

SEQUENTIAL LINEAR INTEGER PROGRAMMING FOR INTEGER OPTIMAL CONTROL WITH TOTAL VARIATION REGULARIZATION*

SVEN LEYFFER¹ AND PAUL MANNS^{1,2,**} 

Abstract. We propose a trust-region method that solves a sequence of linear integer programs to tackle integer optimal control problems regularized with a total variation penalty. The total variation penalty implies that the considered integer control problems admit minimizers. We introduce a local optimality concept for the problem, which arises from the infinite-dimensional perspective. In the case of a one-dimensional domain of the control function, we prove convergence of the iterates produced by our algorithm to points that satisfy first-order stationarity conditions for local optimality. We demonstrate the theoretical findings on a computational example.

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1. INTRODUCTION

Integer optimal control problems cover many practical problems from different fields like traffic light control [23], gas network control [25], and automotive control [21, 29]. Let $\alpha > 0$, $1 \leq d \in \mathbb{N}$, $1 \leq M \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^d$ be a bounded and connected Lipschitz extension domain. We consider the optimization problem

$$\begin{aligned} \min_{v \in L^2(\Omega)} J(v) &:= F(v) + \alpha \text{TV}(v) \\ \text{s.t. } v(x) &\in \{\nu_1, \dots, \nu_M\} \subset \mathbb{Z} \text{ for almost all (a.a.) } x \in \Omega, \end{aligned} \tag{P}$$

where the function $F : L^2(\Omega) \rightarrow \mathbb{R}$ is lower semi-continuous and bounded from below and $\text{TV} : L^1(\Omega) \rightarrow [0, \infty]$ denotes the total variation. While requiring the inclusion $\{\nu_1, \dots, \nu_M\} \subset \mathbb{Z}$ improves the presentation of our arguments, all of our proofs can be adjusted such that the claims also hold for $\{\nu_1, \dots, \nu_M\} \subset \mathbb{R}$. We assume

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that the control values ν_i are in ascending order, *i.e.*, $\nu_i < \nu_{i+1}$ for all $i \in \{1, \dots, M-1\}$. Problems of this form arise from optimal control problems by choosing $F = j \circ S$, where j is some objective and S the control-to-state operator of some underlying controlled process (*e.g.* some ODE or PDE).

We also introduce the modified control problem, where the total variation regularizer is absent as in $(\hat{\mathbf{P}})$:

$$\begin{aligned} & \inf_{v \in L^2(\Omega)} F(v) \\ & \text{s.t. } v(x) \in \{\nu_1, \dots, \nu_M\} \subset \mathbb{Z} \text{ for a.a. } x \in \Omega. \end{aligned} \tag{\hat{\mathbf{P}}}$$

Since the seminal work of Rudin *et al.* [39], regularization by means of total variation has become a widespread algorithmic tool in mathematical imaging; see, for example, [4, 8, 9, 15, 19, 27, 32, 48] and also triggered interest in the control community; see, for example, [16, 28, 33].

Recently, Clason *et al.* analyzed and demonstrated the usefulness of total variation regularization for multi-material topology optimization [10]. They combined a so-called multibang regularizer [11, 12] with a total variation regularizer to promote both that the optimized function is discrete valued in large parts of the domain and that it has large connected components. This approach can be considered as solving a tight relaxation of an integer control problem because the notions of multi-material topology optimization and integer optimal control coincide if the material function of the former may take only discrete values.

Another actively researched technique in the context of integer optimal control is combinatorial integral approximation decomposition [43], where the solution process is split into the solution of a continuous relaxation and the solution of an approximation problem that can be solved efficiently [31, 35, 36, 41, 42]. The major drawback of this technique is that the fractional-valued solution of the relaxation is approximated in the weak-* topology of L^∞ , which often yields highly oscillating control functions. High oscillations hamper meaningful interpretations and implementations of the computed control functions in practice and correspond to high values of the total variation. In particular, if a fractional-valued control v is approximated by discrete-valued ones v^n in the weak-* topology of L^∞ ($v^n \xrightarrow{*} v$), the corresponding sequence of total variation terms tends to infinity; *i.e.*, we have $\text{TV}(v^n) \rightarrow \infty$ regardless of the value of $\text{TV}(v)$ (see, for example, the comments in Sect. 3 of [26] and Sect. 5 of [3]).

Therefore, minimization and constraining of total variation terms have been included in the approximation step of the combinatorial integral approximation decomposition in recent articles [2, 3, 44]. Furthermore, the convex relaxation of the multibang regularizer can be integrated into the combinatorial integral decomposition approach as well; see [34]. Both of these adjustments can considerably reduce the resulting total variation, or switching costs, for a given discretization grid and approximation quality. High oscillations of the resulting control are often inevitable, however, if high approximation quality is desired.

The reason is that the approximation arguments underlying the combinatorial integral approximation decomposition are valid only for the problem class $(\hat{\mathbf{P}})$ under the assumptions of [31] and do not hold for (\mathbf{P}) . In other words, unlike for the relationship between $(\hat{\mathbf{P}})$ and its continuous relaxation, the relationship between (\mathbf{P}) and its continuous relaxation cannot be exploited with the combinatorial integral decomposition approach.

In contrast to $(\hat{\mathbf{P}})$, however, bounded subsets of the feasible set of (\mathbf{P}) are closed with respect to $\{\nu_1, \dots, \nu_M\}$ -valued control functions. To exploit this feature, therefore, in this article we follow a different approach from the relaxation-based approaches. Specifically, we propose to solve (\mathbf{P}) to local optimality (which will be defined below) by solving a sequence of linear integer programs (IPs) in which we can incorporate local gradient information *i.e.* available due to the function space perspective. Our approach is related to solving a discretized and unregularized optimal control problem to global optimality; see [5]. However, the approach in [5] does not scale well if very fine discretization meshes are used. A similar approach to the proposed one was analyzed and demonstrated recently in [24], where the authors take the perspective of modifying sets instead of functions and their algorithm also seeks for a point that satisfies a local stationarity condition. However, there is no guarantee that a feasible point that satisfies this condition exists, and highly oscillating control functions may occur if (a subsequence of) the iterates converge weakly-* to an infeasible (*i.e.*, fractional-valued) limit function. We also

mention the articles [17, 38] in which trust-region-based heuristics and concepts from nonlinear programming are used to obtain points of low objective value for finite-dimensional mixed-integer nonlinear programs.

We argue in Section 4 that our algorithmic approach has similarities to active set methods. The switching structure of the resulting control settles down eventually, and afterwards only the switch locations of a fixed sequence of switches are optimized until a stationary point is reached. We note that the recently proposed methods on switching time optimization [13, 14], which are based on proximal algorithms, have similarities to this approach.

1.1. Contributions

The topological restrictions induced by the discreteness constraint and the TV-regularizer are strong enough such that (P) admits optimal solutions. We exploit this insight to construct a gradient-based descent algorithm that operates on the feasible set of (P). Specifically, we introduce a trust-region subproblem that becomes an IP after discretization. We present an algorithm that solves a sequence of these trust-region subproblems to optimize (P).

We restrict to one-dimensional domains to analyze the asymptotics of the algorithm in function space. We show that limit points of the computed sequence of iterates satisfy a necessary condition for local optimality of both the trust-region subproblem and (P). The necessary condition may be interpreted as a first-order condition, and the algorithm therefore resembles a trust-region algorithm for nonlinear programming.

We provide numerical evidence that validates the theoretical results. The numerical results also show that—for a fixed fine-discretization grid—the method yields results comparable to tackling the discretized problem, a mixed-integer nonlinear program, with integer programming solvers at much lower computational cost.

1.2. Notation

For $p \in [1, \infty)$, $L^p(\Omega)$ denotes the space of p -integrable functions, and $L^\infty(\Omega)$ denotes the space of all essentially bounded functions. We let $\mathbb{N} = \{0, 1, \dots\}$ denote the set of natural numbers including 0. We denote the Lebesgue measure in \mathbb{R}^d by λ . For $k \in \mathbb{N}$, we denote the k -dimensional Hausdorff measure for subsets of Ω by \mathcal{H}^k . A function $u \in L^1(\Omega)$ is of bounded variation if

$$\text{TV}(u) := \sup \left\{ \int_{\Omega} u(x) \nabla \cdot \phi(x) \, dx \mid \phi \in C_c^1(\Omega)^d \text{ and } \sup_{s \in \Omega} \|\phi(s)\| \leq 1 \right\} < \infty,$$

where $C_c^1(\Omega)$ denotes the class of continuously differentiable functions that are compactly supported in Ω and $\|\cdot\|$ denotes Euclidean norm. The space of functions in $L^1(\Omega)$ with bounded variation is called $BV(\Omega)$ and is a Banach space when equipped with the norm $\|u\|_{BV} = \|u\|_{L^1(\Omega)} + \text{TV}(u)$. We denote the $\{0, 1\}$ -valued indicator function of a set A by χ_A . For an optimization problem (Q), we denote its feasible set by $\mathcal{F}_{(Q)}$.

1.3. Structure of the remainder of the paper

We briefly recall that (P) has minimizers, and we introduce our concept of local solutions in Section 2. In Section 3 we present our proposed trust-region algorithm and the trust-region subproblem. In Section 4 we derive a suitable notion of stationarity for locally optimal points and analyze the asymptotics of the trust-region algorithm, both for the case that $\Omega \subset \mathbb{R}$. We also show how the trust-region subproblems can be approximated on uniform grids that discretize Ω . In Section 5 we provide computational experiments that validate our analysis. We give concluding remarks in Section 6.

2. OPTIMAL SOLUTIONS OF TV-REGULARIZED INTEGER CONTROL PROBLEMS

We show the existence of optimal controls for (P) and introduce the concept of locally optimal solutions for (P).

2.1. Existence of optimal solutions

In contrast to problem (\hat{P}) [30, 31, 36] or problems with a regularizer induced by a convex, but not strictly convex, integral superposition operator [12, 24, 34], we will briefly show that the problem (P) has a solution under the following mild assumptions.

Assumption 2.1.

1. Let $F : L^2(\Omega) \rightarrow \mathbb{R}$ be lower semi-continuous.
2. Let F be bounded from below.

Before stating and proving the existence of a minimizer in Proposition 2.3, we briefly state and prove that the feasible set of (P), $\mathcal{F}_{(P)}$, is closed with respect to norm convergence in $L^1(\Omega)$, which will be employed in the derivation of several results in the remainder of this paper.

Lemma 2.2. *Let $(v^n)_n$ be bounded in $BV(\Omega)$. Let v^n be feasible for (P) for all $n \in \mathbb{N}$. Let $v^n \rightarrow v$ in $L^1(\Omega)$. Then v is feasible for (P).*

Proof. Because v^n converges in $L^1(\Omega)$, it has a pointwise almost everywhere (a.e.) convergent subsequence $(v^{n_k})_k$. The claim follows from $v^{n_k}(x) \rightarrow v(x)$ for a.a. $x \in \Omega$ and $v^{n_k}(x) \in \{\nu_1, \dots, \nu_M\}$ for a.a. $x \in \Omega$. \square

We combine Lemma 2.2 with the properties a minimizing sequence of (P) to prove the existence of minimizers. A minimizing sequence of an optimization problem is a sequence of feasible points such that a corresponding sequence of objective values converges to the infimal objective value of the optimization problem or is unbounded below.

Proposition 2.3. *If Assumption 2.1 holds, then there exists a minimizer $\bar{v} \in BV(\Omega)$ of (P).*

Proof. First, we follow the direct method of calculus of variations to deduce the existence of a convergent minimizing sequence. Second, we prove that the limit is feasible.

Because F and α TV are bounded from below, (P) is bounded from below as well. The feasible set is bounded in $L^\infty(\Omega)$ and hence in $L^1(\Omega)$. Therefore, all minimizing sequences for (P) in $BV(\Omega)$ are bounded in $L^1(\Omega)$ and, because of the presence of the term α TV, also in $BV(\Omega)$. The space $BV(\Omega)$ admits a weak-* topology, and we may extract a weakly-* convergent subsequence from a minimizing sequence in $BV(\Omega)$ that satisfies $v^n \rightarrow \bar{v}$ in $L^1(\Omega)$ (and by boundedness in $L^\infty(\Omega)$ and pointwise a.e. convergence of a subsequence also in $L^2(\Omega)$). Thus $F(\bar{v}) \leq \liminf_{n \rightarrow \infty} F(v^n)$. Moreover, $\text{TV}(\bar{v}) \leq \liminf_{n \rightarrow \infty} \text{TV}(v^n)$ follows from the semi-continuity of TV with respect to weak-* convergence in BV or, more generally, because norms of Banach spaces are always weak-* lower semi-continuous if they arise as dual norms of other Banach spaces.

The feasibility of the limit \bar{v} then follows from Lemma 2.2. \square

Remark 2.4. The existence result in Proposition 2.3 can be derived from general results on the existence of minimizers for functionals that are regularized by total variation; see, for example, Chapter 4 & 5 of [1]. The case of binary controls is mentioned explicitly ([6], Cor. 2.6). Because the argument can be stated briefly and shows how the boundedness of the TV-term implies the existence of discrete-valued minimizers, we provide an explicit proof here.

Remark 2.5. Another mode of convergence in $BV(\Omega)$ that lies between weak-* and norm convergence is strict convergence. This means that in addition to $v^{n_k} \rightarrow v$ in $L^1(\Omega)$, it holds that $\text{TV}(v^{n_k}) \rightarrow \text{TV}(v)$. Constructing strictly convergent sequences to local minimizers will be guiding us in our algorithm construction below.

Remark 2.6. The proof uses a weak-* convergent minimizing sequence. We cannot assume uniform convergence or strict convergence for general minimizing sequences. For example, the typewriter sequence¹ is a $\{0, 1\}$ -valued weak*-convergent sequence in $BV(\Omega)$ with limit 0 but TV-value of 2 for all iterates. It is also not pointwise convergent.

Remark 2.7. As is used in the proof of Lemma 2.2, we can always obtain $v^n \rightarrow v$ in $L^p(\Omega)$ for all $p \in [0, \infty)$ for feasible controls v^n by virtue of Lebesgue's dominated convergence theorem because $\|v^n\|_{L^\infty(\Omega)} \leq \max\{|\nu_i| \mid i \in \{1, \dots, m\}\}$ and Ω is a bounded domain. This allows us to only assume lower semicontinuity instead of the usual weak lower semicontinuity of F . If additional fractional-valued controls are considered that are not regularized with a TV-penalty but, for example, with an L^2 -penalty, F has to be weakly lower semicontinuous for the analysis to hold.

2.2. Locally optimal solutions

We start with the definition of our concept of locally optimal solutions.

Definition 2.8. Let v be feasible for (P) and $r > 0$. Then, we say that v is r -optimal if and only if

$$F(v) + \alpha \text{TV}(v) \leq F(\tilde{v}) + \alpha \text{TV}(\tilde{v}) \text{ for all } \tilde{v} \text{ feasible for (P) with } \|v - \tilde{v}\|_{L^1(\Omega)} \leq r.$$

If v is optimal for (P), then v is also r -optimal for all $r > 0$. Thus, r -optimality is a necessary condition for optimality. This optimality concept is similar to the notion of *local minimizers* for finite-dimensional mixed-integer nonlinear programs [38], where optimality of the integer solution in a neighborhood in \mathbb{R}^n is considered.

In a Euclidean space, one can always choose a neighborhood around a feasible integer point *i.e.* small enough such that it does not contain any further feasible integer points. This is not true in the infinite-dimensional case. On the contrary, the situation in the following example is generic.

Example 2.9. We consider the case of an open domain $\emptyset \neq \Omega \subset \mathbb{R}^d$, $M \geq 2$ and $v \in BV(\Omega) \cap \mathcal{F}_{(P)}$. We may consider a ball B and construct $\hat{v} \in \mathcal{F}_{(P)}$ by setting

$$\hat{v}(x) := \begin{cases} \nu_{i+1} & \text{if } x \in B \text{ and } v(x) = \nu_i \text{ for } i < M, \\ \nu_{M-1} & \text{if } x \in B \text{ and } v(x) = \nu_M, \\ v(x) & \text{if } x \in \Omega \setminus B. \end{cases}$$

Then $\|v - \hat{v}\|_{L^1(\Omega)} \leq \max_{i \neq j} |\nu_i - \nu_j| \lambda(B)$, which tends to zero when driving the radius of B to zero.

Remark 2.10. Optimizing in L^1 -neighborhoods and considering the total variation in the objective rather than as a constraint implies that we base our approach in the weak-* topology of the space $BV(\Omega)$. Alternatively, we could have considered the norm topology (with the additional restriction $\text{TV}(\tilde{v} - v) \leq r$) or the strict topology (with the additional restriction $|\text{TV}(\tilde{v}) - \text{TV}(v)| \leq r$) on $BV(\Omega)$.

We believe that the proposed approach is beneficial for several reasons. The norm and strict topology are more restrictive. In particular, for the setting of one-dimensional domains with finitely many integer-valued controls $\text{TV}(\tilde{v} - v) \leq r$ implies that $\tilde{v} - v$ is a constant function for $r < 1$, see also (4.4) below. Moreover, as we will see in the convergence proofs of the proposed algorithmic framework, the regularization with $\alpha \text{TV}(v)$ (which may be interpreted as a *soft constraint*) is sufficient to obtain strict convergence of the produced subsequences without enforcing the strict topology in the constraint of the trust-region subproblem. Therefore, our algorithmic framework allows that nontrivial changes in the switching structure in each subproblem are feasible but the obtained strict convergence implies that the switching structure eventually settles over the iterations, see also our Remark 4.26 after the convergence proofs.

¹see <https://terrytao.wordpress.com/2010/10/02/245a-notes-4-modes-of-convergence/>

3. SEQUENTIAL LINEAR INTEGER PROGRAMMING ALGORITHM

In this section we develop a function space algorithm to compute r -optimal points of (P). We use a trust-region strategy for globalization. The trust-region subproblem and basic properties are provided in Section 3.1, and the algorithm is presented in Section 3.2. We show that, after discretization of the control with piecewise constant functions (discontinuous Galerkin elements of order 0), the trust-region subproblem becomes an IP.

3.1. Trust-region subproblem

The concept of r -optimality as introduced in Definition 2.8 is local in terms of changes of a function v with respect to $\|\cdot\|_{L^1(\Omega)}$. We can find a sufficient condition for an r -optimal point by studying the following trust-region subproblem, where the objective is linearized partially around \tilde{v} ; *i.e.*, we linearize F while using the exact TV term:

$$(\text{TR}(\tilde{v}, \tilde{g}, \Delta)) := \begin{cases} \min_{v \in L^2(\Omega)} (\tilde{g}, v - \tilde{v})_{L^2(\Omega)} + \alpha \text{TV}(v) - \alpha \text{TV}(\tilde{v}) \\ \text{s.t. } \|v - \tilde{v}\|_{L^1(\Omega)} \leq \Delta, \\ v(x) \in \{\nu_1, \dots, \nu_M\} \text{ a.e.} \end{cases} \quad (\text{TR})$$

In (TR), we use $\tilde{g} = \nabla F(\tilde{v})$ if F is differentiable at $v \in \mathcal{F}(\mathbf{P})$. Otherwise, we assume that $\tilde{g} \in \partial F(\tilde{v}) = \{g \in L^2(\Omega) \mid F(v) - F(\tilde{v}) \geq (g, v - \tilde{v})_{L^2(\Omega)}\}$, and in particular $\partial F(\tilde{v}) \neq \emptyset$.

Remark 3.1. We highlight that for the intended use in optimal control a common structure is $F = j \circ S$, where j is some objective functional, for example, of tracking-type, and S is the solution operator of some ODE or PDE. Then $\tilde{g} = \nabla F(\tilde{v})$ can be determined as usual with one solve of the adjoint equation.

Proposition 3.2. *Let $\tilde{v} \in \mathcal{F}(\mathbf{P}) \cap BV(\Omega)$, $\tilde{g} \in L^2(\Omega)$, and $\Delta > 0$. Then, $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$ has a minimizer.*

Proof. It holds that $\tilde{v} \in \mathcal{F}(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$, and thus the feasible set is nonempty. The constraints imply boundedness of the feasible set in $L^1(\Omega)$ and $L^\infty(\Omega)$. Together with the $\text{TV}(v)$ -term, the objective is bounded below.

We consider a minimizing sequence of $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$; *i.e.*, $(v^n)_n \subset \mathcal{F}(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$ with corresponding objective values converging to the infimal value of $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$. The $\text{TV}(v)$ -term implies boundedness of the sequence $(v^n)_n$ in $BV(\Omega)$ and in turn weak-* convergence $v^{n_k} \xrightarrow{*} v^*$ in $BV(\Omega)$ of a subsequence $(v^{n_k})_k$ for some limit $v^* \in BV(\Omega)$. We use $v^{n_k} \rightarrow v^*$ in $L^1(\Omega)$ for a subsequence denoted by the same symbol and Lemma 2.2 to deduce that $v^* \in \mathcal{F}(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$. With the argument from Proposition 2.3 we obtain $v^{n_k} \rightarrow v^*$ in $L^2(\Omega)$ as well. The continuity of $(g, \cdot - \tilde{v})_{L^2(\Omega)}$ and the lower-semicontinuity of TV imply that v^* is a minimizer. \square

Proposition 3.3. *Let $F : L^2(\Omega) \rightarrow \mathbb{R}$ be convex. Then, $\tilde{v} \in \mathcal{F}(\mathbf{P})$ is r -optimal for (P) if the optimal value of $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$ with $\Delta = r$ and some $\tilde{g} \in \partial F(\tilde{v})$ is nonnegative.*

Proof. Let the optimal value of $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$ be nonnegative for $\tilde{g} \in \partial F(\tilde{v})$ and $\Delta > 0$. Then it follows that

$$0 \leq (\tilde{g}, v - \tilde{v})_{L^2(\Omega)} + \alpha \text{TV}(v) - \alpha \text{TV}(\tilde{v})$$

for all feasible $v \in \mathcal{F}(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$. The convexity of F yields

$$(\tilde{g}, v - \tilde{v})_{L^2(\Omega)} \leq F(v) - F(\tilde{v}).$$

Combining both inequalities yields the claim. \square

Proposition 3.3 shows that if F is convex, then the nonnegativity of the optimal objective value of (TR) is a sufficient condition for r -optimality. However, deriving a necessary condition for stationarity is more involved and will be discussed in Section 4.

Next, we introduce an algorithm that solves subproblems of the form (TR) to find an optimal solution of (P).

3.2. Algorithm statement

Our algorithmic approach for solving (P) to r -optimality for some $r > 0$ is formalized in Algorithm 1. The algorithm consists of an outer and an inner loop. In each outer iteration, the trust-region radius is reset, and then the inner loop is executed to compute the next iterate.

The inner loop solves the trust-region subproblem (TR) for a sequence of shrinking trust-region radii until the predicted reduction, the negative objective of (TR), is less than or equal to zero or a sufficient decrease condition is met. If the predicted reduction is less than or equal to zero, Proposition 3.3 implies that the current iterate v^{n-1} is r -optimal with $r = \Delta^{n,k}$ if F is convex. Consequently, Algorithm 1 terminates in this case.

A usual candidate for a sufficient decrease condition for trust-region problems is the inequality

$$\text{ared}(v^{n-1}, \bar{v}^{n,k}) \geq \sigma \text{pred}(v^{n-1}, \Delta^{n,k}) \quad (3.1)$$

for some $\sigma \in (0, 1)$, where

$$\text{ared}(v^{n-1}, \bar{v}^{n,k}) = F(v^{n-1}) + \alpha \text{TV}(v^{n-1}) - F(\bar{v}^{n,k}) - \alpha \text{TV}(\bar{v}^{n,k})$$

denotes the actual reduction achieved by the candidate $\bar{v}^{n,k}$ computed by the trust-region subproblem and the predicted reduction

$$\text{pred}(v^{n-1}, \Delta^{n,k}) = (\tilde{g}^{n-1}, v^{n-1} - \bar{v}^{n,k})_{L^2(\Omega)} + \alpha \text{TV}(v^{n-1}) - \alpha \text{TV}(\bar{v}^{n,k})$$

is the negative objective of (TR($v^{n-1}, \tilde{g}^{n-1}s, \Delta^{n,k}$)). We show in Section 4 for the case $d = 1$ either that the condition (3.1) implies finite termination of the inner loop or that v^{n-1} satisfies a necessary optimality condition. If the inner loop always terminates finitely, the sequence of iterates has weak-* accumulation points that all satisfy the necessary optimality condition.

Remark 3.4. In this work we analyze a function space algorithm. Specifically, Line 6 cannot be implemented exactly. To obtain our numerical results, we solve a sequence of finite-dimensional subproblems that approximate the infinite-dimensional subproblems.

3.3. Subproblems as linear integer programs for DG0 control discretizations

The feasible control functions can take only finitely many values and we may restrict our considerations to a piecewise constant representative for any $v \in \mathcal{F}_{(P)}$ because of the choice $\alpha > 0$. Let \mathcal{T} be a partition of Ω into finitely many polytopes of dimension d with the set of interior facets $\mathcal{E} \subset \mathcal{T} \times \mathcal{T}$. Let $v \in \mathcal{F}_{(P)}$ and $v(x) = \sum_{T \in \mathcal{T}} v_T \chi_T(x)$ for a.a. $x \in \Omega$ with $v_T \in \{\nu_1, \dots, \nu_M\}$, where $\chi_T(x)$ is the characteristic function of $T \in \mathcal{T}$; *i.e.*, v takes the value v_T on grid cell $T \in \mathcal{T}$. Let $\ell_E = \mathcal{H}^{d-1}(E)$ denote the $d - 1$ -dimensional Hausdorff measure of facet $E \in \mathcal{E}$. We define the selector functions for the grid cells that are connected by facet E as T_1 and T_2 ; that is, $E = (T_1(E), T_2(E))$. Moreover, let $[v]_E$ denote the jump height of the function v across the facet E ; *i.e.*, $[v]_E = v_{T_1(E)} - v_{T_2(E)}$.

With this terminology, we can write the quantities required in (TR) as

$$\begin{aligned} \text{TV}(v) &= \sum_{E \in \mathcal{E}} \int_E |[v]| \, d\mathcal{H}^{d-1} = \sum_{E \in \mathcal{E}} \ell_E |[v]_E| = \sum_{E \in \mathcal{E}} \ell_E |v_{T_1(E)} - v_{T_2(E)}|, \\ (\tilde{g}, v - \tilde{v})_{L^2(\Omega)} &= \sum_{T \in \mathcal{T}} (v_T - \tilde{v}_T) \int_T \tilde{g}(x) \, dx, \end{aligned}$$

Algorithm 1 Sequential linear integer programming method (SLIP) to seek r -optimal points of (P)

Input: F differentiable, and satisfying Assumption 2.1.

Input: $\Delta^0 > 0$, $v^0 \in \mathcal{F}_{(\mathbf{P})}$, $\sigma \in (0, 1)$.

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1: for  $n = 1, \dots$  do
2:    $k \leftarrow 0$ 
3:    $\Delta^{n,0} \leftarrow \Delta^0$ 
4:   repeat
5:      $\tilde{g}^{n-1} \leftarrow$  choose element of  $\partial F(v^{n-1})$ .
6:      $\bar{v}^{n,k} \leftarrow$  minimizer of  $(\text{TR}(v^{n-1}, \tilde{g}^{n-1}, \Delta^{n,k}))$ .
7:      $\text{pred}(v^{n-1}, \Delta^{n,k}) \leftarrow (\tilde{g}^{n-1}, v^{n-1} - \bar{v}^{n,k})_{L^2(\Omega)} + \alpha \text{TV}(v^{n-1}) - \alpha \text{TV}(\bar{v}^{n,k})$ 
8:      $\text{ared}(v^{n-1}, \bar{v}^{n,k}) \leftarrow F(v^{n-1}) + \alpha \text{TV}(v^{n-1}) - F(\bar{v}^{n,k}) - \alpha \text{TV}(\bar{v}^{n,k})$ 
9:     if  $\text{pred}(v^{n-1}, \Delta^{n,k}) \leq 0$  then
10:       Terminate. The predicted reduction for  $v^{n-1}$  is zero ( $v^{n-1}$  is  $\Delta^{n,k}$ -optimal if  $F$  is convex).
11:     else if  $\text{ared}(v^{n-1}, \bar{v}^{n,k}) < \sigma \text{pred}(v^{n-1}, \Delta^{n,k})$  then
12:        $k \leftarrow k + 1$ 
13:        $\Delta^{n,k} \leftarrow \Delta^{n,k-1}/2$ .
14:     else
15:        $v^n \leftarrow \bar{v}^{n,k}$ 
16:        $k \leftarrow k + 1$ 
17:     end if
18:   until  $\text{ared}(v^{n-1}, \bar{v}^{n,k-1}) \geq \sigma \text{pred}(v^{n-1}, \Delta^{n,k-1})$  // sufficient decrease achieved
19: end for

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$$\|v - \tilde{v}\|_{L^1(\Omega)} = \sum_{T \in \mathcal{T}} |v_T - \tilde{v}_T| \lambda(T),$$

where λ denotes the Lebesgue measure on \mathbb{R}^d and $\tilde{g} = \nabla F(\tilde{x}_T)$ in the differentiable case. We use these expressions and auxiliary variables to transform (TR) with the ansatz $v = \sum_{T \in \mathcal{T}} v_T \chi_T$ for a fixed partition \mathcal{T} into the IP

$$(\text{TRIP}(\tilde{v}, \tilde{g}, \Delta)) := \left\{ \begin{array}{l} \min_{v_T, u_T, w_E, \omega} \sum_{T \in \mathcal{T}} c_T (v_T - \tilde{v}_T) + \alpha \omega - \alpha \text{TV}(\tilde{v}) \\ \text{s.t. } -u_T \leq v_T - \tilde{v}_T \leq u_T \text{ for all } T \in \mathcal{T}, \\ u_T \geq 0 \text{ for all } T \in \mathcal{T}, \\ \sum_{T \in \mathcal{T}} u_T \lambda(T) \leq \Delta, \\ -w_E \leq \tilde{v}_{T_1(E)} - \tilde{v}_{T_2(E)} \leq w_E \text{ for all } E \in \mathcal{E}, \\ w_E \geq 0 \text{ for all } E \in \mathcal{E}, \\ \sum_{E \in \mathcal{E}} w_E \ell_E \leq \omega, \\ \tilde{v}_T \in \{\nu_1, \dots, \nu_M\} \text{ for all } T \in \mathcal{T}. \end{array} \right. \quad (\text{TRIP})$$

Here, the following real-valued scalar quantities are fixed inputs for the problem $(\text{TRIP}(v, \tilde{g}, \Delta))$.

1. $\tilde{v}_T \in \{\nu_1, \dots, \nu_M\}$ for all $T \in \mathcal{T}$.
2. $c_T := \int_T \tilde{g}(x) \, dx \in \mathbb{R}$ for all $T \in \mathcal{T}$.
3. $\lambda(T) \geq 0$ for all $T \in \mathcal{T}$.

4. $\alpha > 0$.
5. $\text{TV}(\tilde{v}) \geq 0$.
6. $\Delta \geq 0$.
7. $\ell_E \geq 0$ for all $E \in \mathcal{E}$.

Remark 3.5. Generally, one has no need to solve the $(\text{TRIP}(\tilde{v}, \tilde{g}, \Delta))$ exactly. If the upper bound (incumbent solution) of an iterate in the IP satisfies the acceptance criterion of the trust-region step, then the solution algorithm for (TRIP) may terminate early. The efficacy of such a strategy, however, depends on the costs to evaluate F or a surrogate with sufficient accuracy.

Every implementation of (TRIP) requires an additional approximation step; specifically, the objective coefficient $c_T = \int_T \tilde{g}(x) dx$ requires numerical quadrature. In the case of a uniform grid, where also $\ell_E = \ell_F$ holds for all $E, F \in \mathcal{E}$ (for example, a grid of squares), the order of the optimality error of the solution of (TRIP) to the solution of (TRIP^h) , where the coefficients c_T are replaced by coefficients c_T^h obtained with quadrature, is of the same order as the quadrature error.

To see this, let $c_T^h \approx \int_T \tilde{g}(x) dx$ be such that

$$c_T(1 - \varepsilon) \leq c_T^h \leq c_T(1 + \varepsilon)$$

for all grid cells $T \in \mathcal{T}$. Let v_* denote the solution of (TRIP) , and let v_*^h denote the solution of (TRIP^h) . Because ℓ_E is constant, the second term of the objective is an integer multiple of $\alpha \ell_E$. Therefore, it follows that $\alpha \omega_*^h = \text{TV}(v_*^h) = \text{TV}(v_*) = \alpha \omega_*$ if $\varepsilon < \frac{\alpha \ell_E}{|\mathcal{T}| \max_{i,j} |\nu_i - \nu_j|}$. The terms $\sum_T c_T \tilde{v}_T$ as well as $-\alpha \text{TV}(\tilde{v})$ are constant and may be disregarded. Therefore, the objective difference between the optimal solution with and without discretization of c_T is

$$e_h := \left| \sum_{T \in \mathcal{T}} c_T v_{*,T} - \sum_{T \in \mathcal{T}} c_T^h v_{*,T}^h \right|$$

if $\varepsilon < \frac{\alpha \ell_E}{|\mathcal{T}| \max_{i,j} |\nu_i - \nu_j|}$. Using the optimality for both problems and the fact that

$$\left| \sum_{T \in \mathcal{T}} (c_T^h - c_T) v_T \right| \leq |\mathcal{T}| \varepsilon$$

implies the bound

$$e_h \leq |\mathcal{T}| \varepsilon$$

on the objective error of (TRIP) in this case. This means that if we want to obtain an objective error e_h for (TRIP) , we need to enforce an approximation error of $e_h/|\mathcal{T}|$ for the coefficients c_T^h .

Remark 3.6. After discretization of the quantities (TRIP) can be solved general purpose IP solvers. Efficient combinatorial algorithms that exploit the structure of (TRIP) are preferable and subject to ongoing research.

4. ANALYSIS OF ALGORITHM 1 FOR $d = 1$

We analyze Algorithm 1 in function space for $d = 1$. Our analysis builds on properties of the TV-term and geometric constructions that do not generalize easily for $d \geq 2$. We restrict our analysis to $d = 1$ in this work and comment on arguments that do not work for $d \geq 2$ in the respective subsections (see, *e.g.*, Rems. 4.18 and 4.21).

In Section 4.1 we provide results on functions of bounded variation, which we use repeatedly in the remainder. Then, we derive first-order necessary conditions for r -optimal points in Section 4.2, followed by a sufficient decrease condition and asymptotics of the inner loop of Algorithm 1 in Section 4.3. We prove convergence of Algorithm 1 in function space in Section 4.4. Moreover, in Section 4.5 we show how (TRIP) can approximate (TR) on uniform grids.

Our analysis requires the assumption below.

Assumption 4.1. Let $F : L^2(\Omega) \rightarrow \mathbb{R}$ be twice continuously Fréchet differentiable such that for all $\xi \in L^2(\Omega)$, the bilinear form induced by the Hessian

$$\nabla^2 F(\xi) : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$$

satisfies

$$|\nabla^2 F(\xi)(u, w)| \leq C \|u\|_{L^1(\Omega)} \|w\|_{L^1(\Omega)}$$

for some $C > 0$ and all $u, w \in L^2(\Omega)$.

Assumption 4.1 ensures that $\nabla F(v) \in L^2(\Omega)$ for all feasible v and therefore Proposition 3.2. Assumption 4.1 enforces additional regularity on the term F in the objective. For example, the bilinear form induced by the identity $(u, w) \mapsto (u, w)_{L^2(\Omega)}$ does not satisfy the inequality $(u, w)_{L^2(\Omega)} \leq C \|u\|_{L^1(\Omega)} \|w\|_{L^1(\Omega)}$ for all u and w for any $C > 0$ because $L^1(\Omega)$ is not continuously embedded into $L^2(\Omega)$. This means that we restrict to the case $\tilde{g}^{n-1} = \nabla F(v^{n-1})$ in Line 5 in Algorithm 1. We briefly show that Assumption 4.1 nevertheless covers a large set of problems.

Proposition 4.2. Let $z \in L^2(\Omega)$. If $F(v) := \frac{1}{2} \|Kv - z\|_{L^2(\Omega)}^2$ for a bounded linear operator $K : L^1(\Omega) \rightarrow L^2(\Omega)$ and all $v \in L^2(\Omega)$, then Assumption 4.1 is satisfied.

Proof. Let K^* denote the adjoint operator of K . A straightforward calculation shows that $\nabla F(v) = K^*(Kv - z)$ for all $v \in L^2(\Omega)$ and that $\nabla \mathcal{F}^2(\xi)(u, w) = (K^*Ku, w)_{L^2(\Omega)} = (Ku, Kw)_{L^2(\Omega)}$. Next, the Cauchy–Schwarz inequality yields

$$|\nabla \mathcal{F}^2(\xi)(u, w)| \leq \|Ku\|_{L^2(\Omega)} \|Kw\|_{L^2(\Omega)} \leq \|K\|_{1,2}^2 \|u\|_{L^1(\Omega)} \|w\|_{L^1(\Omega)},$$

where $\|K\|_{1,2} = \sup_{\|u\|_{L^1(\Omega)} \leq 1} \|Ku\|_{L^2(\Omega)}$, which is finite by assumption. This shows the claim. \square

Remark 4.3. Many integral operators and solution operators of differential equations have the required regularity. In particular, our computational example satisfies the prerequisites of Proposition 4.2, as is shown in Section 5.

Our main results on the asymptotics of Algorithm 1 are shown under Assumptions 2.1 and 4.1.

4.1. Preliminary results on functions of bounded variation

Feasible points for (P) with finite objective value are functions of bounded variation that may attain only finitely many different values. Thus, they are also so-called *SBV* functions (*S* for special). In particular, their derivatives are absolutely continuous measures with respect to \mathcal{H}^{d-1} [1].

For $d = 1$, *i.e.*, if $\Omega = (t_0, t_f)$ for $t_0, t_f \in \mathbb{R}$, the TV term can be analyzed and characterized with the help of the analysis of the so-called pointwise variation ([1], Sect. 3.2). For $t := (t_1, \dots, t_{N-1})^T$ with $t_i \leq t_{i+1}$ for all

$i \in \{0, \dots, N-1\}$ and the choice $t_N := t_f$ as well as $a := (a_1, \dots, a_N)^T$ we introduce the notation

$$v^{t,a} := \chi_{(t_0, t_1)} a_1 + \sum_{i=1}^{N-1} \chi_{[t_i, t_{i+1})} a_{i+1}. \quad (4.1)$$

We summarize the relationship between v and $v^{t,a}$ in the following proposition.

Proposition 4.4. *Let $v \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$. Then there exist $t \in \mathbb{R}^{N-1}$ and $a = (a_1, \dots, a_N)^T \in \{\nu_1, \dots, \nu_M\}^N$ with $t_0 < \dots < t_N = t_f$ and $a_i \neq a_{i+1}$ for all $i \in \{1, \dots, N-1\}$ such that $v = v^{t,a}$.*

Proof. The claim follows from the analysis of the so-called pointwise variation ([1], Sect. 3.2). In particular, for $v \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$, there exist $N \in \mathbb{N}$, $t_0 < \dots < t_N = t_f$ and $a_1, \dots, a_N \in \{\nu_1, \dots, \nu_M\}$ such that

$$v = \chi_{(t_0, t_1)} a_1 + \sum_{i=1}^{N-1} \chi_{[t_i, t_{i+1})} a_{i+1}, \text{ and} \quad (4.2)$$

$$\text{TV}(v) = \sum_{i=1}^{N-1} |a_{i+1} - a_i| \leq (N-1)(\nu_M - \nu_1) < \infty. \quad (4.3)$$

By dropping t_i and a_i from the vectors t and a if $a_i = a_{i+1}$ we obtain that $a_i \neq a_{i+1}$ for all $i \in \{1, \dots, N-1\}$. \square

Because $a_i \in \{\nu_1, \dots, \nu_M\}$ for all $i \in \{1, \dots, N\}$, it holds that $|a_{i+1} - a_i| \in \{0, \dots, \nu_M - \nu_1\} \subset \mathbb{N}$. This implies that for $v, \tilde{v} \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$, it holds that

$$\text{TV}(\tilde{v}) - \text{TV}(v) \in \mathbb{Z}. \quad (4.4)$$

Note that (4.4) is generally invalid for the multidimensional case $d > 2$.

4.2. First-order necessary conditions for r -optimal points

As we will see below, optimality for the trust-region problem (TR) alone is not sufficient to characterize limit points of the iterates produced by Algorithm 1 because we cannot exclude the case that the trust-region radius contracts to zero. This is due to the fact that the solution of (TR) is constrained by $v(x) \in \{\nu_1, \dots, \nu_M\}$ a.e., which prevents us from proving that for Δ sufficiently small the improvement in the objective of the trust-region subproblem is bounded from below by a fraction of $\|\nabla F(v)\|_{L^2(\Omega)}$ and Δ .

Next, we introduce an auxiliary optimization problem, which is used in our analysis of Algorithm 1 and (TR) and (P). In particular, the auxiliary optimization problem allows us to state a first-order necessary condition for r -optimal points of (P), which we will show is satisfied by the limit points of Algorithm 1. It builds on the step function representation for $v \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$ given by Proposition 4.4. Let $a_1, \dots, a_N \in \{\nu_1, \dots, \nu_M\}$ be given such that $a_i \neq a_{i+1}$ for all $i \in \{1, \dots, N-1\}$. Then, we seek for the solution of the switching location (L) problem

$$\begin{aligned} & \min_{t_1, \dots, t_{N-1}} F(v^{t,a}) \\ & \text{s.t. } v^{t,a} = \chi_{(t_0, t_1)} a_1 + \sum_{i=1}^{N-1} \chi_{[t_i, t_{i+1})} a_{i+1}, \\ & \quad t_{i-1} \leq t_i \text{ for all } i \in \{1, \dots, N\}. \end{aligned} \quad (\text{L})$$

The problem (L) seeks for the optimum switching locations t_i along the interval (t_0, t_f) such that $v^{t,a}$ attains the values a_i in the given order. At switching location t_i , the function $v^{t,a}$ switches from value a_i to a_{i+1} . The

constraint formulation $t_{i-1} \leq t_i$ allows the solution $v^{t,a}$ to actually have fewer switches than given and skip intermediate a_j by choosing $t_{j-1} = t_j$.

Remark 4.5. We use the condition $a_i \neq a_{i+1}$ for all $i \in \{1, \dots, N-1\}$ to avoid redundancy in the formulation, but this assumption is without loss of generality (merge subsequent intervals with coinciding step heights) and does not affect the representable functions v .

We highlight that (L) is a nonlinear program and is, in general, nonconvex, even if F is convex. In addition, differentiability of the objective with respect to the optimization variables of (L) is also not available in the classical sense, which requires extra care when deriving necessary optimality conditions for (L) and (P) below.

Feasible points of (L) may be considered as a subset of feasible points of (P), and (L) admits a minimizer, which is shown in Proposition 4.6.

Proposition 4.6. *Let Assumptions 2.1 hold. Let $a_1, \dots, a_N \in \{\nu_1, \dots, \nu_M\}$ with $a_i \neq a_{i+1}$ for $i \in \{1, \dots, N-1\}$ be given. Then,*

1. *for any feasible $t \in \mathcal{F}_{(L)}$, it follows that $v^{t,a} \in \mathcal{F}_{(P)} \cap BV((t_0, t_f))$, and*
2. *problem (L) admits a minimizer.*

Proof. The first claim is immediate by virtue of (4.3) and $a_i \in \{\nu_1, \dots, \nu_M\} \subset \mathbb{Z}$ for all $i \in \{1, \dots, M\}$. We consider a minimizing sequence $(t^n)_n \in \mathcal{F}_{(L)}$ of switch locations for (L). Because $(t^n)_n \subset \mathcal{F}_{(L)}$ is bounded, it has a cluster point; and we reduce to $t^n \rightarrow t$ (after potentially passing to a subsequence). Clearly, $\text{TV}(v^{s,a}) \leq \sum_{i=1}^{N-1} |a_{i+1} - a_i|$ holds for all $s \in \mathcal{F}_{(L)}$, which implies that $(v^{t^n,a})_n$ is bounded in $BV((t_0, t_f))$. The coordinate-wise convergence $t^n \rightarrow t$ implies that $\chi_{[t_{i-1}^n, t_i^n]} \rightarrow \chi_{[t_{i-1}, t_i]}$ in $L^1((t_0, t_f))$ for all $i \in \{1, \dots, N\}$, which implies $v^{t^n,a} \rightarrow v^{t,a}$ in $L^1((t_0, t_f))$ and in turn in $L^2((t_0, t_f))$. The lower semi-continuity of F implies $F(v^{t,a}) \leq \liminf_{n \rightarrow \infty} F(v^{t^n,a})$, and consequently t minimizes (L), which proves the second claim. \square

Remark 4.7. Because minimizers of (TR) and (P) have finite total variation, the representation of Proposition 4.4 applies.

As noted above, the first-order optimality condition for (L) exhibits a subtlety. If we assume that $\nabla F(v) \in C([t_0, t_f])$ for all v , then the duality between $C([t_0, t_f])$ and the space of Radon measures ensures that differentiation of the objective with respect to the vector $(t_1, \dots, t_{N-1}) \in \mathbb{R}^{N-1}$ for $t_1 < \dots < t_{N-1}$ is well defined. Moreover, using the chain rule, we obtain the first-order condition

$$\nabla F(v)(t_i) = 0 \text{ for } i \in \{1, \dots, N-1\}.$$

However, $\nabla F(v)$ is not continuous in general but only an L^2 -function instead. To provide a first-order condition, we consider so-called Dini derivatives [22]. They allow us to deduce information on $\nabla F(v)$ in a neighborhood of t_i if the expression $\nabla F(v)(t_i)$ is not defined.

Let $v \in \mathcal{F}_{(P)}$, and let t_i be a switching location of v ; *i.e.*, $\lim_{t \uparrow t_i} v(t) - \lim_{t \downarrow t_i} v(t) \in \{\nu_i - \nu_j \mid i, j \in \{1, \dots, M\}\} \setminus \{0\}$. We define $v_i^- := \lim_{t \uparrow t_i} v(t)$, $v_i^+ := \lim_{t \downarrow t_i} v(t)$. We consider the Dini derivatives of the absolutely continuous function $t \mapsto \int_{t_i}^t \nabla F(v)(s) ds$ at t_i :

$$\begin{aligned} \overline{D}_i^+(\nabla F(v)) &:= \limsup_{h \downarrow 0} \frac{1}{h} \int_{t_i}^{t_i+h} \nabla F(v)(s) ds, \\ \overline{D}_i^-(\nabla F(v)) &:= \limsup_{h \downarrow 0} \frac{1}{h} \int_{t_i-h}^{t_i} \nabla F(v)(s) ds, \\ \underline{D}_i^+(\nabla F(v)) &:= \liminf_{h \downarrow 0} \frac{1}{h} \int_{t_i}^{t_i+h} \nabla F(v)(s) ds, \end{aligned}$$

$$\underline{D}_i^-(\nabla F(v)) := \liminf_{h \downarrow 0} \frac{1}{h} \int_{t_i-h}^{t_i} \nabla F(v)(s) \, ds.$$

We note that if t_i is in the Lebesgue set of $\nabla F(v)$, all four terms coincide with $\nabla F(v)(t_i)$.

Lemma 4.8. *Let $v \in \mathcal{F}(\mathbf{P})$, and let t_i be a switching location of v . Let $v_i^- < v_i^+$ and $\underline{D}_i^+(\nabla F(v)) > 0$ or $\overline{D}_i^-(\nabla F(v)) < 0$, or let $v_i^+ < v_i^-$ and $\underline{D}_i^-(\nabla F(v)) > 0$ or $\overline{D}_i^+(\nabla F(v)) < 0$. Then there exist $\varepsilon > 0$ and $h_0 > 0$ such that for all $h \leq h_0$ there exists $d^h \in L^1((t_0, t_f))$ with $\|d^h\|_{L^1((t_0, t_f))} = h$, $v + d^h \in \mathcal{F}(\mathbf{P})$ such that $(\nabla F(v), d^h)_{L^2(\Omega)} \leq -\frac{\varepsilon}{2}h$.*

Proof. Case 1: $v_i^- < v_i^+$. If $\underline{D}_i^+(\nabla F(v)) > \varepsilon$ holds for some $\varepsilon > 0$, then there exists $h_0 > 0$ such that for all $h \leq h_0$ it follows that

$$\begin{aligned} \int_{t_i}^{t_i+h} \nabla F(v)(s)(v_i^- - v(s)) \, ds &= \int_{t_i}^{t_i+h} \nabla F(v)(s)(v_i^- - v(s)) \, ds \\ &= (v_i^- - v_i^+) \int_{t_i}^{t_i+h} \nabla F(v)(s) \, ds \\ &< -\frac{\varepsilon}{2}h. \end{aligned}$$

We choose the function $d^h := \chi_{[t_i, t_i+h)}(v_i^- - v_i^+)$. Similarly, if $\overline{D}_i^-(\nabla F(v)) < -\varepsilon$ holds for some $\varepsilon > 0$, then there exists $h_0 > 0$ such that for all $h \leq h_0$ it follows that

$$\int_{t_i-h}^{t_i} \nabla F(v)(s)(v_i^+ - v(s)) \, ds < -\frac{\varepsilon}{2}h.$$

We choose the function $d^h := \chi_{[t_i-h, t_i)}(v_i^+ - v_i^-)$.

Case 2: $v_i^+ < v_i^-$. If $\overline{D}_i^+(\nabla F(v)) < -\varepsilon$ holds for some $\varepsilon > 0$, then there exists $h_0 > 0$ such that for all $h \leq h_0$ it follows that

$$\int_{t_i}^{t_i+h} \nabla F(v)(s)(v_i^- - v(s)) \, ds < -\frac{\varepsilon}{2}h.$$

We choose the function $d^h := \chi_{[t_i, t_i+h)}(v_i^- - v_i^+)$. Similarly, if $\underline{D}_i^-(\nabla F(v)) > \varepsilon$ holds for some $\varepsilon > 0$, then there exists $h_0 > 0$ such that for all $h \leq h_0$ it holds that

$$\int_{t_i-h}^{t_i} \nabla F(v)(s)(v_i^+ - v(s)) \, ds < -\frac{\varepsilon}{2}h.$$

We choose the function $d^h := \chi_{[t_i-h, t_i)}(v_i^+ - v_i^-)$. □

Lemma 4.8 yields a necessary stationarity condition for local minimizers of (\mathbf{L}) , which we define as follows.

Definition 4.9. Let $v \in \mathcal{F}(\mathbf{P}) \cap BV((t_0, t_f))$ with representation $v = v^{t,a}$ for t and a as given by Proposition 4.4. Let the following conditions hold:

1. $\overline{D}_i^-(\nabla F(v^{t,a})) \geq 0 \geq \underline{D}_i^+(\nabla F(v^{t,a}))$ if $a_i < a_{i+1}$, and
2. $\underline{D}_i^-(\nabla F(v^{t,a})) \leq 0 \leq \overline{D}_i^+(\nabla F(v^{t,a}))$ if $a_{i+1} < a_i$.

Then we say that $v^{t,a}$ is **L**-stationary.

Lemma 4.8 implies that **L**-stationarity is a necessary condition for local minimizers of **(L)** under Assumption 4.1, which is proven below.

Lemma 4.10. *Let Assumption 4.1 hold. Let $a = (a_1, \dots, a_N)^T \in \{\nu_1, \dots, \nu_M\}^N$ with $a_i \neq a_{i+1}$ for $i \in \{1, \dots, N-1\}$ be given. Let t be a local minimizer for **(L)**. Then $v^{t,a}$ is **L**-stationary.*

Proof. We prove the claim with a contrapositive argument. To this end, let $v^{t,a}$ not be **L**-stationary. Then there exists a switching location t_i for which the prerequisites of Lemma 4.8 are satisfied, and we consider $\varepsilon > 0$ and d^h as asserted by Lemma 4.8. We apply Taylor's theorem (see Prop. A.1) and obtain that for some ξ^h in the line segment between $v^{t,a}$ and $v^{t,a} + d^h$ it holds that

$$F(v^{t,a} + d^h) = F(v^{t,a}) + (\nabla F(v^{t,a}), d^h)_{L^2(\Omega)} + \frac{1}{2}F(\xi^h)(d^h, d^h).$$

We use Assumption 4.1 to deduce

$$F(v^{t,a} + d^h) - F(v^{t,a}) \leq (\nabla F(v^{t,a}), d^h)_{L^2(\Omega)} + C\|d^h\|_{L^1(\Omega)}\|d^h\|_{L^1(\Omega)} = (\nabla F(v^{t,a}), d^h)_{L^2(\Omega)} + Ch^2.$$

Inserting the estimate $(\nabla F(v^{t,a}), d^h)_{L^2(\Omega)} \leq -\frac{\varepsilon}{2}h$ from Lemma 4.8, we obtain

$$F(v^{t,a} + d^h) - F(v^{t,a}) \leq -\frac{\varepsilon}{2}h + Ch^2.$$

Because ε and C do not depend on h , we obtain that

$$F(v^{t,a} + d^h) - F(v^{t,a}) < 0$$

holds for all h sufficiently small. We restrict h further such that $t_i + h < t_{i+1}$ and $t_{i-1} < t_i - h$ hold. This restriction gives that the construction of d^h in the proof of Lemma 4.8 satisfies $v^{t,a} + d^h \in \mathcal{F}(\mathbf{L})$. Consequently, t is not locally optimal for **(L)**, which concludes the proof. \square

We now prove that **L**-stationarity is a necessary condition for minimizers of both the trust-region subproblem **(TR)** and the problem **(P)**.

Lemma 4.11. *If $v \in \mathcal{F}(\mathbf{P}) \cap BV((t_0, t_f))$ minimizes $(\mathbf{TR}(v, \nabla F(v), \Delta))$ for some $\Delta > 0$, then v is **L**-stationary.*

Proof. Let $v = v^{t,a}$ for t and a as given by Proposition 4.4. We prove the claim with a contrapositive argument. Let $v^{t,a}$ not be **L**-stationary. Then by virtue of Lemma 4.8 there exist $\varepsilon > 0$ and $h_0 > 0$ such that for all $h \leq h_0$ there exists $d^h \in L^1(\Omega)$ such that $v^{t,a} + d^h \in \mathcal{F}(\mathbf{TR}(v, \nabla F(v), \Delta))$ and $(\nabla F(v^{t,a}), d^h)_{L^2(\Omega)} \leq -\frac{\varepsilon}{2}h$.

Now, we choose $h < \min\{h_0, \Delta, \frac{1}{2} \min_i\{t_{i+1} - t_i\}\}$. The inequality $h \leq \Delta$ implies that the $v^{t,a} + d^h$ are feasible for $(\mathbf{TR}(v, \nabla F(v), \Delta))$. The inequality $h \leq \frac{1}{2} \min_i\{t_{i+1} - t_i\}$ implies that for the d^h constructed in the proof of Lemma 4.8 it follows that $\text{TV}(v^{t,a} + d^h) = \text{TV}(v^{t,a})$.

Consequently, the optimal objective value of $(\mathbf{TR}(v, \nabla F(v), \Delta))$ is bounded from above by $-\frac{\varepsilon}{2}h$, and therefore $v = v^{t,a}$ does not minimize $(\mathbf{TR}(v, \nabla F(v), \Delta))$. This proves the claim. \square

Lemma 4.12. *Let $v \in \mathcal{F}(\mathbf{P}) \cap BV(\Omega)$ be r -optimal for **(P)** for some $r > 0$ with representation $v = v^{t,a}$ for t and a as given by Proposition 4.4. Then t is a local minimizer for **(L)**.*

Proof. Let $0 < h < \frac{1}{\max_{1 \leq i < j \leq M} |\nu_i - \nu_j| N} \min\{\frac{1}{2} \min\{t_{i+1} - t_i\}, r\}$. Then we obtain for all $t_i^h \in (t_i - h, t_i + h)$ for all $i \in \{1, \dots, N-1\}$ that $\text{TV}(v^{t_i^h, a}) = \text{TV}(v)$. Note that by Proposition 4.4 $t_{i+1} - t_i > 0$ for all i , and thus by construction the intervals $(t_i - h, t_i + h)$ do not intersect. Consequently, the r -optimality gives that $F(v^{t_i^h, a}) \geq F(v^{t, a})$. This implies that t is a local minimizer for **(L)**. \square

Remark 4.13. The feasible points $v + d^h \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$ constructed with the help of Lemma 4.8 allow us to bound the optimal value of the trust-region subproblem (TR) away from zero by a fraction of the trust-region radius if v is not optimal. Therefore, they may be considered as the analog of Cauchy points that are used in the analysis of trust-region methods for nonlinear optimizations.

We summarize the relationship of the stationarity concepts in the following theorem.

Theorem 4.14. *Let Assumptions 2.1 hold. Let $v \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$ with representation $v = v^{t,a}$ for t and a as given by Proposition 4.4. Then the following assertions hold.*

1. *If t is optimal for (L) and Assumption 4.1 holds, then v is L-stationary.*
2. *If v is optimal for $(\text{TR}(v, \nabla F(v), \Delta))$ for some $\Delta > 0$, then v is L-stationary.*
3. *If v is r -optimal for some $r > 0$, then t is a local minimizer for (L). If Assumption 4.1 holds, then v is also L-stationary.*
4. *Let F be convex. If v is optimal for $(\text{TR}(v, \nabla F(v), \Delta))$ for some $\Delta > 0$, then v is r -optimal.*

Proof. The first claim follows from Proposition 4.6 and Lemma 4.10. The second claim follows from Lemma 4.11. The third claim follows from Lemma 4.12. The fourth claim follows from Proposition 3.3. \square

Remark 4.15. The construction of the function d^h in the proof of Lemma 4.8 can be considered as the construction of points of sufficient decrease (Cauchy points) for (L); see Remark 4.13. Because of the lack of differentiability of the term TV, we have not found a way to construct such points for the problem (TR), however. Therefore, we cannot prove that v solving $(\text{TR}(v, \nabla F(v), \Delta))$ is a necessary condition for r -optimality.

Remark 4.16. We highlight that the regularity on the Hessian enforced by Assumption 4.1 enters the proof of Lemma 4.10 only to enable the use of Taylor's theorem. The assumption compensates for the fact that perturbations of the switching locations by h imply linear changes in the measure of the preimages $(d^h)^{-1}(\nu_i - \nu_j)$, which result in changes only in the L^2 -norm of $O(h^{\frac{1}{2}})$. However, $O(h^s)$ for $s > \frac{1}{2}$ would be necessary in order to ensure that the remainder term of the first-order Taylor expansion decays fast enough.

We briefly show that L-stationarity indeed generalizes the usual first-order condition for (L).

Proposition 4.17. *Let $v \in \mathcal{F}_{(\mathbf{P})} \cap BV((t_0, t_f))$ be given with representation $v = v^{t,a}$ for t and a as given by Proposition 4.4. Let $\nabla F(v)$ have a representative i.e. continuous at t_i , or let t_i be in the Lebesgue set of $\nabla F(v)$ for all $i \in \{1, \dots, N-1\}$. Then, L-stationarity of v is equivalent to*

$$\nabla F(v)(t_i) = 0 \text{ for all } i \in \{1, \dots, N-1\}.$$

Proof. First, we note that if $\nabla F(v)$ is continuous at t_i , then t_i is a Lebesgue point of $\nabla F(v)$. Moreover, if t_i is a Lebesgue point of v , we obtain that

$$\overline{D_i^+} \nabla F(v) = \overline{D_i^-} \nabla F(v) = \underline{D_i^+} \nabla F(v) = \underline{D_i^-} \nabla F(v) = \nabla F(v)(t_i)$$

by definition of the Lebesgue set; see [46, Chap.3.1]. Then for all $i \in \{1, \dots, N-1\}$, the inequalities in Definition 4.9 are equivalent to $\nabla F(v)(t_i) = 0$. \square

Remark 4.18. We note that the construction of analogs of Cauchy points and stationarity cannot be generalized directly to the multidimensional case $d \geq 2$ and more involved geometric constructions are necessary.

4.3. Sufficient decrease condition and asymptotics of the inner loop

Before continuing with the analysis of the outer loop, we analyze the asymptotics of the inner loop of Algorithm 1. We show that the inner loop terminates finitely unless the current iterate satisfies a necessary

optimality condition (**L**-stationarity) for r -optimality of **(P)**. This is shown in Corollary 4.20, for which we require the preparatory lemma below.

Lemma 4.19. *Let Assumption 4.1 hold. Let $\sigma \in (0, 1)$. Let $v \in \mathcal{F}(\mathbf{P})$. Let $\Delta^k \downarrow 0$. For all $k \in \mathbb{N}$, let v^k be a minimizer of $(\mathbf{TR}(v, \nabla F(v), \Delta^k))$. Then at least one of the following statements holds true.*

1. *The function v is **L**-stationary.*
2. *There exists $k_0 \in \mathbb{N}$ such that the objective value of (\mathbf{TR}) evaluated at v^{k_0} is zero and v is **L**-stationary.*
3. *There exists $k_0 \in \mathbb{N}$ such that*

$$F(v) + \alpha \text{TV}(v) - F(v^{k_0}) - \alpha \text{TV}(v^{k_0}) \geq \sigma \left((\nabla F(v), v - v^{k_0})_{L^2(\Omega)} + \alpha \text{TV}(v) - \alpha \text{TV}(v^{k_0}) \right);$$

i.e., the sufficient decrease condition (3.1) (Line 11 in Algorithm 1) holds.

Proof. We first observe that if the objective value of $(\mathbf{TR}(v, \nabla F(v), \Delta^k))$ evaluated at v^k is zero, then Theorem 4.14 implies that v is **L**-stationary. Next, we assume that the first two claims do not hold true, and we set forth to prove the third. From $\Delta^k \downarrow 0$ and the constraints of (\mathbf{TR}) it follows that $v^k \rightarrow v$ in $L^1(\Omega)$ and in $L^2(\Omega)$. Moreover, we use the lower semi-continuity $\text{TV}(v) \leq \liminf_{k \rightarrow \infty} \text{TV}(v^k)$ to distinguish the following two cases.

Case 1: $0 = \lim_{\ell \rightarrow \infty} \text{TV}(v^{k_\ell}) - \text{TV}(v)$ **for a subsequence** $(v^{k_\ell})_\ell$. Because (4.4) implies $\text{TV}(v^{k_\ell}) - \text{TV}(v) \in \mathbb{Z}$, it follows that $\text{TV}(v^{k_\ell}) = \text{TV}(v)$ for all $\ell \geq \ell_1$ for some $\ell_1 \in \mathbb{N}$. Because the optimal objective of $(\mathbf{TR}(v, \nabla F(v), \Delta^{k_\ell}))$ is negative for all ℓ , it follows that

$$(\nabla F(v), v - v^{k_\ell})_{L^2(\Omega)} > 0$$

for all $\ell \geq \ell_1$. From Taylor's theorem (see Prop. A.1) it follows that

$$F(v) - F(v^k) = \sigma (\nabla F(v), v - v^{k_\ell})_{L^2(\Omega)} + (1 - \sigma) (\nabla F(v), v - v^{k_\ell})_{L^2(\Omega)} + \frac{1}{2} \nabla^2 F(\xi^{k_\ell})(v - v^{k_\ell}, v - v^{k_\ell})$$

for some ξ^{k_ℓ} in the line segment between v and v^{k_ℓ} . Because v is not **L**-stationary by assumption, we can apply Lemma 4.8 and the optimality of v^{k_ℓ} for the problem $(\mathbf{TR}(v, \nabla F(v), \Delta^{k_\ell}))$ as in the proof of Lemma 4.12 to obtain the bounds

$$(1 - \sigma) (\nabla F(v), v - v^{k_\ell})_{L^2(\Omega)} \geq (1 - \sigma) \frac{\varepsilon}{2} \Delta^{k_\ell},$$

and

$$\frac{1}{2} \nabla^2 F(\xi^{k_\ell})(v - v^{k_\ell}, v - v^{k_\ell}) \leq \frac{1}{2} C (\Delta^{k_\ell})^2$$

for constants $\varepsilon > 0$ and $C > 0$ and all ℓ sufficiently large. This implies that the term $(1 - \sigma) (\nabla F(v), v - v^{k_\ell})_{L^2(\Omega)}$ eventually dominates the term $\frac{1}{2} \nabla^2 F(\xi^{k_\ell})(v - v^{k_\ell}, v - v^{k_\ell})$ and that, together with $\text{TV}(v) = \text{TV}(v^{k_\ell})$, Outcome 3 holds with the choice $k_0 := k_\ell$ for some ℓ sufficiently large.

Case 2: $\text{TV}(v) - \text{TV}(v^{k_\ell}) < 0$ **for a subsequence** $(v^{k_\ell})_\ell$ **and all** $\ell \geq \ell_1$ **for some** $\ell_1 \in \mathbb{N}$. In this case we note that (4.4) implies $\text{TV}(v) - \text{TV}(v^{k_\ell}) \leq -1$, giving $\limsup_{\ell \rightarrow \infty} \text{TV}(v) - \text{TV}(v^{k_\ell}) \leq -1$. This gives

$$\limsup_{\ell \rightarrow \infty} \left\{ (\nabla F(v), v - v^{k_\ell})_{L^2(\Omega)} + \alpha \text{TV}(v) - \alpha \text{TV}(v^{k_\ell}) \right\} = \alpha \limsup_{k \rightarrow \infty} \text{TV}(v) - \text{TV}(v^k) = -\alpha < 0,$$

where the first identity follows from the Cauchy–Schwarz inequality and $v^k \rightarrow v$ in $L^2(\Omega)$. However, this means that there exists $k_0 \in \mathbb{N}$ such that v is Δ^{k_0} -optimal, which means that the first claim holds and contradicts the assumption that it does not. This concludes the proof. \square

Corollary 4.20. *Let Assumptions 2.1 and 4.1 hold. Then for all iterations $n = 0, \dots$ executed by Algorithm 1 it follows that one of the following outcomes holds.*

1. *The inner loop terminates after finitely many iterations, with the outcome that the predicted reduction is zero (and the current iterate is L -stationary) or the sufficient decrease condition (3.1) is satisfied.*
2. *The inner loop does not terminate, and the current iterate is L -stationary.*

Proof. This follows from Lemma 4.19 with $\Delta^k = \Delta^{n,k}$ and $v = v^{n-1}$ □

Remark 4.21. The proof of Lemma 4.19 hinges on the fact that for $\text{TV}(v) - \text{TV}(\tilde{v}) < 0$ for feasible v, \tilde{v} we can deduce that $\text{TV}(v) - \text{TV}(\tilde{v}) \leq -1$ from (4.4). This is due to two facts. First, a discrete-valued function of bounded variation is a step function defined on finitely many intervals. Second, considering the step function representation (4.2), we obtain that $\mathcal{H}^{d-1}(E) = \mathcal{H}^0(E) = 1$ for an interior facet (*i.e.*, a point $E = \{x_i\}$ for $i \in \{1, \dots, N-1\}$) that connects the intervals $[x_{i-1}, x_i)$ and $[x_i, x_{i+1})$.

Remark 4.22. Because $\mathcal{H}^{d-1}(E)$ is not the counting measure for $d \geq 2$ but can attain arbitrarily small values instead, (4.4) is in general incorrect for $d \geq 2$, and $\text{TV}(v) - \text{TV}(\tilde{v}) < 0$ does not imply $\text{TV}(v) - \text{TV}(\tilde{v}) < -\varepsilon$ for some $\varepsilon > 0$. Hence, the proof cannot be transferred directly to domains in higher dimensions.

4.4. Asymptotics of Algorithm 1

With the help of the concept of L -stationarity, we are able to characterize the asymptotics of the sequence of iterates generated by Algorithm 1. We recall that we say that a sequence $(v^n)_n \subset BV(\Omega)$ converges strictly to a limit v in $BV(\Omega)$ if $v^n \rightarrow v$ in $L^1(\Omega)$ and $\text{TV}(v^n) \rightarrow \text{TV}(v)$.

Theorem 4.23. *Let Assumptions 2.1 and 4.1 hold. Let the iterates $(v^n)_n$ be produced by Algorithm 1. Then all iterates are feasible for (P), and the sequence of objective values $(J(v^n))_n$ is monotonously decreasing. Moreover, one of the following mutually exclusive outcomes holds:*

1. *The sequence $(v^n)_n$ is finite. The final element v^N of $(v^n)_n$ solves $(\text{TR}(v^N, \nabla F(v^N), \Delta))$ for some $\Delta > 0$, and v^N is L -stationary.*
2. *The sequence $(v^n)_n$ is finite, and the inner loop does not terminate for the final element v^N , which is L -stationary.*
3. *The sequence $(v^n)_n$ has a weak-* accumulation point in $BV((t_0, t_f))$. Every weak-* accumulation point of $(v^n)_n$ is feasible, L -stationary and strict. If the trust-region radii are bounded away from zero, *i.e.*, if $\liminf_{n_\ell \rightarrow \infty} \min_k \Delta^{n_\ell+1,k} > 0$ for a subsequence $(v^{n_\ell})_\ell$ and \bar{v} is a weak-* accumulation point of $(v^{n_\ell})_\ell$, then \bar{v} solves $(\text{TR}(\bar{v}, \nabla F(\bar{v}), \bar{\Delta}))$ for some $\bar{\Delta} > 0$.*

Proof. First, we note that a new iterate v^n is produced (accepted) by the inner loop of Algorithm 1 if the condition $\text{pred}(v^{n-1}, \Delta^{n,k}) > 0$ and the sufficient decrease condition (3.1) are satisfied. Combining these estimates, we have $J(v^n) = F(v^n) + \alpha \text{TV}(v^n) < \text{TV}(v^{n-1}) + \alpha \text{TV}(v^{n-1}) = J(v^{n-1})$. This implies that the sequence $(J(v^n))_n$ is monotonously decreasing. Because the feasible sets of all trust-region subproblems (TR) are included in the feasible set of the problem (P), it follows inductively from Proposition 3.2 that Line 6 produces $\bar{v}^{n,k} \in \mathcal{F}_{(P)}$.

We first consider Outcome 1. We observe that Algorithm 1 terminates with iterate v^{n-1} if and only if for some outer iteration $n \in \mathbb{N}$ and inner iteration $k \in \mathbb{N}$ it holds that $\text{pred}(v^{n-1}, \Delta^{n,k}) = 0$. The definition of $\text{pred}(v^{n-1}, \Delta^{n,k})$ in Lines 6 and 7 implies that $v^N := v^{n-1}$ solves $(\text{TR}(v^N, \nabla F(v^N), \Delta))$ for some $\Delta > 0$.

Next we consider Outcome 2. If the inner loop does not terminate finitely, Lemma 4.19 yields that the current iterate $v^N = v^{n-1}$ is L -stationary.

Therefore, it suffices to assume that Outcome 1 and Outcome 2 do not hold true and to prove that Outcome 3 holds in this case. The absence of Outcome 1 and Outcome 2 implies that for all $n \in \mathbb{N}$ the inner loop terminates after finitely many iterations and Algorithm 1 produces an infinite sequence of iterates $(v^n)_n$.

The sequence $(\text{TV}(v^n))_n$ is bounded from below by zero and from above because the sequence $(\frac{1}{\alpha} J(v^n))_n$ is decreasing and the sequence $(F(v^n))_n$ is bounded from below. Therefore, the sequence $(v^n)_n$ has a weak-* accumulation point in $BV((t_0, t_f))$. Lemma 2.2 yields that all weak-* accumulation points are feasible for (P).

Next we show that weak-* accumulation points are strict. We execute a contrapositive argument and assume that v is a weak-* accumulation point of $(v^n)_n$ with approximating subsequence $v^{n_\ell} \xrightarrow{*} v$ in $BV((t_0, t_f))$ *i.e.* not strict. This implies that there exists a subsequence (for ease of notation also denoted by v^{n_ℓ}) such that $\text{TV}(v) < \liminf_{\ell \rightarrow \infty} \text{TV}(v^{n_\ell})$. We define $\delta := \liminf_{\ell \rightarrow \infty} \text{TV}(v^{n_\ell}) - \text{TV}(v)$. We deduce that $\delta \geq 1$ from (4.4). The sufficient decrease condition (3.1) on acceptance combined with the optimality of the solution of Line 6 gives

$$\begin{aligned} \frac{1}{\sigma} \text{ared}(v^{n_\ell}, v^{n_\ell+1, k}) &\geq (\nabla F(v^{n_\ell}), v^{n_\ell} - v^{n_\ell+1, k})_{L^2(\Omega)} + \alpha \text{TV}(v^{n_\ell}) - \alpha \text{TV}(v^{n_\ell+1, k}) \\ &\geq (\nabla F(v^{n_\ell}), v^{n_\ell} - v)_{L^2(\Omega)} + \alpha \delta \end{aligned} \quad (4.5)$$

for infinitely many elements ℓ of the subsequence and all inner iterations k , for which v is feasible for the trust-region subproblem. For all inner iterations k we obtain the estimate

$$|(\nabla F(v^{n_\ell}), v^{n_\ell} - \tilde{v})_{L^2(\Omega)}| \leq \sqrt{\max_{i,j} |\nu_i - \nu_j|} \sqrt{\Delta^{n_\ell+1, k}} \|\nabla F(v^{n_\ell})\|_{L^2(\Omega)}$$

for all \tilde{v} feasible for the trust-region subproblem by virtue of the Cauchy–Schwarz inequality. Thus there exists k_0 such that for all inner iterations $k \geq k_0$ we have $|(\nabla F(v^{n_\ell}), v^{n_\ell} - \tilde{v})_{L^2(\Omega)}| \leq \frac{1-\sigma}{3-\sigma} \alpha$ and $|F(v^{n_\ell}) - F(\tilde{v})| \leq \frac{1-\sigma}{3-\sigma} \alpha$.

Because $v^{n_\ell} \xrightarrow{*} v$ in $BV((t_0, t_f))$, we have $v^{n_\ell} \rightarrow v$ in $L^2((t_0, t_f))$ and $(\nabla F(v^{n_\ell}), v^{n_\ell} - v)_{L^2(\Omega)} \rightarrow 0$ after potentially restricting to another subsequence, implying that v is feasible in iteration k_0 for all $\ell \geq \ell_0$ for some ℓ_0 large enough. We deduce $\text{ared}(v^{n_\ell}, v^{n_\ell+1, k_0}) \geq \text{pred}(v^{n_\ell}, \Delta^{n_\ell+1, k_0}) - 2\alpha \frac{1-\sigma}{3-\sigma}$ and $\text{pred}(v^{n_\ell}, \Delta^{n_\ell+1, k_0}) \geq \alpha - \alpha \frac{1-\sigma}{3-\sigma}$, implying that $\text{ared}(v^{n_\ell}, v^{n_\ell+1, k_0}) \geq \sigma \text{pred}(v^{n_\ell}, \Delta^{n_\ell+1, k_0})$. Thus a new step is accepted latest in inner iteration k_0 for all $\ell \geq \ell_0$. Combining these results, we obtain that $J(v^{n_\ell+1}) \rightarrow -\infty$ for $\ell \rightarrow \infty$, which contradicts that J is bounded from below. Thus, every weak-* accumulation point of $(v^n)_n$ is strict.

Next we prove that the weak-* and strict limit v solves $(\text{TR}(v, \nabla F(v), \Delta))$ for some $\Delta > 0$ if the trust-region radii upon acceptance of a subsequence of $(v^{n_\ell})_\ell$ are bounded away from zero. We restrict ourselves to such a subsequence and denote it also by $(v^{n_\ell})_\ell$ for ease of notation. A lower bound on the trust-region radii upon acceptance is $\underline{\Delta} := \inf_{\ell \in \mathbb{N}} \min_k \Delta^{n_\ell+1, k} > 0$. Because we may assume that $v^{n_\ell} \rightarrow v$ in $L^2((t_0, t_f))$ by restricting to another subsequence (also denoted by v^{n_ℓ}), it follows that $\nabla F(v^{n_\ell}) \rightarrow \nabla F(v)$ in $L^2((t_0, t_f))$. Moreover, we restrict this subsequence further (also denoted by v^{n_ℓ}) such that $\lim_{\ell \rightarrow \infty} \text{TV}(v^{n_\ell}) = \text{TV}(v)$. Combining these observations with (4.4), we deduce that there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$ we have $\text{TV}(v^{n_\ell}) = \text{TV}(v)$. Using this identity, we obtain $\text{pred}(v, \underline{\Delta}) \leq \text{pred}(v^{n_\ell}, \min_k \Delta^{n_\ell+1, k}) \rightarrow 0$. The sequence of predicted reductions on acceptance of the step $(\text{pred}(v^{n_\ell}, \min_k \Delta^{n_\ell+1, k}))_\ell$ tends to zero because otherwise we would obtain the contradiction $J(v^{n_\ell+1}) \rightarrow -\infty$. This implies that $\text{pred}(v, \underline{\Delta}) = 0$, and consequently v solves $(\text{TR}(v, \nabla F(v), \underline{\Delta}))$ in this case.

Next we prove that v is **L**-stationary. Again we consider a subsequence $(v^{n_\ell})_\ell$ such that $v^{n_\ell} \xrightarrow{*} v$ *i.e.* also strictly convergent. We restrict $(v^{n_\ell})_\ell$ further to a pointwise a.e. convergent subsequence, also denoted by $(v^{n_\ell})_\ell$, such that $\lim_{\ell \rightarrow \infty} \text{TV}(v^{n_\ell}) = \text{TV}(v)$. This implies that there exists $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$ we have $\text{TV}(v^{n_\ell}) = \text{TV}(v)$.

Because $\text{TV}(v) < \infty$, we can use the representation given by Proposition 4.4 and deduce that there are $N \in \mathbb{N}$, $a \in \{\nu_1, \dots, \nu_M\}^N$ with $a_i \neq a_{i+1}$ for all $i \in \{1, \dots, N-1\}$, and $0 = t_0 < \dots < t_N = t_f$ such that

$$v = v^{t, a} = \chi_{(t_0, t_1)} a_1 + \sum_{i=1}^{N-1} \chi_{[t_i, t_{i+1})} a_{i+1}.$$

Having obtained this characterization, we use a contradiction argument to prove that v is **L**-stationary by assuming that the claim does not hold true, invoking Lemma 4.8 and obtaining a contradiction. To this end, we assume that **L**-stationarity is violated for some switching location t_i of v . This implies that the prerequisites of

Lemma 4.8 are satisfied, and we obtain that there exist $\varepsilon > 0$ and $h_0 > 0$ such that for all $h \leq h_0$, there exists $d^h \in L^1((t_0, t_f))$ with $\|d^h\|_{L^1(\Omega)} = h$, $v + d^h \in \mathcal{F}(\mathbf{P})$, and $(\nabla F(v), d^h)_{L^2(\Omega)} \leq -\frac{\varepsilon}{2}h$. Moreover, by construction of d^h in the proof of Lemma 4.8 and choosing $h_0 < \frac{1}{2} \min_i \{t_{i+1} - t_i\}$ it also follows that $\text{TV}(v + d^h) = \text{TV}(v) = \text{TV}(v^{n_\ell})$.

The remaining argument is the following. We first show **Part 1** that we obtain a contradiction if the trust-region radius upon acceptance of the iterates $n_\ell + 1$ is larger than $2h$ for some $h \leq h_0$ infinitely often. Then, we show **Part 2** that the trust-region radius cannot contract to zero for the subsequence $(v^{n_\ell})_\ell$. Note that ε and h_0 chosen above depend on the limit of the subsequence.

Part 1: To establish the contradiction, we assume that trust-region radius upon acceptance of the iterates $n_\ell + 1$ is larger than $2h$ for some $h \leq h_0$ infinitely often. To simplify the argument, we restrict it to the subsequence of $(n_\ell)_\ell$, for ease of notation denoted by the same symbol, such that the trust-region radius upon acceptance of the iterates $n_\ell + 1$ is larger than $2h$. Because $v^{n_\ell} \rightarrow v$ in $L^1(\Omega)$, there exists $\ell_1 \geq \ell_0$ such that for all $\ell \geq \ell_1$ it holds that $\|v^{n_\ell} - v\|_{L^1(\Omega)} \leq h$. Thus, for iteration $n_\ell + 1$ and all inner iterations k , we obtain

$$\min(\text{TR}(v^{n_\ell}, \nabla F(v^{n_\ell}), \Delta^{n_\ell+1, k})) \leq \begin{cases} (\nabla F(v^{n_\ell}), v + d^{\Delta^{n_\ell+1, k}/2} - v^{n_\ell})_{L^2(\Omega)} & \text{if } \Delta^{n_\ell+1, k} \leq 2h_0, \\ (\nabla F(v^{n_\ell}), v + d^{h_0} - v^{n_\ell})_{L^2(\Omega)} & \text{if } \Delta^{n_\ell+1, k} \geq 2h_0. \end{cases}$$

For the accepted iterate, we obtain the estimate

$$\begin{aligned} J(v^{n_\ell}) - J(v^{n_\ell+1}) &\geq \sigma(\nabla F(v^{n_\ell}), v^{n_\ell} - v - d^h)_{L^2(\Omega)} \\ &= -\sigma(\nabla F(v), d^h)_{L^2(\Omega)} + \sigma(\nabla F(v) - \nabla F(v^{n_\ell}), d^h)_{L^2(\Omega)} + \sigma(\nabla F(v^{n_\ell}), v^{n_\ell} - v)_{L^2(\Omega)}. \end{aligned}$$

Because $v^{n_\ell} \rightarrow v$, the last two terms tend to zero, and there exists $\ell_2 \geq \ell_1$ such that for all $\ell \geq \ell_2$ it holds that

$$J(v^{n_\ell}) - J(v^{n_\ell+1}) \geq -\frac{\sigma}{2}(\nabla F(v), d^h)_{L^2(\Omega)} \geq \frac{1}{4}\sigma\varepsilon h.$$

This implies the contradiction $J(v^{n_\ell+1}) \rightarrow -\infty$.

Part 2: We show that the trust-region radii upon acceptance of the iterates $v^{n_\ell+1}$ cannot contract to zero. Let $0 < h \leq h_0$ be given. Then there exists $\ell_3 \geq \ell_0$ such that for all $\ell \geq \ell_3$ it holds that $\|v - v^{n_\ell}\|_{L^2(\Omega)} \leq h^2$ and $\|\nabla F(v) - \nabla F(v^{n_\ell})\|_{L^2(\Omega)} \leq h^2$. Moreover, there exists a constant $c_1 > 0$ that can be chosen independently of h and ℓ such that

$$|(\nabla F(v), d^h)_{L^2(\Omega)} - (\nabla F(v^{n_\ell}), v + d^h - v^{n_\ell})_{L^2(\Omega)}| \leq h^2\|d^h\|_{L^2(\Omega)} + \|\nabla F(v^{n_\ell})\|_{L^2(\Omega)}h^2 \leq c_1h^2. \quad (4.6)$$

Let $\Delta^* \in \{\Delta^{02^{-j}} \mid j \in \mathbb{N}\}$. Let v^* denote a minimizer of the problem $(\text{TR}(v^{n_\ell}, \nabla F(v^{n_\ell}), \Delta^*))$ with the fixed extra constraint $\text{TV}(\tilde{v}) = \text{TV}(v)$. Then Taylor's theorem (see Prop. A.1) and Assumption 4.1 imply

$$F(v^{n_\ell}) - F(v^*) \geq -(\nabla F(v^{n_\ell}), v^* - v^{n_\ell})_{L^2(\Omega)} - c_2\|v^* - v^{n_\ell}\|_{L^1(\Omega)}^2$$

for some $c_2 > 0$. We can choose Δ^* sufficiently small (choose j sufficiently large) such that the inequalities $\Delta^* \leq h_0$ and $(1 - \sigma)\frac{\varepsilon}{4}\Delta^* - (\Delta^*)^2(0.25c_1 + c_2) > 0$ hold true.

To combine these considerations, we choose the $\ell_3 \in \mathbb{N}$ introduced above such that (4.6) holds with the choice $h = \Delta^*$. For all $\ell \geq \ell_3$, we deduce

$$\begin{aligned} -(1 - \sigma)(\nabla F(v^{n_\ell}), v^* - v^{n_\ell})_{L^2(\Omega)} - c_2\|v^* - v^{n_\ell}\|_{L^1(\Omega)}^2 &\geq -(1 - \sigma)(\nabla F(v^{n_\ell}), v^{n_\ell} + d^{\Delta^*/2} - v^{n_\ell})_{L^2(\Omega)} - c_2(\Delta^*)^2 \\ &\geq -(1 - \sigma)(\nabla F(v), d^{\Delta^*/2})_{L^2(\Omega)} - (\Delta^*)^2(0.25c_1 + c_2) \\ &\geq (1 - \sigma)\frac{\varepsilon}{4}\Delta^* - (\Delta^*)^2(0.25c_1 + c_2) \geq 0, \end{aligned}$$

where we have used (4.6) with $h = \Delta^*$ in the second inequality. It follows that

$$(\nabla F(v^{n_\ell}), v^* - v^{n_\ell})_{L^2(\Omega)} \geq (\nabla F(v^{n_\ell}), v + d^{\Delta^*/2} - v^{n_\ell})_{L^2(\Omega)} \geq \frac{\varepsilon}{4} \Delta^* - c_1(\Delta^*)^2 > 0, \quad (4.7)$$

which implies that any solution \hat{v}^* of $(\text{TR}(v^{n_\ell}, \nabla F(v^{n_\ell}), \Delta^*))$ also satisfies $\text{TV}(\hat{v}^*) \leq \text{TV}(v^*) = \text{TV}(v^{n_\ell}) = \text{TV}(v)$. To complete the argument, we distinguish the cases **Case 2a** $\text{TV}(\hat{v}^*) < \text{TV}(v)$ and **Case 2b** $\text{TV}(\hat{v}^*) = \text{TV}(v)$.

Case 2a: If $\text{TV}(\hat{v}^*) < \text{TV}(v^{n_\ell})$, then $\text{TV}(v^{n_\ell}) - \text{TV}(\hat{v}^*) \geq 1$ by (4.4). By virtue of Assumption 4.1 and Proposition A.2, we may choose Δ^* small enough and $\ell_4 \geq \ell_3$ such that for all $\ell \geq \ell_4$ it follows that $|F(\hat{v}^*) - F(v^{n_\ell})| < \frac{1-\sigma}{2}$ and $|\sigma(\nabla F(v^{n_\ell}), \hat{v}^* - v^{n_\ell})_{L^2(\Omega)}| < \frac{1-\sigma}{2}$. We deduce

$$\begin{aligned} \text{ared}(v^{n_\ell}, \hat{v}^*) &= F(\hat{v}^*) - F(v^{n_\ell}) + \text{TV}(v^{n_\ell}) - \text{TV}(\hat{v}^*) \\ &\geq -\frac{1-\sigma}{2} + \text{TV}(v^{n_\ell}) - \text{TV}(\hat{v}^*) && |F(\hat{v}^*) - F(v^{n_\ell})| < \frac{1-\sigma}{2} \\ &\geq -(1-\sigma) - \sigma(\nabla F(v^{n_\ell}), \hat{v}^* - v^{n_\ell})_{L^2(\Omega)} + \text{TV}(v^{n_\ell}) - \text{TV}(\hat{v}^*) && |\sigma(\nabla F(v^{n_\ell}), \hat{v}^* - v^{n_\ell})_{L^2(\Omega)}| < \frac{1-\sigma}{2} \\ &\geq -\sigma(\nabla F(v^{n_\ell}), \hat{v}^* - v^{n_\ell})_{L^2(\Omega)} + \sigma(\text{TV}(v^{n_\ell}) - \text{TV}(\hat{v}^*)) && \text{TV}(v^{n_\ell}) - \text{TV}(\hat{v}^*) \geq 1 \\ &= \sigma \text{pred}(v^{n_\ell}, \hat{v}^*). \end{aligned}$$

Let j be such that $\Delta^* = \Delta^0 2^{-j}$. Then, this argument implies that at the latest the j th iteration of the inner loop is accepted for all $\ell \geq \ell_4$ if $\text{TV}(\hat{v}^*) < \text{TV}(v)$.

Case 2b: If $\text{TV}(\hat{v}^*) = \text{TV}(v)$, then we may consider $v^* = \hat{v}^*$, and for all $\ell \geq \ell_3$ it follows that in the inner iteration k such that $\Delta^0 2^{-k} = \Delta^*$ we have that

$$\begin{aligned} \text{ared}(v^{n_\ell}, v^*) &\geq F(v^{n_\ell}) - F(v^*) \\ &\geq -\sigma(\nabla F(v^{n_\ell}), v^* - v^{n_\ell})_{L^2(\Omega)} - (1-\sigma)(\nabla F(v^{n_\ell}), v^* - v^{n_\ell})_{L^2(\Omega)} - c_2 \|v^* - v^{n_\ell}\|_{L^1(\Omega)}^2 \\ &\geq -\sigma(\nabla F(v^{n_\ell}), v^* - v^{n_\ell})_{L^2(\Omega)} = \sigma \text{pred}(v^{n_\ell}, \Delta^*). \end{aligned}$$

Inserting (4.7), it follows that $\text{pred}(v^{n_\ell}, \Delta^*) \geq \frac{\varepsilon}{4} \Delta^* - c_1(\Delta^*)^2 > 0$. Let j be such that $\Delta^* = \Delta^0 2^{-j}$. The argument above implies that at the latest the j th iteration of the inner loop is accepted for all $\ell \geq \ell_3$ if $\text{TV}(\hat{v}^*) = \text{TV}(v)$.

Combining these cases means that at the latest the j th iteration of the inner loop is accepted for all $\ell \geq \ell_4$, which implies that the trust-region radius upon acceptance is eventually always greater than or equal to Δ^* , which settles **Part 2** and implies $J(v^{n_{\ell+1}}) \rightarrow -\infty$ as established in **Part 1**. This completes the contradiction under the assumption that v is not **L**-stationary. Consequently, v has to be **L**-stationary, which completes the proof. \square

Corollary 4.24. *Under the prerequisites of Theorem 4.23 it holds that the final iterate produced by Algorithm 1 and—in the absence of finite termination—all limit points of the iterates produced by Algorithm 1 are **L**-stationary.*

Proof. This follows from Theorem 4.23 and Theorem 4.14. \square

Remark 4.25. In particular, we obtain that if the trust-region radius is bounded away from zero or the algorithm terminates finitely, then the limit is not only **L**-stationary but also a minimizer of the trust-region subproblem for a strictly positive trust-region radius.

Remark 4.26. This analysis hinges on the fact that after finitely many iterations (and potentially passing to a subsequence), the value of $\text{TV}(v^n)$ does not change anymore for the iterates v^n produced by Algorithm 1.

Once this has happened, Assumption 4.1 and Taylor's theorem are employed to prove that the limit points of $(v^n)_n$ are \mathbf{L} -stationary.

Compared with algorithms for nonlinear optimization, this can be viewed as the active set settling down after finitely many iterations, after which a first-order condition is achieved when restricting to the fixed active set. A fixed active set here corresponds to a fixed sequence of heights that are taken by the resulting control along the interval (t_0, t_f) . While the sequence of heights is fixed, the only quantities that remain to be optimized are the exact locations at which the height changes. In this regard, the problem (\mathbf{L}) is a subproblem that can be solved for a fixed active set, *i.e.*, a fixed switching order. If (\mathbf{L}) can be tackled efficiently with the methods of nonlinear optimization, it can be invoked to ensure that the inner loop terminates finitely.

Remark 4.27. After the total variation has converged, the switching location optimization is closely related to *bang-bang* control, for which a vast literature exists, in particular on sufficient conditions for bang-bang optimal controls [7, 37]. Algorithm 1 may be augmented by using recently developed techniques for switching point optimization (see, *e.g.*, [13, 14, 18, 40, 47]) to improve convergence. In order to determine small modifications that improve the switching structure, the *mode insertion gradient* as proposed in [40, Section 4] may be a beneficial option to replace some of the, presumably more expensive, trust-region subproblem solves.

4.5. Approximation of the trust-region subproblem

For implementations of Algorithm 1, it is important that the trust-region subproblems (TR) can be approximated with (TRIP) by choosing a suitable partition \mathcal{T} of the domain Ω . While a thorough analysis of this relationship and suitable refinement strategies that yield approximation results for discretized versions of Algorithm 1 are beyond the scope of this article, we show an approximation result for one-dimensional domains and a class of uniformly refined equidistant grids that lead to a practical implementation of Algorithm 1.

With a slight abuse of notation, we consider $\Omega := [0, T]$ for simplicity of the presentation. Let $\tilde{g} \in L^\infty([0, T])$, $\Delta > 0$, and let $\tilde{v} \in \mathcal{F}_{(\mathbf{P})} \cap BV([0, T])$ be given. We consider a fixed uniform discretization of $[0, T]$ for $N \in \mathbb{N}$ with $h = N^{-1}$, $N = 2^k$ for some $k \in \mathbb{N}$ and $t_i = ih$ for $i \in \{0, \dots, N\}$. We consider the relationship between the optimization problems

$$\begin{aligned} \min_v \Upsilon(v) &:= \int_0^T \tilde{g}(s)v(s) \, ds + \alpha \text{TV}(v) \\ \text{s.t. } \|v - \tilde{v}\|_{L^1([0, T])} &\leq \Delta, \\ v(s) &\in V \text{ for a.a. } s \in [0, T] \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \min_{v, b_1, \dots, b_N} \Upsilon(v) \\ \text{s.t. } \|v - \tilde{v}\|_{L^1([0, T])} &\leq \Delta, \\ v(s) &= \sum_{i=1}^N b_i \chi_{[t_{i-1}, t_i)}(s) \text{ for a.a. } s \in [0, T] \text{ with } b_1, \dots, b_N \in V. \end{aligned} \tag{4.9}$$

One can easily see that (4.8) is equivalent to $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$ by adding an offset for the two constant terms that depend only on \tilde{v} . Moreover, (4.9) is the instance of (TRIP) derived from $(\text{TR}(\tilde{v}, \tilde{g}, \Delta))$ by choosing the uniform discretization introduced above and the constants as described in Section 3.3. The feasible set of (4.9) is a subset of the feasible set of (4.8), and thus

$$\min (4.8) := \min\{\Upsilon(v) \mid v \text{ feasible for (4.8)}\} \leq \min\{\Upsilon(v) \mid v \text{ feasible for (4.9)}\} =: \min (4.9). \tag{4.10}$$

We consider the following setting. We assume that \tilde{v} is a piecewise constant function *i.e.* (also) defined on an equidistant partition into 2^{k_0} intervals for $k_0 \in \mathbb{N}$. This assumption can (inductively) be satisfied in implementations of Algorithm 1 if v^0 is defined on such a partition and all trust-region subproblems are replaced by IPs of the form (4.9). Moreover, let v be a minimizer of (4.8). Then, there exists $n \in \mathbb{N}$ such that

$$v = \sum_{i=1}^n a_i \chi_{[s_{i-1}, s_i]} \quad (4.11)$$

for $0 = s_0 < \dots < s_n = T$ and $a_1, \dots, a_n \in V$ with $a_i \neq a_{i+1}$ for all $i \in \{1, \dots, n\}$ by virtue of Proposition 4.4.

We consider the setting given by $N, h, \tilde{g}, \tilde{v}, \Delta$, the uniform discretization, (4.8), and (4.9) described above. In this setting, we show the following proposition.

Proposition 4.28. *Let $\tilde{g} \in L^\infty([0, T])$, $\Delta > 0$, and $\tilde{v} \in \mathcal{F}(\mathbb{P}) \cap BV([0, T])$. Let \tilde{v} be a piecewise constant function *i.e.* defined on an equidistant partition into 2^{k_0} intervals for some $k_0 \in \mathbb{N}$. Let v in (4.11) be a minimizer of (4.8). There exist $N_0 \in \mathbb{N}$ such that if $N \geq N_0$, then there exist $(v^h, b_1^h, \dots, b_N^h) \in BV([0, T]) \times V^N$ feasible for (4.9) such that*

$$\min (4.9) \leq \min (4.8) + 2h \|\tilde{g}\|_{L^\infty([0, T])} \sum_{i=1}^n |a_i|. \quad (4.12)$$

Proof. We choose $N_0 \geq 2^{k_0}$ large enough such that for all $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, N_0\}$ with $s_{i-1} < t_j < s_i$.

We define an approximation of v on the equidistant grid t_0, \dots, t_N as follows. For $i \in \{0, \dots, n\}$, we define the indices $j(i)$ iteratively. We set $j(0) = 0$. For $i > 0$, let $j_\ell(i) := \max\{j \in \{1, \dots, N\} \mid t_j \leq s_i\}$ and $j_u(i) := \min\{j \in \{1, \dots, N\} \mid s_i < t_j\}$. We choose $j(i) = j_\ell(i)$ if $i + 1 \leq n$ and

$$\begin{aligned} \int_0^{t_{j_u(i)}} \left| \sum_{k=1}^{i-1} a_k \chi_{[t_{j(k-1)}, t_{j(k)}]}(s) + a_i \chi_{[t_{j(i-1)}, t_{j_\ell(i)}]} + a_{i+1} \chi_{[t_{j_\ell(i)}, t_{j_u(i)}]} - \tilde{v}(s) \right| ds \\ \leq \int_0^{t_{j_u(i)}} \left| \sum_{k=1}^{i-1} a_k \chi_{[t_{j(k-1)}, t_{j(k)}]}(s) + a_i \chi_{[t_{j(i-1)}, t_{j_u(i)}]} - \tilde{v}(s) \right| ds \end{aligned}$$

hold; otherwise we choose $j(i) = j_u(i)$. In particular, it holds that $j(n) = N$. We define

$$v^h := \sum_{i=1}^n a_i \chi_{T_i} \text{ with } T_i := [t_{j(i-1)}, t_{j(i)}] \text{ for all } i \in \{1, \dots, n\}.$$

We note that v^h is constructed on a refinement of the partition on which \tilde{v} is defined. Combining this with the iterative construction of the $j(i)$ and hence the T_i , it follows inductively that

$$\|\tilde{v} - v^h\|_{L^1([0, t_{j(i)}])} \leq \|\tilde{v} - v\|_{L^1([0, t_{j(i)}])}$$

for all $i \in \{1, \dots, n\}$. This proves $\|\tilde{v} - v^h\|_{L^1([0, T])} \leq \Delta$ and hence the feasibility of v^h .

Let $d_{\text{lin}} := \Upsilon(v^h) + \alpha \text{TV}(v^h) - (\Upsilon(v) + \alpha \text{TV}(v))$. Then, we estimate

$$d_{\text{lin}} \leq \left| \int_0^T g(s) \sum_{i=1}^n a_i (\chi_{T_i}(s) - \chi_{[s_{i-1}, s_i]}(s)) ds \right|$$

$$\begin{aligned}
 &\leq \|g\|_{L^\infty((0,T))} \sum_{i=1}^n |a_i| \int_0^T |\chi_{T_i}(s) - \chi_{[s_{i-1}, s_i]}(s)| \, ds \\
 &\leq \|g\|_{L^\infty((0,T))} \sum_{i=1}^n |a_i| \lambda(T_i \Delta[s_{i-1}, s_i]) \\
 &\leq 2h \|g\|_{L^\infty((0,T))} \sum_{i=1}^n |a_i|,
 \end{aligned}$$

where $T_i \Delta[s_{i-1}, s_i]$ denotes the symmetric difference between the sets T_i and $[s_{i-1}, s_i]$. Moreover, the estimate $\lambda(T_i \Delta[s_{i-1}, s_i]) \leq 2h$ in the last inequality holds because the construction of $j(i)$ implies $|t_{j(i)} - s_i| \leq h$ for all $i \in \{0, \dots, n\}$. Moreover, the construction of v^h implies that $\alpha \text{TV}(v^h) \leq \alpha \text{TV}(v)$.

We combine the estimate on d_{lin} and the inequality $\alpha \text{TV}(v^h) \leq \alpha \text{TV}(v)$ with the inequality (4.10) to obtain the estimate (4.12), which closes the proof. \square

Because of the dependence of N_0 in Proposition 4.28 on 2^{k_0} it is conceivable that very fine grids are necessary in an implementation with approximation error control. More work is necessary to develop more sophisticated approximations, also in the one-dimensional case. Due to these considerations, we solve instances of the IP formulation of (4.9) for uniform discretizations for different values of N in our computational experiments below to demonstrate the efficacy of the proposed idea.

5. COMPUTATIONAL EXPERIMENTS

To assess our algorithm computationally, we use the following optimization problem.

$$\min \frac{1}{2} \|Kv - f\|_{L^2((t_0, t_f))}^2 + \alpha \text{TV}(v) \text{ s.t. } v(t) \in \{-2, -1, 0, 1, 2\} \text{ a.e.} \quad (5.1)$$

Here, $t_f - t_0 = 2$, $f \in L^2((t_0, t_f))$ is given, and we use $Kv := k * v$ for a fixed convolution kernel $k \in L^\infty((0, 2))$, where $*$ denotes the convolution operator defined by $(k * v)(t) = \int_{t_0}^t k(t - \tau)v(\tau) \, d\tau$ for $t \in [t_0, t_f]$. Regarding the data we choose $f(t) = 0.4 \cos(2\pi t)$ for $t \in (-1, 1)$ as well as

$$k(t) = -0.1\omega_0 \left(\exp\left(-\frac{\omega_0(t-1)}{\sqrt{2}}\right) \cos\left(\frac{\omega_0(t-1)}{\sqrt{2}} - \frac{\pi}{4}\right) + \exp\left(-\frac{\omega_0(t-1)}{\sqrt{2}}\right) \sin\left(\frac{\omega_0(t-1)}{\sqrt{2}} - \frac{\pi}{4}\right) \right)$$

for $t \in (0, 2)$ and $\omega_0 = \pi$. Note that the same kernel function is used but reported incorrectly in [31, 34].

We perform three experiments. In two of them, we choose $\alpha = 0.0001$ in order to obtain a meaningful comparison with the combinatorial integral approximation decomposition, which is not possible for high values of α because the objective becomes *reduce switching by all costs* in this case, which is fundamentally opposed to the combinatorial integral approximation approach. In general, α allows one to control the trade-off between control switching and the minimization of the (tracking) term F , analogously to regularization of inverse problems with fractional-valued control inputs. We highlight that some care is necessary here because choosing α too large may imply that the resulting control is constant. We leave a rigorous strategy how to determine α to future work. In order to shed some light on this situation, we perform a third computational experiment, where we vary the regularization parameter α for a set of randomly drawn initial values of our implementation of the SLIP method.

We briefly verify that (5.1) satisfies our assumptions.

Proposition 5.1. *Problem (5.1) satisfies Assumptions 2.1 and 4.1.*

Proof. We define $\Omega := (t_0, t_f)$ and $F(v) := \frac{1}{2} \|Kv - f\|_{L^2(\Omega)}^2$ for $v \in L^2(\Omega)$. F is bounded from below by zero. Because the squared norm $\|\cdot\|_{L^2(\Omega)}^2$ and the operator K are continuous (see [45], Thm. 3.1.17), F is lower semi-continuous, and Assumption 2.1 holds.

A straightforward calculation shows that F is continuously differentiable. Because of the Hilbert space setting, the gradient $\nabla F : L^2(\Omega) \rightarrow L^2(\Omega)$ is available and is given as

$$\nabla F(v) = K^*(Kv - f),$$

where K^* denotes the adjoint operator of K . Because K is linear and bounded, this also holds for K^* ; and consequently ∇F is a continuously differentiable affine function.

Young's convolution inequality, $k \in W_c^{1,\infty}(\mathbb{R})$, and the continuous embedding $W^{1,\infty}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ give the continuity of the operator $K : L^1(\Omega) \rightarrow L^2(\Omega)$. Moreover, we obtain

$$|\nabla^2 F(\xi)(u, w)| = |(K^*Ku, w)_{L^2(\Omega)}| = |(Ku, Kw)_{L^2(\Omega)}| \leq \|Ku\|_{L^2(\Omega)} \|Kw\|_{L^2(\Omega)}$$

for all $\xi \in L^2(\Omega)$. Combining these observations with Proposition 4.2 yields that Assumption 4.1 is satisfied. \square

An unregularized variant of (5.1) has been solved after discretization to global optimality in [5]. The authors show that global optimization becomes computationally intractable for fine discretizations, in their case over ≈ 120 intervals. An unregularized variant of (5.1) has been approached with the combinatorial integral approximation decomposition in [31], where high-frequency switching occurs for fine discretization grids. In [34] a different regularization is chosen for (5.1), namely, a convex relaxation of the multibang regularizer introduced in [11]. The combination of the relaxed multibang regularization with approximation algorithms that take switching costs (see [3]) into account in the combinatorial integral approximation can yield a reduced number of switches in [34].

We have run three experiments. In *Experiment 1* with results described in Section 5.2, we compare four variants of the SLIP method for a sequence of refined discretizations. Specifically, we compare the SLIP method's efficiency with and without initialization with the solution from the previous coarser discretization grid and for two different update strategies for the trust-region radius.

In *Experiment 2*, described in Section 5.3, we compare the SLIP method with a MINLP approach of solving (P) for a sequence of refined discretization grids. We also point out briefly the difference from the combinatorial integral approximation decomposition approach. For this experiment, we choose the SLIP variant from *Experiment 1* that gave the best results in terms of the objective on the finest discretization grid.

In *Experiment 3*, described in Section 5.4, we assess the sensitivity of the solution obtained with the SLIP method with respect to different initialization points and regularization parameters α .

We use the SCIP Optimization Suite (version 7.0) [20] with SoPlex as the linear programming solver to solve both the (TRIP) subproblems of the SLIP method and the MINLPs, which are mixed-integer quadratic programs (MIQPs). We note that we have not experienced a qualitative difference in the results when choosing a different IP solver. Both algorithms are executed on a compute server with two Intel(R) Xeon Gold 6130 CPUs (16 cores each) clocked at 2.1 GHz and with 192 GB RAM.

5.1. Discretization

The SLIP method is a function space algorithm and therefore cannot be implemented directly on a computer. Specifically, the solution of the trust-region subproblem (TR) Line 6 cannot be done exactly. Following the considerations in Sections 3.3 and 4.5, we approximate (TR) as follows. For $N \in \{32, 64, 128, 256, 512, 1024, 2048\}$, we define a finest possible equidistant control discretization into N intervals. Thus, for a given N , a control function is an interval-wise constant function on N intervals that equidistantly discretize (t_0, t_f) .

Because $\text{TV}(v)$ is integral for integer-valued control functions v , the term αTV can be implemented with machine precision accuracy. Additionally, terms of the forms

$$F(v) = \frac{1}{2} \|Kv - f\|_{L^2((t_0, t_f))}^2 = \int_{t_0}^{t_f} |(k * v)(t) - f|^2 dt,$$

and

$$(\nabla F(\tilde{v}), v)_{L^2(\Omega)} = (K\tilde{v} - f, Kv)_{L^2(\Omega)} = \int_{\Omega} ((k * \tilde{v})(t) - f(t))(k * v)(t) dt$$

are required to evaluate the objectives of (P) and (TR) for control functions v and \tilde{v} that are interval-wise constant. The accuracy of the evaluation of the integrals on the right-hand sides of these equations depends on the accuracy of the evaluation of the convolutions $k * v$ and $k * \tilde{v}$. We approximate them with fifth-order Legendre–Gauss quadrature rules per interval for a decomposition into 2,048 intervals.

All integrals are using the decomposition into 2048 intervals. For smaller values of N , the control functions are broadcast to functions defined on 2048 intervals in our implementation. This procedure yields that the objectives are consistent over the different discretizations, which allows a comparison of the achieved objective values.

We assemble all of these approximations in the finite-dimensional IP (TRIP), which we then solve with a general-purpose IP solver.

Similarly, we derive corresponding mixed-integer quadratic programs (MIQPs) from (5.1) for the same integral discretizations and the same choices of N . In the cases where we can solve an MIQP to global optimality, the resulting optimal objective value is a lower bound on the result of the SLIP method.

Each value of N yields a smallest possible trust-region radius $\underline{\Delta}^N := (t_f - t_0)/N$. Thus, for given initial trust-region radius Δ^0 and the choice of N , our implementation allows for at most $k_{\max}^N = \log_2(\underline{\Delta}^N/\Delta^0)$ consecutive refinements of the trust-region radius. Our implementation of the SLIP method stops if the sufficient decrease condition (3.1) cannot be satisfied with the trust-region radius $\underline{\Delta}^N$ in the inner loop of the SLIP method.

5.2. Comparison of SLIP configurations

We compare four variants of the SLIP method. Specifically, we run the SLIP method with two different trust-region update strategies and two different ways of initializing the algorithm.

One trust-region update strategy is the trust-region reset strategy (R) *i.e.* defined in Algorithm 1 and uses the reset trust-region radius $\Delta^0 = 0.125$. The other trust-region update strategy (D) works as follows. If a step is accepted, we double the trust-region radius for the next iteration.

One initialization strategy is the use of the control $v^0 \equiv 0$ for all rounding grids (0). The other initialization strategy is to use $v^0 \equiv 0$ for only $N = 32$ and use the resulting control from the optimization with $N/2$ control intervals as initialization for the optimization on N control intervals (P). This strategy corresponds to mesh sequencing.

In total we obtain four algorithm variants, which we denote by SLIP R 0, SLIP R P, SLIP D 0, and SLIP D P. Because the mesh-sequencing strategies SLIP R P and SLIP D P use the results from the previous smaller values of N , we accumulate their compute time over the values $32, \dots, N$ intervals for the run with a control discretization into N intervals.

The objective values achieved are similar for all four algorithm variants. Regarding the final objective value on the finest grid, SLIP R 0 performs slightly better than SLIP R P and SLIP D P, which perform similar to and slightly better than SLIP D 0, respectively. For the compute times, the differences are much larger. Specifically, regarding the compute times for the final objective on the finest grid, SLIP R 0 takes considerably longer than all the other variants with $1.70 \times 10^4 s$. SLIP D P and SLIP R P take $2.31 \times 10^3 s$ and $2.88 \times 10^3 s$, respectively, less than a fifth of the compute time of SLIP R 0. The fastest method is SLIP D 0, which takes only $6.16 \times 10^2 s$.

TABLE 1. Best achieved objectives and corresponding times of the runs of the different variants of the SLIP method. Best values (lowest objective and lowest time) are highlighted using boldfaced font.

N	SLIP R 0		SLIP R P		SLIP D 0		SLIP D P	
	objective	time [s]	objective	time [s]	objective	time [s]	objective	time [s]
32	9.08×10^{-3}	0.354	9.08×10^{-3}	0.354	9.96×10^{-3}	0.680	9.96×10^{-3}	0.680
64	9.17×10^{-3}	1.02	6.17×10^{-3}	1.50	7.74×10^{-3}	1.52	7.16×10^{-3}	1.25
128	7.08×10^{-3}	3.19	5.66×10^{-3}	5.66	6.93×10^{-3}	3.86	6.66×10^{-3}	2.98
256	5.52×10^{-3}	2.35×10^1	5.00×10^{-3}	3.66×10^1	5.19×10^{-3}	2.61×10^1	5.22×10^{-3}	2.40×10^1
512	4.43×10^{-3}	1.69×10^2	4.52×10^{-3}	1.70×10^2	4.44×10^{-3}	1.31×10^2	5.05×10^{-3}	8.67×10^1
1024	4.53×10^{-3}	3.30×10^2	4.49×10^{-3}	3.07×10^2	4.77×10^{-3}	2.52×10^2	4.63×10^{-3}	6.63×10^2
2048	4.34×10^{-3}	1.70×10^4	4.49×10^{-3}	2.88×10^3	4.71×10^{-3}	6.16×10^2	4.45×10^{-3}	2.31×10^3

We have tabulated the objective values and compute times for all variants of SLIP and all N in Table 1. SLIP D 0 is the only strategy in which the optimization on the finest control grid with $N = 2048$ does not take much longer than the computation of the grids until $N = 1024$. The best values achieved in terms of objective and compute time are highlighted in boldfaced font.

5.3. Comparison of SLIP and MINLP solvers

In this experiment, both the SLIP method and the MIQP solves are initialized with the initial control $v^0 \equiv 0$ for all control discretizations. We used the same reset trust-region radius $\Delta^0 = 0.125$ for all control discretizations. This corresponds to variant SLIP R 0 in the preceding section, which showed the best performance in terms of the objective value on the finest grid.

The longest runtime of the SLIP method was 16,985 s for $N = 2048$. The MIQP runs were stopped if they did not complete after five hours (18,000 s). The MIQP run for $N = 32$ was able to find and certify a global minimum after 5670 s. The other MIQP runs did not solve (5.1) to global optimality within the time limit, with smallest reported duality gaps of 100% and more.

For $N = 32, 64,$ and 128 , the MIQP solves with SCIP are able to produce better objective values than the SLIP method, although their compute time is much longer. For example, the best objective achieved for $N = 64$ is 4.733×10^{-3} , which is produced by SCIP for the MIQP formulation after 4,604 s. The objective achieved by the SLIP method is 9.169×10^{-3} , but the SLIP method requires only 4 s. In Table 2 we tabulate the best objectives achieved by the SLIP method and the MIQP formulation as well as the runtime of the SLIP method and the time when the solution with the best objective is produced for the MIQP formulation. For each N the best achieved objective value is highlighted with boldfaced font.

For $N = 32, 64,$ and 128 , the SLIP method produces its final iterate before a better objective is produced by the MIQP solves. In the other runs, the SLIP method produces better objectives in much shorter time horizons than do the MIQP solves. The best objective value over all runs and methods is produced by the SLIP method for $N = 2, 048$ after 16,804 s. To understand these results better, we have visualized in Figure 1 the trajectories of the achieved objectives over (compute) time for both methods and all grids.

To assess the difference from the combinatorial integral approximation decomposition approach, we have removed the total variation term from the objective and run the combinatorial integral approximation decomposition on (5.1). This means that we have solved the continuous relaxation of the control problem (\hat{P}) and then computed a rounding of the resulting fractional-valued controls to $\{-2, -1, 0, 1, 2\}$ -valued controls using the SCARP approach [2, 3]. We have solved the continuous relaxation and performed the rounding for the same control discretizations. When SCARP is used in the rounding step, the total variation of the resulting control is minimal for a given distance to the fractional-valued relaxed control (the prescribed distance is linear in the interval length). The objective values for (5.1) of the discrete-valued controls produced by SCARP are given in the sixth column of Table 2. Most of the time for the combinatorial integral approximation decomposition

TABLE 2. Best achieved objectives and corresponding times of the runs of SLIP and the MIQP solves. For SLIP the final time is reported; for MIQP the time of the computation of the incumbent solution with the best reported objective is reported (both in seconds). The last column reports the objective achieved with the combinatorial integral decomposition using SCARP. Best values (lowest objective) are highlighted using boldfaced font.

N	SLIP		MIQP		SCARP	
	objective	time	objective	time to best objective	objective	time
32	9.081×10^{-3}	$3.543 \times 10^{-1} s$	5.079×10^{-3}	$3.837 \times 10^3 s$	1.839×10^{-2}	$1.775 \times 10^1 s$
64	9.169×10^{-3}	$1.015 s$	4.733×10^{-3}	$4.064 \times 10^3 s^*$	6.369×10^{-3}	$1.775 \times 10^1 s$
128	7.080×10^{-3}	$3.185 s$	5.447×10^{-3}	$1.434 \times 10^4 s^*$	5.551×10^{-3}	$1.778 \times 10^1 s$
256	5.523×10^{-3}	$2.350 \times 10^1 s$	5.513×10^{-3}	$1.644 \times 10^4 s^*$	7.741×10^{-3}	$1.777 \times 10^1 s$
512	4.426×10^{-3}	$1.687 \times 10^2 s$	6.685×10^{-3}	$1.776 \times 10^4 s^*$	1.220×10^{-2}	$1.778 \times 10^1 s$
1024	4.529×10^{-3}	$3.303 \times 10^2 s$	9.153×10^{-3}	$1.680 \times 10^4 s^*$	2.350×10^{-2}	$1.784 \times 10^1 s$
2048	4.339×10^{-3}	$1.698 \times 10^4 s$	2.727×10^{-2}	$1.746 \times 10^4 s^*$	4.610×10^{-2}	$1.800 \times 10^1 s$

* Timeout after $1.8 \times 10^4 s$

approach is spent for the solution of the continuous relaxation, which is computed on the finest grid and independently of the rounding grids that are used for SCARP, which is very fast. Therefore, the timings in the last column of Table 2 do not differ much.

For the coarsest grid, the combinatorial integral approximation decomposition approach produces less switching than does the SLIP method, while the tracking term of the objective is worse. As guaranteed by the theory behind the combinatorial integral decomposition (see, for example, [31, 36]), the value of the tracking term tends to the value of the continuous relaxation with finer control discretizations. This comes at a cost of increased switching behavior; the total variation term in the objective tends to infinity for $N \rightarrow \infty$. For our set of parameters, we observe that the trade-off produced by the combinatorial integral approximation yields a superior result in terms of the objective value for $N = 64$ and $N = 128$ compared with the SLIP method. For higher values of N the increase of the switching implies that the total variation term dominates the tracking term, and an approximately linear increase of the objective value can be observed in the results.

We have plotted the resulting control trajectories computed with the SLIP method for $N = 32$ and $N = 2,048$ in the top row of Figure 2. The corresponding state trajectories and the tracked function f are plotted in the top row of Figure 3. The corresponding trajectories produced by using the combinatorial integral approximation decomposition with SCARP for rounding are shown in the bottom rows.

We also evaluate the numerical results with respect to L-stationarity. We measure L-stationarity by computing $\|(\nabla F(v)(t_i))_i\|$, where the t_i are the switching locations of v , *i.e.*, the t_i where $\lim_{t \uparrow t_i} v(t) \neq \lim_{t \downarrow t_i} v(t)$. The point evaluation of $\nabla F(v)$ at the switching locations of v is meaningful only if $\nabla F(v)$ is a continuous function. We briefly argue that this is always the case for our example in Proposition A.3. For all discretizations, we obtain a downward trend of the L-stationarity measure after the first few iterations. This trend stagnates at about 3×10^{-6} for the finest discretization $N = 2,048$. The stationarity measure is plotted over the iterations for all discretizations in Figure 4. We note that with a fixed available finest discretization as we have with $N = 2,048$, one cannot in general achieve $\|(\nabla F(v)(t_i))_i\| = 0$ even if our computations were performed in exact arithmetic. However, one could fix the order and heights of the switches after the final iteration (thereby fixing also the total variation) and then optimize the switch locations to minimize $F(v)$, which is a nonconvex nonlinear program.

5.4. Sensitivity with respect to initialization and regularization parameter

We choose $N = 512$ and initialize our implementation of the SLIP method with for 25 randomly generated controls (drawn from uniform integer distributions of the feasible control values for each interval) for the

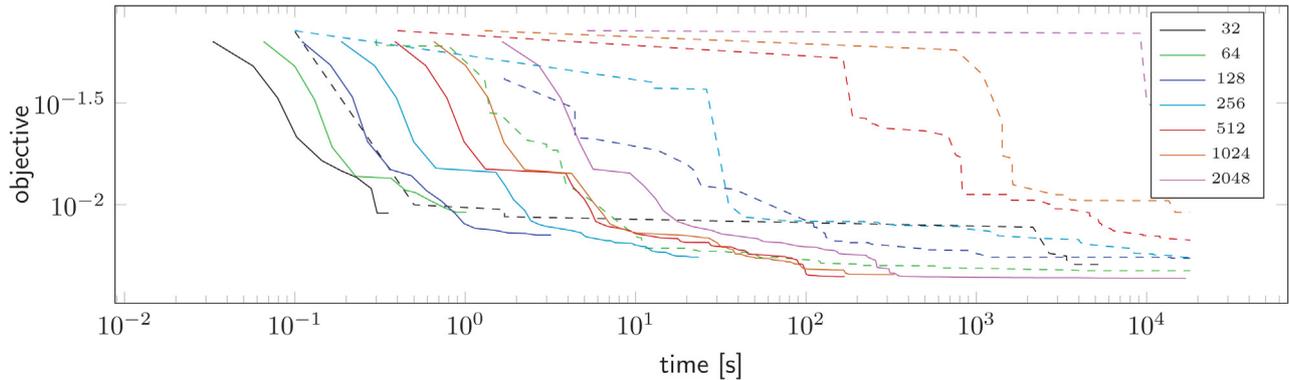


FIGURE 1. Objective values over time for SLIP (solid) and (MIQP) solves (dashed) for $N = 32, \dots, 2048$.

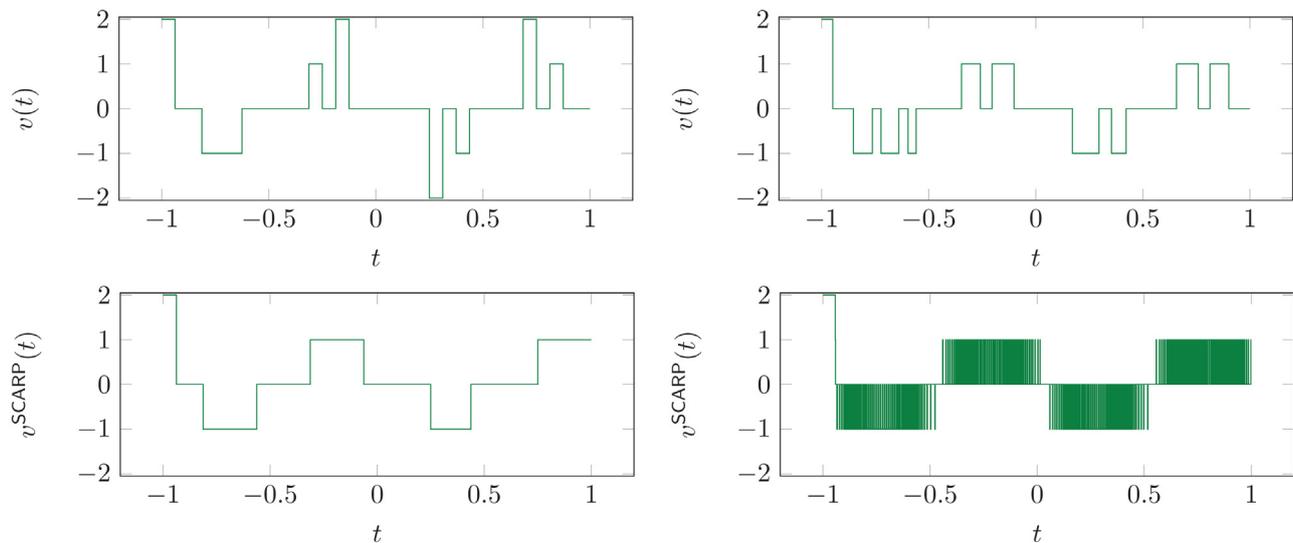


FIGURE 2. Top row: final control trajectories produced by the SLIP method for $N = 32$ (left, objective value 9.081×10^{-3}) and $N = 2,048$ (right, objective value 4.339×10^{-3}). Bottom row: control trajectories produced by the combinatorial integral approximation decomposition approach using SCARP for $N = 32$ (left, objective value 1.839×10^{-2}) and $N = 2,048$ (right, objective value 4.610×10^{-2}).

regularization parameter values $\alpha \in \{10^i \mid i \in \{-5, \dots, 2\}\}$. Again the SLIP implementation is stopped when the trust-region radius cannot be decreased further.

The mean of the achieved \mathbf{L} -stationarity for the final iterate increases for reduced parameters α . Therefore, one may need to choose finer control discretization grids to achieve similar stationarity values for small regularization parameters. The distributions of the achieved values of stationarity among the initialization points per regularization parameter value are shown in Figure 5 (a).

For large values of α , the term αTV dominates the objective and the achieved objective values for different regularization parameters are largely different while the differences become similar for small regularization parameters when the first term dominates. In other words, the algorithm is more sensitive to the initialization point if the regularization parameter is high / switches are penalized hard.

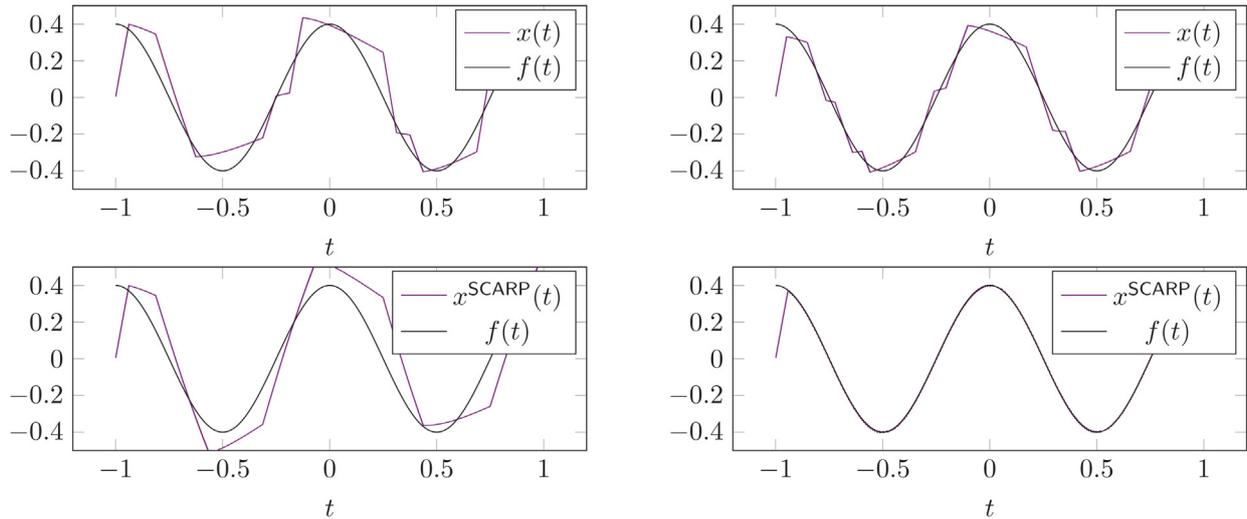


FIGURE 3. Top row: final state trajectories produced by the SLIP method for $N = 32$ (left, objective value 9.081×10^{-3}) and $N = 2,048$ (right, objective value 4.339×10^{-3}) in violet and the tracked function f in black. Bottom row: final state trajectories produced by the combinatorial integral approximation decomposition approach using SCARP for $N = 32$ (left, objective value 1.839×10^{-2}) and $N = 2,048$ (right, objective value 4.610×10^{-2}). Note that $x(0) = 0$ and $f(0) = 0.4$ and thus the tracking error is bounded away from zero.

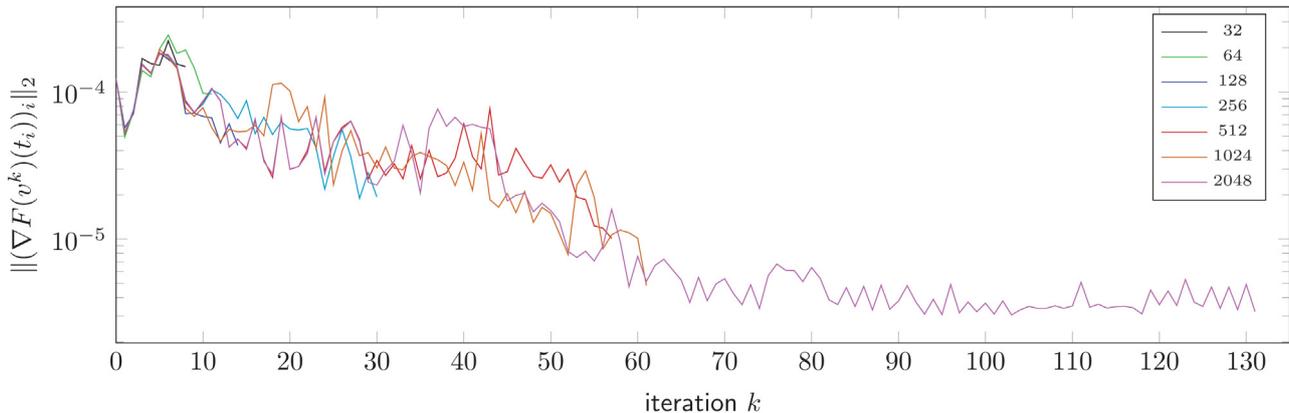


FIGURE 4. \mathbf{L} -stationarity of the controls v^k over the iterations k in the SLIP method for different discretizations. \mathbf{L} -stationarity is measured as the ℓ_2 -norm of $(\nabla F(v^k)(t_i))_i$ for the switching locations t_i of v^k .

In contrast to \mathbf{L} -stationarity, r -optimality for some $r > 0$ cannot be assessed directly for the computed final iterates v^f without solving the infinite-dimensional nonlinear control problem (P) subject to the constraint $\|v - v^f\|_{L^1(\Omega)} \leq r$. In order to at least estimate the performance in this regard, we solve the MINLP that has been used in Section 5.3 with the additional constraint $\|v - v^f\|_{L^1(\Omega)} \leq r \frac{t_f - t_0}{N}$, which can be modeled exactly with linear inequalities and slack variables for our equidistant discretization, for $r \in \{1, 2, 3, 4\}$ and compute the relative differences between the objective values achieved for v^f and the MINLP solution with the additional constraint. The relative difference is the difference divided by the objective value of the MINLP solution. The termination criterion of the MINLP solver is chosen as 10^{-4} for the duality gap (difference between upper and

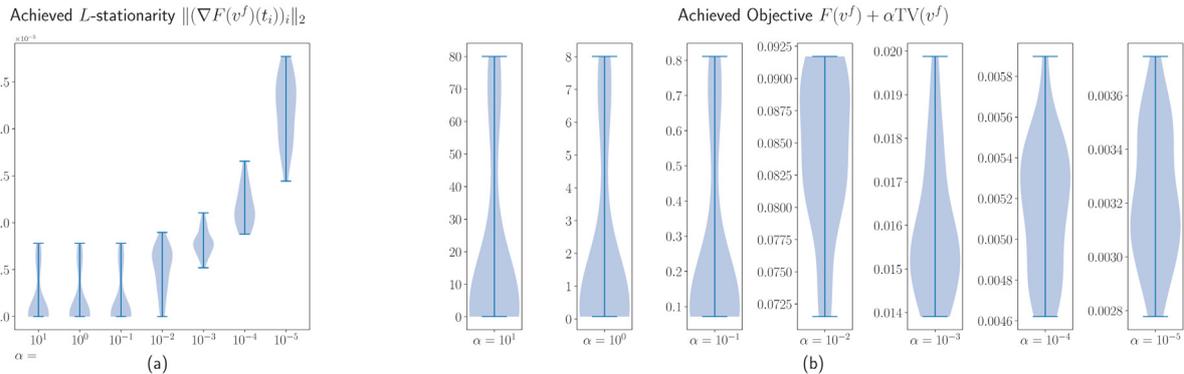


FIGURE 5. Distributions of the achieved **L**-stationarity (left) and objective values (right) of the final iterate v^f for different values of α over randomly drawn initialization points of our SLIP implementation. **L**-stationarity is measured as the ℓ_2 -norm of $(\nabla F(v^f)(t_i))_i$ for the switching locations t_i of v^f .

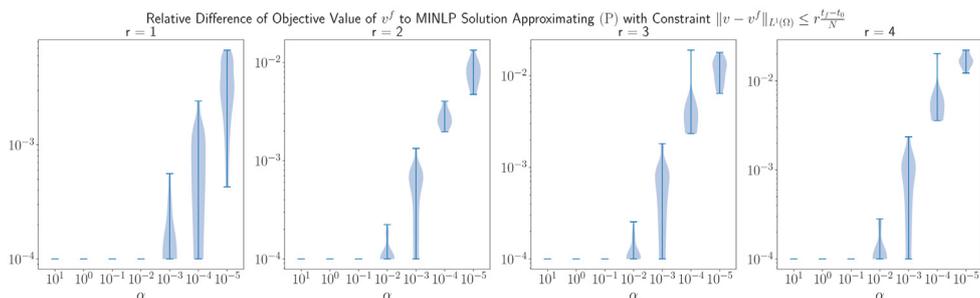


FIGURE 6. Distributions of the achieved relative differences between the objective values achieved for v^f and the MINLP solution with the additional constraint $\|v - v^f\|_{L^1(\Omega)} \leq r \frac{t_f - t_0}{N}$ for $r \in \{1, 2, 3, 4\}$. Values lower than 10^{-4} are set to 10^{-4} because this is the configured duality gap threshold for the MINLP solver.

lower bound on the MINLP solution divided by the value of the lower bound). The results show that within this tolerance, the results produced by our implementation of SLIP are also optimal for the MINLP with the additional constraint $\|v - v^f\|_{L^1(\Omega)} \leq r \frac{t_f - t_0}{N}$ for $\alpha \geq 10^{-2}$. The relative differences increase with increasing values of r , *i.e.* a larger feasible set for the MINLP, and decreasing values of α . This is in line with the higher **L**-stationarity values for decreasing values of α . We provide the achieved relative differences in Figure 6. Relative differences below the duality gap threshold of the MINLP solver are set to the duality gap threshold in the interest of a fair comparison and a clearer presentation.

6. CONCLUSION

Total variation regularization of integer optimal control problems yields existence of minimizers under mild assumptions that are independent of the regularity of the underlying dynamical system. The set of feasible controls of the regularized problems is closed with respect to weak-* and strict convergence in $BV(\Omega)$. The infinite-dimensional vantage point allows for a meaningful concept of local optimizers (r -optimal points) for such problems.

We provide an algorithmic framework that combines a trust-region globalization from nonlinear programming with IPs as subproblems to optimize for r -optimal points. We obtain convergence of the resulting iterates to points that satisfy a first-order necessary condition of r -optimal points for problems with a one-dimensional domain of the control input. However, in order to obtain that \mathbf{L} -stationarity is a first-order necessary condition for both the trust-region subproblem and r -optimal points, a regularity assumption on the Hessian of the objective F and thus the underlying dynamical system is necessary.

While the algorithmic framework is still computationally expensive, we have experienced superior performance in terms of both computational effort and achieved objective value when compared with the application of an off-the-shelf spatial branch-and-bound strategy to a discretized integer optimal control problem. The results show that one can easily achieve a meaningful balance between the first term in the objective (tracking in our example) and the total variation term. However, the algorithmic framework is computationally less attractive than the combinatorial integral decomposition. If fine control discretizations but low switching costs are required or the first step of the combinatorial integral approximation produces a control *i.e.* not close (in norm) to an integer-valued one, the high-frequency switching resulting from the combinatorial integral approximation can be avoided by using the presented algorithmic framework. The computational results, in particular Figure 2, show that although starting from the initial guess *i.e.* identical zero, the algorithm is able to develop a nontrivial switching structure of the resulting control. The computational results hint that, in particular for small values of the regularization parameter α , one may need to choose fine discretizations, thereby hinting that adaptive grid refinement strategies may be necessary for a sophisticated implementation.

In order to save compute time in practice, the numerical results from Experiment 2 suggest that one should not run the trust-region reset strategy for fine grids directly on the finest required grid but run it on refined meshes and initialize it with the solution from the previous run. Alternatively or additionally, one may replace the trust-region reset strategy with a simple update strategy, for which we, however, fail to guarantee the convergence properties obtained in Section 4. Contrary to our intuition the results indicate that a combination of the trust-region update strategy with initialization from the previous grid is not the fastest strategy in general. Moreover we intend to use \mathbf{L} -stationarity directly in the algorithm in future implementations. This means that we need to ensure more regularity of the problem (\mathbf{L}) , in particular differentiability of the reduced objective with respect to the switching locations, and then solve it after the switching structure has settled as is proposed in Remark 4.26.

Due to its similarity to PDE-constrained optimal control problems with bound constraints, the algorithm might be improved by integrating curvature information into the trust-region subproblems, which may lead to more difficult subproblems, however.

APPENDIX A. AUXILIARY RESULTS

We state the version of Taylor's theorem that we require for our analysis.

Proposition A.1. *Let $F : L^2(\Omega) \rightarrow \mathbb{R}$ be twice continuously Fréchet differentiable. Then it holds for all $u, v \in L^2(\Omega)$ that $F(v) = F(u) + (\nabla F(u), v - u)_{L^2(\Omega)} + \frac{1}{2}(v - u, \nabla^2 F(\xi)(v - u))_{L^2(\Omega)}$ for some ξ in the line segment between v and u .*

Proof. This follows from combining Taylor's theorem in Banach spaces (see Sect. 4.5 in [49]) with the intermediate value theorem, which is applicable because F is \mathbb{R} -valued. \square

We state and prove the relationship between $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^1(\Omega)}$ for $\{\nu_1, \dots, \nu_M\}$ -valued controls.

Proposition A.2. *Let $v \in L^\infty(\Omega)$ be $\{\nu_1, \dots, \nu_M\}$ -valued. Then, $\|v\|_{L^2(\Omega)} \leq M_1 \sqrt{\|v\|_{L^1(\Omega)}}$, where $M_1 := \max_i |\nu_i|$. Let $v_1, v_2 \in L^\infty(\Omega)$ be $\{\nu_1, \dots, \nu_M\}$ -valued. Then, $\|v_1 - v_2\|_{L^2(\Omega)} \leq M_2 \sqrt{\|v_1 - v_2\|_{L^1(\Omega)}}$, where $M_2 := \max_{i,j} |\nu_i - \nu_j|$.*

Proof. The first claim follows from the inequality $(\int_{\Omega} |v(s)|^2 ds)^{\frac{1}{2}} \leq (\int_{\Omega} \max_i |\nu_i| |v(s)| ds)^{\frac{1}{2}}$. The second claim follows analogously. \square

We state and prove the regularity of $\nabla F(v)$ for our computational example, in which we use a convolution kernel of $W^{1,\infty}(\mathbb{R})$ -regularity.

Proposition A.3. *In the setting of Section 5 it holds for all $v \in L^2((t_0, t_f))$ that*

$$\nabla F(v) = K^*Kv - K^*f \in C([t_0, t_f]).$$

Proof. The formula $\nabla F(v) = K^*Kv - K^*f$ follows from basic derivative calculus. Because $k \in W_c^{1,\infty}(\mathbb{R})$, it holds that $Kv = (k * v)\chi_{(t_0, t_f)}$ is a continuous function. Moreover, the adjoint of the convolution is the cross-correlation, which is a convolution with the kernel $s \mapsto k(-x)$ instead of k . This implies that K^*Kv , K^*f , and in turn $\nabla F(v)$ are continuous functions. \square

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