ON THE LENGTH OF UNCOMPLETABLE WORDS
IN UNAMBIGUOUS AUTOMATA

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Abstract. This paper deals with uncomplete unambiguous automata. In this setting, we investigate the minimal length of uncompletable words. This problem is connected with a well-known conjecture formulated by A. Restivo. We introduce the notion of relatively maximal row for a suitable monoid of matrices. We show that, if M is a monoid of $\{0,1\}$-matrices of dimension n generated by a set S, then there is a matrix of M containing a relatively maximal row which can be expressed as a product of $O(n^3)$ matrices of S. As an application, we derive some upper bound to the minimal length of an uncompletable word of an uncomplete unambiguous automaton, in the case that its transformation monoid contains a relatively maximal row which is not maximal. Finally we introduce the maximal row automaton associated with an unambiguous automaton $\mathcal{A}$. It is a deterministic automaton, which is complete if and only if $\mathcal{A}$ is. We prove that the minimal length of the uncompletable words of $\mathcal{A}$ is polynomially bounded by the number of states of $\mathcal{A}$ and the minimal length of the uncompletable words of the associated maximal row automaton.

1. Introduction

Let X be a set of words on an alphabet A and let X^* be its Kleene closure. We say that X is *complete* iff any word of A^* is a factor of some word belonging to X^*. If X is not complete, then any word which is factor of no word of X^* is said to be *uncompletable*.

The notion of uncompletable word plays an important role in several fields of Formal Language Theory and Theory of Codes. For instance, an uncompletable word is necessary to embed a regular code in a maximal regular one (see *e.g.*, [2]).

In [20], A. Restivo conjectured that a finite and uncomplete set X has always an uncompletable word whose length is quadratically bounded by the maximal length of the words of X (see also [3, 19, 21]). Some results related to this problem have been obtained in [10, 13–15, 18].

If one considers the more general case of regular sets, one is naturally led to study uncomplete automata. We say that a word is *uncompletable* in an automaton $\mathcal{A}$ iff it is not the label of any path in the graph of $\mathcal{A}$. The automaton $\mathcal{A}$ is *uncomplete* iff there exists an uncompletable word in $\mathcal{A}$. If X is a regular set, then the

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language X^* is accepted by a (trim) finite state automaton with a unique initial and terminal state. A word is uncompletable in X if and only if it is uncompletable in such an automaton.

We notice that a deterministic transitive automaton is uncomplete if and only if there are a state q and an input letter a such that the transition from q with input a is not defined.

As is well known, any uncomplete deterministic n-state automaton has an uncompletable word of length n^2. On the other side, no polynomial bound exists for the minimal length of an uncompletable word in a non-deterministic finite state automaton [11]. In the more general frame of $\mathbb{Z}$-automata, uncompleteness is undecidable. This is, indeed, a consequence of unsolvability of the mortality problem of matrices (see *e.g.*, [16]).

In this paper, we study uncompletable words of unambiguous finite state automata. For such a class of automata, no polynomial bound for the minimal length of an uncompletable word is known. However, at our knowledge, no unambiguous uncomplete automaton has ever been exhibited, whose shortest uncompletable word has length larger than n^2.

The problem we study in this article is strongly related to that of finding a synchronizing word of minimal length. This problem has several applications in Formal Language Theory and Information Theory. Indeed, synchronizing words have the property of 'resetting' the automaton to a prefixed state, for instance after an error. Thus, it is useful to dispose of a synchronizing word, possibly of small length, in order that eventual previous errors in the input sequence do not affect the successive part of the corresponding output sequence. In this direction, there have been several studies in the literature, related to the famous Černý conjecture introduced in [8] (see *e.g.*, [1, 5–7, 12, 17, 18] for the deterministic case and [4, 5] for the unambiguous case). In [7] several results are proved, concerning some relations between the minimal length of the uncompletable words of a finite set and that of synchronizing word of a finite code.

An unambiguous n-state automaton on the alphabet A can be identified with a morphism $\phi_{\mathcal{A}}$ of the free monoid A^* onto a monoid of $n \times n$ matrices whose entries are 0 or 1 (briefly, $\{0,1\}$-matrices). In the case where the automaton is complete, a characterization of the minimal ideal of such a monoid, in terms of maximality of rows and columns of the matrices, has been given in [9]. This property is an essential tool for constructing synchronizing words of complete synchronizing unambiguous automata of polynomially bounded length (see *e.g.*, [4]).

In this paper, we introduce the notion of relatively maximal row (and column) of a matrix in a monoid of $\{0,1\}$-matrices. This notion is strictly weaker than that of maximality. We show that, if $\mathcal{A}$ is an n-state transitive unambiguous automaton, there is a word w of length $|w| \leq n(n-1)^2/2$ such that $\phi_{\mathcal{A}}(w)$ contains a relatively maximal row and a relatively maximal column.

Successively we prove that, if w is a word such that $\phi_{\mathcal{A}}(w)$ contains a row, which is relatively maximal, then there is an uncompletable word of length less than or equal to $2(n-1)(|w| + n - 1)$, where $|w|$ denotes the length of w. This result includes, as a particular case, the setting of deterministic automata.

Next, we introduce the *maximal row automaton* associated with an unambiguous automaton $\mathcal{A}$. It is a deterministic automaton which is uncomplete if and only if $\mathcal{A}$ is uncomplete. We establish a connection between the minimal lengths of the uncompletable words of $\mathcal{A}$ and of the associated maximal row automaton.

2. Preliminaries

Let A be a finite set, and A^* be the free monoid generated by A. The set A is called *alphabet*. The elements of A and A^* are called *letters* and *words*, respectively. We denote by ε the *empty word*, that is the neutral element of A^*. The *length* of a word w, that is the number of letters occurring in it, is denoted by $|w|$.

An *automaton* is a triple (A, Q, δ), where $A = \{\alpha_1, \alpha_2, \ldots, \alpha_h\}$ is the *input alphabet*, $Q = \{q_1, q_2, \ldots, q_n\}$ is the *set of the states*, $\delta : Q \times A \to \mathcal{P}(Q)$ is the *transition function*. An automaton $\mathcal{A}$ is said to be *deterministic* iff $\mathrm{Card}(\delta(q, \alpha)) \leq 1$ for each $\alpha \in A$ and $q \in Q$. We associate to the automaton $\mathcal{A}$ the *monoid morphism* $\phi_{\mathcal{A}} : (A^*, \circ) \to (\mathbb{N}^{Q \times Q}, \cdot)$, defined by

$$(\phi_{\mathcal{A}}(\alpha))_{i,j} = \begin{cases} 1, & \text{if } q_j \in \delta(q_i, \alpha), \\ 0, & \text{otherwise,} \end{cases}$$

where A^* and $\mathbb{N}^{Q \times Q}$ are equipped with the concatenation product and the row-column product of matrices, respectively. Note that $\phi_{\mathcal{A}}(\varepsilon)$ is equal to the identity matrix I_n. A word $w \in A^*$ is said to be *uncompletable* in A iff $\phi_{\mathcal{A}}(w) = 0$ (the null matrix). This is equivalent to saying that w is not the label of any path in the graph of the automaton $\mathcal{A}$. We say that an unambiguous automaton is *complete* iff no word w of A^* is uncompletable.

We denote by $\{0,1\}^{Q \times Q}$ the set of $n \times n$ matrices whose entries are all 0 or 1. An automaton is said to be *unambiguous* iff $\phi_{\mathcal{A}}(A^*) \subset \{0,1\}^{Q \times Q}$.

For every $m \in \{0,1\}^{Q \times Q}$ and $i, j = 1, 2, \ldots, n$, the symbols m_{i*} and m_{*j} indicate the ith row and the jth column of m, respectively. A nonempty subset $M \subset \{0,1\}^{Q \times Q}$ is said to be *transitive* iff for each $i, j = 1, \ldots, n$ there is an element $m \in M$ with $m_{i,j} = 1$.

An automaton $\mathcal{A}$ is *transitive* iff $\phi_{\mathcal{A}}(A^*)$ is transitive. This is equivalent to say that the underlying graph of the automaton $\mathcal{A}$ is strongly connected.

In the sequel, we assume that all considered automata are transitive, unless differently stated. Observe that actually this is not a restriction. Indeed, if $\mathcal{A}$ is an uncomplete automaton, then the strongly connected components of its graph are graphs of uncomplete transitive automata. Moreover, concatenating in a suitable order uncompletable words of such 'components', one obtains an uncompletable word of $\mathcal{A}$. Thus, the minimal length of an uncompletable word of $\mathcal{A}$ is not larger than the sum of the minimal lengths of the uncompletable words of its components.

Given any two $n \times n$ matrices $A = (A_{i,j})$ and $B = (B_{i,j})$ with non-negative entries, we say that $A \leq B$ iff $A_{i,j} \leq B_{i,j}$ for every $i, j = 1, 2, \ldots, n$. If $A \leq B$ and $A \neq B$, then we write $A < B$. It is not difficult to see that, if $A \leq B$ and $C \geq 0$, then $AC \leq BC$ and $CA \leq CB$. Moreover, for any arbitrary $\{0,1\}$-matrix A and for every $\{0,1\}$-matrix B with coefficient 1 in position (i,j) one has $(BA)_{i*} \geq A_{j*}$.

Let M be a monoid of $(0,1)$-matrices. We say that a vector $\mathbf{a} = (a_1 \, a_2 \ldots a_n)$ of a matrix of M is a *maximal row* of M iff it is a maximal element in the set of the rows of the matrices of M, with respect to the ordering defined above.

A row $\mathbf{a} = (a_1 \, a_2 \ldots a_n) = A_{j*}$ of a matrix of $A \in M$ is said to be *relatively maximal* iff for every $B \in M$ and for each $i = 1, \ldots, n$ with $B_{i,j} = 1$, one has $(BA)_{i*} = A_{j*} = \mathbf{a}$. Roughly speaking, the row A_{j*} is relatively maximal if, in the computation of the products BA, such a row is never summed to another non-null row of A. We notice that, in general, if $A, B \in M$ are such that $B_{i,j} = 1$, then one has $(BA)_{i*} \geq A_{j*}$. Thus, this inequality is strict if and only if A_{j*} is not a relatively maximal row of A.

Maximal and relatively maximal columns are defined symmetrically.

It is trivial that every maximal row is relatively maximal too, while the converse, in general, is not true, as the next example shows.

Example 2.1. Let M be the monoid generated by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We prove that the first row of A, namely $(0\,1\,0)$, is relatively maximal. Observe that, to this aim, it is enough to show that for each $C \in M$ and $k = 1, 2, 3$, one has $(CA)_{k*} \not> (0\,1\,0)$, namely

$$(CA)_{k*} \notin \{(1\,1\,0), (0\,1\,1), (1\,1\,1)\}. \tag{2.1}$$

Easy calculations show that

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = A;$$

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad BA^2 = B^2 A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$ABA = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \le A, \quad AB^2 A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \le A^2; \quad B^3 A = BA.$$

Thus, for every $C \in M$, any row of CA is bounded from above by some row of one of the matrices above, and therefore it contains at most one non-null entry, getting (2.1). However, the row $(0\,1\,0)$ is not maximal, since $(0\,1\,1) > (0\,1\,0)$, and $(0\,1\,1)$ is the first row of B.

Furthermore, note that the monoid M contains the null matrix, since $AB^2AB = 0$. In fact, as a consequence of a result of [9], it would be possible to prove that if a monoid of $\{0, 1\}$-matrices does not contain the null matrix, then all relatively maximal rows are maximal rows.

Remark 2.2. Observe that if $\mathcal{A}$ is a deterministic automaton, then each row of every element of $\phi_{\mathcal{A}}(A)$ contains at most one 1 and has all other entries equal to 0. It is not difficult to show that every non-zero row is maximal and every identically null row is relatively maximal.

Now we recall some results which will be useful in the sequel. The first statement of the next proposition conveys the fact that the length of the shortest path connecting two states in the graph of a transitive n-state automaton is at most $n - 1$.

Proposition 2.3. *(see e.g., ([4], Prop. 2.1) Let $\mathcal{A}$ be an unambiguous n-state automaton and $i, j = 1, \ldots, n$.*

1. *There is a word $w \in A^*$ such that $(\phi_{\mathcal{A}}(w))_{i,j} = 1$ and $|w| \le n - 1$.*
2. *If there are $u \in A^*$ and $h \in \{1, \ldots, n\}$ with $(\phi_{\mathcal{A}}(u))_{h,i} = (\phi_{\mathcal{A}}(u))_{h,j} = 1$, then there exist $v \in A^*$ and $k \in \{1, \ldots, n\}$ such that $(\phi_{\mathcal{A}}(v))_{k,i} = (\phi_{\mathcal{A}}(v))_{k,j} = 1$ and*

$$|v| \le \frac{1}{2}n(n - 1).$$

Proposition 2.4. *Let $\mathcal{A}$ be an n-state automaton and $w \in A^*$ be such that $\phi_{\mathcal{A}}(w)$ contains a relatively maximal row $\mathbf{b} = (\phi_{\mathcal{A}}(w))_{i*}$. Then, for any non-null row $\mathbf{a}$ of $\phi_{\mathcal{A}}(A^*)$, there is a word $v \in A^*$ with $|v| \le n - 1$ and $\mathbf{a}\,\phi_{\mathcal{A}}(vw) = \mathbf{b}$.*

Proof. Let $\mathbf{a} = (a_1\,a_2 \ldots a_n)$. By hypothesis, there is $j \in \{1, \ldots, n\}$ with $a_j = 1$. By Condition (1) of Proposition 2.3, there exists a word $v \in A^*$ with $|v| \le n - 1$ and $(\phi_{\mathcal{A}}(v))_{j,i} = 1$. Thus,

$$\mathbf{a}\,\phi_{\mathcal{A}}(vw) \ge (\phi_{\mathcal{A}}(w))_{i*} = \mathbf{b}.$$

Since $\mathbf{b}$ is relatively maximal, it follows that $\mathbf{a}\,\phi_{\mathcal{A}}(v\,w) = \mathbf{b}$, getting the assertion. $\square$

3. The main results

We begin with the following technical proposition, which extends ([4], Lem. 4.2) to our context.

Proposition 3.1. *Let $\mathcal{A}$ be an n-state unambiguous automaton, $u \in A^*$ be a word with $\phi_{\mathcal{A}}(A) \ne \{0\}$, and $\mathbf{a}$ be a row of $\phi_{\mathcal{A}}(u)$, not relatively maximal. Then there exists a word $z \in A^*$ of length less than or equal to $n(n - 1)/2$, such that $\phi_{\mathcal{A}}(zu)$ has a row $\mathbf{a}'$ with $\mathbf{a}' > \mathbf{a}$.*

Proof. Let $\mathbf{a} = (\phi_{\mathcal{A}}(u))_{j*}$. Since $\mathbf{a}$ is not relatively maximal, there exist a word $w \in A^*$ and $i \in \{1, 2, \ldots, n\}$ with $(\phi_{\mathcal{A}}(w\,u))_{i*} > (\phi_{\mathcal{A}}(u))_{j*}$ and $(\phi_{\mathcal{A}}(w))_{i,j} = 1$. Now we claim that there exists $j' \ne j$ with $(\phi_{\mathcal{A}}(w))_{i,j'} = 1$

and $(\phi_{\mathcal{A}}(u))_{j'*} \neq \mathbf{0}$. Otherwise, for every $j' \neq j$ either $(\phi_{\mathcal{A}}(w))_{i,j'} = 0$ or $(\phi_{\mathcal{A}}(u))_{j'*} = \mathbf{0}$, and a not difficult calculation shows that $(\phi_{\mathcal{A}}(wu))_{i*} = (\phi_{\mathcal{A}}(u))_{j*}$, a contradiction. By Condition (2) of Proposition 2.3, there is a word $z \in \phi_{\mathcal{A}}(A^*)$ with $(\phi_{\mathcal{A}}(z))_{i,j} = (\phi_{\mathcal{A}}(z))_{i,j'} = 1$ and $|z| \leq n(n-1)/2$. Let now $\mathbf{a}' = (\phi_{\mathcal{A}}(zu))_{i*}$. One has

$$
\begin{aligned}
(\phi_{\mathcal{A}}(zu))_{i*} &= \sum_{l=1}^{n}(\phi_{\mathcal{A}}(z))_{i,l}(\phi_{\mathcal{A}}(u))_{l*} \\
&\geq (\phi_{\mathcal{A}}(z))_{i,j}(\phi_{\mathcal{A}}(u))_{j*} + (\phi_{\mathcal{A}}(z))_{i,j'}(\phi_{\mathcal{A}}(u))_{j'*} = \mathbf{a} + (\phi_{\mathcal{A}}(u))_{j'*}.
\end{aligned}
$$

Since $(\phi_{\mathcal{A}}(u))_{j'*} \neq \mathbf{0}$, for at least an index $l \in \{1,\dots,n\}$ we get $(\phi_{\mathcal{A}}(zu))_{i,l} = a_l + 1 > a_l$, namely $\mathbf{a}' > \mathbf{a}$. This ends the proof. $\qquad\square$

Remark 3.2. Note that an analogous of Proposition 3.1 holds even if it is dealt with columns instead of rows.

The next result deals with the existence of relatively maximal rows and columns of small length and with a common upper bound, and extends ([4], Prop. 4.3).

Proposition 3.3. *Let $\mathcal{A}$ be an n-state unambiguous automaton. Then there are two words $u, v \in A^*$ such that $\phi_{\mathcal{A}}(u)$ (resp., $\phi_{\mathcal{A}}(v)$) contains a relatively maximal row $\mathbf{a} = (a_1\, a_2 \dots a_n)$ (resp., column $\mathbf{b} = (b_1\, b_2 \dots b_n)$), $a_1 = b_1 = 1$, and*

$$|uv| \leq \frac{1}{2}n(n-1)^2. \tag{3.1}$$

Proof. By iterated applications of Proposition 3.1, one can obtain words u, v of length less than or equal to $n(n-1)^2/2$, such that $\phi_{\mathcal{A}}(u)$ (resp., $\phi_{\mathcal{A}}(v)$) contains a relatively maximal row (resp., column). In order to find words u, v satisfying the stronger inequality (3.1), we need a more elaborate argument.

We construct two finite sequences of words of A^*, $u_1, u_2, \dots, u_t$ and $v_1, v_2, \dots, v_s$, and corresponding rows $\mathbf{a}^{(i)}$ of $\phi_{\mathcal{A}}(u_i)$ and columns $\mathbf{b}^{(j)}$ of $\phi_{\mathcal{A}}(v_j)$.

At the first step, put $u_1 = \varepsilon$, $\mathbf{a}^{(1)} = (1\, 0 \dots 0) = (\phi_{\mathcal{A}}(u_1))_{1*}$. Now suppose we have found words $u_1, u_2, \dots u_i$ and rows $\mathbf{a}^{(1)} < \mathbf{a}^{(2)} < \dots < \mathbf{a}^{(i)}$ of $\phi_{\mathcal{A}}(u_1), \phi_{\mathcal{A}}(u_2), \dots, \phi_{\mathcal{A}}(u_i)$ respectively, such that $|u_i| \leq \dfrac{i-1}{2}n(n-1)$. If $\mathbf{a}^{(i)}$ is relatively maximal, then we take $t = i$ and stop. Otherwise, we apply Proposition 3.1 and Remark 3.2, and we find a word $u_{i+1} \in A^*$ and a row $\mathbf{a}^{(i+1)}$ of $\phi_{\mathcal{A}}(u_{i+1})$, such that $\mathbf{a}^{(i+1)} > \mathbf{a}^{(i)}$ and

$$|u_{i+1}| \leq |u_i| + \frac{1}{2}n(n-1) \leq \frac{i}{2}n(n-1).$$

We observe that, after a finite number of steps, this procedure ends, because the greatest row which can be reached is $(1\,1\dots 1)$, by virtue of unambiguity. Thus, the words $v_1, v_2 \dots v_s$ and the rows $\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \dots, \mathbf{b}^{(s)}$ are defined symmetrically. By construction, one has $a_1^{(t)} = b_1^{(s)} = 1$,

$$|u_t| \leq \frac{t-1}{2}n(n-1), \qquad |v_s| \leq \frac{s-1}{2}n(n-1),$$

and hence

$$|u_t v_s| = |u_t| + |v_s| \leq \frac{1}{2}n(n-1)(t+s-2).$$

Thus, in order to prove the proposition, it is enough to demonstrate that $t + s - 2 \leq n - 1$. For every $q \neq 1$, we get either $a_q^{(t)} = 0$ or $b_q^{(s)} = 0$. Indeed, if $a_q^{(t)} = b_q^{(s)} = 1$, then the (scalar) product between $\mathbf{a}^{(t)}$ and $\mathbf{b}^{(s)}$ should be strictly greater than 1: this is impossible, since the involved automaton is unambiguous. Thus, the

total number of null entries of $\mathbf{a}^{(t)}$ and $\mathbf{b}^{(s)}$ is at least $n-1$. In passing from $\mathbf{a}^{(i-1)}$ to $\mathbf{a}^{(i)}$ (resp., from $\mathbf{b}^{(j-1)}$ to $\mathbf{b}^{(j)}$), the number of null entries becomes strictly smaller. So, after t (resp., s) steps, we get that the row $\mathbf{a}^{(t)}$ (resp., the column $\mathbf{b}^{(s)}$) contains at most $n-t$ (resp., $n-s$) null entries. One derives

$$n - t + n - s \le n - 1,$$

that is, $s + t - 2 \le n - 1$. This ends the proof. $\qquad\qquad\square$

Now we prove that, in our setting, given a non-maximal row and a relatively maximal row, it is possible to combine them with a suitable word of small length, in order to obtain the null vector.

Proposition 3.4. *Let $\mathcal{A}$ be an n-state unambiguous automaton. Assume that there is a word $z \in A^*$ such that $(\phi_{\mathcal{A}}(z))_{p*}$ is a relatively maximal row and $\mathbf{a}$ is a non-maximal row. Then there exists a word $x \in A^*$ of length less than or equal to $n-1$, such that $\mathbf{a}\,\phi_{\mathcal{A}}(xz) = \mathbf{0}$.*

Proof. By hypothesis, there are a matrix $\mu \in \phi_{\mathcal{A}}(A^*)$ and $r \in \{1, 2, \ldots, n\}$, with $\mu_{r*} > \mathbf{a}$, namely $\mu_{r,t} \ge n_{r,t}$ for every $t = 1, 2, \ldots, n$ and there exists $s \in \{1, 2, \ldots, n\}$ with $\mu_{r,s} > a_s$, that is $\mu_{r,s} = 1$ and $a_s = 0$, by unambiguity. Hence

$$\mu_{r*} \ge \mathbf{a} + \mathbf{e}_s, \tag{3.2}$$

where $\mathbf{e}_s$ denotes the vector of $\mathbb{R}^n$ whose sth component is 1 and whose other components are 0. Note that, by (left or right) multiplying both handsides of the inequality in (3.2) by any matrix with non-negative entries, the sign of the inequality remains the same. By transitivity and condition (1) of Proposition 2.3, there is a word $x \in A^*$ with $(\phi_{\mathcal{A}}(x))_{s,p} = 1$ and $|x| \le n - 1$. Observe that

$$\mathbf{e}_s \phi_{\mathcal{A}}(x) = (\phi_{\mathcal{A}}(x))_{s*} \ge \mathbf{e}_p, \tag{3.3}$$

because $(\phi_{\mathcal{A}}(x))_{s,p} = 1$. Furthermore note that, since $\mu_{r,s} = (\phi_{\mathcal{A}}(x))_{s,p} = 1$, then, by unambiguity, it is not difficult to deduce that $(\mu\,\phi_{\mathcal{A}}(x))_{r,p} = 1$. From this and relative maximality of $(\phi_{\mathcal{A}}(z))_{p*}$ it follows that

$$(\phi_{\mathcal{A}}(z))_{p*} = (\mu\,\phi_{\mathcal{A}}(x)\,\phi_{\mathcal{A}}(z))_{r*}. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we obtain

$$(\phi_{\mathcal{A}}(z))_{p*} = (\mu\,\phi_{\mathcal{A}}(x\,z))_{r*} = \mu_{r*}\,\phi_{\mathcal{A}}(x\,z) \ge \mathbf{a}\,\phi_{\mathcal{A}}(x\,z) + \mathbf{e}_s\,\phi_{\mathcal{A}}(x\,z)$$
$$\ge \mathbf{a}\,\phi_{\mathcal{A}}(x\,z) + \mathbf{e}_p\,\phi_{\mathcal{A}}(z) \ge \mathbf{a}\,\phi_{\mathcal{A}}(x\,z) + (\phi_{\mathcal{A}}(z))_{p*}.$$

By the above inequalities, one derives that $\mathbf{a}\,\phi_{\mathcal{A}}(xz) = \mathbf{0}$. This concludes the proof. $\qquad\square$

A consequence of Propositions 2.4 and 3.4 is the following result, which, when there exists a row which is relatively maximal but not maximal, allows us to construct an uncompletable word.

Theorem 3.5. *Let $\mathcal{A}$ be an n-state unambiguouos automaton and $w \in A^*$ be a word such that $\phi_{\mathcal{A}}(w)$ contains a row, which is relatively maximal, but not maximal. Then $\mathcal{A}$ has an uncompletable word v of length $2(n-1)$ $(|w| + n - 1)$.*

Proof. The uncompletable word is constructed as the last element of a sequence of n words $y_1, y_2, \ldots, y_n$ such that for all $i = 1, \ldots, n$ one has

$$(\phi_{\mathcal{A}}(y_i))_{1*} = \ldots = (\phi_{\mathcal{A}}(y_i))_{i*} = \mathbf{0}.$$

Let $\mathbf{b} = (\phi_{\mathcal{A}}(w))_{q*}$ be a relatively maximal row which is not maximal. At the first step, let $y_0 = \varepsilon$ be the empty word. If $(\phi_{\mathcal{A}}(y_0))_{1*} = \mathbf{a}$ is a maximal row, then, by Proposition 2.4 there is a word $v_0 \in A^*$, with $|v_0| \leq n-1$ and $(\phi\mathcal{A}(y_0 v_0 w))_{1*} = \mathbf{b}$. Set

$$z_0 = \begin{cases} y_0 v_0 w, & \text{if } \mathbf{a} \text{ is maximal,} \\ y_0, & \text{otherwise.} \end{cases}$$

Thus, $(\phi_{\mathcal{A}}(z_0))_{1*}$ is not a maximal row. Hence, by virtue of Proposition 3.4, there is a word $x_0 \in A^*$ with

$$(\phi_{\mathcal{A}}(z_0))_{1*} \, \phi_{\mathcal{A}}(x_0) = (\phi_{\mathcal{A}}(z_0 x_0))_{1*} = \mathbf{0}$$

and $|x_0| \leq |w| + n - 1$. We set $y_1 = z_0 x_0$. As one easily checks, one has

$$|y_1| \leq 2(|w| + n - 1).$$

At the second step, proceeding analogously as above, we distinguish the two cases where $(\phi_{\mathcal{A}}(y_1))_{2*}$ is maximal or not. In the first case, we find a word v_1 such that $z_1 = y_1 v_1 w$ is not maximal, in the second case we take $z_1 = y_1$. Again by Proposition 3.4, there is a word $x_1 \in A^*$ with

$$(\phi_{\mathcal{A}}(z_1))_{2*} \, \phi_{\mathcal{A}}(x_1) = (\phi_{\mathcal{A}}(z_1 x_1))_{2*} = \mathbf{0}$$

and $|x_1| \leq |w| + n - 1$. Thus, setting $y_2 = z_1 x_1$ one has

$$|y_2| \leq 4(|w| + n - 1) \quad \text{and} \quad \phi_{\mathcal{A}}(y_2))_{1*} = \phi_{\mathcal{A}}(y_2))_{2*} = \mathbf{0}.$$

Proceeding iteratively, at the n-th step we find a word $y_n \in A^*$ such that

$$|y_n| \leq 2n(|w| + n - 1) \text{ and } (\phi_{\mathcal{A}}(y_n))_{1*} = \ldots = (\phi_{\mathcal{A}}(y_n))_{n*} = \mathbf{0}.$$

This ends the proof. $\qquad\square$

Let $\mathcal{A} = (A, Q, \delta)$ be an unambiguous automaton. We introduce the deterministic automaton $\mathcal{B} = (A, Q', \delta')$, where Q' is the set of all maximal rows of $\phi_{\mathcal{A}}(A^*)$, and the transition function δ' is defined by

$$\delta'(\mathbf{q}, \alpha) = \begin{cases} \mathbf{q} \, \phi_{\mathcal{A}}(\alpha), & \text{if } \mathbf{q} \, \phi_{\mathcal{A}}(\alpha) \in Q', \\ \emptyset, & \text{otherwise,} \end{cases}$$

$\mathbf{q} \in Q'$, $\alpha \in A$. The automaton $\mathcal{B}$ will be called the *maximal row automaton* associated with $\mathcal{A}$. Note that, if $\mathcal{A}$ is uncomplete, then also the associated maximal row automaton is. Indeed, the product of every maximal row of each element of $\phi_{\mathcal{A}}(A^*)$ with the null matrix is equal to the null row.

Now we show that, if $\mathcal{B}$ has an uncompletable word, then so does $\mathcal{A}$, and in particular we establish a connection between the minimal lengths of uncompletable words of $\mathcal{A}$ and $\mathcal{B}$.

Proposition 3.6. *Let $\mathcal{A}$ be an n-state unambiguous automaton, $n_{\mathcal{B}}$ be the minimal length of an uncompletable word of $\mathcal{B}$ and $w \in A^*$ be a word such that $\phi_{\mathcal{A}}(w)$ contains a relatively maximal row. Then $\mathcal{A}$ has an uncompletable word v with*

$$|v| \leq n(|w| + n - 1 + n_{\mathcal{B}}).$$

Proof. Analogously as in Theorem 3.5, we construct an uncompletable word of $\mathcal{A}$. Let u be an uncompletable word of $\mathcal{B}$ of length $n_{\mathcal{B}}$.

At the first step, let $y_0 = \varepsilon$ be the empty word. If $(\phi_{\mathcal{A}}(y_0))_{1*} = \mathbf{a}$ is a maximal row, then, $(\phi_{\mathcal{A}}(y_0))_{1*}\phi_{\mathcal{A}}(w) = (\phi_{\mathcal{A}}(y_0 u))_{1*}$ is not maximal, because u is uncompletable in $\mathcal{B}$. We set

$$
z_0 = \begin{cases} y_0 u, & \text{if } \mathbf{a} \text{ is maximal,} \\ y_0, & \text{otherwise.} \end{cases}
$$

Thus, $(\phi_{\mathcal{A}}(z_0))_{1*}$ is not a maximal row. Hence, by virtue of Proposition 3.4, there is a word $x_0 \in A^*$ with

$$
(\phi_{\mathcal{A}}(z_0))_{1*}\,\phi_{\mathcal{A}}(x_0) = (\phi_{\mathcal{A}}(z_0 x_0))_{1*} = \mathbf{0}
$$

and $|x_0| \leq |w| + n - 1$. We set $y_1 = z_0 x_0$. As one easily checks, one has $|y_1| \leq |w| + n - 1 + n_{\mathcal{B}}$.

At the second step, proceeding analogously as above, we distinguish the two cases where $(\phi_{\mathcal{A}}(y_1))_{2*}$ is a maximal row or not. We set $z_1 = y_1 u$ in the first case, $z_1 = y_1$ in the second case. Again by Proposition 3.4, there is a word $x_1 \in A^*$ with

$$
(\phi_{\mathcal{A}}(z_1))_{2*}\,\phi_{\mathcal{A}}(x_1) = (\phi_{\mathcal{A}}(z_1 x_1))_{2*} = \mathbf{0}
$$

and $|x_1| \leq |w| + n - 1$. Thus, setting $y_2 = z_1 x_1$ one has $|y_2| \leq 2(|w| + n - 1 + n_{\mathcal{B}})$.

By iterating, we find a word $y_n \in A^*$ such that

$$
|y_n| \leq n\left(|w| + n - 1 + n_{\mathcal{B}}\right) \text{ and } \phi_{\mathcal{A}}(y_n) = 0.
$$

$\square$

From Propositions 3.3 and 3.6 one derives immediately the following

Theorem 3.7. *Let $\mathcal{A}$ be an unambiguous uncomplete automaton with n states, $n_{\mathcal{A}}$, $n_{\mathcal{B}}$ be the lengths of the minimal uncompletable words of $\mathcal{A}$ and of the associated maximal row automaton $\mathcal{B}$, respectively. Then,*

$$
n_{\mathcal{B}} \leq n_{\mathcal{A}} \leq n\left(\frac{1}{2}n(n-1)^2 + n - 1 + n_{\mathcal{B}}\right).
$$

4. Conclusions

The description of the set of uncompletable words in a finite state automaton is still an open problem, at least in the unambiguous case, in which it seems that there are not many results in the literature. A problem which arose is to find better estimates, possibly sharp, of the minimal length of an uncompletable word. We showed that the study of relatively maximal rows and columns in monoids of $\{0, 1\}$-matrices, as well as of the maximal row automaton, may be a useful tool in this setting. In view of some analogies with the problem of searching synchronizing words of minimal length in complete unambiguous automata, we conjecture that existence of a polynomial bound for the minimal length of an uncompletable word in an uncomplete unambiguous n-state automaton. In our last results, we have shown that the investigation of unambiguous automata has been brought back to the study of suitable deterministic automata. Even if the number of states of such an automaton may be, in general, exponential, we hope that a deeper knowledge of its structure may lead, in future, to some interesting results.

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