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THE WEAK CIRCULAR REPETITION THRESHOLD OVER LARGE ALPHABETS

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Abstract. The *repetition threshold* for words on n letters, denoted $\mathrm{RT}(n)$, is the infimum of the set of all r such that there are arbitrarily long r-free words over n letters. A repetition threshold for circular words on n letters can be defined in three natural ways, which gives rise to the *weak*, *intermediate*, and *strong* circular repetition thresholds for n letters, denoted $\mathrm{CRT_W}(n)$, $\mathrm{CRT_I}(n)$, and $\mathrm{CRT_S}(n)$, respectively. Currie and the present authors conjectured that $\mathrm{CRT_I}(n) = \mathrm{CRT_W}(n) = \mathrm{RT}(n)$ for all $n \geq 4$. We prove that $\mathrm{CRT_W}(n) = \mathrm{RT}(n)$ for all $n \geq 45$, which confirms a weak version of this conjecture for all but finitely many values of n.

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1. Introduction

Throughout, we use standard definitions and notations from combinatorics on words (see [16]). The word u is a factor of the word w if we can write $w = xuy$ for some (possibly empty) words x, y. If at least one of x, y is nonempty, then we say that u is a *proper factor* of w. For a set of words L, the word u is a *factor* of L if u is a factor of some word in L.

Let $w = w_1 w_2 \cdots w_k$ be a word, where the w_i's are letters. A positive integer p is a *period* of w if $w_{i+p} = w_i$ for all $1 \leq i \leq k - p$. In this case, we say that $|w|/p$ is an *exponent* of w, and the largest such number is called *the* exponent of w. For a real number $r > 1$, a finite or infinite word w is called r-free (r^+-free) if w contains no factors of exponent greater than or equal to r (strictly greater than r, respectively).

Throughout, for every positive integer n, let A_n denote the n-letter alphabet $\{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{n}\}$. For every $n \geq 2$, the *repetition threshold* for n letters, denoted $\mathrm{RT}(n)$, is defined by

$$\mathrm{RT}(n) = \inf\{r > 1\colon \text{ there are arbitrarily long } r\text{-free words over } A_n\}.$$

Essentially, the repetition threshold describes the border between avoidable and unavoidable repetitions in words over an alphabet of n letters. The repetition threshold was first defined by Dejean [13]. Her 1972 conjecture on

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the values of $\mathrm{RT}(n)$ has now been confirmed through the work of many authors [5, 7–9, 13, 18, 19, 22, 23]:

$$\mathrm{RT}(n) = \begin{cases} 2, & \text{if } n = 2; \\ 7/4, & \text{if } n = 3; \\ 7/5, & \text{if } n = 4; \\ n/(n-1), & \text{if } n \geq 5. \end{cases}$$

The last cases of Dejean's conjecture were confirmed in 2011 by Currie and the second author [7], and independently by Rao [23]. However, probably the most important contribution was made by Carpi [5], who confirmed the conjecture for all but finitely many values of n.

Here, we are interested in the notion of a repetition threshold for *circular words* on n letters. Two (linear) words x and y are said to be *conjugates* if there are words u and v such that $x = uv$ and $y = vu$. The conjugates of a word w can be obtained by rotating the letters of w cyclically. For a word w, the *circular word* $\langle w \rangle$ is the set of all conjugates of w. Intuitively, one can think of a circular word as being obtained from a linear word by linking the ends, giving a cyclic sequence of letters. A *circumnavigation* of the circular word $\langle w \rangle$ is a word of the form ava, where a is a letter and av is a conjugate of w.

By the definition of a factor of a set of words, a word is a factor of a circular word $\langle w \rangle$ if and only if it is a factor of some conjugate of w. As for linear words, a circular word is r-free (r^+-free) if it has no factors of exponent greater than or equal to r (strictly greater than r, respectively).

While the factors of an r-free circular word must be r-free as linear words, they need not necessarily be r-free when taken as circular words. For example, the 2-free circular word $\langle \texttt{012021} \rangle$ contains the factor $\texttt{0120}$, and the circular word $\langle \texttt{0120} \rangle$ contains the square factor 00. This means that several equivalent definitions of the repetition threshold $\mathrm{RT}(n)$ give rise to distinct notions of a repetition threshold for circular words.

In this paper, we will be most interested in the *weak circular repetition threshold* for n letters, denoted $\mathrm{CRT_W}(n)$, and defined by

$$\mathrm{CRT_W}(n) = \inf\{\, r > 1\colon \text{ there are arbitrarily long } r\text{-free circular words over } A_n \}.$$

The *intermediate circular repetition threshold* for n letters, denoted $\mathrm{CRT_I}(n)$, is defined by

$$\mathrm{CRT_I}(n) = \inf\{\, r > 1\colon \text{ there are } r\text{-free circular words of every sufficiently large length over } A_n \},$$

and the *strong circular repetition threshold* for n letters, denoted $\mathrm{CRT_S}(n)$, is defined by

$$\mathrm{CRT_S}(n) = \inf\{\, r > 1\colon \text{ there are } r\text{-free circular words of every length over } A_n \}.$$

Evidently, we have

$$\mathrm{RT}(n) \leq \mathrm{CRT_W}(n) \leq \mathrm{CRT_I}(n) \leq \mathrm{CRT_S}(n), \tag{1.1}$$

for all $n \geq 2$. Table 1 contains all of the confirmed and conjectured values of $\mathrm{RT}(n)$, $\mathrm{CRT_W}(n)$, $\mathrm{CRT_I}(n)$, and $\mathrm{CRT_S}(n)$. Note in particular that

$$\mathrm{CRT_W}(2) < \mathrm{CRT_I}(2) < \mathrm{CRT_S}(2),$$

which shows that the three notions of circular repetition threshold are not equivalent.

Table 1. The confirmed and conjectured values of the weak, intermediate, and strong circular repetition thresholds. Confirmed values are in bold font, while conjectured values are in normal font (and coloured red).

n	2	3	4	$n \geq 5$
$\mathrm{RT}(n)$	$\mathbf{2}$	$\mathbf{\frac{7}{4}}$	$\mathbf{\frac{7}{5}}$	$\boldsymbol{\frac{n}{n-1}}$
$\mathrm{CRT_W}(n)$	$\mathbf{2}$	$\mathbf{\frac{7}{4}}$	$\frac{7}{5}$	$\frac{n}{n-1}$
$\mathrm{CRT_I}(n)$	$\mathbf{\frac{7}{3}}$	$\mathbf{\frac{7}{4}}$	$\frac{7}{5}$	$\frac{n}{n-1}$
$\mathrm{CRT_S}(n)$	$\mathbf{\frac{5}{2}}$	$\mathbf{2}$	$\mathbf{\frac{3}{2}}$	$\boldsymbol{\frac{\lceil n/2\rceil+1}{\lceil n/2\rceil}}$

Through the work of several authors [1, 10, 12, 15, 24], all values of the strong circular repetition threshold are known:

$$\mathrm{CRT_S}(n) = \begin{cases} 5/2, & \text{if } n = 2; \\ 2, & \text{if } n = 3; \\ \frac{\lceil n/2\rceil+1}{\lceil n/2\rceil}, & \text{if } n \geq 4. \end{cases}$$

The values of the weak and intermediate circular repetition thresholds are only known exactly for $n = 2$ and $n = 3$:

- $\mathrm{CRT_W}(2) = 2$ (Thue [4]);
- $\mathrm{CRT_I}(2) = 7/3$ (Aberkane and Currie [2], Shur [24]); and
- $\mathrm{CRT_I}(3) = \mathrm{CRT_W}(3) = 7/4$ (Shur [24], Currie *et al.* [10]).

From (1.1) and the known values of $\mathrm{RT}(n)$ and $\mathrm{CRT_S}(n)$, we have

$$\frac{n}{n-1} \leq \mathrm{CRT_W}(n) \leq \mathrm{CRT_I}(n) \leq \frac{\lceil n/2\rceil + 1}{\lceil n/2\rceil},$$

for every $n \geq 5$. These are currently the best known bounds on both $\mathrm{CRT_W}(n)$ and $\mathrm{CRT_I}(n)$ when $n \geq 5$. Currie and the present authors [10] recently made the following conjecture, which strengthens the second statement of an older conjecture of Shur ([24], Conj. 1).

Conjecture 1.1. *For every $n \geq 4$, we have* $\mathrm{CRT_I}(n) = \mathrm{CRT_W}(n) = \mathrm{RT}(n)$.

Here, we prove a weak version (pun intended) of Conjecture 1.1 for all but finitely many values of n.

Theorem 1.2. *For every $n \geq 45$, we have* $\mathrm{CRT_W}(n) = \mathrm{RT}(n) = n/(n-1)$.

The layout of the remainder of the paper is as follows. In Section 2, we summarize the work of Carpi [5] in confirming all but finitely many cases of Dejean's conjecture. In Section 3, we establish Theorem 1.2 with a construction that relies heavily on the work of Carpi. We conclude with a discussion of some related notions of repetition threshold for classes of graphs.

We note that the work of Carpi [5] that we rely on in this paper was also instrumental in the recent proof by Currie and the present authors [11] that for every $n \geq 27$, the number of $n/(n-1)^+$-free words of length k over n letters grows exponentially in k. This speaks to the strength of Carpi's results.

2. Carpi's reduction to ψ_n-kernel repetitions

In this section, let $n \geq 2$ be a fixed integer. Pansiot [22] was first to observe that if a word over the alphabet A_n is $(n-1)/(n-2)$-free, then it can be encoded by a word over the binary alphabet $B = \{\mathtt{0}, \mathtt{1}\}$. For consistency, we use the notation of Carpi [5] to describe this encoding. Let $\mathbb{S}_n$ denote the symmetric group on A_n, and define the morphism $\varphi_n : B^* \to \mathbb{S}_n$ by

$$\varphi_n(\mathtt{0}) = \begin{pmatrix}\mathtt{1} & \mathtt{2} & \cdots & \mathtt{n\text{-}1}\end{pmatrix}; \text{ and}$$
$$\varphi_n(\mathtt{1}) = \begin{pmatrix}\mathtt{1} & \mathtt{2} & \cdots & \mathtt{n}\end{pmatrix}.$$

Now define the map $\gamma_n : B^* \to A_n^*$ by

$$\gamma_n(b_1 b_2 \cdots b_k) = a_1 a_2 \cdots a_k,$$

where

$$a_i \varphi_n(b_1 b_2 \cdots b_i) = \mathtt{1},$$

for all $1 \leq i \leq k$. To be precise, Pansiot proved that if a word $\alpha \in A_n^*$ is $(n-1)/(n-2)$-free, then α can be obtained from a word of the form $\gamma_n(u)$, where $u \in B^*$, by renaming the letters.

Let $u \in B^*$, and let $\alpha = \gamma_n(u)$. Pansiot showed that if α has a factor of exponent greater than $n/(n-1)$, then either the word α itself contains a *short repetition*, or the binary word u contains a *kernel repetition* (see [22] for details). Carpi reformulated this statement so that both types of forbidden factors appear in the binary word u. Let $k \in \{1, 2, \ldots, n-1\}$, and let $v \in B^+$. Then v is called a *k-stabilizing word* (of order n) if $\varphi_n(v)$ fixes the points $\mathtt{1}, \mathtt{2}, \ldots, \mathtt{k}$. Let $\mathrm{Stab}_n(k)$ denote the set of k-stabilizing words of order n. The word v is called a *kernel repetition* (of order n) if it has period p and a factor v' of length p such that $v' \in \ker(\varphi_n)$ and $|v| > \frac{np}{n-1} - (n-1)$. Carpi's reformulation of Pansiot's result is the following.

Proposition 2.1 (Carpi [5], Prop. 3.2)**.** *Let $u \in B^*$. If a factor of $\gamma_n(u)$ has exponent larger than $n/(n-1)$, then u has a factor v satisfying one of the following conditions:*

(i) *$v \in \mathrm{Stab}_n(k)$ and $0 < |v| < k(n-1)$ for some $1 \leq k \leq n-1$; or*
(ii) *v is a kernel repetition of order n.*

Now assume that $n \geq 9$, and define $m = \lfloor (n-3)/6 \rfloor$ and $\ell = \lfloor n/2 \rfloor$. Carpi [5] defines an $(n-1)(\ell+1)$-uniform morphism $f_n : A_m^* \to B^*$ with the following extraordinary property.

Proposition 2.2 (Carpi [5], Prop. 7.3)**.** *Suppose that $n \geq 27$, and let $w \in A_m^*$. Then for every $k \in \{1, 2, \ldots, n-1\}$, the word $f_n(w)$ contains no k-stabilizing word of length smaller than $k(n-1)$.*

We note that Proposition 2.2 was proven by Carpi [5] in the case that $n \geq 30$ in a computation-free manner. The improvement to $n \geq 27$ stated here was achieved later by Currie and the second author [8], using lemmas of Carpi [5] along with a significant computer check.

Proposition 2.2 says that for *every* word $w \in A_m^*$, no factor of $f_n(w)$ satisfies condition (i) of Proposition 2.1. Thus, we need only worry about factors satisfying condition (ii) of Proposition 2.1, *i.e.*, kernel repetitions. To this end, define the morphism $\psi_n : A_m^* \to \mathbb{S}_n$ by $\psi_n(v) = \varphi_n(f_n(v))$ for all $v \in A_m^*$. A word $v \in A_m^*$ is called a *ψ_n-kernel repetition* if it has a period q and a factor v' of length q such that $v' \in \ker(\psi_n)$ and $(n-1)(|v|+1) \geq nq - 3$. Carpi established the following result.

Proposition 2.3 (Carpi [5], Prop. 8.2)**.** *Let $w \in A_m^*$. If a factor of $f_n(w)$ is a kernel repetition, then a proper factor of w is a ψ_n-kernel repetition.*

Remark 2.4. The word "proper" in the statement of Proposition 2.3 was not in the original statement of Carpi ([5], Prop. 8.2), but it is easily verified that Carpi's proof actually proves this stronger statement, which will be necessary for our work.

In other words, if no proper factor of $w \in A_m^*$ is a ψ_n-kernel repetition, then no factor of $f_n(w)$ satisfies condition (ii) of Proposition 2.1. Finally, we note that the morphism f_n is defined in such a way that the kernel of ψ_n has a very simple structure.

Lemma 2.5 (Carpi [5], Lem 9.1)**.** *If $v \in A_m^*$, then $v \in \ker(\psi_n)$ if and only if 4 divides $|v|_a$ for every letter $a \in A_m$.*

3. Constructing $n/(n-1)^+$-free circular words over n letters

In this section, let $n \geq 27$ be a fixed integer, and let $m = \lfloor (n-3)/6 \rfloor$ and $\ell = \lfloor n/2 \rfloor$, as in the previous section. Finally, define $M = 4^{m-2}$. Since $n \geq 27$, we have $m \geq 4$ and $M \geq 16$.

In order to prove Theorem 1.2, we will construct an $n/(n-1)^+$-free circular word of length $M(n-1)(\ell+1)t$ over A_n for every integer $t \geq 1$. We first show that we can restrict our attention to words over the smaller alphabet A_m, just as Carpi did for linear words. We begin with an analogue of Proposition 2.1 for circular words.

Lemma 3.1. *Let $u \in B^* \cap \ker(\varphi_n)$. If a factor of the circular word $\langle \gamma_n(u) \rangle$ has exponent larger than $n/(n-1)$, then the circular word $\langle u \rangle$ has a factor v satisfying one of the following conditions:*

(i) *$v \in \mathrm{Stab}_n(k)$ and $0 < |v| < k(n-1)$ for some $1 \leq k \leq n-1$; or*
(ii) *v is a kernel repetition of order n.*

Proof. Let $u = u_1 u_2 \cdots u_s$, where $u_i \in B$ for all $i \in \{1, \ldots, s\}$. Let $\gamma_n(u) = a_1 a_2 \ldots a_s$, where $a_i \in A_n$ for all $i \in \{1, \ldots, s\}$. First, we claim that for every conjugate $u' = u_j \cdots u_s u_1 \cdots u_{j-1}$ of u, the word $\gamma(u')$ is equal, up to a permutation of the letters, to the corresponding conjugate $a_j \cdots a_s a_1 \cdots a_{j-1}$ of $\gamma_n(u)$. It suffices to show that $\gamma_n(u_2 u_3 \cdots u_s u_1)$ is equal to $a_2 a_3 \cdots a_k a_1$, up to a permutation of the letters. By definition of γ_n, we have

$$a_i \varphi_n(u_1 u_2 \cdots u_i) = \mathtt{1},$$

for all $i \in \{1, \ldots, s\}$. Let $\gamma_n(u_2 u_3 \cdots u_s u_1) = b_2 b_3 \cdots b_s b_1$. Again, by definition of γ_n, we have

$$b_i [\varphi_n(u_1)]^{-1} \varphi_n(u_1 u_2 \ldots u_i) = \mathtt{1},$$

for all $i \in \{2, \ldots, s\}$. Finally, using the definition of γ_n and the fact that $u \in \ker(\varphi_n)$, we have

$$b_1 [\varphi_n(u_1)]^{-1} \varphi_n(u_1) = b_1 [\varphi_n(u_1)]^{-1} \varphi_n(u_1 u_2 \ldots u_s u_1) = \mathtt{1}.$$

Thus, we see that $b_i = a_i \varphi_n(u_1)$ for all $i \in \{1, \ldots, s\}$, and this completes the proof of the claim.

Suppose now that a factor w of the circular word $\langle \gamma_n(u) \rangle$ has exponent larger than $n/(n-1)$. Then, up to permutation of the letters, the word w is a factor of $\gamma_n(u')$ for some conjugate u' of u. By Lemma 2.1, the word u' contains a factor v satisfying either condition (i) or condition (ii). Since u' is a conjugate of u, the word v is a factor of $\langle u \rangle$, and this completes the proof. ☐

The next lemma is an analogue of Proposition 2.3 for circular words.

Lemma 3.2. *Let $w \in A_m^* \cap \ker(\psi_n)$. If the circular word $\langle w \rangle$ contains no ψ_n-kernel repetitions, then the circular word $\langle \gamma_n(f_n(w)) \rangle$ is $n/(n-1)^+$-free.*

Proof. Suppose, towards a contradiction, that $\langle w \rangle$ contains no ψ_n-kernel repetitions, and that $\langle \gamma_n(f_n(w)) \rangle$ contains a factor of exponent greater than $n/(n-1)$. Since $w \in \ker(\psi_n)$, we have $f_n(w) \in \ker(\varphi_n)$. Thus, by

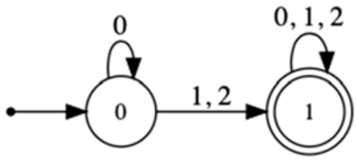

FIGURE 1. The automaton accepting those $(k)_3$ such that the word β has a factor of length k that begins and ends in the letter $\mathtt{2}$.

Lemma 3.1, some factor v of $\langle f_n(w)\rangle$ satisfies either condition (i) or condition (ii) of Lemma 3.1. Now v must be a factor of $f_n(w')$ for some circumnavigation w' of w (*i.e.*, we have $w' = aw''a$, where $a \in A_m$ and aw'' is a conjugate of w). By Proposition 2.2, for every $1 \le k \le n-1$, the word $f_n(w')$ contains no k-stabilizing words of length less than $k(n-1)$. So it must be the case that v is a kernel repetition of order n. Then by Proposition 2.3, some proper factor of w' must be a ψ_n-kernel repetition. Since every proper factor of the circumnavigation w' is a factor of the circular word $\langle w\rangle$, this is a contradiction. So we conclude that the circular word $\langle \gamma_n(f_n(w))\rangle$ is $n/(n-1)^+$-free. □

By Lemma 3.2, in order to construct an $n/(n-1)^+$-free circular word of length $M(n-1)(\ell+1)t$ over A_n, it suffices to construct a word of length Mt over A_m that lies in $\ker(\psi_n)$ and contains no ψ_n-kernel repetitions. Our construction of such a word uses many ideas of Carpi ([5], Sect. 9), and we recommend that the reader reviews this section before proceeding.

Following Carpi, define $\beta = (b_i)_{i\ge 1}$, where

$$b_i = \begin{cases} \mathtt{1}, & \text{if } i \equiv 1 \pmod 3; \\ \mathtt{2}, & \text{if } i \equiv 2 \pmod 3; \\ b_{i/3}, & \text{if } i \equiv 0 \pmod 3). \end{cases}$$

We note that β can also be defined (*c.f.* ([6], Ex. 4)) as the fixed point of the morphism $\tau : A_2^* \to A_2^*$ defined by

$$\mathtt{1} \mapsto \mathtt{121},$$
$$\mathtt{2} \mapsto \mathtt{122}.$$

We need two lemmas concerning the factors of β.

Lemma 3.3. *For every $k \ge 1$, the word β has a factor of length k that begins and ends in the letter* $\mathtt{2}$.

Proof. The statement can be verified by a case-based proof depending on the value of k mod 3, or by using the automatic theorem proving software `Walnut` [20]. We describe the latter approach. After saving the automaton generating the fixed point of τ in the `Word Automata Library` folder as `B.txt`, we use the predicate

```
eval BeginsAndEndsInTwo "?msd_3 (k>=1 & (Ei (B[i]=@2 & B[i+k-1]=@2)))":
```

The automaton for this predicate is illustrated in Figure 1. The automaton clearly accepts all $(k)_3$ such that $k \ge 1$. □

Lemma 3.4 (Carpi [5], Lem 9.2)**.** *Let u be a factor of β with period q. For every $k \ge 0$, if $|u| \ge q + 3^k$, then 3^k divides q.*

For every $t \ge 1$, we define a set of words $\Lambda_t \subseteq A_m^*$, all of length Mt. We define Λ_t by $x_0x_1\cdots x_{Mt-1} \in \Lambda_t$ if and only if $x_i = \max\{a \in A_m \colon\ 4^{a-3} \text{ divides } i\}$ whenever $i \equiv 0 \pmod 4$, and $x_i \in \{\mathtt{1},\mathtt{2},\mathtt{3}\}$ whenever $i \not\equiv 0 \pmod 4$.

We prove two lemmas concerning the words in Λ_t. The proof of the first lemma uses the main idea from Carpi's proof of ([5], Lem. 9.3).

Lemma 3.5. *Let $x \in \Lambda_t$, and let v be a factor of the circular word $\langle x\rangle$. If $v \in \ker(\psi_n)$, then M divides $|v|$.*

Proof. The statement is trivially true if $v = \varepsilon$, so assume $|v| > 0$. Set $|v| = 4^b c$, where 4^b is the maximal power of 4 dividing $|v|$. Suppose, towards a contradiction, that $b \leq m-3$. Since $v \in \ker(\psi_n)$, by Lemma 2.5, we see that 4 divides $|v|$, meaning $b \geq 1$.

Write $x = x_0x_1\cdots x_{Mt-1}$. Then we have $v = x_ix_{i+1}\cdots x_{i+4^bc-1}$ for some $i \geq 1$, with indices taken modulo Mt. Since Mt is divisible by $M = 4^{m-2}$ and $b < m-2$, for all $j \geq 0$ and $k \in \{1, 2, \ldots, b\}$, we see that 4^k divides j if and only if 4^k divides $j \bmod Mt$. Since $b \geq 1$, we have $b+3 \geq 4$, and hence $x_j \geq b+3$ implies $j \equiv 0 \pmod 4$. Thus, by definition, for any $j \in \{i, i+1, \ldots, i+4^bc-1\}$, we have $x_j \geq b+3$ if and only if 4^b divides j. Thus, we have that the sum $\sum_{a=b+3}^{m} |v|_a$ is exactly the number of integers in the set $\{i, i+1, \ldots, i+4^bc-1\}$ that are divisible by 4^b, which is exactly c. Since $v \in \ker(\psi_n)$, by Lemma 2.5, we conclude that 4 divides c, contradicting the maximality of b. □

Lemma 3.6. *Let $x \in \Lambda_t$. For every letter $a \in \{\mathtt{4}, \mathtt{5}, \ldots, \mathtt{m}\}$, the number $|x|_a$ is a multiple of 4.*

Proof. Write $x = x_0x_1 \ldots x_{Mt-1}$, and let $a \in \{\mathtt{4}, \mathtt{5}, \ldots, \mathtt{m}\}$. Since $x \in \Lambda_t$, we have $x_i \geq a$ if and only if 4^{a-3} divides i. Since $|x| = Mt = 4^{m-2}t$, we have

$$\sum_{b=a}^{m} |x|_b = Mt/4^{a-3} = 4^{1+m-a}t.$$

Since $m-a+1 \geq 1$, we see that 4 divides $\sum_{b=a}^{m} |x|_b$. The fact that 4 divides $|x|_a$ now follows by a straightforward inductive argument. □

We are now ready to construct a circular word of length Mt over A_m that belongs to $\ker(\psi_n)$ and contains no ψ_n-kernel repetitions. The proof of the next proposition uses some arguments that were first used by Carpi ([5], Prop. 9.4).

Proposition 3.7. *Suppose that $n \geq 45$. For every integer $t \geq 1$, there is a word $w \in A_m^* \cap \ker(\psi_n)$ of length Mt such that the circular word $\langle w\rangle$ contains no ψ_n-kernel repetitions.*

Proof. Fix $t \geq 1$. Since $n \geq 45$, we have $m \geq 7$ and $M = 4^{m-2} \geq 4^5$. We use these facts frequently. Let u be a factor of β of length $Mt/4$ that begins and ends in the letter 2; such a factor is guaranteed to exist by Lemma 3.3. Let $\sigma : A_2^* \to \{\mathtt{1}, \mathtt{3}\}^*$ be the morphism defined by

$$\mathtt{1} \mapsto \mathtt{1}$$
$$\mathtt{2} \mapsto \mathtt{3}.$$

Write $u\sigma(u) = u_1u_2 \ldots u_{Mt/2}$.

Let $s = 4 - (|u|_\mathtt{2} \bmod 4)$, and define $v = v_1v_2 \cdots v_{Mt/4}$ by

$$v_i = \begin{cases} \mathtt{3}, & \text{if } 1 \leq i \leq s; \\ \mathtt{2}, & \text{if } s+1 \leq i \leq 2s; \\ \mathtt{1}, & \text{if } 2s \leq i \leq Mt/4. \end{cases}$$

Finally, define $w = w_0 w_1 \cdots w_{Mt-1}$ by

$$w_i = \begin{cases} \max\{a \in A_m \colon\ 4^{a-3} \text{ divides } i\}, & \text{if } i \equiv 0 \pmod 4; \\ v_{(i+2)/4}, & \text{if } i \equiv 2 \pmod 4; \\ u_{(i+1)/2}, & \text{if } i \text{ is odd.} \end{cases}$$

Evidently, we have $w \in \Lambda_t$. We will show that $w \in \ker(\psi_n)$, and that the circular word $\langle w \rangle$ contains no ψ_n-kernel repetitions, thus proving the proposition statement.

First, we show that $w \in \ker(\psi_n)$. By Lemma 2.3, it suffices to show that 4 divides $|w|_a$ for all $a \in \{\mathtt{1}, \mathtt{2}, \ldots, \mathtt{m}\}$. Since $w \in \Lambda_t$, by Lemma 3.6, we have that 4 divides $|w|_a$ for all $a \in \{\mathtt{4}, \mathtt{5}, \ldots, \mathtt{m}\}$. Next, note that

$$|w|_{\mathtt{2}} = |u|_{\mathtt{2}} + s = |u|_{\mathtt{2}} + 4 - (|u|_{\mathtt{2}} \bmod 4),$$

which is clearly divisible by 4. Since

$$|w|_{\mathtt{3}} = |\sigma(u)|_{\mathtt{3}} + s = |u|_{\mathtt{2}} + s = |w|_{\mathtt{2}},$$

we see that 4 divides $|w|_{\mathtt{3}}$ as well. Finally, we have

$$|w|_{\mathtt{1}} = |w| - \sum_{b=2}^{m} |w|_b = Mt - \sum_{b=2}^{m} |w|_b.$$

Since 4 divides both M and $|w|_b$ for all $b \geq 2$, we conclude that 4 divides $|w|_{\mathtt{1}}$.

It remains to show that the circular word $\langle w \rangle$ contains no ψ_n-kernel repetitions. Suppose towards a contradiction that some factor X of $\langle w \rangle$ is a ψ_n-kernel repetition of period q. Then by definition, we have

$$nq - 3 \leq (n-1)(|X| + 1). \tag{3.1}$$

Since $m = \lfloor (n-3)/6 \rfloor$, we must also have $n \leq 6m + 8$.

First of all, consider the case $|X| \leq q + 5$. Applying this inequality to the right side of (3.1) and then rearranging, we obtain

$$q \leq 6n - 3.$$

By Lemma 3.5, we must have $q \geq M = 4^{m-2}$. Together with the fact that $n \leq 6m + 8$, this gives

$$4^{m-2} \leq 36m + 45.$$

But this last inequality implies $m \leq 6$, a contradiction.

So we may assume that $|X| \geq q + 6$. Then one has $q + 2 \cdot 3^k \leq |X| < q + 2 \cdot 3^{k+1}$ for some integer $k \geq 1$. Using (3.1) and the inequality $|X| + 1 \leq q + 2 \cdot 3^{k+1}$, we obtain

$$q \leq 2(n-1)3^{k+1} + 3 \leq 3^k(36m + 45). \tag{3.2}$$

Deleting the letters of even index in X, we obtain a factor x of the circular word $\langle u\sigma(u) \rangle$. By Lemma 3.5, we know that 4^{m-2} divides q, so q is certainly divisible by 2, and x has period $q/2$. Since $|X| \geq q + 2 \cdot 3^k$, we have

$$|x| \geq \left\lfloor \frac{|X|}{2} \right\rfloor \geq q/2 + 3^k \geq q/2 + 3. \tag{3.3}$$

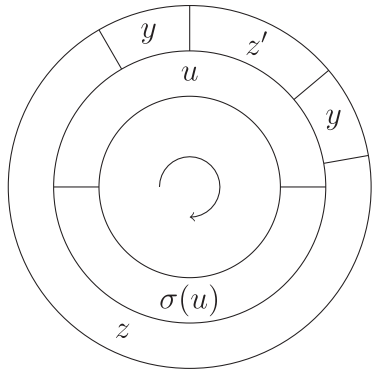

FIGURE 2. The circular word $\langle u\sigma(u)\rangle$ in Case 2 of the proof of Proposition 3.7.

Write $x = x'y = yx''$, where $|x'| = |x''| = q/2$. From (3.3), we see that $|y| \geq 3$.

By construction, we may write $u = \mathtt{2}u'\mathtt{2}$ and $\sigma(u) = \mathtt{3}u''\mathtt{3}$, where $u' \in \{\mathtt{1},\mathtt{2}\}^*$ and $u'' \in \{\mathtt{1},\mathtt{3}\}^*$. So we have

$$u\sigma(u) = \mathtt{2}u'\mathtt{23}u''\mathtt{3}.$$

Thus, in the circular word $\langle u\sigma(u)\rangle$, the factors $\mathtt{23}$ and $\mathtt{32}$ both appear exactly once. Further, every factor of length 3 of u contains a $\mathtt{2}$, and every factor of length 3 of $\sigma(u)$ contains a $\mathtt{3}$. Therefore, either every appearance of y in $\langle u\sigma(u)\rangle$ is contained entirely in u, or every appearance of y in $\langle u\sigma(u)\rangle$ is contained entirely in $\sigma(u)$. Without loss of generality, assume that every appearance of y in $\langle u\sigma(u)\rangle$ is contained entirely in u. Now we consider two cases.

Case 1: The factor x is contained entirely in u.
Then x is a factor of β with period $q/2$. By Lemma 3.5, we know that 4^{m-2} divides q. Recall from (3.3) that $|x| \geq q/2 + 3^k$. Thus, by Lemma 3.4, we have that 3^k divides $q/2$. Therefore, we have $q \geq 4^{m-2}3^k$. Together with (3.2), this gives

$$4^{m-2} \leq 36m + 45.$$

But we have already seen that this inequality implies $m \leq 6$, a contradiction.

Case 2: The factor x contains the entire word $\sigma(u)$ as a factor.
Then we can write $x = yzy$, where $|yz| = q/2$, and z contains the entire word $\sigma(u)$. Let $yzyz'$ be a conjugate of $u\sigma(u)$. See Figure 2 for an illustration of the circular word $\langle u\sigma(u)\rangle$.

Now let $YZYZ'$ be the conjugate of w corresponding to $yzyz'$, *i.e.*, we have that YZ is the length q prefix of X, $|Y| = 2|y|$, $|Z| = 2|z|$, and $|Z'| = 2|z'|$. We claim that $YZ'Y$ is a ψ_n-kernel repetition with period $|YZ'|$. First of all, note that since $yz'y$ is a factor of u, while z contains all of $\sigma(u)$, we have $|Z| > |Z'|$. Next, note that both $YZ \in \ker(\psi_n)$ and $YZYZ' \in \ker(\psi_n)$. Now let $a \in A_m$. By Lemma 2.3, both $|YZ|_a$ and $|YZYZ'|_a$ are multiples of 4. It follows that

$$|YZ'|_a = |YZYZ'|_a - |YZ|_a,$$

is also a multiple of 4, and hence $YZ' \in \ker(\psi_n)$. By the definition of ψ_n-kernel repetition, we have

$$(n-1)(|YZY| + 1) \geq n \cdot |YZ| - 3.$$

Rearranging the above inequality, and then using the fact that $|Z| > |Z'|$, we have

$$(n-2)|Y| + n + 2 \geq |Z| > |Z'|,$$

which implies that $YZ'Y$ is also a ψ_n-kernel repetition. But this is impossible by Case 1. □

Finally, Theorem 1.2 follows directly from the next result.

Proposition 3.8. *Suppose that $n \geq 45$. For every integer $t \geq 1$, there is a word $W \in A_n^*$ of length $M(n-1)(\ell+1)t$ such that the circular word $\langle W \rangle$ is $n/(n-1)^+$-free.*

Proof. Fix $t \geq 1$. By Proposition 3.7, there is a word $w \in A_m^* \cap \ker(\psi_n)$ of length Mt such that the circular word $\langle w \rangle$ contains no ψ_n-kernel repetitions. Since f_n is $(n-1)(\ell+1)$-uniform, and γ_n preserves length, we have $|\gamma_n(f_n(w))| = M(n-1)(\ell+1)t$. By Lemma 3.2, the circular word $\langle \gamma_n(f_n(w)) \rangle$ is $n/(n-1)^+$-free. □

4. Conclusion

We have shown that $\mathrm{CRT_W}(n) = \mathrm{RT}(n) = n/(n-1)$ for all $n \geq 45$. The conjecture that $\mathrm{CRT_W}(n) = \mathrm{RT}(n)$ remains open for all $4 \leq n \leq 44$, and the stronger conjecture that $\mathrm{CRT_I}(n) = \mathrm{RT}(n)$ remains open for all $n \geq 4$.

To conclude, we will place the notion of weak circular repetition threshold in a broader context, and discuss a more general problem. Let G be a graph, and let $f : V(G) \to A_n$ be an n-colouring of G. A word $w \in A_n^*$ is called a *factor* of G if $w = f(v_1)f(v_2)\cdots f(v_k)$ for some path $v_1, v_2, \ldots v_k$ in G that contains no repeated vertices. An n-colouring of the graph G is called r-free if it contains no factors of exponent greater than or equal to r. By this definition, the 2-free colourings of G are exactly the *nonrepetitive colourings* of G, first defined by Alon *et al.* [3]. Nonrepetive colourings have been widely studied in the last two decades. In particular, Dujmović *et al.* [14] recently confirmed what was probably the most important conjecture on nonrepetitive colourings, namely that every planar graph can be nonrepetitively coloured with a bounded number of colours. Their paper also contains an extensive list of references to other work on nonrepetitive colourings and related notions.

To date, most work on r-free colourings has concerned the problem of fixing a number r (most commonly $r = 2$) and determining the minimum number of colours necessary for an r-free colouring of a given graph. Ochem and Vaslet [21] introduced a notion of repetition threshold for classes of graphs, which considers the problem the other way around – for a fixed number of colours, find the smallest value of r such that there is an r-free colouring of a given graph. Formally, the *repetition threshold* for n letters on G, denoted $\mathrm{RT}(n, G)$, is defined by:

$$\mathrm{RT}(n, G) = \inf\{r > 1 : \text{ there is an } r\text{-free } n\text{-colouring of } G\}.$$

For a collection of graphs $\mathcal{G}$, the repetition threshold for $\mathcal{G}$ is defined by $\mathrm{RT}(n, \mathcal{G}) = \sup_{G \in \mathcal{G}} \mathrm{RT}(n, G)$. Note that the strong circular repetition threshold is equivalent to the repetition threshold $\mathrm{RT}(n, \mathcal{C})$, where $\mathcal{C}$ is the collection of cycles.

Ochem and Vaslet determined all values of $\mathrm{RT}(n, \mathcal{T})$, where $\mathcal{T}$ is the collection of all trees. Lužar, Ochem, and Pinlou [17] determined all values of $\mathrm{RT}(n, \mathcal{CP})$ and $\mathrm{RT}(n, \mathcal{CP}_3)$, where $\mathcal{CP}$ is the collection of all caterpillars, and $\mathcal{CP}_3$ is the collection of all caterpillars of maximum degree 3. They also gave upper and lower bounds on $\mathrm{RT}(n, \mathcal{T}_3)$, where $\mathcal{T}_3$ is the collection of all trees of maximum degree 3.

Ochem and Vaslet [21] also defined a notion of repetition threshold for "sufficiently large subdivisions" of all graphs. For a graph G, let $\mathcal{S}(G)$ denote the collection of *subdivisions* of G (*i.e.*, those graphs obtained from G by a sequence of edge subdivisions). For a collection of graphs $\mathcal{G}$, we define the *weak repetition threshold* for $\mathcal{G}$, denoted $\mathrm{WRT}(n, \mathcal{G})$, by

$$\mathrm{WRT}(n, \mathcal{G}) = \sup_{G \in \mathcal{G}} \inf_{G_s \in \mathcal{S}(G)} \mathrm{RT}(n, G_s).$$

The repetition threshold for subdivided graphs, as defined by Ochem and Vaslet, is then equivalent to the weak repetition threshold for the collection $\mathcal{H}$ of all graphs. Ochem and Vaslet proved that

$$\mathrm{WRT}(n,\mathcal{H}) = \begin{cases} 7/3, & \text{if } n=2; \\ 7/4, & \text{if } n=3; \\ 3/2, & \text{if } n \geq 4. \end{cases}$$

For all $n \geq 4$, the lower bound $\mathrm{WRT}(n,\mathcal{H}) \geq 3/2$ follows from the somewhat trivial fact that any n-colouring of a graph with a vertex of degree n must contain a factor of exponent $3/2$. This suggests restricting to classes of graphs with bounded maximum degree, as was done for the repetition thresholds of caterpillars and trees [17]. Let $\mathcal{H}_k$ denote the collection of graphs with maximum degree k. For all $n \geq 2$, it is easy to see that we have

$$\mathrm{WRT}(n,\mathcal{H}_2) = \mathrm{CRT}_{\mathrm{W}}(n),$$

since every graph of maximum degree 2 is a disjoint union of paths and cycles. So by Theorem 1.2, we have $\mathrm{WRT}(n,\mathcal{H}_2) = n/(n-1)$ for all $n \geq 45$. In addition to determining the unknown values of $\mathrm{WRT}(n,\mathcal{H}_2)$, it would be interesting to determine values of $\mathrm{WRT}(n,\mathcal{H}_k)$ for $k \geq 3$.

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