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TWO EQUIVALENT FAMILIES OF LINEAR FULLY COUPLED FORWARD BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate two families of fully coupled linear Forward-Backward Stochastic Differential Equations (FBSDEs) and its applications to optimal Linear Quadratic (LQ) problems. Within these families, one could get same well-posedness of FBSDEs with totally different coefficients. A family of FBSDEs is proved to be equivalent with respect to the Unified Approach. Thus one could get well-posedness of whole family once a member exists a unique solution. Another equivalent family of FBSDEs are investigated by introducing a linear transformation method. Owing to the coupling structure between forward and backward equations, it leads to a highly interdependence in solutions. We are able to decouple FBSDEs into partial coupling, by virtue of linear transformation, without losing the existence and uniqueness to solutions. Moreover, owing to non-degeneracy of transformation matrix, the solution to original FBSDEs is totally determined by solutions of FBSDEs after transformation. In addition, an example of optimal LQ problem is presented to illustrate.

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1. Introduction

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space on which $W = (W_t)_{t\geq 0}$ is defined a standard Brownian motion. We assume $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ to be natural filtration generated by W_t augmented by null sets of $\mathbb{P}$. A general fully coupled FBSDEs takes following form:

$$\begin{cases} \mathrm{d}X(t) = b(t, X(t), Y(t), Z(t))\mathrm{d}t + \sigma(t, X(t), Y(t), Z(t))\mathrm{d}W(t), \\ -\mathrm{d}Y(t) = f(t, X(t), Y(t), Z(t))\mathrm{d}t - Z(t)\mathrm{d}W(t), \\ X_0 = x \quad Y(T) = h[X(T)], \qquad\qquad 0 \leq t \leq T, \end{cases} \tag{1.1}$$

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where b, σ, f are all $\mathcal{F}$-progressively measurable processes defined on appropriate spaces with $W(t)$ being a standard Brownian motion and $h(\cdot)$ is $\mathcal{F}_T$-measurable function.

For technical clarity, we employ following standard assumptions throughout paper.

Assumption 1.1. (H1) Integrability condition holds for coefficient b, σ, f, h on $[0, T]$:

$$E\{(\int_0^T [|b(t,0,0,0)| + |f(t,0,0,0)|]\mathrm{d}t)^2 + \int_0^T |\sigma(t,0,0,0)|^2 \mathrm{d}t + |h(0)|^2\} < \infty$$

(H2) The coefficients b, σ, f and h are uniformly Lipschitz continuous with respect to (x, y, z) and x respectively with a Lipschitz constant $L > 0$.

To get well-posedness of nonlinear FBSDEs (1), the Method of Contract mapping was firstly introduced by Antonelli [3] and later detailed by Pardoux and Tang [13] to solve FBSDEs with relative small time duration T. Afterwards, Ma, Protter and Yong [11] introduced the Four Step Scheme to handle cases of arbitrary duration T, which requires regularity assumption on coefficients in Markovian structure.

To investigate FBSDEs (1) with non-Markovian coefficients, the Method of Continuation was introduced, by Hu and Peng [6]; Peng and Wu [14]; Yong [21], into literature. But as a trade off, it required a Monotonicity conditions as follows:

Assumption 1.2. (H3) Assume that x, y, z are same dimension. Denoting $\theta_i = (x_i, y_i, z_i), i = 1, 2$ and $A(t, \theta_i) = \begin{pmatrix} -f \\ b \\ \sigma \end{pmatrix} (t, \theta_i)$, there exist some constants $\beta_1, \beta_2, \mu > 0$, for any $\hat{x} = x_1 - x_2,\ \hat{y} = y_1 - y_2,\ \hat{z} = z_1 - z_2$, such that

$$\begin{cases} \langle A(t,\theta_1) - A(t,\theta_2), \theta_1 - \theta_2 \rangle \ \leq -\beta_1 |\hat{x}|^2 - \beta_2(|\hat{y}|^2 + |\hat{z}^2|), \\ \\ \langle h(x_1) - h(x_2), x_1 - x_2 \rangle \geq \mu |\hat{x}|^2. \end{cases} \tag{1.2}$$

Recently, a Unified Approach was introduced by Ma, Wu, Zhang and Zhang [10] to investigate nonlinear FBSDEs. This method constructed a unified scheme which aims to summarize all existing methodologies in literature, and provided a series of sufficient and necessary conditions to get the well-posedness of (1).

Core to the Unified Approach, analogue to the Four Step Scheme, is to find a decoupling field $u(t, \cdot)$ such that $Y_t = u(t, X_t)$ on $[0, T]$. It will ultimately leads to well-posedness of (1) that decoupling field $u(t, \cdot)$ is uniformly Lipschitz in its spatial variable. And proving u being uniformly Lipschitz continuous amounts to finding solutions to following "variational FBSDEs":

$$\begin{cases} \nabla X(t) = 1 + \int_0^t \left[b_1 \nabla X(s) + b_2 \nabla Y(s) + b_3 \nabla Z(s)\right] \mathrm{d}s + \int_0^t \left[\sigma_1 \nabla X(s) + \sigma_2 \nabla Y(s) + \sigma_3 \nabla Z(s)\right] \mathrm{d}W(s), \\ \nabla Y(t) = h \nabla X(T) + \int_t^T \left[f_1 \nabla X(s) + f_2 \nabla Y(s) + f_3 \nabla Z(s)\right] \mathrm{d}s - \int_t^T \nabla Z(s) \mathrm{d}W(s), \qquad t \in [0, T], \end{cases} \tag{1.3}$$

where $\nabla \Theta \triangleq \frac{\Theta^1 - \Theta^2}{x_1 - x_2}$ denotes derivative of $\Theta = X, Y, Z$ with respect to initial value x. Note that (3) is a linear fully coupled FBSDEs where coefficients $b_i, \sigma_i, f_i, i = 1, 2, 3$ are bounded in consequence of Lipschitz condition (H2).

A simple example, also a motivation for this paper, often appeared in optimal investment problem [7, 8] and stochastic control problem [12] is of following form:

$$\begin{cases} X(t) = x + \int_0^t [a(s)X(s) + b(s)Y(s) + c(s)Z(s)]\mathrm{d}s + \int_o^t [d(s)X(s) + e(s)Y(s) + f(s)Z(s)]\mathrm{d}W(s), \\ Y(t) = hX(T) + \int_t^T [m(s)X(s) + p(s)Y(s) + q(s)Z(s)\mathrm{d}s] - \int_t^T Z(s)\mathrm{d}W(s), \qquad 0 \le t \le T, \end{cases} \tag{1.4}$$

where $a(\cdot), b(\cdot), c(\cdot), d(\cdot), e(\cdot), f(\cdot), m(\cdot), p(\cdot), q(\cdot)$ are bounded processes and h is an $\mathcal{F}_T$ random variable. (4) is often derived from the Pontryagin's maximum principle when seeking the closed-loop optimal control of linear quadratic (LQ) optimal control problem. In a deterministic LQ problem, it is well known that the control weight in the cost functional must be positive definite, otherwise the optimization problem would not be well-posed (see Anderson and Moore [2], Bensoussan [4]). This kind of linear FBSDEs are widely applied in many areas, such as ordinary differential equations [1, 16], stochastic control [5, 8, 20] and mathematical finance [7, 18, 22]. However, the well-posedness of (4) is not covered by any existing methods despite it is linear, homogeneous and bounded. As we will see in Section 5, the solvability of (4) will be a straight consequence of our results.

In this paper, we aim to get well-posedness of this kind of Linear FBSDEs. In Section 2, we formulate linear fully coupled FBSDEs and introduce the monotonicity conditions for it. In Section 3, we discuss a family of linear FBSDEs which are proved to be equivalent with respect to the Unified Approach. In Section 4, we introduce a linear transformation method to study some FBSDEs which could not be proved to be well posed by the existing methods. In Section 5, we employ the linear transformation to deal with the LQ stochastic control problem.

2. Notations and problem formulation

First, we introduce following spaces:

$$L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{H}) = \{\xi : \Omega \to \mathbb{H} \mid \xi \text{ is } \mathcal{F}_T\text{-measurable bounded variable}\};$$

$$L^\infty_{\mathcal{F}}([0,T];\mathbb{H}) = \{\varphi : [0,T]\times\Omega \to \mathbb{H} \mid \varphi(\cdot) \text{ is } \mathbb{F}\text{-adapted, bounded processes}\}.$$

In stochastic control model, state system is governed by following stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}x(t) = [A(t)x(t) + B(t)u(t)]\mathrm{d}t + [C(t)x(t) + D(t)u(t)]\mathrm{d}W(t), \\ x(0) = x \end{cases} , \tag{2.1}$$

where $x \in R$ and $A(\cdot), B(\cdot), C(\cdot), D(\cdot) \in L^\infty_{\mathcal{F}}([0,T];\mathbb{H})$. Here control process $u(\cdot) \in \mathcal{U}_{ad}$ is an $\mathcal{F}_t$-adapted process and cost function to be minimized is defined by

$$J(u(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^T [\langle R(t)x(t), x(t)\rangle + 2\langle S(t)u(t), x(t)\rangle + \langle N(t)u(t), u(t)\rangle]\mathrm{d}t + \frac{1}{2}\mathbb{E}[\langle Qx(T), x(T)\rangle], \tag{2.2}$$

where $Q \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{H})$ and $R(\cdot), S(\cdot), N(\cdot) \in L^\infty_{\mathcal{F}}([0,T];\mathbb{H})$.
Problem (LQ). An admissible control $u^\star(\cdot) \in \mathcal{U}_{ad}$ is called optimal if it solves

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}_{ad}} J(u(\cdot)). \tag{2.3}$$

The stochastic maximum principle is one of fundamental approaches to solve Problem (LQ) and it gives a necessary condition hold by any optimal solution. We apply the maximum principle [20] to Problem (LQ):

Lemma 2.1. *Let $\bar{u}(\cdot)$ be an optimal control minimizing cost function $\mathcal{J}$ over $\mathcal{U}_{ad}$ and let $\bar{x}(\cdot)$ be corresponding optimal trajectory. Then there exists a pair of processes $(\bar{y}(\cdot), \bar{z}(\cdot)) \in L^\infty_{\mathcal{F}}([0,T];\mathbb{H})$ such that following stochastic Hamiltonian system holds:*

$$\begin{cases} \mathrm{d}\bar{x}(t) = \left[\left(A - BN^{-1}S^\top\right)\bar{x}(t) - BN^{-1}B^\top\bar{y}(t) - BN^{-1}D^\top\bar{z}(t)\right]\mathrm{d}t \\ \qquad\qquad + \left[\left(C - DN^{-1}S^\top\right)\bar{x}(t) - DN^{-1}B^\top\bar{y}(t) - DN^{-1}D^\top\bar{z}(t)\right]\mathrm{d}W(t) \\ -\mathrm{d}\bar{y}(t) = \left[\left(R - SN^{-1}S^\top\right)\bar{x}(t) + \left(A - SN^{-1}B^\top\right)\bar{y}(t) + \left(C - SN^{-1}D^\top\right)\bar{z}(t)\right]\mathrm{d}t - \bar{z}(t)\mathrm{d}W(t) \\ \bar{x}(0) = x, \quad \bar{y}(T) = Q\bar{x}(T) \end{cases} \tag{2.4}$$

And also, optimal control $\bar{u}(\cdot)$ should take forms:

$$\bar{u}(t) = -N^{-1}(t)S^\top(t)\bar{x}(t) - N^{-1}(t)B^\top(t)\bar{y}(t) - N^{-1}(t)D^\top(t)\bar{z}(t), \tag{2.5}$$

where the superscript $(\cdot)^\top$ denotes the transpose of matrix and $N^{-1}(t)$ denotes the inverse of $N(t)$.

Remark 2.2. To relax the restriction of the control weight matrix N being positive definite, in this paper, we postulate it to be bounded and invertible. Moreover, we assume N^{-1} is also bounded.

For more details of the maximum principle and stochastic Hamiltonian system theory, we refer to book Yong and Zhou [20] and the reference therein. Note that stochastic Hamiltonian system (8) is of fully coupled linear FBSDEs. To investigate well-posedness of (8), we rewrite it by following linear fully coupled FBSDEs:

$$\begin{cases} \mathrm{d}X(t) = [b_1(t)X(t) + b_2(t)Y(t) + b_3(t)Z(t)]\mathrm{d}t + [\sigma_1(t)X(t) + \sigma_2(t)Y(t) + \sigma_3(t)Z(t)]\mathrm{d}W(t), \\ -\mathrm{d}Y(t) = [f_1(t)X(t) + f_2(t)Y(t) + f_3(t)Z(t)]\mathrm{d}t - Z(t)\mathrm{d}W(t), \\ X_0 = x \quad Y(T) = hX(T), \qquad\qquad 0 \le t \le T, \end{cases} \tag{2.6}$$

where $b_i(\cdot), f_i(\cdot), \sigma_i(\cdot) \in L^\infty_{\mathcal{F}}([0,T];\mathbb{H}), i = 1,2,3$, $T > 0$ be a fixed time horizon and $h \in L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{H})$.

Remark 2.3. It is a straight consequence that $b_i, \sigma_i, f_i, i = 1,2,3$ and h are bounded since $b(t,\cdot,\cdot,\cdot)$, $\sigma(t,\cdot,\cdot,\cdot)$, $f(t,\cdot,\cdot,\cdot), h(\cdot)$ in (1) hold for the Lipschitz conditions (H2).

For technical clarity, we only consider $\mathbb{H} = \mathcal{R}^1$ for simplification in what follows and multi-dimensional cases can be dealt with similarly. Here and after, we sometimes suppress t also for processes $b_i, \sigma_i, f_i, i = 1,2,3$, for simplicity of notations.

Motivated by Assumption 1.2, we have following monotonicity conditions for linear FBSDEs (10):

Lemma 2.4. *(Monotonicity Conditions)* (i) For $\forall\ x, y, z \in R$ and fixed t,

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ b_1 & b_2 & b_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \le -\beta_1|x|^2 - \beta_2(|y|^2 + |z|^2),$$

where β_1 and β_2 are nonnegative constants. When $\beta_1 > 0, h > 0$, then $\beta_2 \ge 0$; When $\beta_2 > 0$, then $\beta_1 \ge 0, h \ge 0$.

(ii) For $\forall\ x, y, z \in R$ and fixed t,

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ b_1 & b_2 & b_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ge \beta_1|x|^2 + \beta_2(|y|^2 + |z|^2),$$

where β_1 and β_2 are nonnegative constants. When $\beta_1 > 0, h < 0$, then $\beta_2 \ge 0$; When $\beta_2 > 0$, then $\beta_1 \ge 0, h \le 0$.

3. Equivalent coefficient matrix for unified approach

The Unified Approach is one of principal methods to solve well-posedness of (10). By virtue of variational FBSDEs, Ma *et al.* [10] derived a dominating ODE of (10) which takes following form:

$$\mathbf{y}_t = h + \int_t^T F\left(s, \mathbf{y}_s\right) \mathrm{d}s, \qquad 0 \le s \le T, \tag{3.1}$$

where

$$F(s,y) \triangleq f_1(s) + f_2(s)y + y\left[b_1(s) + b_2(s)y\right] + \frac{\left(f_3(s) + b_3(s)y\right) y \left(\sigma_1(s) + \sigma_2(s)y\right)}{1 - \sigma_3(s)y}. \tag{3.2}$$

And the well-posedness of (10) is a straightforward consequence of boundary solutions to (12) according to following lemma.

Lemma 3.1. *(Ma et al. [10]) Linear FBSDEs (10) is well posed if and only if the dominating ODE (11) exists boundary upper/lower solutions $\overline{\mathbf{y}}_t, \underline{\mathbf{y}}_t$ on $[0,T]$:*

$$\overline{\mathbf{y}}_t = \bar{h} + \int_t^T \overline{F}\left(s, \overline{\mathbf{y}}_s\right) \mathrm{d}s, \quad \underline{\mathbf{y}}_t = \underline{h} + \int_t^T \underline{F}\left(s, \underline{\mathbf{y}}_s\right) \mathrm{d}s$$

where $\overline{h}, \underline{h}$ denote upper/lower bound of h and

$$\overline{F}(t,y) := \operatorname{ess\,sup} F(t,y), \quad \underline{F}(t,y) := \operatorname{ess\,inf} F(t,y).$$

It is noted that some coefficients in (12) are symmetric which indicates some different FBSDEs correspond to a same dominating ODE (11).

Denoting $F_s(y) = F(s,y)$, we have

$$F_s(y) = \frac{\left(b_3\sigma_2 - b_2\sigma_3\right) y^3 + \left(b_2 + \sigma_2 f_3 - \sigma_3 f_2 + \sigma_1 b_3 - \sigma_3 b_1\right) y^2 + \left(f_2 + b_1 + f_3\sigma_1 - f_1\sigma_3\right) y + f_1}{1 - \sigma_3 y} \tag{3.3}$$

Note that

$$b_3\sigma_2 - b_2\sigma_3 = -\begin{vmatrix} b_2 & b_3 \\ \sigma_2 & \sigma_3 \end{vmatrix}, \quad \sigma_2 f_3 - \sigma_3 f_2 = \begin{vmatrix} -f_2 & -f_3 \\ \sigma_2 & \sigma_3 \end{vmatrix},$$

$$\sigma_1 b_3 - \sigma_3 b_1 = -\begin{vmatrix} b_1 & b_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}, \quad f_3\sigma_1 - f_1\sigma_3 = \begin{vmatrix} -f_1 & -f_3 \\ \sigma_1 & \sigma_3 \end{vmatrix},$$

where $|\cdot|$ denotes the determinant of matrix.

Rewrite (13) in terms of these determinants, and we have

$$F_s(y) = \frac{-\begin{vmatrix} b_2 & b_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} y^3 + \left(b_2 + \begin{vmatrix} -f_2 & -f_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) y^2 + \left(f_2 + b_1 + \begin{vmatrix} -f_1 & -f_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) y + f_1}{1 - \sigma_3 y} \tag{3.4}$$

Proposition 3.2. *Let* $A = \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ b_1 & b_2 & b_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ *denote coefficient matrix of linear FBSDEs (10), then for* $\forall\ p \in R$, $B = \begin{pmatrix} -f_1 & -f_2 & -f_3+p \\ b_1+\sigma_1 p & b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ *is an equivalent coefficient matrix to* A *with respect to dominating function (12).*

Proof. Note that

$$-\begin{vmatrix} b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_2 & \sigma_3 \end{vmatrix} = b_3\sigma_2 - b_2\sigma_3, \quad \begin{vmatrix} -f_2 & -f_3+p \\ \sigma_2 & \sigma_3 \end{vmatrix} = \sigma_2 f_3 - \sigma_3 f_2 - \sigma_2 p,$$

$$-\begin{vmatrix} b_1+\sigma_1 p & b_3+\sigma_3 p \\ \sigma_1 & \sigma_3 \end{vmatrix} = \sigma_1 b_3 - \sigma_3 b_1, \quad \begin{vmatrix} -f_1 & -f_3+p \\ \sigma_1 & \sigma_3 \end{vmatrix} = f_3\sigma_1 - f_1\sigma_3 - \sigma_1 p,$$

Substituting all equations above into (14), we have

$$\tilde{F}_s(y) = \frac{-\begin{vmatrix} b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_2 & \sigma_3 \end{vmatrix} y^3 + \left(b_2+\sigma_2 p + \begin{vmatrix} -f_2 & -f_3+p \\ \sigma_2 & \sigma_3 \end{vmatrix} - \begin{vmatrix} b_1+\sigma_1 p & b_3+\sigma_3 p \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) y^2}{1-\sigma_3 y}$$
$$+ \frac{\left(f_2 + b_1 + \sigma_1 p + \begin{vmatrix} -f_1 & -f_3+p \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) y + f_1}{1-\sigma_3 y} = F_s(y)$$

This implies the equivalence between A and B in (12). $\square$

Remark 3.3. For multi-dimensional cases, we assume $b_i, f_i, \sigma_i, p \in \mathcal{R}^{m\times m}$. It is not trivial to get

$$-\begin{vmatrix} b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_2 & \sigma_3 \end{vmatrix} = b_3\sigma_2 - b_2\sigma_3.$$

Then it suffices to assume that b_i, f_i, σ_i, p are all symmetric.

Similarly, we could get another family of equivalent coefficient matrix to A.

Corollary 3.4. *Let* $A = \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ b_1 & b_2 & b_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ *denotes the coefficient matrix of linear FBSDEs (10), then for* $\forall\ q \in R$, $C = \begin{pmatrix} -f_1 & -f_2-f_3 q & -f_3 \\ b_1 & b_2+b_3 q & b_3 \\ \sigma_1 - q & \sigma_2+\sigma_3 q & \sigma_3 \end{pmatrix}$ *is an equivalent coefficient matrix to* A *with respect to the dominating function (12).*

Remark 3.5. Note that b_1 and f_2 are symmetric in (13), then it is also an equivalent coefficient matrix for $B = \begin{pmatrix} -f_1 & -f_2-\sigma_1 p & -f_3+p \\ b_1 & b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ in Proposition 3.2 and $C = \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ b_1+f_3 q & b_2+b_3 q & b_3 \\ \sigma_1 - q & \sigma_2+\sigma_3 q & \sigma_3 \end{pmatrix}$ in Corollary 3.4, respectively.

Remark 3.6. In Proposition 3.2, for $\lambda \in R$, the equivalent matrix $B = \begin{pmatrix} -f_1 & -f_2 & -f_3+\lambda \\ b_1+\sigma_1\lambda & b_2+\sigma_2\lambda & b_3+\sigma_3\lambda \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ corresponds to the following FBSDEs:

$$\begin{cases} \mathrm{d}X(t) = [b_1X(t)+b_2Y(t)+b_3Z(t)]\mathrm{d}t + [\sigma_1X(t)+\sigma_2Y(t)+\sigma_3Z(t)]\mathrm{d}\tilde{W}(t), \\ -\mathrm{d}Y(t) = [f_1X(t)+f_2Y(t)+f_3Z(t)]\mathrm{d}t - Z(t)\mathrm{d}\tilde{W}(t), \\ X_0 = x \quad Y(T) = hX(T), \qquad\qquad 0 \le t \le T, \end{cases} \tag{3.5}$$

where $\mathrm{d}\tilde{W}(t) = \mathrm{d}W(t) + \lambda\mathrm{d}t$.

By virtue of Girsanov Transformation, for $\forall t \ge 0$, we have an equivalent probability $\mathbb{Q}$ with respect to $\mathbb{P}$ such that

$$\frac{d\mathbb{Q}\ |\mathcal{F}_t}{d\mathbb{P}\ |\mathcal{F}_t} = \exp\{-\frac{1}{2}\lambda^2 t - \lambda W(t)\},$$

and $\tilde{W}(t)$ is a standard Brownian motion under probability $\mathbb{Q}$.

It is noted that, in Remark 3.6, dominating function (12) has the same structure under equivalent probability measure $\mathbb{P}$ and $\mathbb{Q}$. As a consequence, the well-posedness of FBSDEs under $\mathbb{Q}$ is equivalent to the FBSDEs (10) under $\mathbb{P}$. Next we present the main result for this section.

Theorem 3.7. *For any R-valued bounded process $p, q : [0,T] \mapsto R$, the well-posedness of the following FBSDEs is equivalent to FBSDEs (10):*

$$\begin{cases} \mathrm{d}X(t) = [(b_1+\sigma_1)X(t) + (b_2+\sigma_2 p + b_3 q + \sigma_3 pq)Y(t) + (b_3+\sigma_3 p)Z(t)]\mathrm{d}t \\ \qquad\qquad\qquad\qquad + [(\sigma_1 - q)X(t) + (\sigma_2+\sigma_3 p)Y(t) + \sigma_3 Z(t)]\mathrm{d}W(t), \\ -\mathrm{d}Y(t) = [f_1X(t) + (f_2+f_3q-pq)Y(t) + (f_3-p)Z(t)]\mathrm{d}t - Z(t)\mathrm{d}W(t), \\ X_0 = x \quad Y(T) = hX(T), \qquad\qquad 0 \le t \le T. \end{cases} \tag{3.6}$$

Proof. Note that the coefficient matrix of (16) is as the follows:

$$D = \begin{pmatrix} -f_1 & -f_2 + (-f_3+p)q & -f_3+p \\ b_1+\sigma_1 p & b_2+\sigma_2 p + (b_3+\sigma_3 p)q & b_3+\sigma_3 p \\ \sigma_1 - q & \sigma_2+\sigma_3 q & \sigma_3 \end{pmatrix}.$$

According to Corollary 3.4, D is an equivalent coefficient matrix with respect to $B = \begin{pmatrix} -f_1 & -f_2 & -f_3+p \\ b_1+\sigma_1 p & b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$. It is a straight consequence, by Proposition 3.2, that B is equivalent to $A = \begin{pmatrix} -f_1 & -f_2 & -f_3 \\ b_1 & b_2 & b_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$. It is easy to get that FBSDEs (16) and (10) has the same dominating ODE (11). This completes the proof by Lemma 3.1. □

We can also formulate other equivalent matrix with respect to A, by applying Proposition 3.2, Corollary 3.4 and Remark 3.5 sequentially, and the proof would be similar.

Next we employ the results in this section to investigate FBSDEs (10) with all the coefficients constant. In this case, $\overline{h} = \underline{h} = h \in R$, and

$$\begin{aligned} F(y) &= \overline{F}(t,y) = \underline{F}(t,y) \\ &= \frac{(b_3\sigma_2 - b_2\sigma_3)\, y^3 + (b_2 + \sigma_2 f_3 - \sigma_3 f_2 + \sigma_1 b_3 - \sigma_3 b_1)\, y^2 + (f_2 + b_1 + f_3\sigma_1 - f_1\sigma_3)\, y + f_1}{1 - \sigma_3 y}, \end{aligned}$$

where $b_i, \sigma_i, f_i, i = 1, 2, 3$ are all constants.

By virtue of the Unified Approach, Ma *et al.* [10] presented a necessary and sufficient condition for the existence and uniqueness of solutions to such cases:

Lemma 3.8. *If coefficients $b_i, \sigma_i, f_i, i = 1, 2, 3$ and h are all constants, then linear FBSDEs (10) exists a unique solution for arbitrary $T > 0$ and terminal condition h ($h \neq \frac{1}{\sigma_3}$) if and only if one of following cases holds:*

- *(i)* $h < \frac{1}{\sigma_3}$, $F(h) \leq 0$, *and either* $F(y)$ *has a zero point in* $(-\infty, h]$ *or* $b_3\sigma_2 - b_2\sigma_3 = 0$*;*
- *(ii)* $h > \frac{1}{\sigma_3}$, $F(h) \geq 0$, *and either* $F(y)$ *has a zero point in* $[h, \infty)$ *or* $b_3\sigma_2 - b_2\sigma_3 = 0$*;*
- *(iii)* $h < \frac{1}{\sigma_3}$, $F(h) \geq 0$, *and F has a zero point in* $[h, \frac{1}{\sigma_3}]$*;*
- *(iv)* $h > \frac{1}{\sigma_3}$, $F(h) \leq 0$, *and F has a zero point in* $[\frac{1}{\sigma_3}, h]$*;*

Note that it is difficult to get all zero points of $F(y)$. To make full use of Lemma 3.8, we need to simplify the criterion which is easy to check. Denoting

$$\begin{aligned} \mathcal{L}(y) = (1 - \sigma_3 y)F(y) = (b_3\sigma_2 - b_2\sigma_3)\, y^3 &+ (b_2 + \sigma_2 f_3 - \sigma_3 f_2 + \sigma_1 b_3 - \sigma_3 b_1)\, y^2 \\ &+ (f_2 + b_1 + f_3\sigma_1 - f_1\sigma_3)\, y + f_1, \end{aligned}$$

we present following sufficient condition:

Theorem 3.9. *Assume all coefficients b_i, σ_i, f_i, h are constants in linear FBSDEs (10). Then it is well-posed for arbitrary $T > 0$ if one of following cases hold true:*

- *(i)* $1 - \sigma_3 h > 0$, $b_3\sigma_2 - b_2\sigma_3 \leq 0$ *and* $\mathcal{L}(h) \cdot \sigma_3 \leq 0$*;*
- *(ii)* $1 - \sigma_3 h < 0$, $b_3\sigma_2 - b_2\sigma_3 \geq 0$ *and* $\mathcal{L}(h) \cdot \sigma_3 \leq 0$*;*
- *(iii)* $\sigma_3 > 0$, $\mathcal{L}(\frac{1}{\sigma_3}) \leq 0$ *and* $\mathcal{L}(h) \geq 0$;
- *(iv)* $\sigma_3 < 0$, $\mathcal{L}(\frac{1}{\sigma_3}) \geq 0$ *and* $\mathcal{L}(h) \leq 0$;

Proof. (i) Note that proof is trivial for $b_3\sigma_2 - b_2\sigma_3 = 0$.

For cases of $b_3\sigma_2 - b_2\sigma_3 < 0$, if $\sigma_3 > 0$, we have

$$h < \frac{1}{\sigma_3} \ \text{ and } \ \mathcal{L}(h) \leq 0.$$

It indicates that

$$F(h) = \frac{\mathcal{L}(h)}{1 - \sigma_3 h} \leq 0.$$

Note that $b_3\sigma_2 - b_2\sigma_3 < 0$ is coefficient of y^3 in $\mathcal{L}(y)$. It is easy to get a $\mathcal{L}$ has a zero point in $(-\infty, h]$ which coincides with case (i) of Lemma 3.8.

If $\sigma_3 < 0$, we have

$$h > \frac{1}{\sigma_3}, \ \mathcal{L}(h) \geq 0 \text{ and } \ F(h) = \frac{\mathcal{L}(h)}{1 - \sigma_3 h} \geq 0.$$

And $b_3\sigma_2 - b_2\sigma_3 < 0$ leads to that L has a zero point in $[h, +\infty)$ which corresponds to case (ii) of Lemma 3.8.

(ii) can be proved in a similar way.

(iii) When $\sigma_3 > 0$, we first assume $h < \frac{1}{\sigma_3}$. We can get $F(h) \geq 0$ owing to $\mathcal{L}(h) \geq 0$.

In addition, it is noted that $\mathcal{L}(\frac{1}{\sigma_3}) < 0$. Owing to $\mathcal{L}(\cdot)$ being a continuous function, then there exists a constant $\eta < \frac{1}{\sigma_3}$ such that

$$\mathcal{L}(\eta) \leq 0.$$

It follows that $\mathcal{L}$ has a zero point in $[h, \eta]$ which coincides with case (iii) of Lemma 3.8. Similarly, if $h > \frac{1}{\sigma_3}$, this case coincides with case (iv) of Lemma 3.8.

(iv) can be proved similarly which completes the proof. □

Remark 3.10. However, the monotonicity condition (Lem. 2.4) is a sufficient condition. Actually, Liu and Wu [9] proved that, for (10) with constant coefficients, the monotonicity condition is indeed a special case of the unified approach.

Then, by employing results in this section, we investigate some differences between the monotonicity condition (Lem. 2.4) and the unified approach. For cases satisfying the unified approach while the monotonicity condition do not hold, we apply our results to derive some feasible values of p such that the equivalent coefficient matrix also holds for the monotonicity conditions.

Note that an equivalent coefficient matrix $B = \begin{pmatrix} -f_1 & -f_2 & -f_3 + p \\ b_1 + \sigma_1 p & b_2 + \sigma_2 p & b_3 + \sigma_3 p \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}$ and $C = \begin{pmatrix} -f_1 & -f_2 - f_3 q & -f_3 \\ b_1 & b_2 + b_3 q & b_3 \\ \sigma_1 - q & \sigma_2 + \sigma_3 q & \sigma_3 \end{pmatrix}$ could be transformed into symmetric structure for Lemma 2.4:

$$\hat{B} = \begin{pmatrix} -f_1 & (b_1 - f_2 + \sigma_1 p)/2 & (\sigma_1 - f_3 + p)/2 \\ (b_1 - f_2 + \sigma_1 p)/2 & b_2 + \sigma_2 p & (\sigma_2 + b_3 + \sigma_3 p)/2 \\ (\sigma_1 - f_3 + p)/2 & (\sigma_2 + b_3 + \sigma_3 p)/2 & \sigma_3 \end{pmatrix}$$

and

$$\hat{C} = \begin{pmatrix} -f_1 & (b_1 - f_2 - f_3 q)/2 & (\sigma_1 - f_3 - q)/2 \\ (b_1 - f_2 - f_3 q)/2 & b_2 + b_3 q & (b_3 + \sigma_2 + \sigma_3 q)/2 \\ (\sigma_1 - f_3 - q)/2 & (b_3 + \sigma_2 + \sigma_3 q)/2 & \sigma_3 \end{pmatrix}, \text{ respectively.}$$

Here we present a theorem to determine values of p.

Theorem 3.11. *Equivalent coefficient matrix B holds for monotonicity conditions (Lem. 2.4) if p satisfies one of following criterions:*

(i) $h < 0$, $f_1 < 0$ *and*

$$\begin{cases} \begin{vmatrix} -f_1 & (b_1 - f_2 + \sigma_1 p)/2 \\ (b_1 - f_2 + \sigma_1 p)/2 & b_2 + \sigma_2 p \end{vmatrix} > 0, \\ \begin{vmatrix} -f_1 & (b_1 - f_2 + \sigma_1 p)/2 & (\sigma_1 - f_3 + p)/2 \\ (b_1 - f_2 + \sigma_1 p)/2 & b_2 + \sigma_2 p & (\sigma_2 + b_3 + \sigma_3 p)/2 \\ (\sigma_1 - f_3 + p)/2 & (\sigma_2 + b_3 + \sigma_3 p)/2 & \sigma_3 \end{vmatrix} > 0; \end{cases} \tag{3.7}$$

(ii) $h>0,\ f_1>0$ *and*

$$\begin{cases}\begin{vmatrix} -f_1 & (b_1-f_2+\sigma_1 p)/2 \\ (b_1-f_2+\sigma_1 p)/2 & b_2+\sigma_2 p \end{vmatrix} > 0, \\ \begin{vmatrix} -f_1 & (b_1-f_2+\sigma_1 p)/2 & (\sigma_1-f_3+p)/2 \\ (b_1-f_2+\sigma_1 p)/2 & b_2+\sigma_2 p & (\sigma_2+b_3+\sigma_3 p)/2 \\ (\sigma_1-f_3+p)/2 & (\sigma_2+b_3+\sigma_3 p)/2 & \sigma_3 \end{vmatrix} < 0; \end{cases} \tag{3.8}$$

Proof. (i) Note that $\hat{B}$ is positive definite if (17) and $f_1<0$ hold. Then for $\forall(x,y,z)\in\mathbb{R}^3$, there exist constants $\alpha_1,\alpha_2,\alpha_3>0$ such that

$$\begin{aligned} &\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -f_1 & (b_1-f_2+\sigma_1 p)/2 & (\sigma_1-f_3+p)/2 \\ (b_1-f_2+\sigma_1 p)/2 & b_2+\sigma_2 p & (\sigma_2+b_3+\sigma_3 p)/2 \\ (\sigma_1-f_3+p)/2 & (\sigma_2+b_3+\sigma_3 p)/2 & \sigma_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} -f_1 & -f_2 & -f_3+p \\ b_1+\sigma_1 p & b_2+\sigma_2 p & b_3+\sigma_3 p \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \alpha_1|x|^2+\alpha_2|y|^2+\alpha_3|z|^2, \end{aligned}$$

which completes the proof according to case (ii) of Lemma 2.4.

(ii) can be proved in a similar way. □

Similarly, we can also determine values of q by virtue of $\hat{C}$.

Corollary 3.12. *Equivalent coefficient matrix C holds for monotonicity conditions (Lem. 2.4) if q satisfies one of following criterions:*

(i) $h<0,\ f_1<0$ *and*

$$\begin{cases}\begin{vmatrix} -f_1 & (b_1-f_2-f_3 q)/2 \\ (b_1-f_2-f_3 q)/2 & b_2+b_3 q \end{vmatrix} > 0, \\ \begin{vmatrix} -f_1 & (b_1-f_2-f_3 q)/2 & (\sigma_1-f_3-q)/2 \\ (b_1-f_2-f_3 q)/2 & b_2+b_3 q & (b_3+\sigma_2+\sigma_3 q)/2 \\ (\sigma_1-f_3-q)/2 & (b_3+\sigma_2+\sigma_3 q)/2 & \sigma_3 \end{vmatrix} > 0; \end{cases} \tag{3.9}$$

(ii) $h>0,\ f_1>0$ *and*

$$\begin{cases}\begin{vmatrix} -f_1 & (b_1-f_2-f_3 q)/2 \\ (b_1-f_2-f_3 q)/2 & b_2+b_3 q \end{vmatrix} > 0, \\ \begin{vmatrix} -f_1 & (b_1-f_2-f_3 q)/2 & (\sigma_1-f_3-q)/2 \\ (b_1-f_2-f_3 q)/2 & b_2+b_3 q & (b_3+\sigma_2+\sigma_3 q)/2 \\ (\sigma_1-f_3-q)/2 & (b_3+\sigma_2+\sigma_3 q)/2 & \sigma_3 \end{vmatrix} < 0. \end{cases} \tag{3.10}$$

To illustrate results in this section, here we present an example.

Example 3.13. Consider a linear FBSDEs as follows:

$$\begin{cases} X(t) = x + \int_0^t [X(s) - Y(s) - 2Z(s)]\mathrm{d}s + \int_0^t [2Y(s) + Z(s)]\mathrm{d}W(s), \\ \\ Y(t) = -X(T) + \int_t^T [-2X(s) + Z(s)]\mathrm{d}s - \int_t^T Z(s)\mathrm{d}W(s), \qquad 0 \le t \le T, \end{cases} \tag{3.11}$$

In this example, we have $f_1 = -2,\ f_2 = 0,\ f_3 = 1,\ b_1 = 1,\ b_2 = -1,\ b_3 = -2,\ \sigma_1 = 0,\ \sigma_2 = 2,\ \sigma_3 = 1$. Then the coefficients matrix of (21) is $\begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$ and $h = -1$. According to Lemma 2.4, it is easy to verify that (21) can not match the monotonicity conditions. Note that $h = -1 < \frac{1}{\sigma_3} = 1$, $F(h) = -1 < 0$ and $b_3\sigma_2 - b_2\sigma_3 = -3 < 0$, then there exists a constant $\zeta < h$ such that

$$F(\zeta) > 0,$$

which leads to that $F(\cdot)$ has a zero point in $(-\infty, h]$. FBSDEs (21) is well posed according to case (i) of Lemma 3.8 (Unified Approach).

To find an equivalent coefficient matrix holding for monotonicity conditions Lemma 2.4), it is noted that $h = -1 < 0$ and $f_1 = -2 < 0$. Substituting all coefficients into (17), we can get a feasible interval of p:

$$\begin{cases} 4p - \frac{9}{4} > 0, \\ \\ -2p^3 + 4p^2 + 11p - 8 > 0. \end{cases}$$

Let denote $p = 1$ and (21) can be transformed according to Proposition 3.2:

$$\begin{cases} X(t) = x + \int_0^t [X(s) + Y(s) - Z(s)]\mathrm{d}s + \int_o^t [2Y(s) + Z(s)]\mathrm{d}\tilde{W}(s), \\ \\ Y(t) = -X(T) + \int_t^T -2X(s)\mathrm{d}s - \int_t^T Z(s)\mathrm{d}\tilde{W}(s), \qquad 0 \le t \le T, \end{cases} \tag{3.12}$$

where $\mathrm{d}\tilde{W}(s) = \mathrm{d}W(s) - \mathrm{d}s$ is a standard Brownian motion under probability measure $\mathbb{Q}$. Here $\mathbb{Q}$ is an equivalent probability measure to $\mathbb{P}$ with

$$\frac{d\mathbb{Q}\ |\mathcal{F}_t}{d\mathbb{P}\ |\mathcal{F}_t} = \exp\{-W(t) - \frac{1}{2}t\}.$$

Note that, in new FBSDEs (22), we can verify following relations:

$$\begin{aligned} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= 2x^2 + xy + y^2 + yz + z^2 \\ &= (x + \frac{1}{2}y)^2 + x^2 + (\frac{2}{3}y + \frac{3}{4}z)^2 + \frac{11}{36}y^2 + \frac{7}{16}z^2 \\ &\ge x^2 + \frac{11}{36}y^2 + \frac{7}{16}z^2 \ge x^2 + \frac{1}{6}(y^2 + z^2) \end{aligned}$$

Recall that $h=-1$, by taking $\beta_1=1$ and $\beta_2=\frac{1}{6}$, then the monotonicity conditions (Lem. 2.4) hold according to case (ii) of Lemma 2.4.

4. Linear transformation method

In this section, we consider a linear transformation method for (10) to get another family of FBSDEs. Owing to non-degeneracy of transformation matrix, new FBSDEs after transformation have the same well-posedness of original FBSDEs. Therefore, we could get the well-posedness of original FBSDEs if FBSDEs after transformation is well posed on $[0,T]$.

Let introduce a non-degenerate 2×2 matrix $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}, a_{ij}\in R, i,j=1,2$. Then we consider a following transformation for (10):

$$\begin{pmatrix}\tilde{X}(t)\\ \tilde{Y}(t)\end{pmatrix}=A\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}X(t)\\ Y(t)\end{pmatrix}=\begin{pmatrix}a_{11}X(t)+a_{12}Y(t)\\ a_{21}X(t)+a_{22}Y(t)\end{pmatrix},\quad \forall t\in[0,T].$$

Also,

$$\begin{cases} X(t)=\dfrac{a_{22}\tilde{X}(t)-a_{12}\tilde{Y}(t)}{|A|} \\ Y(t)=\dfrac{-a_{21}\tilde{X}(t)+a_{11}\tilde{Y}(t)}{|A|}, \end{cases}$$

where $|A|$ represents the determinant of A.

Applying *Itô 's* formula to $\tilde{X}_t$ and $\tilde{Y}_t$, original FBSDEs (10) change into following form:

$$\begin{cases} \mathrm{d}\tilde{X}(t)=[\tilde{b}_1\tilde{X}(t)+\tilde{b}_2\tilde{Y}(t)+\tilde{b}_3\tilde{Z}(t)]\mathrm{d}t+[\tilde{\sigma}_1\tilde{X}(t)+\tilde{\sigma}_2\tilde{Y}(t)+\tilde{\sigma}_3\tilde{Z}(t)]\mathrm{d}W(t), \\ -\mathrm{d}\tilde{Y}(t)=[\tilde{f}_1\tilde{X}(t)+\tilde{f}_2\tilde{Y}(t)+\tilde{f}_3\tilde{Z}(t)]\mathrm{d}t-\tilde{Z}(t)\mathrm{d}W(t), \\ \tilde{X}(0)=\dfrac{|A|}{a_{22}}x+\dfrac{a_{12}}{a_{22}}\tilde{Y}(0), Y(T)=\dfrac{a_{21}+a_{22}h}{a_{11}+a_{12}h}\tilde{X}(T), \qquad 0\le t\le T, \end{cases} \tag{4.1}$$

where

$$\tilde{b}_1=\frac{(a_{11}b_3-a_{12}f_3)({a_{21}}^2\sigma_2-a_{21}a_{22}\sigma_1)+(a_{22}+a_{21}\sigma_3)[a_{22}(a_{11}b_1-a_{12}f_1)-a_{21}(a_{11}b_2-a_{12}f_2)]}{|A|(a_{22}+a_{21}\sigma_3)},$$

$$\tilde{b}_2=\frac{(a_{11}b_3-a_{12}f_3)(a_{12}a_{21}\sigma_1-a_{11}a_{21}\sigma_2)+(a_{22}+a_{21}\sigma_3)[a_{11}(a_{11}b_2-a_{12}f_2)-a_{12}(a_{11}b_1-a_{12}f_1)]}{|A|(a_{22}+a_{21}\sigma_3)},$$

$$\tilde{b}_3=\frac{a_{11}b_3-a_{12}f_3}{a_{22}+a_{21}\sigma_3},$$

$$\tilde{\sigma}_1=\frac{(a_{11}\sigma_3+a_{12})({a_{21}}^2\sigma_2-a_{21}a_{22}\sigma_1)+(a_{22}+a_{21}\sigma_3)(a_{11}a_{22}\sigma_1-a_{11}a_{21}\sigma_2)}{|A|(a_{22}+a_{21}\sigma_3)},$$

$$\tilde{\sigma}_2 = \frac{(a_{11}\sigma_3 + a_{12})(a_{12}a_{21}\sigma_1 - a_{11}a_{21}\sigma_2) + (a_{22} + a_{21}\sigma_3)({a_{11}}^2\sigma_2 - a_{11}a_{12}\sigma_1)}{|A|(a_{22} + a_{21}\sigma_3)}, \tag{4.2}$$

$$\tilde{\sigma}_3 = \frac{a_{11}\sigma_3 + a_{12}}{a_{22} + a_{21}\sigma_3},$$

$$\tilde{f}_1 = \frac{(a_{21}b_3 - a_{22}f_3)(a_{21}a_{22}\sigma_1 - {a_{21}}^2\sigma_2) + (a_{22} + a_{21}\sigma_3)[a_{21}(a_{21}b_2 - a_{22}f_2) - a_{22}(a_{21}b_1 - a_{22}f_1)]}{|A|(a_{22} + a_{21}\sigma_3)},$$

$$\tilde{f}_2 = \frac{(a_{21}b_3 - a_{22}f_3)(a_{11}a_{21}\sigma_2 - a_{12}a_{21}\sigma_1) + (a_{22} + a_{21}\sigma_3)[a_{12}(a_{21}b_1 - a_{22}f_1) - a_{11}(a_{21}b_2 - a_{22}f_2)]}{|A|(a_{22} + a_{21}\sigma_3)},$$

$$\tilde{f}_3 = \frac{a_{22}f_3 - a_{21}b_3}{a_{22} + a_{21}\sigma_3},$$

$$\tilde{Z}(t) = a_{21}\sigma_1 X(t) + a_{21}\sigma_2 Y(t) + (a_{21}\sigma_3 + a_{22})Z(t).$$

That is, after transformation, the coefficient matrix has following structure:

$$\begin{aligned}
\widetilde{M} &= \begin{pmatrix} -\tilde{f}_1 & -\tilde{f}_2 & -\tilde{f}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{\sigma}_1 & \tilde{\sigma}_2 & \tilde{\sigma}_3 \end{pmatrix} \\
&= -\tfrac{1}{|A|(a_{22}+a_{21}\sigma_3)} \begin{pmatrix} a_{21}b_3 - a_{22}f_3 & 0 & 0 \\ 0 & a_{11}b_3 - a_{12}f_3 & 0 \\ 0 & 0 & a_{11}\sigma_3 + a_{12} \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 1 & |A| \\ 1 & 1 & |A| \\ 1 & 1 & |A| \end{pmatrix} \begin{pmatrix} a_{21}a_{22}\sigma_1 - a_{21}^2\sigma_2 & 0 & 0 \\ 0 & a_{11}a_{21}\sigma_2 - a_{12}a_{21}\sigma_1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&\quad + \begin{pmatrix} a_{21}b_1 - a_{22}f_1 & a_{21}b_2 - a_{22}f_2 & 0 \\ a_{11}b_1 - a_{12}f_1 & a_{11}b_2 - a_{12}f_2 & 0 \\ a_{11}\sigma_1 & a_{11}\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & O_{2\times 1} \\ O_{1\times 2} & 0 \end{pmatrix},
\end{aligned}$$

where $O_{1\times 2}$ and $O_{2\times 1}$ denote corresponding null vectors.

Remark 4.1. Note that (23) are also linear FBSDEs of which coefficients become much more complicated after transforming. And also, $(\tilde{X}, \tilde{Y}, \tilde{Z})$ are determined by (X, Y, Z) with respect to transformation matrix A.

Recall that

$$\tilde{F}(y) = \frac{-\begin{vmatrix} \tilde{b}_2 & \tilde{b}_3 \\ \tilde{\sigma}_2 & \tilde{\sigma}_3 \end{vmatrix} y^3 + \left(\tilde{b}_2 + \begin{vmatrix} -\tilde{f}_2 & -\tilde{f}_3 \\ \tilde{\sigma}_2 & \tilde{\sigma}_3 \end{vmatrix} - \begin{vmatrix} \tilde{b}_1 & \tilde{b}_3 \\ \tilde{\sigma}_1 & \tilde{\sigma}_3 \end{vmatrix}\right) y^2 + \left(\tilde{f}_2 + \tilde{b}_1 + \begin{vmatrix} -\tilde{f}_1 & -\tilde{f}_3 \\ \tilde{\sigma}_1 & \tilde{\sigma}_3 \end{vmatrix}\right) y + \tilde{f}_1}{1 - \tilde{\sigma}_3 y}.$$

Then, for simplicity of notation, we assume $\frac{a_{11}}{a_{12}} = m$, $\frac{a_{12}}{a_{22}} = c$ and $\frac{a_{21}}{a_{22}} = n$ which leads to

$$A = \begin{pmatrix} m & 1 \\ nc & c \end{pmatrix}.$$

Substituting (24) into $\tilde{F}(y)$, we have

$$\tilde{\mathcal{L}}(y) = \tilde{F}(y)(1 - \tilde{\sigma}_3 y) = \Lambda_0 y^3 + \Lambda_1 y^2 + \Lambda_2 y + \tilde{f}_1, \tag{4.3}$$

where

$$\begin{aligned}
\Lambda_0 &= -\begin{vmatrix} \tilde{b}_2 & \tilde{b}_3 \\ \tilde{\sigma}_2 & \tilde{\sigma}_3 \end{vmatrix} \\
&= \frac{-\begin{vmatrix} b_2 & b_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} {a_{11}}^3 - \left(b_2 + \begin{vmatrix} -f_2 & -f_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) {a_{11}}^2 a_{12} + \left(f_2 + b_1 + \begin{vmatrix} -f_1 & -f_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) a_{11} {a_{12}}^2 - f_1 {a_{12}}^3}{-|A|\ (a_{22} + a_{21}\sigma_3)} \\
&= \frac{-\begin{vmatrix} b_2 & b_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} m^3 - \left(b_2 + \begin{vmatrix} -f_2 & -f_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) m^2 + \left(f_2 + b_1 + \begin{vmatrix} -f_1 & -f_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right) m - f_1}{(nc - mc)\ (c + nc\sigma_3)},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\Lambda_1 &= \tilde{b}_2 + \begin{vmatrix} -\tilde{f}_2 & -\tilde{f}_3 \\ \tilde{\sigma}_2 & \tilde{\sigma}_3 \end{vmatrix} - \begin{vmatrix} \tilde{b}_1 & \tilde{b}_3 \\ \tilde{\sigma}_1 & \tilde{\sigma}_3 \end{vmatrix} \\
&= \frac{-3\begin{vmatrix} b_2 & b_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} m^2 n - \left(b_2 + \begin{vmatrix} -f_2 & -f_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right)(m^2 + 2mn)}{(n-m)(c + nc\sigma_3)} \\
&\qquad + \frac{\left(f_2 + b_1 + \begin{vmatrix} -f_1 & -f_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right)(2m + n) - 3f_1}{(n-m)(c + nc\sigma_3)}, \\
\Lambda_2 &= \tilde{f}_2 + \tilde{b}_1 + \begin{vmatrix} -\tilde{f}_1 & -\tilde{f}_3 \\ \tilde{\sigma}_1 & \tilde{\sigma}_3 \end{vmatrix} \\
&= \frac{3\begin{vmatrix} b_2 & b_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} mn^2 + \left(b_2 + \begin{vmatrix} -f_2 & -f_3 \\ \sigma_2 & \sigma_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right)(n^2 + 2mn)}{(n-m)(c + nc\sigma_3)} \\
&\qquad - \frac{\left(f_2 + b_1 + \begin{vmatrix} -f_1 & -f_3 \\ \sigma_1 & \sigma_3 \end{vmatrix}\right)(2n + m) - 3f_1}{(n-m)(1 + n\sigma_3)}.
\end{aligned}$$

For cases where Theorem 3.9 do not hold, we apply our results to derive some feasible values of c such that FBSDEs after transformation meet requirements of Theorem 3.9.

Proposition 4.2. *Let $\tilde{b}_i, \tilde{f}_i, \tilde{\sigma}_i, \tilde{h}$ be given in (24) and $\tilde{\mathcal{L}}(y)$ take form in (25). Then $A = \begin{pmatrix} m & 1 \\ Nc & c \end{pmatrix}$ is a linear transformation matrix if one of following cases hold:*

(i) $1 - \tilde{\sigma}_3\tilde{h} > 0$, $\tilde{b}_3\tilde{\sigma}_2 - \tilde{b}_2\tilde{\sigma}_3 \leq 0$ *and* $\tilde{\mathcal{L}}(\tilde{h}) \cdot \tilde{\sigma}_3 \leq 0$*;*
(ii) $1 - \tilde{\sigma}_3\tilde{h} < 0$, $\tilde{b}_3\tilde{\sigma}_2 - \tilde{b}_2\tilde{\sigma}_3 \geq 0$ *and* $\tilde{\mathcal{L}}(\tilde{h}) \cdot \tilde{\sigma}_3 \leq 0$*;*
(iii) $\tilde{\sigma}_3 > 0$, $\tilde{\mathcal{L}}(\frac{1}{\tilde{\sigma}_3}) \leq 0$ *and* $\tilde{\mathcal{L}}(\tilde{h}) \geq 0$;
(iv) $\tilde{\sigma}_3 < 0$, $\tilde{\mathcal{L}}(\frac{1}{\tilde{\sigma}_3}) \geq 0$ *and* $\tilde{\mathcal{L}}(\tilde{h}) \leq 0$;

For cases where coefficients do not hold for Theorem 3.9, we choose proper values of c according to Proposition 4.2. Owing to non-degeneracy of transformation matrix A, we can get well-posedness of original

FBSDEs (10) by

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} m & 1 \\ nc & c \end{pmatrix}^{-1} \begin{pmatrix} \tilde{X}(t) \\ \tilde{Y}(t) \end{pmatrix}, \tag{4.5}$$

$$Z(t) = \frac{\tilde{Z}(t) - nc\sigma_1 X(t) - nc\sigma_2 Y(t)}{(nc\sigma_3 + c)}.$$

By virtue of the Linear Transformation Method, we get a family of FBSDEs after transformation which are equivalent to (10). Hence we could find out a representative of such family which has a lower coupling structure of FBSDEs after transformation (23).

Proposition 4.3. *If n is a zero point of function*

$$H(y) = -(b_3\sigma_2 - b_2\sigma_3)\, y^3 + (b_2 + \sigma_2 f_3 - \sigma_3 f_2 + \sigma_1 b_3 - \sigma_3 b_1)\, y^2 - (f_2 + b_1 + f_3\sigma_1 - f_1\sigma_3)\, y + f_1,$$

FBSDEs after transformation is partial coupled in following form:

$$\begin{cases} \tilde{X}(t) = \dfrac{|A|}{a_{22}} x + \dfrac{a_{12}}{a_{22}} \tilde{Y}(0) + \displaystyle\int_0^t [\tilde{b}_1 \tilde{X}(s) + \tilde{b}_2 \tilde{Y}(s) + \tilde{b}_3 \tilde{Z}(s)] \mathrm{d}s + \int_0^t [\tilde{\sigma}_1 \tilde{X}(s) + \tilde{\sigma}_2 \tilde{Y}(s) + \tilde{\sigma}_3 \tilde{Z}(s)] \mathrm{d}W(s), \\ \tilde{Y}(t) = \dfrac{a_{21} + a_{22}h}{a_{11} + a_{12}h} \tilde{X}(T) + \displaystyle\int_t^T [\tilde{f}_2 \tilde{Y}(s) + \tilde{f}_3 \tilde{Z}(s)] \mathrm{d}s - \int_t^T \tilde{Z}(s) \mathrm{d}W(s), \quad 0 \le t \le T. \end{cases} \tag{4.6}$$

Proof. Note that

$$H(\frac{a_{21}}{a_{22}}) = (a_{21}b_3 - a_{22}f_3)(a_{21}a_{22}\sigma_1 - {a_{21}}^2\sigma_2) + (a_{22} + a_{21}\sigma_3)[a_{21}(a_{21}b_2 - a_{22}f_2) - a_{22}(a_{21}b_1 - a_{22}f_1)$$

Thus $H(n) = 0$ leads to

$$\tilde{f}_1 = 0$$

which completes the proof. □

Remark 4.4. Besides for $\tilde{f}_1$, one can also get another partial coupled FBSDEs after transformation except for $\tilde{\sigma}_3 = 0$. Here we need to point out that $\tilde{\sigma}_3 = 0$ contradicts Proposition 4.2 which has no well-posedness.

Compared to (23), (28) is a partial coupled FBSDEs which has more applications in fields of Partial Differential Equation (PDE), stochastic control and other related fields.

5. The applications to Linear Quadratic (LQ) stochastic control problem

In this section, we could employ some techniques in above sections to study the well-posedness of stochastic Hamiltonian system (8) in Problem (LQ).

The coefficients matrix of (8) is:

$$\begin{pmatrix} SN^{-1}S^\top - R & SN^{-1}B^\top - A & SN^{-1}D^\top - C \\ A - BN^{-1}S^\top & -BN^{-1}B^\top & BN^{-1}D^\top \\ C - DN^{-1}S^\top & -DN^{-1}B^\top & -DN^{-1}D^\top \end{pmatrix}.$$

According to the monotonicity conditions (Lem. 2.4), we have following corollary.

Corollary 5.1. *Stochastic Hamiltonian system (8) is well-posed, for* $\forall t \in [0,T]$ *and any bounded process* $N(\cdot)$*, if one of following conditions hold:*

(i) $N(t) > 0$ *and* $S(t)N^{-1}(t)S(t)^\top - R(t) < 0$*;*
(ii) $N(t) < 0$ *and* $S(t)N^{-1}(t)S(t)^\top - R(t) > 0$.

But for cases in which coefficient matrix does not hold for Corollary 5.1, the linear transformation method could be applied to get the well-posedness of stochastic Hamiltonian system (8). Here we present an example to illustrate.

Example 5.2. Let $A(\cdot), B(\cdot), C(\cdot), D(\cdot), R(\cdot), S(\cdot), N(\cdot)$ and Q be constants. We consider following LQ problem on $t \in [0,T]$:

$$\begin{aligned} &\text{Minimize} \qquad J(u(\cdot)) = \tfrac{1}{2}\mathbb{E}\int_0^T \left[x^2(t) + 4u(t)x(t) - u^2(t)\right] \mathrm{d}t - 4\mathbb{E}\left[x^2(T)\right] \\ &\text{Subject to} \quad \begin{cases} \mathrm{d}x(t) = [x(t) + u(t)]\mathrm{d}t + [x(t) + 2u(t)]\mathrm{d}W(t), \\ x(0) = x. \end{cases} \end{aligned} \tag{5.1}$$

Suppose that $\mathcal{U}_{ad}$ is convex and the corresponding Hamiltonian function H is convex for all $t \in [0,T]$ a.s. From Section 3.5 in [20], one could obtain the optimal control taking forms as $\bar{u}(t) = 2\bar{x}(t) + \bar{y}(t) + 2\bar{z}(t)$ and the corresponding optimal trajectory $\bar{x}(t)$. To make sure the optimality of $(\bar{u}, \bar{x})$ holds, one need to show if there exists a pair of process $(\bar{y}(t), \bar{z}(t))$ such that $(\bar{x}(t), \bar{u}(t), \bar{y}(t), \bar{z}(t))$ satisfying the stochastic Hamiltonian system (8). More precisely, the following FBSDEs are well-posed on $t \in [0,T]$ or not:

$$\begin{cases} \bar{x}(t) = x + \int_0^t [3\bar{x}(s) + \bar{y}(s) - 2\bar{z}(s)]\mathrm{d}s + \int_o^t [5\bar{x}(s) + 2\bar{y}(s) + 4\bar{z}(s)]\mathrm{d}W(s), \\ \bar{y}(t) = -4\bar{x}(T) + \int_t^T [5\bar{x}(s) + 3\bar{y}(s) + 5\bar{z}(s)]\mathrm{d}s - \int_t^T \bar{z}(s)\mathrm{d}W(s), \qquad 0 \le t \le T. \end{cases} \tag{5.2}$$

Note that (30) is a fully coupled FBSDEs with constant coefficients matrix $\begin{pmatrix} -5 & -3 & -5 \\ 3 & 1 & 2 \\ 5 & 2 & 4 \end{pmatrix}$.

Unfortunately, we could not get the existence and uniqueness of FBSDEs (30) by any existing methods. Firstly, the monotonicity conditions (Lem. 2.4) does not hold obviously. In addition, for the Unified Approach (Thm. 3.9), we have

$$\sigma_3 = 4 > 0,\ 1 - \sigma_3 Q = 17 > 0, \quad \mathcal{L}(y) = -8y^3 - 23y^2 + 11y + 5,$$

$$\mathcal{L}(Q) = 365 > 0, \quad \mathcal{L}(\frac{1}{\sigma_3}) = 8 > 0.$$

Thus we can not get well-posedness of (30) according to Theorem 3.9.

By employing the linear transformation method, we need to find a proper transformation matrix $A = \begin{pmatrix} m & 1 \\ nc & c \end{pmatrix}$ such that (23) has a unique solution. To lower coupling level of (23), according to Proposition 4.3, we can get

$$H(y) = 8y^3 - 23y^2 - 11y + 5.$$

Thus we take $n=-0.658$ one of zero points of $H(\cdot)$. And, according to Proposition 4.2, to find a triple (m,c) satisfying one of following inequality systems:

$$\begin{cases} 1-\frac{(4m+1)(nc-4c)}{(c+4nc)(m-4)}>0 & (<0),\\ \frac{8m^3-23m^2-11m+5}{(nc-mc)\ (c+4nc)}\leq 0 & (\geq 0),\\ \tilde{\mathcal{L}}(\frac{nc-4c}{m-4})\cdot\frac{4m+1}{c+4nc}\leq 0, \end{cases} \quad \text{or} \quad \begin{cases} \frac{4m+1}{c+4nc}>0 & (<0),\\ \tilde{\mathcal{L}}(\frac{c+4nc}{4m+1})\leq 0 & (\geq 0),\\ \tilde{\mathcal{L}}(\frac{nc-4c}{m-4})\geq 0 & (\leq 0), \end{cases} \tag{5.3}$$

where $\tilde{\mathcal{L}}(\cdot)$ takes form in (25). For simplicity of calculation, we take $m=1, c=1$ by (31), then we have transformation matrix $A=\begin{pmatrix} 1 & 1\\ -0.658 & 1\end{pmatrix}$. This leads to a new FBSDEs after transformation:

$$\begin{cases} \mathrm{d}\tilde{X}(t)=[8.75\tilde{X}(t)-5.11\tilde{Y}(t)+4.29\tilde{Z}(t)]\mathrm{d}t+[-3.87\tilde{X}(t)+1.84\tilde{Y}(t)-3.06\tilde{Z}(t)]\mathrm{d}W(t),\\ \\ \mathrm{d}\tilde{Y}(t)=[-0.69\tilde{Y}(t)+2.26\tilde{Z}(t)]\mathrm{d}t-\tilde{Z}(t)\mathrm{d}W(t), \quad \tilde{X}_0=1.658x+\tilde{Y}(0), \quad \tilde{Y}_0=1.55\tilde{X}(T), \end{cases} \tag{5.4}$$

where $\tilde{Z}(t)=-3.29\bar{x}(t)-1.32\bar{y}(t)-1.63\bar{z}(t)$, $\tilde{h}=1.55$.

Note that new FBSDEs is partial coupled where the coefficient matrix is $\begin{pmatrix} 0 & -0.69 & 2.26\\ 8.75 & -5.11 & 4.29\\ -3.87 & 1.84 & -3.06\end{pmatrix}$. Substituting this into (25), we have

$$\tilde{\mathcal{L}}(y)=-7.76y^3+3.06y^2+18.17y. \tag{5.5}$$

Note that

$$\tilde{\sigma}_3=-3.06, \quad 1-\tilde{\sigma}_3\tilde{h}=5.74>0, \quad b_3\sigma_2-b_2\sigma_3=-7.76<0, \quad \tilde{\mathcal{L}}(\tilde{h})=6.62>0.$$

According to case (i) of Proposition 4.2, (32) exists a unique solution $(\tilde{X}(t),\tilde{Y}(t))$ for $t\in[0,T]$.

Owing to non-degeneracy of $\begin{pmatrix} 1 & 1\\ -0.658 & 1\end{pmatrix}$, we get a unique solution to original FBSDEs (30):

$$\begin{pmatrix}\bar{x}(t)\\ \bar{y}(t)\end{pmatrix}=\begin{pmatrix} 1 & 1\\ -0.658 & 1\end{pmatrix}^{-1}\begin{pmatrix}\tilde{X}(t)\\ \tilde{Y}(t)\end{pmatrix}=\begin{pmatrix} 0.603 & -0.603\\ 0.397 & 0.603\end{pmatrix}\begin{pmatrix}\tilde{X}(t)\\ \tilde{Y}(t)\end{pmatrix},$$

$$\bar{z}(t)=-\frac{\tilde{Z}(t)+3.29\bar{x}(t)+1.32\bar{y}(t)}{1.63},$$

which implies $\bar{u}(t)=2\bar{x}(t)+\bar{y}(t)+2\bar{z}(t)$ optimal control to LQ problem (29), and $\bar{x}(t)$ is the optimal state trajectory.

6. Conclusion

In this paper, we investigate two families of coupled FBSDEs. Although coefficients of these FBSDEs varies a lot, their well-posedness are proved to be equivalent. We firstly prove that, by a series of coefficients matrix, well-posedness to a family of FBSDEs with different structures are invariant. We also illustrate that such family of FBSDEs are all well posed once we get the well-posedness to one member by any existing methods.

Secondly, by introducing the linear transformation method, we get another equivalent family of FBSDEs to investigate. More importantly, we could lower coupling level of original FBSDEs without losing well-posedness which make it possible to solve fully coupled FBSDEs. Owing to non-degeneracy of transformation matrix, the solution to original FBSDEs could be determined by solutions after transformation.

In addition, we employ our results to study stochastic LQ control problem with non-standard coefficients. Besides for stochastic LQ optimal control problems, the linear transformation method could also be applied in other fields, for example, recursive control problem and partial differential equations.

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