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A REVERSE ISOPERIMETRIC INEQUALITY FOR PLANAR (α, β)-CONVEX BODIES

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Abstract. In this paper, we study a reverse isoperimetric inequality for planar convex bodies whose radius of curvature is between two positive numbers $0 \leq \alpha < \beta$, called (α, β)-convex bodies. We show that among planar (α, β)-convex bodies of fixed perimeter, the extremal shape is a domain whose boundary is composed by two arcs of circles of radius α joined by two arcs of circles of radius β.

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1. Introduction

The planar version of the isoperimetric inequality states that the circle encloses the maximal area among all curves of the same length. In this paper, we deal with the reverse problem of finding, among the convex curves of fixed length and satisfying a constraint on the curvature, the one enclosing the minimal area. One of the first results in this framework has been obtained by A. Borisenko and K. Drach in [8]. They proved that for any convex set K whose curvature k satisfies $k \geq \lambda > 0$ the inequality

$$A(K) \geq \frac{P(K)}{2\lambda} - \frac{1}{\lambda^2} \sin\left(\frac{P(K)\lambda}{2}\right), \tag{1.1}$$

holds, where $A(K)$ and $P(K)$ denote the area and the perimeter of the set K respectively. Moreover they studied the equality case proving that the equality is satisfied if and only if K is the intersection of two disks of radius λ, called λ-lune. On the other side R. Chernov, K. Drach and K. Tatarko in [9] proved the reverse isoperimetric inequality for convex sets with the curvature satisfying $k \leq \lambda < \infty$ showing that

$$A(K) \geq \frac{P(K)}{\lambda} - \frac{\pi}{\lambda^2}. \tag{1.2}$$

Also in this case the sets satisfying the equality in the previous inequality have been characterized and shown to be the convex hull of two balls of radius $\frac{1}{\lambda}$.

Keywords and phrases: Shape optimization, reverse isoperimetric inequality, Pontryagin maximum principle, convexity.

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In this paper we generalize the inequality (1.1) to the class of convex sets with curvature radius ρ satisfying $\alpha \le \rho \le \beta$ for given constants $0 \le \alpha < \beta$, by proving that

$$A(K) \ge \frac{1}{2}(\beta+\alpha)(P(K)-2\pi\alpha) + \pi\alpha^2 - (\beta-\alpha)^2 \sin\left(\frac{P(K)-2\pi\alpha}{2(\beta-\alpha)}\right). \tag{1.3}$$

Moreover the equality holds if and only if the boundary of K is composed by two arcs of circles of radius α joined by two arcs of circles of radius β. We explicitly observe that formally for $\beta \to +\infty$ one gets the inequality (1.2).

The idea of the proof stems from the argument used in [8] based on the Pontryagin Maximum Principle to derive the optimality conditions satisfied by a constrained minimizer of the area. Then we exploit these conditions in an analytical way to establish the result, as opposed to the more geometrical analysis done in [8].

It is worth noticing that imposing some constraints on the class of all admissible curves, is necessary, since otherwise the problem can be easily proved to be ill-posed. In the literature different reverse isoperimetric problems have been considered (see for instance [3, 7, 11, 13, 14, 18, 20, 25] for different approaches and different classes of admissible sets).

2. Preliminaries

With the notation $\mathfrak{S}(A, B)$, where A and B are two sets and $\mathfrak{S}$ is one of the standard symbol for a functional space (such as $C^{0,1}$ for Lipschitz functions, $W^{1,1}$ for absolutely continuous, etc.), we mean the space of maps defined on A with values in B. Moreover, for $I = [a, b] \subset \mathbb{R}$ with $\mathfrak{S}_{per}(I, B)$ we denote the $(b-a)$-periodic functions in $\mathfrak{S}(\mathbb{R}, B)$.

2.1. Convex bodies and (α, β)-convexity

In this section we recall some basic properties of convex sets in Euclidean spaces, we introduce the class of competitors for our optimisation problem and prove some useful properties such as regularity and compactness.

Throughout the paper we will denote with B_r the closed ball with radius $r > 0$ centred in the origin. By a convex body we shall mean a compact convex set $K \subset \mathbb{R}^n$ with non-empty interior. With $\mathcal{K}^n$ we will denote the class of convex bodies in $\mathbb{R}^n$. The *support function* of K is the real-valued function defined on the unit sphere $\mathbb{S}^{n-1}$ by

$$h_K(v) := \max_{k \in K} \langle k, v \rangle, \quad v \in \mathbb{S}^{n-1}.$$

We recall that the support function h_K characterises the set K and any function $h : \mathbb{S}^{n-1} \to \mathbb{R}$, such that its 1-homogeneous extension is convex, is the support function of a convex body (*cf.* [22], Sect. 1.7). Moreover $K \in \mathbb{R}^n$ is strictly convex if and only if the 1-homogeneous extension of its support function belongs to $C^1(\mathbb{R}^n \setminus \{0\})$ (see [22], Cor. 1.7.3 & Sect. 2.5).

A convenient way to endow $\mathcal{K}^n$ with a topology is to use the Hausdorff distance between two non-empty compact sets, denoted by $d^H(\cdot,\cdot)$ (*cf.* [22], Sect. 1.8). Indeed, we recall that the perimeter and the area functionals are continuous with respect to the Hausdorff topology on $\mathcal{K}^n$ (see also [19], Thms. 23 and 26). By Lemma 1.8.14 of [22], given $K, M \in \mathcal{K}^n$, we can characterize the Hausdorff distance of K from M in terms of their support functions:

$$d^H(K, M) = \|h_K - h_M\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{2.1}$$

Moreover, by the *Blaschke selection theorem* (*cf.* [22], Thm. 1.8.7), every bounded sequence of convex bodies has a subsequence that converges to a convex body in the Hausdorff topology.

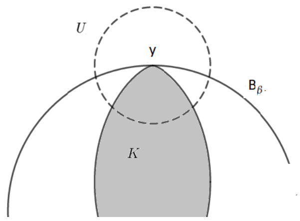

FIGURE 1. A set K locally embeddable in a ball.

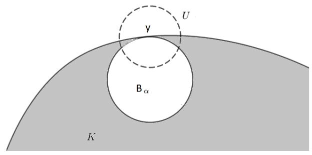

FIGURE 2. A ball locally embeddable in a set K.

Following page 157 of [22] we say that the convex body L is *locally embeddable* in the convex body K if for each point $x \in \partial K$ there are a point $y \in L$ and a neighborhood U of y such that

$$(L \cap U) + x - y \subset K.$$

We are now ready to give the notion of curvature that we are going to use for our result:

Definition 2.1. Let α and β be two real numbers with $0 < \alpha < \beta$. We say that a convex body K is (α, β)-*convex* if K is locally embeddable in B_β and B_α is locally embeddable in K (see Figs. 1 and 2). In the case $\alpha = 0$, we will assume that K is locally embeddable in B_β.

Lemma 2.2. *A convex body K is locally embeddable in B_β if and only of for each point $y \in \partial K$, there exist a point $x \in \partial B_\beta$ and a neighbourhood U of y such that $(K \cap U) + x - y \subset B_\beta$. In particular any convex body K which is locally embeddable in a ball is strictly convex.*

Proof. Assume K is local embeddable in B_β. We fix $y \in \partial K$. Let H_y be a supporting hyperplane of K in y. Let $\nu_y \in \partial B_\beta$ be orthogonal to H_y. By the assumption of local embeddability, there exist $y_0 \in \partial K$ and a neighborhood of y_0 such that $(K \cap U_{y_0}) + \nu_y - y_0 \subset B_\beta$. This implies that $y = y_0$. By contradiction, assume that $y \neq y_0$. Since $K \cap H_y$ is convex, then the segment $[y, y_0]$ is contained in $K \cap H_y$ and thus in the boundary of K. Therefore, for every neighborhood U_{y_0} of y_0, we have that $(K \cap U_{y_0}) + \nu_y - y_0$ is not contained in B_β, that is a contradiction. As a byproduct we also note that any supporting hyperplane H_y has a unique point of intersection with the boundary of K. We deduce that if K is local embeddable in B_β, any point of the boundary of K is supported by a sphere and therefore K is strictly convex (see [10], Sect. 1.2).

For the converse, it is sufficient to apply ([10], Thm. 1.9), to prove that K is local embeddable in B_β. □

Remark 2.3. Let $\alpha > 0$. An equivalent definition of (α, β)-convexity can be given in terms of the following notions of β-convexity and α-concavity, sometimes used in literature. A convex body K is said to be α-*concave* if the ball of radius $1/\alpha$ is locally embeddable in K (*cf.* [9], Def. 1.2). K is said to be β-*convex* if for each point $y \in \partial K$, there exist a point $x \in \partial B_\beta$ and a neighbourhood U of y such that $(K \cap U) + x - y \subset B_\beta$. In view of the previous lemma, this is equivalent to the local embeddability of K in B_β. Therefore K is (α, β)-convex if it is at the same time β-convex and $1/\alpha$-concave.

The concept of local embeddability when K or L is a ball has been studied and used in several contexts. The following lemmata provide two classical regularity properties related to theses concepts.

Lemma 2.4. *If a convex body K is locally embeddable in a ball, then its support function is of class $C^{1,1}$.*

Proof. From the local embeddability of K in a ball, say B, it follows that K is a strictly convex body. Therefore by Theorem 3.2.3 of [22], there exists a convex body $M \in \mathcal{K}^n$ such that $B = K + M$ (*i.e.* K is a *summand* of B) which is equivalent to say that K *slides freely inside* B (*cf.* [22], Thm. 3.2.2). The result follows by the characterization of convex bodies with support function of class $C^{1,1}$ (*cf.* [17], Prop. 2.3). □

Lemma 2.5. *Let K be a convex body. If a ball is locally embeddable in K, then its boundary ∂K is of class $C^{1,1}$.*

Proof. Let B a ball locally embeddable in K. Since B is strictly convex, by Theorem 3.2.3 of [22] there exists a convex body $M \in \mathcal{K}^n$ such that $K = B + M$ (*i.e.* B is a *summand* of K), that is equivalent to ∂K being of class $C^{1,1}$ (*cf.* for example [16], Prop. 2.4.3). □

Remark 2.6. Note that if $\alpha = 0$ the class of (α, β)-convex sets is nothing but the family of β-convex sets introduced in [8]. From the previous lemmata it follows that for $\alpha > 0$, an (α, β)-convex body is a $C^{1,1}$ strictly convex set with support function of class $C^{1,1}$. In the case $\alpha = 0$, we can only say that the support function of its boundary is $C^{1,1}$.

We will mostly work in the two dimensional setting, dealing with planar convex sets, therefore we recall some useful preliminaries results on planar convex geometry. First we note that it is often convenient to work with the so called *parametric support function*, *i.e.* $p_K(t) := h_K \circ \boldsymbol{\sigma}(t)$ where $\boldsymbol{\sigma}(t) = (\cos(t), \sin(t))$ and $t \in [0, 2\pi]$ (note that this is how the support function of a planar convex body is defined in classical literature, *cf.* for example [21, 24]). In the next propositions we recall some well known and useful properties related to the parametric support function of a planar convex body (see also [6] for a recent survey on the subject). We remark that similar results hold under weaker assumption regularity assumptions (*cf.* [15, 22]), but we restrict our attention to what will be sufficient for our purposes.

Proposition 2.7. *Let $K \in \mathcal{K}^2$ be a strictly convex planar body and p_K its parametrized support function. Assume that $p_K \in C^{1,1}_{per}(0, 2\pi)$. Then the radius of curvature of the boundary ∂K, $\rho_K(t)$, satisfies for a.e. $t \in (0, 2\pi)$ the equation*

$$\rho_K(t) = p_K(t) + p_K''(t) \geq 0. \tag{2.2}$$

Viceversa, if $h \in C^{1,1}_{per}(0, 2\pi)$ is a function satisfying (2.2), then there exists a convex body K such that h is its parametric support function.

Proposition 2.8. *Under the same assumptions of Proposition 2.7 the boundary ∂K can be parametrized by*

$$\begin{cases} x(t) = p_K(t)\cos(t) - p_K'(t)\sin(t) \\ y(t) = p_K(t)\sin(t) + p_K'(t)\cos(t) \end{cases}$$

Moreover the perimeter and the area of K can be computed by the following formulae

$$P(K) = \int_0^{2\pi} (p_K(t) + p_K''(t))\,\mathrm{d}t = \int_0^{2\pi} \rho_K(t)\mathrm{d}t,$$

$$A(K) = \frac{1}{2}\int_0^{2\pi} (p_K(t) + p_K''(t))\,p_K(t)\,\mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \rho_K(t)\,p_K(t)\,\mathrm{d}t.$$

The (α,β)-convexity for planar domains can be expressed in terms of parametrized support function. Indeed a convex body $K\in\mathcal{K}^2$ is (α,β)-convex if and only if its parametrized support function p_K satisfies the inequalities

$$\alpha\le p_K(t)+p_K''(t)\le\beta\quad\text{a.e. in }(0,2\pi).$$

It easily follows, by Proposition 2.8, that the perimeter of any (α,β)-convex body in the plane satisfies $2\pi\alpha\le P(K)\le 2\pi\beta$. Moreover this characterization allows us to prove that for planar domains the (α,β)-convexity is preserved by Hausdorff convergence.

Lemma 2.9. *Let $0\le\alpha<\beta<\infty$ and $\{K_n\}_{n\in\mathbb{N}}\subset\mathcal{K}^2$ be a sequence of (α,β)-convex bodies. Assume that the sequence K_n converges to K in the Hausdorff topology. Then K is (α,β)-convex.*

Proof. Let p_{K_n} be the parametric support function of K_n and p_K the parametric support function of K. As K_n is (α,β)-convex body then p_{K_n} is of class $C^{1,1}$ and the radius of curvature of ∂K_n exists almost everywhere. Therefore, the parametric support function satisfies

$$\alpha\le p_{K_n}+p_{K_n}''\le\beta\qquad\text{a.e. in}\quad[0,2\pi].\tag{2.3}$$

By formula 2.1, as K_n converges to K in the Hausdorff distance, p_{K_n} converges to p_K in $L^\infty([0,2\pi])$. Since p_{K_n} is bounded in $L^\infty([0,2\pi])$, we deduce from inequality (2.3) that p_{K_n}'' is bounded in $L^\infty([0,2\pi])$.

By Proposition 2.8

$$p_{K_n}'(t)=-x_n(t)\sin(t)+y_n(t)\cos(t)\,,$$

where $(x_n(t),y_n(t))$ is the parametrization of the boundary of K_n. As $(x_n(t),y_n(t))\in\partial K_n$ and all the K_n are contained in a ball, then p_{K_n}' is bounded in $L^\infty([0,2\pi])$. Therefore p_{K_n}' is bounded in the Sobolev space $W^{1,\infty}([0,2\pi])$. By the Rellich-Kondrachov theorem, there exists $w\in W^{1,\infty}([0,2\pi])$ such that, up to a subsequence, $p_K'\overset{*}{\rightharpoonup}w$ in $W^{1,\infty}([0,2\pi])$ and $p_{K_n}'\to w$ in $L^\infty([0,2\pi])$.

Since p_{K_n} is bounded in $W^{1,\infty}([0,2\pi])$, by using again the Rellich-Kondrachov theorem, there exists $g\in W^{1,\infty}([0,2\pi])$ such that, up to a subsequence, $p_{K_n}\overset{*}{\rightharpoonup}g$ in $W^{1,\infty}([0,2\pi])$ and $p_{K_n}\to g$ in $L^\infty([0,2\pi])$.

Since K_n converges to K in the Hausdorff distance, and p_{K_n} converges to p_K in $L^\infty([0,2\pi])$, we deduce that $g=p_K$, $w=g'$. Thus, we can extract a subsequence still denoted p_{K_n}' such that $p_{K_n}'\overset{*}{\rightharpoonup}p_K'$ in $W^{1,\infty}([0,2\pi])$. This implies that $p_K\in W^{2,\infty}([0,2\pi])$ which means p_K is $C^{1,1}$. As p_K is the parametric support function of K, then p_K and p_K'' are 2π- periodic. Moreover, (2.3) implies

$$\int\alpha\phi\le\int(p_{K_n}+p_{K_n}'')\phi\le\int\beta\phi\quad\text{for all }\phi\text{ smooth and non negative.}$$

Using the weak$-*$ convergence in $W^{2,\infty}([0,2\pi])$, we have

$$\int\alpha\phi\le\int(p_K+p_K'')\phi\le\int\beta\phi\qquad\text{for all }\ \phi\text{ smooth and non negative.}$$

Thus, by the Fundamental lemma in the calculus of variations, we deduce that

$$\alpha\le p_K+p_K''\le\beta\qquad\text{a.e. in }[0,2\pi],$$

that is, K is (α,β)-convex. ∎

We conclude this section with an example of a family of planar (α,β)-convex bodies that will play an important role in the sequel.

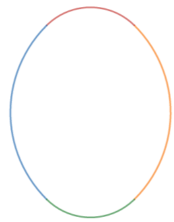

FIGURE 3. An (α, β)-egg

Example 2.10 ((α,β)-eggs)**.** For $\alpha \geq 0$, an example of (α,β)-convex bodies in $\mathcal{K}^2$ is the family of sets that we will call (α,β)-eggs (see Fig. 3). They are symmetric with respect to the Cartesian axes, their boundary is composed by 4 arcs of circles with radii α and β alternatively and their centers are chosen in such a way to ensure the regularity of ∂K (see Lem. 2.5). Given $0 < \alpha < \beta < +\infty$ and $l \in (\alpha\pi, \beta\pi)$, the (α,β)-egg, with perimeter $P = 2l$, can be parametrised as follows. We set $\tau = \frac{1}{2}\frac{l-\pi\alpha}{\beta-\alpha}$, $\kappa_1 = (\beta-\alpha)\cos(\tau)$ and $\kappa_2 = (\beta-\alpha)\sin(\tau)$. We note that $\tau \in \left(0, \frac{\pi}{2}\right)$ and $\kappa_1 \cdot \kappa_2 > 0$. We define the points $\mathbf{c}_1 = (-\kappa_1, 0)$, $\mathbf{c}_2 = (0, \kappa_2)$, $\mathbf{c}_3 = (\kappa_1, 0)$, $\mathbf{c}_4 = (0, -\kappa_2)$. Then the boundary of the (α,β)-egg is parametrised by

$$\boldsymbol{\gamma}(t) = \begin{cases} \mathbf{c}_1 + \beta\boldsymbol{\sigma}(t), & t \in (-\tau, \tau) \\ \mathbf{c}_2 + \alpha\boldsymbol{\sigma}(t), & t \in (\tau, \pi - \tau) \\ \mathbf{c}_3 + \beta\boldsymbol{\sigma}(t), & t \in (\pi - \tau, \pi + \tau) \\ \mathbf{c}_4 + \alpha\boldsymbol{\sigma}(t), & t \in (\pi + \tau, \pi - \tau) \end{cases}.$$

Observe that in the case $\alpha = 0$, the (α,β)-egg set reduces to the β-lune defined in [8]. Indeed the arcs of radius 0 corresponds to the corner points.

Remark 2.11. An (α,β)-egg is an example of convex set whose radius of curvature ρ_K is piecewise constant and assumes alternatively the two values α and β. One could consider in general a wider class of planar (α,β)-convex sets that satisfy this property, *i.e.* considering the class of sets whose boundary is a finite union of arcs of circles with radii α and β. When $\alpha > 0$, due to the regularity of the boundary given by Lemma 2.5, one can easily infer that two consecutive arcs cannot have the same radius of curvature and at least four arcs are needed. It follows therefore that the arcs forming the boundary of K have to be even in number. In the case $\alpha = 0$, since the support function is continuous, it follows the boundary consists of the arcs of the circle of radius β adjoining each other at the corner points.

Example 2.12 ((α,β)-regular N-gon)**.** Given $0 \leq \alpha < \beta < \infty$ and $N \in \mathbb{N}$ with $N \geq 3$, we call (α,β)*-regular* N*-gon* the (α,β)-convex planar set K whose boundary ∂K is made up of $2N$ arcs of circles alternating the radii between α and β and such that the length of all the arcs with the same radius is constant. In order to write the parametrized radius of curvature of a general (α,β)-regular N-gon, K, fix $\sigma, \tau > 0$ such that $N(\sigma + \tau) = 2\pi$ and define, for $i \in \{1, 2, \ldots, 2N\}$,

$$t_i := \begin{cases} \frac{i-1}{2}(\sigma + \tau) + \sigma & \text{if } i \text{ is odd} \\ \frac{i}{2}(\sigma + \tau) & \text{if } i \text{ is even} \end{cases}.$$

The parametrized radius of curvature of K can be written as

$$\rho_K(t) = \begin{cases} \beta, & t \in [t_{2i+1}, t_{2i+2}] \\ \alpha, & t \in [t_{2i+2}, t_{2i+3}] \end{cases}.$$

Let $P(K) = L$ be the perimeter of K. By Proposition 2.8 we easily get

$$P(K) = (\beta\sigma + \alpha\tau)N = L \tag{2.4}$$

and therefore

$$\sigma = \frac{L - \alpha 2\pi}{N(\beta - \alpha)}, \quad \tau = \frac{2\pi\beta - L}{N(\beta - \alpha)}. \tag{2.5}$$

Using (2.2), the parametric support function of K can consequently be written as

$$p_K(t) = \begin{cases} C_1^{2i+1}\cos t + C_2^{2i+1}\sin t + \beta\,, & t \in [t_{2i+1}, t_{2i+2}] \\ C_1^{2i+2}\cos t + C_2^{2i+2}\sin t + \alpha\,, & t \in [t_{2i+2}, t_{2i+3}] \end{cases}, \quad i \in \{0, 1, \ldots, N-1\}.$$

Define $\lambda_j := p_K(t_j)$ for $j \in \{0, 1, \ldots, N-1\}$. The continuity of p_k in any t_j ensures us that

$$C_1^{2i+1} = \frac{(\lambda_{2i+2} - \beta)\sin(t_{2i+1}) - (\lambda_{2i+1} - \beta)\sin(t_{2i+2})}{\sin(t_{2i+2} - t_{2i+1})},$$

$$C_2^{2i+1} = \frac{(\lambda_{2i+2} - \beta)\cos(t_{2i+1}) - (\lambda_{2i+1} - \beta)\cos(t_{2i+2})}{\sin(t_{2i+1} - t_{2i+2})},$$

$$C_1^{2i} = \frac{(\lambda_{2i} - \alpha)\sin(t_{2i+1}) - (\lambda_{2i+1} - \alpha)\sin(t_{2i})}{\sin(t_{2i+1} - t_{2i+2})},$$

$$C_2^{2i} = \frac{(\lambda_{2i} - \alpha)\cos(t_{2i+1}) - (\lambda_{2i+1} - \alpha)\cos(t_{2i})}{\sin(t_{2i} - t_{2i+1})}.$$

2.2. Some easy consequences of the Pontryagin principle

We will reformulate our constrained shape optimisation problem as an optimal control problem and we will exploit the optimality conditions given by the Pontryagin principle. The optimal control approach for shape optimisation problems is classical (see for example the monograph [2] for a wide introduction on the subject and [1] for a more contemporary approach) and recently has been fruitfully applied to deal with constrained optimisation problems for convex domains (see [4, 5]). Here we summarise the elementary notions on control theory and we state the version of Pontryagin optimality conditions suited for our purposes, considering indeed only autonomous problems with periodic phase variables valued in $\mathbb{R}^2$.

Let $I = [a, b] \subset \mathbb{R}$ be a given interval, $J \subset \mathbb{R}$ be a compact set. For given maps $f, g \in C^1(\mathbb{R}^3)$ and $\mathbf{h} \in C^1(\mathbb{R}^3, \mathbb{R}^2)$ consider the problem of minimizing the functional

$$F(\mathbf{x}, u) := \int_a^b f\big(\mathbf{x}(t), u(t)\big)\mathrm{d}t$$

among all pairs $\big(\mathbf{x}(t), u(t)\big) \in W^{1,1}_{per}(I, \mathbb{R}^2) \times L^\infty(I, J)$ that satisfy for almost every $t \in I$ the differential constraint

$$\mathbf{x}'(t) = \mathbf{h}\big(\mathbf{x}(t), u(t)\big) \tag{2.6}$$

as well as the integral constraint

$$G(\mathbf{x}, u) := \int_a^b g\big(\mathbf{x}(t), u(t)\big)\mathrm{d}t = C_0 \tag{2.7}$$

for a given constant C_0. The previous constrained extremal problem is a typical example of *optimal control problem*, u is the so called *control variable*, $\mathbf{x}$ takes the name of *phase variable* and any pair $(\mathbf{x}, u)$ that satisfies (2.6) will be called a *controlled process*. A controlled process that minimises (locally in a $C(I)$-neighbourhood of $\mathbf{x}$) the functional $F(\mathbf{x}, u)$ among the controlled processes satisfying (2.7) will be called an *optimal process* for $F(\mathbf{x}, u)$ under (2.6) and (2.7). Following Euler's terminology, integral constraints of the type (2.7) are often named *isoperimetric constraints* and we will follow this convention, motivated by the fact that in the next section we will rephrase a geometrical isoperimetric problem as an optimal control problem and (2.7) will play exactly the role of the constraint on the perimeter. As it is customary, we will use the self-explanatory notations $\nabla_{\mathbf{x}} f$, $\nabla_{\mathbf{x}} g$, $\nabla_{\mathbf{x}} \mathbf{h}$, $\partial_u f$, $\partial_u g$, $\partial_u \mathbf{h}$ and so on, to denote the partial derivatives of f, g, $\mathbf{h}$.

Theorem 2.13 (Pontryagin Principle)**.** *Let $(\mathbf{x}, u) \in W^{1,1}_{per}(I, \mathbb{R}^2) \times L^\infty(I, J)$ be an optimal process $F(\mathbf{x}, u)$ under (2.6) and (2.7). Then there exist $\lambda \geq 0$, $\mu \in \mathbb{R}$ and $\mathbf{p} \in W^{1,1}(I, \mathbb{R}^2)$ not all of them trivial such that, for almost all $t \in I$,*

$$\dot{\mathbf{p}}(t) = \mathbf{p}(t) \cdot \nabla_{\mathbf{x}} \mathbf{h}\big(\mathbf{x}(t), u(t)\big) + \mu \,\nabla_{\mathbf{x}} g\big(\mathbf{x}(t), u(t)\big) - \lambda \,\nabla_{\mathbf{x}} f\big(\mathbf{x}(t), u(t)\big) \tag{2.8}$$

and the optimal control u satisfies, for all $t \in I$, the optimality condition

$$\begin{aligned} &\mathbf{p}(t) \cdot \mathbf{h}\big(\mathbf{x}(t), u(t)\big) + \mu g\big(\mathbf{x}(t), u(t)\big) - \lambda f\big(\mathbf{x}(t), u(t)\big) \\ &\quad = \max_{v \in J} \left\{ \mathbf{p}(t) \cdot \mathbf{h}\big(\mathbf{x}(t), v\big) + \mu g\big(\mathbf{x}(t), v\big) - \lambda f\big(\mathbf{x}(t), v\big) \right\}. \end{aligned} \tag{2.9}$$

The differential system (2.8) takes the name of *adjoint system* and it is nothing but the Euler-Lagrange equation derived as a stationarity condition on the Lagrangian of the optimal problem (*cf.* [2], Sect. 4.2.2).

We will use a couple of consequences of Theorem 2.13 when applied to constrained problems arising in convex geometry. To this aim in the following corollary we specify the Pontryagin's conditions for optimality in one dimensional control problems with a second order differential constraint.

Corollary 2.14. *Given $f \in C^1(\mathbb{R}^2)$, $g \in C^1(\mathbb{R}^2)$ and a constraint C_0, let the pair $(x, u) \in W^{2,1}_{per}(I, \mathbb{R}) \times L^\infty(I, J)$ be a minimizer of the functional*

$$F(x, u) := \int_a^b f\big(x(t), u(t)\big)\mathrm{d}t$$

among all the admissible pairs satisfying the differential constraint

$$x(t) + \ddot{x}(t) = u(t) \tag{2.10}$$

and the integral constraint

$$\int_a^b g\big(x(t), u(t)\big)\mathrm{d}t = C_0. \tag{2.11}$$

Then there exist $\lambda \geq 0$, $\mu \in \mathbb{R}$ and $p \in W^{1,1}(I, \mathbb{R})$ not all of them trivial such that, for almost all $t \in I$, p is a solution of the equation

$$\ddot{p}(t) + p(t) = \mu\, \partial_x g\big(x(t), u(t)\big) - \lambda\, \partial_x f\big(x(t), u(t)\big) \tag{2.12}$$

and the optimal control u satisfies, for all $t \in I$, the optimality condition

$$\begin{aligned} p(t)u(t) &+ \mu g\big(x(t), u(t)\big) - \lambda f\big(x(t), u(t)\big) \\ &= \max_{v \in J} \left\{ p(t)v + \mu g\big(x(t), v\big) - \lambda f\big(x(t), v\big) \right\}. \end{aligned} \tag{2.13}$$

Proof. The proof easily follows by rewriting the differential constraint as a system of first order equations and applying Theorem 2.13 with the phase variable $\mathbf{x}(t) = (x_1(t), x_2(t))$ that satisfies the system

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = u - x_1 \end{cases}.$$

The functionals involved are independent of the auxiliary variable x_2 and Theorem 2.13 provides the existence of the multipliers exist λ, μ and $\mathbf{p} := (p_1, p_2)$ satisfying the adjoint system

$$\begin{cases} \dot{p_1} = p_2 + \mu\, \partial_x g\big(x(t), u(t)\big) - \lambda\, \partial_x f\big(x(t), u(t)\big) \\ \dot{p_2} = -p_1 \end{cases}$$

and the maximality condition

$$\begin{aligned} p_1(t)x_2(t) + p_2(t)\big(u(t) - x_1(t)\big) &+ \mu g\big(x_1(t), u(t)\big) - \lambda f\big(x_1(t), u(t)\big) \\ &= \max_{v \in J} \left\{ p_1(t)x_2(t) + p_2(t)\big(v - x_1(t)\big) + \mu g\big(x_1(t), v\big) - \lambda f\big(x_1(t), v\big) \right\}. \\ &= p_1(t)x_2(t) - p_2(t)x_1(t) + \max_{v \in J} \left\{ p_2(t)v + \mu g\big(x_1(t), v\big) - \lambda f\big(x_1(t), v\big) \right\}. \end{aligned}$$

These equations are easily seen to be equivalent to (2.12) and (2.13) setting $p = p_2$ and $x = x_1$. □

In the special case when the functional F and the isoperimetric constraint are linear in the control variable u, we can further deduce a *bang-bang* type condition for optimal controls. The following corollary easily follows from the previous one from the optimality condition (2.13) (being linear in the v variable).

Corollary 2.15. *Under the same assumptions of Corollary 2.14, if we further assume that $f(x, u) = a(x)\, u$ and $g(x, u) = b(x)\, u$ with $a, b \in C^1(I)$, then we have*

$$u(t) = \begin{cases} \beta & \text{if} \quad p(t) + \mu\, b(x(t)) - \lambda\, a(x(t)) > 0 \\ \alpha & \text{if} \quad p(t) + \mu\, b(x(t)) - \lambda\, a(x(t)) < 0 \end{cases} \tag{2.14}$$

where $\alpha := \min\{t \,:\, t \in J\}$ and $\beta := \max\{t \,:\, t \in J\}$.

Remark 2.16. Let us remark that if the set

$$S := \{t \in I \;:\; p(t) + \mu\, b(x(t)) - \lambda\, a(x(t)) = 0\}$$

has zero Lebesgue measure, then u is almost everywhere determined by (2.14). This is the case for instance if (x, u) is an optimal control with a a non-singular trajectory (*cf.* [23]).

3. Main result

The main results of the paper are stated in the following theorem.

Theorem 3.1. *Let $0 \leq \alpha < \beta < \infty$. For any $K \in \mathcal{K}^2$, planar (α,β)-convex body such the $2\pi\alpha < P(K) < 2\pi\beta$, the following inequality holds true:*

$$A(K) \geq \frac{1}{2}(\beta+\alpha)(P(K)-2\pi\alpha)+\pi\alpha^2-(\beta-\alpha)^2\sin\left(\frac{P(K)-2\pi\alpha}{2(\beta-\alpha)}\right). \tag{3.1}$$

Moreover the equality holds if and only if K is the (α,β)-egg.

Remark 3.2. Let $0 \leq \alpha < \beta < \infty$ and $L \in (2\pi\alpha, 2\pi\beta)$. Then, modulo proper rigid transformations, the (α,β)-egg is the unique minimizer of the area functional among all the (α,β)-convex bodies in the plane with given perimeter equal to L.

From now on in this section we will implicitly assume that α, β and L are fixed in such a way that $0 \leq \alpha < \beta < \infty$ and $L \in (2\pi\alpha, 2\pi\beta)$. The proof of the main theorem will be a consequence of the following lemmata. In the next one we prove the existence of an optimal set. Its convexity is ensured by Lemma 2.9.

Lemma 3.3. *The shape optimisation problem*

$$\min\left\{A(K) : K \in \mathcal{K}^2 \text{ is an } (\alpha,\beta)-\text{convex body with } P(K)=L\right\} \tag{3.2}$$

admits at least a solution.

Proof. The proof follows by the direct methods of Calculus of Variations. Any minimizing sequence K_n is bounded. Indeed, all the competitors are convex sets with perimeter and area equi-bounded, therefore also their diameters are equi-bounded (see for example [12], Lem. 4.1). By Blaschke selection theorem, up to extracting a subsequence, K_n converges to a convex body K_∞ in the Hausdorff metric. Lemma 2.9 ensures that K_∞ is an admissible set and the conclusion follows by the continuity of the perimeter and the area functionals on $(\mathcal{K}^2, d_H)$. □

In the next lemma we derive the optimality conditions for our minimization problem that is, the ODE (3.3). The argument is similar to that one used in [8]. We observe that a delicate point is to prove that the set of points where the support function p_K is equal to γ is finite.

Lemma 3.4. *Let $K \in \mathcal{K}^2$ be a minimizer for problem (3.2). Then up to eventually translate K, there exists a constant $\gamma \in \mathbb{R}$ such that*

$$\rho_K(t) = \begin{cases} \beta, & \text{for} \quad p_K(t) < \gamma \\ \alpha, & \text{for} \quad p_K(t) > \gamma \end{cases}, \quad \text{a.e. } t \in (0,2\pi), \tag{3.3}$$

where with p_K, with a slight abuse of notation, we denoted the parametrised support function of the eventual translation of K and $\rho_K(t)$ is the radius of curvature of ∂K. Moreover the set $S := \{t \in [0,2\pi) \,:\, p_K(t) = \gamma\}$ is finite.

Proof. We start observing that by Propositions 2.7 we can identify a given admissible set $K \in \mathcal{K}^2$ with its parametric support function p_K and by Proposition 2.8 we can rephrase the minimization problem (3.2) as an optimal control problem. If $K \in \mathcal{K}^2$ is a minimizer for problem (3.2), then the pair given by its support function and its parametric radius of curvature, *i.e.* $(x,u) = (p_K, \rho_K) \in W^{2,1}_{per}(I,\mathbb{R}) \times L^\infty(I,J)$ with $I = (0,2\pi)$

and $J = [\alpha, \beta]$, form indeed an optimal control process for the functional

$$F(x,u) := \frac{1}{2}\int_0^{2\pi} u(t)\,x(t)\,\mathrm{d}t$$

under the differential constraint

$$x(t) + \ddot{x}(t) = u(t)$$

and the isoperimetric one

$$\int_0^{2\pi} u(t)\,\mathrm{d}t = L.$$

We can therefore use Corollary 2.15 to deduce that there exist $\lambda \geq 0$, $\mu \in \mathbb{R}$ and $s \in W^{1,1}(I,\mathbb{R})$ not all of them trivial such that

$$\rho_K(t) = \begin{cases} \beta & \text{if} \quad s(t) + \mu - \frac{\lambda}{2}\,p_K(t) > 0 \\ \alpha & \text{if} \quad s(t) + \mu - \frac{\lambda}{2}\,p_K(t) < 0 \end{cases} \tag{3.4}$$

Moreover, by Corollary 2.14, the multiplier $s(t)$ solves the adjoint equation

$$\ddot{s}(t) + s(t) = -\frac{\lambda}{2}\,\rho_K(t),$$

that, together with the differential constraint written for the pair (p_K, ρ_K), implies that the function $\beta := s + \frac{\lambda}{2}p_K$ is a 2π-periodic solution of the ordinary differential equation $y(t) + \ddot{y}(t) = 0$. Therefore there exist constants c_1 and c_2, such that

$$s(t) + \frac{\lambda}{2}p_K(t) = c_1\cos(t) + c_2\sin(t). \tag{3.5}$$

We can therefore rewrite (3.4), as

$$\rho_K(t) = \begin{cases} \beta & \text{if} \quad c_1\cos(t) + c_2\sin(t) + \mu - \lambda\,p_K(t) > 0 \\ \alpha & \text{if} \quad c_1\cos(t) + c_2\sin(t) + \mu - \lambda\,p_K(t) < 0 \end{cases}$$

We claim that $\lambda \neq 0$. If not, first we observe that by non-triviality condition of the Pontryagin principle, μ and $s(t)$ cannot be simultaneously identically zero. If $s \equiv 0$, then from (3.4) we have

$$\rho_K(t) = \begin{cases} \beta & \text{if} \quad \mu > 0 \\ \alpha & \text{if} \quad \mu < 0 \end{cases}.$$

Being $\mu \neq 0$, K is a circle of radius α or β that is not an admissible set. If instead $s \not\equiv 0$, the adjoint equation (3.5) ensures us that $s(t) = c_1\cos(t) + c_2\sin(t) = A\cos(t+\phi)$, with $A \neq 0$ and ϕ constant. The condition (3.4) becomes

$$\rho_K(t) = \begin{cases} \beta & \text{if} \quad A\cos(t+\phi) + \mu > 0 \\ \alpha & \text{if} \quad A\cos(t+\phi) - \mu < 0 \end{cases}.$$

Therefore, since the equation $A\cos(t+\phi)+\mu=0$ admits at most two solutions in the interval $[0,2\pi)$, it follows that ∂K is the union of at most two arcs of circle with radii α and β. This is impossible for an (α,β)-convex set by the regularity Lemma 2.5 (*cf.* Rem. 2.11). This proves the claim.

Since $\lambda>0$, in a translated coordinate system centered in $\left(\frac{c_1}{\lambda},\frac{c_2}{\lambda}\right)$, the parametric support function of K will change in $p_K(t)-\frac{c_1}{\lambda}\cos(t)-\frac{c_2}{\lambda}\sin(t)$. Therefore (3.4) will read

$$\rho_K(t)=\begin{cases}\beta & \text{if} \quad p_K(t)<\frac{\mu}{\lambda}=:\gamma\\ \alpha & \text{if} \quad p_K(t)>\frac{\mu}{\lambda}=:\gamma\end{cases}$$

We now prove that $S:=\{t\in[0,2\pi]\ :\ p_K(t)=\gamma\}$ is a finite set. Let $t_0\in[0,2\pi]\in S^c$, say $p_K(t_0)>\gamma$, and let (a_0,b_0) be the connected component of the set S^c, containing t_0 (more explicitly we can define $a_0=\inf\{\tilde{t}\ :\ p_K(t)>\gamma\ \forall\, t\in(\tilde{t},t_0)\}$ and $b_0=\sup\{\tilde{t}\ :\ p_K(t)>\gamma\ \forall\, t\in(t_0,\tilde{t})\}$). Observe that by continuity of p_K we deduce that

$$p_K(a_0)=p_K(b_0)=\gamma. \tag{3.6}$$

Moreover we can uniquely solve the equation $p_K+p_K''=\alpha$ in (a_0,b_0), and therefore deduce the existence of two constants C_1 and C_2 such that $p_K(t)=C_1\cos t+C_2\sin t+\alpha$ in $[a_0,b_0]$. We claim that b_0 is an isolated point for S. The same argument could be applied for the left endpoint a_0. By contradiction, let $\{t_m\}_{m\in\mathbb{N}}$ with $t_m>b_0$, $p_K(t_m)=\gamma$ and such that $t_m\to b_0^+$. The regularity of p_K ensures that the left and right derivatives of p_K in b_0 agree. We can write (recalling the explicit expression of p_K in (a_0,b_0))

$$-C_1\sin b_0+C_2\cos b_0=p_K'(b_0^-)=p_K'(b_0^+)=\lim_{m\to\infty}\frac{p_K(t_m)-p_K(b_0)}{t_m-b_0}=0.$$

The last equality, together with (3.6), tells us that the couple (C_1,C_2) solves the following linear system

$$\begin{cases}C_1\cos a_0+C_2\sin a_0=\gamma-\alpha\\ C_1\cos b_0+C_2\sin b_0=\gamma-\alpha\\ -C_1\sin b_0+C_2\cos b_0=0\end{cases},$$

that is solvable only if (imposing the determinant of the full matrix to be zero)

$$(\gamma-\alpha)(\cos(a_0-b_0)-1)=0$$

that in turns implies $\gamma=\alpha$ or $b_0-a_0=2\pi$. The last equality means that x represents a full circle of radius α, that is not an admissible competitor for our problem if we choose $L>2\pi\alpha$. It remains to study the case $\gamma=\alpha$, that leads easily to a contradiction by observing that the only solution of the linear system is $(C_1,C_2)=(0,0)$ and therefore $x(t)=\alpha=\gamma$ for any $t\in(a_0,b_0)$ against the definition of S^c and this proves the claim. An analogous argument can be done when $p_K(t_0)<\gamma$, with β in place of α. Since the connected components of S^c have isolated endpoints, they are finite in number. Finally we have proved that S^c is a finite union of disjoint relatively open intervals in $[0,2\pi]$. Therefore its complement S is a finite union of, possibly degenerate, relatively closed intervals in $[0,2\pi]$. With the same argument as above, it is easy to prove that the interior of S is empty. Indeed it is sufficient to argue by contradiction and use the regularity of p_K at the endpoints of the connected components of S with non empty interior. □

From now on, our technique is completely different from that one of [8]. Indeed we exploit equation (3.3) of Lemma 3.4 in an analytical way. In the next lemma we prove that for an optimal set, the arcs of radii α are congruent to each other, as well as the arcs of radii β.

Lemma 3.5. *Any (α, β)-convex body $K \in \mathcal{K}^2$ that satisfies (3.3) is necessarily an (α, β)-regular N-gon.*

Proof. From Lemma 3.4, we infer that ∂K is the union of a finite number of arcs of circles with radii α and β, being the radius of curvature, ρ_K, a piecewise constant function with a finite number of jumps, assuming only two values. Moreover in any jump point $t \in [0, 2\pi)$ of ρ_K, it holds $p_K(t) = \gamma$. By Remark 2.11 we can easily deduce that the arcs are even in number and the radii alternate between the values α and β. In the case $\alpha = 0$, we will have arcs of radius β, joining each other at corner points.

We can therefore assume that ∂K is made of $2N$ disjoint arcs. The parametric support function of K can be written as

$$p_K(t) = \begin{cases} C_1^{2i+1}\cos t + C_2^{2i+1}\sin t + \beta\,, & t \in [t_{2i+1}, t_{2i+2}] \\ C_1^{2i+2}\cos t + C_2^{2i+2}\sin t + \alpha\,, & t \in [t_{2i+2}, t_{2i+3}] \end{cases} \quad, \quad i \in 0, 1, \ldots, N-1,$$

with $\{t_1 < t_2 < t_3 \cdots < t_{2N+1} = t_1 + 2\pi\}$ and (C_1^{2i+1}, C_2^{2i+1}) are the coordinates of centers of the disks of radius β and (C_1^{2i+2}, C_2^{2i+2}) are the coordinates of centers of the disks of radius α. Imposing the continuity in t_i one gets

$$C_1^{2i+1} = (\gamma - \beta)\frac{\sin t_{2i+2} - \sin t_{2i+1}}{\sin(t_{2i+2} - t_{2i+1})}; \quad C_2^{2i+1} = (\gamma - \beta)\frac{\cos t_{2i+2} - \cos t_{2i+1}}{\sin(t_{2i+1} - t_{2i+2})}. \tag{3.7}$$

For $0 \le i \le N-1$, one gets

$$C_1^{2i+2} = (\gamma - \alpha)\frac{\sin t_{2i+3} - \sin t_{2i+2}}{\sin(t_{2i+3} - t_{2i+2})}; \quad C_2^{2i+2} = (\gamma - \alpha)\frac{\cos t_{2i+3} - \cos t_{2i+2}}{\sin(t_{2i+2} - t_{2i+3})}. \tag{3.8}$$

From (3.7) and (3.8) we easily deduce that $\gamma \ne \alpha$ and $\gamma \ne \beta$, otherwise the arcs of the circles of radius α or β contained in ∂K should lie all on the same circle centered at the origin.

The continuity of the derivative of the parametric support function in t_j ensures us that, for $0 \le j \le 2N-1$,

$$C_1^j \sin(t_{j+1}) - C_2^j \cos(t_{j+1}) = C_1^{j+1}\sin(t_{j+1}) - C_2^{j+1}\cos(t_{j+1}) \tag{3.9}$$

Combining the relations (3.7), (3.8) and (3.9), we can write, for $0 \le i \le N-1$,

$$\begin{cases} (\gamma - \beta)\dfrac{1 - \cos(t_{2i+2} - t_{2i+1})}{\sin(t_{2i+2} - t_{2i+1})} = (\gamma - \alpha)\dfrac{-1 + \cos(t_{2i+3} - t_{2i+2})}{\sin(t_{2i+3} - t_{2i+2})} \\[2ex] (\gamma - \alpha)\dfrac{1 - \cos(t_{2i+3} - t_{2i+2})}{\sin(t_{2i+3} - t_{2i+2})} = (\gamma - \beta)\dfrac{-1 + \cos(t_{2i+4} - t_{2i+3})}{\sin(t_{2i+4} - t_{2i+3})} \end{cases}.$$

Since the function $t \to \frac{1-\cos(t)}{\sin(t)}$ is strictly monotone, from the previous system we infer the existence of two positive constants τ and σ such that $t_{2i+2} - t_{2i+1} = \tau$ and $t_{2i+3} - t_{2i+2} = \sigma$ for any $0 \le i \le N-1$. This proves the claim. ∎

Remark 3.6. Let $K \in \mathcal{K}^2$ be an (α, β)-regular N-gon with perimeter $P(K) = L$, as in the Example 2.12. Suppose that up to a translation of K, the values of the parametric support functions in the points t_j are constants, *i.e.* there exists λ, such that $\lambda_j = \lambda$ for any $j \in \{0, 1, \ldots, N-1\}$. As a byproduct of the proof of the previous lemma, we can explicitly calculate the value of $\lambda = p_K(t_j)$. Indeed, simply by imposing the continuity of the parametric support function in t_i and solving the linear system, we get

$$\lambda = \frac{\beta[1-\cos(\tau)]\sin(\sigma) + \alpha[1-\cos(\sigma)]\sin(\tau)}{[1-\cos(\tau)]\sin(\sigma) + [1-\cos(\sigma)]\sin(\tau)}.$$

Lemma 3.7. *Let $K \in \mathcal{K}^2$ be an (α,β)-regular N-gon with $P(K) = L$ that satisfies (3.3) then*

$$A(K) = \frac{\beta+\alpha}{2}(L - 2\pi\alpha) + \pi\alpha^2 + (\beta-\alpha)^2 \frac{N\big(\cos(\frac{\pi}{N}) - \cos(\frac{\pi(\beta+\alpha)-L}{N(\beta-\alpha)})\big)}{2\sin(\frac{\pi}{N})}. \tag{3.10}$$

Moreover the minimum value for the area functional is realized for $N = 2$, i.e. for the (α,β)-egg.

Proof. Let $K \in \mathcal{K}^2$ be an (α,β)-regular N-gon and using the same notation as in Example 2.12 for its parametric support function and radius of curvature, we can compute

$$\begin{aligned}
2A(K) =& \sum_{i=1}^{N} \int_{t_{2i-1}}^{t_{2i}} \beta\big[C_1^{2i-1}\cos(t) + C_2^{2i-1}\sin(t) + \beta\big]\mathrm{d}t \\
&+ \sum_{i=1}^{N} \int_{t_{2i}}^{t_{2i+1}} \alpha\big[C_1^{2i}\cos(t) + C_2^{2i}\sin(t) + \alpha\big]\mathrm{d}t \\
=& \sum_{i=1}^{N} \Big\{ \beta\left[C_1^{2i-1}(\sin(t_{2i}) - \sin(t_{2i-1})) + C_2^{2i-1}(\cos(t_{2i-1}) - \cos(t_{2i}))\right] \\
&+ \alpha\left[C_1^{2i}(\sin(t_{2i+1}) - \sin(t_{2i})) + C_2^{2i}(\cos(t_{2i}) - \cos(t_{2i+1}))\right] \\
&+ \beta^2(t_{2i} - t_{2i-1}) + \alpha^2(t_{2i+1} - t_{2i}) \Big\}.
\end{aligned}$$

Therefore, replacing the expressions of C_i^j derived in (3.7) and (3.8) in the previous formula and recalling that $t_{2i} - t_{2i+1} = \tau$ and $t_{2i+1} - t_{2i} = \sigma$, we infer

$$A(K) = \beta(\lambda-\beta)N\,\frac{\big[1-\cos(\tau)\big]}{\sin(\tau)} + \alpha(\lambda-\alpha)N\,\frac{\big[1-\cos(\sigma)\big]}{\sin(\sigma)} + \frac{N}{2}(\beta^2\tau + \alpha^2\sigma).$$

Indeed, for the first term we have

$$C_1^{2i-1}(\sin(t_{2i}) - \sin(t_{2i-1})) + C_2^{2i-1}(\cos(t_{2i-1}) - \cos(t_{2i}))$$

$$= (\lambda-\beta)\frac{\sin(t_{2i}) - \sin(t_{2i-1})}{\sin(t_{2i} - t_{2i-1})}[\sin(t_{2i}) - \sin(t_{2i-1})] + (\lambda-\beta)\frac{\cos(t_{2i}) - \cos(t_{2i-1})}{\sin(t_{2i} - t_{2i-1})}[\cos(t_{2i}) - \cos(t_{2i-1})]$$

$$= (\lambda-\beta)\frac{2 - 2\sin(t_{2i})\sin(t_{2i-1}) - 2\cos(t_{2i})\cos(t_{2i-1})}{\sin\tau} = (\lambda-\beta)\frac{2 - 2\cos(t_{2i-1} - t_{2i})}{\sin\tau} = 2(\lambda-\beta)\frac{1-\cos\tau}{\sin\tau}\,.$$

For the second term we have

$$C_1^{2i}(\sin(t_{2i+1})-\sin(t_{2i}))+C_2^{2i}(\cos(t_{2i})-\cos(t_{2i+1}))$$

$$=(\lambda-\alpha)\frac{\sin(t_{2i+1})-\sin(t_{2i})}{\sin(t_{2i+1}-t_{2i})}[\sin(t_{2i+1})-\sin(t_{2i})]+(\lambda-\alpha)\frac{\cos(t_{2i+1})-\cos(t_{2i})}{\sin(t_{2i+1}-t_{2i})}[\cos(t_{2i+1})-\cos(t_{2i})]$$

$$=(\lambda-\alpha)\frac{2-2\sin(t_{2i+1})\sin(t_{2i})-2\cos(t_{2i+1})\cos(t_{2i})}{\sin\tau}=(\lambda-\alpha)\frac{2-2\cos(t_{2i+1}-t_{2i})}{\sin\sigma}=2(\lambda-\alpha)\frac{1-\cos\sigma}{\sin\sigma}\,.$$

Finally, using the value of λ given by Remark 3.6, we get

$$\lambda-\beta=(\alpha-\beta)\frac{1-\cos(\sigma)}{\sin(\sigma)+\sin(\tau)-\sin(\sigma+\tau)}\sin(\tau)$$

and

$$\lambda-\alpha=(\beta-\alpha)\frac{1-\cos(\tau)}{\sin(\sigma)+\sin(\tau)-\sin(\sigma+\tau)}\sin(\sigma).$$

Therefore we can write

$$A(K)=\frac{N}{2}(\beta^2\tau+\alpha^2\sigma)-\frac{N(\beta-\alpha)^2\left[1-\cos(\sigma)\right]\left[1-\cos(\tau)\right]}{\sin(\sigma)+\sin(\tau)-\sin(\sigma+\tau)}. \tag{3.11}$$

Using the elementary relations

$$\sin(a)+\sin(b)+\sin(c)-\sin(a+b+c)=4\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{b+c}{2}\right)\sin\left(\frac{a+c}{2}\right)$$

and

$$\big(1-\cos(a)\big)\big(1-\cos(b)\big)=4\sin^2(a/2)\sin^2(b/2)$$

from (3.11) we get

$$A(K)=\frac{N}{2}(\beta^2\tau+\alpha^2\sigma)-N(\beta-\alpha)^2\frac{\sin\left(\frac{\sigma}{2}\right)\sin\left(\frac{\tau}{2}\right)}{\sin\left(\frac{\sigma+\tau}{2}\right)}.$$

To make explicit the dependence on N in the expression of the area we introduce the auxiliary variable (*cf.* (2.5))

$$\omega:=N\frac{\sigma}{2}=\frac{L-2\pi\alpha}{2(\beta-\alpha)}.$$

Recalling that $N(\sigma+\tau)=2\pi$ we can finally write the area of the (α,β)-regular N-gon K as

$$A(K)=\beta^2\pi-(\beta^2+\alpha^2)\frac{\omega}{2}-(\beta-\alpha)^2\Phi(N,\omega),$$

where we have set

$$\Phi(N,\omega):=N\frac{\sin\left(\frac{\omega}{N}\right)\sin\left(\frac{\pi}{N}-\frac{\omega}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}$$

And this proves (3.10).

To prove that the minimum value of the area is attained when $N=2$, which corresponds to the area of the (α,β)-egg, it is sufficient to prove that

$$\Phi(N,\omega)\le\Phi(2,\omega)=\sin(\omega).$$

To this aim, we observe that $\omega\in[0,\pi]$ and we show that the function

$$f_N(x):=\sin(x)\sin\left(\frac{\pi}{N}\right)-N\sin\left(\frac{\pi-x}{N}\right)\sin\left(\frac{x}{N}\right)$$

is positive for $x\in[0,\pi]$ and $N\ge 2$. We observe that $f_N(0)=f_N(\pi)=0$ and that $f_N(x)$ is symmetric with respect to $x_s=\frac{\pi}{2}$. We claim that f_N is increasing in $[0,\frac{\pi}{2}]$. This implies that f_N is positive on $[0,\pi]$ and therefore that the minimum value of the area is attained when $N=2$. For that, we first observe that the function

$$h_N(x):=\frac{\sin(\frac{\pi-2x}{N})}{\sin(\frac{\pi-2x}{N+1})}$$

is increasing on $(0,\frac{\pi}{2})$. Indeed its derivative

$$h_N'(x)=-2\frac{\sin(\frac{\pi-2x}{N})}{N(N+1)\sin(\frac{\pi-2x}{N+1})}\left[(N+1)\cot\left(\frac{\pi-2x}{N}\right)-N\cot\left(\frac{\pi-2x}{N+1}\right)\right]$$

satisfies $h_N'(x)>0$ on $(0,\frac{\pi}{2})$, since $x\mapsto x\cot x$ is decreasing on $\left(0,\frac{\pi}{2}\right)$ and therefore

$$(N+1)\cot\left(\frac{\pi-2x}{N}\right)-N\cot\left(\frac{\pi-2x}{N+1}\right)<0.$$

The monotonicity of h_N implies that for $N\ge 2$

$$\frac{\sin(\frac{\pi}{N+1})}{\sin(\frac{\pi-2x}{N+1})}\ge\frac{\sin(\frac{\pi}{N})}{\sin(\frac{\pi-2x}{N})}\,.$$

In other words, for any fixed $x\in\left(0,\frac{\pi}{2}\right)$, the sequence $\{a_N\}_{N=2}^{\infty}$ defined by

$$a_N:=\frac{\sin(\frac{\pi}{N})}{\sin(\frac{\pi-2x}{N})}$$

is increasing. Therefore, for $N \geq 2$, it holds $a_N \geq a_2$, that reads as

$$\frac{\sin(\frac{\pi}{N})}{\sin(\frac{\pi-2x}{N})} \geq \frac{\sin(\frac{\pi}{2})}{\sin(\frac{\pi-2x}{2})} = \frac{1}{\cos(x)}.$$

The last inequality is equivalent to say that

$$f_N'(x) = \cos x \sin\left(\frac{\pi}{N}\right) - \sin\left(\frac{\pi-2x}{N}\right) > 0\ , \quad x \in \left(0, \frac{\pi}{2}\right),$$

proving the claim. □

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