}$$

with

$$\rho(x,t) = \int_{\Omega_v} f(x,v,t)\mathrm{d}v, \quad J(x,t) = \int_{\Omega_v} \frac{v}{\sqrt{1+|v|^2}} f(x,v,t)\mathrm{d}v.$$

References

[1] B. Ayuso, J.A. Carrillo and C.-W. Shu, Discontinuous Galerkin methods for the one-dimensional Vlasov–Poisson system. *Kinetic and Related Models* **4** (2011) 955–989.

[2] B. Ayuso, J.A. Carrillo and C.-W. Shu, Discontinuous Galerkin methods for the multi-dimensional Vlasov–Poisson problem. *Math. Models Methods Appl. Sci.* **22** (2012) 1250042.

[3] C.K. Birdsall and A.B. Langdon, Plasma Physics Via Computer Simulation. McGraw-Hill, New York (1985).

[4] Y. Cheng and I.M. Gamba, Numerical study of one-dimensional Vlasov–Poisson equations for infinite homogeneous stellar systems. *Commun. Nonlinear Sci. Numer. Simul.* **17** (2012) 2052–2061.

[5] Y. Cheng, I.M. Gamba, F. Li and P.J. Morrison, Discontinuous Galerkin methods for Vlasov–Maxwell equations. *SIAM J. Numer. Anal.* **52** (2014) 1017–1049.

[6] C.Z. Cheng and G. Knorr, The integration of the Vlasov equation in configuration space. *J. Comput. Phys.* **22** (1976) 330–351.

[7] P.G. Ciarlet, Finite element method for elliptic problems. Noth-Holland, Amsterdam (1978).

[8] B. Cockburn and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework. *Math. Comput.* **52** (1989) 411–435.

[9] B. Cockburn and C.-W. Shu, The Runge–Kutta local projection p1-discontinuous Galerkin finite element method for scalar conservation laws. *ESAIM: M2AN* **25** (1991) 337–361.

[10] G.H. Cottet and P.A. Raviart, On Particle in Cell methods for the Vlasov–Poisson equations. *Transport Theory Stat. Phys.* **15** (1986) 1–31.

[11] M. Crouzeix and V. Thomée, The stability in L_p and W_p^1 of the L_2-projection onto finite element function spaces. *Math. Comput.* **178** (1987) 521–532.

[12] B. Eliasson, Numerical modelling of the two-dimensional Fourier transformed Vlasov–Maxwell system, *J. Comput. Phys.* **190** (2003) 501–522.

[13] B. Eliasson, Numerical simulations of the Fourier-transformed Vlasov–Maxwell system in higher dimensions-theory and applications. *Transport Theory Stat. Phys.* **39** (2011) 387–465.

[14] R.E. Heath, I.M. Gamba, P.J. Morrison and C. Michler, A discontinuous Galerkin method for the Vlasov–Poisson system. *J. Comput. Phys.* **231** (2012) 1140–1174.

[15] G. Jacobs and J.S. Hesthaven, Implicit-explicit time integration of a high-order particle-in-cell method with hyperbolic divergence cleaning. *Comput. Phys. Commun.* **180** (2009) 1760–1767.

[16] A.J. Klimas and W.M. Farrell, A splitting algorithm for Vlasov simulation with filamentation filtration. *J. Comput. Phys.* **110** (1994) 150–163.

[17] M.C. Pinto and M. Mehrenberger, Convergence of an adaptive semi-Lagrangian scheme for the Vlasov–Poisson system. *Numer. Math.* **108** (2008) 407–444.

[18] J.-M. Qiu and C.-W. Shu, Conservative high order semi-Lagrangian finite difference WENO methods for advection in incompressible flow. *J. Comput. Phys.* **230** (2011) 863–889.

[19] C.-W. Shu and S. Osher, Efficient implementation of essentially non-oscillatory shock capturing schemes. *J. Comput. Phys.* **77** (1988) 439–471.

[20] E. Sonnendrücker, J. Roche, P. Bertrand and A. Ghizzo, The semi-Lagrangian method for the numerical resolution of the Vlasov equation. *J. Comput. Phys.* **149** (1999) 201–220.

[21] A. Suzuki and T. Shigeyama, A conservative scheme for the relativistic Vlasov–Maxwell system. *J. Comput. Phys.* **229** (2010) 1643–1660.

[22] H. Yang and F. Li, Discontinuous Galerkin methods for relativitistic Vlasov–Maxwell equations, in preparation.

[23] H. Yang, F. Li and J. Qiu, Dispersion and dissipation errors of two fully discrete discontinuous Galerkin methods. *J. Sci. Comput.* **55** (2013) 552–574.

[24] S.I. Zaki, L.R.T. Gardner and T.J.M. Boyd, A finite element code for the simulation of one-dimensional Vlasov plasmas. I. Theory. *J. Comput. Phys.* **79** (1988) 184–199.

[25] S.I. Zaki, T.J.M. Boyd and L.R.T. Gardner, A finite element code for the simulation of one-dimensional Vlasov plasmas. II. Applications. *J. Comput. Phys.* **79** (1988) 200–208.

[26] Q. Zhang and C-W Shu, Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for scalar conservation laws. *SIAM J. Numer. Anal.* **42** (2004) 641–666.

[27] Q. Zhang and C-W Shu, Error estimates to smooth solutions of Runge–Kutta discontinuous Galerkin methods for symmetrizable systems of conservation laws. *SIAM J. Numer. Anal.* **44** (2006) 1703–1720.

[28] Q. Zhang and C-W Shu, Stability analysis and *a priori* error estimates to the third order explicit Runge–Kutta discontinuous Galerkin method for scalar conservation laws. *SIAM J. Numer. Anal.* **48** (2010) 1038–1063.

[29] X. Zhong and C.-W. Shu, Numerical resolution of discontinuous Galerkin methods for time dependent wave equations. *Comput. Methods Appl. Mech. Eng.* **200** (2011) 2814–2827.