

COMPLEXITY ISSUES OF PERFECT SECURE DOMINATION IN GRAPHS

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Abstract. Let $G = (V, E)$ be a simple, undirected and connected graph. A dominating set S is called a secure dominating set if for each $u \in V \setminus S$, there exists $v \in S$ such that $(u, v) \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G . If further the vertex $v \in S$ is unique, then S is called a perfect secure dominating set (PSDS). The perfect secure domination number $\gamma_{ps}(G)$ is the minimum cardinality of a perfect secure dominating set of G . Given a graph G and a positive integer k , the perfect secure domination (PSDOM) problem is to check whether G has a PSDS of size at most k . In this paper, we prove that PSDOM problem is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs, planar graphs and dually chordal graphs. We propose a linear time algorithm to solve the PSDOM problem in caterpillar trees and also show that this problem is linear time solvable for bounded tree-width graphs and threshold graphs, a subclass of split graphs. Finally, we show that the domination and perfect secure domination problems are not equivalent in computational complexity aspects.

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1. INTRODUCTION

Throughout this paper all graphs $G = (V, E)$ should be finite, simple (*i.e.*, without self-loops and multiple edges), undirected and connected with vertex set V and edge set E . The *open neighbourhood* of v in G is $N_G(v) = \{u \in V | (u, v) \in E(G)\}$ and the *closed neighbourhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. The degree $\deg(v)$ of a vertex v is $|N_G(v)|$. An *induced subgraph* is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D \rangle$. A *clique* is a subset of vertices of G such that every two distinct vertices in the subset are adjacent. An *independent set* is a set of vertices in which no two vertices are adjacent. A vertex v of G is said to be a *pendant vertex* if $\deg(v) = 1$. A vertex v is called *support vertex* if it is adjacent to at least one pendant vertex. A vertex v is called *isolated vertex* if $\deg(v) = 0$. An edge of G is said to be a *pendant edge* if one of its vertices is a pendant vertex. A graph is *planar* if it can be drawn on the plane such that no two edges cross each other. A *star* is a tree on n vertices with one vertex having degree $n - 1$ and the other $n - 1$ vertices having degree 1. A *Comb* is a tree obtained by joining a single pendant edge to each vertex of a path. A *split graph* is a graph in which vertex set can be partitioned into an independent set and a clique. A vertex $u \in N_G[v]$ is a *maximum neighbour* of v in G if $N_G[w] \subseteq N_G[u]$ holds

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for each $w \in N_G[v]$. A vertex ordering v_1, v_2, \dots, v_n is a maximum neighbourhood ordering if for each $i < n$, v_i has a maximum neighbour in $\langle \{v_i, v_{i+1}, \dots, v_n\} \rangle$. A graph is *dually chordal* if it has a maximum neighbourhood ordering. A bipartite graph $G = (X, Y, E)$ is called *tree convex* if there exists a tree $T = (X, F)$ such that, for each y in Y , the neighbours of y induce a subtree in T . When T is a star (comb), G is called *star (comb) convex bipartite graph* [10]. For undefined terminology and notations refer to [18].

A vertex v in G dominates the vertices of its closed neighborhood. A set of vertices $S \subseteq V$ is a *dominating set* (DS) of G if for every vertex $u \in V \setminus S$, there exists at least one vertex $v \in S$ such that $(u, v) \in E$, i.e., $N_G[S] = V$. A vertex $u \in V \setminus S$ is said to be undominated if $N_G(u) \cap S = \emptyset$. The concept of *perfect domination* (PD) has been introduced by P.M. Weichsel *et al.* in [17]. A dominating set S of G is called a *perfect dominating set* (PDS) of G if every vertex in $V \setminus S$ is adjacent to exactly one vertex in S .

The concept of *secure domination* has been introduced by E.J. Cockayne *et al.* in [2] and is studied, for example, in [2, 12, 13]. A set of vertices $S \subseteq V$ is a *secure dominating set* (SDS) of G , if S is a dominating set of G and for each $u \in V \setminus S$, there exists $v \in S$ such that $(u, v) \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G . In this case, we say that u is **S -defended** by v or v **S -defends** u .

The concept of *perfect secure domination* has been introduced by Rashmi *et al.* in [16]. A set $S \subseteq V$ is said to be a *perfect secure dominating set* (PSDS) of G , if S is a dominating set of G and for every vertex $u \in V \setminus S$, there exists a unique vertex $v \in S$ such that $(u, v) \in E$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set in G . In this case, we say that u is **perfectly secure dominated** by v or v **perfectly secure dominates** u . In other words S is not a PSDS of G if any vertex in $V \setminus S$ is S -defended by more than one vertex in S . The minimum cardinality of a DS, PDS, SDS and PSDS respectively are called *domination number* $\gamma(G)$, *perfect domination number* $\gamma_p(G)$, *secure domination number* $\gamma_s(G)$ and *perfect secure domination number* $\gamma_{ps}(G)$ of G . The MINIMUM PERFECT SECURE DOMINATING SET (MPSDS) problem is to find a perfect secure dominating set of minimum cardinality. The computational complexity of perfect secure domination problem has not been studied in the literature.

The following results are proved in [16].

Theorem 1.1. ([16]) For any path P_n with $n \geq 2$, we have $\gamma_{ps}(P_n) = \lceil \frac{3n}{7} \rceil$.

Theorem 1.2. ([16]) For any cycle C_n with $n \geq 4$,

$$\gamma_{ps}(C_n) = \begin{cases} \lceil \frac{3n}{7} \rceil + 1, & \text{if } n \equiv 2 \pmod{7} \\ \lceil \frac{3n}{7} \rceil, & \text{otherwise} \end{cases}$$

Theorem 1.3. ([16]) For the complete bipartite graph $G = K_{r,s}$ with $r \leq s$ we have,

$$\gamma_{ps}(G) = \begin{cases} s, & \text{if } r = 1 \\ 2, & \text{if } r = s = 2 \\ r + s, & \text{otherwise} \end{cases}$$

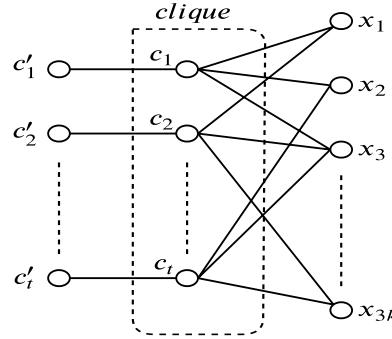
The following observation regarding perfect secure dominating set of a graph will be used throughout this paper.

Observation 1.4. If Q is a support vertices set and S is a minimum PSDS of G with $S \cap Q \neq Q$ then there exists a minimum PSDS S' of G with $|S'| = |S|$ such that $S' \cap Q = Q$.

Proof. Let S be a minimum PSDS of G such that which does not contain a support vertex. Let u be a pendant vertex and v be a support vertex in graph G such that $(u, v) \in E$. Clearly, for the domination property has to be satisfied either $u \in S$ or $v \in S$. If $v \notin S$ then $u \in S$ i.e., v is S -defended by only u . From all such pairs, we can define a minimum PSDS S' such that $S' = (S \setminus \{u\}) \cup \{v\}$. \square

2. COMPLEXITY RESULTS

In this section, we show that the perfect secure domination problem remains NP-complete even when restricted to split, star convex bipartite, comb convex bipartite, planar and dually chordal graphs.

FIGURE 1. Construction of a split graph from an instance of $X3C$.

The decision version of perfect secure domination problem is defined as follows.

Perfect Secure Domination (PSDOM)

INSTANCE : A simple, undirected graph G and a positive integer q .

QUESTION : Does G have a perfect secure dominating set of size at most q ?

2.1. Perfect secure domination in split, star convex and comb convex bipartite graphs

To show that the PSDOM for split, star convex bipartite and comb convex bipartite graphs is NP-complete, we use a well known NP-complete problem, called Exact Cover by 3-Sets ($X3C$) [6], which is defined as follows.

Exact Cover by 3 Sets (X3C)

INSTANCE : A finite set X with $|X| = 3k$ and a collection C of 3-element subsets of X .

QUESTION : Is there a subcollection C' of C such that every element of X appears in exactly one member of C' ?

A variant of $X3C$ in which each element appears in at least two subsets has also been proved as NP-complete [7]. Through out this subsection, we use this variant of $X3C$ problem.

Theorem 2.1. *PSDOM is NP-complete for split graphs.*

Proof. Given a split graph G , a positive integer p and an arbitrary set $S \subseteq V$, we can check in polynomial time whether S is a PSDS of G of size at most p . Hence PSDOM is in NP.

Let $X = \{x_1, x_2, \dots, x_{3k}\}$ and $C = \{c_1, c_2, \dots, c_t\}$ be an arbitrary instance of $X3C$, where $|X| = 3k$, $|C| = t$ and $t \geq 2k$. We now construct an instance of PSDOM for split graph from the given instance of $X3C$ as follows. Construct a graph $G(V, E)$ by creating vertices x_i for each $x_i \in X$, c_i and c'_i for each $c_i \in C$. Add edges (c_i, c'_i) for each $c_i \in C$, (c_j, x_i) if $x_i \in c_j$ and $(c_i, c_j) \forall c_i, c_j \in C$ where $i \neq j$. Let $A = \{c_i : 1 \leq i \leq t\}$ and $B = \{x_i : 1 \leq i \leq 3k\} \cup \{c'_i : 1 \leq i \leq t\}$. It is clear by construction that the graph shown in Figure 1 is a split graph since B forms an independent set and A forms a clique. Clearly $|V| = 2t + 3k$ and $|E| = 4t + \binom{t}{2}$. Next we show that, $X3C$ has a solution if and only if G has PSDS of size at most $k + t$.

Suppose C' is a solution for $X3C$ with $|C'| = k$. Then $S = \{c_i, c'_i : c_i \in C'\} \cup \{c_i : c_i \notin C'\}$. Clearly $|S| \leq k + t$. From the solution of $X3C$, it follows that S dominates all x_i 's and each $c'_i \notin S$ is dominated by c_i . Hence, S is a dominating set and it is a SDS since each $x_i \in X$ is S -defended by c_j where $(x_i, c_j) \in E$ and $(c_j, c'_j) \in S$, each $c'_i \notin S$ is S -defended by c_i . It is also PSDS since each element in $V \setminus S$ is S -defended by exactly one element of S .

Conversely, suppose that S is a PSDS of G such that $|S| \leq k + t$. From Observation 1.4, we can assume that $S \cap C = C$. We have the following lemma.

Lemma 2.2. $|S \cap X| = 0$.

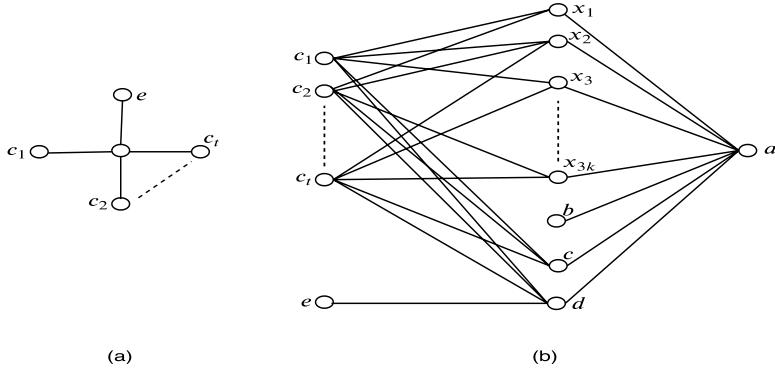


FIGURE 2. (a) Star Graph. (b) Construction of a star convex bipartite graph from an instance of X3C.

Proof. (Proof by contradiction) Let $|S \cap X| = m$. Remaining $3k - m$ vertices of X need at least $\lceil \frac{3k-m}{3} \rceil$ number of c 's to S -defend them. But these c 's can not S -defend their corresponding c' 's. Hence that many c 's must also be part of S then $|S| > t + k$, a contradiction. Therefore $|S \cap X| = 0$. \square

Therefore $C' = \{c_i : c_i, c'_i \in S\}$ is a solution for $X3C$, where $|C'| = k$. \square

Since split graphs form a proper subclass of chordal graphs, the following corollary is immediate from Theorem 2.1.

Corollary 2.3. *PSDOM is NP-complete for chordal graphs.*

Theorem 2.4. *PSDOM is NP-complete for star convex bipartite graphs.*

Proof. Clearly, PSDOM is a member of NP. Let $X = \{x_1, x_2, \dots, x_{3k}\}$ and $C = \{c_1, c_2, \dots, c_t\}$ be an arbitrary instance of $X3C$, where $|X| = 3k$, $|C| = t$ and $t \geq 2k$. We now construct an instance of PSDOM for star convex bipartite graph from the given instance of $X3C$ as follows. Construct a graph $G(V, E)$ by creating vertices x_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, b, c, d and e . Add edges (x_i, a) for each $x_i \in X$, (a, b) , (a, c) , (a, d) and (d, e) . Next add edges (c_i, c) , (c_i, d) for each $c_i \in C$ and (c_j, x_i) if $x_i \in c_j$. Assume $A = \{a, e\} \cup \{c_i : 1 \leq i \leq t\}$, $B = \{x_i : 1 \leq i \leq 3k\} \cup \{b, c, d\}$. The set A induces an independent set, but can be associated with a star with vertex a as central vertex and the neighbors of each element in B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance of $X3C$ in polynomial time. The graph constructed and its associated star is shown in Figure 2. Next we show that, $X3C$ has a solution if and only if G has PSDS of size at most $k + 3$.

Suppose C' is a solution for $X3C$ with $|C'| = k$. Then $S = \{c_i : c_i \in C'\} \cup \{a, c, d\}$. Clearly $|S| \leq k + 3$. Clearly, S is a dominating set and it is a SDS since each $x_i \in X$ is S -defended by c_j where $(x_i, c_j) \in E$, each $c_i \notin S$ is S -defended by c , e is S -defended by d and b is S -defended by a . It is also PSDS since each element in $V \setminus S$ is S -defended by exactly one element of S .

Conversely, suppose that S is a PSDS of G such that $|S| \leq k + 3$ and we have the following lemmas.

Lemma 2.5. $|S \cap \{a, b\}| = 1$.

Proof. (Proof by contradiction) Assume S is a PSDS with $|S| \leq k + 3$ and $|S \cap \{a, b\}| = 2$. If $c \in S$ then there exist at least three vertices of X which can not be S -defended. Otherwise, there exist $t - k$ vertices of C which can not be S -defended, a contradiction. Therefore $|S \cap \{a, b\}| = 1$. \square

Lemma 2.6. $|S \cap \{d, e\}| = 1$.

Proof. The proof is obtained with the similar arguments as in Lemma 2.5. \square

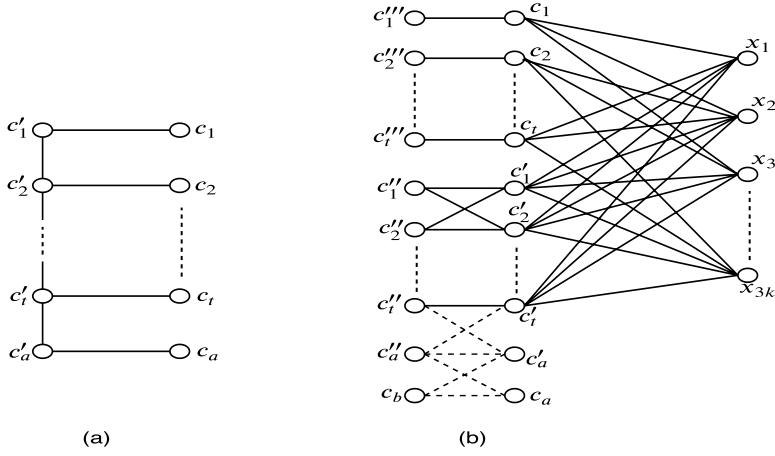


FIGURE 3. (a) Comb Graph. (b) Construction of a comb convex bipartite graph from an instance of X3C.

From Observation 1.4, we can assume that $a, d \in S$. It is clear that $c \in S$ since d can not S -defend the elements of C .

Lemma 2.7. $|S \cap X| = 0$.

Proof. The proof is same as in Lemma 2.2. \square

Therefore $C' = \{c_i : c_i \in S\}$ is a solution for $X3C$, where $|C'| = k$. \square

Theorem 2.8. *PSDOM is NP-complete for comb convex bipartite graphs.*

Proof. Clearly, PSDOM is a member of NP. Let $X = \{x_1, x_2, \dots, x_{3k}\}$ and $C = \{c_1, c_2, \dots, c_t\}$ be an arbitrary instance of $X3C$, where $|X| = 3k$, $|C| = t$ and $t \geq 2k$. We now construct an instance of PSDOM for comb convex bipartite graph from the given instance of $X3C$ as follows. Construct a graph $G(V, E)$ by creating vertices x_i for each $x_i \in X$, c_i, c'_i, c''_i and c'''_i for each $c_i \in C$. Create four more vertices c_a, c'_a, c''_a and c_b if t is odd. Add edges $(c_i, c'''_i), (c'_i, c''_i)$ for each $c_i \in C$ and (c_j, x_i) if $x_i \in c_j$. Next add edges by joining each c'_j to every x_i and also add $(c'_i, c''_j), (c''_i, c'_j)$, where $1 \leq i < t$, $i \bmod 2 = 1$ and $j = i + 1$. If t is odd then add edges $(c'_a, c''_t), (c'_a, c''_a), (c'_a, c_b), (c''_a, c'_t), (c''_a, c_a)$ and (c_a, c_b) .

Assume $A = \{c_i, c'_i : 1 \leq i \leq t\} \cup \{c_a, c'_a\}$, $B = V \setminus A$. The set A induces a comb with elements $\{c'_i : 1 \leq i \leq t\} \cup \{c'_a\}$ as backbone and $\{c_i : 1 \leq i \leq t\} \cup \{c_a\}$ as teeth and the neighbors of each element in B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance of $X3C$ in polynomial time. The graph constructed and its associated comb is shown in Figure 3. Next we show that, $X3C$ has a solution if and only if G has PSDS of size at most $2t + k + 2(t - 2\lfloor \frac{t}{2} \rfloor)$.

Suppose C' is a solution for $X3C$ with $|C'| = k$. Then $S = \{c_i : c_i \in C'\} \cup \{c''_i : 1 \leq i \leq t\} \cup \{c'_i, c''_i : 1 \leq i \leq t, i \bmod 2 = 1\} \cup \{c_a\}$. Clearly, $|S| \leq 2t + k + 2(t - 2\lfloor \frac{t}{2} \rfloor)$. Clearly, S is a dominating set and it is a SDS since each $x_i \in X$ is S -defended by c_j where $(x_i, c_j) \in E$, each $c_i \in C \setminus S$ is S -defended by c'''_i , each $c'_j \in A \setminus S$ is S -defended by c''_{j-1} , if $j = 0 \bmod 2$, each $c''_i \in B \setminus S$ is S -defended by c'_{i-1} and if t is odd then c'_a is S -defended by c''_t , c''_a is S -defended by c'_t and c_b is S -defended by c_a . The set S is also PSDS since each element in $V \setminus S$ is S -defended by exactly one element of S .

Conversely, suppose that S is a PSDS of G such that $|S| \leq 2t + k + 2(t - 2\lfloor \frac{t}{2} \rfloor)$. From Observation 1.4, each $c_i \in S$. These make at least t vertices. Let $S_i = \{c'_i, c''_{i-1}, c'_{i-1}, c''_i\}$ where $1 \leq i \leq t$ and $i \bmod 2 = 0$. Clearly, each $\langle S_i \rangle$ is a cycle graph C_4 . If S contains more than two vertices from a C_4 then each x_i is S -defended by more than one vertex of the C_4 . By Theorem 1.2, S contains atleast two vertices from each C_4 . If t is odd then by

contradiction it can be easily verified that $|S \cap \{c'_t, c''_t, c'_a, c''_a, c_a, c_b\}| = 3$. The total are at least $2t + 2(t - 2\lfloor \frac{t}{2} \rfloor)$ vertices and the following lemmas holds.

Lemma 2.9. $|S \cap X| = 0$.

Proof. The proof is same as in Lemma 2.2. \square

Therefore $C' = \{c_i : c_i \in S, c''_i \in S\}$ is a solution for $X3C$, where $|C'| = k$. \square

The following result is immediate from Theorems 2.4 and 2.8.

Theorem 2.10. *PSDOM is NP-complete for bipartite graphs.*

2.2. Perfect secure domination in planar graphs

To show that the PSDOM is NP-complete for planar graphs, we use a well known NP-complete problem, called Planar Exact Cover by 3-Sets (Planar $X3C$) [14], which is defined as follows.

Planar Exact Cover by 3 Sets (Planar $X3C$)

INSTANCE : A finite set X with $|X| = 3k$ and a collection C of 3-element subsets of X such that (i) every element of X occurs in at most three subsets and such that (ii) the induced graph is planar. (This induced graph H is formed by creating vertices x_i for each $x_i \in X$, c_i for each $c_i \in C$ and by adding edges (c_j, x_i) if $x_i \in c_j$.)

QUESTION : Is there a subcollection C' of C such that every element of X appears in exactly one member of C' ?

Theorem 2.11. *PSDOM is NP-complete for planar graphs.*

Proof. The proof is same as in Theorem 2.1, in which the vertices of C does not form a clique. \square

2.3. Perfect secure domination in dually chordal graphs

To prove the NP-completeness of the PSDOM for dually chordal graphs, we consider the following perfect domination decision problem which has been proved as NP-complete [3].

Perfect Domination (PDOM)

INSTANCE : A simple, undirected graph G and a positive integer p .

QUESTION : Does G have a perfect dominating set of size at most p ?

Theorem 2.12. *PSDOM is NP-complete for dually chordal graphs.*

Proof. Clearly, PSDOM is a member of NP. We now construct an instance of PSDOM for dually chordal graph from the given instance of PDOM as follows. Given an instance $G = (V, E)$ of PDOM, where $V = \{v_1, v_2, \dots, v_n\}$, we construct an instance $G' = (V', E')$ of PSDOM such that $V' = V \cup \{x_1, x_2, x_3\}$ and $E' = E \cup \{(v_i, x_1) : 1 \leq i \leq n\} \cup \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$. Since G' admits a maximum neighbourhood ordering $\{v_1, v_2, \dots, v_n, x_1, x_2, x_3\}$, it is a dually chordal graph and the construction of G' can be accomplished in polynomial time.

Next we show that G has a perfect dominating set of size at most k if and only if G' has a PSDS of size $k + 1$. Suppose D be a perfect dominating set in G of size at most k . Then the set $D \cup \{x_1\}$ is a PSDS of G' of size at most $k + 1$ since x_1 cannot S-defend the vertices of $V(G)$.

On the other hand, let S be a PSDS of G' with $|S| \leq k + 1$. We have the following lemma.

Lemma 2.13. $S \cap \{x_1, x_2, x_3\} = \{x_1\}$.

Proof. (Proof by contradiction) Let $S \cap \{x_1, x_2, x_3\} \neq \{x_1\}$. The following cases are possible.

Case 1 : If $|S \cap \{x_1, x_2, x_3\}| = 3$ then $|S \setminus \{x_1, x_2, x_3\}| \leq k - 2$ and clearly, S is not a PSDS.

Case 2 : If $|S \cap \{x_1, x_2, x_3\}| = 2$ then S is not a PSDS of G' since the $x_i \notin S$ is S -defended by more than one

element of S .

Case 3 : If $S \cap \{x_1, x_2, x_3\} = \{x_2\}$ or $S \cap \{x_1, x_2, x_3\} = \{x_3\}$ then S is not a PSDS of G' since x_1 would be S-defended by more than one vertex.

Therefore, our assumption leads to contradiction. Hence the lemma. \square

Therefore, $S \cap V$ is a perfect dominating set of G of size at most k . \square

3. CATERPILLAR TREE

A *caterpillar* is a tree with the property that the removal of its pendant vertices and incident edges results in a path, which we call the central path. In this section we propose a linear algorithm to compute perfect secure domination number of a caterpillar graph. Let P be the set of pendant vertices and Q be the set of support vertices. Let $N^p(u) = \{v : (u, v) \in T \text{ and } v \in P\}$ be the pendant neighbors of a support vertex u . Also let $N^p[u] = N^p(u) \cup \{u\}$. Our algorithm is given in Algorithm 1.

Lemma 3.1. *Let T be a caterpillar with support vertex set Q and a pendant vertex set P . Also let $T' = T \setminus Q$ be a forest with k trees labelled T_1, T_2, \dots, T_k . If R is an arbitrary PSDS of T such that $R \cap Q = Q$, then*

$$|R| \geq \sum_{u \in Q} |N^p(u)| + \sum_{i=1}^k \gamma_{ps}(T_i).$$

Proof. For each support vertex $u \in Q$, $G[N^p(u)]$ is a complete bipartite graph. Hence, from Theorem 1.3 and Observation 1.4, it is clear that $u \in R$ and $|R \cap N^p[u]| \geq |N^p[u]|$. Clearly, no vertex in the set $V(T') = V(T) \setminus \cup_{u \in Q} N^p[u]$ is securely dominated by the vertices of $\cup_{u \in Q} N^p[u]$. From the fact that each component of T' is a path and $\gamma_{ps}(G) \geq \gamma_s(G)$, it also follows that $|R| \geq \sum_{i=1}^k \gamma_{ps}(T_i)$. Since the sets $\cup_{u \in Q} N^p[u]$ and $V(T')$ are mutually-disjoint the result follows. \square

Algorithm 1 Perfect secure domination number of a caterpillar.

Input: A caterpillar $G(V, E)$.

Output: Perfect secure domination number of G .

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1: if  $G \approx P_n$  then
2:   return  $\lceil \frac{3n}{7} \rceil$ .
3: else
4:   Let  $S = \emptyset$ .
5:   for each vertex  $u \in Q$  do
6:     Let  $S' = |N_G(u) \cap P| - 1$  size subset of  $N_G(u) \cap P$ .
7:      $S = S \cup S' \cup \{u\}$ .
8:   Let graph  $G'(V', E') = \langle V \setminus (P \cup Q) \rangle$ .
9:   Let  $R = \{v : v \in V', |N_{G'}(v)| = 0\}$ .
10:  for each component with  $k$  ( $\geq 2$ ) number of vertices in the graph  $G'$  do
11:    select  $\lceil \frac{3k}{7} \rceil$  vertices as in Theorem 1.1 and include into the set  $S$ .
12:   $S = S \cup R$ .
13: Return  $|S|$ .
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Correctness of the Algorithm 1 : It is easy to verify that, the vertices selected in steps 5 to 7 perfectly secure dominate all the vertices in $\sum_{u \in Q} N^p[u]$ and the vertices selected in steps 9 to 11 perfectly secure dominate the remaining vertices of T i.e., $V(T) \setminus \cup_{u \in Q} N^p[u]$. Hence the set S returned in step 12 of Algorithm 1 is a PSDS of T . From Lemma 3.1, it also follows that $|S| = \gamma_{ps}(T)$ i.e., S is a minimum PSDS of T .

4. THRESHOLD GRAPHS

Threshold graphs have been studied with the following definition [11].

Definition 4.1. A graph $G = (V, E)$ is called a threshold graph if there is a real number T and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(v) \leq T$.

Although several characterizations defined for threshold graphs, we use the following characterization of threshold graphs given in [11] in solving the PSDOM problem.

A graph G is a threshold graph if and only if it is a split graph and, for split partition (C, I) of V where C is a clique and I is an independent set, there is an ordering $\{c_1, c_2, \dots, c_n\}$ of vertices of C such that $N_G[c_1] \subseteq N_G[c_2] \subseteq N_G[c_3] \subseteq \dots \subseteq N_G[c_n]$, and there is an ordering $\{i_1, i_2, \dots, i_m\}$ of the vertices of I such that $N_G(i_1) \supseteq N_G(i_2) \supseteq N_G(i_3) \supseteq \dots \supseteq N_G(i_m)$. Let $T = N_G(c_n) \cap I \setminus N_G(c_{n-1}) \cap I$, $P = \{c_n, c_{n-1}\}$ and T' is a subset of T of size $|T| - 1$.

Theorem 4.2. Let $G(V, E)$ be a threshold graph with split partition (C, I) . Then

$$\gamma_{ps}(G) = \begin{cases} p, & \text{if } G \cong K_{1,p} \\ |P \cup T'|, & \text{if } n > 1, m \geq 1 \text{ and } N_G[c_n] \neq N_G[c_{n-1}] \\ |V|, & \text{otherwise} \end{cases}$$

Proof. If $G \cong K_1$ then $\gamma_{ps}(G) = 1$. Otherwise, let S be a minimum PSDS of G . If $|C| = 1$ i.e., the graph G is in the form $K_{1,p}$ then, by Observation 1.4 and Theorem 1.3, S contains p vertices of G . The following lemmas are used to complete the proof when $|C| > 1$.

Lemma 4.3. If $N_G[c_n] \neq N_G[c_{n-1}]$ then $S = P \cup T'$.

Proof. (Proof by contradiction) Assume $N_G[c_n] \neq N_G[c_{n-1}]$ and $S \neq P \cup T'$, where T' is $|T| - 1$ size subset of T . We have the following cases :

Case 1 : If $|S \cap P| = 0$ then each element of P is S -defended by all the elements of S .

Case 2 : If $|S \cap P| = 1$ then the following cases are possible

Case 2.1 : If $S \cap P = \{c_{n-1}\}$ then clearly S is not a dominating set since the vertex in $(N_G(c_n) \cap I) \setminus S$ is not dominated by any element in S .

Case 2.2 : If $S \cap P = \{c_n\}$ then S is not a PSDS since each vertex in $N_G(c_{n-1}) \cap I$ and c_{n-1} can not be S -defended.

Case 3 : If S does not contain $|T| - 1$ elements from T then $(S \setminus \{x_n\}) \cup \{i_a\}$, where $i_a \in T$ makes all vertices in $(T \setminus \{i_a\}) \setminus S$ undominated.

Case 4 : If $S \cap T = T$ then S is not a PSDS since both c_n and c_{n-1} S -defends all the vertices of G .

Our assumption leads to contradiction. Hence the lemma. \square

Lemma 4.4. If $N_G[c_n] = N_G[c_{n-1}]$ then $S = V$.

Proof. (Proof by contradiction) Assume $N_G[c_n] = N_G[c_{n-1}]$ and $S \neq V$ i.e., $|S| < n + m$. The following cases are possible:

Case 1 : If $|S \cap P| \neq 2$ then the same condition holds as in first two cases of Lemma 4.3.

Case 2 : If $|S \cap P| = 2$ then each element in $V \setminus S$ is S -defended by both the elements of P .

Our assumption leads to contradiction. Hence $|S| \geq n + m$. Therefore $|S| = n + m$. \square

If the threshold graph G is disconnected with k connected components G_1, G_2, \dots, G_k then it is easy to verify that $\gamma_{ps}(G) = \sum_{i=1}^k \gamma_{ps}(G_i)$. Now, the following result is immediate from Theorem 4.2.

Theorem 4.5. PSDOM problem can be solvable in linear time for threshold graphs.

Proof. Since the split partition can be obtained in linear time [11], the result follows. \square

5. BOUNDED TREE-WIDTH GRAPHS

Let G be a graph, T be a tree and v be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T . The pair (T, v) is called a tree-decomposition of G if it satisfies the following three conditions: (i) $V(G) = \bigcup_{t \in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t , (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3 . The width of (T, v) is the number $\max\{|V_t| - 1 : t \in T\}$, and the tree-width $tw(G)$ of G is the minimum width of any tree-decomposition of G . By Courcelle's Theorem, it is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [4]. We show that PSDOM problem can be expressed in CMSOL.

Theorem 5.1 (Courcelle's Theorem). ([4]) Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k , it can be checked in linear-time whether G has property P .

Theorem 5.2. Given a graph G and a positive integer k , PSDOM can be expressed in CMSOL.

Proof. First, we present the CMSOL formula which expresses that the graph G has a dominating set S of size at most k .

$$\text{Dominating}(S) = (\forall p)((\exists q)(q \in S \wedge \text{adj}(p, q)) \vee (p \in S)),$$

where $\text{adj}(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G . $\text{Dominating}(S)$ ensures that for every vertex $p \in V$, either $p \in S$ or p is adjacent to a vertex in S and the cardinality of S is at most k .

Now, by using the above CMSOL formula we can express PSDOM in CMSOL formula as follows.

$$\text{PSDOM}(S) = (|S| \leq k) \wedge \text{Dominating}(S) \wedge (\forall x)((x \in S) \vee ((\forall y)(\forall z)(y \in S, z \in S \wedge \text{adj}(x, y) \wedge \text{adj}(x, z) \wedge \text{Dominating}((S \setminus \{y\}) \cup \{x\}) \rightarrow \neg \text{Dominating}((S \setminus \{z\}) \cup \{x\}))))$$

The above statement states that if an element $x \notin S$ is adjacent to two or more elements of S then it should be S -defended by exactly any one adjacent element of S .

Therefore, PSDOM can be expressed in CMSOL. \square

Now, the following result is immediate from Theorems 5.1 and 5.2.

Theorem 5.3. PSDOM problem can be solvable in linear time for bounded tree-width graphs.

6. COMPLEXITY DIFFERENCE IN DOMINATION AND PERFECT SECURE DOMINATION

Although perfect secure domination is one of the several variants of domination problem, these two differ in computational complexity. In particular, there exist graph classes for which the decision version of the first problem is polynomial-time solvable whereas the decision version of the second problem is NP-complete and vice versa. Similar study has been made between domination and other domination parameters in [8, 15].

The DOMINATION problem is polynomial-time solvable for dually chordal graphs [1, 5], but the PSDOM problem is NP-complete for this class of graphs which is proved in Section 2.3. Now, we construct a new class of graphs in which the MINIMUM PERFECT SECURE DOMINATION problem can be solved trivially, whereas the decision version of the DOMINATION problem is NP-complete, which is defined as follows.

DOMINATION DECISION PROBLEM (DOM)

INSTANCE : A simple, undirected graph G and a positive integer k .

QUESTION : Does there exist a dominating set of size at most k in G ?

Definition 6.1. (GP graph). A graph is GP graph if it can be constructed from a connected graph $G = (V, E)$ where $|V| = n$ and $V = \{v_1, v_2, \dots, v_n\}$, in the following way:

1. Create n path graphs each with 3 vertices such that i^{th} path graph contains vertices $\{a_i, b_i, c_i\}$.

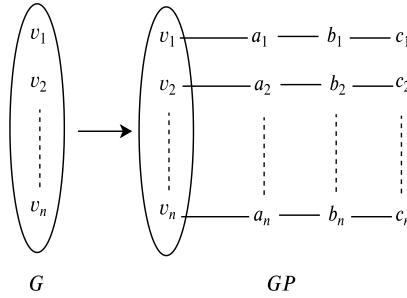


FIGURE 4. An illustration to the construction of GP from G.

2. Add edges $\{(v_i, a_i) : v_i \in V\}$.

General GP graph construction is shown in Figure 4.

Theorem 6.2. *If G' is a GP graph obtained from a graph $G = (V, E)$ ($|V| = n$), then $\gamma_{ps}(G') = 2n$.*

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ be a graph. The construction of $G' = (V', E')$ from G is as follows. Create n copies of P_3 , where a_i, b_i and c_i are the vertices of i^{th} copy of P_3 . Create the edges $\{(v_i, a_i) : 1 \leq i \leq n\}$. It is clear that G' is GP graph. Let $S = V \cup \{b_i : 1 \leq i \leq n\}$. It can be observed that S is a PSDS of G' of size $2n$ and hence $\gamma_{ps}(G') \leq 2n$.

Let S be any PSDS of G' . Clearly, $|S \cap \{b_i, c_i : 1 \leq i \leq n\}| \geq n$ and these vertices can not S-defend any other vertex in V . Therefore, either v_i or a_i , for each i , where $1 \leq i \leq n$ must be included in every PSDS of G' . Hence $\gamma_{ps}(G') \geq 2n$. This completes the proof of the theorem. \square

Lemma 6.3. *Let G' be a GP graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + n$, where $n = |V|$.*

Proof. Suppose D be dominating set of G of size at most k , then it is clear that $D \cup \{b_i : 1 \leq i \leq n\}$ is a dominating set of G' of size at most $n + k$.

Conversely, suppose D' is a dominating set of G' of size at most $n + k$. Then at least one vertex from each of the vertices b_i and c_i must be included in D' . These are at least n vertices. Let $A^* = \{v_i : a_i \in D'\}$. Clearly, $V \cap (A^* \cup D')$ is a dominating set of G of size at most k . Hence the lemma. \square

The following result is well known for the DOMINATION DECISION problem.

Theorem 6.4. ([6]) *The DOMINATION DECISION problem is NP-complete for general graphs.*

From Theorem 6.4 and Lemma 6.3, it follows that DOMINATION DECISION problem is NP-hard. Hence the following theorem.

Theorem 6.5. *The DOMINATION DECISION problem is NP-complete for GP graphs.*

7. CONCLUSION

In this paper, we have shown that PSDOM problem is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs, planar graphs and dually chordal graphs. We also proved that this problem is linear time solvable for threshold graphs, caterpillar trees and bounded tree-width graphs. Further, it is interesting to investigate the algorithmic complexity of PSDOM for other subclasses of chordal graphs namely, strongly chordal graphs, doubly chordal graphs, block graphs, etc. We remark, however, that the two problems, domination and perfect secure domination are not equivalent in computational complexity aspects. A good example is when the input graph is a GP graph, the domination problem is known to be NP-complete

whereas the PSDOM problem is trivially solvable. Thus, there is a scope to study each of these problems on its own for particular graph classes.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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