

IRREDUCIBILITY OF KURAMOTO-SIVASHINSKY EQUATION DRIVEN BY DEGENERATE NOISE

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Abstract. In this paper, we study irreducibility of Kuramoto-Sivashinsky equation which is driven by an additive noise acting only on a finite number of Fourier modes. In order to obtain the irreducibility, we first investigate the approximate controllability of Kuramoto-Sivashinsky equation driven by a finite-dimensional force, the proof is based on Agrachev-Sarychev type geometric control approach. Next, we study the continuity of solving operator for deterministic Kuramoto-Sivashinsky equation. Finally, combining the approximate controllability with continuity of solving operator, we establish the irreducibility of Kuramoto-Sivashinsky equation.

Mathematics Subject Classification. 60H15.

Received January 17, 2021. Accepted February 11, 2022.

1. INTRODUCTION

The Kuramoto-Sivashinsky equation (KSE)

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0$$

was derived independently by Kuramoto *et al.* in [34–36] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky in [41] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on stability of a plane flame front. The KSE is also a mathematical model of reaction-diffusion systems and is related to various pattern formation phenomena where turbulence or chaos appear [2, 21]. The KSE has been studied as a prototypical example for an infinite-dimensional dynamical system [16, 32, 39, 42].

In order to consider a more realistic model phase turbulence in reaction-diffusion systems, it is sensible to consider some kind of stochastic perturbation represented by a noise term in the equation. Stochastic KSE is an important equation, this model has also attracted more and more attentions, a large amount of work has been devoted to the study of stochastic KSE. References [10, 19] studied the existence and uniqueness of solution for stochastic KSE, [20, 45, 46] discussed the attractor of stochastic KSE, [10, 22] obtained the invariant measure of stochastic KSE, [26, 27] proved the null controllability of stochastic KSE, [9] established the large deviation

Keywords and phrases: Irreducibility, Kuramoto-Sivashinsky equation, degenerate noise, approximate controllability, Agrachev-Sarychev method.

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principle of stochastic KSE, [31] established the averaging principle for stochastic KSE. We refer readers to the references therein for more interesting results on stochastic KSE.

In this paper, we consider the irreducibility of stochastic KSE driven by *highly degenerate noise*

$$\begin{cases} u_t + u_{xxxx} + u_{xx} + uu_x = b_1\dot{\beta}_1(t) \sin x + b_2\dot{\beta}_2(t) \cos x + b_3\dot{\beta}_3(t) & \text{in } \mathbb{T} \times (0, T), \\ u(0) = u_0 & \text{in } \mathbb{T}, \end{cases} \quad (1.1)$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\beta_k, k = 1, 2, 3$ are real valued mutually independent standard Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $b_k, k = 1, 2, 3$ are real valued numbers. Denote $H := L^2(\mathbb{T}), V := H^2(\mathbb{T})$. According to Theorem 3.4 of [22], we know

If $u_0 \in H$, (1.1) admits a unique weak solution $u \in C([0, T]; H)$ \mathbb{P} -a.s. Moreover, u is a Markov process in H , which is Feller in H .

Let $u(t, u_0)$ denote the unique solution to (1.1) for the initial value $u_0 \in H$, and we define the transition probabilities $\{P(t, u_0, \cdot) : t \in [0, T], u_0 \in H\}$ by

$$P(t, u_0, \Gamma) := \mathbb{P}\{u(t, u_0) \in \Gamma\}$$

for all Borel sets Γ of H and all $t \in [0, T]$.

Definition 1.1. A family of transition probabilities $\{P(t, u_0, \cdot) : t \in [0, T], u_0 \in H\}$ is called **irreducible in H** if for every $u_0 \in H, t \in [0, T], u_1 \in H$ and $\varepsilon > 0$, we have $P(t, u_0, B_H(u_1, \varepsilon)) > 0$.

Now, we are in a position to present the main result in this paper.

Theorem 1.2. *If $b_k \neq 0$ for $k = 1, 2, 3$, then the family $\{P(t, u_0, \cdot) : t \in [0, T], u_0 \in H\}$ is irreducible in H .*

The main contribution of this paper is the irreducibility of stochastic KSE driven by the highly degenerate noise. The irreducibility of stochastic partial differential equations (SPDEs) has attracted many authors' attention, see [6–8, 18, 22, 23, 43, 44] and references within. However, to the best knowledge of the author, the irreducibility of SPDEs is generally established for non-degenerate noise, the result for the case of degenerate noise is far less studied. The irreducibility is a fundamental concept in stochastic dynamic system, and it plays a crucial role in the research of ergodic theory, moderate deviation principle. More precisely, the main reason for the study of the irreducibility property is its relevance in ergodic theory, and in the analysis of the uniqueness and ergodicity of invariant measures. The property of irreducibility is at the core of the problem, it is sufficient to obtain the ergodic result by adding the strong Feller property, see the classical work [17, 24]. Another application of the irreducibility is to establish moderate deviation principle of SPDEs, in general, the moderate deviation principle is obtained by verifying the irreducibility and some Lyapunov condition for SPDEs, see [43].

The proof of Theorem 1.2 relies on the approximate controllability of KSE (see Thm. 3.3) and the continuity of the mapping *noise* \mapsto *solution* along the controllers (see Prop. 4.2). It is well known that one usually solves a control problem to prove the irreducibility for SPDEs. However, due to the highly degenerate of the noise of (1.1), the control problem of the associated deterministic system is much harder than those in the usual case, the traditional methods and techniques in the existing literature are no longer feasible. The novelty of this paper is that we introduce the Agrachev-Sarychev type geometric control approach to overcome this difficulty. More precisely, with the help of asymptotic property of operator \mathcal{R} and saturating property of subspace of \mathcal{H} , we can prove the approximate controllability of the associated deterministic system with finite-dimensional control.

The control theory is known to be a useful tool in the study of stochastic systems with non-degenerate noise, see [17] and references therein for more interesting results on this topic. In recent years, stochastic system with highly degenerate noise is attracting more and more people's attention, this paper shows that the Agrachev-Sarychev approach is a powerful tool for the study of stochastic systems with highly degenerate noise. Let us mention that the method and framework in this paper are quite general and can be adapted to degenerate problems for more SPDEs.

The rest of this paper is organized as follows: In Section 2, we give some preliminary results which will be used in this paper. Section 3 is devoted to the approximate controllability of KSE driven by a finite-dimensional force. The proof of Theorem 1.2 is given in Section 4.

2. PRELIMINARIES

If $u \in L^2(\mathbb{T})$, it can be written as $u(x) = \sum_{k \in \mathbb{Z}} u_k e^{ikx}$ with $u_k \in \mathbb{C}$ and $u_{-k} = \overline{u_k}$, the norm in $H^s(\mathbb{T})$ is defined by $\|\cdot\|_s$:

$$\|u\|_s^2 := \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s |u_k|^2.$$

For $s = 0$, $\|\cdot\|_0 := \|\cdot\|$.

For any $s \geq 0$ and $u = \sum_{k \in \mathbb{Z}} u_k e^{ikx} \in H^s(\mathbb{T})$, we define

$$\partial_x^s u := \sum_{k \in \mathbb{Z}} |k|^s u_k e^{ikx}.$$

In this paper, we will use the following inequalities.

Lemma 2.1. *If $a, b \in \mathbb{R}$, $p > 0$, it holds that*

$$(|a| + |b|)^p \leq \begin{cases} |a|^p + |b|^p & 0 < p \leq 1, \\ 2^{p-1}(|a|^p + |b|^p) & p > 1. \end{cases}$$

Lemma 2.2. *(Gronwall inequality) Let $y(t)$ be a nonnegative function and $a \in \mathbb{R}$, $f \in L^1_{loc}(\mathbb{R})$, if $y' \leq -ay + f$, we have*

$$y(t) \leq y(s)e^{-a(t-s)} + \int_s^t e^{-a(t-\tau)} f(\tau) d\tau.$$

Lemma 2.3. *(Young inequality) Let $a, b \in [0, +\infty)$ and $\varepsilon > 0$, then we have*

$$ab \leq \varepsilon^{-p} \frac{a^p}{p} + \varepsilon^q \frac{b^q}{q},$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2.4. *(See [1]) Let $0 \leq s_1 < s_2$, for any $\varepsilon > 0$ and $u \in H^{s_2}(\mathbb{T})$, there exists a constant $C(\varepsilon) > 0$ such that*

$$\|u\|_{s_1} \leq \varepsilon \|\partial_x^{s_2} u\| + C(\varepsilon) \|u\|.$$

Lemma 2.5. *(See [1]) Let $s > \frac{1}{2}$, then for any $u \in H^s(\mathbb{T})$, we have*

$$\|u\|_{L^\infty(\mathbb{T})} \leq C \|u\|_s.$$

3. APPROXIMATE CONTROLLABILITY OF KSE

We first consider the approximate controllability of the following KSE driven by a finite-dimensional force:

$$\begin{cases} u_t + u_{xxxx} + u_{xx} + uu_x = h(x, t) + \eta(x, t) & \text{in } \mathbb{T} \times (0, T), \\ u(0) = u_0 & \text{in } \mathbb{T}, \end{cases} \quad (3.1)$$

where the function η plays the role of the control.

Definition 3.1. For $s \geq 0$, we shall say that (3.1) is approximately controllable by \mathcal{H} -valued control if for any initial point $u_0 \in H^s(\mathbb{T})$, any target $u_1 \in H^s(\mathbb{T})$, and any number $\varepsilon > 0$, there is a control $\eta \in L^2(0, T; \mathcal{H})$ and a unique solution u of problem (3.1) defined on the interval $[0, T]$ such that

$$\|u(T) - u_1\|_s \leq \varepsilon.$$

Let \mathcal{H} be a finite-dimensional subspace of $H^{s+4}(\mathbb{T})$. We define a non-decreasing sequence of finite-dimensional subspaces:

$$\mathcal{H}_0 := \mathcal{H}, \quad \mathcal{H}_j := \mathcal{F}(\mathcal{H}_{j-1}), \quad j \geq 1, \quad \mathcal{H}_\infty := \bigcup_{j=1}^{\infty} \mathcal{H}_j,$$

where

$$\mathcal{F}(\mathcal{H}) := \text{span} \left\{ \eta - \sum_{i=1}^N u_i \partial_x u_i : \eta, u_i \in \mathcal{H}, i = 1, 2, \dots, N \right\}.$$

Definition 3.2. We say that \mathcal{H} is saturating if \mathcal{H}_∞ is dense in $H^s(\mathbb{T})$.

Now, we are in a position to present the main result in this section.

Theorem 3.3. *Let $T > 0, s > \frac{1}{2}$, $u_0 \in H^s(\mathbb{T})$ and $h \in L^2(0, T; H^{s-2}(\mathbb{T}))$. If \mathcal{H} is saturating, then (3.1) is approximately controllable by \mathcal{H} -valued control.*

The proof of Theorem 3.3 is based on the idea from the works of Agrachev and Sarychev [3–5], who studied the approximate controllability of the 2D Navier-Stokes and Euler systems by finite-dimensional forces. The Agrachev-Sarychev method is also applied to other equations, see [11, 40] and the literatures therein. The controllability of KSE has attracted many authors' attention, see [12–15, 25–30].

Here, we give examples of saturating spaces. \mathcal{I} is called a generator if $\mathcal{I} \subset \mathbb{Z}$ is a finite symmetric set containing 0 and any integer is a linear combination of elements of \mathcal{I} with integer coefficients. We define the space

$$\mathcal{H}(\mathcal{I}) = \text{span} \{ \sin(mx), \cos(mx) : m \in \mathcal{I} \}.$$

Proposition 3.4. *If \mathcal{I} is a generator, then the space $\mathcal{H}(\mathcal{I})$ is saturating. In particular, $\mathcal{E} := \text{span} \{ \sin x, \cos x, 1 \}$ is saturating.*

The proof of Proposition 3.4 is done later in Section 3.4.

3.1. Well-posedness of generalised KSE

We consider the following generalised KSE:

$$\begin{cases} u_t + (u + \zeta)_{xxxx} + (u + \zeta)_{xx} + (u + \zeta)(u + \zeta)_x = \varphi & \text{in } \mathbb{T} \times (0, T), \\ u(0) = u_0 & \text{in } \mathbb{T}. \end{cases} \quad (3.2)$$

For any $T > 0$ and $s \geq 0$, we define the space

$$\mathcal{X}_{T,s} := C([0, T]; H^s(\mathbb{T})) \cap L^2(0, T; H^{s+2}(\mathbb{T}))$$

endowed with the norm

$$\|u\|_{\mathcal{X}_{T,s}} := \|u\|_{C([0,T];H^s(\mathbb{T}))} + \|u\|_{L^2(0,T;H^{s+2}(\mathbb{T}))}.$$

Define the operator $P = -\partial_x^4$ with domain $\mathcal{D}(P) = H^4(\mathbb{T})$, we know that P generates a semigroup $\{S(t)\}_{t \geq 0}$ in $L^2(\mathbb{T})$. Moreover, we have the following important property:

Lemma 3.5. *For any $T > 0$ and $s \geq 0$, the semigroup satisfies that*

$$\begin{aligned} \|S(\cdot)y_0\|_{\mathcal{X}_{T,s}} &\leq C\|y_0\|_s, \quad \forall y_0 \in H^s(\mathbb{T}), \\ \left\| \int_0^\cdot S(\cdot - \tau)f(\tau)d\tau \right\|_{\mathcal{X}_{T,s}} &\leq C\|f\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}, \quad \forall f \in L^2(0,T;H^{s-2}(\mathbb{T})), \end{aligned} \quad (3.3)$$

where $C = C(T, s)$ is a positive constant that increases in the first variable.

The proof of Lemma 3.5 is done later in Section 3.5.

Proposition 3.6. *For any $T > 0$ and $s > \frac{1}{2}$, let $u_0 \in H^s(\mathbb{T})$, $\zeta \in L^4(0, T; H^{s+2}(\mathbb{T}))$ and $\varphi \in L^2(0, T; H^{s-2}(\mathbb{T}))$. Then system (3.2) admits a unique solution $u \in \mathcal{X}_{T,s}$. Moreover, let \mathcal{R} be the mapping taking a triple (u_0, ζ, φ) to the solution of (3.2), for $\hat{u}_0 \in H^s(\mathbb{T})$ and $\hat{\varphi} \in L^2(0, T; H^{s-2}(\mathbb{T}))$, we have*

$$\|\mathcal{R}(u_0, 0, \varphi) - \mathcal{R}(\hat{u}_0, 0, \hat{\varphi})\|_{\mathcal{X}_{T,s}} \leq C(\|u_0 - \hat{u}_0\|_s + \|\varphi - \hat{\varphi}\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}). \quad (3.4)$$

Proof of Proposition 3.6. Before we prove Proposition 3.6, we introduce the following important mathematical setting: Throughout the paper, the letter C denotes unessential positive constant whose value may change in different occasions, which may vary from line to line. We will write the dependence of constant on parameters explicitly if it is essential.

First, we consider the existence and uniqueness of the solution for (3.2).

Let $t \in [0, \theta] \subset [0, T]$ and $v \in \mathcal{X}_{\theta,s}$, we set

$$\Phi(v)(t) = S(t)u_0 + \int_0^t S(t - \tau)F(v)(\tau)d\tau,$$

where

$$F(v) = -\partial_x^2 v - \partial_x^4 \zeta - \partial_x^2 \zeta - (v + \zeta)\partial_x(v + \zeta) + \varphi.$$

Since $H^s(\mathbb{T})(s > \frac{1}{2})$ is a Banach algebra, we get

$$\|(v + \zeta)\partial_x(v + \zeta)\|_{s-2} \leq C\|(v + \zeta)^2\|_{s-1} \leq C\|(v + \zeta)^2\|_s \leq C\|v + \zeta\|_s^2 \leq C(\|v\|_s^2 + \|\zeta\|_s^2),$$

this implies that

$$\|F(v)\|_{s-2} \leq C(\|v\|_s + \|\zeta\|_{s+2} + \|v\|_s^2 + \|\zeta\|_s^2 + \|\varphi\|_{s-2}).$$

According to this estimate, we have

$$\begin{aligned} \|\Phi(v)\|_{L^2(0,\theta;H^{s-2}(\mathbb{T}))} &\leq C\theta^{\frac{1}{2}}(\|v\|_{\mathcal{X}_{\theta,s}} + \|v\|_{\mathcal{X}_{\theta,s}}^2) \\ &\quad + C(\|\zeta\|_{L^2(0,T;H^{s+2}(\mathbb{T}))} + \|\zeta\|_{L^4(0,T;H^s(\mathbb{T}))}^2 + \|\varphi\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}) \\ &\leq C\theta^{\frac{1}{2}}(\|v\|_{\mathcal{X}_{\theta,s}} + \|v\|_{\mathcal{X}_{\theta,s}}^2) \\ &\quad + C(\|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))} + \|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))}^2 + \|\varphi\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}). \end{aligned}$$

It follows from (3.3) that

$$\begin{aligned} \|\Phi(v)\|_{\mathcal{X}_{\theta,s}} &\leq C(\|u_0\|_s + \|F(v)\|_{L^2(0,\theta;H^{s-2}(\mathbb{T}))}) \\ &\leq C_1(\|u_0\|_s + \|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))} + \|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))}^2 + \|\varphi\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}) \\ &\quad + C_2\theta^{\frac{1}{2}}(\|v\|_{\mathcal{X}_{\theta,s}} + \|v\|_{\mathcal{X}_{\theta,s}}^2), \end{aligned} \tag{3.5}$$

where C_1, C_2 depend on T but independent of θ .

For any $v_1, v_2 \in \mathcal{X}_{\theta,s}$, the same argument shows that

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{\mathcal{X}_{\theta,s}} &\leq C\|F(v_1) - F(v_2)\|_{L^2(0,\theta;H^{s-2}(\mathbb{T}))} \\ &\leq C_3[\theta^{\frac{1}{2}}(1 + \|v_1\|_{\mathcal{X}_{\theta,s}} + \|v_2\|_{\mathcal{X}_{\theta,s}}) + \theta^{\frac{1}{4}}\|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))}]\|v_1 - v_2\|_{\mathcal{X}_{\theta,s}}. \end{aligned} \tag{3.6}$$

Set

$$R := 2C_1(\|u_0\|_s + \|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))} + \|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))}^2 + \|\varphi\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}).$$

If $v, v_1, v_2 \in B_{\mathcal{X}_{\theta,s}}(R) := \{u \in \mathcal{X}_{\theta,s} \mid \|u\|_{\mathcal{X}_{\theta,s}} \leq R\}$, we can choose θ sufficiently small such that

$$\begin{aligned} C_2\theta^{\frac{1}{2}}(1 + R) &\leq \frac{1}{2}, \\ C_3[\theta^{\frac{1}{2}}(1 + 2R) + \theta^{\frac{1}{4}}\|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))}] &\leq \frac{1}{2}, \end{aligned}$$

then according to (3.5) and (3.6), we have

$$\begin{aligned} \|\Phi(v)\|_{\mathcal{X}_{\theta,s}} &\leq \frac{R}{2} + C_2\theta^{\frac{1}{2}}(1 + R)\|v\|_{\mathcal{X}_{\theta,s}} \leq R, \\ \|\Phi(v_1) - \Phi(v_2)\|_{\mathcal{X}_{\theta,s}} &\leq \frac{1}{2}\|v_1 - v_2\|_{\mathcal{X}_{\theta,s}}. \end{aligned}$$

This implies that Φ has a unique fixed point $u \in B_{\mathcal{X}_{\theta,s}}(R)$ which will be the solution of (3.2) for small θ .

To obtain the global existence, we will prove the following a priori estimate

$$\|u\|_{\mathcal{X}_{T,s}} \leq C, \tag{3.7}$$

where $C = C(T, \|u_0\|_s, \|\zeta\|_{L^4(0,T;H^{s+2}(\mathbb{T}))}, \|\varphi\|_{L^2(0,T;H^{s-2}(\mathbb{T}))})$.

For $s = 0$, taking the scalar product in $L^2(\mathbb{T})$ of equation (3.2) with u , noting the facts that

$$\begin{aligned} \int_{\mathbb{T}} (\zeta u)_x u dx &= - \int_{\mathbb{T}} \zeta u u_x dx = \frac{1}{2} \int_{\mathbb{T}} \zeta_x u^2 dx \leq C \|\zeta\|_2 \|u\|^2, \\ \int_{\mathbb{T}} \zeta \zeta_x u dx &\leq \|\zeta \zeta_x\| \|u\| \leq C \|\zeta\|_2^2 \|u\|, \end{aligned}$$

we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\partial_x^2 u\|^2 &\leq C(\|\partial_x^2 u\| \|u\| + \|\zeta\|_2 \|\partial_x^2 u\| + \|\zeta\|_2 \|u\|^2 + \|\zeta\|_2^2 \|u\| + \|\varphi\|_{-2} \|u\|_2) \\ &\leq \frac{1}{2} \|\partial_x^2 u\|^2 + C(1 + \|\zeta\|_2) \|u\|^2 + C(1 + \|\zeta\|_2^4 + \|\varphi\|_{-2}^2). \end{aligned}$$

By Gronwall's inequality, (3.7) holds for $s = 0$.

For $s > 0$, we can prove (3.7) by induction. More precisely, let k be any nonnegative integer, assume that if we have (3.7) for $s = k$, we claim that (3.7) holds for $k < s \leq k + 1$.

To this purpose, taking the scalar product in $L^2(\mathbb{T})$ of equation (3.2) with $\partial_x^{2s} u$, we can obtain that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_x^s u\|^2 + \|\partial_x^{s+2} u\|^2 \\ &\leq C(\|\partial_x^{s+2} u\| \|\partial_x^s u\| + \|\zeta\|_{s+2} \|\partial_x^{s+2} u\| - (u u_x + (u\zeta)_x, \partial_x^{2s} u) + \|\zeta\|_{s+2}^2 \|\partial_x^s u\| + \|\varphi\|_{s-2} \|u\|_{s+2}) \\ &\leq \frac{1}{4} \|\partial_x^{s+2} u\|^2 + C\|\partial_x^s u\|^2 + C(1 + \|u\|^2 + \|\zeta\|_{s+2}^4 + \|\varphi\|_{s-2}^2) - (u u_x + (u\zeta)_x, \partial_x^{2s} u). \end{aligned} \quad (3.8)$$

Our task is reduced to estimate $(u u_x + (u\zeta)_x, \partial_x^{2s} u)$. Applying Young's inequality in Lemma 2.3 and interpolation inequality in Lemma 2.4, we have

$$\begin{aligned} |(u\zeta)_x, \partial_x^{2s} u| &\leq |(\partial_x^{s+1}(u\zeta), \partial_x^s u)| \\ &\leq \|u\zeta\|_{s+1} \|\partial_x^s u\| \\ &\leq C\|u\|_{s+1} \|\zeta\|_{s+1} \|\partial_x^s u\| \\ &\leq C(\|\partial_x^{s+2} u\| + \|u\|) \|\zeta\|_{s+1} \|\partial_x^s u\| \\ &\leq \frac{1}{8} \|\partial_x^{s+2} u\|^2 + C(1 + \|\zeta\|_{s+1}^2) \|\partial_x^s u\|^2 + C\|\zeta\|_{s+1}^2 \|u\|^2. \end{aligned} \quad (3.9)$$

We will prove that for any $k < s \leq k + 1$, k is any nonnegative integer, it holds that

$$|(u u_x, \partial_x^{2s} u)| \leq \frac{1}{8} \|\partial_x^{s+2} u\|^2 + C. \quad (3.10)$$

Indeed, if $k = 0$, $0 < s \leq 1$, then $2s + 1 \leq s + 2$, then for any $\varepsilon > 0$, it holds that

$$\begin{aligned} |(u u_x, \partial_x^{2s} u)| &\leq C\|u\|_{L^\infty(\mathbb{T})} \|u\| \|\partial_x^{2s+1} u\| \\ &\leq C\|u\|_1 \|u\| \|u\|_{s+2} \\ &\leq C\|u\| (\varepsilon \|\partial_x^{s+2} u\|^2 + C(\varepsilon) \|u\|^2), \end{aligned}$$

where we have applied the interpolation inequality in Lemma 2.4 to obtain $\|u\|_1 \|u\|_{s+2} \leq \varepsilon \|\partial_x^{s+2} u\|^2 + C(\varepsilon) \|u\|^2$. According to (3.7) for $s = 0$, we can choose ε small enough such that (3.10) holds.

If $k \geq 1$, $k < s \leq k + 1$, we have $2s - k + 1 \leq s + 2$, then it follows that

$$\begin{aligned} |(uu_x, \partial_x^{2s} u)| &\leq C |(\partial_x^k(u^2), \partial_x^{2s-k+1} u)| \\ &\leq C \|u\|_k^2 \|u\|_{s+2} \\ &\leq C \|u\|_k^4 + \frac{1}{8} \|u\|_{s+2}^2 \\ &\leq \frac{1}{8} \|\partial_x^{s+2} u\|^2 + C(\|u\|_k^4 + \|u\|^2). \end{aligned}$$

Due to the assumption that (3.7) holds for $s = k$, the above estimate implies that (3.10) holds.

Combining estimates (3.8)–(3.10), we conclude that

$$\frac{d}{dt} \|\partial_x^s u\|^2 \leq C(1 + \|\zeta\|_{s+2}^4) \|\partial_x^s u\|^2 + C(1 + \|\zeta\|_{s+2}^4 + \|\varphi\|_{s-2}^2).$$

Applying Gronwall's inequality again, we can obtain (3.7) for $k < s \leq k + 1$. This implies that (3.7) holds for any $s \geq 0$. Namely, we get the existence of the solution on $[0, T]$.

For the uniqueness, let $u_1, u_2 \in \mathcal{X}_{T,s}$ be two solutions of (3.2), then $v = u_1 - u_2$ satisfies

$$\begin{cases} v_t + v_{xxxx} + v_{xx} + u_1 v_x + v u_{2x} + (\zeta v)_x = 0 & \text{in } \mathbb{T} \times (0, T), \\ v(0) = 0 & \text{in } \mathbb{T}. \end{cases}$$

Repeating the above arguments, we get that

$$\frac{d}{dt} \|v\|^2 \leq C(1 + \|\zeta\|_2 + \|u_1\|_2 + \|u_2\|_2) \|v\|^2,$$

which implies that $v \equiv 0$ in view of the Gronwall's inequality.

Finally, we prove (3.4). To this purpose, let \hat{u} be the solution of (3.2) with $(\hat{u}_0, 0, \hat{\varphi})$ and $w = u - \hat{u}$, then w is a solution of problem

$$\begin{cases} w_t + w_{xxxx} + w_{xx} + uw_x + w \hat{u}_x = \eta & \text{in } \mathbb{T} \times (0, T), \\ w(0) = w_0 & \text{in } \mathbb{T}, \end{cases} \quad (3.11)$$

where $w_0 = u_0 - \hat{u}_0$ and $\eta = \varphi - \hat{\varphi}$. Applying similar methods as above, we can obtain that

$$\begin{aligned} \frac{d}{dt} \|w\|^2 + \|\partial_x^2 w\|^2 &\leq C(1 + \|u\|_2 + \|\hat{u}\|_2) \|w\|^2 + C\|\eta\|_{-2}^2, \\ \frac{d}{dt} \|\partial_x^s w\|^2 + \|\partial_x^{s+2} w\|^2 &\leq C(1 + \|u\|_{s+2}^2 + \|\hat{u}\|_{s+2}^2) \|\partial_x^s w\|^2 + C(\|\eta\|_{s-2}^2 + \|w\|^2), \quad s > 0. \end{aligned}$$

Thus, (3.4) follows from Gronwall's inequality.

This ends the proof of Proposition 3.6. \square

3.2. Asymptotic property

The following asymptotic property plays an important role in the proof of approximate controllability.

Proposition 3.7. *Let $T > 0$ and $s > \frac{1}{2}$. For any $u_0, \eta \in H^{s+2}(\mathbb{T})$, $\zeta \in H^{s+4}(\mathbb{T})$, $h \in L^2(0, T; H^{s-2}(\mathbb{T}))$, there is a number $\delta_0 \in (0, 1)$ such that for any $\delta \in (0, \delta_0)$, the following limit holds*

$$\mathcal{R}_\delta(u_0, \delta^{-\frac{1}{2}}\zeta, h + \delta^{-1}\eta) \longrightarrow u_0 + \eta - \zeta\zeta_x \quad \text{in } H^s(\mathbb{T}) \quad \text{as } \delta \rightarrow 0.$$

Proof. Let us take any $\delta > 0$ and consider the equation

$$u_t + (u + \delta^{-\frac{1}{2}}\zeta)_{xxxx} + (u + \delta^{-\frac{1}{2}}\zeta)_{xx} + (u + \delta^{-\frac{1}{2}}\zeta)(u + \delta^{-\frac{1}{2}}\zeta)_x = h + \delta^{-1}\eta. \quad (3.12)$$

By Proposition 3.6, (3.12) has a unique solution in $\mathcal{X}_{T,s}$. Make a time substitution and consider the functions

$$\begin{aligned} w(t) &:= u_0 + t(\eta - \zeta\zeta_x), \\ v(t) &:= u(\delta t) - w(t), \end{aligned}$$

where $t \leq 1 \wedge \delta^{-1}T$. Then v is a solution of the following system

$$\begin{cases} v_t + \delta(v + w + \delta^{-\frac{1}{2}}\zeta)_{xxxx} + \delta(v + w + \delta^{-\frac{1}{2}}\zeta)_{xx} \\ \quad + \delta(v + w + \delta^{-\frac{1}{2}}\zeta)(v + w + \delta^{-\frac{1}{2}}\zeta)_x - \zeta\zeta_x = \delta h, \\ v(0) = 0. \end{cases} \quad (3.13)$$

Taking the scalar product in $L^2(\mathbb{T})$ of equation (3.13) with v , we can obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 + \delta \|\partial_x^2 v\|^2 &\leq \delta \|v\| \|\partial_x^2 v\| + \delta \|w\|_2 \|\partial_x^2 v\| + \delta^{\frac{1}{2}} \|\zeta\|_4 \|v\| + C\delta \|w\|_1^2 \|v\| + C\delta \|w\|_2 \|v\|^2 \\ &\quad + C\delta^{\frac{1}{2}} \|\zeta\|_2 \|v\|^2 + C\delta^{\frac{1}{2}} \|w\|_1 \|\zeta\|_1 \|v\| + C\delta \|h\|_{-2} (\|v\| + \|\partial_x^2 v\|) \\ &\leq \frac{\delta}{2} \|\partial_x^2 v\|^2 + C\delta^{\frac{1}{2}} (1 + \|h\|_{-2}^2 + \|v\|^2), \end{aligned}$$

this implies that

$$\frac{d}{dt} \|v\|^2 \leq C\delta^{\frac{1}{2}} (1 + \|h\|_{-2}^2 + \|v\|^2).$$

Applying Gronwall's inequality, we have

$$\|v(t)\|^2 \leq C\delta^{\frac{1}{2}} \quad \text{for } 0 < t \leq 1 \wedge (\delta^{-1}T). \quad (3.14)$$

Then taking the scalar product in $L^2(\mathbb{T})$ of equation (3.13) with $\partial_x^{2s} v$, it follows that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_x^s v\|^2 + \delta \|\partial_x^{s+2} v\|^2 \\ &\leq \delta \|\partial_x^s v\| \|\partial_x^{s+2} v\| + \delta \|w\|_{s+2} \|\partial_x^{s+2} v\| + \delta^{\frac{1}{2}} \|\zeta\|_{s+4} \|\partial_x^s v\| + N(w, \zeta, v) + \delta \|h\|_{s-2} (\|\partial_x^{s+2} v\| + \|v\|) \\ &\leq \frac{\delta}{4} \|\partial_x^{s+2} v\|^2 + C\delta^{\frac{1}{2}} (1 + \|h\|_{s-2}^2 + \|\partial_x^s v\|^2) + N(w, \zeta, v), \end{aligned}$$

where

$$N(w, \zeta, v) = (\delta v v_x + \delta w w_x + \delta(wv)_x + \delta^{\frac{1}{2}}(\zeta v)_x + \delta^{\frac{1}{2}}(\zeta w)_x, \partial_x^{2s} v).$$

The first term in $N(w, \zeta, v)$ can be estimated as follows:

Since $H^s(\mathbb{T})(s > \frac{1}{2})$ is an algebra, it follows from the interpolation inequality in Lemma 2.4 that

$$\begin{aligned} |(\delta v v_x, \partial_x^{2s} v)| &= \frac{\delta}{2} |(\partial_x^s(v^2), \partial_x^{s+1} v)| \\ &\leq C\delta \|v^2\|_s \|\partial_x^{s+1} v\| \\ &\leq C\delta \|v\|_s^2 \|\partial_x^{s+1} v\| \\ &\leq C\delta (\|\partial_x^s v\|^2 + \|v\|^2) (\|\partial_x^{s+2} v\| + \|v\|) \\ &\leq \frac{\delta}{8} \|\partial_x^{s+2} v\|^2 + C\delta (\|\partial_x^s v\|^4 + \|v\|^4 + 1). \end{aligned}$$

Therefore, for any $s > \frac{1}{2}$,

$$|(\delta v v_x, \partial_x^{2s} v)| \leq \frac{\delta}{8} \|\partial_x^{s+2} v\|^2 + C\delta (\|\partial_x^s v\|^4 + 1).$$

By the similar method as above, we can deduce that

$$|(\delta(wv)_x, \partial_x^{2s} v)| \leq \frac{\delta}{8} \|\partial_x^{s+2} v\|^2 + C\delta (\|\partial_x^s v\|^2 + 1).$$

Next, we consider $(\delta^{\frac{1}{2}}(\zeta v)_x, \partial_x^{2s} v)$. Since $s > \frac{1}{2}$, with the help of the interpolation inequality in Lemma 2.4, we have

$$\begin{aligned} |(\delta^{\frac{1}{2}}(\zeta v)_x, \partial_x^{2s} v)| &= |\delta^{\frac{1}{2}}(\partial_x^s(\zeta v), \partial_x^{s+1} v)| \\ &\leq C\delta^{\frac{1}{2}} \|\zeta\|_s \|v\|_s \|\partial_x^{s+1} v\| \\ &\leq C\delta^{\frac{1}{2}} (\|\partial_x^s v\| + \|v\|) (\delta^{\frac{1}{4}} \|\partial_x^{s+2} v\| + \delta^{-\frac{1}{4}} \|\partial_x^s v\|) \\ &\leq \frac{\delta}{8} \|\partial_x^{s+2} v\|^2 + C\delta^{\frac{1}{4}} (\|\partial_x^s v\|^2 + 1). \end{aligned}$$

Then, consideration is given to the remaining terms in $N(w, \zeta, v)$,

$$\begin{aligned} |(\delta w w_x, \partial_x^{2s} v)| + |(\delta^{\frac{1}{2}}(\zeta w)_x, \partial_x^{2s} v)| &\leq C\delta^{\frac{1}{2}} (\|w\|_{s+1}^2 + \|w\|_{s+1} \|\zeta\|_{s+1}) \|\partial_x^s v\| \\ &\leq C\delta^{\frac{1}{2}} (\|\partial_x^s v\|^2 + 1). \end{aligned}$$

Combining the above estimates, we have

$$\frac{d}{dt} \|\partial_x^s v\|^2 \leq C\delta^{\frac{1}{4}} (1 + \|h\|_{s-2}^2 + \|\partial_x^s v\|^4).$$

Proceeding as in the proof of Proposition 2.4 in [37], we can choose $\delta_0 \in (0, 1)$ so small such that for any $\delta < \delta_0$, $s > \frac{1}{2}$

$$\|\partial_x^s v(t)\|^2 \leq C\delta^{\frac{1}{4}} \quad \text{for } 0 < t \leq 1 \wedge (\delta^{-1}T). \quad (3.15)$$

According to (3.14) and (3.15), we have

$$\|v(1)\|_s^2 \leq C\delta^{\frac{1}{4}},$$

namely,

$$\|u(\delta) - (u_0 + \eta - \zeta\zeta_x)\|_s^2 \leq C\delta^{\frac{1}{4}}.$$

The proof of Proposition 3.7 is complete. \square

3.3. Proof of Theorem 3.3

The proof of Theorem 3.3 is analogous to Theorem 3.3 in [37], the proof is divided into four steps.

Step 1. Controllability in small time to $u_0 + \mathcal{H}_0$. Let us assume for the moment that $u_0 \in H^{s+2}(\mathbb{T})$. For any $\eta \in \mathcal{H}_0$, applying Proposition 3.7 for the couple $(\eta, 0)$, we see that

$$\mathcal{R}_\delta(u_0, 0, h + \delta^{-1}\eta) \rightarrow u_0 + \eta \text{ in } H^s(\mathbb{T}) \text{ as } \delta \rightarrow 0.$$

It follows from the above limit that for any $\varepsilon > 0$, there is a time $0 < \theta_1 < T$ such that $\hat{\eta} = \theta_1^{-1}\eta \in L^2(0, T; \mathcal{H})$ and

$$\|\mathcal{R}_{\theta_1}(u_0, h + \hat{\eta}) - u_0 - \eta\|_s < \varepsilon.$$

Step 2. Controllability in small time to $u_0 + \mathcal{H}_N$. We argue by induction. Assume that the approximate controllability of (3.1) to the set $u_0 + \mathcal{H}_{N-1}$ is already proved. Let $\eta_1 \in \mathcal{H}_N$ be of the form

$$\eta_1 = \eta - \sum_{m=1}^n \zeta_m \zeta_{mx} \tag{3.16}$$

for some integer $n \geq 1$ and vectors $\eta, \zeta_1, \dots, \zeta_n \in \mathcal{H}_{N-1}$. Applying Proposition 3.7 for the couple $(0, \zeta_1)$, we see that

$$\mathcal{R}_\delta(u_0, \delta^{-\frac{1}{2}}\zeta_1, h) \rightarrow u_0 - \zeta_1 \zeta_{1x} \text{ in } H^s(\mathbb{T}) \text{ as } \delta \rightarrow 0. \tag{3.17}$$

Using the equality

$$\mathcal{R}_\delta(u_0 + \delta^{-\frac{1}{2}}\zeta_1, 0, h) = \mathcal{R}_\delta(u_0, \delta^{-\frac{1}{2}}\zeta_1, h) + \delta^{-\frac{1}{2}}\zeta_1$$

and the limit (3.17), we obtain

$$\|\mathcal{R}_\delta(u_0 + \delta^{-\frac{1}{2}}\zeta_1, 0, h) - u_0 + \zeta_1 \zeta_{1x} - \delta^{-\frac{1}{2}}\zeta_1\|_s \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Combining this with the fact that $\eta, \zeta_1 \in \mathcal{H}_{N-1}$, the induction hypothesis, and Proposition 3.6, we can find a small time $\theta_2 > 0$ and a control $\hat{\eta}_1 \in L^2(0, T; \mathcal{H})$ such that

$$\|\mathcal{R}_{\theta_2}(u_0, h + \hat{\eta}_1) - u_0 - \eta + \zeta_1 \zeta_{1x}\|_s < \varepsilon.$$

Iterating this argument successively for the vectors ζ_2, \dots, ζ_n , we construct a small time $\theta > 0$ and a control $\hat{\eta}_1 \in L^2(0, T; \mathcal{H})$ satisfying

$$\|\mathcal{R}_\theta(u_0, h + \hat{\eta}_1) - u_0 - \eta + \zeta_1 \zeta_{1x} + \dots + \zeta_n \zeta_{nx}\|_s = \|\mathcal{R}_\theta(u_0, h + \hat{\eta}_1) - u_0 - \eta_1\|_s < \varepsilon,$$

where we used (3.16). This proves the approximate controllability in small time to any point in $u_0 + \mathcal{H}_N$.

Step 3. Global controllability in small time. Now let $u_1 \in H^s(\mathbb{T})$ be arbitrary. As \mathcal{H}_∞ is dense in $H^s(\mathbb{T})$, there is an integer $N \geq 1$ and point $\hat{u}_1 \in u_0 + \mathcal{H}_N$ such that

$$\|u_1 - \hat{u}_1\|_s < \frac{\varepsilon}{2}. \quad (3.18)$$

By the results of Steps 1 and 2, for any $\varepsilon > 0$, there is a time $\theta > 0$ and a control $\hat{\eta} \in L^2(0, T; \mathcal{H})$ satisfying

$$\|\mathcal{R}_\theta(u_0, h + \hat{\eta}) - \hat{u}_1\|_s < \frac{\varepsilon}{2}.$$

Combining this with (3.18), we get approximate controllability in small time to u_1 . Due to Proposition 3.6 and the fact that the space $H^{s+2}(\mathbb{T})$ is dense in $H^s(\mathbb{T})$, we can also obtain small time approximate controllability starting from $u_0 \in H^s(\mathbb{T})$.

Step 4. Global controllability in fixed time T . Applying the result of Step 3, for any $\varepsilon > 0$, there is a time $T_1 > 0$ and a control $\eta_1 \in L^2(0, T_1; \mathcal{H})$ satisfying

$$\|\mathcal{R}_{T_1}(u_0, h + \eta_1) - u_1\|_s < \frac{\varepsilon}{2}.$$

Take $v_1 = \mathcal{R}_{T_1}(u_0, h + \eta_1)$. According to Proposition 3.6, we can find $\tau > 0$ such that for $t \in [0, \tau]$,

$$\|\mathcal{R}_t(v_1, h) - v_1\|_s < \frac{\varepsilon}{2}.$$

Define a control function

$$\eta_2(t) = \begin{cases} \eta_1(t) & t \in [0, T_1], \\ 0 & t \in (T_1, T_1 + \tau], \end{cases}$$

then, it follows that

$$\|\mathcal{R}_{T_1+t}(u_0, h + \eta_2) - u_1\|_s < \varepsilon, \quad \forall t \in [0, \tau].$$

If $T_1 + \tau \geq T$, then the proof is complete. Otherwise, take $v_2 = \mathcal{R}_{T_1+\tau}(u_0, h + \eta_2)$, by the result of Step 3, there is a time $T_2 > 0$ and a control $\eta_3 \in L^2(0, T_2; \mathcal{H})$ satisfying

$$\|\mathcal{R}_{T_2}(v_2, h + \eta_3) - u_1\|_s < \frac{\varepsilon}{2}.$$

Take $v_3 = \mathcal{R}_{T_2}(v_2, h + \eta_3)$, applying again Proposition 3.6, for the same τ , if $t \in [0, \tau]$, we have

$$\|\mathcal{R}_t(v_3, h) - v_3\|_s < \frac{\varepsilon}{2}.$$

Define the control function

$$\eta_4(t) = \begin{cases} \eta_2(t) & t \in [0, T_1 + \tau], \\ \eta_3(t - T_1 - \tau) & t \in (T_1 + \tau, T_1 + T_2 + \tau], \\ 0 & t \in (T_1 + T_2 + \tau, T_1 + T_2 + 2\tau], \end{cases}$$

we can see that

$$\|\mathcal{R}_{T_1+T_2+\tau+t}(u_0, h + \eta_4) - u_1\|_s < \varepsilon, \quad \forall t \in [0, \tau].$$

Again, if $T_1 + T_2 + 2\tau \geq T$, then the proof is complete. Otherwise, after a finite number of iterations, we complete the proof of Theorem 3.3.

3.4. Proof of Proposition 3.4

The proof is divided into four steps.

Step 1. We show that

$$\sin(2mx), \cos(2mx) \in \mathcal{H}_1 \quad \text{for } m \in \mathcal{I}.$$

Indeed, the case $m = 0$ is obvious. If $m \neq 0$, we have

$$\begin{aligned} \sin(2mx) &= 2 \sin(mx) \cos(mx) = \frac{2}{m} \sin(mx) [\sin(mx)]_x \in \mathcal{H}_1, \\ \cos(2mx) &= \cos^2(mx) - \sin^2(mx) \\ &= [\cos(mx) - \sin(mx)][\cos(mx) + \sin(mx)] \\ &= -\frac{1}{m} [\cos(mx) - \sin(mx)][\cos(mx) - \sin(mx)]_x \in \mathcal{H}_1. \end{aligned}$$

Step 2. We prove

$$\cos(l+m)x \in \mathcal{H}_1 \quad \text{for } l, m \in \mathcal{I}.$$

Indeed, if $l+m=0$, $\cos(l+m)x \in \mathcal{H}_0 \subset \mathcal{H}_1$.

If $l-m=0$, by the result in step 1, we have $\cos(l+m)x \in \mathcal{H}_1$.

If $|l| \neq |m|$, since

$$\cos(l+m)x = \cos(lx) \cos(mx) - \sin(lx) \sin(mx),$$

our task is reduced to show that $\cos(lx) \cos(mx), \sin(lx) \sin(mx) \in \mathcal{H}_1$.

Taking

$$u_1(x) = \sin(lx) + \cos(mx) \in \mathcal{H}_0,$$

$$u_2(x) = \sin(mx) + \cos(lx) \in \mathcal{H}_0,$$

it follows that

$$\begin{aligned} u_1(x)u_{1x}(x) &= [\sin(lx) + \cos(mx)][l \cos(lx) - m \sin(mx)] \\ &= l \sin(lx) \cos(lx) - m \sin mx \cos(mx) - m \sin(lx) \sin(mx) + l \cos(lx) \cos(mx) \\ &= \frac{l}{2} \sin(2lx) - \frac{m}{2} \sin(2mx) - m \sin(lx) \sin(mx) + l \cos(lx) \cos(mx), \\ u_2(x)u_{2x}(x) &= [\sin(mx) + \cos(lx)][m \cos(mx) - l \sin(lx)] \\ &= m \sin(mx) \cos(mx) - l \sin lx \cos(lx) - l \sin(lx) \sin(mx) + m \cos(lx) \cos(mx) \\ &= \frac{m}{2} \sin(2mx) - \frac{l}{2} \sin(2lx) - l \sin(lx) \sin(mx) + m \cos(lx) \cos(mx). \end{aligned} \tag{3.19}$$

By the definition of \mathcal{H}_1 and the results in Step 1, we can obtain that

$$\begin{aligned} u_1(x)u_{1x}(x) - \frac{l}{2}\sin(2lx) + \frac{m}{2}\sin(2mx) &\in \mathcal{H}_1, \\ u_2(x)u_{2x}(x) - \frac{m}{2}\sin(2mx) + \frac{l}{2}\sin(2lx) &\in \mathcal{H}_1. \end{aligned}$$

According to (3.19), we can get the expressions of $\cos(lx)\cos(mx)$ and $\sin(lx)\sin(mx)$, this means that $\cos(lx)\cos(mx), \sin(lx)\sin(mx) \in \mathcal{H}_1$.

Step 3. We prove

$$\sin(l+m)x \in \mathcal{H}_1 \quad \text{for } l, m \in \mathcal{I}.$$

Indeed, if $l+m=0$, $\sin(l+m)x \in \mathcal{H}_0 \subset \mathcal{H}_1$.

If $l-m=0$, by the result in step 1, we have $\sin(l+m)x \in \mathcal{H}_1$.

If $|l| \neq |m|$, since

$$\sin(l+m)x = \sin(lx)\cos(mx) + \sin(mx)\cos(lx),$$

our task is reduced to show that $\sin(lx)\cos(mx), \sin(mx)\cos(lx) \in \mathcal{H}_1$.

Taking

$$\begin{aligned} u_3(x) &= \sin(lx) + \sin(mx) \in \mathcal{H}_0, \\ u_4(x) &= \cos(lx) + \cos(mx) \in \mathcal{H}_0, \end{aligned}$$

proceeding as in (3.19), we have

$$\begin{aligned} u_3(x)u_{3x}(x) &= \frac{l}{2}\sin(2lx) + \frac{m}{2}\sin(2mx) + m\sin(lx)\cos(mx) + l\sin(mx)\cos(lx), \\ u_4(x)u_{4x}(x) &= -\frac{l}{2}\sin(2lx) - \frac{m}{2}\sin(2mx) - l\sin(lx)\cos(mx) - m\sin(mx)\cos(lx). \end{aligned} \tag{3.20}$$

Since

$$\begin{aligned} u_3(x)u_{3x}(x) - \frac{l}{2}\sin(2lx) - \frac{m}{2}\sin(2mx) &\in \mathcal{H}_1, \\ u_4(x)u_{4x}(x) + \frac{l}{2}\sin(2lx) + \frac{m}{2}\sin(2mx) &\in \mathcal{H}_1, \end{aligned}$$

It follows from (3.20) that $\sin(lx)\cos(mx), \sin(mx)\cos(lx) \in \mathcal{H}_1$.

Step 4. Note that for any $m \in \mathcal{I}$, $\sin(-mx), \cos(-mx) \in \mathcal{H}$, thus by the results in Step 2 and Step 3, we have

$$\cos(l \pm m)x, \sin(l \pm m)x \in \mathcal{H}_1 \quad \text{for } l, m \in \mathcal{I}.$$

Since \mathcal{I} is a generator, repeating the above steps, we derive

$$\text{span}\{\sin(mx), \cos(mx) : m \in \mathbb{Z}\} \subset \mathcal{H}_\infty,$$

this implies that $\mathcal{H}(\mathcal{I})$ is saturating.

The proof of Proposition 3.4 is complete.

3.5. Proof of Lemma 3.5

Let y be the solution to the system

$$\begin{cases} y_t + y_{xxxx} = f & \text{in } \mathbb{T} \times (0, T), \\ y(0) = y_0 & \text{in } \mathbb{T}. \end{cases} \quad (3.21)$$

Taking the scalar product in $L^2(\mathbb{T})$ of equation (3.21) with $y + \partial_x^{2s}y$, it follows that

$$\frac{1}{2} \frac{d}{dt} (\|y\|^2 + \|\partial_x^s y\|^2) + \|y_{xx}\|^2 + \|\partial_x^{s+2} y\|^2 = (y + \partial_x^{2s} y, f),$$

this yields

$$\|y(t)\|_s^2 + \int_0^t \|\partial_x^{s+2} y(r)\|^2 dr \leq C(\|y_0\|_s^2 + \int_0^t (y + \partial_x^{2s} y, f) dr).$$

Since

$$C \int_0^t (y + \partial_x^{2s} y, f) dr \leq C \int_0^t (\|y(r)\|^2 + \|f(r)\|^2 + \|\partial_x^{s-2} f(r)\|^2) dr + \frac{1}{2} \int_0^t \|\partial_x^{s+2} y(r)\|^2 dr,$$

we have

$$\|y(t)\|_s^2 + \int_0^t \|\partial_x^{s+2} y(r)\|^2 dr \leq C[\|y_0\|_s^2 + \int_0^t (\|y(r)\|^2 + \|f(r)\|_{s-2}^2) dr]. \quad (3.22)$$

By applying Gronwall's inequality, we have

$$\|y\|_{C([0,T];H^s(\mathbb{T}))} \leq C(\|y_0\|_s + \|f\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}).$$

Plugging this estimate into (3.22), it is shown that

$$\|y\|_{L^2(0,T;H^{s+2}(\mathbb{T}))} \leq C(\|y_0\|_s + \|f\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}).$$

Combining the above estimates, we conclude that

$$\|y\|_{\mathcal{X}_{T,s}} \leq C(\|y_0\|_s + \|f\|_{L^2(0,T;H^{s-2}(\mathbb{T}))}).$$

If we take $f = 0$ in (3.21), the solution y to (3.21) is $S(t)y_0$, it satisfies the first estimate in (3.3).

If we take $y_0 = 0$ in (3.21), the solution y to (3.21) is $\int_0^t S(t-s)f(s)ds$, it satisfies the second estimate in (3.3).

4. PROOF OF THEOREM 1.2

In this section, we will establish the irreducibility of stochastic KSE.

4.1. Continuity of solving operator for deterministic KSE

We consider the following integral equation:

$$u(t) - u_0 + \int_0^t (u_{xxxx} + u_{xx} + uu_x)(r)dr = g(t). \quad (4.1)$$

For $s \in \mathbb{R}$, we define spaces

$$X^s := C([0, T]; H^s(\mathbb{T})), \quad V_s := \{\zeta \in X^s \mid \zeta(0) = 0\}.$$

By the same argument as in Exercise 2.1.27 of [33], we know that (4.1) has a unique solution $u \in C([0, T], L^2(\mathbb{T})) \cap L^2(0, T; H^2(\mathbb{T}))$ if $u_0 \in L^2(\mathbb{T})$ and $g \in V_2$.

Proposition 4.1. *For any $T > 0$, the solution of equation (4.1) satisfies the following estimates:*

(i). *If $u_0 \in L^2(\mathbb{T})$ and $g \in V_2$,*

$$\sup_{0 \leq t \leq T} \|u(t)\|^2 + \int_0^T \|u(s)\|_2^2 ds \leq C(T, \|u_0\|, \|g\|_{V_2}). \quad (4.2)$$

(ii). *If $u_0 \in H^2(\mathbb{T})$ and $g \in V_4$,*

$$\sup_{0 \leq t \leq T} \|u(t)\|_2^2 + \int_0^T \|u(s)\|_4^2 ds \leq C(T, \|u_0\|_2, \|g\|_{V_4}). \quad (4.3)$$

Proof. (i). Making the substitution $u(t) = y(t) + g(t)$, then y satisfies the equation

$$\begin{cases} y_t + y_{xxx} + y_{xx} + (y + g)(y + g)_x = -g_{xxxx} - g_{xx} & \text{in } \mathbb{T} \times (0, T), \\ y(0) = u_0 & \text{in } \mathbb{T}. \end{cases} \quad (4.4)$$

Multiplying (4.4) with y , we have

$$\frac{1}{2} \frac{d}{dt} \|y\|^2 + \|y_{xx}\|^2 = -(y_{xx}, y) - ((yg)_x, y) - (gg_x + g_{xxxx} + g_{xx}, y).$$

The right hand side can be estimated as follows:

$$\begin{aligned} |(y_{xx}, y)| &\leq \frac{1}{4} \|y_{xx}\|^2 + \|y\|^2, \\ |((yg)_x, y)| &= \left| \int_{\mathbb{T}} g y y_x dx \right| = \frac{1}{2} \left| \int_{\mathbb{T}} g_x y^2 dx \right| \leq C \|g\|_2 \|y\|^2, \\ |(gg_x + g_{xxxx} + g_{xx}, y)| &\leq |(gg_x + g_{xx}, y)| + |(g_{xx}, y_{xx})| \\ &\leq \frac{1}{4} \|y_{xx}\|^2 + C \|y\|^2 + C(\|g\|_1^4 + \|g\|_2^2). \end{aligned}$$

Thus, we can obtain that

$$\frac{d}{dt} \|y\|^2 + \|y_{xx}\|^2 \leq C(\|g\|_2 + 1) \|y\|^2 + C(\|g\|_1^4 + \|g\|_2^2).$$

It follows from Gronwall's inequality that

$$\sup_{0 \leq t \leq T} \|y(t)\|^2 + \int_0^T \|y_{xx}(s)\|^2 ds \leq C(T, \|u_0\|, \|g\|_{V_2}). \quad (4.5)$$

Due to the fact that $u(t) = y(t) + g(t)$, we can obtain (4.2).

(ii). Multiplying (4.4) with y_{xxxx} , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|y_{xx}\|^2 + \|y_{xxxx}\|^2 \\ &= -(y_{xx}, y_{xxxx}) - (yy_x, y_{xxxx}) - ((yg)_x, y_{xxxx}) - (gg_x + g_{xxx} + g_{xx}, y_{xxxx}) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

According to Lemma 2.3, the following holds

$$\begin{aligned} |I_1| &\leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon) \|y_{xx}\|^2, \\ |I_2| &\leq \|y\| \|y_x\|_{L^\infty} \|y_{xxxx}\| \\ &\leq C \|y\| \|y\|_2 \|y_{xxxx}\| \\ &\leq C \|y\|^2 \|y_{xxxx}\| + C \|y\| \|y_{xx}\| \|y_{xxxx}\| \\ &\leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon) (\|y\|^4 + \|y\|^2 \|y_{xx}\|^2), \\ |I_3| &\leq C \|y\|_1 \|g\|_1 \|y_{xxxx}\| \\ &\leq C \|y\| \|g\|_1 \|y_{xxxx}\| + C \|y_{xx}\| \|g\|_1 \|y_{xxxx}\| \\ &\leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon) \|g\|_1^2 (\|y\|^2 + \|y_{xx}\|^2), \\ |I_4| &\leq C (\|g\|_1^2 + \|g\|_4) \|y_{xxxx}\| \\ &\leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon) (\|g\|_1^4 + \|g\|_4^2). \end{aligned}$$

Taking ε sufficiently small, it follows that

$$\frac{d}{dt} \|y_{xx}\|^2 + \|y_{xxxx}\|^2 \leq C(1 + \|y\|^2 + \|g\|_1^2) \|y_{xx}\|^2 + C(\|y\|^4 + \|g\|_1^2 \|y\|^2 + \|g\|_1^4 + \|g\|_4^2).$$

Applying (4.5) and Gronwall's inequality, we can obtain that

$$\sup_{0 \leq t \leq T} \|y_{xx}(t)\|^2 + \int_0^T \|y_{xxxx}(s)\|^2 ds \leq C(T, \|u_0\|_2, \|g\|_{V_4}).$$

This implies (4.3).

The proof of Proposition 4.1 is complete. □

Proposition 4.2. *Let $u = F(g)$ be the solution of (4.1), then*

(i). *If $u_0 \in L^2(\mathbb{T})$, the map $F : V_2 \rightarrow X^0$, $g \mapsto u$ is continuous;*

(ii). *If $u_0 \in H^2(\mathbb{T})$, the map $F : V_4 \rightarrow X^2$, $g \mapsto u$ is continuous.*

Proof. (i). For any fixed $R > 0$, let $g_i \in V_2$, $\|g_i\|_{V_2} \leq R$ ($i = 1, 2$) and u_i be the solution of (4.1) with $g = g_i$. Define $y_i := u_i - g_i$ ($i = 1, 2$), $g := g_1 - g_2$, $y := y_1 - y_2$, then y satisfies the following equation:

$$\begin{cases} y_t + y_{xxxx} + y_{xx} + u_1 u_{1x} - u_2 u_{2x} = -g_{xxxx} - g_{xx} & \text{in } \mathbb{T} \times (0, T), \\ y(0) = 0 & \text{in } \mathbb{T}. \end{cases} \quad (4.6)$$

Multiplying (4.6) with y yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y\|^2 + \|y_{xx}\|^2 &= -(y_{xx}, y) - (u_1 u_{1x} - u_2 u_{2x}, y) - (g_{xxxx} + g_{xx}, y) \\ &\leq \varepsilon \|y_{xx}\|^2 + C(\varepsilon) (\|y\|^2 + \|g\|_2^2) - (u_1 u_{1x} - u_2 u_{2x}, y). \end{aligned}$$

It remains only to estimate $(u_1 u_{1x} - u_2 u_{2x}, y)$. Since

$$\begin{aligned} |(u_1 u_{1x} - u_2 u_{2x}, y)| &= |(u_1(y+g)_x + (y+g)u_{2x}, y)| \\ &\leq |(u_1 y_x, y)| + |(u_1 g_x, y)| + |(y u_{2x}, y)| + |(g u_{2x}, y)|, \end{aligned}$$

it is shown that

$$\begin{aligned} |(u_1 y_x, y)| + |(y u_{2x}, y)| &\leq C \int_{\mathbb{T}} (|u_1| + |u_2|) |y| |y_x| dx \\ &\leq C (\|u_1\| + \|u_2\|) \|y\| \|y_x\|_{L^\infty(\mathbb{T})} \\ &\leq C (\|u_1\| + \|u_2\|) \|y\| \|y\|_2 \\ &\leq C (\|u_1\| + \|u_2\|) \|y\|^2 + C (\|u_1\| + \|u_2\|) \|y\| \|y_{xx}\| \\ &\leq \varepsilon \|y_{xx}\|^2 + C(\varepsilon) (1 + \|u_1\|^2 + \|u_2\|^2) \|y\|^2, \end{aligned}$$

$$\begin{aligned} |(u_1 g_x, y)| + |(g u_{2x}, y)| &\leq C \int_{\mathbb{T}} (|u_1| + |u_2|) |g_x| |y| dx + \int_{\mathbb{T}} |g u_{2x} y| dx \\ &\leq C (\|u_1\| + \|u_2\|) \|g\|_2 \|y\| + C \|g\|_1 \|u_2\| \|y_x\| \\ &\leq C (\|u_1\| + \|u_2\|) \|g\|_2 \|y\| + C \|g\|_1 \|u_2\| (\|y\| + \|y_{xx}\|) \\ &\leq C (\|u_1\| + \|u_2\|) \|g\|_2 \|y\| + C \|g\|_1 \|u_2\| \|y_{xx}\| \\ &\leq \varepsilon \|y_{xx}\|^2 + C \|y\|^2 + C(\varepsilon) (\|u_1\|^2 + \|u_2\|^2) \|g\|_2^2. \end{aligned}$$

Combining the above estimates and choosing $0 < \varepsilon \ll 1$, we get

$$\begin{aligned} \frac{d}{dt} \|y\|^2 + \|y_{xx}\|^2 &\leq C(1 + \|u_1\|^2 + \|u_2\|^2) \|y\|^2 + C(1 + \|u_1\|^2 + \|u_2\|^2) \|g\|_2^2 \\ &\leq C(T, \|u_0\|, R) \|y\|^2 + C(T, \|u_0\|, R) \|g\|_2^2, \end{aligned}$$

here we have used (4.2). It follows from Gronwall's inequality that

$$\|y\|_{X^0}^2 \leq C(T, \|u_0\|, R) \|g\|_{V_2}^2,$$

which leads to

$$\|u_1 - u_2\|_{X^0} \leq C(T, \|u_0\|, R) \|g_1 - g_2\|_{V_2}.$$

(ii). For any fixed $R > 0$, let $g_i \in V_4$, $\|g_i\|_{V_4} \leq R$ ($i = 1, 2$).

Multiplying (4.6) with y_{xxxx} , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|y_{xx}\|^2 + \|y_{xxxx}\|^2 &= -(y_{xx}, y_{xxxx}) - (u_1 u_{1x} - u_2 u_{2x}, y_{xxxx}) - (g_{xxxx} + g_{xx}, y_{xxxx}) \\ &\leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon) (\|y_{xx}\|^2 + \|g\|_4^2) - (u_1 u_{1x} - u_2 u_{2x}, y_{xxxx}). \end{aligned}$$

It is sufficient to estimate $(u_1 u_{1x} - u_2 u_{2x}, y_{xxxx})$. Proceeding as in the proof (i), we can obtain that

$$\begin{aligned}
 & |(u_1 u_{1x} - u_2 u_{2x}, y_{xxxx})| \\
 & \leq |(u_1 y_x, y_{xxxx})| + |(u_1 g_x, y_{xxxx})| + |(y u_{2x}, y_{xxxx})| + |(g u_{2x}, y_{xxxx})| \\
 & \leq (\|u_1\|_{L^\infty(\mathbb{T})} \|y_x\| + \|u_1\|_{L^\infty(\mathbb{T})} \|g_x\| + \|y\|_{L^\infty(\mathbb{T})} \|u_{2x}\| + \|g\|_{L^\infty(\mathbb{T})} \|u_{2x}\|) \|y_{xxxx}\| \\
 & \leq C[(\|u_1\|_1 + \|u_2\|_1) \|y\|_1 + (\|u_1\|_1 + \|u_2\|_1) \|g\|_1] \|y_{xxxx}\| \\
 & \leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon)[(\|u_1\|_1^2 + \|u_2\|_1^2) \|y\|_1^2 + (\|u_1\|_1^2 + \|u_2\|_1^2) \|g\|_1^2] \\
 & \leq \varepsilon \|y_{xxxx}\|^2 + C(\varepsilon)[(\|u_1\|_1^2 + \|u_2\|_1^2)(\|y\|^2 + \|y_{xx}\|^2) + (\|u_1\|_1^2 + \|u_2\|_1^2) \|g\|_1^2].
 \end{aligned}$$

Taking ε small enough, we arrive at

$$\frac{d}{dt} \|y_{xx}\|^2 + \|y_{xxxx}\|^2 \leq C(1 + \|u_1\|_1^2 + \|u_2\|_1^2) \|y_{xx}\|^2 + C(1 + \|u_1\|_1^2 + \|u_2\|_1^2) (\|g\|_4^2 + \|y\|^2).$$

According to the results in (i), we have

$$\|y(t)\|^2 \leq C(T, \|u_0\|, R) \|g\|_{V_2}^2 \leq C(T, \|u_0\|, R) \|g\|_{V_4}^2.$$

Therefore, it follows that

$$\frac{d}{dt} \|y_{xx}\|^2 + \|y_{xxxx}\|^2 \leq C(T, \|u_0\|_1, R) \|y_{xx}\|^2 + C(T, \|u_0\|_1, R) \|g\|_{V_4}^2.$$

Applying Gronwall's inequality, we get

$$\|y\|_{X^2}^2 \leq C(T, \|u_0\|_2, R) \|g\|_{V_4}^2,$$

this implies

$$\|u_1 - u_2\|_{X^2} \leq C(T, \|u_0\|_2, R) \|g_1 - g_2\|_{V_4}.$$

The proof of Proposition 4.2 is complete. \square

4.2. Proof of Theorem 1.2

Denote $H := L^2(\mathbb{T})$, $V := H^2(\mathbb{T})$, $\mathcal{E} := \text{span}\{\sin x, \cos x, 1\}$. Set $E := C([0, T]; \mathcal{E})$. We know that the norms $\|\cdot\|_E$ and $\|\cdot\|_{V_2}$ are equivalent for the functions in E . We define $\xi(t) := b_1 \beta_1(t) \sin x + b_2 \beta_2(t) \cos x + b_3 \beta_3(t)$.

For any $u_0 \in H$, $t > 0$, we have $u^{u_0}(t) \in V$ a.s., where u^{u_0} is the solution to (1.1) with initial date u_0 . Since u^{u_0} is Markov in H , for any $u_1 \in H$, $T > 0$, $\varepsilon > 0$,

$$\begin{aligned}
 \mathbb{P}(\|u^{u_0}(T) - u_1\| < \varepsilon) &= \int_V \mathbb{P}(\|u^{u_0}(T) - u_1\| < \varepsilon \mid u^{u_0}(t) = v) \mathbb{P}(u^{u_0}(t) \in dv) \\
 &= \int_V \mathbb{P}(\|u^v(T-t) - u_1\| < \varepsilon) \mathbb{P}(u^{u_0}(t) \in dv)
 \end{aligned}$$

To prove that $\mathbb{P}(\|u^{u_0}(T) - u_1\| < \varepsilon) > 0$, it is sufficient to prove that for any $T > 0$, $u_0 \in V$,

$$\mathbb{P}(\|u^{u_0}(T) - u_1\| < \varepsilon) > 0. \tag{4.7}$$

Now, we prove that (4.7) holds for any $T > 0, u_0 \in V$. We denote the solution to (4.1) with u_0 and g by $u^{u_0}(\cdot, g)$, by this notation, the solution of (1.1) can be written as $u^{u_0}(\cdot, \xi)$. Let $u_0 \in V$ and $u_1 \in H$, since V is dense in H , there exists a function $\tilde{u}_1 \in V$ such that $\|\tilde{u}_1 - u_1\| < \frac{\varepsilon}{4}$. According to Theorem 3.3, (3.1) is approximately controllable by \mathcal{E} -valued control, for any $\varepsilon > 0$, there exists a control $\eta \in L^2(0, T; \mathcal{E})$ such that

$$\|u^{u_0}(T, \xi_1) - \tilde{u}_1\|_2 < \frac{\varepsilon}{4},$$

where $\xi_1(t) := \int_0^t \eta(s) ds$, this leads to the fact

$$\|u^{u_0}(T, \xi_1) - u_1\| < \frac{\varepsilon}{2},$$

This implies that

$$\mathbb{P}(\|u^{u_0}(T, \xi) - u_1\| < \varepsilon) \geq \mathbb{P}(\|u^{u_0}(T, \xi) - u^{u_0}(T, \xi_1)\| < \frac{\varepsilon}{2}).$$

It follows from Proposition 4.2 that there exists a positive constant $\delta > 0$ such that when $\|\xi - \xi_1\|_E < \delta$,

$$\|u^{u_0}(T, \xi) - u^{u_0}(T, \xi_1)\| < \frac{\varepsilon}{2},$$

thus, we can obtain that

$$\mathbb{P}(\|u^{u_0}(T, \xi) - u_1\| < \varepsilon) \geq \mathbb{P}(\|\xi - \xi_1\|_E < \delta).$$

Since the support of $\mathcal{L}(\xi)$ is E , we have

$$\mathbb{P}(\|\xi - \xi_1\|_E < \delta) > 0,$$

then,

$$\mathbb{P}(\|u^{u_0}(T, \xi) - u_1\| < \varepsilon) > 0.$$

The proof of Theorem 1.2 is complete.

Acknowledgements. Peng Gao would like to thank the referees and the editors for their careful comments and useful suggestions. Peng Gao would like to thank the financial support of the China Scholarship Council (No. 201806625036) and the hospitality of CNRS and IMJ, Université Paris Diderot-Paris 7 during his visit from December 2018 to November 2019. Peng Gao is supported by the Fundamental Research Funds for the Central Universities (2412020FZ022).

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