CONTROLLABILITY PROBLEMS FOR THE HEAT EQUATION
WITH VARIABLE COEFFICIENTS ON A HALF-AXIS

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Abstract. In the paper, the problems of controllability and approximate controllability are studied for the heat equation $w_t = \frac{1}{\rho}(kw_x)_x + \gamma w$, $x > 0$, $t \in (0,T)$, controlled by the Dirichlet boundary condition. Control is considered in $L^\infty(0,T)$. It is proved that each initial state of this system is approximately controllable to any its end state in a given time $T > 0$.

1. INTRODUCTION

Controllability problems for the heat equation with constant and variable coefficients were studied in a number of papers (see, *e.g.*, [1–4, 9–12, 14, 18–22, 24, 25]). However, there are much fewer papers where these problems were investigated for *the heat equation with **constant** coefficients in unbounded domains* (see, *e.g.*, [2, 9, 10, 19, 20, 23]), and it seems there are no papers where these problems were investigated for *the heat equation with **variable** coefficients in unbounded domains*.

In the paper, the controllability problems for the heat equation with variable coefficients on a half-axis controlled by the Dirichlet boundary condition are studied.

Let $T > 0$ be a constant, ρ, k, γ, and w^0 be functions of $x \in [0, +\infty)$, and u be a function of $t \in [0, T]$. We consider the following heat equation

$$w_t = \frac{1}{\rho}(kw_x)_x + \gamma w, \qquad\qquad x \in (0, +\infty), \ t \in (0,T), \qquad\qquad (1.1)$$

controlled by the Dirichlet boundary condition

$$w(0, \cdot) = u, \qquad\qquad t \in (0,T), \qquad\qquad (1.2)$$

under the initial condition

$$w(\cdot, 0) = w^0, \qquad\qquad x \in (0, +\infty). \qquad\qquad (1.3)$$

Here ρ, k, γ, and w^0 are given functions, $u \in L^\infty(0,T)$ is a control, and $w : [0,+\infty) \times [0,T] \to \mathbb{R}$ is the state of the system. We assume $\rho, k \in C^1[0,+\infty)$ are positive on $[0,+\infty)$, $(\rho k) \in C^2[0,+\infty)$, $(\rho k)'(0) = 0$. Consider the even extensions of ρ, k, γ. Throughout the paper we will denote these extensions by the same symbols ρ, k, γ, respectively. In addition, for

$$\sigma(x) = \int_0^x \sqrt{\rho(|\xi|)/k(|\xi|)}\, \mathrm{d}\xi, \quad x \in \mathbb{R}, \tag{1.4}$$

we assume

$$\sigma(x) \to +\infty \quad \text{as } x \to +\infty. \tag{1.5}$$

Moreover, we assume

$$Q_2(\rho, k) - \gamma \in L^\infty(0,+\infty) \bigcap C^1[0,+\infty) \tag{1.6}$$

and

$$\sqrt{\frac{\rho}{k}}\, (Q_2(\rho, k) - \gamma)\, \sigma \in L^1(0,+\infty), \tag{1.7}$$

where $Q_2(\rho, k) = \sqrt{k/\rho}\,(Q_1(\rho, k))' + (Q_1(\rho, k))^2$, $(Q_1(\rho, k)) = \sqrt{k/\rho}(k\rho)'/(4k\rho)$.

To formulate the main results of the paper, we need some notations.

Let $\Omega = (0,+\infty)$ or $\Omega = \mathbb{R}$. Let $\mathcal{D}(\Omega)$ be the space of finite infinitely differentiable functions whose support is finite and is contained in Ω. For $\varphi \in L^2_{\mathrm{loc}}(\Omega)$ we consider $\varphi' \in \mathcal{D}'(\Omega)$. If $\varphi \in L^2_{\mathrm{loc}}(\Omega)$, we define the derivative $\mathbb{D}_{\rho k}$ by the rule

$$\mathbb{D}_{\rho k}\varphi = \sqrt{\frac{k}{\rho}}\varphi' + Q_1(\rho, k)\varphi.$$

If, in addition, $\mathbb{D}_{\rho k}\varphi \in L^2_{\mathrm{loc}}(\Omega)$ and $(\mathbb{D}_{\rho k}\varphi)' \in L^2_{\mathrm{loc}}(\Omega)$ (the derivative $(\cdot)'$ is considered in $\mathcal{D}'(\Omega)$), we can consider $\mathbb{D}^2_{\rho k}\varphi$. Then $\varphi'' \in \mathcal{D}'(\Omega)$ and

$$\mathbb{D}^2_{\rho k}\varphi = \frac{1}{\rho}\big(k\varphi'\big)' + Q_2(\rho, k)\varphi.$$

Obviously, $\mathbb{D}^m_{\rho k}\varphi = \varphi^{(m)}$ if $\rho = k = 1$, $m = 0, 1, 2$. Here and further we use the notation $s = \overline{k,m}$ for $s \in \mathbb{Z}$ such that $k \le s \le m$ if $k \le m$ and for $s \in \emptyset$ if $k > m$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}$.

Let

$$L^2_\rho(\Omega) = \{f \in L^2_{\mathrm{loc}}(\Omega) \mid \sqrt{\rho}f \in L^2(\Omega)\}$$

with the norm

$$\|f\|_{L^2_\rho(\Omega)} = \|\sqrt{\rho}f\|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2 \rho(x)\, \mathrm{d}x\right)^{1/2}, \quad f \in L^2_\rho(\Omega).$$

For $p = 0, 1, 2$, consider the space

$$\mathring{\mathbb{H}}^p = \{\varphi \in L^2_{\mathrm{loc}}(0, +\infty) \mid (\forall m = \overline{0, p} \ \mathbb{D}^m_{\rho k}\varphi \in L^2_\rho(0, +\infty)) \text{ and } (\forall m = \overline{0, p-1} \ (\mathbb{D}^m_{\rho k}\varphi)(0^+) = 0)\}$$

with the norm

$$[\![\varphi]\!]^{p\circ} = \left(\sum_{m=0}^{p} \binom{p}{m} \left(\|\mathbb{D}^m_{\rho k}\varphi\|_{L^2_\rho(\Omega)} \right)^2 \right)^{1/2}, \quad \varphi \in \mathring{\mathbb{H}}^p,$$

and the dual space $\mathring{\mathbb{H}}^{-p} = \left(\mathring{\mathbb{H}}^p \right)^*$ with the norm associated with the strong topology of this space. Evidently, $\mathring{\mathbb{H}}^0 = \mathring{\mathbb{H}}^{-0} = L^2_\rho(0, +\infty)$. We have

$$\langle\!\langle \mathbb{D}_{\rho k} f, \varphi \rangle\!\rangle^\circ = - \langle\!\langle f, \mathbb{D}_{\rho k}\varphi \rangle\!\rangle^\circ, \quad f \in \mathring{\mathbb{H}}^{-m}, \ \varphi \in \mathring{\mathbb{H}}^{m+1}, \ m = 0, 1,$$

where $\langle\!\langle g, \psi \rangle\!\rangle^\circ$ denotes the value of a distribution $g \in \mathring{\mathbb{H}}^{-p}$ on a test function $\psi \in \mathring{\mathbb{H}}^p$, $p = 0, 1, 2$. In particular, we have

$$\langle\!\langle g, \psi \rangle\!\rangle^\circ = \langle g, \psi \rangle_{L^2_\rho(0,+\infty)} = \int_0^\infty g(x)\psi(x)\rho(x)\,\mathrm{d}x, \quad g \in \mathring{\mathbb{H}}^0, \ \psi \in \mathring{\mathbb{H}}^0.$$

Thus, we can rewrite equation (1.1) in the form

$$w_t = \mathbb{D}^2_{\rho k}w - qw, \quad t \in (0, T), \tag{1.8}$$

where $\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^p w : [0, T] \to \mathring{\mathbb{H}}^{-2p}$, $p = 0, 1$; $w^0 \in \mathring{\mathbb{H}}^0$;

$$q = Q_2(\rho, k) - \gamma. \tag{1.9}$$

Due to (1.6), $q \in L^\infty(0, +\infty) \cap C^1[0, +\infty)$. Note that q is defined on $\mathbb{R}$ and $q \in C^1(-\infty, 0] \cup C^1[0, +\infty)$, but q' may have a jump at $x = 0$.

A state $w^0 \in \mathring{\mathbb{H}}^0$ is said to be *controllable to a state* $w^T \in \mathring{\mathbb{H}}^0$ *in a given time* $T > 0$ if there exists a control $u \in L^\infty(0, T)$ such that there is a unique solution w to (1.1)–(1.3), and $w(\cdot, T) = w^T$.

A state $w^0 \in \mathring{\mathbb{H}}^0$ is said to be *approximately controllable to a state* $w^T \in \mathring{\mathbb{H}}^0$ *in a given time* $T > 0$ if for each $\varepsilon > 0$ there exists a control $u_\varepsilon \in L^\infty(0, T)$ such that there is a unique solution w_ε to (1.1)–(1.3) with $u = u_\varepsilon$ and $[\![w_\varepsilon(\cdot, T) - w^T]\!]^{0\circ} \leq \varepsilon$.

Eventually, we can formulate the main results of the paper.

Theorem 1.1. *If a state* $w^0 \in \mathring{\mathbb{H}}^0$ *is controllable to the state* $w^T = 0$ *in a time* $T > 0$, *then* $w^0 = 0$.

Theorem 1.2. *Each state* $w^0 \in \mathring{\mathbb{H}}^0$ *is approximately controllable to any state* $w^T \in \mathring{\mathbb{H}}^0$ *in a given time* $T > 0$.

In Section 2, these results are reformulated in Theorems 2.7 and 2.8 in terms more convenient for our investigation.

In the case of constant coefficients ($\rho = k = 1, \gamma = 0$), the result of Theorem 1.1 (and 2.7) has been obtained earlier in [9]. This result is similar to that of the paper [19].

In the case of constant coefficients ($\rho = k = 1, \gamma = 0$), the result of this Theorem 1.2 (and 2.8) has been also obtained earlier in [9].

To study control system (1.1)–(1.3), we use the transformation operator $\widetilde{\mathbb{T}}$ and the modified Sobolev spaces $\widetilde{\mathbb{H}}^s$ where standard derivatives $(\mathrm{d}/\mathrm{d}x)^m$ are replaced by $\mathbb{D}_{\rho k}^m$, $m = \overline{0, |s|}$, $s = \overline{-2, 2}$. The operator $\widetilde{\mathbb{T}} : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$ together with the spaces $\widetilde{\mathbb{H}}^s$, $s = \overline{-2, 2}$, associated with the equation data (ρ, k, γ) have been introduced and studied in [5–8]. The definitions of $\widetilde{\mathbb{T}}$, $\widetilde{\mathbb{H}}^s$, and $\widetilde{H}^s$ are recalled below in Section 2.

The operator $\widetilde{\mathbb{T}}$ is a continuous one-to-one mapping between the spaces $\widetilde{H}^s$ and $\widetilde{\mathbb{H}}^s$. Moreover, it is one-to-one mapping between the set of the solutions to (1.1)–(1.3) with constant coefficients ($\rho = k = 1$, $\gamma = 0$) where $u = u^{110} \in L^\infty(0, T)$ and the set of the solutions to this problem with variable coefficients ρ, k, γ where $u = u^{\rho k \gamma} \in L^\infty(0, T)$ (see Thms. 3.3 and 3.6). The proofs of the main results of the paper are based on the application of Theorems 3.3 and 3.6 proved in Section 3. The control system with variable coefficients ρ, k, γ replicates the controllability properties of the control system with constant coefficients ($\rho = k = 1$, $\gamma = 0$) and *vice versa.*

The last result also holds true for the wave equation on a half-axis [5–8]. However, the proofs are essentially different because of entirely different nature of the heat and wave equations. Applying the operator $\widetilde{\mathbb{T}}^{-1}$ to a solution to the equation with variable coefficients ρ, k, γ and a control $u \in L^\infty(0, T)$, we obtain a solution to the equation with the constant coefficients $\rho = k = 1$, $\gamma = 0$ and a control $v \in L^\infty(0, T)$ different from the control u. To find and to estimate the control v we have to solve an integral equation of the form

$$v(t) = f(t) + \int_0^t P(t - \xi) v(\xi) \, \mathrm{d}\xi, \quad t \in [0, T]. \tag{1.10}$$

In the case of the wave equation, it was proved that f and P are bounded on $[0, T]$ [5–8]. Therefore, the integral operator in the right-hand side of (1.10) is of the Hilbert–Schmidt type. Hence, the Fredholm alternative together with the generalised Gronwall theorem can be applied to solve (1.10) in $L^2(0, T)$ and estimate the solution v in $L^\infty(0, T)$ when we deal with the wave equation [5–8]. However, in the case of the heat equation, it is proved that f and $\sqrt{(\cdot)} P$ are bounded on $[0, T]$ (hence, $P(\xi) = O(1/\sqrt{\xi})$ as $\xi \to 0^+$). Here $\sqrt{(\cdot)}$ is the symbol of the function $\xi \mapsto \sqrt{\xi}$. That is why the integral operator in the right-hand side of (1.10) is not of the Hilbert–Schmidt type, and the Fredholm alternative is not applicable in the general case. Trying to use the Banach fixed-point theorem, we see that it is applicable only for a small interval, and it is not applicable for an arbitrary interval. That is why we use the method of successive approximations to construct a solution to (1.10) on $[0, T]$. Then we split the interval $[0, T]$ into appropriately small intervals and apply the Banach fixed-point theorem in L^2-space on each interval to prove the uniqueness of the solution (see Lem. 3.5).

Since the control system with variable coefficients ρ, k, γ replicates the controllability properties of the control system with constant coefficients ($\rho = k = 1$, $\gamma = 0$), we obtain the controllability properties of the first control system from the controllability properties of the second one by applying the operator $\widetilde{\mathbb{T}}$, *i.e.* we obtain Theorems 2.7 and 2.8 by applying Theorems 3.3 and 3.6, in Section 2.

The obtained results are illustrated by examples in Section 4.

2. Notations and main results

Let us give definitions of the spaces used in the paper.

For $p = 0, 1, 2$, consider the Sobolev spaces

$$H^p = \left\{ \varphi \in L^2_{\mathrm{loc}}(\mathbb{R}) \mid \forall m = \overline{0, p} \ \varphi^{(m)} \in L^2(\mathbb{R}) \right\}$$

with the norm

$$\|\varphi\|^p = \left(\sum_{m=0}^p \binom{p}{m} \left(\left\| \varphi^{(m)} \right\|_{L^2(\mathbb{R})} \right)^2 \right)^{1/2}, \quad \varphi \in H^p,$$

and $H^{-p} = (H^p)^*$ with the norm associated with the strong topology of this space. We have $H^0 = L^2(\mathbb{R}) = (H^0)^* = H^{-0}$. $\langle f, \varphi \rangle$ denotes the value of a distribution $f \in H^{-p}$ on a test function $\varphi \in H^p$, $p = 0, 1, 2$.

Let $\widetilde{H}^n$ be the subspace of all odd distributions in H^n, $n = \overline{-2, 2}$. One can see that $\widetilde{H}^n$ is a closed subspace of H^n, $n = \overline{-2, 2}$.

In [6–8], the following modified Sobolev spaces have been introduced and studied. For $p = 0, 1, 2$, consider

$$\mathbb{H}^p = \left\{ \varphi \in L^2_\rho(\mathbb{R}) \mid \forall m = \overline{0, p} \ \mathbb{D}^m_{\rho k} \varphi \in L^2_\rho(\mathbb{R}) \right\}$$

with the norm

$$[\![\varphi]\!]^p = \left(\sum_{m=0}^p \binom{p}{m} \left(\left\| \mathbb{D}^m_{\rho k} \varphi \right\|_{L^2_\rho(\mathbb{R})} \right)^2 \right)^{1/2}, \quad \varphi \in \mathbb{H}^p,$$

and the dual space $\mathbb{H}^{-p} = (\mathbb{H}^p)^*$ with the norm associated with the strong topology of this space. $\langle\!\langle f, \varphi \rangle\!\rangle$ denotes the value of a distribution $f \in \mathbb{H}^{-p}$ on a test function $\varphi \in \mathbb{H}^p$, $p = 0, 1, 2$. Evidently, $\mathbb{H}^0 = \mathbb{H}^{-0} = (\mathbb{H}^0)^* = L^2_\rho(\mathbb{R})$ and

$$\langle\!\langle f, \varphi \rangle\!\rangle = \langle f, \varphi \rangle_{L^2_\rho(\mathbb{R})} = \int_{-\infty}^{\infty} f(x)\varphi(x)\rho(x)\,\mathrm{d}x, \quad f \in \mathbb{H}^0, \ \varphi \in \mathbb{H}^0.$$

Put

$$\langle\!\langle \mathbb{D}_{\rho k} f, \varphi \rangle\!\rangle = - \langle\!\langle f, \mathbb{D}_{\rho k} \varphi \rangle\!\rangle, \quad f \in \mathbb{H}^{-p}, \ \varphi \in \mathbb{H}^{p+1}, \ p = 0, 1.$$

For $\rho = k = 1$, we have $\mathbb{H}^m = H^m$, $m = \overline{-2, 2}$. In [6], it has been proved that $\mathbb{H}^m \subset \mathbb{H}^n$ is dense continuous embedding, $-2 \leq n \leq m \leq 2$, and $\mathcal{D} \subset \mathbb{H}^p \subset \mathbb{H}^{-p} \subset \mathcal{D}'$ are dense continuous embeddings, $p = 0, 1, 2$, where $\mathcal{D} = \mathcal{D}(\mathbb{R})$. However, the relation between the Schwartz space $\mathcal{S}$ and $\mathbb{H}^p$ essentially depends on ρ and k. For example, if $\rho = k$ then

$$\varphi \in \mathbb{H}^p \Leftrightarrow \sqrt{\rho}\varphi \in H^p, \quad p = \overline{-2, 2}.$$

If $\rho(x) = k(x) = \cosh x$, $x \in \mathbb{R}$, then

$$\mathcal{S} \not\subset \mathbb{H}^p \quad \text{and} \quad \mathbb{H}^{-p} \not\subset \mathcal{S}', \quad p = 0, 1, 2.$$

If $\rho(x) = k(x) = 1/\cosh x$, $x \in \mathbb{R}$, then

$$\mathcal{S} \subset \mathbb{H}^p \quad \text{and} \quad \mathbb{H}^{-p} \subset \mathcal{S}', \quad p = 0, 1, 2.$$

Let $\widetilde{\mathbb{H}}^n$ be the subspace of all odd distributions in $\mathbb{H}^n$, $n = \overline{-2, 2}$.

Remark 2.1. The restriction of a function from $\widetilde{\mathbb{H}}^p$ to $[0, +\infty)$ belongs to $\overset{\circ}{\mathbb{H}}{}^p$ for $p = 0, 1$. However, there exist functions from the space $\widetilde{\mathbb{H}}^2$ whose restrictions do not belong to the space $\overset{\circ}{\mathbb{H}}{}^2$ because each function from $\overset{\circ}{\mathbb{H}}{}^2$ and its derivative are equal to 0 at $x = 0$, but the derivative of an odd function may not equal 0 at $x = 0$. But the odd extension of a function from $\overset{\circ}{\mathbb{H}}{}^p$ belongs to $\widetilde{\mathbb{H}}^p$, $p = 0, 1, 2$ (see [6]).

The transformation operator $\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$ is used to investigate system (1.1)–(1.3). The operators $\mathbf{S}$ and $\widetilde{\mathbf{T}}_r$ have been introduced and studied in [6, 7].

Theorem 2.2 (see [6, 7]). *The following assertions hold.*

(i)   *The operator $\widetilde{\mathbb{T}}$ is an isomorphism of $\widetilde{H}^m$ and $\widetilde{\mathbb{H}}^m$, $m = -2, -1, 0$.*
(ii) $\widetilde{\mathbb{T}}\delta' = \sqrt[4]{(\rho k)(0)}\mathbb{D}_{\rho k}\delta$.
(iii)   *If $g \in \widetilde{H}^0$ and $g(0^+) \in \mathbb{R}$, then $\left(\widetilde{\mathbb{T}}g\right)(0^+) \in \mathbb{R}$ and*

$$\left(\mathbb{D}_{\rho k}^2 - q\right)\widetilde{\mathbb{T}}g - 2\sqrt{(\rho k)(0)}\left(\widetilde{\mathbb{T}}g\right)(0^+)\mathbb{D}_{\rho k}\delta = \widetilde{\mathbb{T}}\left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}g - 2g(0^+)\delta'\right).$$

(iv)   *If $f \in \widetilde{\mathbb{H}}^0$ and $f(0^+) \in \mathbb{R}$, then $\left(\widetilde{\mathbb{T}}^{-1}f\right)(0^+) \in \mathbb{R}$ and*

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\widetilde{\mathbb{T}}^{-1}f - 2\left(\widetilde{\mathbb{T}}^{-1}f\right)(0^+)\delta' = \widetilde{\mathbb{T}}^{-1}\left(\left(\mathbb{D}_{\rho k}^2 - q\right)f - 2\sqrt{(\rho k)(0)}f(0^+)\mathbb{D}_{\rho k}\delta\right).$$

Here δ is the Dirac distribution.

2.1. Controllability results

Consider control system (1.1)–(1.3). From Remark 2.1, it follows that there exist distributions from $\overset{\circ}{\mathbb{H}}^{-2}$ which cannot be extended to the space $\widetilde{\mathbb{H}}^{-2}$. However, the derivative $\mathbb{D}_{\rho k}^2 f_+ \in \overset{\circ}{\mathbb{H}}^{-2}$ of a function $f_+ \in \overset{\circ}{\mathbb{H}}^0$ can be extended to the space $\widetilde{\mathbb{H}}^{-2}$ according to the following theorem.

Theorem 2.3 ([7], Thm. 3.10). *Let $f_+ \in \overset{\circ}{\mathbb{H}}^0$, $\varphi \in \widetilde{\mathbb{H}}^2$ and f be the odd extension of f_+. If $f(0^+) \in \mathbb{R}$, then the distribution $\mathbb{D}_{\rho k}^2 f_+ \in \overset{\circ}{\mathbb{H}}^{-2}$ can be extended to the odd distribution $F \in \widetilde{\mathbb{H}}^{-2}$ and*

$$\langle\langle F, \varphi\rangle\rangle = \left\langle\left\langle \mathbb{D}_{\rho k}^2 f, \varphi\right\rangle\right\rangle + 2\sqrt{(\rho k)(0)}f(0^+)\left(\mathbb{D}_{\rho k}\varphi\right)(0).$$

Let us explain this theorem by a simple example. Let $f_+ \in C^2[0, +\infty)$, and let f be its odd extension. We have $f(x) = f_+(x)H(x) - f_+(-x)H(-x)$, $x \in \mathbb{R}$, where H is the Heaviside function ($H(x) = 1$ if $x \in [0, +\infty)$ and $H(x) = 0$ otherwise). The function f has a jump at $x = 0$. Then $f''(x) = F(x) + 2f_+(0^+)\delta'(x)$, where $F(x) = f_+''(x)H(x) - f_+''(-x)H(-x)$, $x \in \mathbb{R}$. For any $\varphi \in \widetilde{H}^2$, we have

$$\langle F, \varphi\rangle_{L^2(\mathbb{R})} = \langle f'', \varphi\rangle_{L^2(\mathbb{R})} + 2f_+(0^+)\varphi'(0).$$

This formula is generalised in Theorem 2.3.

Let $w(\cdot, t), w^0 \in \overset{\circ}{\mathbb{H}}^0$ and let $W(\cdot, t), W^0$ be their odd extensions with respect to x, respectively, $t \in [0, T]$. If w is a solution to control system (1.1)–(1.3), then taking into account Theorem 2.3 and (1.8), we see that W is a solution to the following system

$$W_t = \mathbb{D}_{\rho k}^2 W - qW - 2\sqrt{(\rho k)(0)}u\mathbb{D}_{\rho k}\delta, \qquad\qquad \text{on } \mathbb{R} \times (0, T), \qquad\qquad (2.1)$$

$$W(\cdot, 0) = W^0, \qquad\qquad \text{on } \mathbb{R}, \qquad\qquad (2.2)$$

where q is defined by (1.9), $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^p W : [0, T] \to \widetilde{\mathbb{H}}^{-2p}$, $p = 0, 1$, $W^0 \in \widetilde{\mathbb{H}}^0$, δ is the Dirac distribution with respect to x. Consider also the steering condition

$$W(\cdot, T) = W^T, \qquad \text{on } \mathbb{R}, \qquad\qquad (2.3)$$

where $W^T \in \widetilde{\mathbb{H}}^0$.

Let $W(\cdot, t), W^0 \in \widetilde{\mathbb{H}}^0$ and let $w(\cdot, t), w^0$ be their restrictions to $(0, +\infty)$ with respect to x, respectively, $t \in [0, T]$. If W is a solution to control system (2.1), (2.2), then due to Corollary 3.4 (see below Section 3),

$$w(0, \cdot) = W(0^+, \cdot) = u \qquad \text{a.e. on } (0, T) \tag{2.4}$$

and w is a solution to control system (1.1)–(1.3).

Thus control systems (1.1)–(1.3) and (2.1), (2.2) are equivalent. Therefore, we will further consider control system (2.1), (2.2) instead of original system (1.1)–(1.3).

Consider also the control system with the simplest heat operator (system (2.1)–(2.3) with $\rho = k = 1$, $\gamma = 0$)

$$Z_t = Z_{\xi\xi} - 2u\delta', \qquad\qquad \text{on } \mathbb{R} \times (0, T), \tag{2.5}$$

$$Z(\cdot, 0) = Z^0, \qquad\qquad \text{on } \mathbb{R}, \tag{2.6}$$

$$Z(\cdot, T) = Z^T, \qquad\qquad \text{on } \mathbb{R}, \tag{2.7}$$

where $u \in L^\infty(0, T)$ is a control, $\left(\frac{d}{dt}\right)^p Z : [0, T] \to \widetilde{H}^{-2p}$, $p = 0, 1$, $Z^0, Z^T \in \widetilde{H}^0$, δ is the Dirac distribution with respect to ξ.

This system was investigated in [9]. In particular, it has been proved therein that

$$Z(0^+, \cdot) = u, \qquad \text{a.e. on } (0, T). \tag{2.8}$$

Let $T > 0$ and $W^0 \in \widetilde{\mathbb{H}}^0$. Let $\widetilde{\mathcal{R}}_T^{\rho k \gamma}(W^0)$ be the set of all states $W^T \in \widetilde{\mathbb{H}}^0$ for which there exists a control $u^{\rho k \gamma} \in L^\infty(0, T)$ such that there exists a unique solution W to (2.1)–(2.3) with $u = u^{\rho k \gamma}$.

Definition 2.4. A state $W^0 \in \widetilde{\mathbb{H}}^0$ is said to be controllable to a state $W^T \in \widetilde{\mathbb{H}}^0$ with respect to system (2.1), (2.2) in a given time $T > 0$ if $W^T \in \widetilde{\mathcal{R}}_T^{\rho k \gamma}(W^0)$.

In other words, a state $W^0 \in \widetilde{\mathbb{H}}^0$ is said to be controllable to a state $W^T \in \widetilde{\mathbb{H}}^0$ with respect to system (2.1), (2.2) in a given time $T > 0$ if there exists a control $u^{\rho k \gamma} \in L^\infty(0, T)$ such that there exists a unique solution W to (2.1), (2.2) with $u = u^{\rho k \gamma}$ and $W(\cdot, T) = W^T$.

Definition 2.5. A state $W^0 \in \widetilde{\mathbb{H}}^0$ is said to be approximately controllable to a state $W^T \in \widetilde{\mathbb{H}}^0$ with respect to system (2.1), (2.2) in a given time $T > 0$ if $W^T \in \overline{\widetilde{\mathcal{R}}_T^{\rho k \gamma}(W^0)}$, where the closure is considered in the space $\widetilde{\mathbb{H}}^0$.

In other words, a state $W^0 \in \widetilde{\mathbb{H}}^0$ is said to be approximately controllable to a state $W^T \in \widetilde{\mathbb{H}}^0$ with respect to system (2.1), (2.2) in a given time $T > 0$ if for each $\varepsilon > 0$, there exists $u_\varepsilon^{\rho k \gamma} \in L^\infty(0, T)$ such that there exists a unique solution $W_\varepsilon^{\rho k \gamma}$ to (2.1), (2.2) with $u = u_\varepsilon^{\rho k \gamma}$ and $\left[\!\left[W_\varepsilon^{\rho k \gamma}(\cdot, T) - W^T \right]\!\right]^0 < \varepsilon$.

Theorems 3.3 and 3.6 (see below Sect. 3) imply

Theorem 2.6. *Let $T > 0$, $W^0 \in \widetilde{\mathbb{H}}^0$, $Z^0 = \widetilde{\mathbb{T}}^{-1} W^0$. Then*

(i) $\widetilde{\mathcal{R}}_T^{\rho k \gamma}\left(W^0\right) = \widetilde{\mathbb{T}}\left(\widetilde{\mathcal{R}}_T^{110}\left(Z^0\right)\right).$

(ii)   *A state Z^0 is controllable to a state Z^T with respect to system (2.5), (2.6) in a time T iff a state W^0 is controllable to a state W^T with respect to system (2.1), (2.2) in this time T.*

(iii)   *A state Z^0 is approximately controllable to a state Z^T with respect to system (2.5), (2.6) in a time T iff a state W^0 is approximately controllable to a state W^T with respect to system (2.1), (2.2) in this time T.*

Thus the control system (2.1), (2.2) with variable coefficients ρ, k, γ replicates the controllability properties of the control system (2.5), (2.6) with constant coefficients ($\rho = k = 1$, $\gamma = 0$) and *vice versa*.

Theorem 2.7. *If a state $W^0 \in \widetilde{\mathbb{H}}^0$ is controllable to the state $W^T = 0$ with respect to system (2.1), (2.2) in a time $T > 0$, then $W^0 = 0$.*

In the case $\rho = k = 1$, $\gamma = 0$ this theorem has been proved in [9]. By using Theorem 2.6, we get Theorem 2.7.

Theorem 2.8. *Each state $W^0 \in \widetilde{\mathbb{H}}^0$ is approximately controllable to any target state $W^T \in \widetilde{\mathbb{H}}^0$ with respect to system (2.1), (2.2) in a given time $T > 0$.*

In the case $\rho = k = 1$, $\gamma = 0$ this theorem has been proved in [9]. Applying Theorem 2.6, we get Theorem 2.8.

Using the algorithm given in Theorem 5.2 of [9], we can construct piecewise constant controls solving the approximate controllability problem for system (2.5), (2.6). Thus, due to Theorem 3.3, we obtain controls solving the approximate controllability problem for system (2.1), (2.2) (see below Sect. 3).

3. THE TRANSFORMATION OPERATOR $\widetilde{\mathbb{T}}$

In this section, we describe some properties of the operator $\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$. Then we apply it to control system (2.1), (2.2).

First, consider the operator $\mathbf{S} : H^{-2} \to \mathbb{H}^{-2}$ introduced and studied in [6, 7].

Theorem 3.1 (see [6, 7])**.** *The following assertions hold.*

(i)   *The operator $\mathbf{S}$ is an isometric isomorphism of H^m and $\mathbb{H}^m$, $m = \overline{-2, 2}$.*
(ii) $\mathbb{D}_{\rho k} \mathbf{S}\psi = \mathbf{S}\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi$, $\psi \in H^m$, $m = \overline{-1, 2}$.
(iii) $\langle\langle f, \varphi \rangle\rangle = \langle \mathbf{S}^{-1} f, \mathbf{S}^{-1}\varphi \rangle$, $f \in \mathbb{H}^{-m}$, $\varphi \in \mathbb{H}^m$, $m = \overline{0, 2}$.
(iv) $\mathbf{S}\delta = \sqrt[4]{(\rho k)(0)}\delta$.

In particular, we have

$$\mathbf{S}\psi = \frac{\psi \circ \sigma}{\sqrt[4]{\rho k}}, \quad \psi \in H^0, \qquad \text{and} \qquad \mathbf{S}^{-1}\varphi = \left(\sqrt[4]{(\rho k)}\varphi \right) \circ \sigma^{-1}, \quad \varphi \in \mathbb{H}^0,$$

where $\psi \circ \sigma = \psi(\sigma(x))$, σ is defined by (1.4). It follows from (1.4), (1.5) that σ is an odd increasing invertible function and $\sigma(x) \to \pm\infty$ as $x \to \pm\infty$.

Now, consider the operator $\widetilde{\mathbf{T}}_r : \widetilde{H}^{-2} \to \widetilde{H}^{-2}$ for

$$r(\lambda) = q \circ \sigma^{-1} = (Q_2(\rho, k) - \gamma) \circ \sigma^{-1}. \tag{3.1}$$

Due to (1.6) and (1.7), we get

$$r \in L^\infty(0, +\infty) \cap C^1[0, +\infty) \quad \text{and} \quad \lambda r \in L^1(0, +\infty). \tag{3.2}$$

This operator has been introduced and studied in [6, 7]. In particular, we have

$$\left(\widetilde{\mathbf{T}}_r g\right)(\lambda) = g(\lambda) + \mathrm{sgn}\,\lambda \int_{|\lambda|}^\infty K(|\lambda|, \xi) g(\xi) \mathrm{d}\xi, \qquad\qquad \lambda \in \mathbb{R}, \ g \in \widetilde{H}^0,$$

$$\left(\widetilde{\mathbf{T}}_r^{-1} f\right)(\xi) = f(\xi) + \mathrm{sgn}\,\xi \int_{|\xi|}^\infty L(|\xi|, \lambda) f(\lambda) \mathrm{d}\lambda, \qquad\qquad \xi \in \mathbb{R}, \ f \in \widetilde{H}^0,$$

where, according to Chapter 3 of [17], the kernel $K \in C^2(\Omega)$ is a unique solution to the system

$$
\begin{cases}
K_{y_1 y_1} - K_{y_2 y_2} = r(y_1) K & \text{on } \Omega, \\
K(y_1, y_1) = \dfrac{1}{2} \displaystyle\int_{y_1}^{\infty} r(\xi) \mathrm{d}\xi, & y_1 > 0, \\
\displaystyle\lim_{y_1 + y_2 \to \infty} K_{y_1}(y_1, y_2) = \lim_{y_1 + y_2 \to \infty} K_{y_2}(y_1, y_2) = 0 & \text{on } \Omega,
\end{cases}
\tag{3.3}
$$

$\Omega = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 > y_1 > 0\}$, and the kernel $L \in C^2(\Omega)$ is determined by the equation

$$
L(y_1, y_2) + K(y_1, y_2) + \int_{y_1}^{y_2} L(y_1, \xi) K(\xi, y_2) \mathrm{d}\xi = 0 \qquad \text{on } \Omega.
\tag{3.4}
$$

The operator $\widetilde{\mathbf{T}}_r$ is the extension to $\widetilde{H}^{-2}$ of the well-known transformation operator of the Sturm–Liouville problem (see, $e.g.$, [17], Chap. 3). First this extension was realised in [5] for

$$
|r(\lambda)| \le A e^{-|\lambda|}, \qquad \lambda \in \mathbb{R},
\tag{3.5}
$$

where $A > 0$ is a constant. Then some properties of this extension were studied in the general case in [15, 16]. Finally, the complete description of the extension and its application to the wave equation with variable coefficients have been given in [6, 7].

Theorem 3.2 (see [6, 7]). *The following assertions hold.*

(i)   *The operator $\widetilde{\mathbf{T}}_r$ is an automorphism of $\widetilde{H}^m$, $m = \overline{-2, 2}$.*

(ii)   *If $g \in \widetilde{H}^0$ and $g(0^+) \in \mathbb{R}$, then $\left(\widetilde{\mathbf{T}}_r g \right)(0^+) \in \mathbb{R}$ and*

$$
\left(\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} - r \right) \widetilde{\mathbf{T}}_r g - 2 \left(\widetilde{\mathbf{T}}_r g \right)(0^+) \delta' = \widetilde{\mathbf{T}}_r \left(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} g - 2 g(0^+) \delta' \right).
$$

(iii)   *If $f \in \widetilde{H}^0$ and $f(0^+) \in \mathbb{R}$, then $\left(\widetilde{\mathbf{T}}_r^{-1} f \right)(0^+) \in \mathbb{R}$ and*

$$
\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \widetilde{\mathbf{T}}_r^{-1} f - 2 \left(\widetilde{\mathbf{T}}_r^{-1} f \right)(0^+) \delta' = \widetilde{\mathbf{T}}_r^{-1} \left(\left(\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} - r \right) f - 2 f(0^+) \delta' \right).
$$

(iv) $\widetilde{\mathbf{T}}_r \delta' = \delta'$.

We also need the following estimates proved in Chapter 3 of [17]:

$$
|K(y_1, y_2)| \le M_0 \sigma_0 \left(\frac{y_1 + y_2}{2} \right) \qquad \text{on } \Omega,
\tag{3.6}
$$

$$
|K_{y_1}(y_1, y_2)| \le \frac{1}{4} \left| r \left(\frac{y_1 + y_2}{2} \right) \right| + M_1 \sigma_0 \left(\frac{y_1 + y_2}{2} \right) \qquad \text{on } \Omega,
\tag{3.7}
$$

where $M_0 > 0$, $M_1 > 0$ are constants, and

$$
\sigma_0(x) = \int_x^{\infty} |r(\xi)| \mathrm{d}\xi, \qquad x > 0.
\tag{3.8}
$$

Further we consider the application of the transformation operator $\widetilde{\mathbb{T}}$ to a control system.

Theorem 3.3. *Let Z be a solution to* (2.5), (2.6) *with $u = u^{110}$, where $u^{110} \in L^\infty(0,T)$, $Z^0 \in \widetilde{H}^0$. Let $W(\cdot, t) = \left(\widetilde{\mathbb{T}}Z \right)(\cdot, t)$, $t \in [0,T]$. Then W is a solution to* (2.1), (2.2) *with $W^0 = \widetilde{\mathbb{T}}Z^0$ and with the control $u = u^{\rho k \gamma}$,*

$$u^{\rho k \gamma}(t) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(u^{110}(t) + \int_0^\infty K(0,x) Z(x,t) \mathrm{d}x \right), \qquad t \in [0,T], \tag{3.9}$$

where K is a solution to (3.3), *r is defined by* (3.1). *In addition,* (2.4) *is valid and*

$$[\![W(\cdot, t)]\!]^0 \le C_0 \| Z(\cdot, t) \|^0, \qquad t \in [0,T], \tag{3.10}$$

$$\| u^{\rho k \gamma} \|_{L^\infty(0,T)} \le B_0(T) \| u^{110} \|_{L^\infty(0,T)} + C_1 \left\| Z^0 \right\|^0, \tag{3.11}$$

where $C_0 > 0$, $C_1 > 0$ are constants independent of T,

$$B_0(T) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(1 + 2 M_0 \sqrt{2 \sigma_0(0) R} \sqrt{\frac{T+2}{\pi}} \right),$$

M_0 is the constant from (3.6), *σ_0 is defined by* (3.8),

$$R = \int_0^\infty \xi |r(\xi)| \mathrm{d}\xi. \tag{3.12}$$

Proof. The proof of the first part of the theorem is similar to the proof of the first part of the corresponding theorem in [6, 7] ([6], Thm. 4.1, [7], Thm. 6.1). Using Theorem 2.2 (iii), we get that W is a solution to (2.1), (2.2) with $W^0 = \widetilde{\mathbb{T}}Z^0$ and with the control given by (3.9).

Using (2.8) and (3.9), we obtain

$$W(0^+, t) = \left(\widetilde{\mathbb{T}}Z \right)(0^+, t) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(Z(0^+, t) + \int_0^\infty K(0,x) Z(x,t) \mathrm{d}x \right) = u^{\rho k \gamma}(t), \quad t \in [0,T].$$

Thus, (2.4) is valid.

Applying Theorem 2.2 (i), we conclude that there exists a constant $C_0 > 0$ such that (3.10) holds.

Let us estimate $u^{\rho k \gamma}$. With regard to (3.6) and (3.9), we get

$$\| u^{\rho k \gamma} \|_{L^\infty(0,T)} \le \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(\| u^{110} \|_{L^\infty(0,T)} + M_0 \sqrt{\sigma_0(0) R} \| Z(\cdot, t) \|^0 \right), \qquad t \in [0,T]. \tag{3.13}$$

In [9], the following formula has been obtained

$$(\mathcal{F}Z)(\sigma, t) = e^{-t\sigma^2} (\mathcal{F}Z^0)(\sigma) - \sqrt{\frac{2}{\pi}} i\sigma \int_0^t e^{-(t-\xi)\sigma^2} u^{110}(\xi) \mathrm{d}\xi, \qquad \sigma \in \mathbb{R}, \ t \in [0,T],$$

where $\mathcal{F} : H^0 \to H^0$ is the Fourier transform operator. From here, it follows that

$$\| (\mathcal{F}Z)(\cdot, t) \|^0 \le \left\| \mathcal{F}Z^0 \right\|^0 + 2 \sqrt{\frac{2}{\pi}} \| u^{110} \|_{L^\infty(0,T)} \sqrt{T+2}, \qquad t \in [0,T].$$

Therefore,

$$\|Z(\cdot,t)\|^0 \le \|Z^0\|^0 + 2\sqrt{\frac{2}{\pi}}\|u^{110}\|_{L^\infty(0,T)}\sqrt{T+2}, \qquad t \in [0,T]. \tag{3.14}$$

With regard to (3.13) and (3.14), we get

$$\|u^{\rho k\gamma}\|_{L^\infty(0,T)} \le \frac{1}{\sqrt[4]{(\rho k)(0)}}\left(\|u^{110}\|_{L^\infty(0,T)} + M_0\sqrt{\sigma_0(0)R}\left(\|Z^0\|^0 + 2\sqrt{\frac{2}{\pi}}\|u^{110}\|_{L^\infty(0,T)}\sqrt{T+2}\right)\right).$$

The theorem is proved. $\qquad\square$

Corollary 3.4. *Let W be a solution to (2.1), (2.2) with $u = u^{\rho k\gamma}$, where $W^0 \in \widetilde{\mathbb{H}}^0$ and $u \in L^\infty(0,T)$. Then $W(0^+,\cdot) = u$ a.e. on $[0,T]$, i.e. (2.4) holds.*

Proof. Put $Z(\cdot,t) = \left(\widetilde{\mathbb{T}}^{-1}W\right)(\cdot,t)$, $t \in [0,T]$. Applying the operator $\widetilde{\mathbb{T}}^{-1}$ to equation (2.1) and taking into account Theorem 2.2 (iv), we get

$$Z_t(\cdot,t) = Z_{\xi\xi}(\cdot,t) - 2Z(0^+,t)\delta' + 2\sqrt{(\rho k)(0)}\left(W(0^+,t) - u^{\rho k\gamma}(t)\right)\widetilde{\mathbb{T}}^{-1}\mathbb{D}_{\rho k}\delta, \quad t \in [0,T].$$

Using Theorem 2.2 (ii), we obtain

$$Z_t(\cdot,t) = Z_{\xi\xi}(\cdot,t) - 2\left(Z(0^+,t) - \sqrt[4]{(\rho k)(0)}W(0^+,t) + \sqrt[4]{(\rho k)(0)}u^{\rho k\gamma}(t)\right)\delta', \quad t \in [0,T].$$

Therefore, Z is a solution to system (2.5), (2.6) with $Z^0 = \widetilde{\mathbb{T}}^{-1}W^0$ and with the control $u = u^{110}$,

$$u^{110}(t) = Z(0^+,t) - \sqrt[4]{(\rho k)(0)}W(0^+,t) + \sqrt[4]{(\rho k)(0)}u^{\rho k\gamma}(t), \qquad t \in [0,T].$$

Taking into account (2.8), we conclude that $W(0^+,t) = u^{\rho k\gamma}(t)$, $t \in [0,T]$. $\qquad\square$

To prove Theorem 3.6, we need the following lemma.

Lemma 3.5. *Let*

$$|f(t)| \le N_0, \qquad\qquad t \in [0,T], \tag{3.15}$$

$$|P(t)| \le \frac{N_1}{\sqrt{\pi t}}, \qquad\qquad t \in (0,T], \tag{3.16}$$

where $N_0 > 0$ and $N_1 > 0$ are constants. Then there exists a unique solution $v \in L^\infty(0,T)$ to equation

$$v(t) = f(t) + \int_0^t v(\xi)P(t-\xi)\mathrm{d}\xi, \qquad t \in [0,T], \tag{3.17}$$

and

$$\|v\|_{L^\infty(0,T)} \le N_0\left(1 + 2N_1\sqrt{\frac{T}{\pi}}e^{N_1^2 T}\right). \tag{3.18}$$

Proof. For any $v \in L^\infty(0,T)$, we have

$$\left| \int_0^t v(\xi) P(t-\xi) d\xi \right| \leq \|v\|_{L^\infty(0,T)} \frac{N_1}{\sqrt{\pi}} \int_0^t \frac{d\xi}{\sqrt{t-\xi}} \leq \frac{2N_1}{\sqrt{\pi}} \sqrt{t} \|v\|_{L^\infty(0,T)}$$

$$\leq \frac{N_1}{\Gamma\left(\frac{3}{2}\right)} T^{\frac{1}{2}} \|v\|_{L^\infty(0,T)}, \qquad t \in [0,T], \tag{3.19}$$

according to (3.16).

Consider the operator $\mathbf{A}: L^\infty(0,T) \to L^\infty(0,T)$ acting by the rule

$$(\mathbf{A}v)(t) = f(t) + \int_0^t v(\xi) P(t-\xi) d\xi, \qquad t \in [0,T], \ D(\mathbf{A}) = L^\infty(0,T). \tag{3.20}$$

Then (3.17) takes the form

$$v = \mathbf{A}v. \tag{3.21}$$

Set

$$v_0 = f, \qquad v_n = \mathbf{A}v_{n-1}, \qquad n \in \mathbb{N}. \tag{3.22}$$

Let us prove that

$$|(v_n - v_{n-1})(t)| \leq N_0 \frac{N_1^n}{\Gamma(\frac{n+2}{2})} t^{\frac{n}{2}}, \qquad n \in \mathbb{N}, \ t \in [0,T]. \tag{3.23}$$

We prove (3.23) by induction. The base case follows from (3.15), (3.19), (3.22). The induction step:

$$|(v_n - v_{n-1})(t)| \leq \int_0^t |(v_{n-1} - v_{n-2})(\xi)| P(t-\xi) d\xi \leq N_0 \frac{N_1^{n-1}}{\Gamma(\frac{n+1}{2})} \frac{N_1}{\sqrt{\pi}} \int_0^t \frac{\xi^{\frac{n-1}{2}}}{\sqrt{t-\xi}} d\xi, \qquad n \in \mathbb{N}, \ t \in [0,T], \tag{3.24}$$

according to the induction hypothesis and (3.16).

For $n \in \mathbb{N}$, we have

$$\int_0^t \frac{\xi^{\frac{n-1}{2}}}{\sqrt{t-\xi}} d\xi = t^{\frac{n}{2}} \int_0^1 \tau^{\frac{n-1}{2}} (1-\tau)^{-\frac{1}{2}} d\tau = t^{\frac{n}{2}} B\left(\frac{n+1}{2}, \frac{1}{2}\right) = t^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}, \qquad t \in [0,T].$$

With regard to (3.24), for $n \in \mathbb{N}$, we get

$$|(v_n - v_{n-1})(t)| \leq \frac{N_0 N_1^n}{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})} t^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \frac{N_0 N_1^n}{\Gamma\left(\frac{n+2}{2}\right)} t^{\frac{n}{2}}, \qquad t \in [0,T], \tag{3.25}$$

Thus, the proof of (3.23) is complete.

Due to (3.25), the series $\sum_{n=1}^\infty (v_n - v_{n-1})$ converges in $L^\infty(0,T)$. Therefore, there exists $v \in L^\infty(0,T)$ such that

$$\|v_n - v\|_{L^\infty(0,T)} \longrightarrow 0 \quad \text{as } n \to \infty, \tag{3.26}$$

because

$$v_n = v_0 + \sum_{p=1}^{n}(v_p - v_{p-1}), \quad n \in \mathbb{N}. \tag{3.27}$$

Moreover, taking into account (3.15), (3.25), and (3.27), we obtain

$$\|v\|_{L^\infty(0,T)} \le N_0\left(1 + \sum_{n=1}^{\infty}\frac{N_1^n}{\Gamma\left(\frac{n+2}{2}\right)}T^{\frac{n}{2}}\right) = N_0\left(1 + \sum_{k=1}^{\infty}\frac{N_1^{2k}}{k!}T^k + \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{N_1^{2k+1}2^{k+1}}{(2k+1)!!}T^{\frac{2k+1}{2}}\right)$$

$$\le N_0\left(1 + e^{N_1^2 T} - 1 + 2N_1\sqrt{\frac{T}{\pi}}e^{N_1^2 T}\right) = N_0\left(1 + 2N_1\sqrt{\frac{T}{\pi}}\right)e^{N_1^2 T},$$

i.e. estimate (3.18) holds for v.

Now let us prove that v is a solution to (3.21). Taking into account (3.19) and (3.26), we get

$$|(v_n - \mathbf{A}v)(t)| \le \left|\int_0^t (v_{n-1} - v)(\xi)P(t-\xi)\mathrm{d}\xi\right| \le \frac{N_1}{\Gamma\left(\frac{3}{2}\right)}T^{\frac{1}{2}}\|v_{n-1} - v\|_{L^\infty(0,T)} \longrightarrow 0 \quad \text{as } n \to \infty, \ t \in [0,T].$$

Hence (3.21) holds, *i.e.* v is a solution to (3.20), therefore it is a solution to (3.17).

Finally, let us prove that the solution v to (3.20) is unique. To this aid, it is sufficient to prove that equation

$$y(t) = \int_0^T y(\xi)P(t-\xi)\mathrm{d}\xi, \qquad t \in [0,T], \tag{3.28}$$

has only trivial solution in $L^\infty(0,T)$. Choose any $n \in \mathbb{N}$ such that

$$n > \frac{N_1^2}{\Gamma^2\left(\frac{3}{2}\right)}T \tag{3.29}$$

and put $T_k = \frac{kT}{n}$, $k = \overline{0,n}$. We have $0 = T_0 < T_1 < T_2 < \cdots < T_{n-1} < T_n = T$ and $T_k - T_{k-1} = \frac{T}{n}$, $k = \overline{1,n}$.

Let $y \in L^\infty(0,T)$ be a solution to (3.28). If $t \in [T_0, T_1] = \left[0, \frac{T}{n}\right]$, then

$$y(t) = \int_{T_0}^t y(\xi)P(t-\xi)\mathrm{d}\xi, \qquad t \in [T_0, T_1], \tag{3.30}$$

and

$$\left|\int_{T_0}^t y(\xi)P(t-\xi)\mathrm{d}\xi\right| \le \frac{N_1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{T_1}\|y\|_{L^\infty(0,T_1)} = \frac{N_1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\frac{T}{n}}\|y\|_{L^\infty(0,T_1)}, \quad t \in [T_0, T_1],$$

according to (3.19). Due to (3.29), the integral operator on the right-hand side of (3.28) is a contraction mapping in $L^\infty(0,T)$. Due to the Banach fixed-point theorem, equation (3.28) has a unique solution in $L^\infty(T_0, T_1)$. Therefore, $y = 0$ on (T_0, T_1). Repeating this reasoning for the intervals $(T_1, T_2]$, $(T_2, T_3]$,..., $(T_{n-1}, T_n]$ (one by one) and equations

$$y(t) = \int_{T_{k-1}}^t y(\xi)P(t-\xi)\mathrm{d}\xi, \qquad t \in [T_{k-1}, T_k], \ k = \overline{2,n},$$

we conclude that $y = 0$ on $(0, T)$. Thus, the solution v to (3.20) (and (3.17)) is unique in $L^\infty(0, T)$.

The Lemma 3.5 is proved. $\qquad\square$

Theorem 3.6. *Let W be a solution to* (2.1), (2.2) *with $u = u^{\rho k \gamma}$, where $u^{\rho k \gamma} \in L^\infty(0, T)$, $W^0 \in \widetilde{\mathbb{H}}^0$. Let $Z(\cdot, t) = \left(\widetilde{\mathbb{T}}^{-1} W \right)(\cdot, t)$, $t \in [0, T]$. Then Z is a solution to* (2.5), (2.6) *with $Z^0 = \widetilde{\mathbb{T}}^{-1} W^0$ and with the control $u = u^{110}$,*

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) + \int_0^\infty L(0, x) \left(\boldsymbol{S}^{-1} W \right)(x, t) \mathrm{d}x, \qquad t \in [0, T], \tag{3.31}$$

where L is determined by (3.4). *In addition,*

$$\|Z(\cdot, t)\|^0 \le C_2 \, [\![W(\cdot, t)]\!]^0, \qquad t \in [0, T], \tag{3.32}$$

$$\|u^{110}\|_{L^\infty(0,T)} \le B_1(T) \left(\|u^{\rho k \gamma}\|_{L^\infty(0,T)} + C_3 \, [\![W^0]\!]^0 \right), \tag{3.33}$$

where $C_2 > 0$, $C_3 > 0$ are constants independent of T,

$$B_1(T) = \sqrt[4]{(\rho k)(0)} e^{M_0^2 \sigma_0^2(0) T} \left(1 + 2 M_0 \sigma_0(0) \sqrt{\frac{T}{\pi}} \right),$$

M_0 is the constant from (3.6), *σ_0 is defined by* (3.8).

Proof. Applying Theorems 2.2 (i), 2.2 (iv) and Corollary 3.4, we obtain (3.31) and (3.32). Let us prove (3.33).

From (3.31), it follows that

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) + \int_0^\infty L(0, x) \left(\widetilde{\mathbf{T}}_r Z \right)(x, t) \mathrm{d}x$$

$$= \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) + \int_0^\infty L(0, x) \left(Z(x, t) + \int_x^\infty K(x, \xi) Z(\xi, t) \mathrm{d}\xi \right) \mathrm{d}x, \qquad t \in [0, T].$$

With regard to (3.4), we get

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) - \int_0^\infty K(0, x) Z(x, t) \mathrm{d}x, \qquad t \in [0, T].$$

(In fact, it is relation (3.9) from Thm. 3.3.)

In [9], it has been proved that

$$Z(x, t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x) + \sqrt{\frac{2}{\pi}} x \int_0^t u^{110}(\xi) \frac{e^{-\frac{x^2}{4(t-\xi)}}}{(2(t-\xi))^{3/2}} \mathrm{d}\xi, \qquad x \in \mathbb{R}, \ t \in [0, T].$$

Then we obtain

$$u^{110}(t) = \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma}(t) - \int_0^\infty K(0, x) \left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x) \right) \mathrm{d}x$$

$$- \sqrt{\frac{2}{\pi}} \int_0^\infty K(0, x) x \int_0^t u^{110}(\xi) \frac{e^{-\frac{x^2}{4(t-\xi)}}}{(2(t-\xi))^{3/2}} \mathrm{d}\xi \mathrm{d}x, \qquad t \in [0, T]. \tag{3.34}$$

Put

$$f(t) = \sqrt[4]{(\rho k)(0)}u^{\rho k\gamma}(t) - \int_0^\infty K(0,x)\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x)\right) dx, \qquad t \in [0,T], \qquad (3.35)$$

$$P(t) = -\sqrt{\frac{2}{\pi}}\frac{1}{(2t)^{3/2}}\int_0^\infty K(0,x)xe^{-\frac{x^2}{4t}} dx, \qquad t \in (0,T]. \qquad (3.36)$$

Then (3.34) takes the form (3.17). Since

$$\left(\mathcal{F}_{x\to\sigma}\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x)\right)\right)(\sigma) = e^{-t\sigma^2}\left(\mathcal{F}Z^0\right)(\sigma), \qquad \sigma \in \mathbb{R},\ t \in [0,T],$$

we have

$$\left\|\frac{e^{-\frac{(\cdot)^2}{4t}}}{\sqrt{4\pi t}} * Z^0\right\|^0 = \left\|e^{-t(\cdot)^2}\mathcal{F}Z^0\right\|^0 \leq \left\|\mathcal{F}Z^0\right\|^0 = \left\|Z^0\right\|^0, \qquad t \in [0,T].$$

As it is known ([17], Chap. 3), $K(y_1, y_2) = 0$ when $y_2 < y_1$. Therefore, taking into account (3.6) and (3.32), we get

$$\left|\int_0^\infty K(0,x)\left(\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} * Z^0(x)\right) dx\right| \leq \frac{1}{\sqrt{2}}\left\|K(0,\cdot)\right\|_{L^2(0,+\infty)}\left\|Z^0\right\|^0 \leq M_0 C_2\sqrt{\sigma_0(0)R}\left[\!\left] W^0\right]\!\right]^0, \qquad t \in [0,T],$$

where R is defined by (3.12). Thus,

$$|f(t)| \leq \sqrt[4]{(\rho k)(0)}\|u^{\rho k\gamma}\|_{L^\infty(0,T)} + M_0 C_2\sqrt{\sigma_0(0)R}\left[\!\left] W^0\right]\!\right]^0, \qquad t \in [0,T]. \qquad (3.37)$$

We also have

$$|P(t)| \leq \sqrt{\frac{2}{\pi}}\frac{M_0\sigma_0(0)}{(2t)^{3/2}}\int_0^\infty xe^{-\frac{x^2}{4t}} dx = \frac{M_0\sigma_0(0)}{\sqrt{\pi t}}, \qquad t \in (0,T], \qquad (3.38)$$

according to (3.6).

Taking into account (3.37) and (3.38) and applying Lemma 3.5, we conclude that there exists a unique solution $u^{110} \in L^\infty(0,T)$ to equation (3.17) (and, consequently, (3.34)). Moreover,

$$\|u^{110}\|_{L^\infty(0,T)} \leq \left(\sqrt[4]{(\rho k)(0)}\|u^{\rho k\gamma}\|_{L^\infty(0,T)} + M_0 C_2\sqrt{\sigma_0(0)R}\left[\!\left] W^0\right]\!\right]^0\right)\left(1 + 2M_0\sigma_0(0)\sqrt{\frac{T}{\pi}}\right)e^{M_0^2\sigma_0^2(0)T}.$$

The theorem is proved. $\qquad \square$

Thus, Theorems 3.3 and 3.6 allow us to conclude that system (2.1), (2.2) with variable coefficients ρ, k, γ replicates controllability properties of the system of this type with the constant coefficients $\rho = k = 1$, $\gamma = 0$ and *vice versa*.

Using the algorithm given in [9] of Theorem 5.2, we can construct piecewise constant controls $u_{N,l}^{110}$, $N,l \in \mathbb{N}$, solving the approximate controllability problem for system (2.5), (2.6). We can also find approximate states

$Z_{N,l}^T$, $N, l \in \mathbb{N}$, such that $\|Z^T - Z_{N,l}^T\|^0 \to 0$ as $N \to \infty$ and $l \to \infty$ in the form

$$Z_{N,l}^T(\xi) = \frac{e^{-\frac{\xi^2}{4T}}}{2\sqrt{\pi T}} * Z^0(\xi) + \sqrt{\frac{2}{\pi}} \xi \int_0^T e^{-\frac{\xi^2}{4\tau}} \frac{u_{N,l}^{110}(T-\tau)}{(2\tau)^{3/2}} d\tau, \qquad N, l \in \mathbb{N}, \ \xi \in \mathbb{R}.$$

Thus, due to Theorem 3.3, we obtain controls solving the approximate controllability problem for system (2.1), (2.2) in the form

$$u_{N,l}^{\rho k \gamma}(t) = \frac{1}{\sqrt[4]{(\rho k)(0)}} \left(u_{N,l}^{110}(t) + \int_0^\infty K(0, \xi) Z_{N,l}(\xi, t) d\xi \right), \qquad N, l \in \mathbb{N}, \ t \in [0, T],$$

where

$$Z_{N,l}(\xi, t) = \frac{e^{-\frac{\xi^2}{4t}}}{2\sqrt{\pi t}} * Z^0(\xi) + \sqrt{\frac{2}{\pi}} \xi \int_0^t e^{-\frac{\xi^2}{4\tau}} \frac{u_{N,l}^{110}(t-\tau)}{(2\tau)^{3/2}} d\tau, \qquad N, l \in \mathbb{N}, \ \xi \in \mathbb{R}, \ t \in [0, T].$$

Moreover, $u_{N,l}^{\rho k \gamma} \in L^\infty(0, T)$ according to Theorem 3.3. Besides, we get $W_{N,l}^T = \widetilde{\mathbb{T}} Z_{N,l}^T$ and $\left\| W^T - W_{N,l}^T \right\|^0 \to 0$ as $N \to \infty$ and $l \to \infty$.

4. EXAMPLES

Example 4.1. Consider system (1.1)–(1.3), where

$$k(x) = (4 + x^2)^2, \quad \rho(x) = 4 + x^2, \quad \gamma(x) = \frac{9}{4} + \frac{|x|}{8} - \frac{3}{4 + x^2} - \frac{\sqrt{4 + x^2}}{8}, \quad x \in \mathbb{R}.$$

Then

$$Q_2(\rho, k) = \frac{24 + 9x^2}{4(4 + x^2)} \quad \text{and} \quad q(x) = Q_2(\rho, k) - \gamma(x) = \frac{\sqrt{4 + x^2} - |x|}{8}, \quad x \in \mathbb{R}.$$

According to (1.4), we have

$$\sigma(x) = \operatorname{sgn} x \ln \left(\frac{|x|}{2} + \sqrt{1 + \frac{x^2}{4}} \right), \quad x \in \mathbb{R}, \quad \text{and} \quad \sigma^{-1}(\lambda) = 2 \sinh \lambda, \quad \lambda \in \mathbb{R}.$$

Let us consider system (2.5)–(2.7) with

$$Z^0(x) = \operatorname{sgn} x e^{-\frac{|x|}{2}}, \quad Z^T(x) = \operatorname{sgn} x e^{-\frac{2|x|-T}{4}}, \quad x \in \mathbb{R}.$$

It is easy to see that the state Z^0 is controllable to the state Z^T with respect to system (2.5), (2.6) in the time T with the control

$$u^{110}(t) = e^{\frac{t}{4}}, \quad t \in [0, T],$$

and

$$Z(x, t) = \operatorname{sgn} x e^{-\frac{2|x|-t}{4}}, \quad x \in \mathbb{R}, \ t \in [0, T],$$

is the unique solution to this system.

Now consider system (2.1)–(2.3) with given q. According to Theorem 2.6 (ii), the state $W^0 = \widetilde{\mathbb{T}} Z^0$ is controllable to the state $W^T = \widetilde{\mathbb{T}} Z^T$ with respect to system (2.1), (2.2) in the time T. Moreover, due to Theorem 3.3, a control solving controllability problem for system (2.1)–(2.3) is defined by (3.9).

Let us find W^0, W^T and $u^{\rho k \gamma}$ explicitly. Due to (3.1), $r(\lambda) = q \circ \sigma^{-1} = \frac{1}{4} e^{-\lambda}$, $\lambda > 0$. The kernel of the transformation operator $\widetilde{\mathbf{T}}_r$ was found in [5, 15] for this r. We get

$$K(y_1, y_2) = \frac{e^{-\frac{y_1 + y_2}{2}}}{4} \frac{I_1\left(\sqrt{e^{-\frac{y_1}{2}}\left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}}\right)}\right)}{\sqrt{e^{-\frac{y_1}{2}}\left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}}\right)}}, \qquad y_2 > y_1 > 0, \tag{4.1}$$

where I_1 is the modified Bessel function $I_n(x) = i^{-n} J_n(ix)$, $n = \overline{0, \infty}$, J_1 is the Bessel function. Therefore, due to (3.9)

$$u^{\rho k \gamma}(t) = \frac{1}{2^{3/2}} \left(e^{\frac{t}{4}} + \int_0^\infty \frac{e^{-\frac{x}{2}}}{4} \frac{I_1\left(\sqrt{1 - e^{-\frac{x}{2}}}\right)}{\sqrt{1 - e^{-\frac{x}{2}}}} e^{-\frac{2x - t}{4}} dx \right), \qquad t \in [0, T]. \tag{4.2}$$

Replacing $\sqrt{1 - e^{-\frac{x}{2}}}$ by y, we have

$$\frac{1}{4} e^{\frac{t}{4}} \int_0^\infty e^{-x} \frac{I_1\left(\sqrt{1 - e^{-\frac{x}{2}}}\right)}{\sqrt{1 - e^{-\frac{x}{2}}}} dx = e^{\frac{t}{4}} \int_0^1 (1 - y^2) I_1(y) dy, \qquad t \in [0, T].$$

Integrating by parts and taking into account that $I_1(y) = I_0'(y)$ and $y I_0(y) = (y I_1(y))'$, we obtain from here

$$\frac{1}{4} e^{\frac{t}{4}} \int_0^\infty e^{-x} \frac{I_1\left(\sqrt{1 - e^{-\frac{x}{2}}}\right)}{\sqrt{1 - e^{-\frac{x}{2}}}} dx = e^{\frac{t}{4}} (2 I_1(1) - 1), \qquad t \in [0, T]. \tag{4.3}$$

Continuing (4.2), we get

$$u^{\rho k \gamma}(t) = \frac{I_1(1)}{\sqrt{2}} e^{\frac{t}{4}}, \qquad t \in [0, T]. \tag{4.4}$$

According to the definition of the operator $\widetilde{\mathbb{T}}$, we have

$$W(x, t) = \left(\mathbf{S} \widetilde{\mathbf{T}}_r Z\right)(x, t) = \frac{\left(\widetilde{\mathbf{T}}_r Z(\cdot, t)\right) \circ \sigma(x)}{\sqrt[4]{(\rho k)(x)}} = \frac{\left(\widetilde{\mathbf{T}}_r Z(\cdot, t)\right)\left(\operatorname{sgn} x \ln\left(\frac{|x|}{2} + \sqrt{1 + \frac{x^2}{4}}\right)\right)}{(4 + x^2)^{3/4}}, \quad x \in \mathbb{R}, \ t \in [0, T].$$

Due to the definition of the operator $\widetilde{\mathbf{T}}_r$, we get

$$\left(\widetilde{\mathbf{T}}_r Z\right)(\lambda, t) = \operatorname{sgn} \lambda e^{-\frac{2|\lambda| - t}{4}} + \operatorname{sgn} \lambda \int_{|\lambda|}^\infty \frac{e^{-\frac{|\lambda| + x}{2}}}{4} \frac{I_1\left(\sqrt{e^{-\frac{|\lambda|}{2}}\left(e^{-\frac{|\lambda|}{2}} - e^{-\frac{x}{2}}\right)}\right)}{\sqrt{e^{-\frac{|\lambda|}{2}}\left(e^{-\frac{|\lambda|}{2}} - e^{-\frac{x}{2}}\right)}} e^{-\frac{2x - t}{4}} dx, \quad \lambda \in \mathbb{R}, \ t \in [0, T].$$

Replacing $\sqrt{e^{-\frac{|\lambda|}{2}}\left(e^{-\frac{|\lambda|}{2}}-e^{-\frac{x}{2}}\right)}$ by y in the integral, we obtain

$$
\left(\widetilde{\mathbf{T}}_r Z\right)(\lambda, t) = \operatorname{sgn}\lambda e^{-\frac{2|\lambda|-t}{4}} + \operatorname{sgn}\lambda e^{\frac{2|\lambda|+t}{4}}\int_0^{e^{-\frac{|\lambda|}{2}}}(e^{-|\lambda|}-y^2)I_1(y)\mathrm{d}y
$$

$$
= \operatorname{sgn}\lambda e^{-\frac{2|\lambda|-t}{4}} + \operatorname{sgn}\lambda e^{\frac{t}{4}}\left(2I_1\left(e^{-\frac{|\lambda|}{2}}\right)-e^{-\frac{|\lambda|}{2}}\right) = 2e^{\frac{t}{4}}\operatorname{sgn}\lambda I_1\left(e^{-\frac{|\lambda|}{2}}\right),\quad \lambda\in\mathbb{R},\ t\in[0,T].
$$

Since $\operatorname{sgn}\lambda = \operatorname{sgn}x$, we have

$$
W(x,t) = \frac{2e^{\frac{t}{4}}}{(4+x^2)^{3/4}}\operatorname{sgn}x I_1\left(e^{-\frac{1}{2}\ln\left(\frac{|x|}{2}+\sqrt{1+\frac{x^2}{4}}\right)}\right)
$$

$$
= \frac{2e^{\frac{t}{4}}}{(4+x^2)^{3/4}}\operatorname{sgn}x I_1\left(\left(\frac{|x|}{2}+\sqrt{1+\frac{x^2}{4}}\right)^{-1/2}\right),\quad x\in\mathbb{R},\ t\in[0,T].
$$

Therefore,

$$
W^0(x) = \frac{2}{(4+x^2)^{3/4}}\operatorname{sgn}x I_1\left(\left(\frac{|x|}{2}+\sqrt{1+\frac{x^2}{4}}\right)^{-1/2}\right),\qquad x\in\mathbb{R},\tag{4.5}
$$

$$
W^T(x) = \frac{2e^{\frac{T}{4}}}{(4+x^2)^{3/4}}\operatorname{sgn}x I_1\left(\left(\frac{|x|}{2}+\sqrt{1+\frac{x^2}{4}}\right)^{-1/2}\right),\qquad x\in\mathbb{R}.\tag{4.6}
$$

Thus, control (4.4) solves the controllability problem for system (2.1)–(2.3) in the time $T>0$, where the initial and the steering conditions are defined by (4.5) and (4.6).

Example 4.2. Consider system (1.1)–(1.3), where

$$
k(x) = \frac{4+x^2}{3+|x|},\quad \rho(x) = (4+x^2)(3+|x|),\quad \gamma(x) = \frac{12-|x|^3}{(3+|x|)^3(4+x^2)^2},\quad x\in\mathbb{R},\quad u = u^{\rho k\gamma}.
$$

Let us analyse whether the state $w^0 = 0$ is approximately controllable to the state

$$
w^T(x) = \frac{4}{\sqrt{2\pi(4+x^2)}}e^{\frac{1}{4}-\frac{x^2(|x|+6)^2}{16T}}\sin\frac{x(|x|+6)}{2\sqrt{2T}},\quad x\in(0,+\infty)
$$

in the time $T=1$. One can easily obtain

$$
Q_2(\rho,k) = \frac{12-|x|^3}{(3+|x|)^3(4+x^2)^2},\quad x\in\mathbb{R}.
$$

Hence, $q(x)=0$. We have

$$
\sigma(x) = \frac{1}{2}x(|x|+6),\quad x\in\mathbb{R},\quad\text{and}\quad \sigma^{-1}(\lambda) = \operatorname{sgn}\lambda\left(\sqrt{2|\lambda|+9}-3\right),\quad \lambda\in\mathbb{R}.
$$

Consider system (2.1)–(2.3), where

$$q(x) = 0, \quad W^0(x) = 0, \quad W^T(x) = \frac{4}{\sqrt{2\pi(4 + x^2)}} e^{\frac{1}{4} - \frac{x^2(|x|+6)^2}{16T}} \sin \frac{x(|x|+6)}{2\sqrt{2T}}, \quad x \in \mathbb{R}. \tag{4.7}$$

Due to (3.1), $r = 0$ on $\mathbb{R}$. Therefore, $\widetilde{\mathbf{T}}_r = \mathrm{Id}$, and the transformation operator $\widetilde{\mathbb{T}}$ takes the form $\widetilde{\mathbb{T}} = \mathbf{S}$. Denote

$$Z(\cdot, t) = \left(\widetilde{\mathbb{T}}^{-1} W\right)(\cdot, t) = \left(\mathbf{S}^{-1} W\right)(\cdot, t), \quad t \in [0, T], \quad Z^0 = \widetilde{\mathbb{T}}^{-1} W^0 = \mathbf{S}^{-1} W^0, \quad Z^T = \widetilde{\mathbb{T}}^{-1} W^T = \mathbf{S}^{-1} W^T.$$

Due to Theorem 3.6, Z is the solution to system (2.5)–(2.7), where

$$u = u^{110} = \sqrt[4]{(\rho k)(0)} u^{\rho k \gamma} = 2u^{\rho k \gamma}, \quad Z^0(\xi) = 0, \quad Z^T(\xi) = \frac{4}{\sqrt{2\pi}} e^{\frac{1}{4} - \frac{\xi^2}{4T}} \sin \frac{\xi}{\sqrt{2T}}, \quad \xi \in \mathbb{R}.$$

This system was considered in Example 8.3 in [9]. Controls solving the approximate controllability problem for system (2.5)–(2.7) have been found in the form

$$u^{110}_{N,l} = \sum_{p=0}^{N} U^N_{p,l}, \quad N \in \mathbb{N},$$

where $U^N_{p,l} \in \mathbb{R}$ is a constant, $p = \overline{0, N}$, l depends on N, $N \in \mathbb{N}$. The end states have been found in the form

$$Z^T_{N,l}(\xi) = -x \int_0^T \frac{e^{-\frac{\xi^2}{4(T-\tau)}}}{(2(T-\tau))^{3/2}} u^{110}_{N,l}(\tau) \mathrm{d}\tau, \quad \xi \in \mathbb{R}.$$

(A) $N = 2$, $l = 10$,
a $\approx -25.76502227592589$,
b $\approx 66.49038006690556$,
c $\approx -53.1923040535244$.

(B) $N = 4$, $l = 500$,
d $\approx -2085114770.47268$,
e $\approx 8449110930.700284$,
f $\approx -12841531823.85922$,
g $\approx 8676290996.825712$,
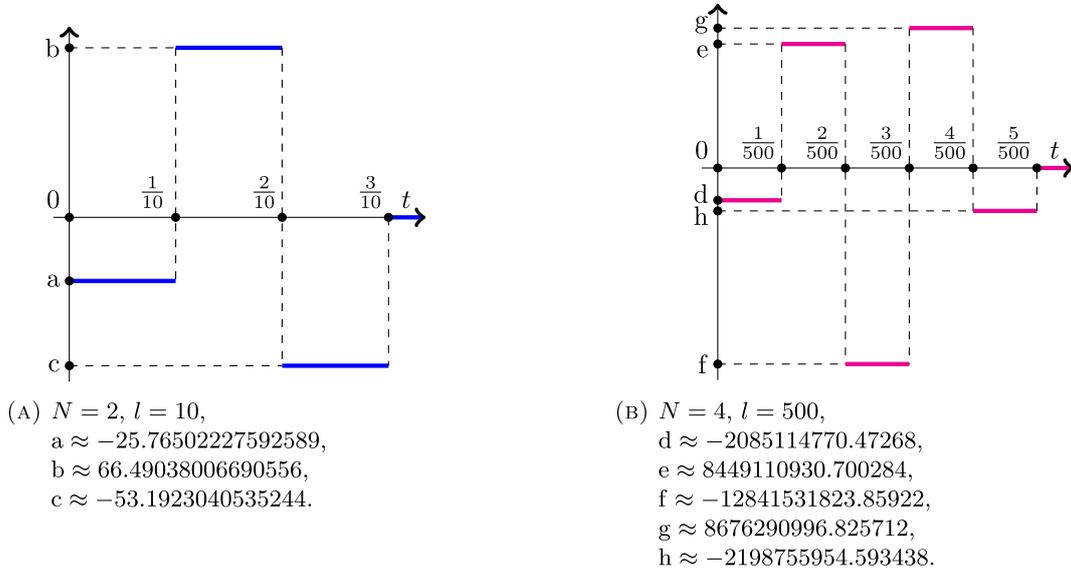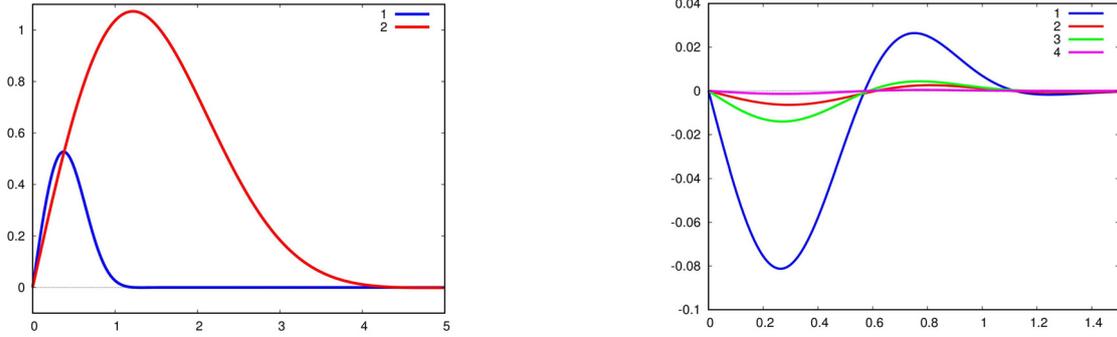h $\approx -2198755954.593438$.

FIGURE 1. The controls $u^{\rho k \gamma}_{N,l}$.

(A) ① The given function $W^T(x)$; ② The function $Z^T = \widetilde{\mathbb{T}}^{-1}W^T$.

(B) The difference $W^T - W^T_{N,l}$ in the cases: ① $N = 2$, $l = 10$; ② $N = 2$, $l = 100$; ③ $N = 4$, $l = 50$; ④ $N = 4$, $l = 500$.

FIGURE 2. The influence of the controls $u = u^{\rho k\gamma}_{N,l}$ on the end state $W^T_{N,l}$ of the solution to (2.1)–(2.3) with (4.7).

In addition, it has been proved that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \exists l \in \mathbb{N} \quad \left\| Z^T - Z^T_{N,l} \right\|^0 \le \varepsilon.$$

Applying Theorem 3.3, we conclude that controls $u^{\rho k\gamma}_{N,l} = \frac{1}{2}u^{110}_{N,l}$ solve the approximate controllability problem for the given system. Moreover, for ε, N, and l mentioned above, we have $\left[\!\left[W^T - W^T_{N,l} \right]\!\right]^0 \le C_0\varepsilon$, where

$$W^T_{N,l}(x) = \frac{1}{\sqrt{4 + x^2}} Z^T_{N,l}\left(\frac{1}{2}x(|x| + 6)\right), \quad x \in \mathbb{R},$$

and $C_0 > 0$ is the constant from estimate (3.10). The graphs of $u^{\rho k\gamma}_{N,l}$ and $W^T_{N,l}$ see in Figures 1, 2.

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