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EDGE-BIPANCYCLICITY IN CONDITIONAL EDGE-FAULTY K-ARY N-CUBES*

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Abstract. The class of k-ary n-cubes represents the most commonly used interconnection topology for parallel and distributed computing systems. In this paper, we consider the faulty k-ary n-cube with even $k \geq 4$ and $n \geq 2$ such that each vertex of the k-ary n-cube is incident with at least two healthy edges. Based on this requirement, we investigate the fault-tolerant capabilities of the k-ary n-cube with respect to the edge-bipancyclicity. We prove that in the k-ary n-cube Q_n^k, every healthy edge is contained in fault-free cycles of even lengths from 6 to $|V(Q_n^k)|$, even if the Q_n^k has up to $4n-5$ edge faults and our result is optimal with respect to the number of edge faults tolerated.

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1. Introduction

A multiprocessor system and an interconnection network (networks for short) have an underlying topology, which is usually presented by a graph, where nodes represent processors and links represent communication links between processors. Study of the properties of the graph is an important part of the study of the system. The k-ary n-cube, denoted by Q_n^k, has been one of the most common graphs for the system because of its desirable properties, such as ease of implementation and its ability to reduce message latency by exploiting the communication locality found in many parallel applications [5, 6]. For example, the underlying topology of the Cray T3D [8], the iWarp [13] and the IBM Blue Gene [1] is based on k-ary n-cubes.

The problem of embedding a guest graph into a host graph attracts much attention [2, 14, 16]. For an undirected simple graph G, we denote the vertex set and the edge set by $V(G)$ and $E(G)$, respectively. G is said to be pancyclic if it contains a cycle of every length from its girth to $|V(G)|$. Bipancyclicity is essentially a restriction of the concept of pancyclicity to bipartite graphs whose cycles are necessarily of even length [18]. Furthermore, a bipancyclic graph is said to be edge-bipancyclic if every edge lies in a cycle of every even length. The edge-bipancyclicity of various graphs has attracted a great deal of attention in recent years [7, 10–12, 15].

In the system (or the network) where the nodes and their links are likely to fail, it is important to consider the fault tolerance (robustness) of the system (or the network). The fault tolerance of large systems (or networks) is usually a measure of the extent to which the system (or the network) can retain its original nature in the

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event of a certain number of nodes of failure and/or links failure. In [12], Li *et al.* studied the problem of cycles embedding in k-ary n-cubes with faulty vertices and edges. They presented the following result.

Theorem 1.1 ([12])**.** *Given an integer $n \geq 2$ and an even integer $k \geq 4$, let Q_n^k be a k-ary n-cube with f_v faulty vertices and f_e faulty edges. If $f_v + f_e \leq 2n-3$, then each healthy edge of Q_n^k lies in a cycle of every even length from 4 to $k^n - 2f_v$.*

In the event of a random edge failure, it is very unlikely that all of the edges incident with a single vertex fail simultaneously. This reason has motivated research on the conditional faulty k-ary n-cube in which each vertex is incident with at least two healthy edges. In [3], Ashir and Stewart studied the problem of embedding hamiltonian cycles in k-ary n-cubes with conditional edge-faults. They presented the following result.

Theorem 1.2 ([3])**.** *Let $n \geq 2$ and let $k \geq 4$ be even. Then the conditional faulty Q_n^k with at most $4n-5$ faulty edges is hamiltonian.*

In [18], Xiang *et al.* determined the bipancyclicity of the k-ary n-cube with conditional edge-faults.

Theorem 1.3 ([18])**.** *Let $n \geq 2$ and let $k \geq 4$ be even. Then the conditional faulty Q_n^k with at most $4n-5$ faulty edges is bipancyclic.*

In this paper, we consider conditional faulty Q_n^k with even $k \geq 4$ and prove that every healthy edge is contained in a fault-free cycle of every even length from 6 to $|V(Q_n^k)|$, even if the Q_n^k has up to $4n-5$ edge faults. We also give an example showing that the upper bound $4n-5$ is sharp.

2. Terminology

The k-ary n-cube, denoted by Q_n^k ($k \geq 2$ and $n \geq 2$), is a graph consisting of k^n vertices, each of which has the form $u = u_{n-1}u_{n-2}\ldots u_0$, where $0 \leq u_i \leq k-1$ for $0 \leq i \leq n-1$. Two vertices $u = u_{n-1}u_{n-2}\ldots u_0$ and $v = v_{n-1}v_{n-2}\ldots v_0$ are adjacent if and only if there exists an integer $j \in \{0, 1, \ldots, n-1\}$, such that $u_j = v_j \pm 1 (\mathrm{mod}\ k)$ and $u_i = v_i$, for $i \in \{0, 1, \ldots, n-1\} \setminus \{j\}$. Such an edge (u, v) is called a *j-dimensional edge*. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. Obviously, Q_n^k is bipartite if k is even. So, Q_n^k (even $k \geq 4$) doesn't contain any fault-free cycle of odd length. For any two distinct vertices u and v of such Q_n^k, set

$$\delta(u, v) = \begin{cases} 1, & \text{if } u \text{ and } v \text{ are in different parts;} \\ 0, & \text{if } u \text{ and } v \text{ are in the same part.} \end{cases}$$

We can partition Q_n^k along the dimension j, by deleting all the j-dimensional edges, into k disjoint subcubes, $Q_n^k[0],\ Q_n^k[1], \ldots, Q_n^k[k-1]$ (abbreviated as $Q[0],\ Q[1], \ldots, Q[k-1]$, if there is no ambiguity). It is clear that each $Q[i]$ is isomorphic to Q_{n-1}^k for $0 \leq i \leq k-1$. Let $u = u_{n-1}u_{n-2}\ldots u_{j+1}iu_{j-1}\ldots u_0$ be a vertex of $Q[i]$. The vertex $v = u_{n-1}u_{n-2}\ldots u_{j+1}i'u_{j-1}\ldots u_0$ of $Q[i']$ is denoted by $n^{i'}(u)$, where $0 \leq i' \leq k-1$ and $i' \neq i$. Obviously, $n^{i'}(u)$ and u are adjacent if and only if $i' = i \pm 1$. For $0 \leq p, q \leq k-1$, we use $Q_n^k[p, q]$ (abbreviated as $Q[p, q]$, if there is no ambiguity) to denote the subgraph of Q_n^k which is induced by $\{u : u \in V(Q[i]), i = p, p+1, \ldots, q-1, q\}$. Then $Q[p, p] = Q[p]$ and $Q[p, p-1] = Q_n^k$.

The set of edges incident with a vertex v in a graph G is denoted by $\partial_G(v)$. Let $d_G(v)$ be the number of edges of G incident with v and let F be a set of faulty edges of Q_n^k. We call F a conditional faulty edge set of Q_n^k if each vertex of $Q_n^k - F$ is incident with at least two edges. Throughout this paper, we denote by F a conditional faulty edge set of Q_n^k, and by $\mathcal{F}^i$ the set of faulty i-dimensional edges. Then $F = \bigcup_{i=0}^{n-1} \mathcal{F}^i$. For $0 \leq j \leq k-1$, set $F_j = F \cap E(Q[j])$. For graph-theoretical terminology and notation not defined here we follow [4].

3. Some lemmas

The purpose of this section is to give some results that are useful for the proof of the main theorem. In Q_2^k, every vertex has the form $v_1v_0 = ab$, where $0 \le a, b \le k-1$. For convenience, we write ab as $v_{a,b}$.

For $0 \le i, j \le k-1$, Row$[i:j]$ is the subgraph of Q_2^k induced by $\{v_{a,b} : a = i, i+1, \dots, j,\ 0 \le b \le k-1\}$; Col$[i:j]$ is the subgraph of Q_2^k induced by $\{v_{a,b} : 0 \le a \le k-1, b = i, i+1, \dots, j\}$. Instead of Row$[i:i]$ and Col$[j:j]$, we simply write Row$[i]$ and Col$[j]$. For $1 \le k_1, k_2 \le k-1$, the subgraph of Q_2^k induced by $\{v_{a,b} : 0 \le a \le k_1 - 1, 0 \le b \le k_2 - 1\}$ is denoted by Grid(k_1, k_2). A vertex of Grid(k_1, k_2) is called a corner vertex if its degree in Grid(k_1, k_2) is 2.

For convenience of discussion, we define the following subsets of $E(Q_2^k)$.

For $0 \le i \le k-1$, Row$(i : i+1) = \{(v_{i,j}, v_{i+1,j}) : 0 \le j \le k-1\}$.

For $0 \le j \le k-1$, Col$(j : j+1) = \{(v_{i,j}, v_{i,j+1}) : 0 \le i \le k-1\}$.

For $0 \le l \le k-1$, $M_0^l = \{(v_{l,j}, v_{l,j+1}) : j = 0, 2, \dots, k-2\}$ and $M_1^l = \{(v_{l,j}, v_{l,j+1}) : j = 1, 3, \dots, k-1\}$. Then M_0^l and M_1^l are two edge-disjoint perfect matchings of Row$[l]$. Clearly, $|M_0^l| = |M_1^l| = k/2$ and $M_0^l \cup M_1^l = E(\text{Row}[l])$.

Lemma 3.1 ([9]). *Let s and t be any two distinct vertices of Grid(m, n) with $m, n \ge 2$. If $\delta(s,t) = 1$ and s be a corner vertex of Grid(m, n), then there exists a hamiltonian st-path of Grid(m, n).*

Lemma 3.2 ([9]). *Let s and t be any two distinct vertices of Grid(m, n) with $m, n \ge 4$. If $\delta(s,t) = 1$, then there exists a hamiltonian st-path of Grid(m, n).*

In [14], Stewart and Xiang considered the hamiltonian path embeddings in Q_n^k with even $k \ge 4$. They presented the following results.

Lemma 3.3 ([14]). *Let s and t be any two distinct vertices of Row$[p:q]$, where $0 \le p < q \le k-1$. If $\delta(s,t) = 1$, then there exists a hamiltonian st-path P of Row$[p:q]$. Moreover, each of the following holds:*

(i) P contains at least one edge of Row$[r]$, for every $r = p, q$.

(ii) P contains at least two nonadjacent edges of Row$[p]$ or Row$[q]$.

(iii) If $k \ge 6$, then P contains at least two nonadjacent edges of Row$[r]$, for every $r = p, q$.

(iv) If $k = 4$ and $s, t \notin V(\text{Row}[p])$ (resp. $s, t \notin V(\text{Row}[q])$), then P contains two nonadjacent edges of Row$[p]$ (resp. Row$[q]$).

Lemma 3.4 ([14]). *Given an integer $n \ge 2$ and an even integer $k \ge 4$, let Q_n^k be a k-ary n-cube with f_v faulty nodes and f_e faulty edges. If $f_v + f_e \le 2n-2$ and s, t are distinct healthy vertices such that $\delta(s,t) = 1$(resp. $\delta(s,t) = 0$), then then there exists an st-path of length at least $k^n - 2f_v - 1$(resp. $k^n - 2f_v - 2$).*

Lemma 3.5 ([9]). *Given an even $k \ge 4$, let s, t and x be any three distinct vertices in Row$[0:p]$ of Q_2^k, where $1 \le p \le k-1$. If $\delta(x,s) = 1$ and $\delta(s,t) = 0$, then there exists a hamiltonian st-path of Row$[0:p] - x$.*

Lemma 3.6 ([12]). *Let $e \in E(\text{Row}[0:p])$, where $1 \le p \le k-1$. Then the following hold.*

(i) Row$[0:p]$ has two distinct hamiltonian cycles containing e such that one contains M_α^0 and M_β^p, and the other contains $M_{1-\alpha}^0$ and $M_{1-\beta}^p$, where $\alpha, \beta \in \{0, 1\}$.

(ii) e lies in a cycle of each even length from 4 to $(p+1)k$.

Lemma 3.7 ([12]). *Let v^* be a faulty vertex in Row $[0]$ of Row$[0:p]$. If $p \ge 2$, then each edge of Row$[0:p] - v^*$ lies in a cycle of every length from 4 to $(p+1)k - 2$.*

Lemma 3.8 ([12]). *Let e^* be a faulty edge in Row$[0]$ of Row$[0:p]$. If $p \ge 3$, then the faulty Row$[0:p]$ is edge-bipancyclic.*

We define the following paths in Row$[i : i+1]$. Let $i \le a \le i+1,\ 0 \le b \ne m \le k-1$. If $a = i$ then let $\bar{a} = i+1$, and if $a = i+1$ then let $\bar{a} = i$.

$$C_m^+(v_{a,b}, v_{\bar{a},b}) = v_{a,b}v_{a,b+1}v_{a,b+2}\cdots v_{a,m-1}v_{a,m}v_{\bar{a},m}v_{\bar{a},m-1}v_{\bar{a},m-2}\cdots v_{\bar{a},b+1}v_{\bar{a},b}.$$

$$C_m^-(v_{a,b}, v_{\bar{a},b}) = v_{a,b}v_{a,b-1}v_{a,b-2}\cdots v_{a,m+1}v_{a,m}v_{\bar{a},m}v_{\bar{a},m+1}v_{\bar{a},m+2}\cdots v_{\bar{a},b-1}v_{\bar{a},b}.$$

In addition, if $m = b$, we define $C_b^+(v_{a,b}, v_{\bar{a},b}) = C_b^-(v_{a,b}, v_{\bar{a},b}) = (v_{a,b}, v_{\bar{a},b})$.

Lemma 3.9. *Let $(v_{0,0}, v_{0,1})$ be a faulty edge in Row$[0 : p]$ and let s, t be any two vertices of Row$[0 : p]$. If $\delta(s, t) = 1$, then there is a hamiltonian st-path of Row$[0 : p] - (v_{0,0}, v_{0,1})$ that contains at least two nonadjacent edges of Row$[0]$ and one edge of Row$[p]$ or two nonadjacent edges of Row$[p]$ and one edge of Row$[0]$.*

Proof. Set $s = v_{i,j}$ and $t = v_{i',j'}$, where $0 \le i, i' \le p$ and $0 \le j, j' \le k - 1$. We prove the assertion by induction on p. It may be readily checked that the lemma holds for $p = 1$, Suppose that $p \ge 2$ and the assertion holds for Row$[0 : p - 1]$. It is enough to consider Row$[0 : p]$. Recall that $s = v_{i,j}$ and $t = v_{i',j'}$.

Suppose first that $0 \le i, i' \le p - 1$. By the induction hypothesis, there is a hamiltonian st-path P^1 of Row$[0 : p-1] - (v_{0,0}, v_{0,1})$ that contains at least one edge of Row$[p-1]$. It is easy to obtain the path as required. Suppose next that $0 \le i \le p-1$ and $i' = p$. By the induction hypothesis, there is a hamiltonian $sv_{p-1,j'-1}$-path P^1 of Row$[0 : p-1] - (v_{0,0}, v_{0,1})$. Then $P = P^1 \cup$ Row$[p] + (v_{p-1,j'-1}, v_{p,j'-1}) - (v_{p,j'-1}, t)$ is as required. Suppose now that $i = i' = p$. By the induction hypothesis, there is a hamiltonian $v_{p-1,j+1}v_{p-1,j'+1}$-path P^1 of Row$[0 : p - 1] - (v_{0,0}, v_{0,1})$. Then $P = P^1 \cup$ Row$[p] - \{(s, v_{p,j+1}), (t, v_{p,j'+1})\} + \{(v_{p,j+1}, v_{p-1,j+1}), (v_{p,j'+1}, v_{p-1,j'+1})\}$ is as required. □

According to the proof of Lemma 3.9, we have the following Corollary.

Corollary 3.10. *Let $(v_{0,0}, v_{0,1})$ be a faulty edge in Row$[0 : p]$ and let s, t be any two vertices of Row$[0 : p]$. If $\delta(s, t) = 1$, then there is a hamiltonian st-path P of Row$[0 : p] - (v_{0,0}, v_{0,1})$. Moreover, P contains at least two nonadjacent edges of Row$[r]$ if $(s, t) \notin E($Row$[r])$, where $r \in \{0, p\}$.*

Lemma 3.11. *Let $(v_{0,0}, v_{0,1})$ be a faulty edge of Row$[0 : p]$ in Q_2^k.*

(i) If $p \ge 2$, then the faulty Row$[0 : p]$ is edge-bipancyclic.

(ii) If $p = 1$, then each healthy edge of Row$[0] \cup$ Row$[1] - (v_{1,0}, v_{1,1})$ lies in a cycle of every even length from 4 to $2k$.

(iii) If $p = 1$ and $k = 4$, then $(v_{1,0}, v_{1,1})$ lies in a cycle of every even length from 4 to $2k$.

(iv) If $p = 1$, then each edge of Row$(0 : 1)$ lies in a cycle of every even length from 4 to $k + 2$ and the $k + 2$-cycle contains at least $k/2$ edges of Row$[r]$, for every $r = 0, 1$.

Proof. Choose a healthy edge e of Row$[0 : p]$. If $p \ge 3$, then, by Lemma 3.8, the faulty Row$[0 : p]$ is edge-bipancyclic. Suppose that $p = 2$. If e is not incident with $v_{0,0}$, then, by Lemma 3.7, e lies in a cycle of every even length from 4 to $3k - 2$ in Row$[0 : 2] - v_{0,0}$. By Lemma 3.9, e lies in a healthy cycle of length $3k$. Thus, there is a cycle of each even length from 4 to $3k$ that contains e. If e is incident with $v_{0,0}$, then $e \in \{(v_{0,0}, v_{1,0}), (v_{0,0}, v_{0,k-1})\}$.

Set $C_i = \langle v_{0,0}, v_{0,k-1}, C_i^-(v_{0,k-1}, v_{1,k-1}), v_{1,k-1}, v_{1,0}, v_{0,0}\rangle$, $i = k-1, k-2, \ldots, 1$. Note that C_i is a cycle of length $2(k - i + 1)$ and $e \in E(C_i)$. So e lies in a cycle of each even length from 4 to $2k$. For any edge $(v_{1,j}, v_{1,j+1})$ of the cycle $\langle v_{0,0}, C_1^-(v_{0,0}, v_{1,0}), v_{1,0}, v_{0,0}\rangle$, where $j \in \{1, 3, \ldots, k - 1\}$, replace $(v_{1,j}, v_{1,j+1})$ with the paths $\langle v_{1,j}, v_{2,j}, v_{2,j+1}, v_{1,j+1}\rangle$, and so on, so as to obtain healthy cycles of lengths $2k + 2, 2k + 4, \ldots, 3k$. Therefore, e lies in a cycle of each even length from 4 to $3k$. Then the statement (i) holds. By the structure of Row$[0 : 1] - (v_{0,0}, v_{0,1})$, it is easy to verify that the statements (ii)-(iv) hold. □

Choose any dimension $r \in \{0, 1, \ldots, n - 1\}$. Partition Q_n^k ($n \ge 3$ and even $k \ge 4$) along the dimension r into k disjoint subcubes $Q[0], Q[1], \ldots, Q[k - 1]$. Let $|F| \le 4n - 5$ and $p \in \{0, 1, \ldots, k - 1\}$. We have the following lemmas.

Lemma 3.12. *Given a fault-free edge (u, v) of $Q[0]$, if $|F_i| \le 2n - 5$, for $i = 0, 1, \ldots, p$, then each of the following holds.*

(*i*) *There is a hamiltonian uv-path P of $Q[0,p]-F$ such that $|E(P\cap Q[p])|=k^{n-1}-1$.*
(*ii*) *There is a fault-free uv-path of every odd length from 1 to $(p+1)k^{n-1}-1$ in $Q[0,p]-F$.*

Proof. We prove the assertion by induction on p. According to Theorem 1.1, the assertion is true for $p=0$. Suppose that $p\geq 1$ and the assertion holds for $Q[0:p-1]$. Then there is a hamiltonian uv-path P_1 of $Q[0,p-1]-F$ such that $|E(P_1\cap Q[p-1])|=k^{n-1}-1$. It is enough to consider $Q[0:p]$.

Recall that $|F|\leq 4n-5$. Since $\lceil\frac{k^{n-1}-1}{2}\rceil>4n-5$, there is an edge $(s,t)\in E(P_1\cap Q[p-1])$ such that $(s,n^p(s)),(t,n^p(t)),(n^p(s),n^p(t))\notin F$. Observe that $|F_p|\leq 2n-5=2(n-1)-3$. By Theorem 1.1, there is a fault-free $n^p(s)n^p(t)$-path P_2 in $Q[p]$ of every odd length l_2 with $1\leq l_2\leq k^{n-1}-1$. Thus, $P=P_1\cup P_2+\{(s,n^p(s)),(t,n^p(t))\}-(s,t)$ is a uv-path of every odd length $l=|E(P_1)|+l_2+1$ with $pk^{n-1}+1\leq l\leq(p+1)k^{n-1}-1$. By the induction hypothesis, there is a fault-free uv-path of every odd length from 1 to $pk^{n-1}-1$ in $Q[0,p-1]$. Hence, a path of a specified length can be constructed. If $|E(P_2)|=k^{n-1}-1$, then P is a hamiltonian uv-path and $|E(P\cap Q[p])|=|E(P_2)|=k^{n-1}-1$. Thus the proof is complete. □

Lemma 3.13 ([17])**.** *Given an even integer $k\geq 4$, let s^* and t^* be two adjacent vertices of Q_n^k and let s and t be any two vertices of $Q_n^k-\{s^*,t^*\}$ such that $\delta(s,t)=1$. Then there exists a hamiltonian st-path of $Q_n^k-\{u^*,v^*\}$.*

Lemma 3.14 ([17])**.** *Given an even integer $k\geq 4$, let s^* be a vertex of Q_n^k and let s and t be any two distinct vertices of $Q_n^k-s^*$ such that $\delta(s,t)=0$ and $\delta(s,s^*)=1$. Then there exists a hamiltonian st-path of $Q_n^k-s^*$.*

Given a graph G, let S and T be two subsets of $V(G)$. An (S,T)-*path* is a path which starts at a vertex of S, ends at a vertex of T, and whose internal vertices belong to neither S nor T. By induction and Lemma 3.13, we can get the following lemma easily.

Lemma 3.15. *Let $S=\{s^1,s^2\}\subset V(Q[p])$ such that $\delta(s^1,s^2)=0$. Then there exists a set $T=\{t^1,t^2\}\subset V(Q[q])$ such that $\delta(s^1,t^1)=1$, $\delta(t^1,t^2)=0$ and there are two vertex-disjoint (S,T)-paths in $Q[p,q]$ that contain all vertices of $Q[p,q]$, where $0\leq p\leq q\leq k-1$.*

Lemma 3.16. *Let $s\in V(Q[0])$ and $t^1,t^2\in V(Q[j])$ such that $\delta(s,t^1)=1$, $\delta(t^1,t^2)=0$ and $t^1\neq t^2$. Then there exists a hamiltonian t^1t^2-path of $Q[0,j]-s$, where $0\leq j\leq k-1$.*

Proof. According to Lemma 3.14, the assertion is true for $j=0$. Suppose that $j\geq 1$ and the assertion holds for $Q[0,j-1]$. It is enough to consider $Q[0,j]$.

Let $S=\{t^1,t^2\}$. By Lemma 3.15, there exists a set $T=\{t^3,t^4\}\subset V(Q[1])$ such that $\delta(t^1,t^3)=1$, $t^3\neq t^4$, $\delta(t^3,t^4)=0$ and there are two vertex-disjoint (S,T)-paths P^1 and P^2 in $Q[1,j]$ that contain all vertices of $Q[1,j]$. By Lemma 3.14, there is a hamiltonian $n^0(t^3)n^0(t^4)$-path P^3 of $Q[0]-s$. Then $\bigcup_{r=1}^3 P^r+\{(t^3,n^0(t^3)),(t^4,n^0(t^4))\}$ is as required. □

Lemma 3.17 ([17])**.** *Let $u,v\in V(Q[0,p])$ such that $\delta(u,v)=1$. If F_i is a conditional faulty edge set of $Q[i]$ and $f_i\leq 4n-10$ for $i=0,1,\ldots,p$, then there exists a hamiltonian uv-path of $Q[p,q]-F$.*

4. Edge-bipancyclicity of conditional faulty k-ary 2-cubes

In this section, we will prove the following result:

Theorem 4.1. *Given an even integer $k\geq 4$, let F be a conditional faulty edge set of Q_2^k with $|F|\leq 3$. Then, for any healthy e of Q_2^k, e lies in a cycle of each even length from 6 to k^2 in Q_2^k-F.*

We will investigate the edge-bipancyclicity of the conditional faulty k-ary 2-cube with at most 3 faulty edges. It is enough to consider $|F|=3$. Suppose that $|\mathcal{F}^1|\geq|\mathcal{F}^0|$. Then $2\leq|\mathcal{F}^1|\leq 3$. In the following lemmas, suppose that $k\geq 4$ is an even integer and $e=(s,t)$ is a healthy edge of Q_2^k. Set $s=v_{i,j}$ and $t=v_{i',j'}$, where $0\leq i,j,i',j'\leq k-1$. Let C_1 and C_2 be two cycles. Then denote by $C_1\bigtriangleup C_2$ the graph induced by the edges of $E(C_1)\bigtriangleup E(C_2)$, where $E(C_1)\bigtriangleup E(C_2)$ denotes the symmetric difference of $E(C_1)$ and $E(C_2)$. We consider the following cases.

Case 1. $|\mathcal{F}^1| = 2$.

In this case, $|\mathcal{F}^0| = 1$. Without loss of generality, we assume that $\mathcal{F}^0 = \{(v_{0,0}, v_{0,1})\}$ and $\mathcal{F}^1 = \{(v_{i_1,j_1}, v_{i_1+1,j_1}), (v_{i_2,j_2}, v_{i_2+1,j_2})\}$, where $0 \leq i_1, j_1, i_2, j_2 \leq k-1$.

Case 1.1. $i_1 = i_2$ (see Lem. 4.2).

Case 1.2. $i_1 = 0$ and $i_2 = k-1$ (see Lem. 4.3).

Case 1.3. $1 \leq i_1 \leq k-2$ and $i_2 = k-1$ (see Lem. 4.4).

Case 1.4. $1 \leq i_1 < i_2 \leq k-2$ (see Lem. 4.5).

Case 2. $|\mathcal{F}^1| = 3$.

In this case, $|\mathcal{F}^0| = 0$. Let $\mathcal{F}^1 = \{(v_{i_1,j_1}, \ v_{i_1+1,j_1}), (v_{i_2,j_2}, v_{i_2+1,j_2}), (v_{i_3,j_3}, v_{i_3+1,j_3})\}$, where $0 \leq i_1, j_1, i_2, j_2, i_3, j_3 \leq k-1$.

Case 2.1. $i_1 = i_2 = i_3 = 0$ (see Lem. 4.6).

Case 2.2. $i_2 = i_3 = i_1 - 1$ (see Lem. 4.7).

Case 2.3. $i_2 = i_3 \neq i_1 \pm 1$ (see Lem. 4.8).

Case 2.4. $i_1 = i_2 + 1$ and $i_2 = i_3 + 1$ (see Lem. 4.9).

Case 2.5. $i_1 \notin \{i_2, i_3, i_2 \pm 1, i_3 \pm 1\}$ and $i_2 \neq i_3$ (see Lem. 4.10).

Lemma 4.2. *If $i_1 = i_2$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. Suppose first that $i_1 = i_2 \in \{0, k-1\}$. Without loss of generality, we assume that $i_1 = i_2 = k-1$. Note that Q_2^k−Row$(k-1:0)$ is isomorphic to Row$[0:k-1]$. By Lemma 3.8, the faulty Row$[0:k-1]$ is edge-bipancyclic. Then e lies in a cycle of each even length from 6 to k^2 in $Q_2^k - F$. Suppose next that $1 \leq i_1 = i_2 \leq k-2$.

Case 1. $e \in E(\text{Row}[0:i_1])$ or $e \in E(\text{Row}[i_1+1:k-1])$.

If $e \in E(\text{Row}[i_1+1:k-1])$, then $e \in E(\text{Row}[i_1+1:0])$. By symmetry, it is enough to consider the case that $e \in E(\text{Row}[0:i_1])$ in the following.

Case 1.1. e lies in a cycle of each even length from 6 to $(i_1+1)k$ in Row$[0:i_1] - (v_{0,0}, v_{0,1})$.

By Lemma 3.6(*i*), Row$[0:i_1]$ has a hamiltonian cycle C containing e and M_0^0.

Suppose first that $i_1 + 1 \leq k-2$. By Lemma 3.6(*ii*), there is a $v_{k-1,0}v_{k-1,1}$-path P^1 of every odd length $l^1 \in [1, (k-1-i_1)k-1]$ in Row$[i_1+1:k-1]$. We may obtain a cycle $C \cup P^1 + \{(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{k-1,1})\} - (v_{0,0}, v_{0,1})$ of every even length $l = |E(C)| + l^1 + 1 \in [(i_1+1)k+2, k^2]$.

Suppose next that $i_1 + 1 = k-1$. If $e \notin M_0^0$, let $C_n = \langle v_{0,n}, v_{k-1,n}, v_{k-1,n+1}, v_{0,n+1}, v_{0,n}\rangle$, for $n = 0, 2, \ldots, k-2$. Then $C \triangle C_0 \triangle C_2 \triangle \ldots \triangle C_n$ is a cycle of every even length from $(k-1)k+2 = k^2-k+2$ to k^2 that contains e. If $e \in M_0^0$, then $e \in E(\text{Row}[0])$. By Lemma 3.6(*i*), Row$[0:k-3]$ has a hamiltonian cycle C' containing M_0^0 and M_α^{k-3}, where $\alpha \in \{0,1\}$. Without loss of generality, we assume that $\alpha = 0$. Let $C'' = C' \cup \text{Row}[k-1] + \{(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{k-1,1})\} - \{(v_{0,0}, v_{0,1}), (v_{k-1,0}, v_{k-1,1})\}$ and $C_n = \langle v_{k-3,n}, v_{k-2,n}, v_{k-2,n+1}, v_{k-3,n+1}, v_{k-3,n}\rangle$, for $n = 0, 2, \ldots, k-2$. Then $C'' \triangle C_0 \triangle C_2 \triangle \ldots \triangle C_n$ is a cycle of every even length from k^2-k+2 to k^2 that contains e.

Case 1.2. Row$[0:i_1] - (v_{0,0}, v_{0,1})$ has not any cycle containing e of every even length from 6 to $(i_1+1)k$.

By Lemma 3.11, $i_1 = 1$ and $e \in \text{Row}(0:1) \cup \{(v_{1,0}, v_{1,1})\}$. Observe that $i_1 + 1 = 2 \leq k-2$. By using a similar construction in the second paragraph of Case 1.1, we may obtain a cycle of every even length from $2k+2$ to k^2 in Row$[2:1]$. As $2k+2 < k^2-2k+4$, it remains to show that there is a cycle of every even length from 6 to k^2-2k+4.

If $e = (v_{1,0}, v_{1,1})$, then, by Lemma 3.6(*ii*), there is a $v_{k-1,0}v_{k-1,1}$-path P_1^1 of every odd length $l_1^1 \in [1, k^2 - 2k-1]$ in Row$[2:k-1]$. So $\langle v_{1,0}, v_{1,1}, v_{0,1}, v_{k-1,1}, P_1^1, v_{k-1,0}, v_{0,0}, v_{1,0}\rangle$ is a cycle of every even length $l_1^1 + 5 \in [6, k^2-2k+4]$.

If $e = (v_{0,j}, v_{1,j}) \in \text{Row}(0:1)$, then, by Lemma 3.11(*iii*), e lies in a cycle C of every even length from 4 to $k+2$ and C contains $k/2$ edges of Row$[0]$ when $|E(C)| = k+2$. Suppose that $|E(C)| = k+2$. Choose an edge (v_{0,j^1}, v_{0,j^1+1}) of C. By Lemma 3.6(*ii*), there is a $v_{k-1,j^1}v_{k-1,j^1+1}$-path P_2^1 of every odd length $l_2^1 \in [1, k^2-2k-1]$ in Row$[2:k-1]$. Then $C \cup P_2^1 + \{(v_{0,j^1}, v_{k-1,j^1}), (v_{0,j^1+1}, \ v_{k-1,j^1+1})\} - (v_{0,j^1}, v_{0,j^1+1})$ is a cycle of every even length $|E(C)| + l_2^1 + 2 \in [6, k^2 - k + 2]$ that contains e. Note that $k^2 - 2k + 4 < k^2 - k + 2$. Thus e lies in a cycle of every even length from 6 to $k^2 - 2k + 4$.

Case 2. $e \in \text{Row}(i_1 : i_1 + 1)$.

Now $e = (v_{i_1,j}, v_{i_1+1,j})$. By symmetry, we may assume that $k/2 \le i_1 \le k - 2$. Then $2 \le i_1 \le k - 2$. If either $(v_{i_1,j-1}, v_{i_1+1,j-1}) \notin F$ or $(v_{i_1,j+1}, v_{i_1+1,j+1}) \notin F$, then e lies in a 4-cycle. Otherwise, e lies in a 6-cycle $\langle v_{i_1,j}, v_{i_1,j-1}, v_{i_1,j-2}, v_{i_1+1,j-2}, v_{i_1+1,j-1}, v_{i_1+1,j}, v_{i_1,j}\rangle$. Then there is a cycle C^1 of length $l^1 \in \{4, 6\}$ containing e in $\text{Row}[i_1 : i_1 + 1] - F$. Choose an edge $(v_{i_1,r}, v_{i_1,r+1})$ of C^1. In the following, we will construct a cycle of every even length l with $6 \le l \le k^2$ that contains e in $Q_2^k - F$.

Case 2.1. $l^1 + 2 \le l \le i_1 k + 4$.

Suppose that there exists a $v_{i_1-1,r}v_{i_1-1,r+1}$-path P^2 of every odd length $l^2 \in [1, i_1 k - 1]$ in $\text{Row}[0 : i_1 - 1] - F$. Then e lies in a cycle $C^1 \cup P^2 + \{(v_{i_1,r}, v_{i_1-1,r}), (v_{i_1,r+1}, v_{i_1-1,r+1})\} - (v_{i_1,r}, v_{i_1,r+1})$ of every even length $l = l^1 + l^2 + 2$ with $l^1 + 2 \le l \le i_1 k + l^1$. Recall that $l^1 \in \{4, 6\}$. Then e lies in a cycle of every even length from $l^1 + 2$ to $i_1 k + 4$.

Otherwise, Lemma 3.11 implies that $i_1 = 2$, $(v_{i_1,r}, v_{i_1,r+1}) = (v_{2,0}, v_{2,1})$ and $k \ge 6$. Observe that $(v_{3,0}, v_{3,1}) \in E(C^1)$. By Lemma 3.6($ii$), there exists a $v_{4,0}v_{4,1}$-path P^3 of every odd length $l^3 \in [1, (k-4)k - 1]$ in $\text{Row}[4 : k - 1]$. Then e lies in a cycle $C^1 \cup P^3 + \{(v_{3,0}, v_{4,0}), (v_{3,1}, v_{4,1})\} - (v_{3,0}, v_{3,1})$ of every even length $l = l^1 + l^3 + 2$ with $l^1 + 2 \le l \le (k-4)k + l^1$. As $i_1 = 2$ and $k \ge 6$, it follows that $i_1 k + 4 \le (k-4)k + l^1$. So e lies in a cycle of every even length from $l^1 + 2$ to $i_1 k + 4$.

Case 2.2. $i_1 k + 6 \le l \le k^2 - 2k$.

In this case $i_1 + 1 \le k - 2$. Lemma 3.6(i) implies that $\text{Row}[0 : i_1 - 1]$ has a hamiltonian cycle C^2 containing $(v_{i_1-1,r}, v_{i_1-1,r+1})$ and M_0^0. Then $C^3 = C^1 \cup C^2 + \{(v_{i_1-1,r}, v_{i_1,r}),\ (v_{i_1-1,r+1}, v_{i_1,r+1})\} - \{(v_{i_1-1,r}, v_{i_1-1,r+1}), (v_{i_1,r}, v_{i_1,r+1})\}$ is a cycle containing e and $|E(C^3)| = i_1 k + l^1$. Using the path $\langle v_{k-1,0}, v_{0,0}, C^3 - (v_{0,0}, v_{0,1}), v_{0,1}, v_{k-1,1}\rangle$, it is not difficult to construct the required cycles.

Case 2.3. $k^2 - 2k + 2 \le l \le k^2$.

Clearly, we can obtain a healthy k^2-cycle containing e easily.

Let $C^4 = (\text{Row}[i_1] \cup \text{Row}[i_1 + 1]) \triangle C^1$. Then $e \in E(C^4)$, $|E(C^4)| \ge 2k - 2$ and we may choose any edge $(v_{i_1+1,j^1}, v_{i_1+1,j^1+1})$ of $C^4 \cap \text{Row}[i_1 + 1]$ such that $j^1 \ne 0$. We may easily verify that there exist healthy cycles containing e of even lengths from $k^2 - 2k + 2$ to $k^2 - 2$ in $C^4 \cup \text{Row}[i_1 + 2 : i_1 - 1] + \{(v_{i_1+1,j^1}, v_{i_1+2,j^1}), (v_{i_1+1,j^1+1} v_{i_1+2,j^1+1})\}$. ☐

Lemma 4.3. *If $i_1 = 0$ and $i_2 = k - 1$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. Recall that $(s, t) = (v_{i,j}, v_{i',j'})$ and $\mathcal{F}^1 = \{(v_{i_1,j_1}, v_{i_1+1,j_1}), (v_{i_2,j_2}, v_{i_2+1,j_2})\}$. We distinguish three cases. In each case, we will construct a cycle of every even length l with $6 \le l \le k^2$ that contains e in $Q_2^k - F$.

Case 1. $(s, t) \in E(\text{Row}[0])$.

Clearly $(s, t) = (v_{0,j}, v_{0,j'})$. Without loss of generality, we may assume that $j' = j + 1$.

Case 1.1. $6 \le l \le k^2 - 2k + 6$.

As either $i_1 \notin \{j, j + 2\}$ or $i_1 \notin \{j', j' - 2\}$. Without loss of generality, assume $i_1 \notin \{j, j + 2\}$. So $C^1 = \langle s, t, v_{0,j+2}, v_{1,j+2}, v_{1,j'}, v_{1,j}, s\rangle$ is a 6-cycle containing e. By Lemma 3.6(ii), there is a $v_{2,j}v_{2,j+1}$-path P^1 of every odd length $l^1 \in [1, k^2 - 2k - 1]$ in $\text{Row}[2 : k - 1]$. So $C^1 \cup P^1 + \{(v_{1,j}, v_{2,j}), (v_{1,j+1}, v_{2,j+1})\} - (v_{1,j}, v_{1,j+1})$ is a cycle of each even length $6 + l^1 + 1 \in [8, k^2 - 2k + 6]$ containing e.

Case 1.2. $k^2 - 2k + 8 \le l \le k^2$.

Suppose first that $(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1}) \notin F$ or $(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{k-1,1}) \notin F$. Without loss of generality, assume that $(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1}) \notin F$. Then we can verify that $\text{Row}[0] \cup \text{Row}[1 : k - 1] + \{(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1})\}$ contains the cycles as required.

Suppose next that $|\{(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1})\} \cap F| = |\{(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{k-1,1})\} \cap F| = 1$. That is $j_1, j_2 \in \{0, 1\}$. Noting that F is a conditional faulty edge set of Q_2^k and $(v_{0,0}, v_{0,1}) \in F$, it can be seen that $j_1 \ne j_2$ and so $j_1 = 0, j_2 = 1$ or $j_1 = 1, j_2 = 0$. Without loss of generality, assume that $j_1 = 1, j_2 = 0$. Clearly, $\text{Row}[0] \cup \text{Row}[1 : k - 1] + \{(v_{0,0}, v_{1,0}), (v_{0,1}, v_{k-1,1})\}$ contains the cycles as required

Case 2. $(s,t) \in \text{Row}(0:1)$.

In this case, $(s,t) = (v_{0,j}, v_{1,j})$.

Case 2.1. Both $\langle s, v_{0,j+1}, v_{1,j+1}\rangle$ and $\langle s, v_{0,j-1}, v_{1,j-1}\rangle$ are all faulty.

Without loss of generality, assume that $(s, v_{0,j-1})$, $(v_{0,j+1}, v_{1,j+1}) \in F$. Then $s = v_{0,1}$ and $t = v_{1,1}$. Thus $\langle s, C^+_{j^1}(s,t), t, s\rangle$ is a healthy cycle containing e of each even length from 6 to $2k$, where $j^1 = 3, 4, \ldots, k-1, 0$. Therefore, it is not difficult to verify that $\langle s, C^+_0(s,t), t, s\rangle \cup \text{Row}[2:k-1] + \{(t, v_{2,j}), (v_{1,j+1}, v_{2,j+1})\}$ contains the cycles as required.

Case 2.2. Either $\langle s, v_{0,j+1}, v_{1,j+1}\rangle$ is fault-free or $\langle s, v_{0,j-1}, v_{1,j-1}\rangle$ is fault-free.

We only consider the case that $\langle s, v_{0,j+1}, v_{1,j+1}\rangle$ is fault-free since the proof for the other case is similar. Clearly, $1 \le j \le k-1$. By Lemma 3.6(ii), there is a $tv_{1,j+1}$-path P^1 of every odd length from 3 to k^2-k-1 in $\text{Row}[1:k-1]$. Then $\langle s, v_{0,j+1}, v_{1,j+1}, P^1, t, s\rangle$ is a cycle containing e of every even length from 6 to k^2-k+2.

Suppose first that $j_2 \notin \{0,1\}$. Lemma 3.6(ii) implies that there is a $v_{k-1,0}v_{k-1,1}$-path P^2 of every odd length from 1 to k^2-2k-1 in $\text{Row}[2:k-1]$. Then $\text{Row}[0] \cup \text{Row}[1] \cup P^2 + \{(s,t), (v_{0,j+1}, v_{1,j+1}), (v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{k-1,1})\} - \{(s, v_{0,j+1}), (t, v_{1,j+1}), (v_{0,0}, v_{0,1})\}$ is a cycle of every even length from $2k+2$ to k^2. As $2k+2 < k^2-k+4$, it follows that e lies in a cycle of every even length from k^2-k+4 to k^2.

Suppose next that $j_2 \in \{0,1\}$. By the structure of $\text{Row}[0:1]-F$, we may choose two vertices $v_{0,a}$ and $v_{1,b}$ such that $\delta(v_{0,a}, v_{1,b}) = 1$, $(v_{0,a}, v_{k-1,a}) \notin F$ and there is a hamiltonian $v_{0,a}v_{1,b}$-path P of $\text{Row}[0:1]-F$ passing through (s,t). Clearly, we can construct a cycle of every even length from k^2-k+4 to k^2.

Case 3. $(s,t) \in E(\text{Row}[1:k-1])$.

The healthy cycles containing e of even lengths from 6 to k^2-2 can be obtained easily. It remains to construct a cycle of length k^2 containing e in $Q^k_2 - F$.

Case 3.1. $(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1}) \notin F$ or $(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{k-1,1}) \notin F$.

Without loss of generality, assume that $(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1}) \notin F$. Suppose first that $(s,t) \in E(\text{Row}[1])$. It is easy to construct a hamiltonian xy-path P^1 containing (s,t) in $\text{Row}[0:1]-F$, where $\delta(x,y)=1$. Using the path P^1, we can obtain a healthy k^2-cycle containing e. Suppose next that $(s,t) \in E(\text{Row}[2:k-1]) \cup \text{Row}(1:2)$. By Lemma 3.6($i$), $\text{Row}[1:k-1]$ contains a hamiltonian cycle C^1 containing e and $(v_{1,0}, v_{1,1})$. Then we can verify that $\text{Row}[0] \cup C^1 + \{(v_{0,0}, v_{1,0}), (v_{0,1}, v_{1,1})\}$ contains the cycle as required.

Case 3.2. $(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{1,1}) \in F$ or $(v_{0,0}, v_{1,0}), (v_{0,1}, v_{k-1,1}) \in F$.

Without loss of generality, assume that $(v_{0,0}, v_{k-1,0}), (v_{0,1}, v_{1,1}) \in F$. Furthermore, we may assume that $\delta(s, v_{0,0}) = 0$ and $\delta(t, v_{0,1}) = 0$. Suppose first that $t = v_{1,0}$ or $s = v_{k-1,1}$. If $t = v_{1,0}$, then, Corollary 3.10 implies that there is a hamiltonian $v_{k-1,1}s$-path P^1_2 of $\text{Row}[2:k-1]-t$. Then $\langle s, P^1_2, v_{k-1,1}, v_{0,1}, v_{0,2}, \ldots, v_{0,k-1}, v_{0,0}, t, s\rangle$ is as required. If $s = v_{k-1,1}$, then the proof is similar. Suppose next that $t \ne v_{1,0}$ and $s \ne v_{k-1,1}$. Recall that $s = v_{i,j}$, $t = v_{i',j'}$ and $\delta(s, v_{0,0}) = 0$. Then $i+j$ is even and $i'+j'$ is odd.

Case 3.2.1. $i = i'$.

Case 3.2.1.1. $j' < j$.

In this case, $0 \le j' \le k-2$.

Suppose first that $j' = 0$. Observe that i' is odd. As $t \ne v_{1,0}$, we have that $3 \le i = i' \le k-1$. Recall that $s \ne v_{k-1,1}$. Then $t \ne v_{k-1,0}$ and so $3 \le i' = i \le k-3$. By Lemmas 3.1 and 3.2, there is a hamiltonian $v_{i'-1,1}s$-path P^2 of $\text{Row}[1:k-2] \cap \text{Col}[1:k-1]$. Let $P^3 = \langle v_{i'-1,0}, v_{i'-2,0}, \ldots, v_{0,0}, v_{0,k-1}, v_{0,k-2}, \ldots, v_{0,1}, v_{k-1,1}, v_{k-1,2}, \ldots, v_{k-1,k-1}, v_{k-1,0}, v_{k-2,0}, \ldots, t\rangle$. Then $\langle s, P^2, v_{i'-1,1}, v_{i'-1,0}, P^3, t, s\rangle$ is as required.

Suppose next that $1 \le j' \le k-2$. Let $G_1 = \text{Row}[1:k-1] \cap \text{Col}[1:j-1]$ and $G_2 = \text{Row}[1:k-1] \cap \text{Col}[j:0]$.

If $j-1 = 1$, then $j = 2$, $j' = 1$ and $i = i'$ is even. So $2 \le i' \le k-2$. Lemma 3.1 implies that there is a hamiltonian $v_{0,2}v_{0,k-1}$-path P^4_1 of $\text{Row}[0:1] \cap \text{Col}[2:k-1]$. By Lemmas 3.1 and 3.2, there is a hamiltonian $v_{i'+1,2}s$-path P^4_2 of $\text{Row}[2:k-1] \cap \text{Col}[2:0]$. Then $\langle s, P^4_2, v_{i'+1,2}, v_{i'+1,1}, v_{i'+2,1}, \ldots, v_{k-1,1}, v_{0,1}, v_{0,2}, P^4_1, v_{0,k-1}, v_{0,0}, v_{1,0}, v_{1,1}, v_{2,1}, \ldots, t, s\rangle$ is as required (see Fig. 1a).

If $j-1 \ge 2$, then, Lemma 3.1 implies that there is a hamiltonian $sv_{1,0}$-path P^5_1 of G_2 and there is a hamiltonian $tv_{k-1,1}$-path P^5_2 of G_1. Then $\langle s, P^5_1, v_{1,0}, v_{0,0}, v_{0,k-1}, v_{0,k-2}, \ldots, v_{0,1}, v_{k-1,1}, P^5_2, t, s\rangle$ is as required (see Fig. 1b).

Case 3.2.1.2. $j < j'$.

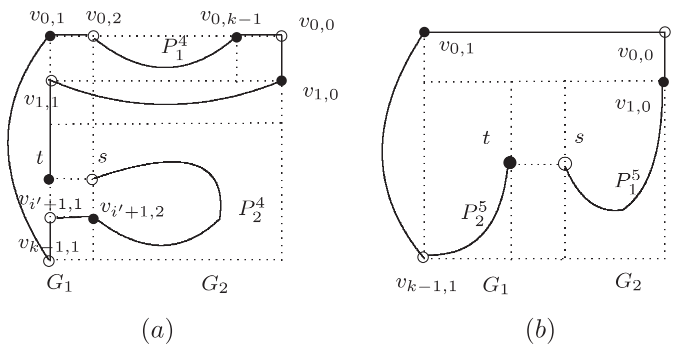

FIGURE 1. A hamiltonian st-path.

In this case, $0 \le j \le k-2$.

Suppose first that $j = 0$. Observe that $i = i'$ is even and so $2 \le i = i' \le k-2$. If $j' = 1$, let $G_1 = \text{Row}[1 : k-1] \cap \text{Col}[1 : k-2]$ and $G_2 = \text{Row}[1 : k-1] \cap \text{Col}[k-1 : 0]$. Then we can obtain a required path by using a similar construction in the fifth paragraph of Case 3.2.1.1. If $j' = k-1$, then, Lemma 3.5 implies that there is a hamiltonian $sv_{k-1,1}$-path P_1^6 of $\text{Row}[i : k-1] - t$. If $i' \ge 4$, then, by Lemma 3.3, there is a hamiltonian $v_{1,0}v_{i'-1,k-1}$-path P_2^6 of $\text{Row}[1 : i'-1]$. If $i' = 2$, let $P_2^6 = \text{Row}[1] - (v_{1,0}, v_{1,k-1})$. Then $\langle s, P_1^6, v_{k-1,1}, v_{0,1}, v_{0,2}, \ldots, v_{0,k-1}, v_{0,0}, v_{1,0}, P_2^6, v_{i'-1,k-1}, t, s\rangle$ is as required.

Suppose next that $1 \le j \le k-2$.

If $i = i' = 1$, then j is odd and j' is even. If $j = 1$, then $s = v_{1,1}$. Lemma 3.1 implies that there is a hamiltonian $v_{k-1,k-1}t$-path P_1^7 of $\text{Row}[1 : k-1] \cap \text{Col}[2 : k-1]$. Then $\langle s, v_{2,1}, \ldots, v_{k-1,1}, v_{0,1}, v_{0,2}, \ldots, v_{0,k-1}, v_{0,0}, v_{1,0}, \ldots, v_{k-1,0}, v_{k-1,k-1}, P_1^7, t, s\rangle$ is as required. If $j \ge 3$, then, Lemma 3.5 implies that there is a hamiltonian $sv_{0,2}$-path P_2^7 of $\text{Col}[2 : j'] - t$. By Lemma 3.1, there is a hamiltonian $v_{1,0}v_{k-1,1}$-path P_3^7 of $\text{Row}[1 : k-1] \cap \text{Col}[0 : 1]$. As j' is even, we have that $j' \le k-2$. Lemma 3.3 implies that there is a hamiltonian $v_{0,k-1}v_{1,j'+1}$-path P_4^7 of $\text{Col}[j'+1 : k-1]$. Then $\langle s, P_2^7, v_{0,2}, v_{0,1}, v_{k-1,1}, P_3^7, v_{1,0}, v_{0,0}, v_{0,k-1}, P_4^7, v_{1,j'+1}, t, s\rangle$ is as required.

If $2 \le i = i' \le k-2$. Lemma 3.5 implies that there is a hamiltonian $sv_{k-1,1}$-path P_5^7 of $\text{Row}[i : k-1] - t$ and there is a hamiltonian $v_{0,2}v_{i'-1,j'}$-path P_6^7 of $\text{Row}[0 : i'-1] - v_{0,1}$. Then $\langle s, P_5^7, v_{k-1,1}, v_{0,1}, v_{0,2}, P_6^7, v_{i'-1,j'}, t, s\rangle$ is as required.

If $i = i' = k-1$, then Lemma 3.5 implies that there is a hamiltonian $sv_{k-1,1}$-path P_7^7 of $\text{Row}[k-2, k-1] - t$. By Lemma 3.5, there is a hamiltonian $v_{0,2}v_{0,j'}$-path P_8^7 of $\text{Row}[0 : k-3] - v_{0,1}$. Then $\langle s, P_7^7, v_{k-1,1}, v_{0,1}, v_{0,2}, P_8^7, v_{0,j'}, t, s\rangle$ is as required.

Case 3.2.2. $i \ne i'$.

Using Lemmas 3.3 and 3.5, the proof of this case is immediate. □

Lemma 4.4. *If $1 \le i_1 \le k-2$ and $i_2 = k-1$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. We distinguish four cases. In each case, we will construct a cycle of every even length l with $6 \le l \le k^2$ that contains e in $Q_2^k - F$. Recall that $e = (s,t) = (v_{i,j}, v_{i',j'})$ and $\mathcal{F}^1 = \{(v_{i_1,j_1}, v_{i_1+1,j_1}), (v_{i_2,j_2}, v_{i_2+1,j_2})\}$.

Case 1. $(s,t) \in E(\text{Row}[0 : i_1])$.

Case 1.1. $6 \le l \le (i_1+1)k$.

By Lemma 3.11, e lies in cycles of even lengths from 4 to $(i_1+1)k$ in $\text{Row}[0 : i_1] - F$ or $i_1 = 1$ and $e \in \text{Row}(0 : 1) \cup \{(v_{1,0}, v_{1,1})\}$. Then we only need to consider the case that $i_1 = 1$ and $e \in \text{Row}(0 : 1) \cup \{(v_{1,0}, v_{1,1})\}$. Suppose first that $e \in \text{Row}(0 : 1)$. It is easy to construct the cycles as required. Suppose next that $e = (s,t) = (v_{1,0}, v_{1,1})$.

Claim 1. $e = (s,t)$ lies in a cycle of every even length from 6 to $2k$ in $\text{Row}[i_1 : i_1+1] - F$.

Note that $i_1 = 1$. If $j_1 \ge 2$, it is easy to verify that there exist healthy cycles of lengths $6, 8, \ldots, 2k-2$. If $j_1 \in \{0, 1\}$, without loss of generality, assume that $j_1 = 1$. Then $\langle s, t, C_{j1}^+(t, v_{2,1}), v_{2,1}, v_{2,0}, s\rangle$ is a healthy cycle of every even lengths from 6 to $2k-2$, where $j^1 = 2, 3, \ldots, k-1$. Noting that $e \notin M_1^1$ and $|M_1^1| = k/2 \geq 2$. It follows that we may choose an edge $(v_{1,j^2}, v_{1,j^2+1}) \in M_1^1$ such that (v_{1,j^2}, v_{1,j^2+1}) and (v_{1,j_1}, v_{2,j_1}) are nonadjacent. Then $\text{Row}[1] \cup \text{Row}[2] + \{(v_{1,j^2}, v_{2,j^2}), (v_{1,j^2+1}, v_{2,j^2+1})\} - \{(v_{1,j^2}, v_{1,j^2+1}), (v_{2,j^2}, v_{2,j^2+1})\}$ is a cycle containing e of length $2k$. Therefore, this claim holds.

By Claim 1, e lies in a cycle of every even length from 6 to $2k = (i_1+1)k$.

Case 1.2. $(i_1+1)k + 2 \leq l \leq k^2$.

Suppose first that $i_1 + 1 \leq k-2$. Using Lemma 3.9, it is easy to verify that there exists a cycle containing e of every even length from $(i_1+1)k+2$ to k^2. Suppose next that $i_1 + 1 = k-1$. Then $(i_1+1)k+2 = k^2-k+2$ and $(s,t) \in E(\text{Row}[0:k-2])$.

If $(s,t) \in E(\text{Row}[0:k-3])$, then, Lemma 3.9 implies that there is a hamiltonian st-path P^3 of $\text{Row}[0:k-3] - F$. Choose an edge $(v_{k-3,j^3}, v_{k-3,j^3+1})$ of P^3. By Claim 1, there is a cycle containing $(v_{k-3,j^3}, v_{k-3,j^3+1})$ in $\text{Row}[k-2:k-1] - F$ of every even length from 6 to $2k$. Clearly, we can construct healthy cycles of lengths from k^2-k+2 to k^2. If $(s,t) \in E(\text{Row}[k-2]) \cup \text{Row}(k-3:k-2)$, without loss of generality, assume that $(s,t) \notin M_0^{k-2}$. By Lemma 3.6(i), $\text{Row}[1:k-2]$ has a hamiltonian cycle C^2 containing e, M_0^{k-2} and M_α^1, where $\alpha \in \{0,1\}$. We can construct a healthy cycles of lengths from k^2-k+2 to k^2-2. Lemma 3.9 implies that there is a hamiltonian st-path P^5 of $\text{Row}[0:k-2] - (v_{0,0}, v_{0,1})$. It is easy to see that there exists a cycle containing e of length k^2.

Case 2. $(s,t) \in \text{Row}(i_1 : i_1+1) \cup \text{Row}(k-1:0)$.

In this case, $j = j'$. Noting that $(v_{i,j+1}, v_{i',j+1}) \notin F$ or $(v_{i,j-1}, v_{i',j-1}) \notin F$, without loss of generality, assume that $(v_{i,j+1}, v_{i',j+1}) \notin F$. Then we can easily construct the required cycles in $Q_2^k - \text{Row}(i:i') + \{(s,t), (v_{i,j+1}, v_{i',j+1})\}$.

Case 3. $(s,t) \in E(\text{Row}[i_1+1:k-1])$.

Suppose first that $i_1+1 \leq k-2$. Then the required cycles can be constructed easily. Suppose next that $i_1 + 1 = k-1$. By Claim 1, e lies in a cycle of every even length from 6 to $2k$ in $\text{Row}[k-2:k-1] - F$. Clearly, we may obtain the cycles as required. □

Lemma 4.5. *If $1 \leq i_1 < i_2 \leq k-2$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. We distinguish three cases. In each case, we will construct a cycle containing e of every even length l with $6 \leq l \leq k^2$ in $Q_2^k - F$. Recall that $e = (s,t) = (v_{i,j}, v_{i',j'})$ and $\mathcal{F}^1 = \{(v_{i_1,j_1}, v_{i_1+1,j_1}), (v_{i_2,j_2}, v_{i_2+1,j_2})\}$.

Case 1. $(s,t) \in E(\text{Row}[0:i_1])$ or $(s,t) \in E(\text{Row}[i_2+1:k-1])$.

If $(s,t) \in E(\text{Row}[i_2+1:k-1])$, then $(s,t) \in E(\text{Row}[i_2+1:0])$. It is similar to the case that $(s,t) \in E(\text{Row}[0:i_1])$. So we only need to consider the case $(s,t) \in E(\text{Row}[0:i_1])$ in the following.

Case 1.1. $i_1 + 1 \leq i_2 - 1$.

In this case, $k \geq 6$.

Case 1.1.1. $e = (s,t)$ lies in a cycle of every even length from 6 to $(i_1+1)k$ in $\text{Row}[0:i_1] - F$.

In this case, it remains to construct the cycles containing e of even lengths from $(i_1+1)k+2$ to k^2.

Suppose first that $i_2 + 1 \leq k-2$. Then it is easy to verify that we can obtain a cycle containing e of every even length from $(i_1+1)k+2$ to k^2. Suppose next that $i_2+1 = k-1$. Then $i_2 = k-2$. Suppose that $(s,t) \notin E(\text{Row}[i_1])$. Using Corollary 3.10, we may construct a cycle C_1^1 containing e of every even length from $(i_1+1)k+2$ to k^2-k in $\text{Row}[0:k-2] - F$ and C_1^1 contains M_0^{k-2} if $|E(C_1^1)| = k^2-k$. Furthermore, we can obtain the cycles of even lengths from k^2-k+2 to k^2. If $(s,t) \in E(\text{Row}[i_1])$, then, Lemma 3.6(1) implies that $\text{Row}[0:i_1]$ has a hamiltonian cycle C^2 containing e and M_0^0. Observe that $(v_{0,0}, v_{0,1}) \in M_0^0$ and $e \notin M_0^0$. Then we can construct a cycle of every even length from $(i_1+2)k+2$ to k^2.

Case 1.1.2. There is no cycle containing e of every even length from 4 to $(i_1+1)k$ in $\text{Row}[0:i_1] - F$.

By Lemma 3.11, $i_1 = 1$ and $e = (s,t) \in \text{Row}(0:1) \cup \{(v_{1,0}, v_{1,1})\}$. By Lemma 3.9, there is a hamiltonian st-path P^1 of $\text{Row}[0:1] - F$ that contains at least one edge of $\text{Row}[0]$. Note that $i_1 + 1 \leq i_2 - 1$. It is easy to construct cycles containing e of even lengths from k^2-2k+2 to k^2. If $(s,t) = (v_{1,0}, v_{1,1})$, then $j = 0$. Let $P^4 = \langle s, t, v_{0,1}, v_{k-1,1}, v_{k-1,0}, v_{0,0}, s\rangle$. If $(s,t) \in \text{Row}(0:1)$, let $P^4 = \langle s, t, v_{1,j+1}, v_{0,j+1}, v_{k-1,j+1}, v_{k-1,j}, s\rangle$.

We can observe that P^4 is a fault-free 6-cycle containing e and $(v_{k-1,j}, v_{k-1,j+1}) \in E(P^4)$. It remains to construct cycles of even lengths from 8 to k^2-2k containing e.

Suppose first that $i_2+1 \le k-2$. Note that P^4 contains $(v_{k-1,j}v_{k-1,j+1})$, $i_1+1 \le i_2-1$ and $i_2+1 \le k-2$. Then it is easy to construct a cycle of each even length from 8 to k^2-2k containing e. Suppose next that $i_2+1 = k-1$. If $(s,t) \in \text{Row}(0:1)$, then, by Lemma 3.11$(iv)$, $e=(s,t)$ lies in a cycle of every even length from 8 to $k+2$ in $\text{Row}[0:1]-F$ and the $k+2$-cycle C^1 contains $k/2$ edges of Row[1]. Recall that $k \ge 6$. So we can obtain cycles containing e of even lengths from 8 to k^2-2k in $\text{Row}[0:k-2]-F$. If $(s,t)=(v_{1,0},v_{1,1})$, then, by Claim 1 in Lemma 4.3, (s,t) lies in a cycle C^2 of every even length from 6 to $2k$ in $\text{Row}[1:2]-F$. According to the proof of Claim 1 in Lemma 4.3, $(v_{2,0},v_{2,1}) \in E(C^2)$. Lemma 3.6$(ii)$ implies that there is a $v_{3,0}v_{3,1}$-path P_2^6 in $\text{Row}[3:k-2]$ of every odd length from 1 to $(k-4)k-1$. Then $C^2 \cup P_2^6 + \{(v_{2,0},v_{3,0}),(v_{2,1},v_{3,1})\} - (v_{2,0},v_{2,1})$ is a cycle containing e of every even length from $2k+2$ to k^2-2k.

Case 1.2. $i_1+1=i_2$.

Case 1.2.1. $i_2+1 \le k-2$.

In this case, $k \ge 6$. By Lemma 3.6(i), $\text{Row}[0:i_1]$ has a hamiltonian cycle C^1 containing e, M_0^0 and $M_\alpha^{i_1}$, where $\alpha \in \{0,1\}$. Clearly, we can verify that e lies in a cycle of every even length from k^2-2k+2 to k^2.

It is enough to construct the cycles containing e of even lengths from 6 to k^2-2k.

Suppose that e lies in a cycle of every even length from 4 to $(i_1+1)k$ in $\text{Row}[0:i_1]-F$. It is easy to verify that there exists a fault-free cycle of every even length from 6 to k^2-2k containing e in $Q_2^k - V(\text{Row}[i_1+1])$. Suppose next that there is not any cycle containing e of every even length from 4 to $(i_1+1)k$ in $\text{Row}[0:i_1]-F$. By Lemma 3.11, $i_1=1$ and $(s,t) \in \text{Row}(0:1) \cup \{(v_{1,0},v_{1,1})\}$. Note that $i_2=i_1+1=2$.

If $(s,t) \in \text{Row}(0:1)$, then, Lemma 3.11$(iv)$ implies that (s,t) lies in a cycle of every even length from 6 to $k+2$ in $\text{Row}[0:1]-F$ and the $k+2$-cycle C^2 contains $k/2$ edges of Row[0]. We can construct a cycle of every even length from 6 to k^2-2k containing e in $\text{Row}[3:1]-F$. Suppose that $(s,t)=(v_{1,0},v_{1,1})$. We can construct the required cycles in $\text{Row}[1] \cup \text{Row}[3:k-1] \cup \langle s,t,v_{0,1},v_{k-1,1},v_{k-1,0},v_{0,0},s\rangle$.

Case 1.2.2. $i_2+1=k-1$.

Clearly, $i_1=k-3$ and $i_2=k-2$.

If e lies in a cycle of every even length from 4 to k^2-2k in $\text{Row}[0:k-3]-F$, then it remains to construct cycles containing e of even lengths from k^2-2k+2 to k^2. By Lemma 3.6(i), $\text{Row}[0:k-3]$ has a hamiltonian cycle C^1 containing (s,t), M_0^0 and M_α^{k-3}, where $\alpha \in \{0,1\}$. So we can construct the required cycles using the cycle C^1.

If there is not any cycle containing e of every even length from 4 to k^2-2k in $\text{Row}[0:k-3]-F$, then, by Lemma 3.11, $i_1=1$ and so $i_2=i_1+1=2$. As $i_2+1=k-1$, we have that $k=4$. Thus it is easy to construct a cycle of every even length from 6 to $k^2=16$ containing e.

Case 2. $(s,t) \in E(\text{Row}[i_1+1:i_2])$.

If $i_1+1 \le i_2-1$, then $k \ge 6$. It is easy to verify that there exist the cycles as required. If $i_1+1=i_2$, then $(s,t)=(v_{i,j},v_{i',j'}) \in E(\text{Row}[i_1+1])$. Without loss of generality, assume that $j<j'$. By Claim 1 of Lemma 4.3, e lies in a cycle of every even length from 6 to $2k$ in $\text{Row}[i_1:i_1+1]-F$. Suppose first that $i_2+1 \le k-2$. Then we can construct a cycle containing e of every even length from $2k+2$ to k^2. Suppose next that $i_2+1=k-1$. Then $i_1=k-3$, $i_2=k-2$ and $(s,t) \in E(\text{Row}[k-2])$. By Claim 1 in Lemma 3.4, there exists a $2k$-cycle C^2 containing e in $\text{Row}[k-2:k-1]-F$ such that C^2 contains at least two nonadjacent edges of $\text{Row}[k-1]$. Then, we can obtain cycles containing e of even lengths from $2k+2$ to k^2.

Case 3. $(s,t) \in (\text{Row}(i_1:i_1+1) \cup \text{Row}(i_2:i_2+1) \cup \text{Row}(k-1:0))$.

If $(s,t) \in \text{Row}(k-1:0)$, then $(s,t) \in E(\text{Row}[i_2+1:0])$. Similar to Case 1, e lies in cycles of even lengths from 6 to k^2. If $(s,t) \in \text{Row}(i_2:i_2+1)$, then, by symmetry, it is similar to the case that $(s,t) \in \text{Row}(i_1:i_1+1)$. So we only need to consider the case that $(s,t) \in \text{Row}(i_1:i_1+1)$. Clearly $(s,t)=(v_{i_1,j},v_{i_1+1,j})$.

Suppose first that $i_1+1 \le i_2-1$. We can obtain healthy cycles of even lengths from 6 to k^2 containing e. Suppose next that $i_1+1=i_2$. Then, $(s,t) \in \text{Row}(i_1:i_2)$.

Claim 1. (s,t) lies in a cycle of every even length from 4 to k in $\text{Row}[i_1:i_1+1]-(v_{i_1,j_1},v_{i_2,j_1})$.

Without loss of generality, assume that $j_1=0$ and $1 \le j \le k/2$. Clearly, $\langle s, C_{k-1}^+(s,t), t,\ s\rangle$ is a cycle containing e of every even length from 4 to $2(k-j)$. Noting that $2(k-j) \ge k$, then this claim holds.

By Claim 1, (s,t) lies in a cycle of every even length from 4 to k. As either $j_1 \neq j-1$ or $j_1 \neq j+1$, without loss of generality, assume that $j_1 \neq j+1$. We may verify that there exists a healthy cycle containing e of each even length from $k+2$ to k^2 in $\mathrm{Row}[i_2] \cup \mathrm{Row}[i_2+1 : i_1] + \{(s,t), (v_{i_1,j+1}, v_{i_2,j+1})\}$. □

Next, we consider the case that $|\mathcal{F}^1| = 3$. In this case, $|\mathcal{F}^0| = 0$. Let $\mathcal{F}^1 = \{(v_{i_1,j_1}, v_{i_1+1,j_1}), (v_{i_2,j_2}, v_{i_2+1,j_2}), (v_{i_3,j_3}, v_{i_3+1,j_3})\}$, where $0 \leq i_1, j_1, i_2, j_2, i_3, j_3 \leq k-1$. Recall that $e = (s,t) = (v_{i,j}, v_{i',j'})$.

Lemma 4.6. *If $i_1 = i_2 = i_3 = 0$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. If $(s,t) \in E(\mathrm{Row}[1:0])$, then, by Lemma 3.6($ii$), this lemma follows. So we only need to consider the case that $(s,t) \in \mathrm{Row}(0:1)$. If $k = 4$, then we may easily verify that there exist cycles as required. In the following, we prove that the lemma holds for $k \geq 6$.

Suppose first that there exists $j^1 \in \{j-1, j+1\}$ such that $j^1 \notin \{j_1, j_2, j_3\}$. Obviously, we can construct the required cycles in $\mathrm{Row}[1:0] + \{(s,t), (v_{0,j^1}, v_{1,j^1})\}$.

Suppose next that $j-1, j+1 \in \{j_1, j_2, j_3\}$. As $k \geq 6$, we have that $j-2 \neq j+2$. Then either $j-2 \notin \{j_1, j_2, j_3\}$ or $j+2 \notin \{j_1, j_2, j_3\}$. Without loss of generality, assume that $j+2 \notin \{j_1, j_2, j_3\}$. So $C^1 = \langle s, v_{0,j+1}, v_{0,j+2}, v_{1,j+2}, v_{1,j+1}, t, s\rangle$ is a 6-cycle containing e.

By Lemma 3.6(ii), there is a $v_{2,j}v_{2,j+1}$-path P^5 in $\mathrm{Row}[2 : k-1]$ of every odd length from 1 to $(k-2)k-1$. Then $C^1 \cup P^5 + \{(t, v_{2,j}), (v_{1,j+1}, v_{2,j+1})\} - (t, v_{1,j+1})$ is a cycle containing e of every even length from 8 to $k^2 - 2k + 6$. Lemma 3.3 implies that there is a hamiltonian $v_{2,j+1}v_{k-3,j+1}$-path P^6 of $\mathrm{Row}[2 : k-3]$. By Lemma 3.6(ii), there is a $v_{k-1,j+1}v_{k-2,j+1}$-path P^7 in $\mathrm{Row}[k-2 : k-1]$ of every odd length from 1 to $2k-1$. Then $\mathrm{Row}[0] \cup \mathrm{Row}[1] \cup P^6 \cup P^7 + \{(s,t), (v_{0,j+1}, v_{k-1,j+1}), (v_{1,j+1}, v_{2,j+1}), (v_{k-3,j+1}, v_{k-2,j+1})\} - \{(s, v_{0,j+1}), (t, v_{1,j+1})\}$ is a cycle containing e of every even length from $k^2 - 2k + 2$ to k^2. □

Lemma 4.7. *If $i_2 = i_3 = i_1 - 1$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. Without loss of generality, assume that $i_1 = 0$. So $i_2 = i_3 = k-1$.

Suppose first that $(s,t) \in E(\mathrm{Row}[0])$. By Claim 1 of Lemma 4.3, $e = (s,t)$ lies in a cycle of every even length from 6 to $2k$ in $\mathrm{Row}[0:1] - F$. Choose an edge (v_{0,j^1}, v_{0,j^1+1}) of $\mathrm{Row}[0] - (s,t)$ such that $j_1 \notin \{j^1, j^1+1\}$. Then it is easy to construct healthy cycles containing e of even lengths from $2k+2$ to k^2 in $\mathrm{Row}[0] \cup \mathrm{Row}[1 : k-1] + \{(v_{0,j^1}, v_{1,j^1}), (v_{0,j^1+1}, v_{1,j^1+1})\}$.

Suppose next that $(s,t) \in \mathrm{Row}(0:1)$. By Claim 1 of Lemma 4.4, (s,t) lies in a cycle of every even length from 4 to k in $\mathrm{Row}[0:1] - (v_{0,j_1}, v_{1,j_1})$. Noting that either $j-1 \neq j^1$ or $j+1 \neq j^1$, without loss of generality, assume that $j+1 \neq j^1$. Clearly, there exists a cycle containing e of every even length from $k+2$ to k^2.

Suppose now that $(s,t) \in E(\mathrm{Row}[1 : k-1])$. It is easy to verify that there exist the cycles as required. It remains to consider $(s,t) \in \mathrm{Row}(k-1 : 0)$.

If $\{j-1, j+1\} \neq \{j^2, j^3\}$, without loss of generality, assume that $j-1 \notin \{j^2, j^3\}$. We can construct required cycles in $\mathrm{Row}[1:0] + \{(s,t), (v_{k-1,j-1}, v_{0,j-1})\}$. If $\{j-1, j+1\} = \{j^2, j^3\}$, then $(v_{k-1,j+2}, v_{0,j+2}) \notin F$. So $C^2 = \langle t, s, v_{k-1,j+1}, v_{k-1,j+2}, v_{0,j+2}, v_{0,j+1}, t\rangle$ is a 6-cycle containing e. By Lemma 3.6(ii), there is a $v_{k-2,j}v_{k-2,j+1}$-path P^2 in $\mathrm{Row}[1 : k-2]$ of every odd length from 1 to $k^2 - 2k - 1$. Then $C^2 \cup P^2 + \{(s, v_{k-2,j}), (v_{k-1,j+1}, v_{k-2,j+1})\} - (s, v_{k-1,j+1})$ is a cycle containing e of every even length from 8 to $k^2 - 2k + 6$. Since either $j-1 \neq j_1$ or $j+1 \neq j_1$, without loss of generality, assume that $j+1 \neq j_1$. Then $(v_{0,j+1}, v_{1,j+1}) \notin F$. Obviously, we can obtain a fault-free cycle containing e of every even length from $k^2 - 2k + 8$ to k^2 in $\mathrm{Row}[0] \cup \mathrm{Row}[1 : k-1] + \{(s,t), (v_{0,j+1}, v_{1,j+1})\}$. □

Lemma 4.8. *If $i_2 = i_3 \neq i_1 \pm 1$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. Without loss of generality, assume that $i_2 = i_3 = k-1$. Then $1 \leq i_1 \leq k-3$. Recall that $e = (s,t) = (v_{i,j}, v_{i',j'})$.

Suppose that $(s,t) \in E(\mathrm{Row}[0 : k-1])$. As $1 \leq i_1 \leq k-3$, it is easy to construct the cycles as required. We only need to consider the case that $(s,t) \in \mathrm{Row}(k-1 : 0)$.

Suppose first that $\{j-1, j+1\} \neq \{j_2, j_3\}$. Without loss of generality, assume that $j+1 \notin \{j_2, j_3\}$. Then we can obtain the required cycles in $\mathrm{Row}[0 : i_1] \cup \mathrm{Row}[i_1+1 : k-1] + \{(s,t), (v_{k-1,j+1}, v_{0,j+1})\}$.

Suppose next that $\{j-1, j+1\} = \{j_2, j_3\}$. Without loss of generality, assume that $j = 0$ and so $\{j_2, j_3\} = \{1, k-1\}$. Now $(s,t) = (v_{k-1,0}, v_{0,0})$. For every $n_1 = 2, 3, \ldots, k-2$, $C^+_{n_1}(s,t) + (s,t)$ is a cycle containing e of every length from 6 to $2k-2$. Let $C^1 = C^+_{k-2}(s,t) + (s,t)$.

Observe that $i_1 \geq 1$. For every $n_1 = 0, 2, \ldots, k-4$, replace (v_{0,n_1}, v_{0,n_1+1}) of C^1 with the path $\langle v_{0,n_1}, v_{1,n_1}, v_{1,n_1+1}, v_{0,n_1+1}\rangle$ so as to obtain cycles containing e of even lengths from $2k$ to $3k-4$. Let $C^2 = C^1 \cup \text{Row}[1] + \{(t, v_{1,0}), (v_{0,1}, v_{1,1})\} - \{(t, v_{0,1}), (v_{1,0}, v_{1,1})\}$. Then C^2 is a cycle containing e of length $3k-2$. Obviously, C^2 contains M^1_1.

If $i_1 \geq 2$, then, for every edge $(v_{1,j^1}, v_{1,j^1+1}) \in M^1_1$ of C^2, replace (v_{1,j^1}, v_{1,j^1+1}) with the paths $\langle v_{1,j^1}, v_{2,j^1}, \ldots, v_{n_2,j^1}, v_{n_2,j^1+1}, v_{n_2-1,j^1+1}, \ldots, v_{1,j^1+1}\rangle$ for $n_2 = 2, 3, \ldots, i_1$, so as to obtain cycles containing e of even lengths from $3k$ to $(i_1+2)k-2$. As $i_1+1 \leq k-1$, using a similar way, we may obtain cycles containing e of even lengths from $(i_1+2)k$ to k^2-2. Note that $i_1 \geq 1$ and $i_1+1 \leq k-2$. We may easily construct a cycle containing e of length k^2. □

Lemma 4.9. *If $i_1 = i_2 + 1$ and $i_2 = i_3 + 1$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. Without loss of generality, assume that $i_1 = 0$. Then $i_2 = k-1$ and $i_3 = k-2$. Recall that $e = (s,t) = (v_{i,j}, v_{i',j'})$.

Case 1. $(s,t) \in E(\text{Row}[0])$ or $(s,t) \in E(\text{Row}[k-1])$.

Without loss of generality, assume that $(s,t) \in E(\text{Row}[0])$. According to the proof of Claim 1 in Lemma 4.3, (s,t) lies in a cycle C^1 in $\text{Row}[k-1:0] - F$ of every even length from 6 to $2k$ and C^1 contains at least two nonadjacent edges of $\text{Row}[k-1]$ if $|E(C^1)| = 2k$. Clearly, it is easy to construct cycles containing e of even lengths from $2k+2$ to k^2 in $\text{Row}[1:0] - F$.

Case 2. $(s,t) \in E(\text{Row}[1:k-2])$.

Lemma 3.6(ii) implies that e lies in a cycle of every even length from 6 to k^2-2k in $\text{Row}[1:k-1]$. Without loss of generality, assume that $(s,t) \notin E(\text{Row}[k-2])$. Furthermore, we assume that $(s,t) \notin M^1_0$. By Lemma 3.6(i), $\text{Row}[1:k-2]$ has a hamiltonian cycle C^1 containing (s,t), M^1_0 and M^{k-2}_α, where $\alpha \in \{0,1\}$. Then we may construct cycles containing e of even lengths from k^2-k+2 to k^2.

Case 3. $(s,t) \in \text{Row}(0:1)$ or $(s,t) \in \text{Row}(k-2:k-1)$.

By symmetry, we may assume that $(s,t) \in \text{Row}(0:1)$. Now, $(s,t) = (v_{0,j}, v_{1,j})$. By Claim 1 of Lemma 4.4, (s,t) lies in a cycle of every even length from 4 to k in $\text{Row}[0:1] - F$. As either $j-1 \neq j_1$ or $j+1 \neq j_1$, without loss of generality, assume that $j+1 \neq j_1$. It is easy to verify that there exist healthy cycles containing e of even lengths from $k+2$ to k^2 in $\text{Row}[0] \cup \text{Row}[1:k-1] + \{(s,t), (v_{0,j+1}, v_{1,j+1})\}$.

Case 4. $(s,t) \in \text{Row}(k-1:0)$.

By symmetry, we may assume that $(s,t) \in \text{Row}(k-1:0)$. Now, $(s,t) = (v_{k-1,j}, v_{0,j})$. By Claim 1 of Lemma 4.4, (s,t) lies in a cycle of every even length from 4 to k in $\text{Row}[k-1:0] - F$. As either $j-1 \neq j_2$ or $j+1 \neq j_2$, without loss of generality, assume that $j+1 \neq j_2$. Then $C^1 = \text{Row}[0] \cup \langle t, s, v_{k-1,j+1}, v_{0,j+1}\rangle - (t, v_{0,j+1})$ is a cycle containing e of length $k+2$.

Choose an edge (v_{0,j^1}, v_{0,j^1+1}) of C^1 such that $j_1 \notin \{j^1, j^1+1\}$. By Lemma 3.6($ii$), there is a $v_{1,j^1}v_{1,j^1+1}$-path P^1 of every odd length from 1 to k^2-2k-1 in $\text{Row}[1:k-2]$. Then $C^2 = C^1 \cup P^1 + \{(v_{0,j^1}, v_{1,j^1}), (v_{0,j^1+1}, v_{1,j^1+1})\} - (v_{0,j^1}, v_{0,j^1+1})$ is a cycle containing e of every even length from $k+4$ to k^2-k+2.

Let $C^3 = \text{Row}[k-1] \cup C^2 - (s, v_{k-1,j+1})$. Then C^3 is a cycle containing e of every even length from $2k+2$ to k^2. As $2k+2 < k^2-k+4$, we have that e lies in a cycle of every even length from k^2-k+4 to k^2. □

Lemma 4.10. *If $i_1 \notin \{i_2, i_3, i_2 \pm 1, i_3 \pm 1\}$ and $i_2 \neq i_3$, then e lies in a healthy cycle of each even length from 6 to k^2.*

Proof. Suppose first that $i_2 = i_3 + 1$. Without loss of generality, assume that $i_2 = 0$. Then $i_3 = k-1$ and $2 \leq i_1 \leq k-3$. Clearly, $k \geq 6$. Note that $i_2 + 1 \leq i_1 - 1$ and $i_1 + 1 \leq i_3 - 1$. Then we can easily construct the cycles as required. Suppose next that $i_p \notin \{i_q, i_q \pm 1\}$, for any $p \neq q \in \{1,2,3\}$. That is every pair of edges in F are nonadjacent. We may verify that e lies in a healthy cycle of each even length from 6 to k^2. □

By Lemmas 4.2–4.10, we have Theorem 4.1.

5. Edge-bipancyclicity of conditional faulty k-ary n-cubes

In this section, we will prove our main result:

Theorem 5.1. *Given an integer $n \geq 2$ and an even integer $k \geq 4$, let F be a conditional faulty edge set of Q_n^k with $|F| \leq 4n-5$. Then, for any healthy e of Q_n^k, e lies in a cycle of each even length from 6 to k^n in $Q_n^k - F$.*

We prove the assertion by induction on n. According to Theorem 4.1, the assertion is true for $n = 2$. Suppose that $n \geq 3$ and the assertion holds for Q_{n-1}^k. It is enough to consider $|F| = 4n-5$. Suppose that $|\mathcal{F}^0| \geq |\mathcal{F}^i|$, for $i = 1, 2, \ldots, n-1$. Then $|\mathcal{F}^0| \geq \lceil \frac{4n-5}{n} \rceil \geq 3$. Partition Q_n^k along the dimension 0 into k disjoint subcubes $Q[0],\ Q[1], \ldots, Q[k-1]$. Now, $|F_i| \leq |F| - |\mathcal{F}^0| \leq 4n-8$ for $i = 0, 1, \ldots, k-1$.

Suppose that $e = (s, t)$ is a healthy edge of Q_n^k. Assume that $s \in V(Q[i])$ and $t \in V(Q[j])$, where $0 \leq i, j \leq k-1$.

Lemma 5.2. *Given an integer $2 \leq p \leq k-1$, let $0 \leq i \leq j \leq p$, $|F_{i'}| \leq 2n-5$ for $i' = 0, 1, \ldots, p$ and $|F \cap E(Q[0:p])| \leq 4n-6$.*

(i) If $i = j$, then e lies in healthy cycles of even lengths from 4 to $(p+1)k^{n-1}$.

(ii) If $i = j-1$, then e lies in healthy cycles of even lengths from 6 to $(p+1)k^{n-1}$.

Proof. Suppose first that $i = j$. By Lemma 3.12(ii), e lies in a cycle of every even length from 4 to $(p-i+1)k^{n-1}$ in $Q[i:p]$. If $i = 0$, then the statement (i) follows. Suppose that $i > 0$. According to the proof of Lemma 3.12, there is a hamiltonian st-path P^1 of $Q[i:p] - F$ such that $|E(P^1 \cap Q[i])| \geq k^{n-1} - 2$. As $\frac{k^{n-1}-2}{2} > 4n-6$, we may choose an edge (u, v) of $P^1 \cap Q[i]$ such that $(u, n^{i-1}(u)), (v, n^{i-1}(v)) \notin F$. By Lemma 3.12($ii$), there is an $n^{i-1}(u)n^{i-1}(v)$-path P^2 in $Q[0:i-1] - F$ of every odd length from 1 to $ik^{n-1} - 1$. Then $P^1 \cup P^2 + \{(s,t), (u, n^{i-1}(u)), (v, n^{i-1}(v))\} - (u, v)$ is a cycle containing e of every even length from $(p-i+1)k^{n-1} + 2$ to $(p+1)k^{n-1}$.

Suppose next that $i = j-1$. If for any edge $(s, s_1) \in E(Q[i])$ which is incident with s, either $\langle s, s_1, n^j(s_1)\rangle$ is fault-free or $\langle t, n^j(s_1), s_1\rangle$ is fault-free, then, without loss of generality, assume that $\langle s, s_1, n^j(s_1)\rangle$ is fault-free. By Lemma 3.12(ii), there is an ss_1-path P^3 in $Q[0:i] - F$ of every odd length from 1 to $(i+1)k^{n-1} - 1$ and there is a $tn^j(s_1)$-path P^4 in $Q[j:p] - F$ of every odd length from 3 to $(p-j+1)k^{n-1} - 1$. Then $\langle s, P^3, s_1, n^j(s_1), P^4, t, s\rangle$ is a cycle containing e of every even length from 6 to $(p+1)k^{n-1}$.

If for any edge $(s, s_1) \in E(Q[i])$ which is incident with s, both $\langle s, s_1, n^j(s_1)\rangle$ and $\langle t, n^j(s_1), s_1\rangle$ are all faulty, then $\langle s, s_1, n^j(s_1), t\rangle$ contains at least one faulty edge. If $\langle s, s_1, n^j(s_1), t\rangle$ contains at least two faulty edges, then $2(2n-2) \leq |F \cap E(Q[0:p])| \leq 4n-6$, a contradiction. Then there exists an edge $(s, s_2) \in E(Q[i])$ such that $\langle s, s_2, n^j(s_2), t\rangle$ contains exactly one faulty edge. In this case $(s_2, n^j(s_2)) \in F$.

Recall that $p \geq 2$. Then either $e \in E(Q[0:p-1])$ or $e \in E(Q[1:p])$. Without loss of generality, assume that $e \in E(Q[0:p-1])$. Let $\mathcal{P} = \{\langle s, s', n^j(s'), t\rangle : (s, s') \in E(Q[i])\}$. Then $\mathcal{P}$ contains at least $|\mathcal{P}| = 2n-2$ faulty edges and so there are at most $(4n-6) - (2n-2) = 2n-4$ faulty edges outside $\mathcal{P}$. Since $d_{Q[i]}(s_2) - (2n-4) = 2$, we may choose an edge (s_2, s_3) of $Q[i] - (s_2, s)$ such that $\langle s_2, s_3, n^j(s_3), n^j(s_2)\rangle$ and $(n^j(s_3), n^{j+1}(s_3))$ are all fault-free. Clearly, $C^1 = \langle s, s_2, s_3, n^j(s_3), n^j(s_2), t, s\rangle$ is a 6-cycle containing e. Similarly, we may choose an edge (s_3, s_4) of $Q[i] - (s_3, s_2)$ such that $\langle s_3, s_4, n^j(s_4), n^j(s_3)\rangle$ and $\langle n^j(s_4), n^{j+1}(s_4), n^{j+1}(s_3)\rangle$ are all fault-free. Then $C^2 = C^1 \cup \langle s_3, s_4, n^j(s_4), n^j(s_3)\rangle - (s_3, n^j(s_3))$ is a 8-cycle containing e.

By Lemma 3.12(ii), there is an $n^{j+1}(s_3)n^{j+1}(s_4)$-path P^5 in $Q[j+1] - F$ of every odd length from 1 to $k^{n-1} - 1$. Then $C^2 \cup P^5 + \{(n^j(s_3), n^{j+1}(s_3)), (n^j(s_4), n^{j+1}(s_4))\} - (n^j(s_3), n^j(s_4))$ is a cycle containing e of every even length from 10 to $k^{n-1} + 8$.

Lemma 3.17 implies that there is a hamiltonian ss_4-path P^6 of $Q[i] - F$. Then $\langle s, P^6, s_4, n^j(s_4), n^{j+1}(s_4), P^5, n^{j+1}(s_3), n^j(s_3), n^j(s_2), t, s\rangle$ is a cycle containing e of every even length from $k^{n-1} + 6$ to $2k^{n-1} + 4$.

Lemma 3.17 implies that there is a hamiltonian $tn^j(s_4)$-path P^7 of $Q[j]-F$. Choose an edge $(x_1,y_1)\in P^7$ such that $\langle x_1,n^{j+1}(x_1),n^{j+1}(y_1),y_1\rangle$ is healthy. By Lemma 3.12(ii), there is an $n^{j+1}(x_1)n^{j+1}(y_1)$-path P^8 in $Q[j+1:p]-F$ of every odd length from 1 to $(p-j)k^{n-1}-1$. Then $P^6\cup P^7\cup P^8+\{(s,t),(s_4,n^j(s_4)),(x_1,n^{j+1}(x_1)),(y_1,n^{j+1}(y_1))\}-(x_1,y_1)$ is a cycle containing e of every even length from $2k^{n-1}+2$ to $(p-i+1)k^{n-1}$. If $i=0$, then the proof is complete. If $i>0$, similar to the proof above, we can obtain the cycles as required. □

By induction and Lemma 3.17, we can get the following lemma easily.

Lemma 5.3. *Let $0\le i_1\le j_1\le q\le k-1$ and let $F_{i'}$ be a conditional faulty edge set of $Q[i']$ and $|F_{i'}|\le 4n-10$ for $i'=0,1,\dots,q$. For any two distinct vertices $u\in V(Q[i_1])$ and $u\in V(Q[j_1])$ such that $\delta(u,v)=1$, if $0<i_1=j_1=q$, then there exists a hamiltonian uv-path P of $Q[0,q]-F$ such that $|E(P\cap Q[q])|=k^{n-1}-2$; otherwise there exists a hamiltonian uv-path P of $Q[0,q]-F$ such that $|E(P\cap Q[q])|=k^{n-1}-1$.*

Lemma 5.4. *Let $u,v\in V(Q[p,q])$ be any two distinct vertices such that $\delta(u,v)=1$, and let $(x,y),(x,z)$ be two distinct healthy edges in $E(Q[r])$, where $x\notin\{u,v\}$ and $p\le r\le q$. If $|F_{i'}|\le 2n-6$ for $i'=p,p+1,\dots,q$ and $|F\cap E(Q[p,q])|\le 2n-2$, then there exists a hamiltonian uv-path P of $Q[p,q]-F$ passing through (x,y) and (x,z).*

Proof. Set $F'_r=F_r\cup(\partial_{Q[r]}(x)\setminus\{(x,y),(x,z)\})$. Then $|F'_r|\le(2n-6)+(2n-4)=4n-10$. As $|F_{i'}|\le 2n-5$ for $i'=p,p+1,\dots,q$, it is easy to see that $F_{i'}$ is a conditional faulty edge set of $Q[i']$ and F'_r is a conditional faulty edge set of $Q[r]$. Note that $|(F\cap E(Q[p,q]))\cup F'_r|\le(2n-2)+(2n-4)=4n-6$. By Lemma 5.3, there exists a hamiltonian uv-path P of $Q[p,q]-(F\cup F'_r)$. Clearly, P is also a hamiltonian path of $Q[p,q]-F$ and $(x,y),(x,z)\in E(P)$. □

Lemma 5.5. *Let $(u,v)\in E(Q[p])$ and let $(x,y),(x,z)$ be two distinct healthy edges in $E(Q[p])$, where $x\notin\{u,v\}$. If $|F_p|\le 2n-5$, then there exists a hamiltonian uv-path P of $Q[p]-F$ passing through (x,y) and (x,z).*

Proof. Set $F'_p=F_p\cup(\partial_{Q[i]}(x)\setminus\{(x,y),(x,z)\})$. Then $|F'_p|\le(2n-5)+(2n-4)=4n-9$. By the induction hypothesis, there exists a hamiltonian uv-path P of $Q[p]-(F\cup F'_p)$. Clearly, P is also a hamiltonian path of $Q[p]-F_p$ and $(x,y),(x,z)\in E(P)$. □

In the following, we complete the proof of the main result. We break the proof into the following eight lemmas depending on the distribution of edge faults in the conditional faulty k-ary n-cube. The basic strategy for each lemma is to divide the k-ary n-cube into some coalitions of subcubes, obtain cycles in these coalitions by applying the above lemmas to these coalitions, and construct the required cycles in the conditional faulty k-ary n-cube by joining cycles in different coalitions.

First, we consider the case that $|F_r|\le 2n-5$ for $r=0,1,\dots,k-1$.

Lemma 5.6. *If $|F_r|\le 2n-5$ for $r=0,1,\dots,k-1$, then e lies in healthy cycles of even lengths from 6 to k^n.*

Proof. Recall that $e=(s,t)$ and $s\in V(Q[i]),\ t\in V(Q[j])$. According to Lemma 3.12(ii), the assertion is true for $i=j$. Suppose that $i\ne j$. Without loss of generality, we assume that $i=j-1$. For any edge $(s,s_1)\in E(Q[i])$ which is incident with s, if either $\langle s,s_1,n^j(s_1)\rangle$ is fault-free or $\langle t,n^j(s_1),s_1\rangle$ is fault-free, then we can obtain required cycles by using a similar construction in the second paragraph of Lemma 5.2. If both $\langle s,s_1,n^j(s_1)\rangle$ and $\langle t,n^j(s_1),s_1\rangle$ are all faulty for every edge $(s,s_1)\in E(Q[i])$, let $\mathcal{P}=\{\langle s,s',n^j(s'),t\rangle:(s,s')\in E(Q[i])\}$. Then $\mathcal{P}$ contains at least $|\mathcal{P}|=2n-2$ faulty edges and so there are at most $(4n-5)-(2n-2)=2n-3$ faulty edges outside $\mathcal{P}$. If every path in $\mathcal{P}$ contains at least two faulty edges, then $2(2n-2)\le|F|=4n-5$, a contradiction. Then there exists a path $\langle s,s_2,n^j(s_2),t\rangle\in\mathcal{P}$ which contains exactly one faulty edge. In this case $(s_2,n^j(s_2))\in F$.

Note that either $e\in E(Q[0:k-2])$ or $e\in E(Q[1:k-1])$. Without loss of generality, assume that $e\in E(Q[0:k-2])$.

Case 1. There is an edge (s_2,s_3) of $Q[i]-(s_2,s)$ such that $\langle s_2,s_3,n^j(s_3),n^j(s_2)\rangle$ and $(n^j(s_3),n^{j+1}(s_3))$ are all fault-free.

Obviously, $C^1 = \langle s, s_2, s_3, n^j(s_3), n^j(s_2), t, s\rangle$ is a 6-cycle containing e.

If there exists an edge (s_3, s_4) of $Q[i] - s_2$ such that $\langle s_3, s_4, n^j(s_4), n^j(s_3)\rangle$ and $\langle n^j(s_4), n^{j+1}(s_4), n^{j+1}(s_3)\rangle$ are all fault-free, then we can obtain required cycles by using a similar construction in the forth to seventh paragraphs of Lemma 5.2. Otherwise, we claim that there are at least $2n-3$ path in $\mathcal{P}$ such that each path contains exactly 1 faulty edge. In fact, s and s_3 have at most 1 common neighbor in $Q[i] - s_2$. Observe that $(4n-5) - [d_{Q[i]-s_2}(s_3) - 1] = 2n-1$ and $d_{Q[s]} = 2n-2$. Then $(2n-2) - [(2n-1) - (2n-2)] = 2n-3$. Therefore this claim holds. As $2n-3 > 1$, there is a path $\langle s, s_1, n^j(s_1), t\rangle \in \mathcal{P}$ containing exactly 1 faulty edge, where $s_1 \neq s_2$. Clearly, $(s_1, n^j(s_1)) \in F$.

As $d_{Q[i]}(s_1) = 2n-2 > 2 + [(4n-5) - (|\mathcal{P}|-1) - d_{Q[i]-s_2}(s_3)] = 3$, we may choose an edge $(s_1, s_1') \in E(Q[i] - \{s, s_3\})$ such that both $\langle s_1, s_1', n^j(s_1'), n^j(s_1)\rangle$ and $(n^j(s_1'), n^{j+1}(s_1'))$ are all fault-free. By the definition of Q_n^k, s_1' and s_3 are adjacent to at most two common vertices in $Q[i]$. Then exists an edge (s_1', s_2'') of $Q[i] - s_1$ such that $\langle s_1', s_1'', n^j(s_1''), n^j(s_1')\rangle$ and $\langle n^j(s_1''), n^{j+1}(s_1''), n^{j+1}(s_1')\rangle$ are all fault-free because $d_{Q[i]}(s_1') = 2n-2 > |\{s_1\}| + 2$. Using similar proofs in the forth to seventh paragraphs of Lemma 5.2, we may obtain required cycles.

Case 2. There is not any edge (s_2, s_3) of $Q[i] - (s_2, s)$ such that $\langle s_2, s_3, n^j(s_3), n^j(s_2)\rangle$ and $(n^j(s_3), n^{j+1}(s_3))$ are all fault-free.

In this case, each path in $\mathcal{P}$ contains exactly one faulty edge. Then, for any path $\langle s, s_1, n^j(s_1), t\rangle \in \mathcal{P}$ such that $s_1 \neq s_2$, we have that $(s_1, n^j(s_1)) \in F$. Similar to Case 1, we can obtain the cycles as required. ☐

Now, we consider the case that $|F_r| \geq 2n-4$, for some $r \in \{0, 1, \ldots, k-1\}$. Without loss of generality, we assume that $r = 0$. Then $|F_r| \leq 2n-4$ for $r = 1, 2, \ldots, k-1$ and so F_r is a conditional faulty edge set of $Q[r]$.

Lemma 5.7. *If* $|F_0| \leq 4n-9$, F_0 *is a conditional faulty edge set of* $Q[0]$ *and* $|F_r| \leq 2n-5$ *for* $r = 1, 2, \ldots, k-1$, *then* e *lies in healthy cycles of even lengths from* 6 *to* k^n.

Proof. Clearly, $|\mathcal{F}^0| \leq (4n-5) - (2n-4) = 2n-1$. we distinguish three cases

Case 1. $(s, t) \in E(Q[0])$.

As $|F_0| \leq 4n-9 = 4(n-1) - 5$ and F_0 is a conditional faulty edge set of $Q[0]$, by induction, we can construct the cycles as required.

Case 2. $(s, t) \in E(Q[1 : k-1])$.

Lemma 5.2 implies that (s, t) lies in a cycle in $Q[1 : k-1] - F$ of every even length from 6 to $k^n - k^{n-1}$. Observe that either $(s, t) \in E(Q[1 : k-2])$ or $(s, t) \in E(Q[2 : k-1])$. Without loss of generality, assume that the former case holds. It is easy to construct healthy cycles containing e of even lengths from $k^n - k^{n-1} + 2$ to k^n.

Case 3. $s \in V(Q[0])$ and $t \in V(Q[1]) \cup V(Q[k-1])$.

By symmetry, it is enough to consider the case that $s \in V(Q[0])$ and $t \in V(Q[1])$. Set $\mathcal{P} = \{\langle s, s', n^1(s'), t\rangle : (s, s') \in E(Q[i])\}$. Clearly, $|\mathcal{P}| = 2n-2$.

Suppose first that there is a path $\langle s, s_1, n^1(s_1), t\rangle \in \mathcal{P}$ which is healthy. By the induction hypothesis and Lemma 3.12(ii), we can obtain the cycles as required. Suppose next that every path in $\mathcal{P}$ is faulty. We can observe that there exists a path $\langle s, s_2, n^1(s_2), t\rangle \in \mathcal{P}$ which contains exactly 1 faulty edge. If $(n^1(s_2), t) \in F$, then, using a similar proof above, we may construct the cycles as required.

Case 3.1. $(s_2, n^1(s_2)) \in F$.

Case 3.1.1. There is an edge (s_2, s_3) of $Q[0] - (s_2, s)$ such that both $\langle s_2, s_3, n^1(s_3), n^1(s_2)\rangle$ and $(n^1(s_3), n^2(s_3))$ are all fault-free.

Obviously, $C^1 = \langle s, s_2, s_3, n^1(s_3), n^1(s_2), t, s\rangle$ is a 6-cycle containing e.

Case 3.1.1.1. There exists an edge (s_3, s_4) of $Q[0] - (s_3, s_2)$ such that $\langle s_3, s_4, n^1(s_4), n^1(s_3)\rangle$ and $\langle n^1(s_4), n^2(s_4), n^2(s_3)\rangle$ are all fault-free.

In this case, similar to the proofs in the fifth paragraph of Lemma 5.2, there exists a fault-free cycle containing e of every even length from 8 to $2k^{n-1} + 8$.

If there exists a path $\langle s, s_5, n^1(s_5), t\rangle \in \mathcal{P}$ such that $(s_5, n^1(s_5)) \notin F$, let $F' = F \setminus \{(s, s_5), (t, n^1(s_5))\}$. Clearly, we may obtain a fault-free cycle containing e of every even length from $k^{n-1} + 4$ to k^n. If for every path

$\langle s, s_6, n^1(s_6), t\rangle \in \mathcal{P}$ such that $(s_6, n^1(s_6)) \in F$, then there are at least $2n-2$ faulty 0-dimensional edges between $Q[0]$ and $Q[1]$ and so $|F_0| \leq (4n-5)-(2n-2) = 2n-3$.

Suppose first that $|F_0| \leq 2n-4$. Then $|F_0| \leq 4n-10$. By similar constructions in the sixth and seventh paragraphs of Lemma 5.2, we may obtain a cycle containing e of every even length from $2k^{n-1}+2$ to k^n.

Suppose next that $|F_0| = 2n-3$. Then $F \subset F_0 \cup E(\mathcal{P})$. Choose an edge $(x,y) \in F_0$ and a vertex z of $Q[0]-\{s,x,y\}$ such that $\delta(s,z)=1$ and $(z,n^1(z)) \notin F$. Set $F_0' = F \setminus \{(x,y)\}$. By Lemma 3.17, there is a hamiltonian sz-path P_1^9 of $Q[0]-F_0'$. Similar to the seventh paragraph of Lemma 5.2, there is a $tn^1(z)$-path P_2^9 in $Q[1,k-2]-F$ of every odd length from $k^{n-1}+2$ to $k^n-2k^{n-1}-1$. If $(x,y) \notin E(P_1^9)$, then we may choose an edge $(x_1,y_1) \in E(P_1^9)$. If $(x,y) \in E(P_1^9)$, let $(x_1,y_1)=(x,y)$. By Lemma 3.12(ii), there is an $n^{k-1}(x_1)n^{k-1}(y_1)$-path P_3^9 in $Q[k-1]$ of every odd length from 1 to $k^{n-1}-1$. Therefore $P_1^9 \cup P_2^9 \cup P_3^9 + \{(s,t),(x_1,n^{k-1}(x_1))(y_1, n^{k-1}(y_1)),(z,n^1(z))\} - (x_1,y_1)$ is a cycle containing e of every even length from $2k^{n-1}+4$ to k^n.

Case 3.1.1.2. There is not any edge (s_3,s_4) of $Q[0]-(s_3,s_2)$ such that $\langle s_3, s_4, n^1(s_4),\ n^1(s_3)\rangle$ and $\langle n^1(s_4), n^2(s_4), n^2(s_3)\rangle$ are all fault-free.

In this case, similar to the second paragraph of Case 1 in Lemma 5.6, there is a path $\langle s, s_5, n^1(s_5), t\rangle \in \mathcal{P}$ which contains exactly 1 faulty edge, where $s_5 \neq s_2$. If $(n^1(s_5),t) \in F$ or $(s_5,n^1(s_5)) \in F$, then we may obtain the cycles as required. If $(s,s_5) \in F$, set $F_0' = F_0 \setminus \{(s,s_5)\}$. Note that $\langle s,t,n^1(s_5),s_5\rangle$ is fault-free. It is easy to construct a healthy cycle containing e of every even length from 8 to k^n.

Case 3.1.2. There is not any edge (s_2,s_3) of $Q[0]-(s_2,s)$ such that $\langle s_2,s_3,n^1(s_3),n^1(s_2)\rangle$ and $(n^1(s_3),n^2(s_3))$ are all fault-free.

Now, every path in $\mathcal{P}$ contains exactly 1 faulty edge. Choose a path $\langle s,s_8,n^1(s_8),\ t\rangle \in \mathcal{P}$ such that $s_8 \neq s_2$. Similar to Case 3.1.1.2, we may construct the cycles containing e of every even length from 8 to k^n. Suppose first that $(t,n^1(s_8)) \in F$ or $(s_8,n^1(s_8)) \in F$. It is easy to verify that there is a 6-cycle containing e. Suppose next that $(s,s_8) \in F$. Since $4n-5 = |\mathcal{P}| + d_{Q[0]-s}(s_2)$, $\langle s, n^{k-1}(s), n^{k-1}(s_8), s_8\rangle$ is healthy. So $\langle s,n^{k-1}(s),n^{k-1}(s_8),s_8,n^1(s_8),t,s\rangle$ is a 6-cycle containing e.

Case 3.2. $(s,s_2) \in F$.

Set $F_0' = F_0 \setminus \{(s,s_2)\}$. Observe that $\langle s,t,n^1(s_2),s_2\rangle$ is healthy. We can obtain a fault-free cycle containing e of every even length from 8 to k^n. It remains to construct a 6-cycle containing e. Suppose first that $\langle s,n^{k-1}(s),n^{k-1}(s_2),s_2\rangle$ is healthy. Then $\langle s,n^{k-1}(s),n^{k-1}(s_2),s_2,\ n^1(s_2),t,s\rangle$ is a 6-cycle containing e. Suppose next that $\langle s,n^{k-1}(s),n^{k-1}(s_2),s_2\rangle$ contains at least 1 faulty edge.

Case 3.2.1. For every healthy edge (s,s_3) of $Q[0]$, $(s_3,n^1(s_3)) \notin F$ or $(t,n^1(s_3)) \notin F$.

If $(s_3,n^1(s_3)) \notin F$, then, similar to the proof in the second paragraph of Case 3, we can obtain the 6-cycle as required. If $(s_3,n^1(s_3)) \in F$, then $(t,n^1(s_3)) \notin F$. Similar to Case 3.1, we may construct the 6-cycle containing e.

Case 3.2.2. For every healthy edge (s,s_3) of $Q[0]$, $(s_3,n^1(s_3)),(t,n^1(s_3)) \in F$.

Set $\mathcal{P}' = \{P:\ P$ is an ss_2-path of length 3 in $Q[0]\}$. Clearly, any two paths in $\mathcal{P}'$ are internally disjoint. According to the definition of Q_n^k, $|\mathcal{P}'| = 2n-4$. If there is a fault-free ss_2-path $P^5 \in \mathcal{P}'$, then $\langle s,P^5,s_2,n^1(s_2),t,s\rangle$ is a 6-cycle containing e. If every path in $\mathcal{P}'$ is faulty, then $\mathcal{P}'$ contains at least $2n-4$ faulty edges. Recall that F_0 is a conditional faulty edge set of $Q[0]$. Then there exist two healthy edges (s,x_1) and (s,x_2) of $Q[0]$. Note that $(x_1,n^1(x_1)),(x_2,n^1(x_2)),(t,n^1(x_1)),(t,n^1(x_2)) \in F$.

Set $\mathcal{T} = \{P:\ P$ is a $tn^1(x_1)$-path of length 3 in $Q[1]\}$ and $f_1 = |F \cap E(\langle s,n^{k-1}(s),\ n^{k-1}(s_2),s_2\rangle)|$. Since $(4n-5)-f_1-|\mathcal{P}'|-|\{(s,s_2)\}|-|\{(x_1,n^1(x_1)),(x_2,n^1(x_2)),(t,\ n^1(x_1))\}| \leq (4n-5)-1-(2n-4)-1-3 = 2n-6$, it can be seen that there are at most $2n-6$ faulty edges in $\mathcal{T}$. So there are at least $|\mathcal{T}|-(2n-6)=2$ healthy paths in $\mathcal{T}$. Combining this with the fact that there is at most 1 neighbor y_1 of x_1 in $Q[0]-s$ such that $(x_1,y_1) \in E(\mathcal{P}')$, there exists a fault-free path $\langle t,w_1,w_2,n^1(x_1)\rangle$ in $\mathcal{T}$ such that $\langle w_2,n^0(w_2),x_1\rangle$ is fault-free. Then $\langle s,x_1,n^0(w_2),w_2,w_1,t,s\rangle$ is a 6-cycle containing e. □

Lemma 5.8. *If $|F_0| = 4n-8$ and F_0 is a conditional faulty edge set of $Q[0]$, then e lies in healthy cycles of even lengths from 6 to k^n.*

Proof. Clearly, $|\mathcal{F}^0| = (4n-5)-(4n-8) = 3$ and $|F_r| = 0$ for $r = 1,2,\dots,k-1$.

Claim 1. There exists an edge $(x,y) \in F_0$ such that (x,y) is adjacent to at most one edge in $\mathcal{F}^0$.

By contradiction. Suppose that every faulty edge of $Q[0]$ is adjacent to at least two faulty edges of $\mathcal{F}^0$. Choose a faulty edge $(s_1,t_1) \in F_0$. If there exist two nonadjacent edges in F_0, then $|\mathcal{F}^0| \geq 4$, a contradiction. So, any two faulty edges in F_0 are adjacent. Note that there is no cycle of length 3 in $Q[0]$. Then all the faulty edges in F_0 are incident with a vertex z of $Q[0]$. As F_0 is a conditional faulty edge set of $Q[0]$, it can be seen that there are at most $2n-4$ faulty edges incident with z. Observe that $4n-8 > 2n-4$, a contradiction. Therefore, this claim holds.

By Claim 1, without loss of generality, we assume that $(x, n^1(x)), (y, n^1(y)) \notin F$. As $|F_1| = 0$, we have that $(n^1(x), n^1(y)) \notin F$. Set $F_0' = F_0 \setminus \{(x,y)\}$. Then $|F_0'| = 4n-9 = 4(n-1)-5$. By the induction hypothesis, there is a hamiltonian xy-path P^0 of $Q[0]-F_0'$. Clearly, $(x,y) \notin E(P^0)$, which implies that P^0 is a hamiltonian path of $Q[0]-F_0$.

Case 1. $(s,t) \in E(Q[0])$.

By the induction hypothesis, (s,t) lies in a hamiltonian cycle C_1^1 of $Q[0]-F_0'$. If $(x,y) \notin E(C_1^1)$, choose an edge (s_1,t_1) of $C_1^1-(s,t)$ such that $(s_1,n^1(s_1)), (t_1,n^1(t_1)) \notin F$. If $(x,y) \in E(C_1^1)$, let $(s_1,t_1) = (x,y)$. Then we can construct a fault-free cycle containing e of every even length from $k^{n-1}+2$ to k^n. It remains to construct a cycle containing e of every length from 6 to k^{n-1}. Recall that F_0 is a conditional faulty edge set of $Q[0]$. We may choose two healthy edges $(s,s'), (t,t')$ of $Q[0]$. Then $\langle s', s, t, t'\rangle$ is a healthy path of $Q[0]$. Set

$E_1 = \{(s', n^1(s')), (n^1(s'), n^2(s')), (n^1(s), n^2(s)), (t, n^1(t))\}$,

$E_2 = \{(s', n^{k-1}(s')), (n^{k-1}(s'), n^{k-2}(s')), (n^{k-1}(s), n^{k-2}(s)), (t, n^{k-1}(t))\}$,

$E_3 = \{(s, n^1(s)), (n^1(t), n^2(t)), (t', n^1(t')), (n^1(t'), n^2(t'))\}$,

$E_4 = \{(s, n^{k-1}(s)), (n^{k-1}(t), n^{k-2}(t)), (t', n^{k-1}(t')), (n^{k-1}(t'), n^{k-2}(t'))\}$.

Clearly, E_1, E_2, E_3 and E_4 are pairwise disjoint edge sets. As $|\mathcal{F}^0| = 3$, it follows that there exists a fault-free edge set in $\{E_1, E_2, E_3, E_4\}$. By symmetry, we assume that E_1 is healthy. Then $C^1 = \langle s, t, n^1(t), n^1(s), n^1(s'), s', s\rangle$ is a 6-cycle containing $e = (s,t)$. By Lemma 3.12(*ii*), there is an $n^2(s')n^2(s)$-path P^1 in $Q[2]$ of every odd length from 1 to $k^{n-1}-1$. So $C^1 \cup P^1 + \{(n^1(s'), n^2(s')), (n^1(s), n^2(s))\} - (n^1(s), n^1(s'))$ is a cycle containing e of every even length from 8 to $k^{n-1}+6$.

Case 2. $(s,t) \in E(Q[1, k-1])$.

We may choose an edge $(x_1,y_1) \in E(P^0)$ such that $(x_1, n^1(x_1)), (y_1, n^1(y_1)), (x_1, n^{k-1}(x_1)), (y_1, n^{k-1}(y_1)) \notin F$ and $(n^{k-1}(x_1), n^{k-1}(y_1)) \neq (s,t)$ because $\lceil \frac{k^{n-1}-1}{2} \rceil = \frac{k^{n-1}}{2} > |\mathcal{F}^0|+1$. Lemma 5.2 implies that e lies in a cycle in $Q[1,k-1]-F$ of every even length from 6 to $k^n - k^{n-1}$.

Suppose first that $(s,t) \in E(Q[2,k-1])$. Note that $|\mathcal{F}^0| = 3 < 2n-2$. By Lemma 5.4, there is a hamiltonian st-path P^1 of $Q[2,k-1]-F$ passing through $(n^{k-1}(x_1), n^{k-1}(y_1))$. Lemma 3.12(*ii*) implies that there is an $n^1(x)n^1(y)$-path P^2 in $Q[1]$ of every odd length from 1 to $k^{n-1}-1$. Then $P^0 \cup P^1 \cup P^2 + \{(s,t), (x, n^1(x)), (y, n^1(y)), (x_1, n^{k-1}(x_1)), (y_1, n^{k-1}(y_1))\} - \{(x_1,y_1), (n^{k-1}(x_1), n^{k-1}(y_1))\}$ is a cycle containing e of every even length from $k^n - k^{n-1}+2$ to k^n.

Suppose next that $(s,t) \in E(Q[1,k-2])$. If $(s,t) \neq (n^1(x), n^1(y))$, then, similar to the above argument, there is a cycle containing e of every even length from $k^n - k^{n-1}+2$ to k^n. If $(s,t) = (n^1(x), n^1(y))$, then, Lemma 3.13 implies that there is a hamiltonian $n^1(x)n^1(y)$-path P^5 of $Q[1]-\{s,t\}$. Choose an edge $(x_2,y_2) \in E(P^5)$ such that $(x_2, n^2(x_2)), (y_2, n^2(y_2)) \notin F$. By Lemma 3.12(*ii*), there is an $n^{k-1}(x_2)n^{k-1}(y_2)$-path P^6 in $Q[2,k-1]-F$ of every odd length from 1 to $k^n - 2k^{n-1}-1$. Then $P^0 \cup P^5 \cup P^6 + \{(s,t), (x, n^1(x)), (y, n^1(y)), (x_1, n^1(x_1)), (y_1, n^1(y_1)), (x_2, n^{k-1}(x_2)), (y_2, n^{k-1}(y_2))\} - \{(x_1, y_1), (x_2, y_2)\}$ is a cycle containing e of every even length from $k^n - 2k^{n-1}+2$ to k^n. Thus, e lies in a cycle of every even length from $k^n - k^{n-1}+2$ to k^n.

Case 3. $s \in V(Q[0])$ and $t \in V(Q[1]) \cup V(Q[k-1])$.

Case 3.1. $6 \leq l \leq k^n - 2k^{n-1}+4$.

Suppose that $t \in V(Q[1])$. Recall that F_0 is a conditional faulty edge set of $Q[0]$. We may choose a fault-free path $\langle s_3, s_2, s, s_1, s_4\rangle$ of $Q[0]-F_0$.

Suppose first that either $(s_1, n^1(s_1)) \notin \mathcal{F}^0$ or $(s_2, n^1(s_2)) \notin \mathcal{F}^0$. Without loss of generality, assume that $(s_1, n^1(s_1)) \notin \mathcal{F}^0$. It is easy to verify that e lies in a cycle of every even length from 6 to $k^n - 2k^{n-1}+4$. Suppose next that $(s_1, n^1(s_1)), (s_2, n^1(s_2)) \in \mathcal{F}^0$. Let $E_1 = \{(s_3, n^1(s_3)), (n^1(s_3), n^2(s_3)), (n^1(s_2), n^2(s_2))\}$ and

$E_2 = \{(s_4, n^1(s_4)), (n^1(s_4),\ n^2(s_4)), (n^1(s_1), n^2(s_1))\}$. Note that $E_1 \cap E_2 = \emptyset$ and $|\mathcal{F}^0| = 3$. Then either E_1 is healthy or E_2 is healthy. Without loss of generality, assume that E_1 is healthy. Then it is easy to construct the cycles as required. If $t \in V(Q[k-1])$, then the proof is similar.

Case 3.2. $k^n - 2k^{n-1} + 6 \leq l \leq k^n$.

Case 3.2.1. $s \in \{x, y\}$.

Without loss of generality, assume that $s = x$. Then P^0 is a hamiltonian sy-path of $Q[0] - F_0$. Suppose that $t \in V(Q[1])$. Thus we can obtain a healthy cycle containing e of every even length from $k^n - 2k^{n-1} + 2$ to k^n in $Q[0] \cup Q[1, k-1] + \{(s,t), (y, n^1(y))\}$. If $t \in V(Q[k-1])$, then $t = n^{k-1}(x)$. Choose two edges $(n^1(y), y_1) \in E(Q[1])$ and $(t, t_1) \in E(Q[k-1])$ such that $(y^1, n^2(y_1)), (t_1, n^{k-2}(t_1)) \notin F$. By Lemma 3.12($ii$), there is an $n^1(y)y_1$-path P^3 in $Q[1]$ of every odd length from 1 to $k^{n-1} - 1$ and there is a tt_1-path P^4 in $Q[k-1]$ of every odd length from 1 to $k^{n-1} - 1$. Lemma 3.17 implies that there is a hamiltonian $n^{k-2}(t_1)n^2(y_1)$-path P^5 of $Q[2, k-2] - F$. Then $\langle s, t, P^4, t_1, n^{k-2}(t_1), P^5, n^2(y_1), y_1, P^3, n^1(y), y, P^0, s\rangle$ is a cycle containing e of every even length from $k^n - 2k^{n-1} + 4$ to k^n.

Case 3.2.2. $s \notin \{x, y\}$.

We may choose an edge $(s, s_1) \in E(Q[0])$ such that $s_1 \notin \{x, y\}$ and s_1 is incident with at most 1 faulty edge in $\mathcal{F}^0$ since $d_{Q[0]}(s) - |\{(x,y)\}| = 2n - 3 \geq |\mathcal{F}^0|$. By the induction hypothesis, there exists a hamiltonian ss_1-path P_1^0 of $Q[0] - F_0'$. If $(x, y) \notin E(P_1^0)$, choose an edge $(u, v) \in E(P_1^0)$ such that $(u, n^1(u)), (v, n^1(v)) \notin F$. If $(x, y) \in E(P_1^0)$, let $(u, v) = (x, y)$.

Case 3.2.2.1. $t \in V(Q[1])$.

If $(s_1, n^1(s_1)) \notin \mathcal{F}^0$, then, by Lemma 3.13, there is a hamiltonian $tn^1(s_1)$-path P^1 of $Q[1] - \{n^1(u), n^1(v)\}$. Choose an edge $(x_1, y_1) \in E(P^0 - \{(s, s_1), (u, v)\})$ such that $(x_1, n^{k-1}(x_1)), (y_1, n^{k-1}(y_1)) \notin F$. We may obtain the cycles as required.

If $(s_1, n^1(s_1)) \in \mathcal{F}^0$, then $(s_1, n^{k-1}(s_1)) \notin \mathcal{F}^0$. Observe that $2n - 2 \geq 4$ and $|\mathcal{F}^0 \cap E(Q[1, k-1])| \leq 3 - |\{(s_1, n^1(s_1))\}| = 2$. We may choose two edges $(n^{k-1}(s_1), s_2) \in E(Q[k-1])$ and $(t, t_1) \in E(Q[1] - \{n^1(u), n^1(v)\})$ such that $(s_2, n^{k-2}(s_2)), (t_1, n^2(t_1)) \notin \mathcal{F}^0$. By Lemma 3.12($ii$), there is an $n^{k-1}(s_1)s_2$-path P^3 in $Q[k-1]$ of every odd length from 1 to $k^{n-1} - 1$. By Theorem 1.1, there is an $n^1(u)n^1(v)$-path in $Q[1] - \{t, t_1\}$ of every odd length from 1 to $k^{n-1} - 5$. Lemma 3.13 implies that there exists a hamiltonian $n^1(u)n^1(v)$-path of $Q[1] - \{t, t_1\}$. Therefore, there is an $n^1(u)n^1(v)$-path P^4 of every odd length from 1 to $k^{n-1} - 3$ in $Q[1] - \{t, t_1\}$. By Lemma 3.17, there is a hamiltonian $n^2(t_1)n^{k-2}(s_2)$-path P^5 of $Q[2, k-2] - F$. Then $P_1^0 \cup P^3 \cup P^4 \cup P^5 + \{(s,t), (t, t_1), (s_1, n^{k-1}(s_1)), (u, n^1(u)), (v, n^1(v)), (s_2, n^{k-2}(s_2)), (t_1, n^2(t_1))\} - (u, v)$ is a cycle containing e of every even length from $k^n - 2k^{n-1} + 6$ to k^n.

Case 3.2.2.2. $t \in V(Q[k-1])$.

Suppose first that $(s_1, n^1(s_1)) \notin \mathcal{F}^0$. Recall that $t \in V(Q[k-1])$. The proof is similar to the proof of the case that $(s_1, n^{k-1}(s_1)) \notin \mathcal{F}^0$ and $t \in V(Q[1])$. Suppose next that $(s_1, n^1(s_1)) \in \mathcal{F}^0$. Then $(s_1, n^{k-1}(s_1)) \notin \mathcal{F}^0$. It is easy to verify that there exist the required cycles in $Q[0] \cup Q[1, k-1] + \{(s,t), (s_1, n^{k-1}(s_1)), (u, n^1(u)), (v, n^1(v))\}$. ☐

Lemma 5.9. *If there exists a vertex* $x \in V(Q[0])$ *which is incident with* $2n - 3$ *edges in* F_0 *and* $|F_0| \leq 4n - 9$, *then* e *lies in a healthy cycle of length* k^n.

Proof. Clearly, $|F_r| \leq |F| - (2n-3) - |\mathcal{F}^0| \leq 2n - 5$, for $r = 1, 2, \ldots, k-1$. In addition, for any $w \in V(Q[r])$, $r = 1, 2, \ldots, k-1$, we have $d_{Q[r]}(w) - |F_r| \geq 3$. So F_r is a conditional faulty edge set of $Q[r]$. Note that x is incident with exactly $d_{Q[0]}(x) - (2n-3) = 1$ healthy edge of $Q[0]$. Then either $(x, n^1(x)) \notin F$ or $(x, n^{k-1}(x)) \notin F$. Recall that $e = (s, t)$, $s \in V(Q[i])$ and $t \in V(Q[j])$.

Case 1. There is an edge $(x, y) \in F_0$ which is adjacent to at most one edge in $\mathcal{F}^0$.

Clearly, either $\{(x, n^1(x)), (y, n^1(y))\} \cap \mathcal{F}^0 = \emptyset$ or $\{(x, n^{k-1}(x)), (y, n^{k-1}(y))\} \cap \mathcal{F}^0 = \emptyset$. Without loss of generality, we assume that the former case holds. Set $F_0' = F_0 \setminus \{(x, y)\}$. Then $|F_0'| \leq |F \setminus \{(x,y)\}| - |\mathcal{F}^0| \leq 4n - 9 - 1 = 4n - 10$. For any $w' \in V(Q[0] - x)$, we have $d_{Q[0]}(w') - [|F| - (2n-3) - |\mathcal{F}^0|] \geq 3$. Observe that there is at most one faulty edge in $\partial_{Q[0]}(x)$ that is incident with w'. Then w' is incident with at least $3 - 1 = 2$ healthy edges of $Q[0]$. Thus F_0' is a conditional faulty edge set of $Q[0]$. By the induction hypothesis, there is a hamiltonian xy-path P^0 of $Q[0]-F'_0$. Clearly, P^0 is also a hamiltonian path of $Q[0]-F_0$. If $(n^1(x),n^1(y))\in F$, let $F'=F\setminus\{(n^1(x),n^1(y))\}$. Otherwise, let $F'=F$. Note that $|F\cap E(Q[1,k-1])|\le 2n-2$ and $|F\cap E(Q[r])|\le 2n-5$, for $r=1,2,\dots,k-1$.

Case 1.1. $s,t\in V(Q[0])$.

By the induction hypothesis, there is a hamiltonian st-path P_1^1 of $Q[0]-F'_0$. Then we can construct a hamiltonian cycle as required.

Case 1.2. $s,t\in V(Q[1,k-1])$.

Case 1.2.1. $\{n^1(x),n^1(y)\}\ne\{s,t\}$.

Suppose first that $i=j=1$ or $2\le i,j\le k-1$. It is easy to verify that there exists the cycle as required. Suppose next that $i=1$ and $j=2$. If there exists a neighbor w of s in $Q[1]-\{(n^1(x),n^1(y))\}$ such that $(w,n^2(w))\notin F$, set $F'_1=F_1\setminus\{(s,w)\}$. By Lemma 5.5, there is a hamiltonian sw-path P_5^2 of $Q[1]-F'_1$ passing through $(n^1(x),n^1(y))$. Clearly, we may construct a hamiltonian cycle containing e. Suppose that every edge (s,w) of $Q[1]-\{(n^1(x),n^1(y))\}$ such that $(w,n^2(w))\in F$. As $d_{Q[1]}(s)-|\{(n^1(x),n^1(y))\}|=2n-3$, we have that $|F_1|\le(4n-5)-(2n-3)-(2n-3)=1$. Choose a vertex w' of $Q[1]-\{s,n^1(x),n^1(y)\}$ such that $\delta(s,w')=1$ and $(w',n^2(w'))\notin F$. Using Lemma 5.4 and a similar proof as above, we may obtain the cycle as required.

Case 1.2.2. $\{n^1(x),n^1(y)\}=\{s,t\}$.

Without loss of generality, assume $s=n^1(x)$. Suppose that $\{(x,n^{k-1}(x)),\,(y,n^{k-1}(y))\}\cap\mathcal{F}^0=\emptyset$. Note that $\{n^{k-1}(x),n^{k-1}(y)\}\ne\{s,t\}$. Similar to the proof above, we may obtain a path as required. So we consider the case that either $(x,n^{k-1}(x))$ or $(y,n^{k-1}(y))$ is faulty below. We only consider the case that $(x,n^{k-1}(x))\in\mathcal{F}^0$. Then $(y,n^{k-1}(y)),(x,s)\notin\mathcal{F}^0$.

If there exists an edge (s,w^1) of $Q[1]-t$ such that $(w^1,n^2(w^1))\notin F$, by Lemma 5.5, there is a hamiltonian sw^1 path P_6^2 of $Q[1]-F$ passing through (s,t). By Lemma 5.3, there is a hamiltonian $n^2(w^1)n^{k-1}(y)$-path P_7^2 of $Q[2,k-1]-F$. Then $P^0\cup P_6^2\cup P_7^2+\{(x,s),(y,n^{k-1}(y)),(w^1,n^2(w^1))\}$ is as required. If every edge (s,w^1) of $Q[1]-t$ such that $(w^1,n^2(w^1))\in F$, then $|F_1|\le 1$. Choose a vertex w^2 of $Q[1]-t$ such that $\delta(s,w^2)=1$ and $(w^2,n^2(w^2))\notin F$. Then we can construct the cycle as required.

Case 1.3. $s\in V(Q[0])$ and $t\in V(Q[1])\cup V(Q[k-1])$.

If $s\in\{x,y\}$, it is easy to construct a healthy hamiltonian cycle containing e. We consider $s\notin\{x,y\}$ below. Choose a vertex w of $Q[0]-\{x,y\}$ such that $(w,n^1(w))$, $(w,n^{k-1}(w))\notin F$ and $\delta(s,w)=1$. By Lemma 5.3, there is a hamiltonian sw-path P_2^3 of $Q[0]-F'_0$. Clearly, $(x,y)\in E(P_2^3)$. Without loss of generality, write P_2^3 as $\langle s,P_{2(1)}^3,x,y,P_{2(2)}^3\ ,w\rangle$.

If $|\{(x,n^{k-1}(x)),(y,\,n^{k-1}(y))\}\cap\mathcal{F}^0|=0$, then, we only need to consider the case that $j=1$. Clearly, we may obtain a required cycle in $Q[0]\cup Q[1,k-1]+\{(s,t),(w,n^1(w)),(x,\,n^{k-1}(x)),(y,n^{k-1}(y))\}$. Now we consider the case that $|\{(x,n^{k-1}(x)),(y,n^{k-1}(y))\}\cap\mathcal{F}^0|=1$ and $j=1$. Since there are at most $|F|-(2n-3)-1=2n-3$ faulty edges outside $F_0\cup\{(x,n^{k-1}(x)),(y,\,n^{k-1}(y))\}$ and $d_{Q[1]}(v)=2n-2$, we have that there exists at least one neighbor t_1 of t in $V(Q[1])$ such that $\{(t,t_1),(t_1,n^2(t_1))\}\notin F$.

Suppose first that there exist at least 2 distinct neighbors t_1,t_2 of t in $V(Q[1])$ such that $\{(t,t_1),(t,t_2),(t_1,n^2(t_1)),(t_2,n^2(t_2))\}\notin F$. Set $F'_1=F_1\cup(\partial_{Q[1]}(v)\setminus\{(t,t_1),(t,t_2)\})$. Then $|F'_1|\le(2n-5)+(2n-4)=4n-9=4(n-1)-5$. By the induction hypothesis, there is a hamiltonian $n^1(x)n^1(y)$-path P_7^3 of $Q[1]-F'_1$. Clearly, $(t,t_1),(t,t_2)\in E(P_7^3)$. As $s\notin\{x,y\}$, we have that $t\notin\{n^1(x),n^1(y)\}$. Write P_7^3 as $\langle n^1(x),P_{7(1)}^3,t_1,t,P_{7(2)}^3,n^1(y)\rangle$ By Lemma 5.3, there is a hamiltonian $n^2(t_1)n^{k-1}(w)$-path P_8^3 of $Q[2,k-1]-F$. So $\langle s,P_{2(1)}^3,x,n^1(x),P_{7(1)}^3,t_1,n^2(t_1),P_8^3,n^{k-1}(w),w,P_{2(2)}^3,y,n^1(y),P_{7(2)}^3,t,s\rangle$ is as required.

Suppose next that there exists only one neighbor t_1 of t in $V(Q[1])$ such that $\{(t,t_1),\ (t_1,n^2(t_1))\}\notin F$. Let $T=\{t':(t,t')\in E(Q[1]),\ (t,t')\in F\text{ or }(t',n^2(t'))\in F\}$. Then $|T|=d_{Q[1]}(t)-1=2n-3$. Note that $|F|=4n-5$. Then $|F_0|=2n-3$ and $|F\cap E(Q[2,k-1])|=0$. Clearly, for any vertex $w\notin T\cup\{x,y,t\}$, w is not incident with any faulty 0-dimensional edge. In addition, if $w\in V(Q[1,k-1])$, then w is not incident with any faulty edge. Choose edges $(s,s_1),(s,s_2)\in E(P^0)$. Without loss of generality, we write P^0 as $\langle x,P_{(1)}^0,s_1,s,s_2,P_{(2)}^0,y\rangle$. Obviously, $\{t,t_1\}\ne\{n^1(x),n^1(y)\}$.

If $s_1\ne x$, using a similar proof as above, we may obtain the cycle as required. If $s_1=x$, then $\delta(n^1(y),t)=0$. As $n^1(y)\ne t$, we have that $(n^1(x),n^1(y))\notin F$. Choose a vertex z of $Q[1]-n^1(x)$ such that $(n^1(y),z)\in E(Q[1])$

and $z \notin T$. Clearly, $(n^1(y), z), (z, n^2(z)) \notin F$. Lemma 5.5 implies that there is a hamiltonian $n^1(x)t$-path P_5^1 of $Q[1] - F$ passing through $(n^1(x), n^1(y))$ and $(n^1(y), z)$, By Lemma 5.3, there is a hamiltonian $n^2(z)n^{k-1}(s_2)$-path P_6^1 of $Q[2, k-2]$. Then $\langle s, x, n^1(x), n^1(y), y, P_{(2)}^0, s_2, n^{k-1}(s_2), P_6^1, n^2(z), z, P_5^1 - \{n^1(x), n^1(y)\}, t, s\rangle$ is as required.

Case 2. Any edge $(x, y') \in F_0$ is adjacent to at least two edges in $\mathcal{F}^0$.

Claim 1. $|\{(x, n^1(x)), (x, n^{k-1}(x))\} \cap \mathcal{F}^0| = 1$ and $|\{(y', n^1(y')), (y', n^{k-1}(y'))\} \cap \mathcal{F}^0| = 1$.

If $(x, n^1(x)), (x, n^{k-1}(x)) \notin \mathcal{F}^0$, then $(y', n^1(y')), (y', n^{k-1}(y')) \in \mathcal{F}^0$. So $2(2n-3) \leq |\mathcal{F}^0| \leq |F| - (2n-3) = 2n-2$, a contradiction. Therefore, $|\{(x, n^1(x)), (x, n^{k-1}(x))\} \cap \mathcal{F}^0| = 1$ and so $|\{(y', n^1(y')), (y', n^{k-1}(y'))\} \cap \mathcal{F}^0| \geq 1$. As $2n-2 = |\{x\}| + (2n-3) \leq |\mathcal{F}^0| \leq |F| - (2n-3) = 2n-2$, we have that $|\mathcal{F}^0| = 2n-2$. Then $|\{(y', n^1(y')), (y', n^{k-1}(y'))\} \cap \mathcal{F}^0| = 1$. Therefore, this claim holds.

By Claim 1, without loss of generality, assume that $(x, n^1(x)) \notin \mathcal{F}^0$. Obviously, $F \cap E(Q[1, k-1]) = \emptyset$. Choose an edge (x, y) in F_0. Set $F_0' = F_0 \setminus \{(x, y)\}$. By the induction hypothesis, there is a hamiltonian xy-path P^2 of $Q[0] - F_0'$. Clearly, P^2 is also a hamiltonian path of $Q[0] - F_0$.

Suppose that $s, t \in V(Q[0])$. It is easy to verify that there exists a healthy hamiltonian cycle containing e. Therefore, it remains to consider the case that either $s \notin V(Q[0])$ or $t \notin V(Q[0])$. We distinguish two cases.

Case 2.1. $(y, n^1(y)) \notin \mathcal{F}^0$.

Since $(x, n^1(x)), (y, n^1(y)) \notin F$ and $F \cap E(Q[1, k-1]) = \emptyset$, we can easily obtain the cycle required.

Case 2.2. $(y, n^1(y)) \in \mathcal{F}^0$.

In this case, $(y, n^{k-1}(y)) \notin \mathcal{F}^0$.

Case 2.2.1. $s, t \in V(Q[1, k-1])$.

If $i = j$, then it is easy to construct the cycle as required. We only consider the case that $i \neq j$. Without loss of generality, assume that $1 \leq i < j \leq k-1$. If $\delta(n^1(x), s) = 1$, then we can obtain a healthy hamiltonian cycle containing e. Suppose that $\delta(n^1(x), s) = 0$.

Suppose first that $s = n^1(x)$. Then $j = 2$. Choose a vertex $t^1 \in V(Q[2] - t)$ such that $\delta(t^1, t) = 0$. By Lemma 3.16, there is a hamiltonian tt^1-path P_1^5 of $Q[1, j] - s$. By Lemma 5.3, there is a hamiltonian $n^{j+1}(t^1)n^{k-1}(y)$-Path P_2^5 of $Q[j+1, k-1]$. Then $\langle s, x, P^2, y, n^{k-1}(y), P_2^5, n^{j+1}(t^1), t^1, P_1^5, t, s\rangle$ is as required. Suppose next that $s \neq n^1(x)$ and $t \neq n^{k-1}(y)$. Using P^2 and Lemma 3.16, it is easy to construct the cycle as required.

Case 2.2.2. $s \in V(Q[0])$ and $t \in V(Q[1]) \cup V(Q[k-1])$.

Without loss of generality, assume that $\delta(s, x) = 1$. If $s \in \{x, y\}$, then it is easy to obtain a cycle as required. If $s \notin \{x, y\}$, we may choose a vertex w of $Q[0] - \{s, x, y\}$ such that $\delta(s, w) = 1$ and $(w, n^1(w)), (w, n^{k-1}(w)) \notin \mathcal{F}^0$. By Lemma 5.3, there is a hamiltonian sw-path P_1^8 of $Q[0] - F_0'$. Clearly, $x \notin \{s, w\}$ and so $(x, y) \in E(P_1^8)$.

Suppose first that $t \in V(Q[1])$, then, by Lemma 3.13, there is a hamiltonian $n^1(w)t$-path P_2^8 of $Q[1] - \{n^1(x), z\}$ where $(n^1(x), z) \in E(Q[1])$ and $z \neq t$. By Lemma 5.3, there is a hamiltonian $n^2(z)n^{k-1}(y)$-path P_3^8 of $Q[2, k-1]$. Then $P_1^8 \cup P_2^8 \cup P_3^8 + \{(s, t), (x, n^1(x)), (y, n^{k-1}(y)), (n^1(x), z), (z, n^2(z)), (w, n^1(w))\} - (x, y)$ is as required.

Suppose next that $t \in V(Q[k-1])$. As $s \notin \{x, y\}$, we have that $t \neq n^{k-1}(y)$, using a similar proof as above, there exists a cycle as required. ☐

Lemma 5.10. *If there exists a vertex $x \in V(Q[0])$ which is incident with $2n-3$ edges in F_0 and $|F_0| \leq 4n-9$, then e lies in healthy cycles of even lengths from 6 to $k^n - 2$.*

Proof. Clearly, $|F_r| \leq |F| - (2n-3) - |\mathcal{F}^0| \leq 2n-5$, for $r = 1, 2, \ldots, k-1$. Recall that $e = (s, t)$, $s \in V(Q[i])$ and $t \in V(Q[j])$.

Case 1. $(s, t) \in E(Q[0])$.

Suppose that $x \notin \{s, t\}$. Let $f_v = |\{x\}|$ and $f_e = |F_0| - (2n-3)$. Then $f_e \leq 2n-6$ and so $f_v + f_e \leq 2n-5$. By Theorem 1.1, e lies in a cycle of every even length from 4 to $k^{n-1} - 2$ in $Q[0] - F_0$. Clearly, we can construct a healthy cycle containing e of every even length from k^{n-1} to $k^n - 2$. Now we consider the case that $x \in \{s, t\}$.

Claim 1. If $|F_0| \leq 4n-8$, then then e lies in cycles of even lengths from 6 to k^n.

Without loss of generality, we assume that $x = s$. Then (s, t) is the only one healthy edge which is incident with s in $Q[0]$. As F is a conditional faulty edge set of Q_n^k and s is incident with $2n-3$ faulty edges in F_0, it implies that either $(s, n^1(s)) \notin F$ or $(s, n^{k-1}(s)) \notin F$. Without loss of generality, we assume that $(s, n^1(s)) \notin F$. Thus $\langle n^1(s), s, t\rangle$ is fault-free.

If either $(t, n^1(t)) \notin F$ or $(s, n^{k-1}(s)), (t, n^{k-1}(t)) \notin F$, then it is easy to verify that there exists a healthy cycle containing e of even length from 6 to $k^n - k^{n-1} + 2$. Note that there are $2n-3$ faulty edges incident with s in F_0 and $|\mathcal{F}^0| \leq 2n-2$. We may choose a faulty edge $(s, s_1) \in F_0$ such that s_1 is incident with at most one edge in $\mathcal{F}^0$. Set $F_0' = F_0 \setminus \{(s, s_1)\}$. Then $|F_0'| \leq 4n-9$. By the induction hypothesis, there is a hamiltonian ss_1-path P^1 of $Q[0] - F_0'$. Clearly, $(s,t) \in E(P^1)$. Thus, using the path P^1, we can obtain a healthy cycle containing e of every even length from $k^n - k^{n-1} + 2$ to k^n.

It remains to consider the case that $(t, n^1(t)) \in F$ and $\{(s, n^{k-1}(s)), (t, n^{k-1}(t))\} \cap F \neq \emptyset$. Suppose first that $(n^1(s), n^1(t)) \notin F$. We claim that we may choose an edge (t, t_1) of $Q[0] - s$ such that $\langle t, t_1, n^1(t_1), n^1(t)\rangle$ is healthy. In fact, there exist $2n-3$ candidate edges in $E(Q[0]-s)$ and at most $(4n-5) - (2n-3) - |\{(t, n^1(t))\}| - |\{(s, n^{k-1}(s)), (t, n^{k-1}(t))\} \cap F| \leq 2n-4$ faulty edges block candidates. Furthermore, one faulty edge blocks at most one candidate. Therefore, this claim holds. Similarly, we may choose an edge $(s, s_1) \in F_0$ such that $\langle s_1, n^1(s_1), n^1(s)\rangle$ is healthy. Now, $\langle s, t, t_1, n^1(t_1), n^1(t), n^1(s), s\rangle$ is a 6-cycle containing e. Clearly, it is east to construct a cycle containing e of every even length from 8 to k^n.

Suppose next that $(n^1(s), n^1(t)) \in F$. Set $\mathcal{C} = \{\langle t, t_1, t_2, n^1(t_2), n^1(s)\rangle : \langle s, t, t_1, t_2, s\rangle$ is a 4-cycle in $Q[0]\}$. Obviously, $(s, t_2) \in F_0$. By the definition of Q_n^k, (s,t) lies in $2n-4$ 4-cycles in $Q[0]$. Thus $|\mathcal{C}| = 2n-4$. Observe that $(4n-5) - (2n-3) - |\{(t, n^1(t)), (n^1(s), n^1(t))\}| - |\{(s, n^{k-1}(s)), (t, n^{k-1}(t))\} \cap F| \leq 2n-5$. Therefore, there exists a path $\langle t, t_1, t_2, n^1(t_2), n^1(s)\rangle \in \mathcal{C}$ which is healthy. Now, $\langle s, t, t_1, t_2, n^1(t_2), n^1(s), s\rangle$ is a 6-cycle containing e. Using this cycle, we can obtain the cycles containing e of every even length from 8 to k^n. Therefore, this claim holds.

By Claim 1, e lies in a cycle of every even length from 6 to $k^n - 2$.

Case 2. $(s,t) \in E(Q[1, k-1])$.

By Lemma 5.2, e lies in a cycle of every even length from 6 to $k^n - k^{n-1}$ in $Q[1, k-1] - F$. By Lemma 5.3, there is a hamiltonian st-path P^1 of $Q[1, k-1] - F$ such that $|E(P^1 \cap (Q[1] \cup Q[k-1]))| \geq 2k^{n-1} - 3$. We may choose an edge (s_1, t_1) of $P^1 \cap (Q[1] \cup Q[k-1])$ such that $\langle s_1, n^0(s_1), n^0(t_1), t_1\rangle$ is healthy and $x \notin \{n^0(s_1), n^0(t_1)\}$ since $\lceil \frac{2k^{n-1}-3}{2} \rceil = k^{n-1} - 1 > 4n-5$. Then, we may obtain a cycle containing e of every even length from $k^n - k^{n-1} + 2$ to $k^n - 2$.

Case 3. $s \in V(Q[1])$ and $t \in V(Q[1]) \cup V(Q[k-1]))$.

Without loss of generality, we assume that $t \in V(Q[1])$.

Case 3.1. $s \neq x$.

Let $\mathcal{P} = \{\langle s, s_1, n^1(s_1), t\rangle : (s, s_1) \in E(Q[0])$ and $s_1 \neq x\}$. Then $|\mathcal{P}| \geq 2n-3$.

Suppose first that there is a path $\langle s, s_1, n^1(s_1), t\rangle \in \mathcal{P}$ such that $(s_1, n^1(s_1)), (t, n^1(s_1)) \notin F$. Therefore, it is easy to verify that there exist the cycles as required.

Suppose next that every path in $\mathcal{P}$ contains at least 1 faulty edge which is not in F_0. As there are at most $2n-2$ faulty edges not in F_0 and $|\mathcal{P}| \geq 2n-3$, we have that there is at most 1 path in $\mathcal{P}$ which contains 2 faulty edges which are not in F_0. Note that s is incident to at least 2 healthy edges of $Q[0]$ and $2 > 1$. It is east to see that we may choose a healthy edge $(s, s_2) \in E(Q[0])$ such that either $(s_2, n^1(s_2)) \notin F$ or $(t, n^1(s_2)) \notin F$. We will construct a cycle of every even length l with $6 \leq l \leq k^n - 2$ that contains e in $Q_n^k - F$.

Case 3.1.1. $6 \leq l \leq k^n - 2k^{n-1} + 6$.

If $(s_2, n^1(s_2)) \notin F$, then, it is easy to verify that e lies in a cycle of every even length from 6 to $k^n - 2k^{n-1} + 6$. If $(s_2, n^1(s_2)) \in F$, then $(t, n^1(s_2)) \notin F$. We may choose an edge $(s_2, s_3) \in E(Q[0] - \{s, x\})$ such that both $\langle s_2, s_3, n^1(s_3), n^1(s_2)\rangle$ and $\langle n^1(s_2), n^2(s_2), n^2(s_3), n^1(s_3)\rangle$ are healthy because $d_{Q[0]}(s_2) - 2 = 2n-4 > (4n-5) - (2n-3) - |\mathcal{P}|$. Clearly, we can construct the cycles as required.

Case 3.1.2. $k^n - 2k^{n-1} + 8 \leq l \leq k^n - 2$.

If $(s, x) \notin E(Q[0])$, then $|F_0| = 2n-3$ and $|\mathcal{P}| = 2n-2$. Clearly, $F \subset E(Q[0,1])$ because $4n-5 = (2n-3) + |\mathcal{P}|$. Choose an edge $(x, y) \in F_0$ and a vertex $w \in V(Q[0] - \{s, x, y\})$ such that $(w, n^1(w)) \notin F$ and $\delta(s, w) = 1$. Set $F_0' = F_0 \setminus \{(x, y)\}$. Then $|F_0'| \leq 4n-10$. By Lemma 3.13, there is a hamiltonian sw-path P^6 of $Q[0] - F_0'$ and there is a hamiltonian $tn^1(w)$-path P^7 of $Q[1] - F$. Obviously, $(x, y) \in E(P^6)$. Then we can construct the required cycles.

If $(s, x) \in E(Q[0])$, then $|\mathcal{P}| = 2n-3$ and so $(4n-5)-(2n-3)-|\mathcal{P}| = 1$.

Suppose first that $(s, x) \notin F_0$. since $2n-3 > 1$, we may choose an edge $(x, y_1) \in F_0$ such that $(y_1, n^1(y_1)) \notin F$. Set $F_0' = F_0 \setminus \{(x, y_1)\}$. Then $|F_0'| \leq 4n-10$. By Lemma 5.3, there is a hamiltonian xy_1-path P_1^8 of $Q[0]-F_0'$. Clearly, $(x, s) \in E(P_1^8)$. As $|F_1| \leq 2n-5$ and $\delta(t, n^1(y_1)) = 0$, Lemma 3.4 implies that there is a $tn^1(y_1)$-path P_2^8 in $Q[1]-F_1$ of length $k^{n-1}-2$. Choose an edge (u, v) of P_2^8 such that $(u, n^2(u)), (v, n^2(v)), (n^2(u), n^2(v)) \notin F$. By Lemma 3.12$(ii)$, there is an $n^2(u)n^2(v)$-path P_3^8 in $Q[2, k-1]-F$ of every odd length from 1 to $k^n - 2k^{n-1} - 1$. Then $P_1^8 \cup P_2^8 \cup P_3^8 + \{(s, t), (y_1, n^1(y_1)), (u, n^2(u)), (v, n^2(v))\} - x - (u, v)$ is a cycle containing e of every even length from $2k^{n-1}$ to $k^n - 2$. Noting that $2k^{n-1} < k^n - 2k^{n-1} + 8$, we see that e lies in a cycle of every even length from $k^n - 2k^{n-1} + 8$ to $k^n - 2$.

Suppose next that $(s, x) \in F_0$. There exists an edge $(x, y_2) \in E(Q[0])$ such that $(x, y_2) \notin F_0$. Set $F_0' = F_0 \setminus \{(x, s)\}$. Then $|F_0'| \leq 4n-10$. By Lemma 5.3, there is a hamiltonian xs-path P_1^9 of $Q[0]-F_0'$. Clearly, $(x, y_2) \in E(P_1^9)$. Recall that $(4n-5)-(2n-3)-|\mathcal{P}| = 1$. Then either $(y_2, n^1(y_2)) \notin F$ or $(y_2, n^{k-1}(y_2)) \notin F$. If $(y_2, n^1(y_2)) \notin F$, then, using a similar proof as above, we may obtain the cycle as required. If $(y_2, n^1(y_2)) \in F$, then $(y_2, n^{k-1}(y_2)) \notin F$. It is easy to construct the cycles as required.

Case 3.2. $s = x$.

Claim 2. If $|F_0| \leq 4n-8$, then e lies in cycles of even lengths from 6 to k^n.

Case 3.2.1. $k^n - 2k^{n-1} + 8 \leq l \leq k^n$.

As $|\mathcal{F}^0| \leq (4n-5)-(2n-3) = 2n-2$, we may choose an edge $(s, y_1) \in F_0$ such that either $(y_1, n^1(y_1)) \notin \mathcal{F}^0$ or $(y_1, n^{k-1}(y_1)) \notin \mathcal{F}^0$. Set $F_0' = F_0 \setminus \{(s, y_1)\}$. Then $|F_0'| \leq 4n-9$. By the induction hypothesis, there is a hamiltonian sy_1-path P_1^1 of $Q[0]-F_0'$. So we can construct the required cycles easily.

Case 3.2.2. $6 \leq l \leq k^n - 2k^{n-1} + 6$.

Choose a healthy edge (s, y_2) of $Q[0]$. If $(y_2, n^1(y_2)) \notin F$, then we may obtain the cycles as required.

Case 3.2.2.1. $(y_2, n^1(y_2)) \in F$ and $(t, n^1(y_2)) \notin F$.

Suppose first that $(n^1(y_2), n^2(y_2)) \notin F$. If there is an edge $(y_2, y_3) \in E(Q[0]-s)$ such that both $\langle y_2, y_3, n^1(y_3), n^1(y_2)\rangle$ and $\langle n^1(y_3), n^2(y_3), n^2(y_2)\rangle$ are healthy, then the required cycles can be constructed. Otherwise, for any faulty edge $(s, y_4) \in F_0$, $\langle s, n^{k-1}(s), n^{k-1}(y_4), y_4, n^1(y_4), t, s\rangle$ is a healthy 6-cycle containing e because $4n-5 = (2n-3) + |\{(y_2, n^1(y_2))\}| + d_{Q[0]-s}(y_2)$. Thus it is easy to verify that e lies in a cycle of every even length from 6 to $k^n - 2k^{n-1} + 6$.

Suppose next that $(n^1(y_2), n^2(y_2)) \in F$. Since $2n-3 > (4n-5)-(2n-3)-|\{(y_2, n^1(y_2)), (n^1(y_2), n^2(y_2))\}|$, we may choose an edge $(s, y_5) \in F_0$ such that $\langle n^{k-1}(s),\ n^{k-1}(y_5), y_5, n^1(y_5), t\rangle$ is healthy. If $(s, n^{k-1}(s)) \notin F$, then, it is easy to construct the cycles as required. Suppose that $(s, n^{k-1}(s)) \in F$. As $(t, n^1(y_2))$ lies in $2n-4$ 4-cycles in $Q[1]$ and $2n-4 > (4n-5)-(2n-3)-|\{(y_2, n^1(y_2)), (n^1(y_2), n^2(y_2)), (s,\ n^{k-1}(s))\}| = 2n-5$, we may choose a cycle $\langle t, z_1, z_2, n^1(y_2), t\rangle$ of $Q[1]$ such that both $\langle t, z_1, z_2, n^0(z_2), y_2\rangle$ and $\langle z_1, n^2(z_1), n^2(z_2), z_2\rangle$ are healthy. Clearly, we may obtain a cycle containing e of every even length from 6 to $k^n - 2k^{n-1} + 6$.

Case 3.2.2.2. $(y_2, n^1(y_2)), (t, n^1(y_2)) \in F$.

Since $2n-3 > (4n-5)-(2n-3)-|\{(y_2, n^1(y_2)), (t, n^1(y_2))\}|$, we may choose an edge $(s, y_6) \in F_0$ such that $\langle n^{k-1}(s), n^{k-1}(y_6), y_6, n^1(y_6), t\rangle$ is healthy. Using a similar proof as the second paragraph of Case 3.2.2.1, we may construct the cycles as required. Therefore, this claim holds.

By Claim 2, e lies in a cycle of every even length from 6 to $k^n - 2$. □

Lemma 5.11. *If there exists a vertex $x \in V(Q[0])$ which is incident with $2n-3$ edges in F_0 and $|F_0| = 4n-8$, then e lies in healthy cycles of even lengths from 6 to k^n.*

Proof. Clearly, $|\mathcal{F}^0| = 3$ and $|F_r| = 0$, for $r = 1, 2, \ldots, k-1$. Recall that $e = (s, t)$.

Case 1. $(s, t) \in E(Q[0])$.

If $x \in \{s, t\}$, then, by Claim 1 in Lemma 5.10, e lies in a cycle of every even length from 6 to k^n. Now we only consider the case that $x \notin \{s, t\}$. Then we may choose two healthy edges $(s, s'), (t, t') \in E(Q[0])$ such that $s' \neq t$ and $t' \neq s$. Similar to Case 1 of Lemma 5.8, we can construct healthy cycles containing e of even lengths from 6 to $k^n - 2k^{n-1} + 6$.

Recall that F is a conditional faulty edge set of Q_n^k. Then either $(x, n^1(x)) \notin F$ or $(x, n^{k-1}(x)) \notin F$. Without loss of generality, we assume that $(x, n^1(x)) \notin F$. As $2n-3 \geq |\mathcal{F}^0| = 3$, we may choose an edge $(x, y) \in F_0$ such that either $(y, n^1(y)) \notin F$ or $(y, n^{k-1}(y)) \notin F$. Set $F_0' = F_0 \setminus \{(x, y)\}$. Then $|F_0'| = 4n - 9$. By the induction hypothesis, $e = (s, t)$ lies in a hamiltonian C^1 of $Q[0] - F_0'$. Clearly, $(x, y) \in E(C^1)$. It is easy to verify that there exists a healthy cycle containing e of every even length from $k^n - 2k^{n-1} + 8$ to k^n.

Case 2. $(s, t) \in E(Q[1, k-1])$.

Lemma 5.2 implies that e lies in a cycle of every even length from 6 to $k^n - k^{n-1}$. Note that either $(x, n^1(x)) \notin F$ or $(x, n^{k-1}(x)) \notin F$. Without loss of generality, we assume that $(x, n^1(x)) \notin F$.

If there exists an edge $(x, y) \in F_0$ such that $(y, n^1(y)) \notin F$, set $F_0' = F_0 \setminus \{(x, y)\}$. Then $|F_0'| = 4n - 9$. By the induction hypothesis, (x, y) lies in a hamiltonian cycle C^1 of $Q[0] - F_0'$. Similar to the Case 2 of Lemma 5.8, it is easy to verify that e lies in a cycle of every even length from $k^n - k^{n-1} + 2$ to k^n. Suppose that $(y, n^1(y)) \in F$ for each edge $(x, y) \in F_0$. Recall that $2n - 3 \geq |\mathcal{F}^0| = 3$. Then each 0-dimensional edge between $Q[0]$ and $Q[k-1]$ is healthy. Thus $(x, n^{k-1}(x)), (y, n^{k-1}(y)) \notin F$ for every edge $(x, y) \in F_0$. Similar to the above argument, we may obtain the cycles as required.

Case 3. $s \in V(Q[0])$ and $t \in V(Q[1]) \cup V(Q[k-1])$.

Without loss of generality, we assume that $s \in V(Q[0])$ and $t \in V(Q[1])$.

Claim 1. There exists an edge $(x, y) \in F_0$ such that (x, y) is adjacent to at most 1 edge in $\mathcal{F}^0$.

Recall that F is a conditional faulty edge set of Q_n^k. Then x is incident with at most 1 edge in $\mathcal{F}^0$. Suppose first that $(x, n^1(x)), (x, n^{k-1}(x)) \notin F$. There exists an edge $(x, y) \in F_0$ such that y is incident with at most 1 edge in $\mathcal{F}^0$ because $2n - 3 \geq |\mathcal{F}^0| = 3$. So (x, y) is adjacent to at most 1 edge in $\mathcal{F}^0$. Suppose next that x is incident with exactly 1 edge in $\mathcal{F}^0$. Then there exists an edge $(x, y) \in F_0$ such that y is not incident with any edge in $\mathcal{F}^0$ because $2n - 3 > |\mathcal{F}^0| - 1 = 2$. Thus (x, y) is adjacent to at most 1 edge in $\mathcal{F}^0$. Therefore, this claim holds.

If $s = x$, then, by Claim 2 in Lemma 5.10, e lies in a cycle of every even length from 6 to k^n. It remains to consider the case that $s \neq x$. Set $\mathcal{P} = \{\langle s, s_1, n^1(s_1), t\rangle : (s, s_1) \in E(Q[0] - x)\}$. Then $|\mathcal{P}| \geq 2n - 3$.

Case 3.1. There exists a path $\langle s, s_1, n^1(s_1), t\rangle \in \mathcal{P}$ such that $(s, s_1), (s_1, n^1(s_1)) \notin F$.

Lemma 3.12(ii) implies that there is an $n^1(s_1)t$-path P_1^1 in $Q[1, k-1] - F$ of every odd length from 3 to $k^n - k^{n-1} - 1$. Then $\langle s, s_1, n^1(s_1), P_1^1, t, s\rangle$ is a cycle containing e of every even length from 6 to $k^n - k^{n-1} + 2$.

By Claim 1, there exists an edge $(x, y) \in F_0$ such that (x, y) is adjacent to at most 1 edge in $\mathcal{F}^0$. Set $F_0' = F_0 \setminus \{(x, y)\}$. Then $|F_0'| = 4n - 9$. By the induction hypothesis, (s, s_1) lies in a hamiltonian cycle C_1^1 of $Q[0] - F_0'$. Clearly, $(x, y) \in E(C_1^1)$ and $x \notin \{s, s_1\}$.

Case 3.1.1. $(x, n^1(x)), (y, n^1(y)) \notin F$.

If $y \notin \{s, s_1\}$, then (x, y) and (s, s_1) are nonadjacent. Then we can verify that e lies in a healthy cycle of every even length from $k^n - k^{n-1} + 4$ to k^n in $C_1^1 \cup Q[1, k-1] + \{(s, t), (s_1, n^1(s_1)), (x, n^1(x)), (y, n^1(y))\}$.

If $y = s$, set $F_0' = F_0 \setminus \{(x, s)\}$. Then $|F_0'| = 4n - 9$. By the induction hypothesis, (x, s) lies in a hamiltonian cycle C_2^1 of $Q[0] - F_0'$. So we can construct a cycle containing e of every even length from $k^n - k^{n-1} + 4$ to k^n.

If $y = s_1$, then $(s, s_1) = (s, y)$ and $s \notin \{x, y\}$. Choose an edge (s, s_2) of $C_1^1 - (s, y)$. Clearly, (x, y) and (s, s_2) are nonadjacent. Suppose first that either $(s_2, n^1(s_2)) \notin \mathcal{F}^0$ or $(s_2, n^{k-1}(s_2)) \notin \mathcal{F}^0$. Similar to Case 3.2.2.1 of Lemma 5.8, we can construct the cycle as required. Suppose next that $(s_2, n^1(s_2)), (s_2, n^{k-1}(s_2)) \in \mathcal{F}^0$. As $y = s_1$, we have that $\delta(x, s) = 0$. Then $|\mathcal{P}| = 2n - 2$. So $|\mathcal{P}| - |\{s_1, s_2\}| = 2n - 4 > |\mathcal{F}^0| - |\{(s_2, n^1(s_2)), (s_2, n^{k-1}(s_2))\}| = 1$. Thus, we may choose an edge (s, s_3) of $Q[0] - \{s_1, s_2\}$ such that $(s_3, n^1(s_3)) \notin \mathcal{F}^0$. Since $y = s_1$, it follows that $y \notin \{s, s_3\}$. Noting that $\delta(x, s) = 0$, we have that $x \neq s_3$. Recall that $x \neq s$. Therefore, (x, y) and (s, s_3) are nonadjacent. Similar to the case that $y \notin \{s, s_1\}$, we may obtain the cycles as required.

Case 3.1.2. $\{(x, n^1(x)), (y, n^1(y))\} \cap F \neq \emptyset$.

In this case, $(x, n^{k-1}(x)), (y, n^{k-1}(y)) \notin F$. Note that $(s, t), (s_1, n^1(s_1)) \notin F$. Clearly, using the cycle C_1^1, we can obtain the cycle as required.

Case 3.2. There is not any path $\langle s, s_1, n^1(s_1), t\rangle \in \mathcal{P}$ such that $(s, s_1), (s_1, n^1(s_1)) \notin F$.

Clearly, $\mathcal{P}$ contains at least $2n - 3$ faulty edges. In each case, we will construct a cycle of every even length l with $6 \leq l \leq k^n$ that contains e in $Q_n^k - F$.

Case 3.2.1. $6 \leq l \leq k^n - 2k^{n-1} + 6$.

Note that s is incident with at least $(2n-2)-[(4n-8)-(2n-3)]-|\{x\}|=2$ healthy edges which are not incident with x in $Q[0]$. Then we may choose two healthy edge $(s,s_2),(s,s_3)$ of $Q[0]$ such that $x\notin\{s_2,s_3\}$. Clearly, $(s_2,n^1(s_2)),(s_3,n^1(s_3))\in F$. So there is at most $|\mathcal{F}^0|-2=1$ faulty 0-dimensional edge between $Q[1]$ and $Q[2]$. Therefore we may choose a vertex $s_4\in\{s_2,s_3\}$ such that $(n^1(s_4),n^2(s_4))\notin F$. Observe that $|F|-(2n-3)-|F\cap E(\mathcal{P})|\le 1<d_{Q[0]}(s_4)-|\{x,s\}|=2n-4$. We may choose an edge (s_4,s_5) of $Q[0]-\{x,s\}$ such that $\langle s_4,s_5,n^1(s_5),n^2(s_5)\rangle$ is healthy. Clearly, we may obtain the cycles as required.

Case 3.2.2. $k^n-2k^{n-1}+8\le l\le k^n$.

By Claim 1, there exists an edge $(x,y)\in F_0$ such that (x,y) is adjacent to at most 1 edge in $\mathcal{F}^0$.

Suppose first that there is a neighbor s_6 of s in $Q[0]-\{x,y\}$ such that $(s_6,n^1(s_6))\notin F$. Set $F_0'=F_0\setminus\{(x,y),(s,s_6)\}$. Then $|F_0'|\le 4n-9$. By the induction hypothesis, (s,s_6) lies in a hamiltonian C^3 of $Q[0]-F_0'$. Clearly, $(x,y)\in E(C^3)$. Clearly, $s_6\notin\{x,y\}$. Similar to Case 3.1.1, we may obtain the cycles as required.

Suppose next that $(s_6,n^1(s_6))\in F$ for every neighbor s_6 of s in $Q[0]-\{x,y\}$. Note that there are at least $2n-3$ neighbors of s in $Q[0]-\{x,y\}$. Combining this with the fact that $2n-3\ge|\mathcal{F}^0|=3$, it is easy to see that $F\cap E(Q[1,k-1])=\emptyset$ and each 0-dimensional edge between $Q[0]$ and $Q[k-1]$ is fault-free. So we may choose a neighbor s_7 of s in $Q[0]-\{x,y\}$ such that $(s_7,n^{k-1}(s_7))\notin F$. Set $F_0'=F_0\setminus\{(x,y),(s,s_7)\}$. Then $|F_0'|\le 4n-9$. By the induction hypothesis, (s,s_7) lies in a hamiltonian C^4 of $Q[0]-F_0'$. Clearly, $(x,y)\in E(C^1)$. It is easy to obtain required cycles in $C^4\cup Q[1,k-1]+\{(s,t),(s_7,n^{k-1}(s_7)),(x,n^{k-1}(x)),(y,n^{k-1}(y))\}$. ☐

Lemma 5.12. *If there exists a vertex $x\in V(Q[0])$ which is incident with $2n-2$ edges in F_0, then e lies in healthy cycles of even lengths from 6 to k^n.*

Proof. Clearly, $|F_r|\le(4n-5)-(2n-2)-3=2n-6$, for $r=1,2,\ldots,k-1$. Recall that $e=(s,t)$. Note that F is a conditional faulty edge set of Q_n^k. So $(x,n^1(x)),(x,n^{k-1}(x))\notin F$. Observe that there are at most $(4n-5)-(2n-2)=2n-3$ faulty edges outside $Q[0]$. Since $d_{Q[0]}(x)-(2n-3)=1$, there exists an edge (x,y) of $Q[0]$ such that both $\langle n^{k-1}(x),n^{k-1}(y),y,n^1(y),n^1(x)\rangle$ and $\{(n^{k-2}(y),n^{k-1}(y)),(n^1(y),n^2(y))\}$ are all fault-free. We distinguish three cases. In each case, we will construct a cycle of every even length l with $6\le l\le k^n$ that contains e in Q_n^k-F.

Case 1. $(s,t)\in E(Q[0])$.

Clearly, $x\notin\{s,t\}$. Similar to Case 1.1 of Lemma 5.10, there is a cycle containing e of every even length from 6 to k^n-2. We may choose an edge (x,y_1) of $Q[0]-y$ such that either $(y_1,n^1(y_1))\notin\mathcal{F}^0$ or $(y_1,n^{k-1}(y_1))\notin\mathcal{F}^0$ since $(2n-2)-|\{y\}|\ge|\mathcal{F}^0|$. Without loss of generality, we assume that $(y_1,n^{k-1}(y_1))\notin\mathcal{F}^0$. Set $F_0'=F_0\setminus\{(x,y),(x,y_1)\}$. Then $|F_0'|\le|F|-|\mathcal{F}^0|-2\le 4n-10$. By Lemma 3.17, there is a hamiltonian st-path P^3 of $Q[0]-F_0'$. Clearly, $(x,y),(x,y_1)\in E(C^1)$. Similarly, there is a hamiltonian $n^1(x)n^1(y)$-path P^4 of $Q[1,k-2]-F$ and there is a hamiltonian $n^{k-1}(x)n^{k-1}(y_1)$-path P^5 of $Q[k-1]-F$. Then $P^3\cup P^4\cup P^5+\{(s,t),(x,n^1(x)),(x,n^{k-1}(x)),(y,n^1(y)),(y_1,n^{k-1}(y_1))\}-\{(x,y),(x,y_1)\}$ is a cycle containing e of length k^n.

Case 2. $(s,t)\in E(Q[1,k-1])$.

Similar to Case 2 of Lemma 5.10, it is easy to verify that there exists a cycle containing e of every even length from 6 to k^n-2. Then we only need to construct a hamiltonian cycle containing e. Note that $(s,t)\in E(Q[1,k-2])$ or $(s,t)\in E(Q[2,k-1])$. Without loss of generality, we assume that $(s,t)\in E(Q[1,k-2])$.

Case 2.1. There is an edge (x,z) of $Q[0]-y$ such that $(z,n^1(z)),(z,n^{k-1}(z))\notin F$.

Set $F_0'=F_0\setminus\{(x,y),(x,z)\}$. By Theorem 1.2, there is a hamiltonian cycle C^1 of $Q[0]-F_0'$. Clearly, $(x,y),(x,z)\in E(C^1)$. Since $\delta(n^1(y),n^1(z))=0$ and $\delta(s,t)=1$, we have that $\{n^1(y),n^1(z)\}\ne\{s,t\}$. If $n^1(z)\notin\{s,t\}$, then $\{n^1(x),n^1(z)\}\ne\{s,t\}$. Lemma 5.4 implies that there is a hamiltonian st-path P^3 of $Q[1,k-2]-(F\setminus\{(n^1(x),n^1(z))\})$ passing through $(n^1(x),n^1(z))$. By Lemma 3.17, there is a hamiltonian $n^{k-1}(x)n^{k-1}(y)$-path P^4 of $Q[k-1]-F$. Thus $C^1\cup P^3\cup P^4+\{(s,t),(x,n^1(x)),(z,n^1(z)),(x,n^{k-1}(x)),(y,n^{k-1}(y))\}-\{(x,y),(x,z),(n^1(x),n^1(z))\}$ is a cycle containing e of length k^n. If $n^1(y)\notin\{s,t\}$, using a similar proof as above, we can obtain the cycle as required.

Case 2.2. For every edge (x,z') of $Q[0]-y$, either $(z',n^1(z'))\in\mathcal{F}^0$ or $(z',n^{k-1}(z'))\in\mathcal{F}^0$.

As $|\mathcal{F}^0|\le 2n-3$ and $d_{Q[0]}(x)-1=2n-3$, we have that for every edge (x,z') of $Q[0]-y$, $|\mathcal{F}^0\cap\{(z',n^1(z')),(z',n^{k-1}(z'))\}|=1$. Clearly, $F\cap Q[1,k-1]=\emptyset$. We may choose an edge (x,z) of $Q[0]-y$ such that $n^1(z) \notin \{s,t\}$ because $2n-2-1 \geq 3 > |\{s,t\}|$. Set $F_0' = F_0 \setminus \{(x,y),(x,z)\}$. By Theorem 1.2, there is a hamiltonian cycle C^2 of $Q[0]-F_0'$.

Suppose first that $(z, n^1(z)) \notin \mathcal{F}^0$. Observe that $\{n^1(x), n^1(z)\} \neq \{s,t\}$. Similar to Case 2.1, a hamiltonian cycle containing e of $Q_n^k - F$ can be constructed. Suppose next that $(z, n^1(z)) \in \mathcal{F}^0$. Then $(z, n^{k-1}(z)) \notin \mathcal{F}^0$. If $\{n^1(x), n^1(y)\} \neq \{s,t\}$, then we can use a similar construction as in Case 2.1. Suppose that $\{n^1(x), n^1(y)\} = \{s,t\}$. Note that $F \cap Q[1,k-1] = \emptyset$. Using the cycle C^2, it is easy to construct the cycle as required.

Case 3. $s \in V(Q[0])$ and $t \in V(Q[1]) \cup V(Q[k-1])$.

Case 3.1. $6 \leq l \leq k^n - 2$.

Without loss of generality, we assume that $s \in V(Q[0])$ and $t \in V(Q[1])$. Set $\mathcal{P} = \{\langle s_1, n^1(s_1), t\rangle : (s,s_1) \in E(Q[0]-x)\}$. Then $|\mathcal{P}| \geq 2n-3$. Suppose first that there is a path $\langle s_1, n^1(s_1), t\rangle \in \mathcal{P}$ such that $(s_1, n^1(s_1)), (t, n^1(s_1)) \notin F$. It easy to verify that there is a cycle containing e of every even length from 6 to k^n-2. Suppose next that every path in $\mathcal{P}$ contains at least 1 faulty edge. Note that there are at most $(4n-5)-(2n-2) = 2n-3$ faulty edges outside $Q[0]$ and $|\mathcal{P}| \geq 2n-3$. Then $|\mathcal{P}| = 2n-3$ and each path $\langle s_2, n^1(s_2), t\rangle \in \mathcal{P}$ contains exactly 1 faulty edge. We can observe that $(s,s_2), (n^1(s_2), n^2(s_2)) \notin F$, $|F_0| = 2n-2$ and $(s,x) \in E(Q[0])$.

If $(t, n^1(s_2)) \in F$, then $\langle t, s, s_2, n^1(s_2)\rangle$ is fault-free. We may easily verify that there exist the cycles as required. If $(t, n^1(s_2)) \notin F$, then $(s_2, n^1(s_2)) \in F$. Note that both $\langle s_2, s_3, n^1(s_3), n^1(s_2)\rangle$ and $\langle n^1(s_3), n^2(s_3), n^2(s_2), n^1(s_2)\rangle$ are healthy for every edge (s_2, s_3) of $Q[0]-\{s,x\}$. Then we can construct a cycle containing e of every even length from 6 to $k^n - 2k^{n-1}+6$. Recall that $(s,x) \in E(Q[0])$. Choose any edge $(x,y_1) \in E(Q[0]-s)$. $(y_1, n^{k-1}(y_1)) \notin F$ because $F \subset F_0 \cup E(\mathcal{P})$. Set $F_0' = F_0 \setminus \{(x,s),(x,y_1)\}$. Then $|F_0'| \leq |F| - |\mathcal{F}^0| - 2 \leq 4n-10 < 4n-9$. By Theorem 1.2 there is a hamiltonian cycle C^2 of $Q[0]-F_0'$. Clearly, $(x,s),(x,y_1) \in E(C^1)$. Therefore, we may easily verify that e lies in a cycle of every even length from $k^n - 2k^{n-1}+8$ to k^n-2.

Case 3.2. $l = k^n$.

Case 3.2.1. $\delta(s,x) = 1$.

Without loss of generality, we assume that $t \in V(Q[1])$. Suppose first that either $(z', n^1(z')) \in F$ or $(z', n^{k-1}(z')) \in F$ for every edge (x,z') of $Q[0]-y$. Note that $Q[0]-x$ is fault-free. By Lemma 3.14, there is a hamiltonian sy-path P_1^1 of $Q[0]-x$. By Lemma 3.17, there is a hamiltonian $n^1(x)t$-path P_2^1 of $Q[1,k-2]-F$ and there is a hamiltonian $n^{k-1}(x)n^{k-1}(y)$-path P_3^1 of $Q[k-1]-F$. Then $\langle s, P_1^1 - x, y, n^{k-1}(y), P_3^1, n^{k-1}(x), x, n^1(x), P_2^1, t, s\rangle$ is a cycle containing e of length k^n. Suppose next that there is an edge (x,z) of $Q[0]-y$ such that $(z, n^1(z)), (z, n^{k-1}(z)) \notin F$. Set $F_0' = F_0 \setminus \{(x,y),(x,z)\}$. Then $|F_0'| \leq 4n-10$. By Lemma 3.17, there is a hamiltonian sx-path P_4^1 of $Q[0]-F_0'$. Obviously, either $(x,y) \in E(P_4^1)$ or $(x,z) \in E(P_4^1)$. Without loss of generality, assume that $(x,y) \in E(P_1^1)$. Similar to the proof above, we can obtain the cycle as required.

Case 3.2.2. $\delta(s,x) = 0$.

Suppose first that $s = x$. Observe that $|\mathcal{F}^0| \leq (4n-5)-(2n-2) = 2n-3$. Then there is an edge (x,z_1) of $Q[0]-y$ such that either $(z_1, n^1(z_1)) \notin \mathcal{F}^0$ or $(z_1, n^{k-1}(z_1)) \notin \mathcal{F}^0$. Without loss of generality, we assume that $(z_1, n^{k-1}(z_1)) \notin \mathcal{F}^0$. Then $(y, n^1(y)), (z_1, n^{k-1}(z_1)) \notin \mathcal{F}^0$. It is easy to construct a cycle containing e of length k^n.

Suppose next that $s \neq x$.

Case 3.2.2.1. There is an edge (x,z_2) of $Q[0]-y$ such that $(z_2, n^1(z_2)), (z_2, n^{k-1}(z_2)) \notin \mathcal{F}^0$.

Without loss of generality, we assume that $t \in V(Q[1])$. Set $F_0' = F_0 \setminus \{(x,y),(x,z_2)\}$. By Lemma 3.17, there is a hamiltonian sy-path $\langle s, P_1^2, z_2, x, y\rangle$ of $Q[0]-F_0'$. As $s \neq x$, we have that $t \neq n^1(x)$. Lemma 5.4 implies that there is a hamiltonian $n^1(y)t$-path $\langle n^1(y), n^1(x), P_2^2, t\rangle$ of $Q[1]-F_1$.

If there exists an edge $(n^{k-1}(y), w)$ of $Q[k-1]-n^{k-1}(x)$ such that $(w, n^{k-2}(w)) \notin \mathcal{F}^0$, set $F_{k-1}' = F_{k-1} \setminus \{(n^{k-1}(y), w)\}$. By Lemma 5.4, there is a hamiltonian $n^{k-1}(x)$ $n^{k-1}(z_2)$-path $\langle n^{k-1}(x), n^{k-1}(y), w_1, P_{3(1)}^2, n^{k-1}(z_2)\rangle$ of $Q[k-1]-F_{k-1}'$. Let $P_3^2 = P_{3(1)}^2$ and $w_2 = w$. If for every edge $(n^{k-1}(y), w')$ of $Q[k-1]-n^{k-1}(x)$, $(w', n^{k-2}(w')) \in \mathcal{F}^0$, then $|\mathcal{F}^0| = 2n-3$ and so $|F_{k-1}| = 0$. We may choose a vertex w_1 of $Q[k-1]-n^{k-1}(x)$ such that $\delta(n^{k-1}(y), w_1) = 1$ and $(w_1, n^{k-2}(w_1)) \notin \mathcal{F}^0$. By Lemma 3.13, there is a hamiltonian $n^{k-1}(z_2)w_1$-path $P_{3(2)}^2$ of $Q[k-1]-\{n^{k-1}(x), n^{k-1}(y)\}$. Let $P_3^2 = P_{3(2)}^2$ and $w_2 = w_1$.

By Lemma 3.17, there is a hamiltonian $n^2(y)n^{k-2}(w_2)$-path P_4^2 of $Q[2, k-2] - F$. Then $\langle s, P_1^2, z_2, n^{k-1}(z_2), P_3^2, w_2\ , n^{k-2}(w_2), P_4^2, n^2(y), n^1(y), y, n^{k-1}(y),\ n^{k-1}(x), x, n^1(x), P_2^2,\ t, s\rangle$ is a cycle containing e of length k^n.

Case 3.2.2.2. For every edge (x, z') of $Q[0] - y$, either $(z', n^1(z')) \in \mathcal{F}^0$ or $(z', n^{k-1}(z')) \in \mathcal{F}^0$.

Clearly, $|F \cap E(Q[1, k-1])| = 0$ and $|F \cap \{(z', n^1(z')), (z', n^{k-1}(z'))\}| = 1$. Without loss of generality, we assume that there exists an edge (x, z_3) of $Q[0] - y$ such that $(z_3, n^{k-1}(z_3)) \notin \mathcal{F}^0$. Set $F_0' = F_0 \setminus \{(x, y), (x, z_3)\}$. By Lemma 3.17, there is a hamiltonian sy-path $\langle s, P_1^3, z_3, x, y\rangle$ of $Q[0] - F_0'$.

Suppose first that $t \in V(Q[1])$. Choose vertices $w_5 \in V(Q[1] - n^1(y))$ and $w_6 \in V(Q[k-1] - n^{k-1}(x))$ such that $\delta(w_5, t) = (w_6, n^{k-1}(y)) = 1$. Note that $t \neq n^1(x)$. By Lemma 3.13, there is a hamiltonian w_5t-path P_2^3 of $Q[1] - \{n^1(x), n^1(y)\}$ and there is a hamiltonian $w_6n^{k-1}(y)$-path P_3^3 of $Q[k-1] - \{n^{k-1}(x),\ n^{k-1}(z_3)\}$. By Lemma 3.17, there is a hamiltonian $n^2(w_5)n^{k-2}(w_6)$-path P_4^3 of $Q[2, k-2]$. Then $\langle s, P_1^3, z_3, n^{k-1}(z_3), n^{k-1}(x), x, n^1(x), n^1(y), y, n^{k-1}(y), P_3^3, w_6, n^{k-2}(w_6), P_4^3, n^2(w_5)\ , w_5, P_2^3, t, s\rangle$ is a cycle containing e of length k^n.

Suppose now that $t \in V(Q[k-1])$. By Lemma 5.4, there is a hamiltonian $n^{k-1}(z_3)t$-path P^4 of $Q[2, k-1]$ passing through $(n^{k-1}(x), n^{k-1}(y))$. Without loss of generality, we write P^4 as $\langle n^{k-1}(z_3), P_1^4, n^{k-1}(x), n^{k-1}(y), P_2^4, t\rangle$. By Lemma 3.17, there is a hamiltonian $n^1(x)n^1(y)$-path P^5 of $Q[1]$ Then $\langle s, P_1^3, z_3, n^{k-1}(z_3), P_1^4,\ n^{k-1}(x), x, n^1(x), P^5, n^1(y), y,\ n^{k-1}(y), P_2^4, t, s\rangle$ is a cycle containing e of length k^n. ☐

Lemma 5.13. *If $|F_r| = 2n-4$ for some $r \in \{1, 2, \ldots, k-1\}$, then e lies in healthy cycles of even lengths from 6 to k^n.*

Proof. Clearly, $|F_0| = 2n-4$, $|F_{r'}| = 0$ for every $r' \in \{0, 1, \ldots, k-1\} \setminus \{0, r\}$ and $|\mathcal{F}^0| = 3$. Recall that $e = (s, t)$. We distinguish three cases. In each case, we will construct a cycle of every even length l with $6 \le l \le k^n$ that contains e in $Q_n^k - F$.

Case 1. $(s, t) \in E(Q[0])$ or $(s, t) \in E(Q[r])$.

Without loss of generality, assume that $(s, t) \in E(Q[0])$. Furthermore, we assume that $k/2 \le r \le k-1$. Note that $2n-4 \le 4n-10$. It is easy to verify that e lies in a cycle of every even length from 6 to k^n.

Case 2. $(s, t) \in E(Q[1, r-1])$ or $(s, t) \in E(Q[r+1, k-1])$.

Without loss of generality, assume that $(s, t) \in E(Q[1, r-1])$. Then $r \ge 2$. Observe that $|F_{r'}| = 0$ for every $r' \in \{0, 1, \ldots, k-1\} \setminus \{0, r\}$ and $|\mathcal{F}^0| = 3$. We may may easily verify that there exist a cycle containing e of every even length from 6 to $(r-1)k^{n-1}$ in $Q[1, r-1] - F$.

Case 2.1. $(r-1)k^{n-1} + 2 \le l \le (r+1)k^{n-1}$.

By Lemma 5.3, there is a hamiltonian st-path P^3 of $Q[1, r-1] - F$ such that $|E(P^3 \cap Q[1])| \ge k^{n-1} - 2$ and $|E(P^3 \cap Q[r-1])| \ge k^{n-1} - 2$. Then we may choose edges $(u_1, v_1) \in E(P^3 \cap Q[r-1])$ and $(u_2, v_2) \in E(P^3 \cap Q[0])$ such that $\langle u_1, n^r(u_1), n^r(v_1), v_1\rangle$ and $\langle u_2, n^0(u_2), n^0(v_2), v_2\rangle$ are healthy. Observe that $2n-4 < 4n-9$. By induction, we can obtain a cycle containing e of every even length from $(r-1)k^{n-1} + 2$ to $(r+1)k^{n-1}$ in $Q[0, r] - F$.

Case 2.2. $(r+1)k^{n-1} + 2 \le l \le k^n$.

In this case $r \le k-2$. Lemma 5.3 implies that there is a hamiltonian st-path P^4 of $Q[0, r] - F$. It is easy to construct a cycle containing e of every even length from $(r+1)k^{n-1} + 2$ to k^n.

Case 3. $s \in V(Q[m])$ and $t \in V(Q[m+1]) \cup V(Q[m-1])$, where $m \in \{0, r\}$.

By symmetry, we assume that $s \in V(Q[0])$ and $t \in V(Q[1])$. Set $\mathcal{P} = \{\langle s, s_1, n^1(s_1), t\rangle : (s, s_1) \in E(Q[0])\}$. Then $|\mathcal{P}| = 2n-2$.

Case 3.1. $r \ge 2$.

In this case, $|F_1| = 0$. As $2n-2 > |\mathcal{F}^0| = 3$, there exists a path $\langle s, s_1, n^1(s_1), t\rangle \in \mathcal{P}$ such that $(s_1, n^1(s_1)) \notin \mathcal{F}^0$. Recall that $|F_1| = 0$. Thus $(t, n^1(s_1)) \notin F$. Then we may easily verify that $Q[0] \cup Q[1, r-1] + \{(s, t), (s_1, n^1(s_1))\}$ has a healthy cycle containing e of every even length from 6 to rk^{n-1} and $Q[2, 0] \cup Q[1] + \{(s, t), (s_1, n^1(s_1))\}$ has a healthy cycle containing e of every even length from $k^n - k^{n-1} + 4$ to k^n. Similar to Case 2.1, it is easy to verify that there is a cycle containing e from $rk^{n-1} + 2$ to $k^n - k^{n-1} + 2$ in $Q[0, k-1] - F$.

Case 3.2. $r = 1$.

If each path in $\mathcal{P}$ contains at least 2 faulty edge, then $2(2n-2) = 4n-4 \le |F| = 4n-5$, a contradiction. So there exists a path $P^0 = \langle s, s_1, n^1(s_1), t\rangle \in \mathcal{P}$ which contains at most 1 faulty edge.

Case 3.2.1. P^0 is healthy.

In this case, we can easily verify that e lies in a cycle of every even length from 8 to k^n. Then, it is enough to construct a 6-cycle that contains e.

If $\langle s, n^{k-1}(s), n^{k-1}(s_1), s_1\rangle$ is fault-free, then $\langle s, n^{k-1}(s), n^{k-1}(s_1), s_1, n^1(s_1), t, s\rangle$ is a 6-cycle containing e. If $\langle t, n^2(t), n^2(s_1), n^1(s_1)\rangle$ is fault-free, then $\langle s, s_1, n^1(s_1), n^2(s_1), n^2(t), t, s\rangle$ is a 6-cycle containing e.

Now we consider the case that both $\langle s, n^{k-1}(s), n^{k-1}(s_1), s_1\rangle$ and $\langle t, n^2(t), n^2(s_1), n^1(s_1)\rangle$ are faulty. Then $|F \cap E(Q[0,1])| \le (4n-5) - 2 = 4n-7$.

Suppose first that there is a path $\langle s, s_2, n^1(s_2), t\rangle \in \mathcal{P}$ such that $s_2 \ne s_1$ and $(s, s_2), (t,\ n^1(s_2)) \notin F$. Set $\mathcal{A} = \{\langle s_1, s_1', n^1(s_1'), n^1(s_1)\rangle : (s_1, s_1') \in E(Q[0]-s)\}$ and $\mathcal{B} = \{\langle s_2, s_2', n^1(s_2'), n^1(s_2)\rangle : (s_2, s_2') \in E(Q[0]-s)\}$. Then $|\mathcal{A}| = |\mathcal{B}| = 2n-3$. By the definition of Q_n^k, $\mathcal{A} \cup \mathcal{B}$ has at least $2(2n-3)-1 = 4n-7$ internally disjoint paths.

If there exists a healthy path P^5 in $\mathcal{A}$, then $\langle s, s_1, P^5, n^1(s_1), t, s\rangle$ is a 6-cycle containing e. If there exists a healthy path $P^{5\prime}$ in $\mathcal{B}$, then $\langle s, s_2, P^{5\prime}, n^1(s_2), t, s\rangle$ is a 6-cycle containing e. If each path in $\mathcal{A}\cup\mathcal{B}$ is faulty, then $\mathcal{A}\cup\mathcal{B}$ contains at least $4n-7$ faulty edges and so $F \cap E(Q[0,1]) \subset E(\mathcal{A}\cup\mathcal{B})$. Since $|\mathcal{P}| = 2n-2 > 2$, we may choose a fault-free path $\langle s, s_3, n^1(s_3), t\rangle \in \mathcal{P}$ such that $s_3 \notin \{s_1, s_2\}$. Set $\mathcal{D} = \{\langle s_3, s_3', n^1(s_3'), n^1(s_3)\rangle : (s_3, s_3') \in E(Q[0]-s)\}$. Observe that $|\mathcal{D}| = 2n-3 \ge 3 > 2$. We may choose a path $P^6 \in \mathcal{D}$ such that $P^6 \notin \mathcal{A}\cup\mathcal{B}$. Then P^6 is healthy. Therefore, $\langle s, s_3, P^6, n^1(s_3), t, s\rangle$ is a 6-cycle containing e.

Suppose next that every path $\langle s, s_2, n^1(s_2), t\rangle \in \mathcal{P}$ such that either $(s, s_2) \in F$ or $(t, n^1(s_2)) \in F$, where $s_2 \ne s_1$. Then $\mathcal{P}$ contains at least $|\mathcal{P}| - |\{s_1\}| = 2n-3$ faulty edges. Since $|F \cap E(Q[0,1])| - |F \cap E(\mathcal{P})| \le (4n-7) - (2n-3) = 2n-4$ and $|\mathcal{A}| = 2n-3 > 2n-4$, we may choose a healthy path $P^7 \in \mathcal{A}$. Then $\langle s, s_1, P^7, n^1(s_1), t, s\rangle$ is a 6-cycle containing e.

Case 3.2.2. $|F \cap E(P^0)| = 1$.

Suppose first that $(s, s_1) \in F$ or $(t, n^1(s_1)) \in F$. By symmetry, it is enough to consider the case that $(t, n^1(s_1)) \in F$. Set $F_1' = F_1 \setminus \{(t, n^1(s_1))\}$. Then $|F_1'| = 2n-5$. Clearly, we may obtain the cycles as required.

Suppose next that $(s, s_1), (t, n^1(s_1)) \notin F$. Then $(s_1, n^1(s_1)) \in \mathcal{F}^0$. Since $|\mathcal{P}| = 2n-2 > |\mathcal{F}^0| = 3$, we may choose a path $\langle s, s_2, n^1(s_2), t\rangle \in \mathcal{P}$ such that $(s_2, n^1(s_2)) \notin \mathcal{F}^0$. Clearly, $s_2 \ne s_1$. If $|F \cap \{(s, s_2), (t, n^1(s_2))\}| = 1$, then the proof is similar as above. If $|F \cap \{(s, s_2), (t, n^1(s_2))\}| = 0$, then $\langle s, s_2, n^1(s_2), t\rangle$ is healthy. Similar to the Case 3.2.1, we may obtain the cycles as required.

We only need to consider the case that $|F \cap \{(s, s_2), (t, n^1(s_2))\}| = 2$. Clearly, we may easily verify that there is a healthy cycle containing e of every even length from 8 to k^n in $Q[0] \cup Q[1, k-1] + \{(s,t), (s_2, n^1(s_2))\}$. Set $\mathcal{I} = \{\langle s, s', n^1(s'), t\rangle : s' \in V(Q[0] - \{s, s_1, s_2\})\}$ and $\mathcal{J} = \{\langle s_1, s_1', n^1(s_1'), n^1(s_1)\rangle : (s_1, s_1') \in E(Q[0]-s)\}$. Then $\mathcal{I}\cup\mathcal{J}$ has $|\mathcal{I}| + |\mathcal{J}| = (2n-4) + (2n-3) = 4n-7$ internally disjoint paths. Since $(4n-5) - |\{(s_1, n^1(s_1)), (s, s_2), (t, n^1(s_2))\}| = 4n-8 < 4n-7$, there exists a healthy path P^4 in $\mathcal{I}\cup\mathcal{J}$. If $P^4 \in \mathcal{I}$, then the proof is similar to the Case 3.2.1. If $P^4 \in \mathcal{J}$, then $\langle s, s_1, P^4, n^1(s_1), t, s\rangle$ is a 6-cycle containing e. □

By Lemmas 5.6–5.13, Theorem 5.1 holds.

It was proven in [3] that there are configurations of $4n-4$ faulty edges in Q_n^k ($n \ge 2$, even $k \ge 4$), so that even if every vertex is incident with at least 2 healthy edges, there does not exist a Hamiltonian cycle. Such a configuration is obtained by taking a cycle $\langle s, s', t', t, s\rangle$ in Q_n^k and ensuring that every edge in $(\partial_{Q_n^k}(s) \cup \partial_{Q_n^k}(t')) \setminus \{(s, s'), (s,t), (s', t'), (t', t)\}$ is faulty. This amounts to $4n-4$ faults. Clearly, there is no hamiltonian cycle in $Q_n^k - F$ containing (s,t). Consequently, the value of $4n-5$ for the number of faulty edges in Q_n^k in the statement of Theorem 5.1 is optimal.

6. Conclusions

In this paper, we investigate a problem on embedding cycles into k-ary n-cube Q_n^k with faulty edges and show that each healthy edge in Q_n^k ($n \ge 2$, even $k \ge 4$) lies in a fault-free cycle of every even length from 6 to $|V(Q_n^k)|$ inclusive for any conditional fault edge set F with $|F| \leq 4n-5$. The above result shows that the conditional fault-tolerant capability of the k-ary n-cube is nice in terms of the cycle embeddings, which can be used to study the building of terabit routers in the environment of k-ary n-cubes. Our further work is to study the fault panconnectivity of k-ary n-cubes with conditional edge-faults.

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