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TOTAL EDGE–VERTEX DOMINATION

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Abstract. An edge e ev-dominates a vertex v which is a vertex of e, as well as every vertex adjacent to v. A subset $D \subseteq E$ is an edge-vertex dominating set (in simply, ev-dominating set) of G, if every vertex of a graph G is ev-dominated by at least one edge of D. The minimum cardinality of an ev-dominating set is named with ev-domination number and denoted by $\gamma_{ev}(G)$. A subset $D \subseteq E$ is a total edge-vertex dominating set (in simply, total ev-dominating set) of G, if D is an ev-dominating set and every edge of D shares an endpoint with other edge of D. The total ev-domination number of a graph G is denoted with $\gamma_{ev}^t(G)$ and it is equal to the minimum cardinality of a total ev-dominating set. In this paper, we initiate to study total edge-vertex domination. We first show that calculating the number $\gamma_{ev}^t(G)$ for bipartite graphs is NP-hard. We also show the upper bound $\gamma_{ev}^t(T) \leq (n - l + 2s - 1)/2$ for the total ev-domination number of a tree T, where T has order n, l leaves and s support vertices and we characterize the trees achieving this upper bound. Finally, we obtain total ev-domination number of paths and cycles.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set V and the edge set E. For the open neighbourhood of a vertex v in a graph G, the notation $N_G(v)$ is used as $N_G(v) = \{u|uv \in E(G)\}$ and the closed neighborhood of v is used as $N_G[v] = N_G(v) \cup \{v\}$.

The degree of a vertex $v \in G$ is equal to the number of vertices adjacent to v and denoted by $d_G(v)$. A vertex with degree one is named a *leaf*. A vertex adjacent to a leaf is named *support*. The support vertices are classified as *weak support* if they have degree two or *strong support* otherwise. If an edge is incident to a leaf, it is named an *end edge*. If an edge is adjacent to an end edge (different from an end edge), it is named a *support edge*. If diameter of a tree is even, a vertex that is in the middle of the tree is named a *central vertex*. The diameter of a tree T is denoted by $diam(T)$. We denote path, cycle and star of order n, with P_n, C_n and $S_{1,n-1}$ respectively. Let T be a tree and u be a vertex of T. A vertex u of T is adjacent to a P_n if there exists an edge $e \in E(T)$ such that $T - e$ has a P_n as connected component.

A subset $D \subseteq V$ is a dominating set, if every vertex in G either is an element of D or is adjacent to at least one vertex in D. The domination number of a graph G is denoted by $\gamma(G)$ and it is equal to the minimum cardinality of a dominating set in G. By a similar definition, a subset $D \subseteq V$ is a total dominating set, if every

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vertex of V has a neighbor in D. The total domination number of a graph G is denoted by $\gamma_t(G)$ and it is equal to the minimum cardinality of a total dominating set in G [2]. Fundamental notions of domination theory are outlined in the book [3] and studied in thesis [6].

An edge e ev-dominates a vertex v which is a vertex of e, as well as every vertex adjacent to v. A subset $D \subseteq E$ is an edge-vertex dominating set (in simply, ev-dominating set) of G, if every vertex of a graph G is ev-dominated by at least one edge of D. The minimum cardinality of an ev-dominating set is named with ev-domination number and denoted by $\gamma_{ev}(G)$. Edge-vertex domination was introduced by Peters [6] and further studied by Lewis [5]. An improved upper bound of edge-vertex domination number of a tree is obtained in [7] and trees with total domination number equal to the edge-vertex domination number plus one was studied in [4].

A vertex v ve-dominates an edge e which is incident to v, as well as every edge adjacent to e. A set $D \subseteq V$ is a ve-dominating set if all edges of a graph G are ve-dominated by at least one vertex of D [6]. The minimum cardinality of a ve-dominating set is named with ve-domination number and denoted by $\gamma_{ve}(G)$.

Total version of the vertex-edge domination was introduced and studied by Boutrig and Chellali [1]. A subset $D \subseteq V$ is a total vertex-edge dominating set (in simply, total ve-dominating set) of G, if D is a ve-dominating set and every vertex of D has a neighbor in D [1]. The total ve-domination number of a graph G is denoted by $\gamma_{ve}^t(G)$ and it is equal to the minimum cardinality of a total ve-dominating set.

In this paper, similar to the total vertex-edge domination we introduce the total edge-vertex domination. A subset $D \subseteq E$ is a total edge-vertex dominating set (in simply, total ev-dominating set) of G, if D is an ev-dominating set and every edge of D shares an endpoint with other edge of D. The total ev-domination number of a graph G is denoted by $\gamma_{ev}^t(G)$ and it is equal to the minimum cardinality of a total ev-dominating set.

We first show that calculating the number $\gamma_{ev}^t(G)$ for bipartite graphs is NP-hard. We also obtain some relations between total ev-domination number of a tree and other domination parameters ve-domination, total domination and total ve-domination number of a tree. Moreover, we show the upper bound $\gamma_{ev}^t(T) \leq (n - l + 2s - 1)/2$ for the total ev-domination number of a tree T (different from a star), where T has order n, l leaves and s support vertices and we characterize the trees achieving this upper bound. Finally, we obtain the total ev-domination number of paths and cycles.

2. Complexity result

Our aim in this section is to establish the NP-complete result for total edge-vertex domination problem in bipartite graphs.

2.1. Total edge-vertex domination (TOTAL EV-DOM)

Instance. Graph $G = (V, E)$ and positive integer $k \leq |V|$.

Question. Does G has a total ev-dominating set of cardinality at most k?

We will show that this problem is NP-complete by reducing the well-known NP-complete problem Exact-3-Cover (X3C) to TOTAL EV-DOM.

2.2. Exact 3-cover (X3C)

Instance. A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

Question. Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 2.1. *Problem TOTAL EV-DOM is NP-Complete for bipartite graphs.*

Proof. TOTAL EV-DOM is a member of $\mathcal{NP}$, since we can decide in polynomial time whether a set of cardinality at most k is a total ev-dominating set. We now show a polynomial transformation from an instance (X, C) of

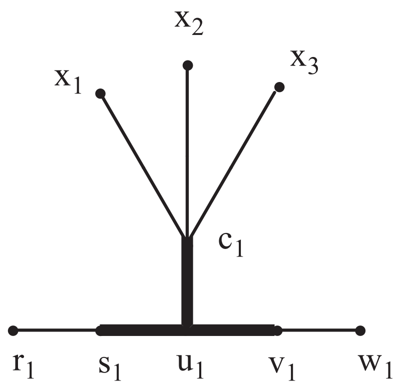

FIGURE 1. A component of the graph G is used for Theorem 2.1.

X3C to an instance G of TOTAL EV-DOM. Let $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{C_1, C_2, ..., C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we create a vertex x_i. For each $C_j \in C$ we build a tree H_j obtained from a path P_5 whose vertices are labeled in order r_j-s_j-u_j-v_j-w_j by adding a vertex c_j and an edge c_ju_j as illustrated in Figure 1. Let $Y = \{c_1, c_2, ..., c_t\}$ and H be the subgraph induced by all $V(H_j)$. Now to obtain a graph G, we add edges c_jx_i if $x_i \in C_j$. Clearly G is a bipartite graph. Set $k = 2t + q$.

Suppose that the instance X, C of X3C has a solution C'. We construct a set S of edges of G as follows: for each $C_j \in C$, we put in S the edges s_ju_j and v_ju_j; for each $C_j \in C'$, we put in S the edge c_ju_j. Since C' exists, its cardinality is precisely q, and so $|S| = 2t + q = k$. Moreover, it is a routine matter to check that S is a total ev-dominating set in G.

Conversely, suppose that G has a total ev-dominating set D of cardinality at most k. To ev-dominate vertices r_j and w_j, set D must contain edges incident with s_j and v_j, respectively. Since D is a total ev-dominating set, we can assume that all v_ju_j's and s_ju_j's belong to D. Observe that $|D \cap E(H)| \geq 2t$ and every vertex of H is total ev-dominated by D. Now, if D contains an edge incident with x_i for some i, then we can replace it by u_pc_p for some $c_p \in N(x_i)$. Hence $|D \cap Y| \leq q$, since $|D| \leq k = 2t + q$. Moreover, using the facts that $|X| = 3q$ and each c_j has exactly three neighbors in X, we deduce that $|D \cap Y| \geq q$ and thus $|D \cap Y| = q$. Consequently, one can easily show that X3C has a solution $C' = \{c_j : c_j \text{ is an endvertex of an edge of } D\}$. □

3. Some relations

Observation 3.1. *[4] For every connected graph G with diameter at least three, there is a $\gamma_{ev}(G)$-set that contains no end edge.*

Observation 3.2. *For every tree T with diameter at least four, there is a $\gamma_{ev}^t(T)$-set that contains no end edge.*

Observation 3.3. *For every tree with diameter at least four, every support edge is contained in a minimum total edge-vertex dominating set.*

It can be seen as an example of minimum total edge-vertex dominating set without one support edge in Figure 2.

There is no total domination set in a one edge graph, so we do not consider a tree with only one edge.

Observation 3.4. *For a nontrivial tree T, $\gamma_{ev}(T) \leq \gamma_{ev}^t(T)$.*

Observation 3.5. *For a nontrivial tree T, $\gamma_{ev}^t(T) \leq \gamma_t(T)$.*

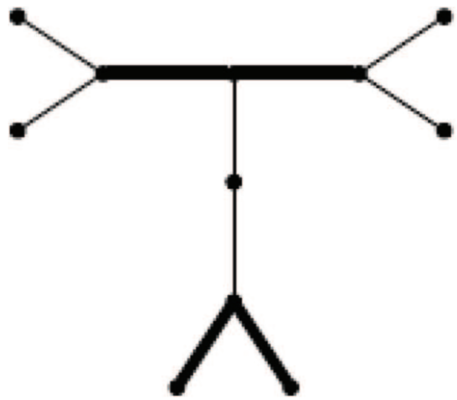

FIGURE 2. Example of minimum total edge-vertex dominating set without one support edge.

Proof. Let D be a $\gamma_t(T)$-set for a tree T. Now we form a set S by adding an edge e for each vertex v of D such that e is incident to v. We show that S is a $\gamma^t_{ev}(T)$-set. If the vertices $u, v \in D$ such that u is adjacent to v, then the edges uv and $vx \in S$ for a neighbor x of v. It is clear that u and v are ev-dominated by uv and vx. Now we assume that a vertex $u \notin D$. Thus u has a neighbor in D, say v and let x be a neighbor of v. Clearly, u is ev-dominated by the edge $vx \in S$. Consequently S is a $\gamma^t_{ev}(T)$-set and $|S| \leq |D|$. □

We note that the difference $\gamma_t(T) - \gamma^t_{ev}(T)$ can be arbitrarily large. Consider a graph G obtained from $m \geq 2$ paths P_5 connected with $m-1$ edges between central vertices of this paths. For the graph G, $\gamma_t(G) = 3m$, $\gamma^t_{ev}(G) = 2m$ and $\gamma_t(G) - \gamma^t_{ev}(G) = m$.

Observation 3.6. *For a nontrivial tree T, $\gamma^t_{ve}(T) \leq \gamma^t_{ev}(T)$.*

Proof. Let D be a $\gamma^t_{ev}(T)$-set for a tree T. We assume that for every pair $uv,\ vx \in D$, u has a neighbor w and x has a neighbor y. Now we form a set S by adding two vertices u, v to it for each pair $uv, vx \in D$. It is clear that the vertices wu, uv, vx and xy can be ve-dominated by only v but it is not total. Consequently S is a $\gamma^t_{ve}(T)$-set and $|S| \leq |D|$. □

We note that the difference $\gamma^t_{ev}(T) - \gamma^t_{ve}(T)$ can be arbitrarily large. Consider a graph G obtained from a star $S_{1,n}$ by attaching n paths P_2, one for each leaf of $S_{1,n}$, by joining one of its leaves to a leaf of $S_{1,n}$. Thus, we obtain that $\gamma^t_{ev}(G) = 2n$, $\gamma^t_{ve}(G) = n + 1$ and $\gamma^t_{ev}(G) - \gamma^t_{ve}(G) = n - 1$.

Lemma 3.7. *For a nontrivial tree T, $\gamma^t_{ev}(T) \leq 2\gamma_{ve}(T)$.*

Proof. Let D be a $\gamma_{ve}(T)$-set for a tree T. Let S be an ev-dominating set by adding two edges e_1 and e_2 for each vertex v of D such that e_1 and e_2 are incident to v. Clearly, every vertex, located on the edges which are ve-dominated by v, is ev-dominated by e_1 and e_2. Thus, S is a total ev-dominating set and $\gamma^t_{ev}(T) \leq 2\gamma_{ve}(T)$. □

We also know that $\gamma_{ve}(T) \leq \gamma_{ev}(T)$ and using this relation we obtain the following corollary.

Corollary 3.8. *For a nontrivial tree T, $\gamma^t_{ev}(T) \leq 2\gamma_{ev}(T)$.*

Let T be a tree with diameter four. Therefore, $\gamma^t_{ev}(T) = 2 = 2\gamma_{ve}(T)$.

We show that a tree T different from a star with order n, number of leaves l and support vertices s has $\gamma^t_{ev}(T) \leq (n - l + 2s - 1)/2$. Furthermore, in order to characterize the trees attaining this bound, we define a family $\mathcal{F}$ of trees $T = T_k$ as follows. Let $T_1 = P_7$ and for a k positive integer, T_{k+1} is a tree recursively obtained from T_k by one of the following operations:

Operation O_1: Attach a vertex by joining it to any support vertex of T_k.

Operation O_2: Attach a path P_3 by joining one of its leaves to the central vertex of T_k.

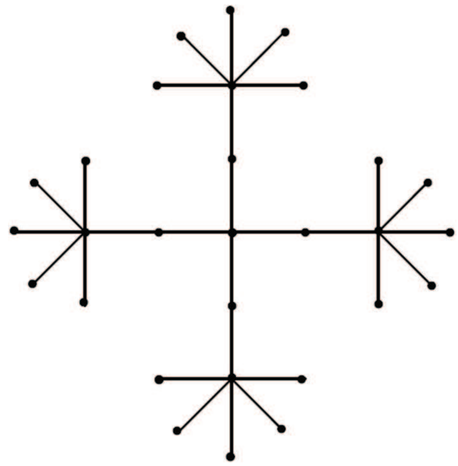

FIGURE 3. A tree $T \in \mathcal{F}$.

Theorem 3.9. *If $T \in \mathcal{F}$, then $\gamma^t_{ev}(T) = (n - l + 2s - 1)/2$.*

Proof. We use induction on the number k operations used to construct the tree $T = T_{k+1}$. If $T = T_1 = P_7$, then $(n - l + 2s - 1)/2 = 4 = \gamma^t_{ev}(P_7)$. Now we assume that the result is true for every $T^{'} = T_k$ of $\mathcal{F}$ obtained by $k-1$ operations with order $n^{'}$, number of leaves $l^{'}$ and support vertices $s^{'}$.

First assume that $T = T_{k+1}$ is obtained from $T^{'}$ by *Operation* O_1. Obviously, $n = n^{'} + 1$, $l = l^{'} + 1$ and $s = s^{'}$. We have $\gamma^t_{ev}(T^{'}) = \gamma^t_{ev}(T)$ by Observation 3.2 and 3.3. By induction on tree $T^{'}$, $\gamma^t_{ev}(T^{'}) = (n^{'} - l^{'} + 2s^{'} - 1)/2 = \big((n-1) - (l-1) + 2s - 1\big)/2 = (n - l + 2s - 1)/2 = \gamma^t_{ev}(T)$.

Assume that $T = T_{k+1}$ is obtained from $T^{'}$ by *Operation* O_2. Let u be the central vertex of $T^{'}$ as a $P_3 : abc$ attached to it with the edge ua. Thus, $n = n^{'} + 3$, $l = l^{'} + 1$ and $s = s^{'} + 1$. Since a $\gamma^t_{ev}(T^{'})$-set can be a $\gamma^t_{ev}(T)$-set by adding the edges ua, ab to it, $\gamma^t_{ev}(T) \le \gamma^t_{ev}(T^{'}) + 2$. Now, let D be $\gamma^t_{ev}(T)$-set. Since the vertex c is a leaf, the edges ua and ab are contained in D. Thus $D - \{ua, ab\}$ is a $\gamma^t_{ev}(T^{'})$-set and $\gamma^t_{ev}(T^{'}) = \gamma^t_{ev}(T) - 2$. Therefore, $\gamma^t_{ev}(T) = \gamma^t_{ev}(T^{'}) + 2$. We use induction on $T^{'}$ and we have $\gamma^t_{ev}(T^{'}) + 2 = (n^{'} - l^{'} + 2s^{'} - 1)/2 + 2 = \big((n-3) - (l-1) + 2(s-1) - 1\big)/2 + 2 = (n - l + 2s - 1)/2 = \gamma^t_{ev}(T)$. □

A member of the family $\mathcal{F}$ is illustrated in Figure 3.

Theorem 3.10. *If T is a nontrivial tree different from a star, then $\gamma^t_{ev}(T) \le (n - l + 2s - 1)/2$ with order n, number of leaves l and support vertices s with equality if and only if $T \in \mathcal{F}$.*

Proof. If $T \in \mathcal{F}$, then $\gamma^t_{ev}(T) = (n - l + 2s - 1)/2$. If $diam(T) = 1$, then $T = P_2$ and there is no total *ev*-dominating set in P_2. If $diam(T) = 2$, then T is a star and $\gamma^t_{ev}(T) = 2$. For a star, the leaves have a common support vertex. Thus we use our upper bound for $diam(T) \ge 3$ and $n \ge 4$ to obtain the best possible upper bound. We want to prove that if T is a tree different from a star with order $n \ge 4$, l leaves and s support vertices, then $\gamma^t_{ev}(T) \le (n - l + 2s - 1)/2$, with equality only for $T \in \mathcal{F}$. We use induction and it is assumed that the result is true for every tree $T^{'} = T_k$ with order $4 \le n^{'} < n$, $l^{'}$ leaves and $s^{'}$ support vertices.

First assume some support vertex of T, for example x, is strong. Let y be a leaf adjacent to x and $T^{'} = T - y$, so we have $n^{'} = n - 1$, $l^{'} = l - 1$ and $s^{'} = s$. If $D^{'}$ be a $\gamma^t_{ev}(D^{'})$-set, then D is also a $\gamma^t_{ev}(T)$-set and by Observation 3.2, $\gamma^t_{ev}(T) \le \gamma^t_{ev}(T^{'})$. Moreover, $\gamma^t_{ev}(T^{'}) \le \gamma^t_{ev}(T)$. Thus, if $T = (n - l + 2s - 1)/2$ then $\gamma^t_{ev}(T^{'}) = (n^{'} - l^{'} + 2s^{'} - 1)/2$ and $T^{'} \in \mathcal{F}$. Thus, $T \in \mathcal{F}$ and it is obtained from $T^{'}$ by *Operation* O_1. So we can assume that every support vertex is weak.

Since we assign $T_1 = P_7$, $diam(T) = 6$, so we assume that $diam(T) \ge 6$ in following cases. We root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r, v be parent of t, u be parent of v, w be parent of u, d be parent of w and e be parent of d in the rooted tree. The subtree induced by a vertex x and its descendants in the rooted tree T is denoted by T_x. Let D be a $\gamma_{ev}^t(T)$-set and $D^{'}$ be a $\gamma_{ev}^t(T^{'})$-set.

Assume that some child of u is a leaf and it is denoted with x. Let $T^{'}=T-x$. The edges wu, uv are contained in $D^{'}$. Thus, $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})$ with $n^{'}=n-1$, $l^{'}=l-1$ and $s^{'}=s-1$. Thus, we obtain $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})=(n^{'}-l^{'}+2s^{'}-1)/2=(n-l+2s-3)/2<(n-l+2s-1)/2$.

Now assume that among the children of u there is a support vertex x, other than v. Let $T^{'}=T-T_v$. Thus $n^{'}=n-2$, $l^{'}=l-1$ and $s^{'}=s-1$. By Observation 3.2, $D^{'}\cup\{uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+1$. Therefore, we obtain $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+1=(n^{'}-l^{'}+2s^{'}-1)/2+1=(n-l+2s-4)/2+1<(n-l+2s-1)/2$.

Now assume that $d_T(u)=2$. Thus, $diam(T)=3$. If $d_T(w)=1$, then $T=P_4$. Now assume that $d_T(w)=2$. Thus, $diam(T)=4$. If $d_T(d)=1$, then $T=P_5$ and $\gamma_{ev}^t(P_5)=2<(5-2+4-1)/2=3$. Now assume that $d_T(d)\geq 2$. Let $T^{'}=T-T_w$. Thus, $D^{'}\cup\{wu,uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ep}^t(T)\leq\gamma_{ev}^t(T^{'})+2$ with $n^{'}=n-4$, $l^{'}=l-1$ and $s^{'}=s-1$. Thus, we have $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2=(n^{'}-l^{'}+2s^{'}-1)/2+2=(n-l+2s-6)/2+2<(n-l+2s-1)/2$.

Now assume that $d_T(w)\geq 3$. First assume that w is adjacent to a path P_2 or P_3 containing k as a neighbor of w . Let $T^{'}=T-T_u$. We have $n^{'}=n-3$, $l^{'}=l-1$ and $s^{'}=s-1$. Thus, $D^{'}\cup\{wu,uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2$. Thus, we have $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2=(n^{'}-l^{'}+2s^{'}-1)/2+2=(n-l+2s-1)/2$. If $d_T(d)=2$ and $T_k=P_3$, then $T^{'}\in\mathcal{F}$ and $D-\{wu,uv\}$ is a $\gamma_{ev}^t(T^{'})$-set and $\gamma_{ev}^t(T)-2\geq\gamma_{ev}^t(T^{'})$. Consequently, if $T=(n-l+2s-1)/2$ then $\gamma_{ev}^t(T^{'})=(n^{'}-l^{'}+2s^{'}-1)/2$ and $T^{'}\in\mathcal{F}$. Thus, $T\in\mathcal{F}$ and it is obtained from $T^{'}$ by *Operation* O_2.

We assume that some child of w different from u is a leaf, say x. We can assume that $d_T(w)=3$. First assume that some child of d, say k, is a leaf. Let $T^{'}=T-T_u$. We have $n^{'}=n-3$, $l^{'}=l-1$ and $s^{'}=s-1$. Thus, $D^{'}\cup\{wu,uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2$. Thus, we have $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2=(n^{'}-l^{'}+2s^{'}-1)/2+2=(n-l+2s-1)/2$.

Suppose that some child of d different from w, say k, is not a leaf. It is enough to consider the cases T_k is equal to T_w or T_k is a path P_2 or P_3. Let $T^{'}=T-T_w$. In this case, $n^{'}=n-5$, $l^{'}=l-2$ and $s^{'}=s-2$. Thus, $D^{'}\cup\{wu,uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2$. Thus, we have $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2=(n^{'}-l^{'}+2s^{'}-1)/2+2=(n-l+2s-4)/2<(n-l+2s-1)/2$.

Now assume that $d_T(d)=2$. Thus, $diam(T)=5$. Let $T^{'}=T-T_d$. We have $n^{'}=n-6$, $l^{'}=l-2$ and $s^{'}=s-2$. Thus, $D^{'}\cup\{wu,uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2$. Thus, we have $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2=(n^{'}-l^{'}+2s^{'}-1)/2+2=(n-l+2s-5)/2<(n-l+2s-1)/2$.

Now assume that $d_T(w)=2$. First assume $d_T(d)\geq 2$. Let $T^{'}=T-T_w$. We have $n^{'}=n-4$, $l^{'}=l-1$ and $s^{'}=s-1$. Thus, $D^{'}\cup\{wu,uv\}$ is a $\gamma_{ev}^t(T)$-set and $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2$. Thus, we have $\gamma_{ev}^t(T)\leq\gamma_{ev}^t(T^{'})+2=(n^{'}-l^{'}+2s^{'}-1)/2+2=(n-l+2s-2)/2<(n-l+2s-1)/2$.

Assume that $d_T(d)=2$. If $d_T(e)=1$, then we obtain that $\gamma_{ev}^t(T)=3<(6-2+4-1)/2=7/2$. This completes the proof. □

We consider a graph T obtained from a P_7 by attaching $n-2$ paths P_3 by joining one of its to central vertex of P_7. Clearly, T is a member of the family $\mathcal{F}$. Thus, we obtain that $\gamma_{ev}^t(T)=(3n+1-n+2n-1)/2=2n$. One may verify that this example is the same graph of Observation 3.6. Now we connect two T with an edge between central vertices of this two same trees. For this case, new order $n^{'}=6n+2$, $l^{'}=2n$ leaves and $s^{'}=2n$ support vertices. Thus, $\gamma_{ev}^t(T)=4n<(n^{'}-l^{'}+2s^{'}-1)/2=(6n+2-2n+4n-1)/2=4n+1/2$. It is clear that our upper bound is tight.

Theorem 3.11. *For the classes of paths P_n with $n\geq 3$ vertices and cycles C_n with n vertices,*

$$\gamma_{ev}^t(P_n)=\gamma_{ev}^t(C_n)=\begin{cases} 2\lfloor n/5\rfloor+1, & \text{if } n\equiv 1 \ (mod\ 5)\\ 2\lceil n/5\rceil, & \text{otherwise}\end{cases}.$$

Proof. Two adjacent edges can ev-dominate five neighbor vertices in a path and cycle. Thus we consider $\gamma_{ev}^t(P_n)$ to *mod* 5. If the order of a path or a cycle is a multiple of five, then $\gamma_{ev}^t(P_n)=\gamma_{ev}^t(C_n)=2n/5$. By Theorem 3.10

we obtain that $\gamma_{ev}^t(P_n) \leq (n-2+4-1)/2 = (n+1)/2$. Now we assume that order n of a path or a cycle is such that $n \equiv 1$ $(mod\, 5)$. In this case, every adjacent five vertices ev-dominated by two neighbor edges of this vertices. Thus, a vertex remains non-dominated and we add an edge to one of these two neighbor edges. Thus, $\gamma_{ev}^t(P_n) = \gamma_{ev}^t(C_n) = 2\lfloor n/5 \rfloor + 1$, if $n \equiv 1$ $(mod\, 5)$. For other cases we obtain $\gamma_{ev}^t(P_n) = \gamma_{ev}^t(C_n) = 2\lceil n/5 \rceil$ by using a similar way. □

4. Open questions

We conclude the paper with some open questions.

Question 4.1. Characterize the trees T with $\gamma_{ev}^t(T) = \gamma_{ve}^t(T)$.

Question 4.2. Characterize the trees T with $\gamma_{ev}^t(T) = \gamma_t(T)$.

Question 4.3. Characterize the trees T with $\gamma_{ev}^t(T) = 2\gamma_{ve}(T)$.

Question 4.4. Characterize the trees T with $\gamma_{ev}^t(T) = 2\gamma_{ev}(T)$.

Question 4.5. Characterize the trees T with $\gamma_{ev}^t(T) = \gamma_{ev}(T)$.

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