ESAIM: COCV 28 (2022) 65
https://doi.org/10.1051/cocv/2022061

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

OPTIMAL CONTROL OF HYPERBOLIC TYPE DISCRETE AND DIFFERENTIAL INCLUSIONS DESCRIBED BY THE LAPLACE OPERATOR

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Abstract. The paper is devoted to the optimization of a first mixed initial-boundary value problem for hyperbolic differential inclusions (DFIs) with Laplace operator. For this, an auxiliary problem with a hyperbolic discrete inclusion is defined and, using locally conjugate mappings, necessary and sufficient optimality conditions for hyperbolic discrete inclusions are proved. Then, using the method of discretization of hyperbolic DFIs and the already obtained optimality conditions for discrete inclusions, the optimality conditions for the discrete approximate problem are formulated in the form of the Euler-Lagrange type inclusion. Thus, using specially proved equivalence theorems, which are the only tool for constructing Euler-Lagrangian inclusions, we establish sufficient optimality conditions for hyperbolic DFIs. Further, the way of extending the obtained results to the multidimensional case is indicated. To demonstrate the above approach, some linear problems and polyhedral optimization with hyperbolic DFIs are investigated.

Mathematics Subject Classification. 34G25, 90C25, 49M25, 49K20, 35L52.

Received April 10, 2021. Accepted September 18, 2022.

1. Introduction

Over the past decades, great progress has been made in various areas of optimal control problems described by ordinary [1, 2, 4, 7, 9, 10, 14, 19, 21, 23, 25, 30, 31, 33, 34, 37, 40] and partial differential equations/inclusions [3, 5, 11–13, 16, 18, 20, 22, 29, 32, 36, 39, 41]. The work [6] investigates boundary value problems for systems of Hamilton-Jacobi-Bellman first-order partial differential equations and variational inequalities, the solutions of which obey the viability constraints. In the paper [17], stabilization in a finite time of homogeneous quasilinear hyperbolic systems with one side controls and with a nonlinear boundary condition on the other hand is considered. Time-independent feedbacks are presented leading to stabilization in a finite time at any time, larger than the optimal time for null controllability of the linearized system if the initial condition is small enough. The paper [38] studies the behavior of solutions of a one-dimensional hyperbolic relaxation system at large times, which can be written in the form of a nonlinear equation of a damped wave. First, the global existence of a unique solution and its decay properties are proved for sufficiently small initial data. In [8], a general uniqueness theorem for nonlinear hyperbolic systems of partial differential equations in one-space

Keywords and phrases: Hyperbolic inclusions, Euler-Lagrange, Laplace operator, Hamiltonian, equivalence, necessary and sufficient.

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dimension is established. The paper [35] considers optimal control problems for hyperbolic equations with controls in the Neumann boundary conditions with pointwise constraints on the control and state functions. New necessary optimality conditions are derived in the form of Pontryagin's pointwise maximum principle for the problem under consideration with state constraints. The approach is based on modern methods of variational analysis. In papers [11, 12, 19, 37, 40] various problems with ordinary differential equations/inclusions are considered. In [11], a strengthened version of Pontryagin's maximum principle is derived for general nonlinear optimal control problems on time scales in a finite-dimensional space. The finite time can be fixed or not, and in the case of general boundary conditions, the corresponding transversality conditions are derived, the proof of which is based on the Ekland's variational principle. The paper [40] develops the theory of tubes for solving differential inclusions within a given set of sets. The concept is defined as the minimum invariant tube with values in the collection. Under certain requirements on the set, the existence and Lipschitz property of "solution tubes" is proved. The paper [19] investigates infinite-dimensional differential inclusions. Sufficient conditions for nonemptiness of the viability kernels of such systems are obtained. In the paper [37], problems of optimal control of dynamical systems controlled by functional differential inclusions of neutral type with constraints are considered. Therein, to develop the method of discrete approximations and use advanced tools of generalized differentiation, a variational analysis of neutral functional-differential inclusions was carried out and new necessary optimality conditions were obtained, both of the Euler-Lagrange and Hamiltonian types. In [10], by using the Baire category method, an existence result is proven under Carath'eodory assumptions for boundary value problem of Dirichlet type with nonconvex ordinary differential inclusions. In [24] on the basis of the diffusion properties of weak solutions of stochastic partial DFIs, some existence theorems and some properties of solutions are given. In [15], an optimal control problem given by hyperbolic type DFIs with boundary conditions and endpoint constraints is studied.

The present paper is dedicated to one of the difficult and interesting fields, where the main goal is to derive sufficient optimality conditions for problems with hyperbolic DFIs in twospatial dimension. Here we obtain the results in the same spirit as in [33], where the Mayer problem with higher order evolution differential inclusions and functional constraints of optimal control theory is studied. Recall that in [33] the obtained transversality and complementary slackness conditions for second-order ordinary differential inclusions play an essential role in further studies, where the results are generalized to a problem with an arbitrary higher-order differential inclusion.

Hyperbolic partial differential equations arise in many physics problems such as string vibration, acoustic modelling, and supersonic fluid flow used in high-speed applications such as rocket motion and supersonic jets. One of the most important hyperbolic equations is the wave equation, which is widely used in modelling electromagnetic waves, seismic waves and shock waves, earthquake engineering, ocean wave propagation and many others. In the mathematical model of hyperbolic partial differential equations, waves propagate at a finite speed, in contrast to parabolic partial differential equations, where the initial value immediately affects the solution at all other points in space. The paper is organized in the following order.

In Section 2, the needed concepts and results, such as, Hamiltonian function H_F and argmaximum sets of a set-valued mapping F, the locally adjoint mappings (LAMs), etc., from the book of Mahmudov [28], are given and the problem with discrete and continuous hyperbolic inclusions are stated. Moreover, some "non-degeneracy condition", that is, the standard condition for convex analysis about the existence of an interior point is formulated.

In Section 3, for a problem with discrete inclusions of hyperbolic type, we prove a theorem on necessary and sufficient conditions for optimality. It is shown that under non-degeneracy condition the necessary conditions are also sufficient for the optimality. In the proof of this theorem, the main role is played by the reduction of the problem posed to the problem of mathematical programming and, as a consequence, the application of its methods.

In Section 4 we introduce difference operators defined on three-point models, and using difference approximations of the Laplace operator and grid functions on a uniform grid, we approximate the problem with hyperbolic DFIs. In what follows, with a skillful definition of an auxiliary set-valued mapping $G(\cdot, x, y, t) : \mathbb{R}^{6n} \rightrightarrows \mathbb{R}^n$ involving the original set-valued mapping $F(\cdot, x, y, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ we reduce this problem to a problem with a discrete inclusion and apply the results obtained in Section 3 and obtain a necessary and sufficient optimality condition for a discrete approximate problem. But, in turn, the latter is possible due to the LAM equivalence theorems that provide the transition from LAM G^* to LAM F^*, which, in our opinion, are the only tool for constructing Euler-Lagrangian inclusions for problem (PC) with hyperbolic DFIs. For this, auxiliary results are obtained on the equivalence of the cone of tangent directions and the argmaximum sets.

In Section 5, we use the results of Section 4 to obtain sufficient optimality conditions for hyperbolic DFIs with Laplace operator. The formulation of sufficient conditions is realized by going over to the formal limit in the discrete-approximate problem, as discrete steps tend to zero. Further, for the generalized initial-boundary value problem, by analogy, we indicate the ways of extending Theorem 5.1 to the multidimensional case. Moreover, in the proof of the optimality condition, the Green's formula is used in the multidimensional case. At the end of section some linear problem with hyperbolic DFIs is considered. Our methodology with a transition to a discrete, and then a discrete-approximate problem guarantees to construct step by step the hyperbolic type Euler-Lagrange adjoint DFIs and the "endpoint" and boundary conditions for the problem with hyperbolic DFIs.

Section 6 is devoted to the problem with hyperbolic polyhedral DFIs. Here, the integrand of the Lagrange functional is also a polyhedral function. The Euler-Lagrange type inclusion is constructed, and necessary and sufficient optimality condition for problem with hyperbolic polyhedral DFIs involving Laplace operator are proved.

2. Necessary facts and and problem statements

Although all the necessary definitions and concepts can be found in the book [28], for the convenience of readers, they are given in this section. Let (u,v) and $\langle u,v\rangle$ be a pair and inner product of elements u,v in n-dimensional Euclidean space $\mathbb{R}^n$, respectively. A set-valued mapping $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is convex if its graph $\operatorname{gph} F=\{(u,v): v\in F(u)\}$ is convex in $\mathbb{R}^{2n}, F$ is convex valued if $F(u)$ is a convex set for each $u\in domF=\{u: F(u)\neq\varnothing\}$. For a set-valued mapping, we introduce the Hamiltonian function and the notation of the argmaximum set:

$$H_F(u,v^*)=\sup_v\{\langle v,v^*\rangle : v\in F(u)\}, v^*\in\mathbb{R}^n;\quad F_A(u;v^*)=\{v\in F(u):\langle v,v^*\rangle = H_F(u,v^*)\}.$$

For a convex set-valued F we let $H_F(u,v^*)=+\infty$, if $F(u)=\varnothing$.

The cone of tangent directions at the point $(u,v)\in \operatorname{gph}F$ will be denoted by $K_F(u,v)\equiv K_{gphF}(u,v)$ and for a convex set-valued mapping F it is defined at a point $(u,v)\in\operatorname{gph}F$ as follows

$$K_{gphF}(u,v)=\{\operatorname{cone}(\operatorname{gph}F-(u,v))\}=\{(\bar u,\bar v):\bar u=\mu(\tilde u-u),\bar v=\mu(\tilde v-v),\mu>0,(\tilde u,\tilde v)\in\operatorname{gph}F\}.$$

The cone $K_A(u,v)$ of tangent directions of nonconvex set $A\subset\mathbb{R}^{2n}$ at $(u,v)\in A$ is defined as the set of $(\bar u,\bar v)$ for which there exists a function $\varphi:(0,+\infty)\to\mathbb{R}^{2n}$ with $\varphi(\gamma)\to 0$ as $\gamma\downarrow 0$ such that $(u,v)+(\bar u,\bar v)+\varphi(\gamma)\in A$ for sufficiently small γ.

Recall that a set-valued mapping $F^*(\cdot;(u,v)):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ defined by

$$F^*(v^*;(u,v))=\{u^*:(u^*,-v^*)\in K_F^*(u,v)\}$$

is called the locally adjoint mapping (LAM) to F at a point (u,v) where $K_F^*(u,v)\equiv K^*_{\operatorname{gph}F}(u,v)$ is the cone dual to the cone $K_F(u,v)$. In what follows, we mainly use the equivalent definition of LAM in terms of the Hamiltonian function A set-valued mapping is called the LAM to "nonconvex" mapping F at a point $(u,v)\in\operatorname{gph}F$:

$$F^*(v^*;(u,v)):=\{u^*: H_F(\tilde u,v^*)-H_F(u,v^*)\le\langle u^*,\tilde u-u\rangle,\forall\tilde u\in\mathbb{R}^n\}, v\in F_A(u;v^*).$$

Since for a convex set-valued mapping $H_F(\cdot, v^*)$ is concave function, by Theorem 2.1 [28, p. 62] the following assertion holds $F^*(v^*;(u,v)) = \partial H_F(u,v^*), v \in F_A(u,v^*)$, where $\partial H_F(u,v^*) = -\partial[-H_F(u,v^*)]$. In fact, the given in the paper notion LAM is closely related to the coderivative concept of Mordukhovich [37], which is essentially different for nonconvex mappings. The notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs and is defined as $D^*F(x,v)(v^*) = \{x^* : (x^*, -v^*) \in N((x,v); \operatorname{gph} F)\}$, where $N((x,v); \operatorname{gph} F)$ is a normal cone at $(x,v) \in \operatorname{gph} F$. In the most interesting settings for the theory and applications, coderivatives are nonconvex-valued and hence are not tangentially/derivatively generated. This is the case of the first coderivative for general finite dimensional set-valued mappings for the purpose of applications to optimal control. The main advantage of the definition of LAM is its simplicity.

The cone $K_A(z)$ of tangent directions of the set A at a point $z = (u,v) \in A \subset \mathbb{R}^{2n}$ is called a local tent [28, p. 120] if for each $\bar{z} \in \operatorname{ri} K_A(z)$ there exists a convex cone $K \subseteq K_A(z)$ and a continuous mapping $\psi(\bar{z})$ defined in a neighborhood of the origin such that (1) $\bar{z} \in \operatorname{ri} K$, $\operatorname{Lin} K = \operatorname{Lin} K_A(z)$, where $\operatorname{Lin} K$ is the linear span of K, (2)$\psi(\bar{z}) = \bar{z} + r(\bar{z}), \|\bar{z}\|^{-1} r(\bar{z}) \to 0$, as $\bar{z} \to 0$, (3)$z + \psi(\bar{z}) \in A$, $\quad \overline{\mathrm{z}} \in K \cap \mathrm{S}_\varepsilon(0)$ for some $\varepsilon > 0$, where $\mathrm{S}_\varepsilon(0)$ is the ball of radius ε and with center the origin.

A function g is said to be proper if it does not take the value $-\infty$ and is not identically equal to $+\infty$.

First, consider the following optimization problem with a discrete analogue of the so-called first mixed hyperbolic DFIs, labelled by (PD):

$$\text{minimize} \sum_{\substack{(x,y,t) \in L_0 \times S_0 \times T_0 \\ (x,y) \neq (0,0),(L,S)}} g(u_{x,y,t}, x, y, t), \tag{2.1}$$

$$\text{(PD)} \quad u_{x+1,y,t} \in Q(u_{x-1,y,t}, u_{x,y-1,t}, u_{x,y,t}, u_{x,y+1,t}, u_{x,y,t+1}, u_{x,y,t-1}, x, y, t), (x,y,t) \in L_1 \times S_1 \times T_1, \tag{2.2}$$

$$u_{x,y,0} = \alpha^1_{xy}, u_{x,y,1} = \alpha^2_{xy}, u_{x,0,t} = \beta_{x0t}, u_{x,S,t} = \beta_{xSt}, u_{0,y,t} = \gamma_{0yt}, u_{L,y,t} = \gamma_{Lyt}, \tag{2.3}$$

where L, S, T are fixed natural numbers, $L_i = \{i, \ldots, L-i\}; S_i = \{i, \ldots, S-i\}; T_i = \{i, \ldots, T-i\}$ $(i = 0,1)$ and $g(\cdot, x, y, t) : \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}, Q(\cdot, x, y, t) : \mathbb{R}^{6n} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, $\alpha_{xy}, \beta_{x0t}, \beta_{xSt}, \gamma_{0yt}, \gamma_{Lyt}$ are fixed vectors, for all x, y, t, respectively. A set of vectors $\{u_{x,y,t}\} = \{u_{x,y,t} : (x,y,t) \in L_0 \times S_0 \times T_0, (x,y) \neq (0,0), \ (L,S)\}$, is called a feasible solution for the problem (2.1)–(2.3) if it satisfies the inclusion (2.2) and initial and boundary conditions (2.3). The problem consists in finding a solution $\{\tilde{u}_{x,y,t}\}$ of a hyperbolic discrete inclusions (2.2) that minimizes (2.1).

Section 4 of the present paper is devoted to the study of a first mixed boundary value problem for hyperbolic DFIs in a two-spatial dimension, containing Laplace operator in a bounded rectangular parallelepiped that is sometimes referred to simply as a cuboid:

$$\text{minimize } J[u(\cdot,\cdot,\cdot)] = \int_0^T \iint_D g(u(x,y,t), x, y, t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{2.4}$$

$$\text{(PC)} \qquad \frac{\partial^2 u(x,y,t)}{\partial t^2} - \Delta u(x,y,t) \in F(u(x,y,t), x, y, t), \quad (x,y,t) \in D \times [0,T], \tag{2.5}$$

$$\begin{aligned} u(x,y,0) = \alpha_1(x,y), \frac{\partial u(x,y,0)}{\partial t} = \alpha_2(x,y), u(x,0,t) = \beta_0(x,t), u(x,S,t) \\ = \beta_S(x,t), u(0,y,t) = \gamma_0(y,t), u(L,y,t) = \gamma_L(y,t), D = [0,L] \times [0,S] \end{aligned} \tag{2.6}$$

where $F(\cdot, x, y, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $g(\cdot, x, y, t)$ is proper convex function, Δ is Laplace's operator: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and α_1, α_2 and $\beta_0, \beta_S, \gamma_0, \gamma_L$ are given continuous functions, $\alpha_1, \alpha_2 : D \to \mathbb{R}^n, \beta_0, \beta_S : [0,L] \times [0,T] \to \mathbb{R}^n; \quad \gamma_0, \gamma_L : [0,S] \times [0,T] \to \mathbb{R}^n; L, S, T$ are real positive numbers. Since hyperbolic inclusion (2.5) is second order in time, values of $u(x,y,t)$ and its partial derivative on t must be specified along the initial time. On the other hand, since equation (2.5) is the second order in space, the values of $u(x,y,t)$ must be specified along boundaries of the region $D \times [0,T]$. In fact, $D \times [0,T]$ is a rectangular parallelepiped that is sometimes referred to simply as a cuboid. We label this problem (PC). The problem consists in finding a solution $\tilde{u}(x,y,t)$ of a first mixed initial-boundary value problem (PC) that minimizes (2.4). For convenience, we assume throughout the context that feasible solutions are classical solutions; let $P_T = D \times (0,T), \Gamma_T = \{(x,y) \in \partial D, 0 < t < T\}$, ∂D is the boundary of $D, D_0 = \{(x,y) \in D, t = 0\}$. Then a function $u(x,y,t) \in C^2(P_T) \cap C^1\left(P_T \cup \Gamma_1 \cup \bar{D}_0\right) \cup \Gamma_T \cup \bar{D}_0)$ satisfying the inclusion (2.5), the initial condition $u(x,y,0) = \alpha_1(x,y), u_t(x,y,0) = \alpha_2(x,y)$ on D_0, and the boundary conditions in Γ_T is called the classical solution of the initial-boundary value problem (2.4)–(2.6). Here $C^2(P_T)$ is the space of continuous functions having in P_T all continuous partial derivatives of the first and second order. It should be noted that the definition of a solution in one sense or another (classical, generalized, almost everywhere, etc.) in no way hinders the implementation of this method for the class of problems under consideration.

With respect to ([28], p.122) $h(\bar{u},u)$ is called a CUA of the function $g : \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm\infty\}$ at a point $u \in \operatorname{dom} g = \{u : |g(u)| < +\infty\}$ if $h(\bar{u},u) \geq \Omega(\bar{u},u)$ for all $\bar{u} \neq 0$ and $h(\cdot,u)$ is a convex closed positive homogeneous function, where

$$\Omega(\bar{u},u) = \sup_{r(\cdot)} \limsup_{\alpha\downarrow 0} (1/\alpha)[g(u+\alpha\bar{u}+r(\alpha)) - g(u)], \alpha^{-1}r(\alpha) \to 0.$$

Here the exterior supremum is taken on all $r(\alpha)$ such that $\alpha^{-1}r(\alpha) \to 0$ as $\alpha \downarrow 0$. In addition, let the function $g(u)$ admit CUA $h(\bar{u},u)$ at point u that are continuous with respect to $\bar{u}$. The latter means that the subdifferentials $\partial g(u) := \partial h(0,u)$ is defined.

For the functions $g(\cdot,x,y,t),(x,y,t) \in D \times [0,T]$ and the set-valued mapping $Q(\cdot,x,y,t)$ we impose the following condition. It should only be noted that, since a non-smooth and nonconvex function cannot be approximated in a neighbourhood of a point of non-smoothness with positively homogeneous functions, we use the concept of CUAs.

Condition N Let in problem (PD) the mapping $Q(\cdot,x,y,t)$ be such that the cone of tangent directions $K_{Q(,x,y,t)}(u_{x-1,y,t},u_{x,y-1,t},u_{x,y,t},u_{x,y+1,t},u_{x,y,t+1},u_{x,y,t-1},u_{x+1,y,t})$ is a local tent. In addition, let the functions $g(\cdot,x,y,t)$ admit CUA $h_{x,y,t}(\bar{u},u_{x,y,t})$ at points $u_{x,y,t}$ that are continuous with respect to $\bar{u}$. The latter means that the subdifferentials $\partial g(u_{x,y,t},x,y,t) := \partial h_{x,y,t}(0,u_{x,y,t})$ are defined. The problem (2.1)–(2.3) is called convex, if the mapping $Q(\cdot,x,y,t)$ is convex and the functions $g(\cdot,x,y,t),,(x,y,t) \in L_0 \times S_0 \times T_0,(x,y) \neq (0,0),(L,S)$ are proper convex functions. The following condition guarantees for the problem (PD) the existence of a standard interior point of convex analysis. As usual, ri A be the relative interior of a set A.

Definition 2.1. The convex problem (PD) satisfies the nondegeneracy condition if for some feasible solution $\{u_{x,y,t}\}$ we have either (a) or (b):

(a) $(u_{x-1,y,t},u_{x,y-1,t},u_{x,y,t},u_{x,y+1,t},u_{x,y,t+1},u_{x,y,t-1},u_{x+1,y,t}) \in \operatorname{rigph} Q(\cdot,x,y,t),(x,y,t) \in L_1 \times S_1 \times T_1$, $u_{x,y,t} \in \operatorname{ridom} g(\cdot,x,y,t)$,

(b) $(u_{x-1,y,t},u_{x,y-1,t},u_{x,y,t},u_{x,y+1,t},u_{x,y,t+1},u_{x,y,t-1},u_{x+1,y,t}) \in \operatorname{intgph} Q(\cdot,x,y,t),(x,y,t) \in L_1 \times S_1 \times T_1$, $(x,y,t) \neq (x_0,y_0,t_0)$, where (x_0,y_0,t_0) is the fixed triple and $g(\cdot,x,y,t)$ are continuous at the point $u_{x,y,t}$.

3. Optimization of first mixed initial-boundary value problem for hyperbolic type discrete inclusions

We first consider the convex problem (PD). In order to use convex programming results, we form the $m = n(L+1)(S+1)(T+1)$ dimensional vector $w = (u_0,u_1,\ldots,u_L) \in \mathbb{R}^m$, where $u_x = (u_{x0},u_1,\ldots,u_{xS}) \in \mathbb{R}^{n(S+1)(T+1)}$ and $u_{xy} = (u_{x,y,0},u_{x,y,1},\ldots,u_{x,y,T}) \in \mathbb{R}^{n(T+1)}$. Let us consider the following convex sets defined in the space $\mathbb{R}^m$:

$M_{x,y,t} = \{w = (u_0,u_1,\ldots,u_L) : (u_{x-1,y,t},u_{x,y-1,t},u_{x,y,t},u_{x,y+1,t},u_{x,y,t+1},u_{x,y,t-1},u_{x+1,y,t}) \in \operatorname{gph} Q(\cdot,x,y,t)\}$,

$$
\begin{aligned}
&(x,y,t)\in L_1\times S_1\times T_1,\\
H_{\alpha 1}&=\left\{w=(u_0,u_1,\ldots,u_L): u_{x,y,0}=\alpha^1_{xy},(x,y)\in L_0\times S_0,(x,y)\neq(0,0),(L,S)\right\},\\
H_{\alpha 2}&=\left\{w=(u_0,u_1,\ldots,u_L): u_{x,y,1}=\alpha^2_{xy},(x,y)\in L_0\times S_0,(x,y)\neq(0,0),(L,S)\right\},\\
H_{\beta 0}&=\left\{w=(u_0,u_1,\ldots,u_L): u_{x,0,t}=\beta_{x0t},(x,t)\in L_0\times T_0,(x,t)\neq(0,0),(L,T)\right\},\\
H_{\beta S}&=\left\{w=(u_0,u_1,\ldots,u_L): u_{x,S,t}=\beta_{xSt},(x,t)\in L_0\times T_0,(x,t)\neq(0,0),(L,T)\right\},\\
H_{\gamma 0}&=\left\{w=(u_0,u_1,\ldots,u_L): u_{0,y,t}=\gamma_{0yt},(y,t)\in S_0\times T_0,(y,t)\neq(0,0),(S,T)\right\},\\
H_{\gamma L}&=\left\{w=(u_0,u_1,\ldots,u_L): u_{L,y,t}=\gamma_{Lyt},(y,t)\in S_0\times T_0,(y,t)\neq(0,0),(S,T)\right\}.
\end{aligned}
$$

Now setting

$$g(w)=\sum_{\substack{(x,y,t)\in L_0\times S_(\times T_0\\ (x,t)\neq(0,0),(L,S)}} g\left(u_{x,y,t},x,y,t\right)$$

we reduce the convex problem (PD) to the following convex minimization problem in the space $\mathbb{R}^m$:

$$\text{minimize } g(w) \text{ subject to } w\in N=\left(\bigcap_{(x,y,t)\in L_1\times S_1\times T_1} M_{x,y,t}\right)\cap H_{\alpha 1}\cap H_{\alpha 2}\cap H_{\beta 0}\cap H_{\beta S}\cap H_{\gamma 0}\cap H_{\gamma L}. \quad (3.1)$$

We apply Theorem 3.4 ([28], p. 99) to the convex minimization problem (3.1). For this, it is necessary to calculate the dual cones $K^*_{M_{x,y,t}}(w), K^*_{H_{\alpha 1}}(w), K^*_{H_{\alpha 2}}(w), K^*_{H_{\beta 0}}(w), K^*_{H_{\beta S}}(w), K^*_{H_{\gamma 0}}(w), K^*_{H_{\gamma L}}(w), w\in N$.

Lemma 3.1. *The dual cone $K^*_{M_{x,y,t}}(w)$ to the cone of tangent directions $K_{M_{x,y,t}}(w)$ has a form:*

$$
\begin{aligned}
K^*_{M_{x,y,t}}(w)=\big\{&w^*=(u^*_0,u^*_1,\ldots,u^*_L):\left(u^*_{x-1,y,t},u^*_{x,y-1,t},u^*_{x,y,t},u^*_{x,y+1,t},u^*_{x,y,t+1},u^*_{x,y,t-1},u^*_{x+1,y,t}\right)\\
&\in K^*_{\mathrm{gphQ}}\left(u_{x-1,y,t},u_{x,y-1,t},u_{x,y,t},u_{x,y+1,t},u_{x,y,t+1},u_{x+1,y,t}\right), u^*_{i,j,k}=0,(i,j,k)\neq(x-1,y,t)\\
&(x,y-1,t),(x,y,t),(x,y+1,t),(x,y,t+1),(x,y,t-1),(x+1,y,t),(x,y,t)\in L_1\times S_1\times T_1\big\}
\end{aligned}
$$

Proof. Let $\bar{w}\in K_{M_{x,y,t}}(w), w\in N$. This means that $w+\mu\bar{w}\in M_{x,y,t}$ for sufficiently small $\mu>0$, which is the same as

$$
\begin{aligned}
&(u_{x-1,y,t}+\mu\bar{u}_{x-1,y,t},u_{x,y-1,t}+\mu\bar{u}_{x,y-1,t},u_{x,y,t}+\mu\bar{u}_{x,y,t},u_{x,y+1,t}+\mu\bar{u}_{x,y+1,t},\\
&u_{x,y,t+1}+\mu\bar{u}_{x,y,t+1},u_{x,y,t-1}+\mu\bar{u}_{x,y,t-1},u_{x+1,y,t}+\mu\bar{u}_{x+1,y,t})\in\operatorname{gph}Q(\cdot,x,y,t).
\end{aligned}
\quad (3.2)
$$

On the other hand, $w^*\in K^*_{M_{x,y,}}(w)$ is equivalent to the condition

$$\langle\bar{w},w^*\rangle=\sum_{(x,y,t)\in L_1\times S_1\times T_1}\left\langle\bar{u}_{x,y,t},u^*_{x,y,t}\right\rangle\geq 0,\bar{w}\in K_{M_{x,y,t}}(w),w\in N,$$

where the components $\bar{u}_{x,y,t}$ of thee vector $\bar{w}$(see (3.2)) are arbitrary. Therefore, the last relation is valid only for $u^*_{i,j,k}=0,(i,j,k)\neq(x-1,y,t),(x,y-1,t),(x,y,t),(x,y+1,t),(x,y,t+1),(x,y,t-1),(x+1,y,t),(x,y,t)\in L_1\times S_1\times T_1$. This ends the proof of the lemma. □

Based on Lemma 3.1 and the idea of the proof of Theorem 3.1 [3] and Proposition 3.2 [32], it is easy to establish the validity of the following theorem, the proof of which is omitted.

Theorem 3.2. *Suppose that $Q(\cdot,x,y,t),(x,y,t)\in L_1\times S_1\times T_1$ are convex set-valued mappings, and $g(\cdot,x,y,t)$ are convex proper functions at the points of some feasible solution $\{u_{x,y,t}\}$. Then for the $\{\tilde{u}_{x,y,t}\}$ to be an optimal solution of the problem (PD) it is necessary that there exist a number $\lambda\in\{0,1\}$ and six vectors $\left\{\psi^*_{x,y,t},\xi^*_{x,y,t},u^*_{x,t},\eta^*_{x,y,t},\varphi^*_{x,y,t},\zeta^*_{x,y,t}\right\}$, not all equal to zero, such that:*

(i) $\left(\psi^*_{x,y,t},\xi^*_{x,y,t},u^*_{x-1,y,t},\eta^*_{x,y,t},\varphi^*_{x,y,t},\zeta^*_{x,y,t}\right)$
$\in Q^*\left(u^*_{x,y,t};\left(\tilde{u}_{x-1,y,t},\tilde{u}_{x,y-1,t},\tilde{u}_{x,y,t},\tilde{u}_{x,y+1,t},\tilde{u}_{x,y,t+1},\tilde{u}_{x,y,t-1},\tilde{u}_{x+1,y,t}\right),x,y,t\right)$
$+\{0\}\times\{0\}\times\left\{\psi^*_{x+1,y,t}+\xi^*_{x,y+1,t}+\eta^*_{x,y-1,t}+\zeta^*_{x,y,t+1}+\varphi^*_{x,y,t-1}-\lambda\partial g\left(\tilde{u}_{x,y,t},x,y,t\right)\right\}\times\{0\}\times\{0\}\times\{0\},$
$(x,y,t)\in L_1\times S_1\times T_1$;

(ii) $u^*_{0,y,t}=0,\eta^*_{x,0,t}=0,\xi^*_{x,S,t}=0,\psi^*_{L,y,t}=0,\zeta^*_{x,y,T}=0,\varphi^*_{x,y,T-1}=0.$

Moreover, under the nondegeneracy condition $\lambda=1$ and therefore, these conditions are also sufficient for the optimality of $\{\tilde{u}_{x,y,t}\}$.

Remark 3.3. It should be noted that, if in the problem (2.1)–(2.3) the functions and mappings are polyhedral, then, according to Lemma 1.22 ([28], p. 23) and Theorem 3.4 ([28], p. 99), the nondegeneracy condition in Theorem 3.2 is superfluous.

Theorem 3.4. *Suppose that conditions* N *are satisfied for the non-convex problem (PD). Then, for the optimality of the solution $\{\tilde{u}_{x,y,t}\}$ in the nonconvex problem with a hyperbolic discrete inclusions, it is necessary that there exist a number $\lambda\in\{0,1\}$ and six of vectors $\left\{\psi^*_{x,y,t},\xi^*_{x,y,t},u^*_{x,t},\eta^*_{x,y,t},\varphi^*_{x,y,t},\zeta^*_{x,y,t}\right\}$, not all zero, satisfying the conditions (i) and (ii) of Theorem 3.2.*

Proof. Here condition N guarantees the fulfillment of the conditions of Theorem 3.24 ([28], p. 133). Therefore, in view of this theorem, we have a necessary condition (may be insufficient), as in Theorem 3.2, written out for a non-convex problem. □

4. Optimization of first mixed problem for hyperbolic type discrete-approximate inclusions

Here, to obtain optimality conditions for the problem (PC), as an intermediate stage, we use the difference derivatives to approximate the problem (PC) and, using Theorem 3.2, we formulate a necessary and sufficient condition for it. We choose steps δ,σ and h on the x,y-and t-axis, respectively, using the grid function $u_{x,y,t}=u_{\delta\sigma h}(x,y,t)$ on a uniform grid on $D\times[0,T]$. We introduce the following difference operators, defined on the three-point models ([29], p. 319), [28], $A_{1\delta}=A_1$ and $A_{2\sigma}=A_2$

$$\begin{aligned}A_1u(x,y,t)&=\frac{u(x+\delta,y,t)-2u(x,y,t)+u(x-\delta,y,t)}{\delta^2}\\A_2u(x,y,t)&=\frac{u(x,y+\sigma,t)-2u(x,y,t)+u(x,y-\sigma,t)}{\sigma^2}\\Bu(x,y,t)&=\frac{u(x,y,t+h)-2u(x,y,t)+u(x,y,t-h)}{h^2}\\&x=\delta,\dots,L-\delta;y=\sigma,\dots,S-\sigma;t=0,h,\dots,T-h.\end{aligned}$$

Therefore, we establish the following difference first mixed hyperbolic type initial-boundary value problem (PDA):

$$\text{minimize}\sum_{\substack{(x,y,t)\in L_0\times S_0\times T_0\\(x,y)\neq(0,0),(L,S)}}\delta\sigma hg(u(x,y,t),x,y,t),\tag{4.1}$$

$$\text{(PDA)} \qquad Bu(x,y,t) - A_1 u(x,y,t) - A_2 u(x,y,t) \in F(u(x,y,t),x,y,t), \tag{4.2}$$

$$u(x,y,0) = \alpha_1(x,y), u(x,y,h) = \alpha_1(x,y) + h\alpha_2(x,y), u(x,0,t) = \beta_0(x,t),$$

$$u(x,S,t) = \beta_S(x,t), u(0,y,t) = \gamma_0(y,t), \quad u_{L,y,t} = \gamma_L(y,t), \quad (x,y,t) \in \mathbb{N} \times \mathfrak{T} \times \mathfrak{R},$$

$$\mathbb{N} = \{\delta, \dots, L-\delta\}; \mathfrak{I} = \{\sigma, \dots, S-\sigma\}; \mathfrak{R} = \{0, h, \dots, T-h\}.$$

Recall that (4.1) is the Riemann sum [29] of the triple integral of a continuous function of three variables g over a bounded rectangular parallelepiped in three-dimensional space.

Introducing a new mapping, we reduce the problem (4.1) and (4.2) to a problem of the form (PD). To do this, rewrite discrete-approximate inclusion (4.2) in the equivalent form, where

$$u(x+\delta,y,t) \in -u(x-\delta,y,t) - \theta^2 u(x,y-\sigma,t) + 2\left(1+\theta^2-\rho^2\right) u(x,y,t)$$
$$-\theta^2 u(x,y+\sigma,t) + \rho^2 u(x,y,t+h) + \rho^2 u(x,y,t-h) - \delta^2 F(u(x,t),x,t), \theta = \frac{\delta}{\sigma}, \rho = \frac{\delta}{h}.$$

Now let us denote

$$\begin{gathered} G(u(x-\delta,y,t), u(x,y-\sigma,t), u(x,y,t), u(x,y+\sigma,t), u(x,y,t+h), u(x,y,t-h), x,y,t) \\ = -u(x-\delta,y,t) - \theta^2 u(x,y-\sigma,t) + 2\left(1+\theta^2-\rho^2\right) u(x,y,t) \\ -\theta^2 u(x,y+\sigma,t) + \rho^2 u(x,y,t+h) + \rho^2 u(x,y,t-h) - \delta^2 F(u(x,y,t),x,y,t) \end{gathered}$$

or more abbreviated

$$G\left(u_1,u_2,u,u_3,u_4,u_5,x,y,t\right) = -u_1 - \theta^2 u_2 + 2\left(1+\theta^2-\rho^2\right) u - \theta^2 u_3 + \rho^2 u_4 + \rho^2 u_5 - \delta^2 F(u,x,y,t) \tag{4.3}$$

and rewrite the problem (PDA) as follows:

$$\begin{gathered} \text{minimize} \sum_{\substack{(x,y,t) \in L_0 \times S_0 \times T_0 \\ (x,y) \neq (0,0),(L,S)}} \delta\sigma h g(u(x,y,t),x,y,t) \\ u(x+\delta,y,t) \in G(u(x-\delta,y,t), u(x,y-\sigma,t), u(x,y,t), \\ u(x,y+\sigma,t), u(x,y,t+h), u(x,y,t-h), x,y,t), \\ u(x,y,0) = \alpha_1(x,y), u(x,y,h) = \alpha_1(x,y) + h\alpha_2(x,y), u(x,0,t) = \beta_0(x,t), \\ u(x,S,t) = \beta_S(x,t), u(0,y,t) = \gamma_0(y,t), u(L,y,t) = \gamma_L(y,t), \quad (x,y,t) \in \mathbb{N} \times \mathfrak{I} \times \mathfrak{R}. \end{gathered} \tag{4.4}$$

According to Theorem 3.2 for optimality of the trajectory $\{\tilde{u}(x,y,t)\}$, in problem (4.4) it is necessary that, there exists a six of vectors $\left\{\hat{\psi}^*(x,y,t), \hat{\xi}^*(x,y,t), \hat{u}^*(x,y,t), \hat{\eta}^*(x,y,t),\ \hat{\varphi}^*(x,y,t), \hat{\zeta}^*(x,y,t)\right\}$ and a number

$\lambda = \lambda_{\delta h \sigma} \in \{0,1\}$ not all zero, such that

$$\begin{aligned} &\left(\hat{\psi}^*(x,y,t), \hat{\xi}^*(x,y,t), \hat{u}^*(x-\delta,y,t), \hat{\eta}^*(x,y,t), \hat{\varphi}^*(x,y,t), \hat{\zeta}^*(x,y,t)\right) \\ &\in G^*\left(\hat{u}^*(x,y,t); (\tilde{u}(x-\delta,y,t), \tilde{u}(x,y-\sigma,t), \tilde{u}(x,y,t), \tilde{u}(x,y+\sigma,t), \tilde{u}(x,y,t+h),\right. \\ &\tilde{u}(x+\delta,y,t)), x,y,t) + \{0\} \times \{0\} \times \left\{\hat{\psi}^*(x+\delta,y,t) + \hat{\xi}^*(x,y+\sigma,t) + \hat{\eta}^*(x,y-\sigma,t)\right. \\ &\left. +\hat{\zeta}(x,y,t+h) + \hat{\varphi}^*(x,y,t-h) - \lambda\delta\sigma h \partial g(\tilde{u}(x,y,t),x,y,t)\right\} \times \{0\} \times \{0\} \times \{0\}; \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\hat{u}^*(0,y,t) = 0, \quad \hat{\eta}^*(x,0,t) = 0, \hat{\xi}^*(x,S,t) = 0, \hat{\psi}^*(L,y,t) = 0, \\ &\hat{\zeta}^*(x,y,T) = 0, \quad \hat{\varphi}^*(x,y,T-h) = 0, \quad (x,y,t) \in \mathbb{N} \times \mathfrak{I} \times \mathfrak{R}. \end{aligned} \tag{4.6}$$

Obviously, to form an optimality condition for the discrete-approximate problem (PDA), one must be able to transform from LAM G^* to LAM F^* in (4.5).

For this we need the following auxiliary results.

Theorem 4.1. *Let $G(\cdot,x,y,t) : \mathbb{R}^{6n} \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, defined as follows*

$$\begin{aligned} &G(u_1,u_2,u,u_3,u_4,u_5,x,y,t) = -u_1 - \theta^2 u_2 + 2\left(1+\theta^2-\rho^2\right) u \\ &- \theta^2 u_3 + \rho^2 u_4 + \rho^2 u_5 - \delta^2 F(u,x,y,t), \theta = \delta/\sigma, \rho = \delta/h. \end{aligned}$$

Moreover, let $G(\cdot,x,y,t)$ be a set-valued mapping such that the cone of tangent directions $K_{G(,,x,y,t)}(u_1,u_2,u,u_3,u_4,u_5,v), (u_1,u_2,u,u_3,u_4,u_5,v) \in \operatorname{gph} G(\cdot,x,y,t)$ is a local tent. Then

$$K_{F(,x,y,t)}\left(u, 2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u - \frac{u_1}{\delta^2} - \frac{u_2}{\sigma^2} - \frac{u_3}{\sigma^2} + \frac{u_4}{h^2} + \frac{u_5}{h^2} - \frac{v}{\delta^2}\right)$$

is a local tent to $\operatorname{gph} F(\cdot,x,y,t)$ and the following inclusions are equivalent:
(1) $(\bar{u}_1,\bar{u}_2,\bar{u},\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{v}) \in K_{G(,x,y,t)}(u_1,u_2,u,u_3,u_4,u_5,v)$,
(2) $\left(\bar{u}, 2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\bar{u} - \frac{\bar{u}_1}{\delta^2} - \frac{\bar{u}_2}{\sigma^2} - \frac{\bar{u}_3}{\sigma^2} + \frac{\bar{u}_4}{h^2} + \frac{\bar{u}_5}{h^2} - \frac{\bar{v}}{\delta^2}\right)$
$\in K_{F(,x,y,t)}\left(u, 2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u - \frac{u_1}{\delta^2} - \frac{u_2}{\sigma^2} - \frac{u_3}{\sigma^2} + \frac{u_4}{h^2} + \frac{u_5}{h^2} - \frac{v}{\delta^2}\right)$

Proof. First let us prove the implication (1) $\Rightarrow$ (2). By the definition of a local tent there exist functions $r_i(\bar{z}), i = 0,1,\ldots,6$ and $r(\bar{z}), \bar{z} = (\bar{u}_1,\bar{u}_2,\bar{u},\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{v})$ such that $r_i(\bar{z})\|\bar{z}\|^{-1} \to 0$, $r(\bar{z})\|\bar{z}\|^{-1} \to 0$ as $\bar{z} \to 0$ and

$$\begin{aligned} &v + \bar{v} + r_0(\bar{z}) \in -(u_1+\bar{u}_1+r_1(\bar{z})) - \theta^2(u_2+\bar{u}_2+r_2(\bar{z})) + 2\left(1+\theta^2-\rho^2\right)(u+\bar{u}+r(\bar{z})) \\ &-\theta^2(u_3+\bar{u}_3+r_3(\bar{z})) + \rho^2(u_4+\bar{u}_4+r_4(\bar{z})) + \rho^2(u_5+\bar{u}_5+r_5(\bar{z})) - \delta^2 F(u+\bar{u}+r(\bar{z}),x,y,t) \end{aligned}$$

for sufficiently small $\bar{z} \in K$, where $K \subseteq \operatorname{ri} K_{G(\cdot,x,y,t)}(z)$ is a convex cone.

Transforming this inclusion, we get

$$2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}$$
$$+2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\bar{u}-\frac{1}{\delta^2}\bar{u}_1-\frac{1}{\sigma^2}\bar{u}_2-\frac{1}{\sigma^2}\bar{u}_3+\frac{1}{h^2}\bar{u}_4+\frac{1}{h^2}\bar{u}_5-\frac{\bar{v}}{\delta^2}$$
$$+2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)r(\bar{z})-\frac{1}{\delta^2}r_1(\bar{z})-\frac{1}{\sigma^2}r_2(\bar{z})-\frac{1}{\sigma^2}r_3(\bar{z})+\frac{1}{h^2}r_4(\bar{z})$$
$$+\frac{1}{h^2}r_5(\bar{z})-\frac{1}{\delta^2}r_0(\bar{z})\in F(u+\bar{u}+r(\bar{z}),x,y,t).$$

Recall that, by the definition of a set-valued mapping G, the inclusion $v\in G\,(u_1,u_2,u,u_3,u_4,u_5\,,\,x,y,t)$ implies

$$2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\in F(u,x,y,t).$$

Therefore, from this relation it is clear that

$$\begin{aligned}&\left(\bar{u},2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\bar{u}-\frac{1}{\delta^2}\bar{u}_1-\frac{1}{\sigma^2}\bar{u}_2-\frac{1}{\sigma^2}\bar{u}_3+\frac{1}{h^2}\bar{u}_4+\frac{1}{h^2}\bar{u}_5-\frac{\bar{v}}{\delta^2}\right)\\ &\in K_{F(,,x,y,t)}\left(u,2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\right).\end{aligned}\tag{4.7}$$

By going in the reverse direction, it is also not hard to see from (4.7) that $(\bar{u}_1,\bar{u}_2,\bar{u},\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{v})$ $\in K_{G(,x,y,t)}\,(u_1,u_2,u,u_3,u_4,u_5,v)$. This completes the proof of the theorem. □

Proposition 4.2. *Let a convex-valued mapping $G(\cdot,x,y,t)$ be defined as in Theorem 3.1. Then $G_A\,(u_1,u_2,u,u_3,u_4,u_5,v^*,x,y,t)=F_A\,(u,-v^*,x,y,t)$ and the following statements for argmaximum sets are equivalent*

(1) $v\in G_A\,(u_1,u_2,u,u_3,u_4,u_5,v^*,x,y,t)$ *;*
(2) $2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\in F_A\,(u;-v^*,x,y,t)\,.$

Proof. First of all, we will find the connection between the Hamilton functions H_G and H_F.

$$\begin{aligned}&H_G\,(u_1,u_2,u,u_3,u_4,u_5,v^*)=\sup\{\langle v,v^*\rangle:v\in G\,(u_1,u_2,u,u_3,u_4,u_5,x,y,t)\}\\ &=\left\langle 2\left(1+\theta^2-\rho^2\right)u-u_1-\theta^2u_2-\theta^2u_3+\rho^2u_4+\rho^2u_5,v^*\right\rangle+\delta^2\sup\{\langle v_1,-v^*\rangle:v_1\in F(u,x,y,t)\}\\ &=\left\langle 2\left(1+\theta^2-\rho^2\right)u-u_1-\theta^2u_2-\theta^2u_3+\rho^2u_4+\rho^2u_5,v^*\right\rangle+\delta^2H_F\,(u,-v^*)\,.\end{aligned}$$

Therefore, keeping in mind this formula for H_G and H_F we get the proof of proposition immediately as follows

$$\begin{aligned}&G_A\,(u_1,u_2,u,u_3,u_4,u_5;v^*,x,y,t)=\{v\in G\,(u_1,u_2,u,u_3,u_4,u_5,x,y,t):\\ &\langle v,v^*\rangle=H_G\,(u_1,u_2,u,u_3,u_4,u_5,v^*)\}=\{v\in G\,(u_1,u_2,u,u_3,u_4,u_5,x,y,t):\\ &\langle v,v^*\rangle=\left\langle 2\left(1+\theta^2-\rho^2\right)u-u_1-\theta^2u_2-\theta^2u_3+\rho^2u_4+\rho^2u_5,v^*\right\rangle+\delta^2H_F\,(u,-v^*)\}\\ &=\left\{2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\in F(u,x,y,t):\right.\end{aligned}$$

$$\left\langle 2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2},-v^*\right\rangle$$
$$=H_F\left(u,-v^*\right)\}=F_A\left(u;-v^*,x,y,t\right). \text{ The needed proof is ended.}$$

□

Theorem 4.3. *Suppose that a mapping $G\left(u_1,u_2,u,u_3,u_4,u_5,x,y,t\right)$ is such that the cones of tangent directions $K_{G(,,x,y,t)}\left(u_1,u_2,u,u_3,u_4,u_5,v\right)$ determine a local tent. Then the following inclusions are equivalent under the conditions that $u_1^*=-v^*, u_2^*=u_3^*=-\left(\delta^2/\sigma^2\right)v^*,\ u_4^*=u_5^*=\left(\delta^2/h^2\right)v^*$:*

(1) $\left(u_1^*,u_2^*,u^*,u_3^*,u_4^*,u_5^*\right)\in G*\left(v^*;\left(u_1,u_2,u,u_3,u_4,u_5,v\right),x,y,t\right),$
$v\in G_A\left(u_1,u_2,u,u_3,u_4,u_5;v^*,x,y,t\right), v^*\in\mathbb{R}^n,$

(2) $\dfrac{u^*}{\delta^2}-2\left(\dfrac{1}{\delta^2}+\dfrac{1}{\sigma^2}-\dfrac{1}{h^2}\right)v^*$
$$\in F^*\left(-v^*;\left(u,2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\right),x,y,t\right),$$
$$2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\in F_A\left(u;-v^*,x,y,t\right).$$

Proof. Suppose that the condition (1) of theorem is fulfilled. By the definition of LAM, this means that in the case of (1) we have

$$\begin{gathered}\langle\bar u_1,u_1^*\rangle+\langle\bar u_2,u_2^*\rangle+\langle\bar u,u^*\rangle+\langle\bar u_3,u_3^*\rangle+\langle\bar u_4,u_4^*\rangle+\langle\bar u_5,u_5^*\rangle-\langle\bar v,v^*\rangle\ge 0,\\ \left(\bar u_1,\bar u_2,\bar u,\bar u_3,\bar u_4,\bar u_5,\bar v\right)\in K_{G(,,x,y,t)}\left(u_1,u_2,u,u_3,u_4,u_5,v\right).\end{gathered}\tag{4.8}$$

Using the equivalence of the inclusions (1) and (2) for cones $K_{G(\cdot,x,y,t)}$ and $K_{F(,,x,y,t)}$, we rewrite this inequality in a convenient form as follows

$$\begin{gathered}\langle\bar u,\omega\rangle-\left\langle 2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\bar u-\frac{\bar u_1}{\delta^2}-\frac{\bar u_2}{\sigma^2}-\frac{\bar u_3}{\sigma^2}+\frac{\bar u_4}{h^2}+\frac{\bar u_5}{h^2}-\frac{\bar v}{\delta^2},\psi\right\rangle\ge 0\\ \left(\bar u,2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\bar u-\frac{\bar u_1}{\delta^2}-\frac{\bar u_2}{\sigma^2}-\frac{\bar u_3}{\sigma^2}+\frac{\bar u_4}{h^2}+\frac{\bar u_5}{h^2}-\frac{\bar v}{\delta^2}\right)\\ \in K_{F(\cdot,x,y,t)}\left(u,2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{u_1}{\delta^2}-\frac{u_2}{\sigma^2}-\frac{u_3}{\sigma^2}+\frac{u_4}{h^2}+\frac{u_5}{h^2}-\frac{v}{\delta^2}\right)\end{gathered}\tag{4.9}$$

where ω and ψ are to be determined. Carrying out the necessary transformations in (4.9), we get

$$\begin{gathered}\langle\bar u_1,\psi\rangle+\left\langle\bar u_2,\theta^2\psi\right\rangle+\left\langle\bar u,\delta^2\omega-2\left(1+\theta^2-\rho^2\right)\psi\right\rangle\\ +\left\langle\bar u_3,\theta^2\psi\right\rangle+\left\langle\bar u_4,-\rho^2\psi\right\rangle-\left\langle\bar u_5,\rho^2\psi\right\rangle-\langle\bar v,-\psi\rangle\ge 0.\end{gathered}\tag{4.10}$$

Then comparing (4.8), (4.10) it is not hard to see that we obtain the following formulas for ω and ψ :

$$\omega=\frac{u^*}{\delta^2}-2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)v^*, u_1^*=-v^*, u_2^*=u_3^*=-\frac{\delta^2}{\sigma^2}v^*, u_4^*=u_5^*=\frac{\delta^2}{h^2}v^*, \psi=-v^*.$$

Thus (4.9) means that

$$\frac{u^*}{\delta^2}-2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)v^*$$

$$\in F^*\left(-v^*;\left(u,2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)u-\frac{1}{\delta^2}u_1-\frac{1}{\sigma^2}u_2-\frac{1}{\sigma^2}u_3+\frac{1}{h^2}u_4+\frac{1}{h^2}u_5-\frac{v}{\delta^2}\right),x,y,t\right).$$

Then from Theorem 3.2 we obtain condition (2), *i.e.*, (1) $\Rightarrow$ (2). By analogy, it can be shown that (2) $\Rightarrow$ (1). As for the conditions concerning the argmaximum sets G_A and F_A, then by Theorem 2.1 ([28], p.62), they simply guarantee the non-emptiness of the LAMs G^* and F^*. The proof of the theorem is completed. □

Remark 4.4. It should be recalled that for the transition from a problem with a discrete approximate inclusion to a continuous one, it is very important to express G^* in terms of F^*, which is apparently the only trick for constructing an Euler-Lagrangian inclusion, for hyperbolic problems (PC). Therefore, a slightly different way of constructing the EulerLagrange inclusion is also important. It consists in the fact that we can calculate $\partial H_{G(\cdot,x,y,t)}(u_1,u_2,u,u_3,u_4,u_5,v^*)$ and express it in terms of $\partial H_{F(,x,y,t)}(u,v^*)$ and, then, using Theorem 2.1 ([28], p.62), calculate F^*.

So, let us return to the Euler-Lagrange-type condition (4.5) and transform it into a more convenient form in what follows. Using conditions, $u_1^*=-v^*, u_2^*=u_3^*=-\left(\delta^2/\sigma^2\right)v^*$, $u_4^*=u_5^*=\left(\delta^2/h^2\right)v^*$, the condition (1) of Theorem 3.4, and the Euler - Lagrange type inclusion (4.5), it is easy to see that

$$\begin{aligned}&\hat{\psi}^*(x,y,t)=-\hat{u}^*(x,y,t),\hat{\xi}^*(x,y,t)=\hat{\eta}^*(x,y,t)=-\left(\delta^2/\sigma^2\right)\hat{u}^*(x,y,t),\\&\hat{\varphi}^*(x,y,t)=\hat{\zeta}^*(x,y,t)=\left(\delta^2/h^2\right)\hat{u}^*(x,y,t).\end{aligned}\tag{4.11}$$

Therefore, introducing these expressions for $\hat{\psi}^*(x,y,t),\hat{\xi}^*(x,y,t),\hat{\eta}^*(x,y,t),\hat{\varphi}^*(x,y,t),\hat{\zeta}^*(x,y,t)$ into the Euler-Lagrange type inclusion (4.5) we have

$$\begin{gathered}\Big(-\hat{u}^*(x,y,t),-\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y,t),\hat{u}^*(x-\delta,y,t),-\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y,t),\frac{\delta^2}{h^2}\hat{u}^*(x,y,t),\\ \frac{\delta^2}{h^2}\hat{u}^*(x,y,t)\Big)\in G^*\left(\hat{u}^*(x,y,t);(\tilde{u}(x-\delta,y,t),\tilde{u}(x,y-\sigma,t),\tilde{u}(x,y,t),\tilde{u}(x,y+\sigma,t),\right.\\ \tilde{u}(x,y,t+h),\tilde{u}(x+\delta,y,t)),x,y,t)+\{0\}\times\{0\}\times\Big\{-\hat{u}^*(x+\delta,y,t)-\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y+\sigma,t)\\ -\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y-\sigma,t)+\frac{\delta^2}{h^2}\hat{u}^*(x,y,t+h)+\frac{\delta^2}{h^2}\hat{u}^*(x,y,t-h)\\ -\lambda\delta\sigma h\partial g(\tilde{u}(x,y,t),x,y,t)\Big\}\times\{0\}\times\{0\}\times\{0\},(x,y,t)\in\mathbb{N}\times\mathfrak{J}\times\mathfrak{R}\end{gathered}$$

or in a more convenient form

$$\begin{aligned}&\Big(-\hat{u}^*(x,y,t),-\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y,t),\hat{u}^*(x-\delta,y,t)+\hat{u}^*(x+\delta,y,t)+\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y+\sigma,t)\\&+\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y-\sigma,t)-\frac{\delta^2}{h^2}\hat{u}^*(x,y,t+h)-\frac{\delta^2}{h^2}\hat{u}^*(x,y,t-h),-\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y,t),\frac{\delta^2}{h^2}\hat{u}^*(x,y,t),\\&\frac{\delta^2}{h^2}\hat{u}^*(x,y,t)\Big)\in G^*\left(\hat{u}^*(x,y,t);(\tilde{u}(x-\delta,y,t),\tilde{u}(x,y-\sigma,t),\tilde{u}(x,y,t),\tilde{u}(x,y+\sigma,t),\right.\\&\tilde{u}(x,y,t+h),\tilde{u}(x+\delta,y,t)),x,y,t)+\{0\}\times\{0\}\times\{-\lambda\delta\sigma h\partial g(\tilde{u}(x,y,t),x,y,t)\}\\&\times\{0\}\times\{0\}\times\{0\},(x,y,t)\in\mathbb{N}\times\mathfrak{J}\times\mathfrak{R}.\end{aligned}\tag{4.12}$$

In order to pass from G^* to F^* here, we apply the second condition (2) of Theorem 4.3. By this condition using the difference operators A_1, A_2 and B from (4.12) we derive that

$$\begin{aligned}&\frac{1}{\delta^2}\left[\hat{u}^*(x-\delta,y,t)+\hat{u}^*(x+\delta,y,t)+\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y+\sigma,t)+\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y-\sigma,t)\right.\\&\left.-\frac{\delta^2}{h^2}\hat{u}^*(x,y,t+h)-\frac{\delta^2}{h^2}\hat{u}^*(x,y,t-h)\right]-2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\hat{u}^*(x,y,t)\\&\in F^*\left(-\hat{u}^*(x,t);(\tilde{u}(x,t),B\tilde{u}(x,y,t)-A_1\tilde{u}(x,y,t)-A_2\tilde{u}(x,y,t)),x,y,t\right)\\&\qquad -\lambda\delta\sigma h\partial g(\tilde{u}(x,y,t),x,y,t),(x,y,t)\in\mathbb{N}\times\mathfrak{I}\times\mathfrak{R}.\end{aligned}$$

Now, it is easy to verify that the left hand side of this inclusion has the following very important representation

$$\begin{aligned}&\frac{1}{\delta^2}\left[\hat{u}^*(x-\delta,y,t)+\hat{u}^*(x+\delta,y,t)+\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y+\sigma,t)+\frac{\delta^2}{\sigma^2}\hat{u}^*(x,y-\sigma,t)\right.\\&\left.-\frac{\delta^2}{h^2}\hat{u}^*(x,y,t+h)-\frac{\delta^2}{h^2}\hat{u}^*(x,y,t-h)\right]-2\left(\frac{1}{\delta^2}+\frac{1}{\sigma^2}-\frac{1}{h^2}\right)\hat{u}^*(x,y,t)\\&=\frac{\hat{u}^*(x+\delta,y,t)-2\hat{u}^*(x,y,t)+\hat{u}^*(x-\delta,y,t)}{\delta^2}+\frac{\hat{u}^*(x,y+\sigma,t)-2\hat{u}^*(x,y,t)-\hat{u}^*(x,y-\sigma,t)}{\sigma^2}\\&-\frac{\hat{u}^*(x,y,t+h)-2\hat{u}^*(x,y,t)-\hat{u}^*(x,y,t-h)}{h^2}=A_1\hat{u}^*(x,y,t)+A_2\hat{u}^*(x,y,t)-B\hat{u}^*(x,y,t).\end{aligned}\tag{4.13}$$

Finally, based on (4.13) for the difference boundary value problem (PDA) the required Euler-Lagrange inclusion consists of the following

$$\begin{aligned}&A_1\hat{u}^*(x,y,t)+A_2\hat{u}^*(x,y,t)-B\hat{u}^*(x,y,t)\\&\in F^*\left(-\hat{u}^*(x,t);(\tilde{u}(x,t),B\tilde{u}(x,y,t)-A_1\tilde{u}(x,y,t)-A_2\tilde{u}(x,y,t)),x,y,t\right)\\&-\lambda\delta\sigma h\partial g(\tilde{u}(x,y,t),x,y,t).\end{aligned}\tag{4.14}$$

Recall that the LAM F^* is a positive homogeneous mapping with respect to the first argument. Thus, dividing the left and right sides of (4.14) by $\delta\sigma h$ and then denoting $-\hat{u}^*(x,y,t)/\delta\sigma h\equiv u^*(x,y,t)$, we finally obtain for the problem (PDA) the following Euler-Lagrange type adjoint inclusion

$$\begin{aligned}&Bu^*(x,y,t)-A_1u^*(x,y,t)-A_2u^*(x,y,t)\\&\in F^*\left(u^*(x,y,t);(\tilde{u}(x,y,t),B\tilde{u}(x,y,t)-A_1\tilde{u}(x,y,t)-A_2\tilde{u}(x,y,t)),x,y,t\right)\\&-\lambda\partial g(\tilde{u}(x,y,t),x,y,t),(x,y,t)\in\mathbb{N}\times\mathfrak{I}\times\mathfrak{R}.\end{aligned}\tag{4.15}$$

On the other hand, taking into account (4.11) the initial-boundary conditions will be rewritten as follows $\hat{u}^*(0,y,t)=0, \hat{u}^*(x,0,t)=0, \hat{u}^*(x,S,t)=0, \hat{u}^*(L,y,t)=0, \hat{u}^*(x,y,T)=0, \hat{u}^*(x,y,T-h)=0$ and as a result we finally get

$$u^*(x,y,T)=0,\frac{1}{h}\left(u^*(x,y,T)-u^*(x,y,T-h)\right)=0,\quad u^*(x,0,t)=u^*(x,S,t)=u^*(0,y,t)=u^*(L,y,t)=0.\tag{4.16}$$

Let us state the result obtained.

Theorem 4.5. *Let $F(\cdot,x,y,t)$ be a convex set-valued mapping, and $g(\cdot,x,y,t)$ be a proper convex function. Then for the optimality of the solution $\{\tilde{u}(x,y,t)\}$ in the discrete-approximate problem (PDA) it is necessary that* *there exist a number $\lambda \in \{0,1\}$ and vectors $\{u^*(x,y,t)\}$, not all zero, satisfying (4.15) and (4.16). In addition, under conditions of nondegeneracy, (4.15) and (4.16) are also sufficient for the optimality of $\{\tilde{u}(x,y,t)\}$.*

Remark 4.6. If we consider the nonconvex problem (2.1)–(2.3) for a first mixed hyperbolic discrete inclusions, then it is easy to see that, under the condition N, the Euler-Lagrange type inclusion (4.15) and the initial-boundary values (4.16) may not be sufficient optimality conditions. This is explained by the fact that when forming the optimality condition in the nonconvex case, instead of Theorem 3.2 [28, p. 98], we use Theorems 3.24 [28, p. 133], where the equality $\lambda = 1$ is optional.

5. Sufficient conditions of optimality for the hyperbolic DFIs

In this section, based on the results obtained in Section 3, we formulate a sufficient optimality condition for the continuous problem (PC). To do this, recall that the limits of $A_1v(x,y,t)$, $A_2v(x,y,t)$ and $Bv(x,y,t)$, are second order partial derivatives with respect to x, y and t, respectively, *i.e.*, $\lim_{\delta\to 0} A_1v(x,y,t) = \frac{\partial^2 v(x,y,t)}{\partial x^2}$, $\lim_{\sigma\to 0} A_2v(x,y,t) = \frac{\partial^2 v(x,y,t)}{\partial y^2}$ and $\lim_{h\to 0} Bv(x,y,t) = \frac{\partial^2 v(x,y,t)}{\partial t^2}$. Thus, setting $\lambda = 1$ in (4.6), (4.7) and passing to the formal limit, as the discrete steps δ, σ and h tend to 0 , in term of Laplace operator we find that:

(i) $\frac{\partial^2 u^*(x,y,t)}{\partial t^2} - \Delta u^*(x,y,t) \in F * \left(u^*(x,y,t); \left(\tilde{u}(x,y,t), \frac{\partial^2 \tilde{u}(x,y,t)}{\partial t^2} - \Delta\tilde{u}(x,y,t)\right), x, y, t\right)$
$-\partial g(\tilde{u}(x,y,t),x,y,t), (x,y,t) \in D \times [0,T]$

(ii) $u^*(x,y,T) = 0, \frac{\partial u^*(x,y,T)}{\partial t} = 0, u^*(x,0,t) = u^*(x,S,t) = u^*(0,y,t) = u^*(L,y,t) = 0.$

In fact, the following condition ensures that the LAM F^* is nonempty [Thm. 2.1 ([28], p. 62)] that:

(iii) $\frac{\partial^2 \tilde{u}(x,y,t)}{\partial t^2} - \Delta\tilde{u}(x,y,t) \in F_A\left(\tilde{u}(x,y,t); u^*(x,y,t), x, y, t\right).$

Here we assume that feasible solutions $u^*(x,y,t)$ are classical solutions satisfying an adjoint hyperbolic inclusion of the Euler-Lagrange type (i) and homogeneous end-point and boundary conditions (ii).

As expected, in the next theorem we prove that conditions (i)–(iii) are sufficient conditions of optimality in the first mixed problem (PC) with hyperbolic DFIs in a two spatial dimension.

Theorem 5.1. *Suppose that $g(\cdot,x,y,t)$ is a continuous convex proper function and $F(\cdot,x,y,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping. Then for the optimality of the solution $\tilde{u}(x,y,t)$ in the first mixed problem (PC) with hyperbolic DFIs and Laplace operator it is sufficient that there exists a classical solutions $u^*(x,y,t)$, satisfying the conditions (i)-(iii).*

Proof. By virtue of the definition of LAM and the definition of the Hamiltonian function H_F, the Euler-Lagrange type (i) gains the form

$$\begin{aligned}
&H_F\left(u(x,y,t); u^*(x,y,t)\right) - H_F\left(\tilde{u}(x,y,t); u^*(x,y,t)\right) \\
&\le \left\langle \frac{\partial^2 u^*(x,y,t)}{\partial t^2} - \Delta u^*(x,y,t), u(x,y,t) - \tilde{u}(x,y,t) \right\rangle \\
&\quad + g(u(x,y,t),x,y,t) - g(\tilde{u}(x,y,t),x,y,t).
\end{aligned}$$

Then under the condition (iii) we get

$$\begin{aligned}
&\left\langle \frac{\partial^2 u(x,y,t)}{\partial t^2} - \Delta u(x,y,t), u^*(x,y,t) \right\rangle - \left\langle \frac{\partial^2 \tilde{u}(x,y,t)}{\partial t^2} - \Delta\tilde{u}(x,y,t), u^*(x,y,t) \right\rangle \\
&\le \left\langle \frac{\partial^2 u^*(x,y,t)}{\partial t^2} - \Delta u^*(x,y,t), u(x,y,t) - \tilde{u}(x,y,t) \right\rangle \\
&\quad + g(u(x,y,t),x,y,t) - g(\tilde{u}(x,y,t),x,y,t)
\end{aligned}$$

or

$$\begin{aligned}&g(u(x,y,t),x,y,t)-g(\tilde{u}(x,y,t),x,y,t)\\&\geq\left\langle\frac{\partial^2(u(x,y,t)-\tilde{u}(x,y,t))}{\partial t^2}-\Delta(u(x,y,t)-\tilde{u}(x,y,t)),u^*(x,y,t)\right\rangle\\&-\left\langle\frac{\partial^2u^*(x,y,t)}{\partial t^2}-\Delta u^*(x,y,t),u(x,y,t)-\tilde{u}(x,y,t)\right\rangle\end{aligned}$$

We rewrite this inequality in the following form

$$\begin{aligned}&g(u(x,y,t),x,y,t)-g(\tilde{u}(x,y,t),x,y,t)\\&\geq\left\langle\frac{\partial^2(u(x,y,t)-\tilde{u}(x,y,t))}{\partial t^2},u^*(x,y,t)\right\rangle-\left\langle\frac{\partial^2u^*(x,y,t)}{\partial t^2},u(x,y,t)-\tilde{u}(x,y,t)\right\rangle\\&+\langle\Delta u^*(x,y,t),u(x,y,t)-\tilde{u}(x,y,t)\rangle-\langle\Delta(u(x,y,t)-\tilde{u}(x,y,t)),u^*(x,y,t)\rangle.\end{aligned}\tag{5.1}$$

Notice that it is easy to check the following formulas

$$\begin{aligned}&\left\langle\frac{\partial^2(u(x,y,t)-\tilde{u}(x,y,t))}{\partial t^2},u^*(x,y,t)\right\rangle-\left\langle\frac{\partial^2u^*(x,y,t)}{\partial t^2},u(x,y,t)-\tilde{u}(x,y,t)\right\rangle\\&=\frac{\partial}{\partial t}\left[\left\langle\frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial t},u^*(x,y,t)\right\rangle-\left\langle\frac{\partial u^*(x,y,t)}{\partial t},u(x,y,t)-\tilde{u}(x,y,t)\right\rangle\right],\end{aligned}\tag{5.2}$$

$$\begin{aligned}&\langle\Delta u^*(x,y,t),u(x,y,t)-\tilde{u}(x,y,t)\rangle-\langle\Delta(u(x,y,t)-\tilde{u}(x,y,t)),u^*(x,y,t)\rangle\\&=\left\langle\frac{\partial^2u^*(x,y,t)}{\partial x^2}+\frac{\partial^2u^*(x,y,t)}{\partial y^2},u(x,y,t)-\tilde{u}(x,y,t)\right\rangle\\&-\left\langle\frac{\partial^2(u(x,y,t)-\tilde{u}(x,y,t))}{\partial x^2}+\frac{\partial^2(u(x,y,t)-\tilde{u}(x,y,t))}{\partial y^2},u^*(x,y,t)\right\rangle\\&=\left\langle\frac{\partial^2u^*(x,y,t)}{\partial x^2},u(x,y,t)-\tilde{u}(x,y,t)\right\rangle-\left\langle\frac{\partial^2(u(x,y,t)-\tilde{u}(x,y,t))}{\partial x^2},u^*(x,y,t)\right\rangle\\&=\frac{\partial}{\partial x}\left[\left\langle u(x,y,t)-\tilde{u}(x,y,t),\frac{\partial u^*(x,y,t)}{\partial x}\right\rangle-\left\langle\frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial x},u^*(x,y,t)\right\rangle\right]\\&+\frac{\partial}{\partial y}\left[\left\langle u(x,y,t)-\tilde{u}(x,y,t),\frac{\partial u^*(x,y,t)}{\partial y}\right\rangle-\left\langle\frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial y},u^*(x,y,t)\right\rangle\right]\end{aligned}\tag{5.3}$$

Then in view of (5.2) and (5.3) integrating the inequality (5.1) over the region $D\times[0,T]$, we have

$$\begin{aligned}&\iiint_D^T[g(u(x,y,t),x,y,t)-g(\tilde{u}(x,y,t),x,y,t)]dxydt\\&\geq\int_0^T\iint_D\frac{\partial}{\partial t}\left[\left\langle\frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial t},u^*(x,y,t)\right\rangle-\left\langle\frac{\partial u^*(x,y,t)}{\partial t},u(x,y,t)-\tilde{u}(x,y,t)\right\rangle\mathrm{d}x\mathrm{d}y\mathrm{d}t\right.\\&+\int_0^T\iint_D\frac{\partial}{\partial x}\left[\left\langle u(x,y,t)-\tilde{u}(x,y,t),\frac{\partial u^*(x,y,t)}{\partial x}\right\rangle-\left\langle\frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial x},u^*(x,y,t)\right\rangle\right]\mathrm{d}x\mathrm{d}y\mathrm{d}t\\&+\int_0^T\iint_D\frac{\partial}{\partial y}\left[\left\langle u(x,y,t)-\tilde{u}(x,y,t),\frac{\partial u^*(x,y,t)}{\partial y}\right\rangle-\left\langle\frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial y},u^*(x,y,t)\right\rangle\mathrm{d}x\mathrm{d}y\mathrm{d}t.(28)\right.\end{aligned}\tag{5.4}$$

For brevity of notation, denoting first, second and third integrals on the right hand-side of (5.4) by J_1, J_2 and J_3, respectively, we deduce

$$\begin{aligned} J_1 &= \int_0^T \iint_D \frac{\partial}{\partial t}\left[\left\langle \frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial t}, u^*(x,y,t)\right\rangle - \left\langle \frac{\partial u^*(x,y,t)}{\partial t}, u(x,y,t)-\tilde{u}(x,y,t)\right\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}t \right.\\ &= \iint_D \left[\left\langle \frac{\partial(u(x,y,T)-\tilde{u}(x,y,T))}{\partial t}, u^*(x,y,T)\right\rangle - \left\langle \frac{\partial u^*(x,y,T)}{\partial t}, u(x,y,T)-\tilde{u}(x,y,T)\right\rangle \mathrm{d}x\mathrm{d}y \right.\\ &- \iint_D \frac{\partial}{\partial t}\left[\left\langle \frac{\partial(u(x,y,0)-\tilde{u}(x,y,0))}{\partial t}, u^*(x,y,t)\right\rangle - \left\langle \frac{\partial u^*(x,y,0)}{\partial t}, u(x,y,0)-\tilde{u}(x,y,0)\right\rangle \mathrm{d}x\mathrm{d}y. \right. \end{aligned}$$

Here since $u(x,y,t), \tilde{u}(x,y,t)$ are feasible, by the initial conditions $u(x,y,0) = \tilde{u}(x,y,0) = \alpha_1(x,y), u_t(x,y,0) = \tilde{u}_t(x,y,0) = \alpha_2(x,y)$ and by endpoint conditions $u^*(x,y,T) = 0$, $u_t^*(x,y,T) = 0($ see $(ii))$ it follows that $J_1 = 0$. Let us evaluate the integral J_2.

$$\begin{aligned} J_2 &= \int_0^T \iint_D \frac{\partial}{\partial x}\left[\left\langle u(x,y,t)-\tilde{u}(x,y,t), \frac{\partial u^*(x,y,t)}{\partial x}\right\rangle - \left\langle \frac{\partial(u(x,y,t)-\tilde{u}(x,y,t))}{\partial x}, u^*(x,y,t)\right\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}t \right.\\ &= \int_0^S \int_0^T \left[\left\langle u(L,y,t)-\tilde{u}(L,y,t), \frac{\partial u^*(L,y,t)}{\partial x}\right\rangle - \left\langle \frac{\partial(u(L,y,t)-\tilde{u}(L,y,t))}{\partial x}, u^*(L,y,t)\right\rangle \mathrm{d}y\mathrm{d}t \right.\\ &- \int_0^S \int_0^T \left[\left\langle u(0,y,t)-\tilde{u}(0,y,t), \frac{\partial u^*(0,y,t)}{\partial x}\right\rangle - \left\langle \frac{\partial(u(0,y,t)-\tilde{u}(0,y,t))}{\partial x}, u^*(0,y,t)\right\rangle\right] \mathrm{d}y\mathrm{d}t. \end{aligned} \tag{5.5}$$

Hence, by analogy, since $u(0,y,t) = \tilde{u}(0,y,t) = \gamma_0(y,t), u(L,y,t) = \tilde{u}(L,y,t) = \gamma_L(y,t)$ and by condition $(ii)u^*(0,y,t) = u^*(L,y,t) = 0$, from equality (5.5) we obtain $J_2 = 0$.

It can be shown in exactly the same way that $J_3 = 0$. Indeed

$$\begin{aligned} J_3 &= \int_0^L \int_0^T \left[\left\langle u(x,S,t)-\tilde{u}(x,S,t), \frac{\partial u^*(x,S,t)}{\partial y}\right\rangle - \left\langle \frac{\partial(u(x,S,t)-\tilde{u}(x,S,t))}{\partial y}, u^*(x,S,t)\right\rangle\right] \mathrm{d}x\mathrm{d}t\\ &- \int_0^L \int_0^T \left[\left\langle u(x,0,t)-\tilde{u}(x,0,t), \frac{\partial u^*(x,0,t)}{\partial y}\right\rangle - \left\langle \frac{\partial(u(x,0,t)-\tilde{u}(x,0,t))}{\partial y}, u^*(x,0,t)\right\rangle\right] \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.6}$$

Therefore, since $u(x,0,t) = \tilde{u}(x,0,t) = \beta_0(x,t), u(x,S,t) = \tilde{u}(x,S,t) = \beta_S(x,t)$ and $u^*(x,0,t) = u^*(x,S,t) = 0$ from equality (5.6) we obtain $J_3 = 0$.

Finally, from inequality (5.4) we have $\int_0^T \iint_D [g(u(x,y,t),x,y,t) - g(\tilde{u}(x,y,t),x,y,t)]dxydt \geq 0$ or $J[u(\cdot,;,)] \geq J[\tilde{u}(\cdot,\cdot,)$, for all feasible $u(\cdot,\cdot,\cdot)$. The theorem is proved. □

Remark 5.2. It is well known [34] that in the problem with ordinary polynomial linear differential operators $Ax = \sum_{k=1}^{s} p_k(t) D^k x$ of the s-th order with variable coefficients $p_k : [0,T] \to \mathbb{R}^1$ and with the operator of derivatives $D^k (k = 1, \ldots, s)$ of the k-th order, the EulerLagrange inclusion includes the adjoint operator $A^* x^*(t) = \sum_{k=1}^{s} (-1)^k D^k [p_k(t) x^*(t)]$, where the sign of the corresponding term does not change for even k. The Euler-Lagrange inclusion (i) shows that the same property holds for the second-order hyperbolic operator.

Remark 5.3. The stated problem (2.4)–(2.6) can be generalized for hyperbolical differential inclusions second order; let D be some bounded convex closed region of $\mathbb{R}^n$ and $Q_T \subset \mathbb{R}^{n+1}$ is a bounded cylinder $Q_T = \{x \in D, 0 < t < T\}$ with height $T > 0$ and base D. Let Γ_T be the lateral surface of the cylinder Q_T, *i.e.* $\Gamma_T = \{x \in \partial D, 0 < t < T\}$, where ∂D is a piecewisesmooth boundary of D and $D_0 = \{x \in D, t = 0\}$ be lower base of Q_T.

Then we can state the remineded first mixed problem as follows

$$\begin{aligned} &\text{minimize } \int_0^T \int_D g(u(x,t),x,t)\mathrm{d}x\mathrm{d}t \\ \text{(PGC)}\quad &\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) \in F(u(x,t),x,t), \quad (x,t)\in Q_T, \Delta u(x,t) = \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2} \\ &u(x,0)=\varphi(x), u_t(x,0)=\psi(x), x\in D; u(x,t)=\chi(x,t),(x,t)\in\Gamma_T, \end{aligned}$$

where $F(\cdot,x,t):\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, $g(\cdot,x,t)$ is proper convex function, φ,ψ and χ are continuous functions. Then a function $u(x,t)\in C^2(Q_T)\cap C^1(Q_T\cup\Gamma_r\cup D_0)$ satisfying the above DFIs and the initial-boundary conditions is called the classical solution of this initial-boundary value problem. For this generalized problem (PGC) in the same spirit can be proved the Theorem 5.1 by analogy, where the condition (ii) has a form $u^*(x,T)=0, u_t^*(x,T)=0, x\in D; u^*(x,t)=0,(x,t)\in\Gamma_T$ and in the proof is used the Green's formula in the multidimensional case:

$$\int_D \left[\langle u(x,t)-\tilde{u}(x,t), \Delta u^*(x,t)\rangle - \langle \Delta(u(x,t)-\tilde{u}(x,t)), u^*(x,t)\rangle\right] dx = 0.$$

Now using Theorem 5.1, consider the following linear problem as an example:

$$\begin{aligned} &\text{minimize } J[u(\cdot,\cdot,\cdot)] = \int_0^T \iint_D g(u(x,y,t),x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\frac{\partial^2 u(x,y,t)}{\partial t^2} - \Delta u(x,y,t) = Au(x,y,t) + Bw(x,u,t), w(x,y,t)\in U, \\ &u(x,y,0)=\alpha_1(x,y), \frac{\partial u(x,y,0)}{\partial t} = \alpha_2(x,y), u(x,0,t)=\beta_0(x,t), u(x,S,t) \\ &=\beta_S(x,t), u(0,y,t)=\gamma_0(y,t), u(L,y,t)=\gamma_L(y,t), D=[0,L]\times[0,S], \end{aligned} \tag{5.7}$$

where A and B are $n\times n$ and $n\times r$ matrices, respectively, $U\subset\mathbb{R}^r$ is a convex closed set, and g is continuously differentiable function of u. It is required to find a control parameter $\tilde{w}(x,y,t)\in U,(x,y,t)\in D\times[0,T]$ that minimizes $J[u(\cdot,,))$, . Here $F(u)=Au+BU$. It is easy to see that

$$F^*(v^*,(u,v)) = \begin{cases} A^*v^*, & -B*v^*\in[\text{cone}(U-w)]^*, \\ \varnothing, & -B*v^*\notin[\text{cone}(U-w)]^*, \\ v=Au+Bw. \end{cases} \tag{5.8}$$

Therefore, using (5.8), and Theorem 5.1, we get

$$\frac{\partial^2 u^*(x,y,t)}{\partial t^2} - \Delta u^*(x,y,t) = A^*u^*(x,y,t) - g_u'(\tilde{u}(x,y,t),x,y,t), \tag{5.9}$$

$$u^*(x,y,T)=0, \frac{\partial u^*(x,y,T)}{\partial t} = 0, u^*(x,0,t)=u^*(x,S,t)=u^*(0,y,t)=u^*(L,y,t)=0. \tag{5.10}$$

Next, we have

$$\langle w-\tilde{w}(x,y,t), B^*u^*(x,y,t)\rangle \le 0, \quad w\in U,$$

which implies the Pontryagin maximum principle [39, p.112]:

$$\langle B\tilde{w}(x,y,t), u^*(x,y,t)\rangle = \max_{w\in U} \langle Bw, u^*(x,y,t)\rangle . \tag{5.11}$$

Thus, we have obtained the following result.

Theorem 5.4. *The solution $\tilde{u}(x,y,t)$ of hyperbolic type linear problem (5.7) corresponding to the control function $\tilde{w}(x,y,t)$ minimizes $J[u(\cdot,\cdot)]$, if there exists a solution $u^*(x,y,t)$ of adjoint equation (5.9), satisfying (5.10), (5.11).*

Corollary 5.5. *Suppose that we have an initial-boundary value problem (2.4)–(2.6), where $F(u,x,t)\equiv U$ is a constant mapping, $U\subset\mathbb{R}^n$ is a convex compact set, $g(\cdot,x,y,t)$ is a continuously differentiable function. Then the control function $\tilde{w}(x,y,t)\in U$ minimizes $J[u(\cdot,\cdot)]$, if the solution $u^*(x,y,t)$ of the adjoint equation*

$$\frac{\partial u^*(x,y,t)}{\partial t} + \Delta u^*(x,y,t) = g'_u(\tilde{u}(x,y,t),x,y,t),$$

with homogeneous end-point and boundary conditions (ii) of Theorem 5.1, satisfies the maximum principle

$$\langle \tilde{w}(x,y,t), u^*(x,y,t)\rangle = \max_{w\in U} \langle w, u^*(x,y,t)\rangle .$$

Proof. It is obvious that $\operatorname{gph} F = \mathbb{R}^n\times U$ and so $K_F(u,v)=\mathbb{R}^n\times \operatorname{cone}(U-v)$, whence

$$K_F*(u,v) = \{0\}\times[\operatorname{cone}(U-v)]^*.$$

Therefore, we find that

$$F^*\left(v^*;(u,v)\right) = \begin{cases} 0, & \text{if } v^*\in[\operatorname{cone}(U-v)]^* \\ \varnothing, & \text{if } v^*\notin[\operatorname{cone}(U-v)]^* \end{cases}$$

Then, using Theorem 5.1, we obtain the required result:

$$\frac{\partial^2 u^*(x,y,t)}{\partial t^2} - \Delta u^*(x,y,t) = -g'_u(\tilde{u}(x,y,t),x,y,t); \langle w-\tilde{w}(x,y,t), u^*(x,y,t)\rangle \le 0, \quad w\in U.$$

Note that the same result can be obtained from (5.9) and (5.11), assuming that A is a zero matrix, B is $n\times n$ unit matrix and $U\subset\mathbb{R}^n$. □

Remark 5.6. In particular, if $n=1, U=[-1,+1]$ and $g(u,x,t)\equiv u$, then by WeierstrassPontryagin maximum condition $\tilde{w}(x,y,t)\cdot u^*(x,y,t)=\max_{-1\le w\le 1} w\cdot u^*(x,y,t)$, whence $\tilde{w}(x,y,t)=\operatorname{sgn} u^*(x,y,t)$, that is $\tilde{w}(x,y,t)=1$, if $u^*(x,y,t)>0$ and $\tilde{w}(x,y,t)=-1$, if $u^*(x,y,t)<0$. Therefore, since $g'_u(u(x,y,t),x,y,t)=1$ the solution $\tilde{u}(x,y,t)$ corresponding to the control function $\tilde{w}(x,y,t)=\pm 1$ minimizes $J[u(;,\cdot)]$, if $u^*(x,y,t)$ is a solution of the adjoint equation $\Delta u^*(x,y,t)-\frac{\partial^2 u^*(x,y,t)}{\partial t^2}=1$.

6. Model of hyperbolic problems with polyhedral set-valued mapping

Now consider the following problem with the so-called hyperbolic polyhedral DFI:

$$
\begin{aligned}
&\text{minimize } J[u(\cdot,\cdot,\cdot)] = \int_0^T \iint_D g(u(x,y,t),x,y,t)\mathrm{d}x\mathrm{d}y\mathrm{d}t,\\
\text{(PPC)}\quad & A\left(\frac{\partial^2 u(x,y,t)}{\partial t^2} - \Delta u(x,y,t)\right) - Bu(x,y,t) \le d, (x,y,t) \in D\times[0,T],\\
& u(x,y,0)=\alpha_1(x,y), u_t(x,y,0)=\alpha_2(x,y), u(x,0,t)=\beta_0(x,t), u(x,S,t)\\
& = \beta_S(x,t), u(0,y,t)=\gamma_0(y,t), u(L,y,t)=\gamma_L(y,t), \quad D=[0,L]\times[0,S],
\end{aligned}
$$

where A, B are $s\times n$ dimensional matrices, d is a s-dimensional column-vector, $g(\cdot,x,y,t)$ is a polyhedral function. Here $F(u)=\{v : Au - Bv \le d\}$. Now we will use Theorem 5.1 for problem (PPC). By analogy formula (2.51) [28, p. 84] we derive that

$$F^*\left(v^*;(\tilde u,\tilde v)\right) = \left\{-A^*q : v^* = -B^*q, q\ge 0, \langle A\tilde u - B\tilde v - d, q\rangle = 0\right\}.$$

Then applying Theorem 5.1 we get

$$
\begin{aligned}
\Delta u^*(x,y,t) - \frac{\partial^2 u^*(x,y,t)}{\partial t^2} - A^*q(x,y,t) \in \partial g(\tilde u(x,y,t),x,y,t), u^*(x,y,t) = -B^*q(x,y,t),\\
q(x,y,t)\ge 0, \left\langle A\left(\frac{\partial^2 \tilde u(x,y,t)}{\partial t^2} - \Delta\tilde u(x,y,t)\right) - B\tilde u(x,y,t) - d, q(x,y,t)\right\rangle = 0,\\
u^*(x,y,T)=0, u_t^*(x,y,T)=0, u^*(x,0,t)=u^*(x,S,t)=u^*(0,y,t)=u^*(L,y,t)=0.
\end{aligned}
\tag{6.1}
$$

Differentiating $u^*(x,y,t) = -B^*q(x,y,t)$ and substituting it into the first relation (6.1), we obtain

$$
\begin{aligned}
B^*\left(\frac{\partial^2 q^*(x,y,t)}{\partial t^2} - \Delta q^*(x,y,t)\right) - A^*q(x,y,t) \in \partial g(\tilde u(x,y,t),x,y,t),\\
\left\langle A\left(\frac{\partial^2 \tilde u(x,y,t)}{\partial t^2} - \Delta\tilde u(x,y,t)\right) - B\tilde u(x,y,t) - d, q(x,y,t)\right\rangle = 0,
\end{aligned}
\tag{6.2}
$$

$$
\begin{aligned}
&B^*q^*(x,y,T)=0, B*q_t^*(x,y,T)=0, B^*q^*(x,0,t)\\
&= B^*q^*(x,S,t)=B^*q^*(0,y,t)=B^*q^*(L,y,t)=0.
\end{aligned}
\tag{6.3}
$$

Thus, we obtain the following theorem

Theorem 6.1. *Let $g(\cdot,x,t):\mathbb{R}^n\to\mathbb{R}^1$ be a polyhedral function and that $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ be a polyhedral set-valued mapping. Then for the optimality of the solution $u(x,y,t),(x,y,t)\in D\times[0,T]$ in problem (PPC) with hyperbolic polyhedral DFIs, it is sufficient that there exists a nonnegative function $q(x,y,t)\ge 0,(x,y,t)\in D\times[0,T]$ satisfying the hyperbolic partial DFI of the Euler-Lagrange type (6.2) and the conjugate initial-boundary condition (6.3).*

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