ESAIM: COCV 28 (2022) 35
https://doi.org/10.1051/cocv/2022030

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

GENERAL INDEFINITE BACKWARD STOCHASTIC LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS

Jingrui Sun[1,*], Jiaqiang Wen[1,**,***] and Jie Xiong[2,****]

Abstract. A general backward stochastic linear-quadratic optimal control problem is studied, in which both the state equation and the cost functional contain the nonhomogeneous terms. The main feature of the problem is that the weighting matrices in the cost functional are allowed to be indefinite, and the cross-product terms in the control and the state processes are present. Necessary and sufficient conditions for the solvability of the problem are obtained, and a characterization of the optimal control in terms of forward-backward stochastic differential equations is derived. By a Riccati equation approach, a general procedure for constructing optimal controls is developed and the value function is obtained explicitly.

Mathematics Subject Classification. 93E20, 49N10, 49N35.

Received September 19, 2021. Accepted April 17, 2022.

1. Introduction

Due to its wide range of applications, the theory of stochastic optimal control developed rapidly in the past few decades. Stochastic control is a natural and effective method that can solve uncertainty, interference, and ambiguity emerging in real-world control problems. As an important class of optimal control problems, the forward stochastic linear-quadratic (LQ, for short) problem has been studied by a lot of researchers (see Wonham [18] and Davis [4], and the references cited therein).

In the historical development of stochastic optimal control, the backward stochastic differential equation (BSDE, for short) plays a central role, which was introduced by Bismut [1] for the linear case and by Pardoux–Peng [10] for the nonlinear situation. Linear BSDEs serve as the adjoint equation of the state equation in the study of the maximum principle of stochastic optimal control problems (see Bismut [1] and Yong–Zhou [19]). Control problems of BSDEs are also attractive and important, not only due to the theoretical level, but also

[*] This author is supported by National Natural Science Foundation of China (Grant No. 11901280) and Guangdong Basic and Applied Basic Research Foundation (Grant No. 2021A1515010031).
[***] This author is supported by National Natural Science Foundation of China (Grant No. 12101291), Natural Science Foundation of Guangdong Province of China (Grant No. 2214050003543), and SUSTech start-up fund (Grant No. Y01286233).
[****] This author is supported partially by National Natural Science Foundation of China (Grants No. 61873325 and 11831010), and SUSTech start-up fund (Grant No. Y01286120).

Keywords and phrases: Backward stochastic differential equation, linear-quadratic, optimal control, Riccati equation, nonhomogeneous term.

[1] Department of Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China.
[2] Department of Mathematics and SUSTech International center for Mathematics, Southern University of Science and Technology, Shenzhen, Guangdong 518055, China.

[**] Corresponding author: wenjq@sustech.edu.cn

their applications in finance; see, for example, Ma–Yong [9], Pham [11], Peng [12], Zhang [20], Wen–Li–Xiong [17], and the references cited therein.

In this paper, we study a class of quadratic control problems for linear BSDEs with nonhomogeneous terms, in which the weighting matrices in the cost functional are allowed to be indefinite and cross-product terms in the control and the state processes are present. To precisely state our problem, let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $W=\{W(t);t\geqslant 0\}$ is defined, where $\mathbb{F}=\{\mathcal{F}_t\}_{t\geqslant 0}$ is usual augmentation of the natural filtration generated by W. For a random variable ξ, we write $\xi\in\mathcal{F}_t$ if ξ is $\mathcal{F}_t$-measurable; and for a stochastic process φ, we write $\varphi\in\mathbb{F}$ if it is progressively measurable with respect to $\mathbb{F}$. Consider the following controlled linear BSDE on a finite horizon $[0,T]$:

$$\begin{cases} \mathrm{d}Y(t)=[A(t)Y(t)+B(t)u(t)+C(t)Z(t)+f(t)]\mathrm{d}t+Z(t)\mathrm{d}W(t), \\ Y(T)=\xi, \end{cases} \tag{1.1}$$

where the coefficients $A,C:[0,T]\to\mathbb{R}^{n\times n}$ and $B:[0,T]\to\mathbb{R}^{n\times m}$ of the *state equation* (1.1) are given bounded deterministic functions; the nonhomogeneous term $f:[0,T]\times\Omega\to\mathbb{R}^n$ is an $\mathbb{F}$-progressively measurable process; and the terminal value ξ belongs to the space

$$L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)\triangleq\Big\{\xi:\Omega\to\mathbb{R}^n \;\Big|\; \xi\in\mathcal{F}_T \text{ and } \mathbb{E}|\xi|^2<\infty\Big\}.$$

The control process u, valued in $\mathbb{R}^m$, is taken from

$$\mathcal{U}\triangleq\Big\{u:[0,T]\times\Omega\to\mathbb{R}^m \;\Big|\; u\in\mathbb{F} \text{ and } \mathbb{E}\int_0^T|u(s)|^2\mathrm{d}s<\infty\Big\}.$$

The criterion for the performance of u is given by the following quadratic functional

$$\begin{aligned} J\big(\xi;u\big)\triangleq\mathbb{E}\Bigg\{&\langle GY(0),Y(0)\rangle+2\langle g,Y(0)\rangle \\ &+\int_0^T\Bigg[\Bigg\langle\begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t)\end{pmatrix}\begin{pmatrix} Y(t) \\ Z(t) \\ u(t)\end{pmatrix},\begin{pmatrix} Y(t) \\ Z(t) \\ u(t)\end{pmatrix}\Bigg\rangle \\ &+2\Bigg\langle\begin{pmatrix} q(t) \\ \rho_1(t) \\ \rho_2(t)\end{pmatrix},\begin{pmatrix} Y(t) \\ Z(t) \\ u(t)\end{pmatrix}\Bigg\rangle\Bigg]\mathrm{d}t\Bigg\}, \end{aligned} \tag{1.2}$$

where the superscript $\top$ denotes the transpose of a matrix; G is a symmetric $n\times n$ constant matrix; g is a constant vector in $\mathbb{R}^n$; and q, ρ_1 and ρ_2 are $\mathbb{F}$-progressively measurable processes; and

$$Q,\quad S=\begin{pmatrix} S_1 \\ S_2\end{pmatrix},\quad R=\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22}\end{pmatrix}$$

are bounded deterministic matrix-valued functions of proper dimensions over $[0,T]$ such that the blocked matrix in the cost functional is symmetric. The backward stochastic optimal control problem of interest is as follows.

Problem (BSLQ). For a given terminal state $\xi\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, find a control $u^*\in\mathcal{U}$ such that

$$J(\xi;u^*)=\inf_{u\in\mathcal{U}}J(\xi;u)\equiv V(\xi). \tag{1.3}$$

Due to the linearity of the backward state equation (1.1) and the quadratic form of the cost (1.2), we call the above problem a *backward stochastic linear-quadratic (BSLQ, for short) optimal control problem.* A process $u^* \in \mathcal{U}$ satisfying (1.3) is called an *optimal control* of Problem (BSLQ) for the terminal state ξ, and the corresponding adapted solution (Y^*, Z^*) of the state equation (1.1) is called an *optimal state process.* The function V is called the *value function* of Problem (BSLQ). When the coefficients f, g, q, ρ_1, ρ_2 vanish, we denote the corresponding Problem (BSLQ) by Problem (BSLQ)^0. The corresponding cost functional and value function are denoted by $J^0(\xi; u)$ and $V^0(\xi)$, respectively.

The BSLQ optimal control problem without nonhomogeneous terms was first studied by Lim–Zhou [7], where all the weighting matrices are assumed to be positive semidefinite and the quadratic cost functional is independent of the cross terms of (Y, Z, u). Applying a forward formulation and a limiting procedure, together with the completion-of-squares technique, a complete solution for such a BSLQ optimal control problem was obtained in [7]. A couple of follow-up works have appeared afterward; see, for instance, Huang–Wang–Xiong [5] and Wang–Wu–Xiong [15] considered BSLQ optimal control problem with partial information; Huang–Wang–Wu [6] investigated a backward mean-field linear-quadratic-Gaussian game with full and partial information; Wang–Xiao–Xiong [16] studied BSLQ optimal control problem with asymmetric information; a dynamic game of linear BSDE systems with mean-field interactions was studied in Du–Huang–Wu [3]; a thorough investigation on BSLQ optimal control problem with random coefficients was further carried out in Sun–Wang [13]; a general mean-field BSLQ optimal control problem was investigated in Li–Sun–Xiong [8]; and based on [7, 8], a theory of optimal control for controllable stochastic linear systems was developed in Bi–Sun–Xiong [2]. It is worth pointing out that, the key point of the above-mentioned works is that they assume the positive/nonnegative definiteness condition imposed on the weighting matrices, and most of their cost functionals are independent of the cross terms in (Y, Z, u) and nonhomogeneous terms are not present.

We say that the stochastic LQ optimal control is *indefinite*, if the weighting matrices in the cost functional $J(\xi, u)$, are not necessarily positive semi-definite. Not assuming the positive definiteness/ semi-definiteness on the weighting matrices will bring great challenge for solving Problem (BSLQ). Recently, Sun–Wu–Xiong [14] considered a homogeneous backward stochastic LQ optimal control problem and obtained the optimal control for the indefinite case. However, their model is not general enough due to the lack of homogeneous terms. For this reason, their results cannot directly apply to solving some related problems, especially the two-person zero-sum Stackelberg game. In this paper, we study a general indefinite BSLQ optimal control problem, in which both the state equation and the cost functional contain nonhomogeneous terms. As we shall see in Section 4, the nonhomogeneous terms bring lots of difficulties when constructing the optimal control of Problem (BSLQ). For example, we need to reconstruct the representation of the solution Z and the optimal control u in terms of X, the solution of the corresponding forward dual process. We shall first derive necessary and sufficient conditions for the existence of optimal controls, and then characterize the optimal control by means of forward-backward stochastic differential equations (FBSDEs, for short). Finally, with this characterization, we develop a general procedure for constructing the optimal control and the value function of Problem (BSLQ).

The rest of the paper is structured as follows. In Section 2, we give some notations firstly and then present a characterization of the optimal control by means of FBSDEs. In Section 3, we simplify Problem (BSLQ) and construct the optimal control under a simple situation. In Section 4, we construct the optimal control of the general Problem (BSLQ). Finally, Section 5 concludes the paper.

2. A characterization of optimal controls in terms of FBSDEs

In this section, we first introduce some notation and then give a characterization of the optimal control in terms of FBSDEs.

Let $\mathbb{R}^{n\times m}$ be the Euclidean space of $n \times m$ real matrices, equipped with the Frobenius inner product

$$\langle M, N\rangle = \mathrm{tr}(M^\top N), \quad M, N \in \mathbb{R}^{n\times m},$$

where $\operatorname{tr}\left(M^\top N\right)$ is the trace of $M^\top N$. The norm induced by the Frobenius inner product is denoted by $|\cdot|$. The identity matrix of size n is denoted by I_n. When no confusion arises, we often suppress the index n and write I instead of I_n. Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of symmetric matrices. For $\mathbb{S}^n$ -valued functions M and N, we write $M \geqslant N$ (respectively, $M > N$) if $M-N$ is positive semidefinite (respectively, positive definite) almost everywhere (with respect to the Lebesgue measure), and write $M \gg 0$ if there exists a constant $\delta > 0$ such that $M \geqslant \delta I_n$. For a subset $\mathbb{H}$ of $\mathbb{R}^{n\times m}$, we denote by $C([0,T];\mathbb{H})$ the space of continuous functions from $[0,T]$ into $\mathbb{H}$, and by $L^\infty(0,T;\mathbb{H})$ the space of Lebesgue measurable, essentially bounded functions from $[0,T]$ into $\mathbb{H}$. Besides the space $L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ introduced previously, the following spaces of stochastic processes will also be frequently used in the sequel:

$$\begin{aligned}
L^2_{\mathbb{F}}(0,T;\mathbb{H}) &= \bigg\{\varphi:[0,T]\times\Omega\to\mathbb{H} \mid \varphi \text{ is } \mathbb{F}\text{-progressively measurable and} \\
&\qquad\qquad \mathbb{E}\int_0^T |\varphi(t)|^2 \mathrm{d}t < \infty\bigg\}, \\
L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{H})) &= \bigg\{\varphi:[0,T]\times\Omega\to\mathbb{H} \mid \varphi \text{ has continuous paths, } \mathbb{F}\text{-adapted and} \\
&\qquad\qquad \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |\varphi(t)|^2\right] < \infty\bigg\}, \\
L^2_{\mathbb{F}}(\Omega;L^1([0,T];\mathbb{H})) &= \bigg\{\varphi:[0,T]\times\Omega\to\mathbb{H} \mid \varphi \text{ is } \mathbb{F}\text{-progressively measurable and} \\
&\qquad\qquad \mathbb{E}\left(\int_0^T |\varphi(t)|\mathrm{d}t\right)^2 < \infty\bigg\}.
\end{aligned}$$

For the coefficients of the state equation (1.1) and the weighting matrices of the cost functional (1.2), we impose the following conditions.

(A1) The coefficients of the state equation (1.1) satisfy

$$A \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B \in L^\infty(0,T;\mathbb{R}^{n\times m}), \quad C \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad f \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)).$$

(A2) The coefficients among the nonhomogeneous term and the weighting matrices in the cost functional (1.2) satisfy

$$\begin{aligned}
&G \in \mathbb{S}^n, \quad Q \in L^\infty(0,T;\mathbb{S}^n), \quad S \in L^\infty(0,T;\mathbb{R}^{(n+m)\times n}), \quad R \in L^\infty(0,T;\mathbb{S}^{n+m}), \\
&g \in \mathbb{R}^n, \quad q \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \quad \rho_1 \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \quad \rho_2 \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m).
\end{aligned}$$

We now present a characterization of the optimal control in terms of forward-backward stochastic differential equations, which will be used to prove the control constructed later to be optimal.

Theorem 2.1. *Let (A1) and (A2) hold and let the terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ be given. A control $u^* \in \mathcal{U}$ is optimal for ξ if and only if the following two conditions hold:*

(i) *$J^0(0;u) \geqslant 0$ for all $u \in \mathcal{U}$.*

(ii) *The adapted solution (X^*, Y^*, Z^*) to the following decoupled FBSDE*

$$\begin{cases} \mathrm{d}X^*(t) = \big[-A^\top(t)X^*(t) + Q(t)Y^*(t) + S_1^\top(t)Z^*(t) + S_2^\top(t)u^*(t) + q(t)\big]\mathrm{d}t \\ \qquad\qquad + \big[-C(t)X^*(t) + S_1(t)Y^*(t) + R_{11}(t)Z^*(t) + R_{12}(t)u^*(t) + \rho_1(t)\big]\mathrm{d}W(t), \\ \mathrm{d}Y^*(t) = \big[A(t)Y^*(t) + B(t)u^*(t) + C(t)Z^*(t) + f(t)\big]\mathrm{d}t + Z^*(t)\mathrm{d}W(t), \quad t \in [0,T], \\ X^*(0) = GY^*(0) + g, \quad Y^*(T) = \xi, \end{cases}$$

satisfies

$$S_2(t)Y^*(t) + R_{21}(t)Z^*(t) - B^\top(t)X^*(t) + R_{22}u^*(t) + \rho_2(t) = 0, \quad t \in [0,T]. \tag{2.1}$$

Proof. Note that $u^* \in \mathcal{U}$ is optimal for ξ if and only if

$$J(\xi; u^* + \varepsilon u) - J(\xi; u^*) \geqslant 0, \quad \forall u \in \mathcal{U},\ \varepsilon \in \mathbb{R}. \tag{2.2}$$

Let $u \in \mathcal{U}$ and $\varepsilon \in \mathbb{R}$ be fixed but arbitrary. Denote by (Y, Z) the adapted solution of the following BSDE,

$$\begin{cases} \mathrm{d}Y(t) = \big[A(t)Y(t) + B(t)u(t) + C(t)Z(t)\big]\mathrm{d}t + Z(t)\mathrm{d}W(t), \quad t \in [0,T], \\ Y(T) = 0, \end{cases}$$

and denote by $(Y^\varepsilon, Z^\varepsilon)$ the adapted solution of

$$\begin{cases} \mathrm{d}Y^\varepsilon(t) = \big[A(t)Y^\varepsilon(t) + B(t)[u^*(t) + \varepsilon u(t)] + C(t)Z^\varepsilon(t) + f(t)\big]\mathrm{d}t + Z^\varepsilon(t)\mathrm{d}W(t), \quad t \in [0,T], \\ Y(T) = \xi. \end{cases}$$

By the uniqueness of the adapted solution of BSDEs, it is clearly that $(Y^\varepsilon, Z^\varepsilon) = (Y^* + \varepsilon Y, Y^* + \varepsilon Z)$. Therefore,

$$\begin{aligned} &J(\xi; u^* + \varepsilon u) - J(\xi, u^*) \\ &= 2\varepsilon\mathbb{E}\Bigg\{\big\langle GY^*(0), Y(0)\big\rangle + \big\langle g, Y(0)\big\rangle \\ &\quad + \int_0^T \left[\left\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y^*(t) \\ Z^*(t) \\ u^*(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} q(t) \\ \rho_1(t) \\ \rho_2(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle \right]\mathrm{d}s \Bigg\} \\ &\quad + \varepsilon^2\mathbb{E}\Bigg\{\big\langle GY(0), Y(0)\big\rangle + \int_0^T \left[\left\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle \right. \\ &= 2\varepsilon\mathbb{E}\Bigg\{\big\langle GY^*(0) + g, Y(0)\big\rangle \\ &\quad + \int_0^T \left[\left\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y^*(t) \\ Z^*(t) \\ u^*(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} q(t) \\ \rho_1(t) \\ \rho_2(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle \right]\mathrm{d}s \Bigg\} \\ &\quad + \varepsilon^2 J^0(0; u). \end{aligned} \tag{2.3}$$

Using the integration by parts formula to $\langle X^*, Y\rangle$, we have

$$
\begin{aligned}
-\big\langle GY^*(0)+g, Y(0)\big\rangle = \mathbb{E}\int_0^T \big[&\langle QY^* + S_1^\top Z^* + S_2^\top u^* + q, Y\rangle \\
&+ \langle S_1Y^* + R_{11}Z^* + R_{12}u^* + \rho_1, Z\rangle + \langle B^\top X^*, u\rangle\big]\mathrm{d}t.
\end{aligned}
\tag{2.4}
$$

Substitute (2.4) in (2.3) yields that

$$
J(\xi; u^* + \varepsilon u) - J(\xi, u^*) = \varepsilon^2 J^0(0,u) + 2\varepsilon\mathbb{E}\int_0^T \langle S_2Y^* + R_{21}Z^* - B^\top X^* + R_{22}u^* + \rho_2, u\rangle \mathrm{d}t.
$$

From the above, it is easy to see that (2.2) holds if and only if (2.1) holds and $J^0(0,u) \geqslant 0$ for every $u \in \mathcal{U}$. $\square$

3. Construction of optimal control: simple situation

In this section, we would like to consider a simple Problem (BSLQ) and construct the optimal control under the following situation:

$$
G = 0, \qquad Q(t) = 0, \qquad R_{12}(t) = R_{21}^\top(t) = 0, \qquad \forall t \in [0,T]. \tag{3.1}
$$

Note that due to the indefiniteness of the weighting matrices, it is not easy to decide whether Problem (BSLQ) admits an optimal control for a given terminal state when $J^0(0;u)$ is merely positive (see Thm. 2.1). However, if $J^0(0;u)$ is uniformly positive, then we are able to overcome the challenge brought by the indefiniteness of the weighting matrices. So in the following, we are going to construct the optimal control of Problem (BSLQ) under the following uniform positivity condition:

(A3) There is a constant $\delta > 0$ such that

$$
J^0(0;u) \geqslant \delta\mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t, \quad \forall u \in \mathcal{U}. \tag{3.2}
$$

We observe that the uniform positivity condition (A3) implies $R_{22} \gg 0$ (see Remark 5.4 of Sun–Wu–Xiong [14]). Moreover, it is easy to see that under the condition (3.1), the initial Problem (BSLQ) is equivalent to minimizing the following cost functional

$$
\begin{aligned}
J\big(\xi; u\big) = \mathbb{E}\Bigg\{ & 2\big\langle g, Y(0)\big\rangle \\
& + \int_0^T \Bigg[\left\langle \begin{pmatrix} 0 & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & 0 \\ S_2(t) & 0 & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle \\
& + 2\left\langle \begin{pmatrix} q(t) \\ \rho_1(t) \\ \rho_2(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle \Bigg] \mathrm{d}s \Bigg\},
\end{aligned}
\tag{3.3}
$$

subject to the initial state equation (1.1). Next, in order to construct the optimal control of Problem (BSLQ), we introduce the following Riccati equation

$$\begin{cases}\dot{\Sigma}(t) - A(t)\Sigma(t) - \Sigma(t)A(t)^\top + \mathcal{B}(t,\Sigma(t))\left[R_{22}(t)\right]^{-1}\mathcal{B}(t,\Sigma(t))^\top \\ \qquad + \mathcal{C}(t,\Sigma(t))[\mathcal{R}(t,\Sigma(t))]^{-1}\Sigma(t)\mathcal{C}(t,\Sigma(t))^\top = 0, \quad t \in [0,T], \\ \Sigma(T) = 0.\end{cases} \tag{3.4}$$

where $\Sigma : [0,T] \to \mathbb{S}^n$ is an $\mathbb{S}^n$-valued function, and

$$\begin{aligned}\mathcal{B}(t,\Sigma(t)) &= B(t) + \Sigma(t)S_2(t)^\top, \\ \mathcal{C}(t,\Sigma(t)) &= C(t) + \Sigma(t)S_1(t)^\top, \\ \mathcal{R}(t,\Sigma(t)) &= I + \Sigma(t)R_{11}(t).\end{aligned} \tag{3.5}$$

Combining (A1)–(A2) and (3.5), we see that the last term in the left-hand side of Riccati equation (3.4) is symmetric.

When there is no risk for confusion, in the following for simplicity presentation, we would like to frequently suppress the argument t from our notations and write $\mathcal{B}(t,\Sigma(t))$, $\mathcal{C}(t,\Sigma(t))$ and $\mathcal{R}(t,\Sigma(t))$ as $\mathcal{B}(\Sigma)$, $\mathcal{C}(\Sigma)$, and $\mathcal{R}(\Sigma)$, respectively. For Riccati equation (3.4), we have the following result concerning the existence and uniqueness, which essentially is Theorem 6.2 of Sun–Wu–Xiong [14].

Proposition 3.1. *Let (A1)–(A3) and (3.1) hold. Then the Riccati equation (3.4) admits a unique positive semidefinite solution $\Sigma \in C([0,T];\mathbb{S}^n)$ such that $\mathcal{R}(\Sigma)$ is invertible a.e. on $[0,T]$ and $\mathcal{R}(\Sigma)^{-1} \in L^\infty(0,T;\mathbb{R}^n)$.*

With the solution Σ to the Riccati equation (3.4), before constructing the optimal control of Problem (BSLQ), we further introduce the following linear BSDE:

$$\begin{cases}\mathrm{d}\varphi(t) = \alpha(t,\Sigma(t))\mathrm{d}t + \beta(t)\mathrm{d}W(t), \quad t \in [0,T], \\ \varphi(T) = \xi,\end{cases} \tag{3.6}$$

where

$$\begin{aligned}\alpha(\Sigma) = &\Big\{ \left[A - \mathcal{B}(\Sigma)R_{22}^{-1}S_2 - \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1\right]\varphi + \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta \\ &- C\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - \Sigma S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - BR_{22}^{-1}\rho_2 - \Sigma S_2^\top R_{22}^{-1}\rho_2 + \Sigma q + f\Big\} \\ = &\Big\{ \left[A - \mathcal{B}(\Sigma)R_{22}^{-1}S_2 - \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1\right]\varphi + \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta \\ &- \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - \mathcal{B}(\Sigma)R_{22}^{-1}\rho_2 + \Sigma q + f\Big\}.\end{aligned} \tag{3.7}$$

In terms of the solution Σ to the Riccati equation (3.4) and the adapted solution (φ,β) to the BSDE (3.6), we now can construct the optimal control of Problem (BSLQ) as follows.

Theorem 3.2. *Let (A1)–(A3) and (3.1) hold. Let (φ,β) be the adapted solution to the BSDE (3.6) and X the solution to the following SDE:*

$$\begin{cases} \mathrm{d}X(t) = \Big\{\big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top - A^\top\big]X \\ \qquad\qquad - \big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi + S_1^\top\mathcal{R}(\Sigma)^{-1}\beta - S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - S_2^\top R_{22}^{-1}\rho_2 + q\Big\}\mathrm{d}t \\ \qquad\qquad - \big[\mathcal{R}(\Sigma)^{-1}\big]^\top\big[\mathcal{C}(\Sigma)^\top X - S_1\varphi - R_{11}\beta - \rho_1\big]\mathrm{d}W(t), \\ X(0) = g. \end{cases} \tag{3.8}$$

Then the optimal control of Problem (BSLQ) for the terminal state ξ is given by

$$u(t) = [R_{22}(t)]^{-1}\big[\mathcal{B}(t,\Sigma(t))^\top X(t) - S_2(t)\varphi(t) - \rho_2(t)\big], \quad t\in[0,T]. \tag{3.9}$$

Proof. Let us define for $t\in[0,T]$,

$$Y(t) = -\Sigma(t)X(t) + \varphi(t), \tag{3.10}$$

$$Z(t) = \mathcal{R}(t,\Sigma(t))^{-1}\big[\Sigma(t)\mathcal{C}(t,\Sigma(t))^\top X(t) - \Sigma(t)S_1(t)\varphi(t) - \Sigma(t)\rho_1(t) + \beta(t)\big]. \tag{3.11}$$

We observe that

$$\begin{aligned} R_{22}u &= \mathcal{B}(\Sigma)^\top X - S_2\varphi - \rho_2 \\ &= (B + \Sigma S_2^\top)^\top X - S_2\varphi - \rho_2 \\ &= B^\top X + S_2(\Sigma X - \varphi) - \rho_2 \\ &= B^\top X - S_2 Y - \rho_2. \end{aligned} \tag{3.12}$$

Furthermore, using (3.9) and (3.11) we obtain

$$\begin{aligned} & S_1^\top Z + S_2^\top u \\ &= S_1^\top\mathcal{R}(\Sigma)^{-1}\big[\Sigma\mathcal{C}(\Sigma)^\top X - \Sigma S_1\varphi - \Sigma\rho_1 + \beta\big] + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top X - S_2^\top R_{22}^{-1}S_2\varphi - S_2^\top R_{22}^{-1}\rho_2 \\ &= \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X - \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi \\ &\quad + S_1^\top\mathcal{R}(\Sigma)^{-1}\beta - S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - S_2^\top R_{22}^{-1}\rho_2, \end{aligned}$$

from which it implies that

$$\begin{aligned} -A^\top X + S_1^\top Z + S_2^\top u + q = &\big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top - A^\top\big]X \\ &- \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi + S_1^\top\mathcal{R}(\Sigma)^{-1}\beta \\ &- S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - S_2^\top R_{22}^{-1}\rho_2 + q. \end{aligned} \tag{3.13}$$

Similarly, using (3.10) and (3.11) we obtain

$$
\begin{aligned}
-C^\top X + S_1 Y + R_{11} Z + \rho_1 &= -\big(C^\top + S_1\Sigma\big)X + S_1\varphi + R_{11}Z + \rho_1 \\
&= \big[R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma - I\big]\,\mathcal{C}(\Sigma)^\top X + \big[I - R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma\big]\,S_1\varphi \\
&\quad + R_{11}\mathcal{R}(\Sigma)^{-1}\beta + (I - R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma)\rho_1.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&R_{11}\mathcal{R}(\Sigma)^{-1} = R_{11}\left(I + \Sigma R_{11}\right)^{-1} = \left(I + R_{11}\Sigma\right)^{-1} R_{11}, \\
&I - R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma = (I + R_{11}\Sigma)^{-1} = \left[\mathcal{R}(\Sigma)^{-1}\right]^\top,
\end{aligned}
$$

we further obtain that

$$
-C^\top X + S_1 Y + R_{11} Z + \rho_1 = -\big[\mathcal{R}(\Sigma)^{-1}\big]^\top \big[\mathcal{C}(\Sigma)^\top X - S_1\varphi - R_{11}\beta - \rho_1\big]. \tag{3.14}
$$

This implies that the solution of (3.8) satisfies the following equation

$$
\begin{cases}
\mathrm{d}X(t) = \big(-A^\top X + S_1^\top Z + S_2^\top u + q\big)\mathrm{d}t + \big(-C^\top X + S_1 Y + R_{11} Z + \rho_1\big)\mathrm{d}W, \\
X(0) = g.
\end{cases} \tag{3.15}
$$

Applying Itô's formula to (3.10), we have

$$
\begin{aligned}
\mathrm{d}Y &= -\dot{\Sigma}X\mathrm{d}t - \Sigma\mathrm{d}X + d\varphi \\
&= \big[\alpha(\Sigma) - \dot{\Sigma}X - \Sigma\big(-A^\top X + S_1^\top Z + S_2^\top u + q\big)\big]\mathrm{d}t \\
&\quad + \big[\beta - \Sigma\big(-C^\top X + S_1 Y + R_{11} Z + \rho_1\big)\big]\mathrm{d}W.
\end{aligned}
$$

Using (3.13) and (3.7), and note that Σ satisfies Riccati equation (3.4), we obtain

$$
\begin{aligned}
&\alpha(\Sigma) - \dot{\Sigma}X - \Sigma\big(-A^\top X + S_1^\top Z + S_2^\top u + q\big) \\
&= \alpha(\Sigma) - \big[\dot{\Sigma} - \Sigma A^\top + \Sigma S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + \Sigma S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X \\
&\quad + \Sigma\big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1} S_2\big]\varphi - \Sigma S_1^\top \mathcal{R}(\Sigma)^{-1}\beta \\
&\quad + \Sigma S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 + \Sigma S_2^\top R_{22}^{-1}\rho_2 - \Sigma q \\
&= \alpha(\Sigma) - \big[A\Sigma - C\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top - BR_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X \\
&\quad + \Sigma\big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1} S_2\big]\varphi - \Sigma S_1^\top \mathcal{R}(\Sigma)^{-1}\beta \\
&\quad + \Sigma S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 + \Sigma S_2^\top R_{22}^{-1}\rho_2 - \Sigma q \\
&= AY + \big[C\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + BR_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X - BR_{22}^{-1}S_2\varphi \\
&\quad - C\mathcal{R}(\Sigma)^{-1}(\Sigma S_1\varphi - \beta) - C\mathcal{R}(\Sigma)^{-1}\Sigma(t)\rho_1(t) - BR_{22}^{-1}\rho_2 + f \\
&= AY + BR_{22}^{-1}\big[\mathcal{B}(\Sigma)^\top X - S_2\varphi - \rho_2\big] + C\mathcal{R}(\Sigma)^{-1}\big[\Sigma\mathcal{C}(\Sigma)^\top X - \Sigma S_1\varphi - \Sigma(t)\rho_1(t) + \beta\big] + f \\
&= AY + Bu + CZ + f.
\end{aligned}
$$

Similarly, using (3.14) and the following relation

$$
\begin{aligned}
\Sigma\left[\mathcal{R}(\Sigma)^{-1}\right]^{\top} &= \Sigma\left(I+R_{11}\Sigma\right)^{-1} = \left(I+\Sigma R_{11}\right)^{-1}\Sigma = \mathcal{R}(\Sigma)^{-1}\Sigma,\\
I-\mathcal{R}(\Sigma)^{-1}\Sigma R_{11} &= I-\left(I+\Sigma R_{11}\right)^{-1}\Sigma R_{11} = \left(I+\Sigma R_{11}\right)^{-1} = \mathcal{R}(\Sigma)^{-1},
\end{aligned}
$$

we have that

$$
\begin{aligned}
&\beta-\Sigma\big(-C^{\top}X+S_1Y+R_{11}Z+\rho_1\big)\\
&=\beta+\Sigma\left[\mathcal{R}(\Sigma)^{-1}\right]^{\top}\left[\mathcal{C}(\Sigma)^{\top}X-S_1\varphi-R_{11}\beta-\rho_1\right]\\
&=\mathcal{R}(\Sigma)^{-1}\left[\Sigma\mathcal{C}(\Sigma)^{\top}X-\Sigma S_1\varphi-\Sigma\rho_1\right]+\left[I-\mathcal{R}(\Sigma)^{-1}\Sigma R_{11}\right]\beta\\
&=\mathcal{R}(\Sigma)^{-1}\left[\Sigma\mathcal{C}(\Sigma)^{\top}X-\Sigma S_1\varphi-\Sigma\rho_1+\beta\right]\\
&=Z.
\end{aligned}
$$

Therefore, the pair (Y,Z) defined by (3.10) and (3.11) satisfies the following BSDE:

$$
\begin{cases}
\mathrm{d}Y(t)=(AY+Bu+CZ+f)\mathrm{d}t+Z\mathrm{d}W,\\
Y(T)=\xi.
\end{cases}
\tag{3.16}
$$

Combining (3.15) and (3.16), we see that the solution X of (3.8), the pair (Y,Z) defined by (3.10) and (3.11) satisfy the following FBSDE

$$
\begin{cases}
\mathrm{d}X(t)=(-A^{\top}X+S_1^{\top}Z+S_2^{\top}u+q)\mathrm{d}t+(-C^{\top}X+S_1Y+R_{11}Z+\rho_1)\mathrm{d}W,\\
\mathrm{d}Y(t)=(AY+Bu+CZ+f)\mathrm{d}t+Z\mathrm{d}W,\\
X(0)=g,\quad Y(T)=\xi.
\end{cases}
\tag{3.17}
$$

In addition, combining (3.12), we have that the control u defined by (3.9) satisfies the following condition

$$
S_2Y-B^{\top}X+R_{22}u+\rho_2=0.
\tag{3.18}
$$

Therefore, from Theorem 2.1, we obtain that u is the (unique) optimal control for the terminal state ξ. □

Based on the above result, we conclude this section with a representation of the value function $V(\xi)$.

Theorem 3.3. *Let (A1)–(A3) and (3.1) hold. Then the value function of Problem (BSLQ) is given by*

$$
\begin{aligned}
V(\xi)=\mathbb{E}\bigg\{&2\langle\varphi(0),g\rangle-\langle\Sigma(0)g,g\rangle\\
&+\int_0^T\Big[-\big\langle\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1,\rho_1\big\rangle-\big\langle R_{22}^{-1}\rho_2,\rho_2\big\rangle+2\big\langle\mathcal{R}(\Sigma)^{-1}\beta,\rho_1\big\rangle\\
&+\big\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta,\beta\big\rangle+2\big\langle S_1^{\top}\mathcal{R}(\Sigma)^{-1}\beta-S_1^{\top}\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-S_2^{\top}R_{22}^{-1}\rho_2+q,\varphi\big\rangle\\
&-\big\langle[S_1^{\top}\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^{\top}R_{22}^{-1}S_2]\varphi,\varphi\big\rangle\Big]\mathrm{d}t\bigg\},
\end{aligned}
\tag{3.19}
$$

where (φ,β) is the adapted solution of BSDE (3.6).

Proof. Let u be the optimal control for the terminal state ξ. Then, by Theorem 2.1, the adapted solution (X,Y,Z) of (3.17) satisfies (3.18). By the definition, we observe that

$$
\begin{aligned}
V(\xi)=J\big(\xi;u\big)=\mathbb{E}\Bigg\{&2\big\langle g,Y(0)\big\rangle\\
&+\int_0^T\Bigg[\Bigg\langle\begin{pmatrix}0 & S_1^\top(t) & S_2^\top(t)\\ S_1(t) & R_{11}(t) & 0\\ S_2(t) & 0 & R_{22}(t)\end{pmatrix}\begin{pmatrix}Y(t)\\ Z(t)\\ u(t)\end{pmatrix},\begin{pmatrix}Y(t)\\ Z(t)\\ u(t)\end{pmatrix}\Bigg\rangle+2\Bigg\langle\begin{pmatrix}q(t)\\ \rho_1(t)\\ \rho_2(t)\end{pmatrix},\begin{pmatrix}Y(t)\\ Z(t)\\ u(t)\end{pmatrix}\Bigg\rangle\Bigg]\mathrm{d}t\Bigg\}\\
=\mathbb{E}\Bigg\{&2\big\langle g,Y(0)\big\rangle+\int_0^T\big[2\left\langle S_1Y,Z\right\rangle+2\left\langle S_2Y,u\right\rangle+\left\langle R_{11}Z,Z\right\rangle+\left\langle R_{22}u,u\right\rangle\\
&+2\left\langle q,Y\right\rangle+2\left\langle \rho_1,Z\right\rangle+2\left\langle \rho_2,u\right\rangle\big]\mathrm{d}t\Bigg\}\\
=\mathbb{E}\Bigg\{&2\big\langle g,Y(0)\big\rangle+\int_0^T\big[\langle S_1^\top Z+S_2^\top u,Y\rangle+\langle S_1Y+R_{11}Z,Z\rangle+\langle S_2Y+R_{22}u,u\rangle\\
&+2\left\langle q,Y\right\rangle+2\langle\rho_1,Z\rangle+2\langle\rho_2,u\rangle\big]\mathrm{d}t\Bigg\}\\
=\mathbb{E}\Bigg\{&\big\langle 2g,Y(0)\big\rangle+\int_0^T\big[\langle S_1^\top Z+S_2^\top u+2q,Y\rangle+\langle S_1Y+R_{11}Z+2\rho_1,Z\rangle+\langle B^\top X+\rho_2,u\rangle\big]\mathrm{d}t\Bigg\}.
\end{aligned}
$$

Take the integration by parts formula to $\langle X,Y\rangle$ implies that

$$
\begin{aligned}
&\mathbb{E}\langle X(T),Y(T)\rangle\\
&=\mathbb{E}\langle g,Y(0)\rangle+\mathbb{E}\int_0^T\big[\langle X,AY+Bu+CZ+f\rangle+\langle -A^\top X+S_1^\top Z+S_2^\top u+q,Y\rangle\\
&\quad+\langle -C^\top X+S_1Y+R_{11}Z+\rho_1,Z\rangle\big]\mathrm{d}t\\
&=\mathbb{E}\langle g,Y(0)\rangle+\mathbb{E}\int_0^T\big[\langle X,Bu+f\rangle+\langle S_1^\top Z+S_2^\top u+q,Y\rangle+\langle S_1Y+R_{11}Z+\rho_1,Z\rangle\big]\mathrm{d}t\\
&=V(\xi)-\mathbb{E}\langle g,Y(0)\rangle-\mathbb{E}\int_0^T\big[-\langle X,f\rangle+\langle q,Y\rangle+\langle\rho_1,Z\rangle+\langle\rho_2,u\rangle\big]\mathrm{d}t.
\end{aligned}
$$

In other words, we have

$$
V(\xi)=\mathbb{E}\langle X(T),Y(T)\rangle+\mathbb{E}\langle g,Y(0)\rangle+\mathbb{E}\int_0^T\big[-\langle X,f\rangle+\langle q,Y\rangle+\langle\rho_1,Z\rangle+\langle\rho_2,u\rangle\big]\mathrm{d}t.
$$

From Theorem 3.2, we see that X also satisfies the equation (3.8) and φ satisfies the equation (3.6). Applying the integration by parts formula to $\langle X,\varphi\rangle$, we obtain that

$$
\begin{aligned}
&\mathbb{E}\langle X(T),\varphi(T)\rangle-\mathbb{E}\langle g,\varphi(0)\rangle\\
&=\mathbb{E}\int_0^T\big\{\big\langle[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top+S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top-A^\top]X,\varphi\big\rangle\\
&\quad-\big\langle[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2]\varphi,\varphi\big\rangle+\langle S_1^\top\mathcal{R}(\Sigma)^{-1}\beta-S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-S_2^\top R_{22}^{-1}\rho_2+q,\varphi\rangle\\
&\quad+\big\langle X,[A-\mathcal{B}(\Sigma)R_{22}^{-1}S_2-\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1]\varphi\big\rangle\\
&\quad+\big\langle X,\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta-\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-\mathcal{B}(\Sigma)R_{22}^{-1}\rho_2+\Sigma q+f\big\rangle\\
&\quad-\big\langle[\mathcal{R}(\Sigma)^{-1}]^\top[\mathcal{C}(\Sigma)^\top X-S_1\varphi-R_{11}\beta-\rho_1],\beta\big\rangle\big\}\,\mathrm{d}t\\
&=\mathbb{E}\int_0^T\big\{\big\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta+\mathcal{R}(\Sigma)^{-1}\rho_1,\beta\big\rangle+\big\langle 2S_1^\top\mathcal{R}(\Sigma)^{-1}\beta-S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-S_2^\top R_{22}^{-1}\rho_2+q,\varphi\big\rangle\\
&\quad+\big\langle X,-\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-\mathcal{B}(\Sigma)R_{22}^{-1}\rho_2+\Sigma q+f\big\rangle-\big\langle[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2]\varphi,\varphi\big\rangle\big\}\,\mathrm{d}t.
\end{aligned}
$$

Note that

$$\mathbb{E}\langle X(T),Y(T)\rangle=\mathbb{E}\langle X(T),\xi\rangle=\mathbb{E}\langle X(T),\varphi(T)\rangle,$$

then we have

$$
\begin{aligned}
V(\xi)&=\mathbb{E}\langle X(T),\varphi(T)\rangle+\mathbb{E}\langle g,Y(0)\rangle+\mathbb{E}\int_0^T\big[-\langle X,f\rangle+\langle q,Y\rangle+\langle\rho_1,Z\rangle+\langle\rho_2,u\rangle\big]\mathrm{d}t\\
&=\mathbb{E}\langle g,\varphi(0)\rangle+\mathbb{E}\langle g,Y(0)\rangle+\mathbb{E}\int_0^T\big[-\langle X,f\rangle+\langle q,Y\rangle+\langle\rho_1,Z\rangle+\langle\rho_2,u\rangle\big]\mathrm{d}t\\
&\quad+\mathbb{E}\int_0^T\Big\{\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta+\mathcal{R}(\Sigma)^{-1}\rho_1,\beta\rangle\\
&\quad+\langle 2S_1^\top\mathcal{R}(\Sigma)^{-1}\beta-S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-S_2^\top R_{22}^{-1}\rho_2+q,\varphi\rangle\\
&\quad+\langle X,-\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1-\mathcal{B}(\Sigma)R_{22}^{-1}\rho_2+\Sigma q+f\rangle\\
&\quad-\big\langle[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2]\varphi,\varphi\big\rangle\Big\}\mathrm{d}t.
\end{aligned}
$$

From Theorem 3.2, we see that Y, Z and u have the following representation

$$
\begin{aligned}
Y(t)&=-\Sigma X+\varphi,\\
Z(t)&=\mathcal{R}(\Sigma)^{-1}\big[\Sigma\mathcal{C}(\Sigma)^\top X-\Sigma S_1\varphi-\Sigma\rho_1+\beta\big],\\
u(t)&=R_{22}^{-1}\big[\mathcal{B}(\Sigma)^\top X-S_2\varphi-\rho_2\big].
\end{aligned}
$$

Moreover, note that

$$
\begin{aligned}
\langle X,\Sigma q\rangle&=\langle\Sigma X-\varphi,q\rangle+\langle\varphi,q\rangle=-\langle Y,q\rangle+\langle\varphi,q\rangle,\\
\big\langle X,\mathcal{B}(\Sigma)R_{22}^{-1}\rho_2\big\rangle&=\big\langle R_{22}^{-1}\left[\mathcal{B}(\Sigma)^\top X-S_2\varphi-\rho_2\right],\rho_2\big\rangle+\big\langle R_{22}^{-1}\left(S_2\varphi+\rho_2\right),\rho_2\big\rangle\\
&=\langle u,\rho_2\rangle+\big\langle R_{22}^{-1}\left[S_2\varphi+\rho_2\right],\rho_2\big\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\left\langle X, \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1\right\rangle &= \left\langle \mathcal{R}(\Sigma)^{-1}\left[\Sigma\mathcal{C}(\Sigma)^\top X - \Sigma S_1\varphi - \Sigma\rho_1 + \beta\right], \rho_1\right\rangle \\
&\quad + \left\langle \mathcal{R}(\Sigma)^{-1}\left[\Sigma S_1\varphi + \Sigma\rho_1 - \beta\right], \rho_1\right\rangle \\
&= \langle Z, \rho_1\rangle + \left\langle \mathcal{R}(\Sigma)^{-1}\left[\Sigma S_1\varphi + \Sigma\rho_1 - \beta\right], \rho_1\right\rangle .
\end{aligned}
$$

Then we obtain that

$$
\begin{aligned}
V(\xi) &= 2\mathbb{E}\langle g, \varphi(0)\rangle - \mathbb{E}\langle \Sigma(0)g, g\rangle \\
&\quad + \mathbb{E}\int_0^T \Big\{ \langle \varphi, q\rangle - \left\langle R_{22}^{-1}\left[S_2\varphi + \rho_2\right], \rho_2\right\rangle - \left\langle \mathcal{R}(\Sigma)^{-1}\left[\Sigma S_1\varphi + \Sigma\rho_1 - \beta\right], \rho_1\right\rangle \Big\}\mathrm{d}t \\
&\quad + \mathbb{E}\int_0^T \Big\{ \left\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta + \mathcal{R}(\Sigma)^{-1}\rho_1, \beta\right\rangle \\
&\quad + \left\langle 2S_1^\top\mathcal{R}(\Sigma)^{-1}\beta - S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - S_2^\top R_{22}^{-1}\rho_2 + q, \varphi\right\rangle \\
&\quad - \left\langle [S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2]\varphi, \varphi\right\rangle \Big\}\mathrm{d}t \\
&= \mathbb{E}\Big\{ 2\langle\varphi(0), g\rangle - \langle\Sigma(0)g, g\rangle + \int_0^T \Big\{ -\left\langle \mathcal{R}(\Sigma)^{-1}\Sigma\rho_1, \rho_1\right\rangle - \left\langle R_{22}^{-1}\rho_2, \rho_2\right\rangle + 2\left\langle \mathcal{R}(\Sigma)^{-1}\beta, \rho_1\right\rangle \\
&\quad + \left\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta, \beta\right\rangle + 2\left\langle S_1^\top\mathcal{R}(\Sigma)^{-1}\beta - S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\rho_1 - S_2^\top R_{22}^{-1}\rho_2 + q, \varphi\right\rangle \\
&\quad - \left\langle [S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2]\varphi, \varphi\right\rangle \Big\}\mathrm{d}t \Big\}.
\end{aligned}
$$

This completes the proof. $\square$

4. Construction of optimal control: general situation

In this section, we construct the optimal control of general Problem (BSLQ) in the situation that without the condition (3.1). For simplicity presentation, we denote

$$
\begin{aligned}
&\mathscr{S}_1 = S_1 - R_{12}R_{22}^{-1}S_2, \quad \mathscr{R}_{11} = R_{11} - R_{12}R_{22}^{-1}R_{21}, \\
&\mathscr{C} = C - BR_{22}^{-1}R_{21}, \qquad v = u + R_{22}^{-1}R_{21}Z, \\
&\widetilde{\rho}_1 = \rho_1 - \rho_2 R_{22}^{-1}R_{21}.
\end{aligned} \tag{4.1}
$$

Using the notations (4.1) and noting that $R_{22} \gg 0$, it is easy to check that the original Problem (BSLQ) is equivalent to the following backward stochastic LQ problem with the state equation

$$
\begin{cases}
\mathrm{d}Y(t) = [A(t)Y(t) + B(t)v(t) + \mathscr{C}(t)Z(t) + f(t)]\mathrm{d}t + Z(t)\mathrm{d}W(t), \qquad t \in [0, T], \\
Y(T) = \xi,
\end{cases} \tag{4.2}
$$

and the cost functional

$$
\begin{aligned}
J\big(\xi;u\big) \triangleq \mathbb{E}\Bigg\{ & \big\langle GY(0),Y(0)\big\rangle + 2\big\langle g,Y(0)\big\rangle \\
& + \int_0^T \Bigg[\Bigg\langle \begin{pmatrix} Q(t) & \mathscr{S}_1^\top(t) & S_2^\top(t) \\ \mathscr{S}_1(t) & \mathscr{R}_{11}(t) & 0 \\ S_2(t) & 0 & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ v(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ v(t) \end{pmatrix} \Bigg\rangle \\
& + 2 \Bigg\langle \begin{pmatrix} q(t) \\ \widetilde{\rho}_1(t) \\ \rho_2(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ v(t) \end{pmatrix} \Bigg\rangle \Bigg] \mathrm{d}t \Bigg\}.
\end{aligned} \tag{4.3}
$$

Moreover, we let $H \in C([0,T];\mathbb{S}^n)$ be the unique solution of the following linear ordinary differential equation

$$
\begin{cases} \dot{H}(t) + H(t)A(t) + A(t)^\top H(t) + Q(t) = 0, \quad t \in [0,T], \\ H(0) = -G. \end{cases}
$$

Applying the integration by parts formula to $\langle HY, Y\rangle$ on $[0,T]$, where Y is the state process determined by (4.2), we have that

$$
\begin{aligned}
& \mathbb{E}\langle H(T)\xi,\xi\rangle + \mathbb{E}\langle GY(0),Y(0)\rangle \\
& = \mathbb{E}\int_0^T \Big[\big\langle (\dot{H} + HA + A^\top H)Y, Y\big\rangle + 2\langle B^\top HY, v\rangle + 2\langle \mathscr{C}^\top HY, Z\rangle + 2\langle HY, f\rangle + \langle HZ, Z\rangle \Big] \mathrm{d}t \\
& = \mathbb{E}\int_0^T \big[-\langle QY,Y\rangle + 2\langle B^\top HY, v\rangle + 2\langle \mathscr{C}^\top HY, Z\rangle + 2\langle HY, f\rangle + \langle HZ, Z\rangle \big] \mathrm{d}t \\
& = \mathbb{E}\int_0^T \Bigg\{ \Bigg\langle \begin{pmatrix} -Q & H\mathscr{C} & HB \\ \mathscr{C}^\top H & H & 0 \\ B^\top H & 0 & 0 \end{pmatrix} \begin{pmatrix} Y \\ Z \\ v \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \Bigg\rangle + 2\langle HY, f\rangle \Bigg\} \mathrm{d}t.
\end{aligned}
$$

Substituting for the term $\mathbb{E}\langle GY(0),Y(0)\rangle$ in the cost functional (4.3) yields that

$$
\begin{aligned}
J\big(\xi;u\big) = \mathbb{E}\Bigg\{ & -\langle H(T)\xi,\xi\rangle + 2\big\langle g, Y(0)\big\rangle \\
& + \int_0^T \Bigg[\Bigg\langle \begin{pmatrix} 0 & (\mathcal{S}_1^H)^\top & (\mathcal{S}_2^H)^\top \\ \mathcal{S}_1^H & \mathcal{R}_{11}^H & 0 \\ \mathcal{S}_2^H & 0 & R_{22} \end{pmatrix} \begin{pmatrix} Y \\ Z \\ v \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \Bigg\rangle + 2 \Bigg\langle \begin{pmatrix} q^H \\ \widetilde{\rho}_1 \\ \rho_2 \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \Bigg\rangle \Bigg] \mathrm{d}s \Bigg\},
\end{aligned}
$$

where

$$
\mathcal{S}_1^H = \mathscr{S}_1 + \mathscr{C}^\top H, \quad \mathcal{S}_2^H = S_2 + B^\top H, \quad \mathcal{R}_{11}^H = \mathscr{R}_{11} + H, \quad q^H = q + Hf.
$$

So, for a given terminal state ξ, minimizing $J(\xi;u)$ subject to (1.1) is equivalent to minimizing the following cost functional

$$
J\big(\xi;u\big) = \mathbb{E}\Big\{ 2\big\langle g, Y(0)\big\rangle
$$

$$+\int_0^T\left[\left\langle\begin{pmatrix}0 & (\mathcal{S}_1^H)^\top & (\mathcal{S}_2^H)^\top\\ \mathcal{S}_1^H & \mathcal{R}_{11}^H & 0\\ \mathcal{S}_2^H & 0 & R_{22}\end{pmatrix}\begin{pmatrix}Y\\ Z\\ v\end{pmatrix},\begin{pmatrix}Y\\ Z\\ v\end{pmatrix}\right\rangle+2\left\langle\begin{pmatrix}q^H\\ \widetilde{\rho}_1\\ \rho_2\end{pmatrix},\begin{pmatrix}Y\\ Z\\ v\end{pmatrix}\right\rangle\right]\mathrm{d}s\Bigg\},$$

subject to the state equation (4.2), which in fact essentially degenerated to the simplified Problem (BSLQ) with the condition (3.1). Therefore, we shall only present the general result, as the proof can be easily given using the same argument in Section 3.

Theorem 4.1. *Under the conditions (A1)–(A3), the following results hold.*

(i) *Let (φ,β) be the adapted solution to the following BSDE*

$$\begin{cases}\mathrm{d}\varphi(t)=\Big\{\big[A-\mathcal{B}^H(\Sigma)R_{22}^{-1}S_2^H-\mathcal{C}^H(\Sigma)[\mathcal{R}^H(\Sigma)]^{-1}\Sigma S_1^H\big]\varphi+\mathcal{C}^H(\Sigma)[\mathcal{R}^H(\Sigma)]^{-1}\beta\\ \qquad\qquad -\mathcal{C}^H(\Sigma)[\mathcal{R}^H(\Sigma)]^{-1}\Sigma\widetilde{\rho}_1-\mathcal{B}^H(\Sigma)R_{22}^{-1}\rho_2+\Sigma q^H+f\Big\}\mathrm{d}t+\beta(t)\mathrm{d}W(t),\quad t\in[0,T],\\ \varphi(T)=\xi,\end{cases}\tag{4.4}$$

and let X the solution to the following SDE

$$\begin{cases}\mathrm{d}X(t)=\Big\{\big[(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma[\mathcal{C}^H(\Sigma)]^\top+(S_2^H)^\top R_{22}^{-1}[\mathcal{B}^H(\Sigma)]^\top-A^\top\big]X\\ \qquad\qquad -\big[(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma S_1^H+(S_2^H)^\top R_{22}^{-1}S_2^H\big]\varphi+(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\beta\\ \qquad\qquad -(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma\widetilde{\rho}_1-(S_2^H)^\top R_{22}^{-1}\rho_2+q^H\Big\}\mathrm{d}t\\ \qquad\qquad -\big[\mathcal{R}^H(\Sigma)^{-1}\big]^\top\big[[\mathcal{C}^H(\Sigma)]^\top X-S_1^H\varphi-R_{11}\beta-\widetilde{\rho}_1\big]\mathrm{d}W(t),\quad t\in[0,T],\\ X(0)=g.\end{cases}$$

Then the optimal control of Problem (BSLQ) for the terminal state ξ is given by

$$\begin{aligned}u=R_{22}^{-1}\Big\{&\big[[\mathcal{B}^H(\Sigma)]^\top-R_{21}[\mathcal{R}^H(\Sigma)]^{-1}\Sigma[\mathcal{C}^H(\Sigma)]^\top\big]X\\ &+\big[R_{21}[\mathcal{R}^H(\Sigma)]^{-1}\Sigma S_1^H-S_2^H\big]\varphi+R_{21}[\mathcal{R}^H(\Sigma)]^{-1}\Sigma\widetilde{\rho}_1-R_{21}[\mathcal{R}^H(\Sigma)]^{-1}\beta-\rho_2\Big\},\end{aligned}$$

where Σ is the unique positive semidefinite solution of the following Riccati equation,

$$\begin{cases}\dot{\Sigma}-A\Sigma-\Sigma A^\top+\mathcal{B}^H(\Sigma)R_{22}^{-1}\left[\mathcal{B}^H(\Sigma)\right]^\top+\mathcal{C}^H(\Sigma)\left[\mathcal{R}^H(\Sigma)\right]^{-1}\Sigma\left[\mathcal{C}^H(\Sigma)\right]^\top=0,\\ \Sigma(T)=0.\end{cases}$$

(ii) *The value function of Problem (BSLQ) is given by*

$$
\begin{aligned}
V(\xi) = \mathbb{E}\bigg\{ & -\langle H(T)\xi, \xi\rangle + 2\langle \varphi(0), g\rangle - \langle \Sigma(0)g, g\rangle \\
& + \int_0^T \Big\{ -\big\langle [\mathcal{R}^H(\Sigma)]^{-1}\Sigma\widetilde{\rho}_1, \widetilde{\rho}_1\big\rangle - \big\langle R_{22}^{-1}\rho_2, \rho_2\big\rangle \\
& + 2\big\langle \mathcal{R}(\Sigma)^{-1}\beta, \widetilde{\rho}_1\big\rangle + \big\langle R_{11}[\mathcal{R}^H(\Sigma)]^{-1}\beta, \beta\big\rangle \\
& + 2\big\langle (S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\beta - (S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma\widetilde{\rho}_1 - (S_2^H)^\top R_{22}^{-1}\rho_2 + q^H, \varphi\big\rangle \\
& - \big\langle \big[(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma S_1^H + (S_2^H)^\top R_{22}^{-1}S_2^H\big]\varphi, \varphi\big\rangle\Big\}\mathrm{d}t\bigg\},
\end{aligned}
$$

where (φ, β) *is the adapted solution of BSDE* (4.4).

5. Conclusion

In this paper, we have investigated an indefinite backward stochastic linear-quadratic optimal control problem with deterministic nonhomogeneous coefficients and have developed a general procedure for constructing optimal controls. The necessary and sufficient conditions of Problem (BSLQ) are derived for the solvability of the problem, and a characterization of the optimal control in terms of forward-backward stochastic differential equations has been presented. The optimal control and the value function of Problem (BLSQ) are given out clearly. The results obtained in this paper provide insight into some related topics, especially into the study of zero-sum backward stochastic differential games. In our future publication, we hope to report some relevant results along this line.

Acknowledgements. The authors would like to thank the associate editor and the anonymous referees for their insightful comments that improved the quality of this paper.

References

[1] J. Bismut, Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.* **44** (1973) 384–404.

[2] X. Bi, J. Sun and J. Xiong, Optimal control for controllable stochastic linear systems. *ESAIM: COCV* **26** (2020) 98.

[3] K. Du, J. Huang and Z. Wu, Linear quadratic mean-field-game of backward stochastic differential systems. *Math. Control Relat. Fields* **8** (2018) 653–678.

[4] M. Davis, Linear Estimation and Stochastic Control. Chapman and Hall, London (1977).

[5] J. Huang, G. Wang and J. Xiong, A maximum principle for partial information backward stochastic control problems with applications. *SIAM J. Control Optim.* **48** (2009) 2106–2117.

[6] J. Huang, S. Wang and Z. Wu, Backward mean-field linear-quadratic-Gaussian (LQG) games: full and partial information. *IEEE Trans. Automat. Control* **61** (2016) 3784–3796.

[7] A.E.B. Lim and X.Y. Zhou, Linear-quadratic control of backward stochastic differential equations. *SIAM J. Control Optim.* **40** (2001) 450–474.

[8] X. Li, J. Sun and J. Xiong, Linear quadratic optimal control problems for mean-field backward stochastic differential equations. *Appl. Math. Optim.* **80** (2019) 223–250.

[9] J. Ma and J. Yong, Forward-backward stochastic differential equations and their applications. Lecture Notes in Math., Springer-Verlag, New York (1999).

[10] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.* **14** (1990) 55–61.

[11] H. Pham, Continuous-Time Stochastic Control and Optimization with Financial Applications. Springer-Verlag, Berlin (2009).

[12] S. Peng, Backward stochastic differential equation, nonlinear expectation and their applications, in Proceedings of the International Congress of Mathematicians (2010), Vol. I, pp. 393–432.

[13] J. Sun and H. Wang, Linear-quadratic optimal control for backward stochastic differential equations with random coefficients. *ESAIM: COCV* **27** (2021) 46.

[14] J. Sun, Z. Wu and J. Xiong, Indefinite Backward Stochastic Linear-Quadratic Optimal Control Problems. Preprint arXiv.org/abs/2104.04747v1.

[15] G. Wang, Z. Wu and J. Xiong, Partial information LQ optimal control of backward stochastic differential equations, in Proceedings of the 10th World Congress on Intelligent Control and Automation. IEEE (2012), pp. 1694–1697.
[16] G. Wang, H. Xiao and J. Xiong, A kind of LQ non-zero sum differential game of backward stochastic differential equation with asymmetric information. *Automatica* **97** (2018) 346–352.
[17] J. Wen, X. Li and J. Xiong, Weak closed-loop solvability of stochastic linear quadratic optimal control problems of Markovian regime switching system. *Appl. Math. Optim.* **84** (2021) 535–565.
[18] W. Wonham, On a matrix Riccati equation of stochastic control. *SIAM J. Control* **6** (1968) 681–697.
[19] J. Yong and X.Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, New York (1999).
[20] J. Zhang, Backward Stochastic Differential Equations: From Linear to Fully Nonlinear Theory. Springer-Verlag, New York (2017).