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EXTENDED MEAN-FIELD CONTROL PROBLEM WITH PARTIAL OBSERVATION*

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Abstract. We study an extended mean-field control problem with partial observation, where the dynamic of the state is given by a forward-backward stochastic differential equation of McKean-Vlasov type. The cost functional, the state and the observation all depend on the joint distribution of the state and the control process. Our problem is motivated by the recent popular subject of mean-field games and related control problems of McKean-Vlasov type. We first establish a necessary condition in the form of Pontryagin's maximum principle for optimality. Then a verification theorem is obtained for optimal control under some convex conditions of the Hamiltonian function. The results are also applied to studying linear-quadratic mean-filed control problem in the type of scalar interaction.

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1. Introduction

The stochastic differential equations (SDEs) of McKean-Vlasov type were introduced by Kac [23] in 1956 as a stochastic model for the Vlasov-kinetic equation of plasma. In recent years, mean-filed games have become very popular subjects since the pioneering work of Lasry and Lions [24–26] and simultaneously Caines, Huang and Malhamé [20]. Since then, the research of mean-field models has wide applications in many fields like finance and economics. The related McKean-Vlasov type stochastic control problems attract the attentions of many researchers, see for example Andersson, Djehiche [3], Buckdahn, Djehiche and Li [10], Carmona and Delarue [16], Li and Liu [29], Meyer-Brandis *et al.* [33], Shen *et al.* [40], Tembine *et al.* [42] and the references therein. The readers are referred to the monographs of Carmona and Delarue [17] and Bensoussan *et al.* [6] for an overview of McKean-Vlasov type control problems.

The above mentioned literatures about McKean-Vlasov type stochastic control problems are considered under the assumption that the stochastic noises are observed completely. However, in the real-world, usually controllers can only get partial information at most cases. Thus the stochastic control problems with partial observation are extensively studied, see *e.g.* Bensoussan [4], Tang [41], Wu [48], Xiong [49] as well as Wang, Wu and Xiong [45, 46]. We refer the readers to Caines and Kizikale [14], Huang, Caines and Malhamé [21], Şen and Caines

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[38, 39] for the investigation of mean-filed games with partial observations. Concerning the mean-field control problems with partial observation, the readers are referred to Hafayed, Abbas and Abba [19], Li and Fu [28], Ma and Liu [31], Wang *et al.*, [43], Wang *et al.* [47] and the reference therein. These papers are all focus on the mean-filed interaction of scalar type. Recently, Buckdahn *et al.* [12] studied mean-filed non-Markovian stochastic optimal control problems with partial observation, where the coefficients depend on the conditional law of the state. Moreover, in their continued work [9], Buckdahn, Chen and Li introduced the partial derivative with respect to the measure and considered a general mean-filed stochastic optimal control problems with partial observation, where they do not need any regularity of the coefficients neither in the control variable nor with respect to the law of control process.

To the best of our knowledge, the control problems for partially observed forward-backward stochastic differential equations (FBSDEs) of mean-field type is quite a new topic, and only some special cases have been solved. For example, Li and Liu [29] considered an optimal control problem for fully coupled FBSDEs of mean-filed type but without partial observation; Meherrem and Hafayed [32] studied stochastic optimal control problem for general McKean-Vlasov-type FBSDEs driven by Teugels martingales, associated with some Lévy process having moments of all orders, and an independent Brownian motion. Ma and Liu [31] introduced a linear quadratic optimal control problems for partially observed FBSDEs of mean-filed type; Wang *et al.* [43] investigated an optimal control problem derived by mean-field FBSDE with noisy observation, where the drift coefficients of the state equation and the observation equation are linear with respect to the state and its expectation. Partially observed optimal control problems for FBSDEs with scalar type mean-filed interaction were studied by Liu and Fu [28].

We mention that, except the paper by Buckdahn *et al.* [9], the mean-filed interaction are given only through the distribution of state of the problem. However, in many practical applications, it is necessary to study the extended case where the interactions are given through the joint distributions of state and control. For example, motivated by the problems involving the minimization of variance, Yong [50] studied linear quadratic optimal control problems for mean-field SDEs, in which both the state and the cost functional allow to depend on the expected value of the control, then the feedback optimal control is obtained through two Riccati equations. Motivated by certain application to economics such as production of an exhaustible resource, Graber [18] extended the work of Yong [50] to study linear quadratic control problems for mean-field SDEs with common noise involving the expected value of the control. Along this direction, Li *et al.* [30] investigated linear quadratic control problems for mean-field backward stochastic differential equations (BSDEs). Let us mention that the works of Graber [18], Li *et al.* [30], Yong [50] are all focused on linear quadratic problem including expected value of the control but not the distribution of the control. For the case that with nonlinear dynamic and joint distribution of state and control, it is recently studied by Acciaio, Backhoff-Veraguas and Carmona [1] and Pham and Wei [37]. In fact, Pham and Wei [37] studied the close-loop feedback control for such mean-field SDEs through dynamic programming principle and related Bellman equations, and they also give applications to mean variance portfolio selection and a systemic risk model. Without the restriction on close-loop feedback control, Acciaio *et al.* [1] established the stochastic maximum principle for the extended control problems of mean-field SDEs *via* a probabilistic approach, and they also study the weak formulation and the applications to optimal liquidation with market impact.

In view of the wide applications in finance and economics of above extended mean-field control system, the purpose of this paper is to study the maximum principle of the extended mean-field control problem with partial observation, where the state and the observation both depend on the joint distribution of the state and the control process. More precisely, the state dynamic is given by the following mean-field type forward-backward system

$$\begin{cases} \mathrm{d}x_t = f(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}t + \sigma(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}W_t + \bar{\sigma}(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}\bar{W}_t^v, \\ -\mathrm{d}y_t = g(t,x_t,y_t,z_t,\bar{z}_t,v_t,\mathcal{L}(x_t,y_t,z_t,\bar{z}_t,v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ x(0) = x_0, \quad y(T) = \Phi(x_T,\mathcal{L}(x_T)), \end{cases} \tag{1.1}$$

and $Y(\cdot)$ is the observation process given by

$$\begin{cases} \mathrm{d}Y_t = h(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}t + \mathrm{d}\bar{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{1.2}$$

where $v(\cdot)$ is a control process adapted to the filtration generated by the observation process $Y(\cdot)$. Throughout the paper, for any given process $\{v_t\}_{t\in[0,T]}$, we denote it briefly as $v(\cdot)$. In light of the nonlinear filtering theory (see [4, 12]), we assume that there exists a reference probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ on which $(W(\cdot), Y(\cdot))$ is a multi-dimensional standard Brownian motion. The symbol $\mathcal{L}$ stands for the law of the given random element under $\mathbb{P}$. By inserting (1.2) into (1.1), we get

$$\begin{cases} \mathrm{d}x_t = [f(t, x_t, v_t, \mathcal{L}(x_t, v_t)) - \bar{\sigma}(t, x_t, v_t, \mathcal{L}(x_t, v_t))h(t, x_t, v_t, \mathcal{L}(x_t, v_t))]\,\mathrm{d}t \\ \qquad\quad + \sigma(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}W_t + \bar{\sigma}(t, x_t, v_t, \mathcal{L}(x_t, v_t))\mathrm{d}Y_t, \\ -\mathrm{d}y_t = g(t, x_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ x(0) = x_0, \quad y(T) = \Phi(x_T, \mathcal{L}(x_T)). \end{cases} \tag{1.3}$$

Under suitable assumptions, equation (1.3) has a unique strong solution $(x(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot))$ for each $v(\cdot) \in \mathcal{U}_{ad}$ (see Rem. 2.2), where $\mathcal{U}_{ad}$ denotes the set of admissible partially observed controls defined clearly in Section 2. If we introduce

$$\rho_t = \exp\left\{\int_0^t h(s, x_s, v_s, \mathcal{L}(x_s, v_s))\mathrm{d}Y_s - \frac{1}{2}\int_0^t |h(s, x_s, v_s, \mathcal{L}(x_s, v_s))|^2\mathrm{d}s\right\} \tag{1.4}$$

and define a probability measure $\mathbb{P}^v$ s.t. $\mathrm{d}\mathbb{P}^v = \rho_T\mathrm{d}\mathbb{P}$, then under suitable assumptions on h (*e.g.* h is bounded), according to Girsanov's theorem, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v, x(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot), Y(\cdot), W(\cdot), \bar{W}^v(\cdot))$ is a weak solution of system (1.1)–(1.2). Then our associated cost functional can be given by (see *e.g.* [41, 46] for the case without mean-field term)

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l(t, \rho_t, x_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t + \chi(\rho_T, x_T, \mathcal{L}(x_T)) + \gamma(y_0)\right], \tag{1.5}$$

where $\mathbb{E}^v$ stands for the expectation w.r.t. the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$. Our partially observed optimal control problem is to seek $u(\cdot) \in \mathcal{U}_{ad}$ s.t. $J(u(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot))$. The aim is to establish Pontryagin's maximum principle and verification theorem which will give respectively the necessary condition and sufficient condition for the optimality.

Let us summarize the difficulties of above problem and our contributions. In this paper, we study an extended mean-field control problem with partial observation, where the dynamic of the state is given by an FBSDE of extended McKean-Vlasov type and the state is partially observed *via* a process whose dynamic is also in extended McKean-Vlasov type. Thus, the model of our paper is novel, it contains the partial observation structure and the joint distribution of the state and the control, which leads to several difficulties. The main difficulties and innovations of this paper are as follows:

(I) The first difficulty we meet to get the maximum principle for our problem is the partial observation structure. In our paper, inspired by Wang, Wu and Xiong [45, 46], we consider that the state and observation are defined on a reference probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (see (1.1) and (1.2)), but the cost functional is defined on probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v)$ (see (1.5)). In this kind of model, because of the interdependence of control process and observation processes, we can not use the classical method to construct adjoint process and variational equations.

To solve this problem, we adopt the methods of Tang [41]. On the one hand, we transform the original partial observation problem to classical problem on the reference probability space by Girsanov's transformation and the dimensional extension, then we construct new adjoint processes and variational equations for the state and the observation. On the other hand, due to the application of Girsanov's transformation, the coefficients l and χ in cost functional (1.5) will be multiplied by ρ (see (2.8)), which leads to the necessity of high order estimates and high order convergence results of variational equations when we derive the variational inequality, see Lemma 3.2, Lemma 3.4 and the proof of Lemma 3.6.

(II) The second difficulty is the joint distribution dependence of state dynamic which is a mean-field FBSDE. On the one hand, when we take the variation of control $u(\cdot)$, the joint distribution of $u(\cdot)$ and $(x(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot))$ will change accordingly, which leads to the failure of classical variational method to be applied. To solve this problem, we need use the L-derivative w.r.t. probability measure, especially the partial L-derivatives because of the dependence of joint distribution. In this case, we can obtain new adjoint equations and variation equations, which are both mean-field FBSDEs (see (3.11) and (3.13)), and we give the existence and uniqueness of solutions for variation equations and adjoint equations, see Theorem 3.1 and Remark 3.8.
On the other hand, as we mentioned in (I), we need high order estimates of variation equations which can compose a mean-field FBSDE, see (3.1). Fortunately, it is possible to obtain the high order estimates of (3.1) since this mean-filed FBSDE is not fully coupled. However, due to the existence of mean-field term, it requires subtle calculations and skills, especially for the higher order estimates of the mean-filed backward equation, see the explanation in the first paragraph of the proof of Lemma 3.4, Appendix A.3. Indeed, we first establish the L^2 estimates of the variational equations and then use it to obtain the desired L^p estimates. Finally, with the help of high order estimates, we can obtain the related variational inequality which allows us to establish the Pontryagin's maximum principle under the reference probability space.

(III) The third difficulty is how to obtain the verification theorem. We emphasize that due to the partial observation structure, when transferring the original control problem to associate equivalent problem, the coefficients l and χ in the original cost functional (1.5) will be multiplied by ρ. So it causes that the convexity assumptions of Hamiltonian function can not be satisfied if l and χ in the original cost functional (1.5) do not depend on ρ, see Remark 4.2 and Remark 5.1. However, one can observe that if l and χ are allowed to depend on ρ, then the convexity assumptions (see (H.3) and (H.4)) may hold. That is the reason why our l and χ depend on ρ in (1.5).
Moreover, due to the existence of the joint distribution, we should introduce a new convexity assumption of the Hamiltonian function, see (H.4), then we can establish the verification theorem of extended mean-field control problem under such new convexity assumption. Furthermore, to illustrate the verification theorem, we also give a linear quadratic example which provides an optimal control.

The organization of this paper is as follows. In Section 2, we formulate the extended mean-field problem with partial observation. We also review some preliminaries about L-derivative. In Section 3, we establish a new type Pontryagin's stochastic maximum principle. Section 4 provides a verification theorem under new convexity assumption. Section 5 considers two kinds of examples, scalar interaction model and linear-quadratic model. In the appendix, we give the detailed proofs of some lemmas of Section 3.

2. Formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, here we denote $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$. Suppose that $(W(\cdot), Y(\cdot))$ is a standard $\mathbb{R}^m \times \mathbb{R}^d$ valued Brownian motion defined on above reference probability space, where $Y(\cdot)$ is the observation process. Let $\mathbb{F}^W := \{\mathcal{F}_t^W\}_{0\le t\le T}$ and $\mathbb{F}^Y := \{\mathcal{F}_t^Y\}_{0\le t\le T}$ be the natural filtration generated by $W(\cdot)$ and $Y(\cdot)$ respectively, and augmented by all $\mathbb{P}$-null sets, and let $\mathbb{F} := \{\mathcal{F}_t^{W,Y}\}_{0\le t\le T}$ be the natural filtration generated by $(W(\cdot), Y(\cdot))$ and augmented by all $\mathbb{P}$-null sets.

For a given filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\le t\le T}$, we denote by $\mathbb{H}^{2,n}_{\mathbb{G}}$ the space of all $\mathbb{R}^n$-valued, $\mathbb{G}$-progressively measurable processes $\eta(\cdot)$ on $[0,T]$ such that $\mathbb{E}\int_0^T |\eta_t|^2 dt < +\infty$. We shall also denote by $\mathbb{S}^{2,n}_{\mathbb{G}}$ the set of all continuous processes $\eta(\cdot) \in \mathbb{H}^{2,n}_{\mathbb{G}}$ such that $\mathbb{E}\left[\sup_{t\in[0,T]} |\eta_t|^2\right] < +\infty$.

Let (E,d) be a Polish space. The σ-field $\mathcal{E}$ equipping E is assumed to be the Borel σ-field $\mathcal{B}(E)$. We use the notation $\mathcal{P}_2(E)$ for the space of probability measures with finite second moments over E. Then we can define the 2-Wasserstein distance $W_2(\mu,\mu^{'})$ on $\mathcal{P}_2(E)$ by

$$W_2(\mu,\mu^{'}) := \inf\left\{\left(\int_{E\times E} d(x,y)^2 \pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{2}}; \quad \pi \in \mathcal{P}_2(E\times E) \text{ with marginals } \mu \text{ and } \mu^{'}\right\}.$$

Then $(\mathcal{P}_2(E), W_2)$ is a Polish space. Moreover, one can systematically equip $\mathcal{P}_2(E)$ with its Borel σ-field and characterize real valued Borel measurable functions on $\mathcal{P}_2(E)$, for more details, see [[17], Chap. 5]. Noting that if ξ,ξ' are E-valued random variables of order 2 (An E-valued random variable ξ is order of 2 means that $\mathbb{E}[d(\xi_0,\xi)^2] < \infty$ for one, and hence for all $\xi_0 \in E$), we have

$$W_2(\mathcal{L}(\xi),\mathcal{L}(\xi')) \le [\mathbb{E}|\xi-\xi'|^2]^{1/2},$$

where we recall that $\mathcal{L}(\xi)$ stands for the law of ξ under $\mathbb{P}$. Moreover, if E is Euclidean space, then by applying Corollary 5.4 of [17] and Hölder's inequality, one also have

$$|\mathbb{E}\xi - \mathbb{E}\xi'| \le W_1(\mathcal{L}(\xi),\mathcal{L}(\xi')) \le W_2(\mathcal{L}(\xi),\mathcal{L}(\xi')) \le [\mathbb{E}|\xi-\xi'|^2]^{1/2}.$$

We consider the following extended mean-filed type FBSDE:

$$\begin{cases} \mathrm{d}x_t = f(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}t + \sigma(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}W_t + \bar{\sigma}(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}\bar{W}^v_t, \\ -\mathrm{d}y_t = g(t,x_t,y_t,z_t,\bar{z}_t,v_t,\mathcal{L}(x_t,y_t,z_t,\bar{z}_t,v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ x(0) = x_0, \quad y(T) = \Phi(x_T,\mathcal{L}(x_T)), \end{cases} \tag{2.1}$$

where the coefficients $f : [0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^k) \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^k) \to \mathbb{R}^{n\times m}$, $\bar{\sigma} : [0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^k) \to \mathbb{R}^{n\times d}$, and $g : [0,T]\times\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^{l\times m}\times\mathbb{R}^{l\times d}\times\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^{l\times m}\times\mathbb{R}^{l\times d}\times\mathbb{R}^k) \to \mathbb{R}^l$ are measurable functions. Here $v(\cdot)$ is a control process belonging to $\mathcal{U}_{ad}$ which is the set of $\mathbb{F}^Y$-progressively measurable processes $v(\cdot)$ taking values in a closed-convex set $U \in \mathbb{R}^k$ such that $\sup_{t\in[0,T]}\mathbb{E}\left[|v_t|^4\right] < +\infty$.

In our problem, the state process $(x(\cdot),y(\cdot),z(\cdot),\bar{z}(\cdot))$ can not be directly observed. Instead, we can observe a related process $Y(\cdot)$ which is governed by the following SDE

$$\begin{cases} \mathrm{d}Y_t = h(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}t + \mathrm{d}\bar{W}^v_t, \\ Y_0 = 0, \end{cases} \tag{2.2}$$

where $h : [0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times\mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^k) \to \mathbb{R}^d$ is a measurable function. For each $v(\cdot) \in \mathcal{U}_{ad}$, as explained in the introduction, if we define the stochastic process $\rho(\cdot)$ as the solution of the following SDE

$$\begin{cases} \mathrm{d}\rho_t = \rho_t h(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}Y_t, \\ \rho_0 = 1, \end{cases} \tag{2.3}$$

the system (2.1)–(2.2) has a weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^v, x(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot), Y(\cdot), W(\cdot), \bar{W}^v(\cdot))$, where $\mathrm{d}\mathbb{P}^v = \rho_T \mathrm{d}\mathbb{P}$, see Remark 2.2 for more details. Then the cost functional is given by

$$J(v(\cdot)) = \mathbb{E}^v \left[\int_0^T l(t, \rho_t, x_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t)) \mathrm{d}t + \chi(\rho_T, x_T, \mathcal{L}(x_T)) + \gamma(y_0) \right], \tag{2.4}$$

where $\mathbb{E}^v$ is the expectation w.r.t. the probability measure $\mathbb{P}^v$. Note that, in above framework, y_0 is deterministic, so it is not necessary to have $\mathcal{L}(y_0)$ in γ.

Now, let us formulate our extended mean-field partially observed control problem.

Problem (EMFPOC). Find $u(\cdot) \in \mathcal{U}_{ad}$ such that $J(u(\cdot)) = \inf_{v(\cdot) \in \mathcal{U}_{ad}} J(v(\cdot))$ subject to (2.1)–(2.4).

Then $u(\cdot)$ is called an optimal partially observed control of **Problem (EMFPOC)**.

2.1. Partial L-differentiability of functions of measures

Due to the appearance of the joint distribution of the control and state process in our extended mean-field partially observed control problem, we use the concept of L-derivative w.r.t. probability measure introduced by Lions, see *e.g.* [15–17]. For the convenience of the reader, in this subsection, we briefly recall the definition of L-derivative and the concept of joint differentiability for functions depending upon a point in $\mathbb{R}^q$ and a probability measure in $\mathcal{P}_2(\mathbb{R}^p)$. We refer the readers to [17] for more details.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which is rich enough in the sense that for every $\mu \in \mathcal{P}_2(\mathbb{R}^p)$, there is a random variable $X \in L^2(\Omega; \mathbb{R}^p)$ with law μ (*i.e.* $\mathbb{P}_X = \mu$). Let us consider a function $f : \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^p) \ni (x, \mu) \to f(x, \mu) \in \mathbb{R}$. We call f is jointly L-differentiable at (x, μ) if there exists $X \in L^2(\Omega; \mathbb{R}^p)$ with $\mathbb{P}_X = \mu$ such that the lifting $\widetilde{f} : \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p) \ni (x, X) \to f(x, \mathbb{P}_X) \in \mathbb{R}$ is jointly Fréchet differentiable at (x, X) and we denote $[D\widetilde{f}](x, X)$ as the Fréchet derivative of $\widetilde{f}$. Thanks to self-duality of L^2 spaces, $[D\widetilde{f}](x, X)$ can be viewed as an element $D\widetilde{f}(x, X)$ of $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p)$ in the sense that

$$[D\widetilde{f}](x, X)(Y) = \mathbb{E}[D\widetilde{f}(x, X) \cdot Y] \quad \text{for all} \quad Y \in \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p).$$

Then we can introduce the partial derivatives in x and μ of f, respectively as $\mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^p) \ni (x, \mu) \to \partial_x f(x, \mu) \in \mathbb{R}^q$ and $\mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^p) \ni (x, \mu) \to \partial_\mu f(x, \mu)(\cdot) \in L^2(\mathbb{R}^p, \mu; \mathbb{R}^p)$. The partial Fréchet derivative of $\widetilde{f}$ in the direction X is given by $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p) \ni (x, X) \to D_X \widetilde{f}(x, X) = \partial_\mu f(x, \mathbb{P}_X)(X) \in L^2(\Omega; \mathbb{R}^p)$. Thus the random variable $D\widetilde{f}(x, X)$ can be represented as

$$D\widetilde{f}(x, X) = (\partial_x f(x, \mathbb{P}_X)(X), \partial_\mu f(x, \mathbb{P}_X)(X)).$$

We call the functions $\partial_x f(\cdot, \mathbb{P}_X)(\cdot)$ and $\partial_\mu f(\cdot, \mathbb{P}_X)(\cdot)$ which is defined on $\mathbb{R}^q \times \mathbb{R}^p$ and valued, respectively, on $\mathbb{R}^q, \mathbb{R}^p$, the partial L-derivatives of f at $(x, \mathbb{P}_X)$. We often use the fact that joint continuous differentiability in the two arguments is equivalent to the partial differentiability in each of the two arguments together with the joint continuity of the partial derivatives (see *e.g.* assumption (H1) in next subsection). Here, the joint continuity of $\partial_x f$ means the joint continuity w.r.t the Euclidean distance on $\mathbb{R}^q$ and the 2-Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^p)$. The joint continuity of $\partial_\mu f$ is understood as the joint continuity of mapping $(x, X) \to \partial_\mu f(x, \mathbb{P}_X)(X)$ from $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^p)$ to $L^2(\Omega; \mathbb{R}^p)$.

The above discussions can be applied to the coefficients of **Problem (EMFPOC)** (see [1] for similar discussions). For example, let ξ be a generic element of $\mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$ and $\mu \in \mathcal{P}_2(\mathbb{R}^n)$ and $\nu \in \mathcal{P}_2(\mathbb{R}^k)$ be the marginals of ξ. For each fixed $t \in [0, T]$, if the function $f : \mathbb{R}^q \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k) \ni (x, \xi) \to f(x, \xi) \in \mathbb{R}$ is jointly differentiable at (x, ξ), there exists a pair of random variable $(X, V) \in L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^k)$ with $\mathcal{L}(X, V) = \xi$ such that the lifting function $\widetilde{f}$ of f given by

$$\widetilde{f} : \mathbb{R}^q \times L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^k) \ni (x, X, V) \mapsto \widetilde{f}(x, X, V) = f(x, \mathcal{L}(X, V)).$$

is Fréchet differentiable at (x, X, V). The Fréchet derivative $[D\widetilde{f}](x, X, V)$ of the lifting function $\widetilde{f}$ at (x, X, V) can be viewed as an element $D\widetilde{f}(x, X, V)$ of $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^n \times \mathbb{R}^k)$ which can be represented as

$$D\widetilde{f}(x, X, V) = (\partial_x f(x, \mathcal{L}(X, V))(X, V), \partial_\mu f(x, \mathcal{L}(X, V))(X, V), \partial_\nu f(x, \mathcal{L}(X, V))(X, V)),$$

where the functions $\partial_x f(\cdot, \mathcal{L}(X, V))(\cdot, \cdot), \partial_\mu f(\cdot, \mathcal{L}(X, V))(\cdot, \cdot), \partial_\nu f(\cdot, \mathcal{L}(X, V))(\cdot, \cdot)$, are partial L-derivatives of f at $(x, \mathcal{L}(X, V))$ which are defined on $\mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R}^k$ and valued, respectively, on $\mathbb{R}^q, \mathbb{R}^n, \mathbb{R}^k$. Similar discussions can be applied to the coefficients $\sigma, \bar{\sigma}, g, h, l, m$ and γ.

Finally, we introduce the following notations. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a copy of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any random variable (x, v) over $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $(\tilde{x}, \tilde{v})$ are independent copy of of (x, v), but defined over $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The expectation $\tilde{\mathbb{E}}[\cdot] = \int_{\tilde{\Omega}} (\cdot) \mathrm{d}\tilde{\mathbb{P}}$ acts only over the variables endowed with a tilde.

2.2. Assumptions and reformulation of the problem

We denote η as a generic element of $\mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k)$, let $\mu_1 \in \mathcal{P}_2(\mathbb{R}^n)$, $\mu_2 \in \mathcal{P}_2(\mathbb{R}^l)$, $\mu_3 \in \mathcal{P}_2(\mathbb{R}^{l\times m})$, $\mu_4 \in \mathcal{P}_2(\mathbb{R}^{l\times d})$, $\mu_5 \in \mathcal{P}_2(\mathbb{R}^k)$ be the marginal distribution of η, and $\xi \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$ be the joint margin distribution of η on the first and the fifth components (*i.e.* joint distribution of μ_1 and μ_5). Throughout the paper, we give the following standing assumptions (similar assumptions are given in [1, 16]).

(H.1) The functions $f, \sigma, \bar{\sigma}, h$ are differentiable w.r.t. (x, v) for each fixed $\xi \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$, $\bar{\sigma}$, h are uniformly bounded. For each $t \in [0, T]$ the mappings $(x, v, \xi) \mapsto \partial_x(f, \sigma, \bar{\sigma}, h)(t, x, v, \xi)$ and $(x, v, \xi) \mapsto \partial_v(f, \sigma, \bar{\sigma}, h)(t, x, v, \xi)$ are continuous . The functions $f, \sigma, \bar{\sigma}, h$ are L-differentiable w.r.t. ξ and the mappings

$$\begin{aligned}\mathbb{R}^{n+k} \times L^2(\Omega; \mathbb{R}^{n+k}) \ni (x, v, (X, \beta)) \mapsto &\partial_{\mu_1}(f, \sigma, \bar{\sigma}, h)(t, x, v, \mathcal{L}(X, \beta))(X, \beta) \\ &\in L^2(\Omega; \mathbb{R}^{n\times n} \times \mathbb{R}^{l\times m\times n} \times \mathbb{R}^{l\times d\times n} \times \mathbb{R}^{d\times n})\end{aligned}$$

and

$$\begin{aligned}\mathbb{R}^{n+k} \times L^2(\Omega; \mathbb{R}^{n+k}) \ni (x, v, (X, \beta)) \mapsto &\partial_{\mu_5}(f, \sigma, \bar{\sigma}, h)(t, x, v, \mathcal{L}(X, \beta))(X, \beta) \\ &\in L^2(\Omega; \mathbb{R}^{n\times k} \times \mathbb{R}^{l\times m\times k} \times \mathbb{R}^{l\times d\times k} \times \mathbb{R}^{d\times k})\end{aligned}$$

are continuous for each $t \in [0, T]$. For each fixed $\eta \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k)$, the function g is differentiable w.r.t. $(x, y, z, \bar{z}, v)$, and for each $t \in [0, T]$ and $\psi = x, y, z, \bar{z}, v$, the mappings $(x, y, z, \bar{z}, v, \eta) \mapsto \partial_\psi g(x, y, z, \bar{z}, v, \eta)$ are continuous. Moreover, the function g is L-differentiable w.r.t. η, such that for $j = 1, 2, 3, 4, 5$, and $q = n + l + l \times m + l \times d + k$, $q_1 = l \times n$, $q_2 = l \times l$, $q_3 = l \times (l \times m)$, $q_4 = l \times (l \times d)$, $q_5 = l \times k$, the following mappings from $\mathbb{R}^q \times L^2(\Omega; \mathbb{R}^q)$, respectively, to $L^2(\Omega; \mathbb{R}^{q_j})$,

$$(x, y, z, \bar{z}, v, (X, Y, Z, \bar{Z}, \beta)) \mapsto \partial_{\mu_j} g(t, x, y, z, \bar{z}, v, \mathcal{L}(X, Y, Z, \bar{Z}, \beta)(X, Y, Z, \bar{Z}, \beta)$$

are continuous for each $t \in [0, T]$. The function Φ is differentiable w.r.t. x, and the mappings $(x, \mu_1) \mapsto \partial_x \Phi(x, \mu_1)$ is continuous for any $t \in [0, T]$. The function Φ is L-differentiable w.r.t. μ_1, and the mapping

$$\mathbb{R}^n \times L^2(\Omega; \mathbb{R}^n) \ni (x, X) \mapsto \partial_{\mu_1} \Phi(x, \mathcal{L}(X))(X) \in L^2(\Omega; \mathbb{R}^{l\times n})$$

is continuous. The derivatives $\partial_x(f, \sigma, \bar{\sigma}, \Phi, h, g)$, $\partial_y g$, $\partial_z g$, $\partial_{\bar{z}} g$ and $\partial_v(f, \sigma, \bar{\sigma}, g, h)$ are uniformly bounded, and

$$\int_{\mathbb{R}^n} |\partial_{\mu_1} \Phi(x, \mu_1)(x')|^2 \mathrm{d}\mu_1(x'), \quad \int_{\mathbb{R}^{n+k}} |\partial_{\mu_i}(f, \sigma, \bar{\sigma}, h)(t, x, v, \xi)(x', v')|^2 \mathrm{d}\xi(x', v'), \quad i = 1, 5$$

as well as for $j = 1, 2, 3, 4, 5$

$$\int_{\mathbb{R}^{n\times l\times (l\times m)\times (l\times d)\times k}} |\partial_{\mu_j} g(t, x, y, z, \bar{z}, v, \eta)(x', y', z', \bar{z}', v')|^2 \mathrm{d}\eta(x', y', z', \bar{z}', v')$$

are uniformly bounded. Moreover, $(f, \sigma, \bar{\sigma})(t, 0, 0, \delta_0)$ and $g(t, 0, 0, 0, 0, \delta_0)$ are uniformly bounded where δ_0 is the Dirac measure at 0.

(H.2) For each fixed $\eta \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k)$, the function l is differentiable w.r.t. $(\rho, x, y, z, \bar{z}, v)$. For each $t \in [0, T]$ and $\psi = \rho, x, y, z, \bar{z}, v$, the mappings $(\rho, x, y, z, \bar{z}, v, \eta) \mapsto \partial_\psi l(\rho, x, y, z, \bar{z}, v, \eta)$ are continuous. The function l is L-differentiable w.r.t. η, such that for $j = 1, 2, 3, 4, 5$, and $\bar{q} = 1 + n + l + l \times m + l \times d + k$, $\bar{q}_1 = n$, $\bar{q}_2 = l$, $\bar{q}_3 = l \times m$, $\bar{q}_4 = l \times d$, $\bar{q}_5 = k$, the following mappings, from $\mathbb{R}^{\bar{q}} \times L^2(\Omega; \mathbb{R}^q)$, respectively, to $L^2(\Omega; \mathbb{R}^{\bar{q}_j})$,

$$(\rho, x, y, z, \bar{z}, v, (X, Y, Z, \bar{Z}, \beta)) \mapsto \partial_{\mu_j} l(t, \rho, x, y, z, \bar{z}, v, \mathcal{L}(X, Y, Z, \bar{Z}, \beta)(X, Y, Z, \bar{Z}, \beta)$$

are continuous. Similarly, the function χ is differentiable w.r.t. x, the function γ is differentiable w.r.t. y, and the mappings $(\rho, x, \mu_1) \mapsto \partial_x \chi(\rho, x, \mu_1)$ and $y \mapsto \partial_y \gamma(y)$ are continuous. The function χ is L-differentiable w.r.t. μ_1, and the mapping

$$\mathbb{R}^{1+n} \times L^2(\Omega; \mathbb{R}^n) \ni (\rho, x, X) \mapsto \partial_{\mu_1} \chi(\rho, x, \mathcal{L}(X))(X) \in L^2(\Omega; \mathbb{R}^n)$$

is continuous. Moreover, there exist a constant $C > 0$ such that for $j = 1, 2, 3, 4, 5$, $\psi = \rho, x, y, z, \bar{z}, v$, it follows that

$$\begin{cases} \|\partial_{\mu_j} l(t, \rho, x, y, z, \bar{z}, v, \eta)(x', y', z', \bar{z}', v')\|_{L^2(\eta)} + |\partial_\psi l(t, \rho, x, y, z, \bar{z}, v, \eta)| \\ \qquad \le C(1 + |\rho| + |x| + |y| + |z| + |\bar{z}| + |v| + W_2(\eta, \delta_0)), \\ \|\partial_{\mu_1} \chi(x, \rho, \mu_1)(x')\|_{L^2(\mu_1)} + |\partial_x \chi(\rho, x, \mu_1)| \le C(1 + |\rho| + |x| + W_2(\mu_1, \delta_0)), \\ |\partial_y \gamma(y)| \le L(1 + |y|), \end{cases}$$

where $\|\partial_{\mu_j} l(t, \rho, x, y, z, \bar{z}, v, \eta)(x', y', z', \bar{z}', v')\|^2_{L^2(\eta)}$ denotes that

$$\int_{\mathbb{R}^{n\times l\times (l\times m)\times (l\times d)\times k}} |\partial_{\mu_j} l(t, \rho, x, y, z, \bar{z}, v, \eta)(x', y', z', \bar{z}', v')|^2 \mathrm{d}\eta(x', y', z', \bar{z}', v')$$

and $\|\partial_{\mu_1} \chi(\rho, x, \mu_1)(x')\|^2_{L^2(\mu_1)} := \int_{\mathbb{R}^n} |\partial_{\mu_1} \chi(\rho, x, \mu_1)(x')|^2 \mathrm{d}\mu_1(x')$. Moreover, we suppose that $l(t, 0, 0, 0, 0, 0, 0, \delta_0)$ is uniformly bounded.

Remark 2.1. In assumption (H.1), the continuity of the mapping

$$\begin{aligned} \mathbb{R}^{n+k} \times L^2(\Omega; \mathbb{R}^{n+k}) \ni (x, v, (X, \beta)) \mapsto & \partial_{\mu_1}(f, \sigma, \bar{\sigma}, h)(t, x, v, \mathcal{L}(X, \beta))(X, \beta) \\ & \in L^2(\Omega; \mathbb{R}^{n\times n} \times \mathbb{R}^{l\times m\times n} \times \mathbb{R}^{l\times d\times n} \times \mathbb{R}^{d\times n}) \end{aligned}$$

means that for $(x, v, (X, \beta)) \in \mathbb{R}^{n+k} \times L^2(\Omega; \mathbb{R}^{n+k})$ and $(y, u, (Y, \alpha)) \in \mathbb{R}^{n+k} \times L^2(\Omega; \mathbb{R}^{n+k})$ such that $|y - x|^2 + |u - v|^2 + \mathbb{E}|Y - X|^2 + \mathbb{E}|\alpha - \beta|^2 \to 0$, we have

$$\mathbb{E}\big|\partial_{\mu_1}(f, \sigma, \bar{\sigma}, h)(t, y, u, \mathcal{L}(Y, \alpha))(Y, \alpha) - \partial_{\mu_1}(f, \sigma, \bar{\sigma}, h)(t, x, v, \mathcal{L}(X, \beta))(X, \beta)\big|^2 \to 0.$$

Other continuity assumptions of the related mappings can be understood similarly.

Remark 2.2. Under assumption (H.1), one can show that system (2.1)–(2.2) has a weak solution. Indeed, by inserting (2.2) into (2.1), we get

$$\begin{cases} \mathrm{d}x_t = [f(t,x_t,v_t,\mathcal{L}(x_t,v_t)) - \bar\sigma(t,x_t,v_t,\mathcal{L}(x_t,v_t))h(t,x_t,v_t,\mathcal{L}(x_t,v_t))]\,\mathrm{d}t \\ \qquad\quad + \sigma(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}W_t + \bar\sigma(t,x_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}Y_t, \\ -\mathrm{d}y_t = g(t,x_t,y_t,z_t,\bar z_t,v_t,\mathcal{L}(x_t,y_t,z_t,\bar z_t,v_t))\mathrm{d}t - z_t\mathrm{d}W_t - \bar z_t\mathrm{d}Y_t, \\ x(0) = x_0, \quad y(T) = \Phi(x_T,\mathcal{L}(x_T)). \end{cases} \tag{2.5}$$

Noticing that the first equation in (2.5) is a forward SDE of McKean-Vlasov type. By applying Theorem 4.21 of [17] (see also Prop. 1.2 of [22]), we know that for each given $v(\cdot)\in\mathcal{U}_{ad}$, it has a unique solution $x(\cdot)\in\mathbb{S}^{2,n}_{\mathbb{F}}$. Moreover, by recalling that $\sup_{t\in[0,T]}\mathbb{E}|v_t|^4<+\infty$ and the uniformly boundedness of $(f,\sigma,\bar\sigma)(t,0,0,\delta_0)$, one can show that $\mathbb{E}\Big[\sup_{t\in[0,T]}|x_t|^p\Big]<+\infty$, for any $2\le p\le 4$ (see Prop. 1.2 of [22]).

Once we solved the first equation, *i.e.* $x(\cdot)$ is given now, then the second equation in (2.5) will be a mean-field BSDE. By applying similar methods in the proof of Theorem 4.23 of [17] (the dependence on the distribution of $z(\cdot)$ will not arise additional difficulties, see *e.g.* [7]), one can show that it has a unique solution $(y(\cdot),z(\cdot),\bar z(\cdot))\in\mathbb{S}^{2,l}_{\mathbb{F}}\times\mathbb{H}^{2,l\times m}_{\mathbb{F}}\times\mathbb{H}^{2,l\times d}_{\mathbb{F}}$. Moreover, similar to the proof of Lemma 3.4 in the appendix, one can show that $\mathbb{E}\big[\sup_{t\in[0,T]}|y_t|^p+(\int_0^T|z_t|^2\mathrm{d}t)^{p/2}+(\int_0^T|\bar z_t|^2\mathrm{d}t)^{p/2}\big]<+\infty$, for $2\le p\le 4$.

Finally, recalling the definition of $\rho(\cdot)$ (see (1.4) or (2.3)) and the boundednees of h, it follows that $\rho(\cdot)$ is a martingale satisfying $\mathbb{E}\Big[\sup_{t\in[0,T]}(|\rho_t|^p+|\rho_t|^{-p}\Big]<+\infty$, for any $p\ge 1$. Define $\mathrm{d}\mathbb{P}^v=\rho_T\mathrm{d}\mathbb{P}$, then $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}^v,x(\cdot),y(\cdot),z(\cdot),\bar z(\cdot),Y(\cdot),W(\cdot),\bar W^v(\cdot))$ is a weak solution of system (2.1)–(2.2), according to Girsanov's theorem.

Let us now reformulate the cost functional (2.4). According to Bayes' formula, the cost functional defined as in (2.4) can be rewritten as (noticing that $\gamma(y_0)$ is deterministic)

$$J(v(\cdot))=\mathbb{E}\Big[\int_0^T\rho_t l(t,\rho_t,x_t,y_t,z_t,\bar z_t,v_t,\mathcal{L}(x_t,y_t,z_t,\bar z_t,v_t))\mathrm{d}t+\rho_T\chi(\rho_T,x_T,\mathcal{L}(x_T))+\gamma(y_0)\Big]. \tag{2.6}$$

We mention that, under assumptions (H.1)–(H.2), we have $|J(v(\cdot))|<+\infty$, *i.e.* the above cost functional is well defined. This can be obtained easily from the assumptions on the coefficients l,m,γ and the integrability property of $\rho(\cdot),x(\cdot),y(\cdot),z(\cdot),\bar z(\cdot)$ (see Rem. 2.2).

We introduce the following notations for dimensional extension

$$\begin{aligned} &X:=\begin{pmatrix}\rho\\ x\end{pmatrix},\quad X_0:=\begin{pmatrix}1\\ x_0\end{pmatrix},\quad X^1:=\begin{pmatrix}\rho^1\\ x^1\end{pmatrix},\quad \Sigma(t,X,v,\xi):=\begin{pmatrix}0\\ \sigma(t,x,v,\xi)\end{pmatrix},\\ &\bar\Sigma(t,X,v,\xi):=\begin{pmatrix}\rho h(t,x,v,\xi)\\ \bar\sigma(t,x,v,\xi)\end{pmatrix},\quad F(t,X,v,\xi):=\begin{pmatrix}0\\ f(t,x,v,\xi)-\bar\sigma(t,x,v,\xi)h(t,x,v,\xi)\end{pmatrix},\\ &G(t,X,y,z,\bar z,v,\eta):=g(t,x,y,z,\bar z,v,\eta),\quad L(t,X,y,z,\bar z,v,\eta):=\rho l(t,\rho,x,y,z,\bar z,v,\eta),\\ &M(X,\mu_1):=\rho\chi(\rho,x,\mu_1). \end{aligned}$$

Then equations (2.1) and (2.3) can be compressed into the following form

$$\begin{cases} \mathrm{d}X_t = F(t,X_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}t+\Sigma(t,X_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}W_t+\bar\Sigma(t,X_t,v_t,\mathcal{L}(x_t,v_t))\mathrm{d}Y_t \\ -\mathrm{d}y_t = G(t,X_t,y_t,z_t,\bar z_t,v_t,\mathcal{L}(x_t,y_t,z_t,\bar z_t,v_t))\mathrm{d}t-z_t\mathrm{d}W_t-\bar z_t\mathrm{d}Y_t, \\ X(0)=X_0,\quad y(T)=\Phi(x_T,\mathcal{L}(x_T)), \end{cases} \tag{2.7}$$

and the cost functional (2.6) can be represented as

$$J(v(\cdot)) = \mathbb{E}\Big[\int_0^T L(t, X_t, y_t, z_t, \bar{z}_t, v_t, \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, v_t))\mathrm{d}t + M(X_T, \mathcal{L}(x_T)) + \gamma(y_0)\Big]. \tag{2.8}$$

Then **Problem (EMFPOC)** becomes to the following equivalent minimization problem: to minimize $J(v(\cdot))$ over $v(\cdot) \in \mathcal{U}_{ad}$ subject to (2.7) and (2.8).

3. Stochastic maximum principle

In this section, we will derive a necessary condition for optimal control in type of Pontryagin's stochastic maximum principle. For simplicity, we set $n = l = k = m = d = 1$. The arguments hold similarly for the multi-dimensional case.

3.1. Variational equations

Let $u(\cdot)$ be an optimal control and $(X(\cdot), y(\cdot), z(\cdot), \bar{z}(\cdot))$ be the corresponding trajectory. Let $v(\cdot)$ be such that $u(\cdot) + v(\cdot) \in \mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is convex, then, for any $0 \le \varepsilon \le 1$, $u^\varepsilon(\cdot) \triangleq u(\cdot) + \varepsilon v(\cdot)$ is also in $\mathcal{U}_{ad}$. To simplify symbols, we set

$$\begin{aligned}
&\xi_t := \mathcal{L}(x_t, u_t), \quad \eta_t := \mathcal{L}(x_t, y_t, z_t, \bar{z}_t, u_t), \quad \theta_t := (X_t, u_t, \xi_t), \quad \theta_t^{'} := (x_t, u_t, \xi_t),\\
&\alpha_t := (x_t, y_t, z_t, \bar{z}_t, u_t), \quad \Theta_t := (X_t, y_t, z_t, \bar{z}_t, u_t, \eta_t), \quad \Theta_t^{'} := (x_t, y_t, z_t, \bar{z}_t, u_t, \eta_t) = (\alpha_t, \eta_t).
\end{aligned}$$

One can check that, for $i = 1, 5$, $j = 1, 2, 3, 4, 5$ and $\psi = x, y, z, \bar{z}, v$,

$$\begin{aligned}
&\partial_X F(t, \theta_t) = \begin{pmatrix} 0 & 0 \\ 0 & \partial_x f(t, \theta_t^{'}) - \partial_x \bar{\sigma}(t, \theta_t^{'}) h(t, \theta_t^{'}) - \bar{\sigma}(t, \theta_t^{'}) \partial_x h(t, \theta_t^{'}) \end{pmatrix},\\
&\partial_v F(t, \theta_t) = \begin{pmatrix} 0 \\ \partial_v f(t, \theta_t^{'}) - \partial_v \bar{\sigma}(t, \theta_t^{'}) h(t, \theta_t^{'}) - \bar{\sigma}(t, \theta_t^{'}) \partial_v h(t, \theta_t^{'}) \end{pmatrix},\\
&\partial_{\mu_i} F(t, \theta_t)(\cdot) = \begin{pmatrix} 0 \\ \partial_{\mu_i} f(t, \theta_t^{'})(\cdot) - \partial_{\mu_i} \bar{\sigma}(t, \theta_t^{'})(\cdot) h(t, \theta_t^{'}) - \bar{\sigma}(t, \theta_t^{'}) \partial_{\mu_i} h(t, \theta_t^{'})(\cdot) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&\partial_X \Sigma(t, \theta_t) = \begin{pmatrix} 0 & 0 \\ 0 & \partial_x \sigma(t, \theta_t^{'}) \end{pmatrix}, \quad \partial_v \Sigma(t, \theta_t) = \begin{pmatrix} 0 \\ \partial_v \sigma(t, \theta_t^{'}) \end{pmatrix}, \quad \partial_{\mu_i} \Sigma(t, \theta_t)(\cdot) = \begin{pmatrix} 0 \\ \partial_{\mu_i} \sigma(t, \theta_t^{'})(\cdot) \end{pmatrix},\\
&\partial_X \bar{\Sigma}(t, \theta_t) = \begin{pmatrix} h(t, \theta_t^{'}) & \rho \partial_x h(t, \theta_t^{'}) \\ 0 & \partial_x \bar{\sigma}(t, \theta_t^{'}) \end{pmatrix}, \partial_v \bar{\Sigma}(t, \theta_t) = \begin{pmatrix} \rho \partial_v h(t, \theta_t^{'}) \\ \partial_v \bar{\sigma}(t, \theta_t^{'}) \end{pmatrix}, \partial_{\mu_i} \bar{\Sigma}(t, \theta_t)(\cdot) = \begin{pmatrix} \rho \partial_{\mu_i} h(t, \theta_t^{'})(\cdot) \\ \partial_{\mu_i} \bar{\sigma}(t, \theta_t^{'})(\cdot) \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
&\partial_X G(t, \Theta_t) = (0, \partial_x g(t, \Theta_t^{\prime})), \quad \partial_{\mu_j} G(t, \Theta_t)(\cdot) = \partial_{\mu_j} g(t, \Theta_t^{\prime})(\cdot), \quad \partial_\psi G(t, \Theta_t) = \partial_\psi g(t, \Theta_t^{\prime})\\
&\partial_X L(t, \Theta_t) = (l(t, \Theta_t) + \rho \partial_\rho l(t, \Theta_t), \rho \partial_x l(t, \Theta_t)),\\
&\partial_{\mu_j} L(t, \Theta_t)(\cdot) = \rho \partial_{\mu_j} l(t, \Theta_t)(\cdot), \quad \partial_\psi L(t, \Theta_t) = \rho \partial_\psi l(t, \Theta_t),\\
&\partial_X M(X, \mathcal{L}(x_T)) = (\chi(X, \mathcal{L}(x_T)) + \rho \partial_\rho \chi(X, \mathcal{L}(x_T)), \rho \partial_x \chi(X, \mathcal{L}(x_T))),\\
&\partial_{\mu_1} M(X, \mathcal{L}(x_T))(\cdot) = \rho \partial_{\mu_1} \chi(X, \mathcal{L}(x_T))(\cdot).
\end{aligned}$$

Recalling that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a copy of $(\Omega, \mathcal{F}, \mathbb{P})$. For any random variable x over $(\Omega, \mathcal{F}, \mathbb{P})$, $\tilde{x}$ denotes an independent copy of x defined over $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The expectation $\tilde{\mathbb{E}}[\cdot]$ acts only over the variables endowed with a tilde. Then we can introduce the following variational equations

$$\begin{cases} \mathrm{d}X_t^1 = \Big(\partial_X F(t,\theta_t)X_t^1 + \partial_v F(t,\theta_t)v_t + \tilde{\mathbb{E}}[\partial_{\mu_1}F(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}F(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\,\mathrm{d}t \\ \qquad + \Big(\partial_X \Sigma(t,\theta_t)X_t^1 + \partial_v \Sigma(t,\theta_t)v_t + \tilde{\mathbb{E}}[\partial_{\mu_1}\Sigma(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}\Sigma(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\,\mathrm{d}W_t \\ \qquad + \Big(\partial_X \bar{\Sigma}(t,\theta_t)X_t^1 + \partial_v \bar{\Sigma}(t,\theta_t)v_t + \tilde{\mathbb{E}}[\partial_{\mu_1}\bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}\bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\,\mathrm{d}Y_t, \\ -\,\mathrm{d}y_t^1 = \Big(\partial_x G(t,\Theta_t)x_t^1 + \partial_y G(t,\Theta_t)y_t^1 + \partial_z G(t,\Theta_t)z_t^1 + \partial_{\bar{z}} G(t,\Theta_t)\bar{z}_t^1 + \partial_v G(t,\Theta_t)v_t \\ \qquad + \tilde{\mathbb{E}}[\partial_{\mu_1}G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_2}G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{y}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_3}G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{z}_t^1] \\ \qquad + \tilde{\mathbb{E}}[\partial_{\mu_4}G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{\bar{z}}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{v}_t]\Big)\mathrm{d}t - z_t^1\mathrm{d}W_t - \bar{z}_t^1\mathrm{d}Y_t, \\ X_0^1 = 0, \quad y_T^1 = \partial_x\Phi(x_T,\mathcal{L}(x_T))x_T^1 + \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\tilde{x}_T)\tilde{x}_T^1], \end{cases} \tag{3.1}$$

where we used the notation $X_t^1 := \begin{pmatrix} \rho_t^1 \\ x_t^1 \end{pmatrix}$, and $\tilde{\alpha}_t := (\tilde{x}_t, \tilde{y}_t, \tilde{z}_t, \tilde{\bar{z}}_t, \tilde{u}_t)$ is an independent copy of $\alpha_t := (x_t, y_t, z_t, \bar{z}_t, u_t)$.

Theorem 3.1. *Let assumptions (H.1)–(H.2) hold, then mean-field FBSDE (3.1) admits a unique solution* $(X^1(\cdot), y^1(\cdot), z^1(\cdot), \bar{z}^1(\cdot)) \in \mathbb{S}_{\mathbb{F}}^{2,1+n} \times \mathbb{S}_{\mathbb{F}}^{2,l} \times \mathbb{H}_{\mathbb{F}}^{2,l\times m} \times \mathbb{H}_{\mathbb{F}}^{2,l\times d}$ *satisfying that for any* $2 \le p \le 4$ *and* $0 < \varepsilon_0 \le p$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|x_t^1|^p + \sup_{t\in[0,T]}|\rho_t^1|^{p-\varepsilon_0} + \sup_{t\in[0,T]}|y_t^1|^p + (\int_0^T |z_t^1|^2 dt)^{p/2} + (\int_0^T |\bar{z}_t^1|^2 dt)^{p/2}\right] < +\infty. \tag{3.2}$$

Proof. The first equation of (3.1) can be decomposed into the following two equations

$$\begin{cases} \mathrm{d}\rho_t^1 = \Big(\rho_t^1 h(t,\theta_t') + \rho_t\partial_x h(t,\theta_t')x_t^1 + \rho_t\partial_v h(t,\theta_t')v_t \\ \qquad\qquad + \rho_t\tilde{\mathbb{E}}[\partial_{\mu_1}h(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \rho_t\tilde{\mathbb{E}}[\partial_{\mu_5}h(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\mathrm{d}Y_t, \\ \rho_0^1 = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \mathrm{d}x_t^1 = \Big((\partial_x f(t,\theta_t') - \bar{\sigma}(t,\theta_t')\partial_x f(t,\theta_t') - h(t,\theta_t')\partial_x\bar{\sigma}(t,\theta_t'))x_t^1 \\ \qquad + (\partial_v f(t,\theta_t') - \bar{\sigma}(t,\theta_t')\partial_v h(t,\theta_t') - h(t,\theta_t')\partial_v\bar{\sigma}(t,\theta_t'))v_t \\ \qquad + \tilde{\mathbb{E}}[\partial_{\mu_1}f(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] - \bar{\sigma}(t,\theta_t')\tilde{\mathbb{E}}[\partial_{\mu_1}h(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] \\ \qquad - h(t,\theta_t')\tilde{\mathbb{E}}[\partial_{\mu_1}\bar{\sigma}(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}f(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t] \\ \qquad - \bar{\sigma}(t,\theta_t')\tilde{\mathbb{E}}[\partial_{\mu_5}h(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t] - h(t,\theta_t')\tilde{\mathbb{E}}[\partial_{\mu_5}\bar{\sigma}(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\,\mathrm{d}t \\ \qquad + \Big(\partial_x\sigma(t,\theta_t')x_t^1 + \partial_v\sigma(t,\theta_t')v_t + \tilde{\mathbb{E}}[\partial_{\mu_1}\sigma(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}\sigma(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\,\mathrm{d}W_t \\ \qquad + \Big(\partial_x\bar{\sigma}(t,\theta_t')x_t^1 + \partial_v\bar{\sigma}(t,\theta_t')v_t + \tilde{\mathbb{E}}[\partial_{\mu_1}\bar{\sigma}(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5}\bar{\sigma}(t,\theta_t')(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big)\,\mathrm{d}Y_t, \\ x_0^1 = 0. \end{cases} \tag{3.4}$$

Noticing that $x^1(\cdot)$ does not depend on $\rho^1(\cdot)$, it means that (3.3) and (3.4) are partially coupled. This allows us to solve (3.4) first and then solve (3.3). Indeed, similar to the proof of Theorem 4.21 of [17] and Proposition 1.2 of [22], under assumption (H.1), we can show that equation (3.4) has a unique solution $x^1(\cdot) \in \mathbb{S}_{\mathbb{F}}^{2,n}$ satisfying $\mathbb{E}\left[\sup_{t\in[0,T]}|x_t^1|^p\right] < +\infty$, for $2 \le p \le 4$. Concerning equation (3.3), from the boundedness of the derivatives of h and the integrability of $x^1(\cdot), \rho(\cdot), v(\cdot)$, one can find that equation (3.3) is a standard SDE with uniformly Lipschitz coefficients, then it has a unique solution $\rho^1(\cdot)$ satisfying $\mathbb{E}\left[\sup_{t\in[0,T]}|\rho_t^1|^p\right] < +\infty$, for $2 \le p < 4$.

Now let us focus on the second equation of (3.1) which is a mean-field BSDE. Noticing the boundedness of $\partial_y G, \partial_z G, \partial_{\bar z} G$ and the followings inequalities

$$
\begin{aligned}
\left|\tilde{\mathbb{E}}[\partial_{\mu_2} G(t,\Theta_t)(\tilde\alpha_t)\tilde y_t^1]\right| &\le \left(\tilde{\mathbb{E}}|\partial_{\mu_2} G(t,\Theta_t)(\tilde\alpha_t)|^2\right)^{1/2}\left(\tilde{\mathbb{E}}|\tilde y_t^1|^2\right)^{1/2} \le C\left(\tilde{\mathbb{E}}|\tilde y_t^1|^2\right)^{1/2},\\
\left|\tilde{\mathbb{E}}[\partial_{\mu_3} G(t,\Theta_t)(\tilde\alpha_t)\tilde z_t^1]\right| &\le \left(\tilde{\mathbb{E}}|\partial_{\mu_3} G(t,\Theta_t)(\tilde\alpha_t)|^2\right)^{1/2}\left(\tilde{\mathbb{E}}|\tilde z_t^1|^2\right)^{1/2} \le C\left(\tilde{\mathbb{E}}|\tilde z_t^1|^2\right)^{1/2},\\
\left|\tilde{\mathbb{E}}[\partial_{\mu_4} G(t,\Theta_t)(\tilde\alpha_t)\tilde{\bar z}_t^1]\right| &\le \left(\tilde{\mathbb{E}}|\partial_{\mu_4} G(t,\Theta_t)(\tilde\alpha_t)|^2\right)^{1/2}\left(\tilde{\mathbb{E}}|\tilde{\bar z}_t^1|^2\right)^{1/2} \le C\left(\tilde{\mathbb{E}}|\tilde{\bar z}_t^1|^2\right)^{1/2},
\end{aligned}
$$

where we have used the assumption (see (H.1)) that, for $j=2,3,4$,

$$
\int_{\mathbb{R}^{n\times l\times(l\times m)\times(l\times d)\times k}} |\partial_{\mu_j} g(t,x,y,z,\bar z,v,\eta)(x',y',z',\bar z',v')|^2 \mathrm{d}\eta(x',y',z',\bar z',v')
$$

are uniformly bounded. A careful inspection of the proof of Lemma 3.1 of [8], or Theorem 3.1 of [13] and Theorem 4.23 of [17], above bounded properties allow us to use the arguments in [8, 13, 17] to prove our mean-field BSDE has a unique solution $(y^1(\cdot),z^1(\cdot),\bar z^1(\cdot))\in \mathbb{S}_{\mathbb{F}}^{2,l}\times\mathbb{H}_{\mathbb{F}}^{2,l\times m}\times\mathbb{H}_{\mathbb{F}}^{2,l\times d}$. Moreover, one can show that, for $2\le p\le 4$, $\mathbb{E}\big[\sup_{t\in[0,T]}|y_t^1|^p + (\int_0^T |z_t^1|^2\mathrm{d}t)^{p/2} + (\int_0^T |\bar z_t^1|^2)^{p/2}\mathrm{d}t\big] < +\infty$. Indeed, this will need subtle analysis, but the procedures are almost the same as the proof of Lemma 3.4, so we omit it here. □

Now let us denote $(X^\varepsilon(\cdot),y^\varepsilon(\cdot),z^\varepsilon(\cdot),\bar z^\varepsilon(\cdot))$ as the trajectory corresponding to $u^\varepsilon(\cdot)$. We set

$$
\begin{aligned}
x_t^{\varepsilon,1} &= \frac{x_t^\varepsilon - x_t}{\varepsilon} - x_t^1, \quad \rho_t^{\varepsilon,1} = \frac{\rho_t^\varepsilon - \rho_t}{\varepsilon} - \rho_t^1, \quad X_t^{\varepsilon,1} = \frac{X_t^\varepsilon - X_t}{\varepsilon} - X_t^1 = \begin{pmatrix} \rho_t^{\varepsilon,1} \\ x_t^{\varepsilon,1} \end{pmatrix},\\
y_t^{\varepsilon,1} &= \frac{y_t^\varepsilon - y_t}{\varepsilon} - y_t^1, \quad z_t^{\varepsilon,1} = \frac{z_t^\varepsilon - z_t}{\varepsilon} - z_t^1, \quad \bar z_t^{\varepsilon,1} = \frac{\bar z_t^\varepsilon - \bar z_t}{\varepsilon} - \bar z_t^1,
\end{aligned}
\tag{3.5}
$$

and the following notations will be used in the sequel of the paper

$$
\begin{aligned}
x_t^{\lambda,\varepsilon} &:= x_t + \lambda(x_t^\varepsilon - x_t) = x_t + \lambda\varepsilon(x_t^1 + x_t^{\varepsilon,1}), \quad u_t^{\lambda,\varepsilon} := u_t + \lambda(u_t^\varepsilon - u_t) = u_t + \lambda\varepsilon v_t,\\
y_t^{\lambda,\varepsilon} &:= y_t + \lambda(y_t^\varepsilon - y_t) = y_t + \lambda\varepsilon(y_t^1 + y_t^{\varepsilon,1}), \quad z_t^{\lambda,\varepsilon} := z_t + \lambda(z_t^\varepsilon - z_t) = z_t + \lambda\varepsilon(z_t^1 + z_t^{\varepsilon,1}),\\
\bar z_t^{\lambda,\varepsilon} &:= \bar z_t + \lambda(\bar z_t^\varepsilon - \bar z_t) = \bar z_t + \lambda\varepsilon(\bar z_t^1 + \bar z_t^{\varepsilon,1}), \quad \alpha_t^{\lambda,\varepsilon} := (x_t^{\lambda,\varepsilon}, y_t^{\lambda,\varepsilon}, z_t^{\lambda,\varepsilon}, \bar z_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon}),\\
\Theta_t^{\lambda,\varepsilon} &:= (X_t^{\lambda,\varepsilon}, y_t^{\lambda,\varepsilon}, z_t^{\lambda,\varepsilon}, \bar z_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon}, \mathcal{L}(x_t^{\lambda,\varepsilon}, y_t^{\lambda,\varepsilon}, z_t^{\lambda,\varepsilon}, \bar z_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon})) = (\rho_t^{\lambda,\varepsilon}, \alpha_t^{\lambda,\varepsilon}, \mathcal{L}(\alpha_t^{\lambda,\varepsilon})),\\
(\Theta_t')^{\lambda,\varepsilon} &:= (x_t^{\lambda,\varepsilon}, y_t^{\lambda,\varepsilon}, z_t^{\lambda,\varepsilon}, \bar z_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon}, \mathcal{L}(x_t^{\lambda,\varepsilon}, y_t^{\lambda,\varepsilon}, z_t^{\lambda,\varepsilon}, \bar z_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon})) = (\alpha_t^{\lambda,\varepsilon}, \mathcal{L}(\alpha_t^{\lambda,\varepsilon})),\\
(\Theta_t)^\varepsilon &:= (X_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, \bar z_t^\varepsilon, u_t^\varepsilon, \mathcal{L}(x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, \bar z_t^\varepsilon, u_t^\varepsilon)) = (\rho_t^\varepsilon, \alpha_t^\varepsilon, \mathcal{L}(\alpha_t^\varepsilon)),\\
(\Theta_t')^\varepsilon &:= (x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, \bar z_t^\varepsilon, u_t^\varepsilon, \mathcal{L}(x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, \bar z_t^\varepsilon, u_t^\varepsilon)) = (\alpha_t^\varepsilon, \mathcal{L}(\alpha_t^\varepsilon)).
\end{aligned}
$$

The following expansion is useful for the proof of following lemmas. For given $(x,v)\in L^2(\Omega;\mathbb{R}^{n\times k})$ and $(\bar x,\bar v)\in L^2(\Omega;\mathbb{R}^{n\times k})$, it follows that (by denoting $x^\lambda := \bar x + \lambda x, v^\lambda := \bar v + \lambda v$)

$$
\begin{aligned}
&\varphi(x+\bar{x}, v+\bar{v}, \mathcal{L}(x,v)) - \varphi(\bar{x},\bar{v},\mathcal{L}(\bar{x},\bar{v})) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(x^\lambda, v^\lambda, \mathcal{L}(x^\lambda, v^\lambda))\mathrm{d}\lambda \\
&= \int_0^1 \partial_x\varphi(x^\lambda, v^\lambda, \mathcal{L}(x^\lambda, v^\lambda))\cdot x\mathrm{d}\lambda + \int_0^1 \partial_v\varphi(x^\lambda, v^\lambda, \mathcal{L}(x^\lambda, v^\lambda))\cdot v\mathrm{d}\lambda \\
&\quad + \int_0^1 \tilde{\mathbb{E}}[\partial_\mu\varphi(x^\lambda, v^\lambda, \mathcal{L}(x^\lambda, v^\lambda))(\widetilde{x^\lambda}, \widetilde{v^\lambda})\cdot \tilde{x}]\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}[\partial_\nu\varphi(x^\lambda, v^\lambda, \mathcal{L}(x^\lambda, v^\lambda))(\widetilde{x^\lambda}, \widetilde{v^\lambda})\cdot \tilde{v}]\mathrm{d}\lambda.
\end{aligned}
\tag{3.6}
$$

We have following lemmas for variation equations whose proofs will be given in the appendix.

Lemma 3.2. *Suppose assumptions (H.1) and (H.2) hold, then we have*

$$
\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}|X_t^{\varepsilon,1}|^2 = \lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}\left(|\rho_t^{\varepsilon,1}|^2 + |x_t^{\varepsilon,1}|^2\right) = 0.
$$

Moreover, for any $2\le p\le 4$ and $0<\varepsilon_0\le p$, it follows that

$$
\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}(|x_t^{\varepsilon,1}|^p + |\rho_t^{\varepsilon,1}|^{p-\varepsilon_0}) = 0.
$$

Lemma 3.3. *Suppose assumptions (H.1) and (H.2) hold, then we have*

$$
\mathit{lim}_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^2 + \int_0^T(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right] = 0. \tag{3.7}
$$

Moreover, we have the following p-order convergence result.

Lemma 3.4. *Suppose assumptions (H.1) and (H.2) hold, then for any $2\le p\le 4$,*

$$
\mathit{lim}_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + \left(\int_0^T(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}\right] = 0. \tag{3.8}
$$

Remark 3.5. We also have the following high-order boundedness result (see (A.14), (A.20) and (A.49) in the appendix), for any $2\le p\le 4$ and $0<\varepsilon_0\le p$, there exists a constant C independent of ε such that

$$
\mathbb{E}\left[\sup_{0\le t\le T}(|x_t^{\varepsilon,1}|^p + |y_t^{\varepsilon,1}|^p + |\rho_t^{\varepsilon,1}|^{p-\varepsilon_0}) + \left(\int_0^T(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}\right] \le C. \tag{3.9}
$$

3.2. Variational inequality and maximum principle

Let us first compute the Gâteaux derivative of the cost functional.

Lemma 3.6. *The functional $u(\cdot) \mapsto J(u(\cdot))$ is Gâteaux differentiable in the direction $v(\cdot)$, and its derivative is given by*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u(\cdot)+\varepsilon v(\cdot))\Big|_{\varepsilon=0} \\
=&\mathbb{E}\int_0^T \Big[\partial_X L(t,\Theta_t)X_t^1 + \partial_y L(t,\Theta_t)y_t^1 + \partial_z L(t,\Theta_t)z_t^1 + \partial_{\bar z} L(t,\Theta_t)\bar z_t^1 + \partial_v L(t,\Theta_t)v_t^1 \\
&\qquad + \tilde{\mathbb{E}}[\partial_{\mu_1} L(t,\Theta_t)(\tilde\alpha_t)\tilde x_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_2} L(t,\Theta_t)(\tilde\alpha_t)\tilde y_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_3} L(t,\Theta_t)(\tilde\alpha_t)\tilde z_t^1] \\
&\qquad + \tilde{\mathbb{E}}[\partial_{\mu_4} L(t,\Theta_t)(\tilde\alpha_t)\tilde{\bar z}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5} L(t,\Theta_t)(\tilde\alpha_t)\tilde v_t]\Big]\mathrm{d}t \\
&\quad + \mathbb{E}\Big[\partial_X M(X_T,\mathcal{L}(x_T))X_T^1 + \tilde{\mathbb{E}}[\partial_{\mu_1} M(X_T,\mathcal{L}(x_T))(\tilde x_T)\tilde x_T^1]\Big] + \partial_y \gamma(y_0) y_0^1.
\end{aligned} \tag{3.10}$$

Proof. Recall that the cost functional is defined by (2.8), using notations above Lemma 3.2, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(u(\cdot)+\varepsilon v(\cdot))\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{J(u^\varepsilon(\cdot)) - J(u(\cdot))}{\varepsilon} \\
=&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left\{\int_0^T [L(t,\Theta_t^\varepsilon) - L(t,\Theta_t)]\mathrm{d}t + M(X_T^\varepsilon,\mathcal{L}(x_T^\varepsilon)) - M(X_T,\mathcal{L}(x_T)) + \gamma(y_0^\varepsilon) - \gamma(y_0)\right\}.
\end{aligned}$$

Let us focus only on the term $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\int_0^T [L(t,\Theta_t^\varepsilon) - L(t,\Theta_t)]\mathrm{d}t$, and other terms can be tackled in a similar way. By using similar expansion as (3.6), we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\int_0^T [L(t,\Theta_t^\varepsilon) - L(t,\Theta_t)]\mathrm{d}t \\
=&\mathbb{E}\int_0^T \Big[\partial_X L(t,\Theta_t)X_t^1 + \partial_y L(t,\Theta_t)y_t^1 + \partial_z L(t,\Theta_t)z_t^1 + \partial_{\bar z} L(t,\Theta_t)\bar z_t^1 + \partial_v L(t,\Theta_t)v_t \\
&\qquad + \tilde{\mathbb{E}}[\partial_{\mu_1} L(t,\Theta_t)(\tilde\alpha_t)\tilde x_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_2} L(t,\Theta_t)(\tilde\alpha_t)\tilde y_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_3} L(t,\Theta_t)(\tilde\alpha_t)\tilde z_t^1] \\
&\qquad + \tilde{\mathbb{E}}[\partial_{\mu_4} L(t,\Theta_t)(\tilde\alpha_t)\tilde{\bar z}_t^1] + \tilde{\mathbb{E}}[\partial_{\mu_5} L(t,\Theta_t)(\tilde\alpha_t)\tilde v_t]\Big]\mathrm{d}t + \lim_{\varepsilon\to 0}\mathbb{E}\int_0^T \Delta_L^\varepsilon(t)\mathrm{d}t,
\end{aligned}$$

where

$$
\begin{aligned}
\Delta_L^\varepsilon(t) :=& \int_0^1 \Big[\big[\partial_v L(t,(\Theta_t)^{\lambda,\varepsilon}) - \partial_v L(t,\Theta_t)\big] v_t \\
&+ \sum_{\psi = X,y,z,\bar z} \Big[\partial_\psi L(t,(\Theta_t)^{\lambda,\varepsilon})(\psi_t^1 + \psi_t^{\varepsilon,1}) - \partial_\psi L(t,\Theta_t)\psi_t^1\Big] \Big] \mathrm{d}\lambda \\
&+ \int_0^1 \Big[\tilde{\mathbb{E}}[\partial_{\mu_1} L(t,(\Theta_t)^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})(\tilde x_t^1 + \widetilde{x_t^{\varepsilon,1}})] - \tilde{\mathbb{E}}[\partial_{\mu_1} L(t,\Theta_t)(\tilde\alpha_t)\tilde x_t^1]\Big] \mathrm{d}\lambda \\
&+ \int_0^1 \Big[\tilde{\mathbb{E}}[\partial_{\mu_2} L(t,(\Theta_t)^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})(\tilde y_t^1 + \widetilde{y_t^{\varepsilon,1}})] - \tilde{\mathbb{E}}[\partial_{\mu_2} L(t,\Theta_t)(\tilde\alpha_t)\tilde y_t^1]\Big] \mathrm{d}\lambda \\
&+ \int_0^1 \Big[\tilde{\mathbb{E}}[\partial_{\mu_3} L(t,(\Theta_t)^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})(\tilde z_t^1 + \widetilde{z_t^{\varepsilon,1}})] - \tilde{\mathbb{E}}[\partial_{\mu_3} L(t,\Theta_t)(\tilde\alpha_t)\tilde z_t^1]\Big] \mathrm{d}\lambda \\
&+ \int_0^1 \Big[\tilde{\mathbb{E}}[\partial_{\mu_4} L(t,(\Theta_t)^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})(\tilde{\bar z}_t^1 + \widetilde{\bar z_t^{\varepsilon,1}})] - \tilde{\mathbb{E}}[\partial_{\mu_4} L(t,\Theta_t)(\tilde\alpha_t)\tilde{\bar z}_t^1]\Big] \mathrm{d}\lambda \\
&+ \int_0^1 \tilde{\mathbb{E}}\left[\left(\partial_{\mu_5} L(t,(\Theta_t)^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}}) - \partial_{\mu_5} L(t,\Theta_t)(\tilde\alpha_t)\right)\tilde v_t\right] \mathrm{d}\lambda.
\end{aligned}
$$

To prove (3.10), we only need to show that $\lim_{\varepsilon\to 0} \mathbb{E}\int_0^T \Delta_L^\varepsilon(t)\mathrm{d}t = 0$. By noticing Lemmas 3.2 and 3.4, it is sufficient to show the uniformly integrability of $\Delta_L^\varepsilon(t)$. Let us now only focus on the uniformly integrability of the term $\partial_X L(t,(\Theta_t)^{\lambda,\varepsilon})(X_t^1 + X_t^{\varepsilon,1})$, the argument for other terms will be similar. From assumption (H.2), one can check that $\partial_X L(t,(\Theta_t)^{\lambda,\varepsilon})(X_t^1 + X_t^{\varepsilon,1})$ is dominated by

$$
C(|\rho_t^1| + |\rho_t^{\varepsilon,1}|)\Lambda + C(|\rho_t^1| + |\rho_t^{\varepsilon,1}|)^2\Lambda^{1/2} + C(|\rho_t^1| + |\rho_t^{\varepsilon,1}|)(|x_t^1| + |x_t^{\varepsilon,1}|)\Lambda^{1/2}
$$

where

$$
\Lambda := \sum_{\psi=\rho,x,y,z,\bar z} (1 + |\psi_t|^2 + \varepsilon^2|\psi_t^1|^2 + \varepsilon^2|\psi_t^{\varepsilon,1}|^2) + \sum_{\psi=x,y,z,\bar z} [1 + \mathbb{E}|\psi_t|^2 + \varepsilon^2\mathbb{E}|\psi_t^1|^2 + \varepsilon^2\mathbb{E}|\psi_t^{\varepsilon,1}|^2],
$$

Then from Lemmas 3.2 and 3.4, estimate (3.9) and $\mathbb{E}[\sup_{t\in[0,T]}|\rho_t^1|^p] < +\infty$, for $2 \le p < 4$, as well as the following inequalities, for $\psi = \rho, x, y, z, \bar z$, $\phi = x, y, z, \bar z$, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\mathbb{E}\int_0^T (|\rho_t^1| + |\rho_t^{\varepsilon,1}|)\cdot|\psi_t^{\varepsilon,1}|^2\mathrm{d}t \le \Big(\mathbb{E}\sup_{t\in[0,T]}(|\rho_t^1| + |\rho_t^{\varepsilon,1}|)^3\Big)^{\frac13}\left(\mathbb{E}\left(\int_0^T |\psi_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{\frac32}\right)^{\frac23} \to 0, \\
&\mathbb{E}\int_0^T (|\rho_t^1| + |\rho_t^{\varepsilon,1}|)\cdot\mathbb{E}|\phi_t^{\varepsilon,1}|^2\mathrm{d}t \le \mathbb{E}\sup_{t\in[0,T]}(|\rho_t^1| + |\rho_t^{\varepsilon,1}|)\cdot\int_0^T \mathbb{E}|\phi_t^{\varepsilon,1}|^2\mathrm{d}t \to 0,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\int_0^T (|\rho_t|+|\rho_t^1|+|\rho_t^{\varepsilon,1}|)^2|\psi_t^{\varepsilon,1}|\mathrm{d}t \le \Big(\mathbb{E}\sup_{t\in[0,T]}(|\rho_t|+|\rho_t^1|+|\rho_t^{\varepsilon,1}|)^3\Big)^{\frac{2}{3}}\Big(\mathbb{E}\Big(\int_0^T|\psi_t^{\varepsilon,1}|^2\mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{1}{3}}\to 0,\\
&\mathbb{E}\int_0^T (|\rho_t|+|\rho_t^1|+|\rho_t^{\varepsilon,1}|)^2\cdot\mathbb{E}|\phi_t^{\varepsilon,1}|\mathrm{d}t \le \mathbb{E}\sup_{t\in[0,T]}(|\rho_t|+|\rho_t^1|+|\rho_t^{\varepsilon,1}|)^2\cdot\int_0^T\mathbb{E}|\phi_t^{\varepsilon,1}|\mathrm{d}t\to 0,\\
&\mathbb{E}\int_0^T|\rho_t^{\varepsilon,1}|\cdot(1+|\psi_t|^2+|\psi_t^{\varepsilon,1}|^2)\mathrm{d}t\le\Big(\mathbb{E}\sup_{t\in[0,T]}|\rho_t^{\varepsilon,1}|^3\Big)^{\frac{1}{3}}\Big(\mathbb{E}\Big(\int_0^T(1+|\psi_t|^2+|\psi_t^{\varepsilon,1}|^2)\mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{2}{3}}\to 0,\\
&\mathbb{E}\int_0^T|\rho_t^{\varepsilon,1}|^2\cdot(1+|\psi_t|+|\psi_t^{\varepsilon,1}|)\mathrm{d}t\le\Big(\mathbb{E}\sup_{t\in[0,T]}|\rho_t^{\varepsilon,1}|^3\Big)^{\frac{2}{3}}\Big(\mathbb{E}\Big(\int_0^T(1+|\psi_t|+|\psi_t^{\varepsilon,1}|)\mathrm{d}t\Big)^{\frac{3}{2}}\Big)^{\frac{1}{3}}\to 0,
\end{aligned}
$$

we can get the uniformly integrability of $\partial_X L(t,(\Theta_t)^{\lambda,\varepsilon})(X_t^1+X_t^{\varepsilon,1})$. The proof is complete. □

Remark 3.7. From above proof, one can find that we need p-order ($p>2$) estimates of both the states and variational states, the second-order estimates are not enough.

To derive the maximum principle, we introduce the following adjoint equation which is a mean-field FBSDE (recalling the notations at the beginning of Sect. 3.1).

$$
\begin{cases}
-\mathrm{d}p_t = \Big\{\partial_X^\top F(t,\theta_t)p_t + \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\langle\partial_{\mu_1}F(t,\tilde{\theta}_t)(x_t,v_t),\tilde{p}_t\rangle]\end{pmatrix} + \partial_X^\top\Sigma(t,\theta_t)k_t + \partial_X^\top\bar{\Sigma}(t,\theta_t)\bar{k}_t\\
\qquad + \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\langle\partial_{\mu_1}\Sigma(t,\tilde{\theta}_t)(x_t,v_t),\tilde{k}_t\rangle]\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\langle\partial_{\mu_1}\bar{\Sigma}(t,\tilde{\theta}_t)(x_t,v_t),\tilde{\bar{k}}_t\rangle]\end{pmatrix} - \begin{pmatrix}0\\ \partial_x G(t,\Theta_t)\end{pmatrix}q_t\\
\qquad - \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\partial_{\mu_1}G(t,\tilde{\Theta}_t)(\alpha_t)\cdot\tilde{q}_t]\end{pmatrix} + \partial_X^\top L(t,\Theta_t) + \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\partial_{\mu_1}L(t,\tilde{\Theta}_t)(\alpha_t)]\end{pmatrix}\Big\}\mathrm{d}t - k_t\mathrm{d}W_t - \bar{k}_t\mathrm{d}\bar{Y}_t,\\
\mathrm{d}q_t = \Big[\partial_y G(t,\Theta_t)q_t + \tilde{\mathbb{E}}[\partial_{\mu_2}G(t,\tilde{\Theta}_t)(\alpha_t)\cdot\tilde{q}_t] - \partial_y L(t,\Theta_t) - \tilde{\mathbb{E}}[\partial_{\mu_2}L(t,\tilde{\Theta}_t)(\alpha_t)]\Big]\mathrm{d}t\\
\qquad + \Big[\partial_z G(t,\Theta_t)q_t + \tilde{\mathbb{E}}[\partial_{\mu_3}G(t,\tilde{\Theta}_t)(\alpha_t)\cdot\tilde{q}_t] - \partial_z L(t,\Theta_t) - \tilde{\mathbb{E}}[\partial_{\mu_3}L(t,\tilde{\Theta}_t)(\alpha_t)]\Big]\mathrm{d}W_t\\
\qquad + \Big[\partial_{\bar{z}} G(t,\Theta_t)q_t + \tilde{\mathbb{E}}[\partial_{\mu_4}G(t,\tilde{\Theta}_t)(\alpha_t)\cdot\tilde{q}_t] - \partial_{\bar{z}} L(t,\Theta_t) - \tilde{\mathbb{E}}[\partial_{\mu_4}L(t,\tilde{\Theta}_t)(\alpha_t)]\Big]\mathrm{d}Y_t,\\
p_T = \partial_X^\top M(X_T,\mathcal{L}(x_T)) + \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\partial_{\mu_1}M(\tilde{X}_T,\mathcal{L}(x_T))(x_T)]\end{pmatrix}\\
\qquad - \begin{pmatrix}0\\ \partial_x\Phi(x_T,\mathcal{L}(x_T))\end{pmatrix}q_T - \begin{pmatrix}0\\ \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(\tilde{x}_T,\mathcal{L}(x_T))(x_T)\cdot\tilde{q}_T]\end{pmatrix},\\
q_0 = -\partial_y\gamma(y_0).
\end{cases}
\tag{3.11}
$$

Remark 3.8. Mean-field FBSDE (3.11) is partially coupled. In fact, the equation of $q(\cdot)$ does not depend on $(p(\cdot),k(\cdot),\bar{k}(\cdot))$ and it is a mean-field SDE with uniformly Lipschitz coefficients which can be uniquely solved. The equation of $p(\cdot)$ is a mean-field BSDE whose coefficients satisfy random Lipschitz condition (by noticing that the coefficients related to $\bar{\Sigma}$ is not uniformly Lipschitz due to the apperance of ρ). With assumptions (H.1)–(H.2), by noticing $\mathbb{E}\left[\sup_{t\in[0,T]}(|\rho_t|^p+|\rho_t|^{-p}\right]<+\infty$, for any $p\ge 1$, we can combine the classical methods of BSDEs with random Lipschitz coefficients (see *e.g.* Thm. 5.23 [35]) and the methods of mean-field BSDEs with uniformly Lipschitz coefficients (*e.g.* Thm. 4.23 of [17] and [7])), to show that $(p(\cdot),k(\cdot),\bar{k}(\cdot))$ can be uniquely solved.

Now, we define the following Hamiltonian function H mapping from $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k \times \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k) \times \mathbb{R}^{n+1} \times \mathbb{R}^l \times \mathbb{R}^{(n+1)\times m} \times \mathbb{R}^{(n+1)\times d}$ to $\mathbb{R}$,

$$\begin{aligned} H(t,X,y,z,\bar z,v,\eta,p,q,k,\bar k) = \langle F(t,X,v,\xi),p\rangle - \langle G(t,X,y,z,\bar z,v,\eta),q\rangle \\ + \mathrm{tr}[k^\top \Sigma(t,X,v,\xi)] + \mathrm{tr}[\bar k^\top \bar\Sigma(t,X,v,\xi)] + L(t,X,y,z,\bar z,v,\eta), \end{aligned} \tag{3.12}$$

where $\xi \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^k)$ is the joint margin distribution of $\eta \in \mathcal{P}_2(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l\times m} \times \mathbb{R}^{l\times d} \times \mathbb{R}^k)$ on the first and the fifth components. Then we can rewrite the adjoint equation (3.11) as

$$\begin{cases} -\mathrm{d}p_t = \left[\partial_X H(t,\Theta_t;p,q,k,\bar k) + \begin{pmatrix} 0 \\ \tilde{\mathbb{E}}[\partial_{\mu_1} H(t,\tilde\Theta_t;\tilde p,\tilde q,\tilde k,\tilde{\bar k})(\alpha_t)] \end{pmatrix}\right]\mathrm{d}t - k_t \mathrm{d}W_t - \bar k_t \mathrm{d}Y_t \\ \mathrm{d}q_t = -\left[\partial_y H(t,\Theta_t;p,q,k,\bar k) + \tilde{\mathbb{E}}[\partial_{\mu_2} H(t,\tilde\Theta_t;\tilde p,\tilde q,\tilde k,\tilde{\bar k})(\alpha_t)]\right]\mathrm{d}t \\ \qquad - \left[\partial_z H(t,\Theta_t;p,q,k,\bar k) + \tilde{\mathbb{E}}[\partial_{\mu_3} H(t,\tilde\Theta_t;\tilde p,\tilde q,\tilde k,\tilde{\bar k})(\alpha_t)]\right]\mathrm{d}W_t \\ \qquad - \left[\partial_{\bar z} H(t,\Theta_t;p,q,k,\bar k) + \tilde{\mathbb{E}}[\partial_{\mu_4} H(t,\tilde\Theta_t;\tilde p,\tilde q,\tilde k,\tilde{\bar k})(\alpha_t)]\right]\mathrm{d}Y_t, \\ p_T = \partial_X^\top M(X_T,\mathcal{L}(x_T)) + \begin{pmatrix} 0 \\ \tilde{\mathbb{E}}[\partial_{\mu_1} M(\tilde X_T,\mathcal{L}(x_T))(x_T)] \end{pmatrix} \\ \qquad - \begin{pmatrix} 0 \\ \partial_x \Phi(x_T,\mathcal{L}(x_T)) \end{pmatrix} q_T - \begin{pmatrix} 0 \\ \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(\tilde x_T,\mathcal{L}(x_T))(x_T)\cdot \tilde q_T] \end{pmatrix}, \\ q_0 = -\partial_y \gamma(y_0). \end{cases} \tag{3.13}$$

We now state the Pontryagin's stochastic maximum principle for optimal control of our **Problem (EMFPOC)**.

Theorem 3.9. *Let (H.1) and (H.2) hold, if $u(\cdot)$ is an optimal control and $(X(\cdot),y(\cdot),z(\cdot),\bar z(\cdot))$ is the corresponding trajectory, and $(p(\cdot),q(\cdot),k(\cdot),\bar k(\cdot))$ is corresponding adjoint process satisfying (3.13), we have*

$$\mathbb{E}\left[\left(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar k_t) + \tilde{\mathbb{E}}[\partial_{\mu_5} H(t,\tilde\Theta_t,\tilde p_t,\tilde q_t,\tilde k_t,\tilde{\bar k}_t)(\alpha_t)]\right)(v-u_t)\Big|\mathcal{F}_t^Y\right] \ge 0,\ \forall v\in U,\ \textit{a.s. a.e.} \tag{3.14}$$

where we recall

$$\alpha_t := (x_t,y_t,z_t,\bar z_t,u_t) \quad \eta_t := \mathcal{L}(\alpha_t), \quad \Theta_t := (X_t,y_t,z_t,\bar z_t,u_t,\eta_t) = (\rho_t,\alpha_t,\eta_t).$$

Proof. Let us first apply Itô's formula to $\langle X_t^1, p_t\rangle$, we derive

$$
\begin{aligned}
\mathbb{E}\langle p_T, X_T^1\rangle =&\mathbb{E}\int_0^T \Big[\langle \partial_v F(t,\theta_t)v_t, p_t\rangle + \partial_x G(t,\Theta_t)q_t x_t^1 - \partial_X L(t,\Theta_t)X_t^1 \\
&+ \langle \partial_v \Sigma(t,\theta_t)v_t, k_t\rangle + \langle \partial_v \bar{\Sigma}(t,\theta_t)v_t, \bar{k}_t\rangle \\
&+ \tilde{\mathbb{E}}[\partial_{\mu_1} G(t,\Theta_t)(\alpha_t)\tilde{q}_t]x_t^1 - \tilde{\mathbb{E}}[\partial_{\mu_1} L(t,\tilde{\Theta}_t)(\alpha_t)]x_t^1 \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_5} F(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t], p_t\rangle + \langle \tilde{\mathbb{E}}[\partial_{\mu_5} \Sigma(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t], k_t\rangle \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_5} \bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t], \bar{k}_t\rangle \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_1} F(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{x}_t^1], p_t\rangle - \tilde{\mathbb{E}}[\langle \partial_{\mu_1} F(t,\tilde{\theta}_t)(x_t,v_t), \tilde{p}_t\rangle]x_t^1 \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_1} \Sigma(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{x}_t^1], k_t\rangle - \tilde{\mathbb{E}}[\langle \partial_{\mu_1} \Sigma(t,\tilde{\theta}_t)(x_t,v_t), \tilde{k}_t\rangle]x_t^1 \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_1} \bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{x}_t^1], \bar{k}_t\rangle - \tilde{\mathbb{E}}[\langle \partial_{\mu_1} \bar{\Sigma}(t,\tilde{\theta}_t)(x_t,v_t), \tilde{\bar{k}}_t\rangle]x_t^1\Big]\mathrm{d}t.
\end{aligned}
$$

By noting that "tilde random variables" are independent copies of the "nontilde variables", we apply Fubini's theorem and obtain

$$
\begin{aligned}
&\mathbb{E}\left[\langle \tilde{\mathbb{E}}[\partial_{\mu_1} F(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{x}_t^1], p_t\rangle - \tilde{\mathbb{E}}[\langle \partial_{\mu_1} F(t,\tilde{\theta}_t)(x_t,v_t), \tilde{p}_t\rangle]x_t^1\right] = 0,\\
&\mathbb{E}\left[\langle \tilde{\mathbb{E}}[\partial_{\mu_1} \Sigma(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{x}_t^1], k_t\rangle - \tilde{\mathbb{E}}[\langle \partial_{\mu_1} \Sigma(t,\tilde{\theta}_t)(x_t,v_t), \tilde{k}_t\rangle]x_t^1\right] = 0,\\
&\mathbb{E}\left[\langle \tilde{\mathbb{E}}[\partial_{\mu_1} \bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{x}_t^1], \bar{k}_t\rangle - \tilde{\mathbb{E}}[\langle \partial_{\mu_1} \bar{\Sigma}(t,\tilde{\theta}_t)(x_t,v_t), \tilde{\bar{k}}_t\rangle]x_t^1\right] = 0.
\end{aligned}
$$

Then by recalling p_T in (3.13), we have

$$
\begin{aligned}
&\mathbb{E}\Big[\partial_X M(X_T,\mathcal{L}(x_T))X_T^1 + \tilde{\mathbb{E}}[\partial_{\mu_1} M(X_T,\mathcal{L}(x_T))(\tilde{x}_T)\tilde{x}_T^1] \\
&\qquad -\partial_x \Phi(x_T,\mathcal{L}(x_T))q_T x_T^1 - \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\tilde{x}_T)q_T]x_T^1\Big] \\
=&\mathbb{E}\int_0^T \Big[\langle \partial_v F(t,\theta_t)v_t, p_t\rangle + \partial_x G(t,\Theta_t)q_t x_t^1 - \partial_X L(t,\Theta_t)X_t^1 \\
&+ \langle \partial_v \Sigma(t,\theta_t)v_t, k_t\rangle + \langle \partial_v \bar{\Sigma}(t,\theta_t)v_t, \bar{k}_t\rangle \\
&+ \tilde{\mathbb{E}}[\partial_{\mu_1} G(t,\Theta_t)(\alpha_t)\tilde{q}_t]x_t^1 - \tilde{\mathbb{E}}[\partial_{\mu_1} L(t,\tilde{\Theta}_t)(\alpha_t)]x_t^1 \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_5} F(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t], p_t\rangle + \langle \tilde{\mathbb{E}}[\partial_{\mu_5} \Sigma(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t], k_t\rangle \\
&+ \langle \tilde{\mathbb{E}}[\partial_{\mu_5} \bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t], \bar{k}_t\rangle\Big]\mathrm{d}t.
\end{aligned}
\tag{3.15}
$$

Similarly, by applying Itô's formula to $\langle y_t^1, q_t\rangle$, and with the help of Fubini's theorem, we have

$$
\begin{aligned}
&\mathbb{E}[q_T y_T^1 - q_0 y_0^1] \\
=&\mathbb{E}\Big[\partial_y \gamma(y_0)y_0^1 + \partial_x \Phi(x_T,\mathcal{L}(x_T))q_T x_T^1 + \tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\tilde{x}_T)\tilde{x}_T^1]q_T\Big] \\
=&-\mathbb{E}\int_0^T \Big[\partial_x G(t,\Theta_t)x_t^1 q_t + \partial_v G(t,\Theta_t)v_t q_t + \partial_y L(t,\Theta_t)y_t^1 + \partial_z L(t,\Theta_t)z_t^1 + \partial_{\bar{z}} L(t,\Theta_t)\bar{z}_t^1 \\
&+ \tilde{\mathbb{E}}[\partial_{\mu_1} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{x}_t^1]q_t + \tilde{\mathbb{E}}[\partial_{\mu_5} G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{v}_t]q_t \\
&+ \tilde{\mathbb{E}}[\partial_{\mu_2} L(t,\Theta_t)(\alpha_t)]y_t^1 + \tilde{\mathbb{E}}[\partial_{\mu_3} L(t,\Theta_t)(\alpha_t)]z_t^1 + \tilde{\mathbb{E}}[\partial_{\mu_4} L(t,\Theta_t)(\alpha_t)]\bar{z}_t^1\Big]\mathrm{d}t.
\end{aligned}
\tag{3.16}
$$

Now from Lemma 3.6, (3.15), (3.16) and Fubini's theorem, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}J(u(\cdot)+\varepsilon v(\cdot))\Big|_{\varepsilon=0}=\lim_{\varepsilon\to 0}\frac{J(u^\varepsilon(\cdot))-J(u(\cdot))}{\varepsilon}\\
=&\mathbb{E}\int_0^T\Big[\langle\partial_v F(t,\theta_t)v_t,p_t\rangle+\langle\tilde{\mathbb{E}}[\partial_{\mu_5}F(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t],p_t\rangle+\langle\partial_v\Sigma(t,\theta_t)v_t,k_t\rangle\\
&\quad+\langle\tilde{\mathbb{E}}[\partial_{\mu_5}\Sigma(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t],k_t\rangle+\langle\partial_v\bar{\Sigma}(t,\theta_t)v_t,\bar{k}_t\rangle+\langle\tilde{\mathbb{E}}[\partial_{\mu_5}\bar{\Sigma}(t,\theta_t)(\tilde{x}_t,\tilde{v}_t)\tilde{v}_t],\bar{k}_t\rangle\\
&\quad-\partial_v G(t,\Theta_t^{'})v_t q_t-\tilde{\mathbb{E}}[\partial_{\mu_5}G(t,\Theta_t)(\tilde{\alpha}_t)\tilde{v}_t]q_t+\partial_v L(t,\Theta_t)v_t+\tilde{\mathbb{E}}[\partial_{\mu_5}L(t,\Theta_t)(\tilde{\alpha}_t)\tilde{v}_t]\Big]\mathrm{d}t\\
=&\mathbb{E}\int_0^T\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)v_t\mathrm{d}t.
\end{aligned}
$$

Since $u(\cdot)$ is optimal, we have $J(u^\varepsilon(\cdot))\geq J(u(\cdot))$, for any $\varepsilon\geq 0$, which yields that

$$
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}J(u(\cdot)+\varepsilon v(\cdot))\Big|_{\varepsilon=0}=\lim_{\varepsilon\to 0}\frac{J(u^\varepsilon(\cdot))-J(u(\cdot))}{\varepsilon}\geq 0,
$$

and then for arbitrary $v(\cdot)$ such that $u(\cdot)+v(\cdot)\in\mathcal{U}_{ad}$, it follows that

$$
\begin{aligned}
0\leq&\mathbb{E}\int_0^T\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)v_t\mathrm{d}t\\
=&\mathbb{E}\int_0^T\mathbb{E}\Big[\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p},\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)v_t\Big|\mathcal{F}_t^Y\Big]\mathrm{d}t.
\end{aligned}\tag{3.17}
$$

Since $\mathcal{U}_{ad}$ is convex, for any given $v(\cdot)\in\mathcal{U}_{ad}$, we may choose the perturbation $u^\varepsilon(\cdot)=u(\cdot)+\varepsilon(v(\cdot)-u(\cdot))$, which is still in $\mathcal{U}_{ad}$, and then from (3.17), we have for any $v(\cdot)\in\mathcal{U}_{ad}$,

$$
\mathbb{E}\int_0^T\mathbb{E}\Big[\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)(v_t-u_t)\Big|\mathcal{F}_t^Y\Big]\mathrm{d}t\geq 0.\tag{3.18}
$$

For any given $v\in U$ (deterministic), we set

$$
A:=\Big\{(t,\omega)\Big|\mathbb{E}\Big[\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)(v-u_t)\Big|\mathcal{F}_t^Y\Big]<0\Big\},
$$

by choosing a modification if necessary, we know that A is a $\mathbb{F}^Y$-progressively measurable set. Now, we take $v_1(\cdot)=vI_AI_{E_\delta}+u(\cdot)I_{A^c}I_{E_\delta^c}$, where $E_\delta\subset[0,T]$ is any Borel measurable set with $|E_\delta|=\delta$, then we know $v_1(\cdot)\in\mathcal{U}_{ad}$. From (3.18) we get

$$
\mathbb{E}\int_0^T\Big[\mathbb{E}\Big[\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)\Big)(v-u_t)\Big|\mathcal{F}_t^Y\Big]I_A\Big]I_{E_\delta}\mathrm{d}t\geq 0,
$$

where we used the fact that I_A is $\mathcal{F}_t^Y$-adapted. Thus, from Lebesgue differentiation theorem, we have

$$
\mathbb{E}\Big[\mathbb{E}\Big[\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)(v-u_t)\Big|\mathcal{F}_t^Y\Big]I_A\Big]\geq 0\ a.e.,
$$

which together with the definition of A indicates that $\mathbb{E}\Big[I_A(t,\omega)\Big] = 0$, *a.e.* Consequently, we deduce that $I_A = 0$, *a.s. a.e.*, which yields that for any $v \in U$,

$$\mathbb{E}\left[\Big(\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t) + \tilde{\mathbb{E}}[\partial_{\mu_5} H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big)(v-u_t)\Big|\mathcal{F}_t^Y\right] \geq 0 \quad a.s.\ a.e. \tag{3.19}$$

By noticing that the left hand side of (3.19) is continuous w.r.t. v , we obtain (3.14). □

Remark 3.10. Inspired by Proposition 4.6 of [16], our Pontryagin's stochastic maximum principle, *i.e.* Theorem 3.9 can be generalized to the case that U is an open set which maybe non-convex. In this case, following similar methods of of [16], we can show that

$$\mathbb{E}\left[\partial_v H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t) + \tilde{\mathbb{E}}[\partial_{\mu_5} H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\Big|\mathcal{F}_t^Y\right] = 0, \text{ a.s. a.e.}$$

4. Verification theorem

We have established Pontryagin's maximum principle which gives a necessary condition for the optimal control, see (3.14) in Theorem 3.9. In this section we will show that (3.14) is also a sufficient condition for optimality under the following convexity assumptions.

(H.3) The function γ is convex. The function M is convex in the sense that

$$M(\check{X},\check{\mu}_1) - M(X,\mu_1) \geq \langle \partial_X M(X,\mu_1), \check{X}-X\rangle + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_1} M(X,\mu_1)(\tilde{x}), \tilde{\check{x}} - \tilde{x}\rangle\big]$$

for all $X, \check{X} \in \mathbb{R}^{n+1}$, and $\mu_1, \check{\mu}_1 \in \mathcal{P}_2(\mathbb{R}^n)$, and any $\tilde{\check{x}}$, $\tilde{x}$ which are $\mathbb{R}^n$-dimensional square integrable random variables in $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ satisfying $\mathcal{L}(\tilde{\check{x}}) = \check{\mu}_1$, $\mathcal{L}(\tilde{x}) = \mu_1$.

(H.4) The Hamiltonian function H satisfies the following convexity condition: for any $t \in [0,T]$ and $\Pi := (p,q,k,\bar{k}) \in \mathbb{R}^{n+1}\times\mathbb{R}^l\times\mathbb{R}^{(n+1)\times m}\times\mathbb{R}^{(n+1)\times d}$, for any $\check{\Lambda} := (\check{X},\check{y},\check{z},\check{\bar{z}},\check{v})$ and $\Lambda := (X,y,z,\bar{z},v)$ which belong $\mathbb{R}^{n+1}\times\mathbb{R}^l\times\mathbb{R}^{l\times m}\times\mathbb{R}^{l\times d}\times\mathbb{R}^k$, any $\check{\eta},\eta \in \mathcal{P}_2(\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^{l\times m}\times\mathbb{R}^{l\times d}\times\mathbb{R}^k)$, and any $(\tilde{\check{x}},\tilde{\check{y}},\tilde{\check{z}},\tilde{\check{\bar{z}}},\tilde{\check{v}})$ (resp. $(\tilde{x},\tilde{y},\tilde{z},\tilde{\bar{z}},\tilde{v})$) which are $\mathbb{R}^n\times\mathbb{R}^l\times\mathbb{R}^{l\times m}\times\mathbb{R}^{l\times d}\times\mathbb{R}^k$-dimensional square integrable random variables in $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ satisfying $\mathcal{L}(\tilde{\check{x}},\tilde{\check{y}},\tilde{\check{z}},\tilde{\check{\bar{z}}},\tilde{\check{v}}) = \check{\eta}$ (resp. $\mathcal{L}(\tilde{x},\tilde{y},\tilde{z},\tilde{\bar{z}},\tilde{v}) = \eta$), it holds that

$$\begin{aligned}
&H(t,\Lambda',\eta',\Pi) - H(t,\Lambda,\eta,\Pi)\\
\geq&\langle \partial_X H(t,\Lambda,\eta,\Pi), \check{X}-X\rangle + \langle \partial_y H(t,\Lambda,\eta,\Pi), \check{y}-y\rangle + \langle \partial_z H(t,\Lambda,\eta,\Pi), \check{z}-z\rangle\\
&+ \langle \partial_{\bar{z}} H(t,\Lambda,\eta,\Pi), \check{\bar{z}}-\bar{z}\rangle + \langle \partial_v H(t,\Lambda,\eta,\Pi), \check{v}-v\rangle + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_1} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{x}}-\tilde{x}\rangle\big]\\
&+ \tilde{\mathbb{E}}\big[\langle \partial_{\mu_2} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{y}}-\tilde{y}\rangle\big] + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_3} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{z}}-\tilde{z}\rangle\big]\\
&+ \tilde{\mathbb{E}}\big[\langle \partial_{\mu_4} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{\bar{z}}}-\tilde{\bar{z}}\rangle\big] + \tilde{\mathbb{E}}\big[\langle \partial_{\mu_5} H(t,\Lambda,\eta,\Pi)(\tilde{\alpha}), \tilde{\check{v}}-\tilde{v}\rangle\big].
\end{aligned}$$

Theorem 4.1. *Suppose (H.1)-(H.4) are satisfied. Let $u(\cdot) \in \mathcal{U}_{ad}$ be an admissible control, $(X(\cdot),y(\cdot),z(\cdot),\bar{z}(\cdot))$ be the corresponding trajectory, and $\Pi(\cdot) := (p(\cdot),q(\cdot),k(\cdot),\bar{k}(\cdot))$ be the corresponding adjoint process satisfying (3.13). If (3.14) holds, then $u(\cdot)$ is an optimal control, i.e. $J(u(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot))$.*

Proof. Let $\check{u}(\cdot) \in \mathcal{U}_{ad}$ be any admissible control and $(\check{X}(\cdot),\check{y}(\cdot),\check{z}(\cdot),\check{\bar{z}}(\cdot))$ be corresponding trajectory, $\Pi(\cdot) := (\check{p}(\cdot),\check{q}(\cdot),\check{k}(\cdot),\check{\bar{k}}(\cdot))$ be corresponding adjoint processes satisfying (3.13). We recall the notations at the beginning of subsection 3.1 and we use similar notations for the variables with the symbol "check" above. On the one hand, by denoting $H(t) := H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)$ and applying Itô's formula respectively to $\langle X_t - \check{X}_t, p_t\rangle$ and

$\langle y_t - \check{y}_t, q_t\rangle$ (recall (2.7) and (3.13)), we have

$$\mathbb{E}\langle X_T - \check{X}_T, p_T\rangle = -\mathbb{E}\int_0^T \Big[\langle X_t - \check{X}_t, \partial_X H(t)\rangle + \langle X_t - \check{X}_t, \tilde{\mathbb{E}}[\partial_{\mu_1} H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\rangle\Big]\,\mathrm{d}t$$
$$+\mathbb{E}\int_0^T \big[\langle F(t,\theta_t)-F(t,\check{\theta}_t),p_t\rangle + \langle \Sigma(t,\theta_t)-\Sigma(t,\check{\theta}_t),k_t\rangle + \langle \bar{\Sigma}(t,\theta_t)-\bar{\Sigma}(t,\check{\theta}_t),\bar{k}_t\rangle\big]\,\mathrm{d}t,$$

and

$$\begin{aligned}\langle y_0 - \check{y}_0, q_0\rangle =&\mathbb{E}\int_0^T \Big[\langle y_t-\check{y}_t, \partial_y H(t)\rangle + \langle y_t - \check{y}_t, \tilde{\mathbb{E}}[\partial_{\mu_2} H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\rangle\Big]\,\mathrm{d}t\\ &+\mathbb{E}\int_0^T \Big[\langle z_t-\check{z}_t, \partial_z H(t)\rangle + \langle z_t - \check{z}_t, \tilde{\mathbb{E}}[\partial_{\mu_3} H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\rangle\Big]\,\mathrm{d}t\\ &+\mathbb{E}\int_0^T \Big[\langle \bar{z}_t-\check{\bar{z}}_t, \partial_z H(t)\rangle + \langle \bar{z}_t - \check{\bar{z}}_t, \tilde{\mathbb{E}}[\partial_{\mu_4} H(t,\tilde{\Theta}_t,\tilde{p}_t,\tilde{q}_t,\tilde{k}_t,\tilde{\bar{k}}_t)(\alpha_t)]\rangle\Big]\,\mathrm{d}t\\ &+\mathbb{E}\int_0^T \langle q_t, G(t,\Theta_t)-G(t,\check{\Theta}_t)\rangle\mathrm{d}t.\end{aligned}$$

On the other hand, from assumption (H.3), by using Fubini's theorem and (3.13), we have

$$\begin{aligned}&\mathbb{E}\left[M(X_T,\mathcal{L}(x_T)) - M(\check{X}_T,\mathcal{L}(\check{x}_T))\right]\\ \le&\mathbb{E}\Big[\langle \partial_X M(X_T,\mathcal{L}(x_T)), X_T-\check{X}_T\rangle + \tilde{\mathbb{E}}[\langle \partial_{\mu_1} M(X_T,\mathcal{L}(x_T))(\tilde{x}_T), \tilde{x}_T - \tilde{\check{x}}_T\rangle\Big]\\ =&\mathbb{E}\Big[\langle \partial_X M(X_T,\mathcal{L}(x_T)), X_T-\check{X}_T\rangle + \tilde{\mathbb{E}}[\langle \partial_{\mu_1} M(\tilde{X}_T,\mathcal{L}(x_T))(x_T), x_T - \check{x}_T\rangle]\Big]\\ =&\mathbb{E}[\langle X_T - \check{X}_T, p_T\rangle],\end{aligned}$$

and similarly,

$$\gamma(y_0)-\gamma(\check{y}_0) \le \langle \partial_y\gamma(y_0), y_0-\check{y}_0\rangle = -\langle y_0-\check{y}_0,q_0\rangle.$$

Next, from assumption (H.4), we have

$$\begin{aligned}&H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t) - H(t,\check{\Theta}_t,\check{p}_t,\check{q}_t,\check{k}_t,\check{\bar{k}}_t)\\ \le&\langle \partial_X H(t), X_t-\check{X}_t\rangle + \langle \partial_y H(t), y_t - \check{y}_t\rangle + \langle \partial_z H(t), z_t-\check{z}_t\rangle + \langle \partial_{\bar{z}} H(t), \bar{z}_t - \check{\bar{z}}_t\rangle\\ &+\tilde{\mathbb{E}}[\partial_{\mu_1} H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)(\tilde{\alpha}_t)(\tilde{X}_t-\tilde{\check{X}}_t)] + \tilde{\mathbb{E}}[\partial_{\mu_2} H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)(\tilde{\alpha}_t)(\tilde{y}_t-\tilde{\check{y}}_t)]\\ &+\tilde{\mathbb{E}}[\partial_{\mu_3} H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)(\tilde{\alpha}_t)(\tilde{z}_t-\tilde{\check{z}}_t)] + \tilde{\mathbb{E}}[\partial_{\mu_4} H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)(\tilde{\alpha}_t)(\tilde{\bar{z}}_t-\tilde{\check{\bar{z}}}_t)]\\ &+\langle \partial_v H(t), u_t-\check{u}_t\rangle + \tilde{\mathbb{E}}[\partial_{\mu_5} H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)(\tilde{\alpha}_t)(\tilde{u}_t-\tilde{\check{u}}_t)].\end{aligned}$$

Then, it follows from (2.8) that

$$
\begin{aligned}
&J(u(\cdot))-J(\check{u}(\cdot))\\
=&\mathbb{E}\left[M(X_T,\mathcal{L}(x_T))-M(\check{X}_T,\mathcal{L}(\check{x}_T))+\gamma(y_0)-\gamma(\check{y}_0)+\int_0^T\big(L(t,\Theta_t)-L(t,\check{\Theta}_t)\big)\,\mathrm{d}t\right]\\
=&\mathbb{E}\left[M(X_T,\mathcal{L}(x_T))-M(\check{X}_T,\mathcal{L}(\check{x}_T))+\gamma(y_0)-\gamma(\check{y}_0)\right]\\
&+\mathbb{E}\int_0^T\Big(H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)-H(t,\check{\Theta}_t,\check{p}_t,\check{q}_t,\check{k}_t,\check{\bar{k}}_t)\Big)\,\mathrm{d}t+\mathbb{E}\int_0^T\langle G(t,\Theta_t)-G(t,\check{\Theta}_t),q_t\rangle\mathrm{d}t\\
&-\mathbb{E}\int_0^T\left[\langle F(t,\theta_t)-F(t,\check{\theta}_t),p_t\rangle+\langle\Sigma(t,\theta_t)-\Sigma(t,\check{\theta}_t),k_t\rangle+\langle\bar{\Sigma}(t,\theta_t)-\bar{\Sigma}(t,\check{\theta}_t),\bar{k}_t\rangle\right]\mathrm{d}t.
\end{aligned}
$$

With the help of above inequalities and Fubini's theorem, we have

$$
\begin{aligned}
J(u(\cdot))-J(\check{u}(\cdot))\leq&\,\mathbb{E}\int_0^T\Big(\langle\partial_vH(t),u_t-\check{u}_t\rangle+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\Theta_t,p_t,q_t,k_t,\bar{k}_t)(\tilde{\alpha}_t)(\tilde{u}_t-\tilde{\check{u}}_t)]\Big)\,\mathrm{d}t\\
=&\,\mathbb{E}\int_0^T\mathbb{E}\left[\Big(H_v(t,\Theta_t,p,q,k,\bar{k})+\tilde{\mathbb{E}}[\partial_{\mu_5}H(t,\tilde{\Theta}_t,\tilde{p},\tilde{q},\tilde{k},\tilde{\bar{k}})(\alpha_t)]\Big)(u_t-\check{u}_t)\Big|\mathcal{F}_t^Y\right]\mathrm{d}t\leq 0,
\end{aligned}
$$

where the last inequality comes from the maximum principle (3.14). Thus, $u(\cdot)$ is optimal. □

Remark 4.2. In this paper, we consider an optimal control problem with partially observable information in which the state is governed by a nonlinear mean-field type FBSDE. The cost functional depending on the joint distribution of the state and the control is defined on probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}^v)$. The structure of our problem is inspired by Wang, Wu and Xiong [45], and we adopt the method of Tang [41] in our paper, the main feature of this method is that we reformulate the cost functional by Bayes' formula, which transforms the cost functional into the one defined on the reference probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ (see (2.6) or (2.8)) and in additional, the term ρ is multiplied. This method is different from the one of Wang, Wu and Xiong [45].

In this formulation, the cost functional (2.6) contains ρ. If the coefficients l and χ in the cost functional (2.6) do not depend on ρ, then one can check that the mappings $(\rho,x,y,z,\bar{z},v,\eta)\mapsto\rho l(t,x,y,z,\bar{z},v,\eta)$ and $(\rho,x)\mapsto\rho\chi(x,\mu_1)$ are usually not convex. Fortunately, if we allow l and χ depend on ρ, then it is possible to make sense that the convexity assumptions (H.3)-(H.4) hold (one simple example is that $l(\rho,x,y,z,\bar{z},\eta)=\frac{1}{\rho}(|x|^2+|y|^2+|z|^2+|\bar{z}|^2+|v|^2+|\eta|^2)$ and $\chi(\rho,x,\mu_1)=\frac{1}{\rho}(|x|^2+|\mu|^2)$), then we can get the verification theorem.

5. Examples

In this section, we will give two examples to illustrate our results. Comparing with the existed literatures concerning McKean-Vlasove type stochastic control problems and partial observation problems, one can find that our maximum principle can deduce the related results in [1, 3, 16, 18, 19, 28–31, 41, 46, 50] in the case of convex control domain. This means that our results are actually an extension of the classical case.

5.1. Scalar interactions

The case of scalar interactions is of particular interest and has been widely investigated, see *e.g.* [3, 10, 27–31, 50]. Usually it can be dealt by using standard calculus without using L-derivatives. In this subsection, we will derive Pontryagin's maximum principle in scalar type interactions by using our Theorem 3.9.

In order to simplify notations, we suppose that $n = l = k = m = d = 1$. We set $\bar{\sigma}(t,x,v,\xi) = \bar{\sigma}_t$ and $h(t,x,v,\xi) = h_t$, where $\bar{\sigma}_t$ and h_t are bounded deterministic functions depending only on t. We also set

$$
\begin{aligned}
&f(t,x,v,\xi) = f_0(x,v,\int \varphi^f \mathrm{d}\xi), \quad \sigma(t,x,v,\xi) = \sigma_0(x,v,\int \varphi^\sigma \mathrm{d}\xi), \quad \Phi(x,\mu) = \Phi_0(x,\int \phi^\Phi \mathrm{d}\mu),\\
&g(t,x,y,z,\bar{z},v,\eta) = g_0(x,y,z,\bar{z},v,\int \psi^g \mathrm{d}\eta), \quad l(t,\rho,x,y,z,\bar{z},v,\eta) = l_0(\rho,x,y,z,\bar{z},v,\int \psi^l \mathrm{d}\eta),\\
&\chi(\rho,x,\mu) = \chi_0(\rho,x,\int \phi^\chi \mathrm{d}\mu),
\end{aligned}
$$

where f_0, σ_0 are defined on $\mathbb{R}\times U\times\mathbb{R}$, $\varphi^f, \varphi^\sigma$ are defined on $\mathbb{R}\times U$, Φ_0 is defined on $\mathbb{R}\times\mathbb{R}$, ϕ^Φ is defined on $\mathbb{R}$, g_0 is defined on $\mathbb{R}^4\times U\times\mathbb{R}$, l_0 is defined on $\mathbb{R}^5\times U\times\mathbb{R}$, and ψ^g, ψ^l are defined on $\mathbb{R}^4\times U$, χ_0 is defined on $\mathbb{R}^3$, ϕ^χ is defined on $\mathbb{R}$. By noticing that for functions $b:\mathbb{R}^d\to\mathbb{R}$, $r:\mathbb{R}\to\mathbb{R}$ and any $\vartheta\in L^2(\Omega;\mathbb{R}^d)$, we consider $R(\mathbb{P}_\vartheta) := b(\mathbb{E}[r(\vartheta)])$, then we have $\partial_\mu R(\mathbb{P}_\vartheta, y) = b'(\mathbb{E}[r(\vartheta)])(\partial_y r)(y)$, here b' denotes the derivative of b. By setting $\alpha_t = (x_t,y_t,z_t,\bar{z}_t,u_t)$, $\Xi_t^g := \mathbb{E}[\psi^g(x_t,y_t,z_t,\bar{z}_t,u_t)]$ and $\Xi_t^l := \mathbb{E}[\psi^l(x_t,y_t,z_t,\bar{z}_t,u_t)]$, then the adjoint equation (3.11) becomes

$$
\begin{cases}
-\mathrm{d}p_{1,t} =[h_t\bar{k}_{1,t} + l_0(t,\rho_t,\alpha_t,\Xi_t^l) + \rho_t\partial_\rho l_0(t,\rho_t,\alpha_t,\Xi_t^l)]\mathrm{d}t - k_{1,t}\mathrm{d}W_t - \bar{k}_{1,t}\mathrm{d}\bar{Y}_t,\\
\quad p_{1,T} =\chi_0(\rho_T,x_T,\mathbb{E}[\phi^\chi(x_T)]) + \rho_T\partial_\rho\chi_0(\rho_T,x_T,\mathbb{E}[\phi^\chi(x_T)]),
\end{cases} \tag{5.1}
$$

$$
\begin{cases}
-\mathrm{d}p_{2,t} =\Big\{p_{2,t}\partial_x f_0(x_t,u_t,\mathbb{E}[\varphi^f(x_t,u_t)]) + \tilde{\mathbb{E}}[\tilde{p}_{2,t}\partial_\zeta f_0(\tilde{x}_t,\tilde{u}_t,\mathbb{E}[\varphi^f(x_t,u_t)])]\cdot\partial_x\varphi^f(x_t,u_t)\\
\qquad + k_{2,t}\partial_x\sigma_0(x_t,u_t,\mathbb{E}[\varphi^\sigma(x_t,u_t)]) + \tilde{\mathbb{E}}[\tilde{k}_{2,t}\partial_\zeta\sigma_0(\tilde{x}_t,\tilde{u}_t,\mathbb{E}[\varphi^\sigma(x_t,u_t)])]\cdot\partial_x\varphi^\sigma(x_t,u_t)\\
\qquad - q_t\partial_x g_0(t,\alpha_t,\Xi_t^g) - \tilde{\mathbb{E}}[\tilde{q}_t\partial_{\zeta_1}g_0(t,\tilde{\alpha}_t,\Xi_t^g)]\cdot\partial_x\psi^g(\alpha_t) + \rho_t\partial_x l_0(t,\rho_t,\alpha_t,\Xi_t^l)\\
\qquad + \tilde{\mathbb{E}}[\rho_t\partial_{\zeta_1}l_0(t,\rho_t,\tilde{\alpha}_t,\Xi_t^l)]\cdot\partial_x\psi^l(\alpha_t)\Big\}\mathrm{d}t - k_{2,t}\mathrm{d}W_t - \bar{k}_{2,t}\mathrm{d}\bar{Y}_t,\\
p_{2,T} =\rho_T\partial_x\chi_0(\rho_T,x_T,\mathbb{E}[\phi^\chi(x_T)]) + \tilde{\mathbb{E}}[\rho_T\partial_\zeta\chi(\rho_T,x_T,\mathbb{E}[\phi^\chi(x_T)])]\cdot\partial_x\phi^\chi(x_T)\\
\qquad - \partial_x\Phi_0(x_T,\mathbb{E}[\phi^\Phi(x_T)])q_T - \tilde{\mathbb{E}}[\tilde{q}_t\partial_\zeta\Phi(x_T,\mathbb{E}[\phi^\Phi(x_T)])]\cdot\partial_x\phi^\Phi(x_T),
\end{cases} \tag{5.2}
$$

$$
\begin{cases}
\mathrm{d}q_t =\Big[\partial_y g_0(t,\alpha_t,\Xi_t^g)q_t + \tilde{\mathbb{E}}[\tilde{q}_t\partial_{\zeta_2}g_0(t,\tilde{\alpha}_t,\Xi_t^g)]\cdot\partial_y\psi^g(\alpha_t) - \rho_t\partial_y l_0(t,\rho_t,\alpha_t,\Xi_t^l)\\
\qquad - \rho_t\tilde{\mathbb{E}}[\partial_{\zeta_2}l_0(t,\rho_t,\tilde{\alpha}_t,\Xi_t^l)]\cdot\partial_y\psi^l(\alpha_t)\Big]\mathrm{d}t + \Big[\partial_z g_0(t,\alpha_t,\Xi_t^g)q_t - \rho_t\partial_z l_0(t,\rho_t,\alpha_t,\Xi_t^l)\\
\qquad + \tilde{\mathbb{E}}[\tilde{q}_t\partial_{\zeta_3}g_0(t,\tilde{\alpha}_t,\Xi_t^g)]\cdot\partial_z\psi^g(\alpha_t) - \rho_t\tilde{\mathbb{E}}[\partial_{\zeta_3}l_0(t,\rho_t,\tilde{\alpha}_t,\Xi_t^l)]\cdot\partial_z\psi^l(\alpha_t)\Big]\mathrm{d}W_t\\
\qquad + \Big[\partial_{\bar{z}}g_0(t,\alpha_t,\Xi_t^g)q_t + \tilde{\mathbb{E}}[\tilde{q}_t\partial_{\zeta_4}g_0(t,\tilde{\alpha}_t,\Xi_t^g)]\cdot\partial_{\bar{z}}\psi^g(\alpha_t)\\
\qquad - \rho_t\partial_{\bar{z}}l_0(t,\rho_t,\alpha_t,\Xi_t^l) - \rho_t\tilde{\mathbb{E}}[\partial_{\zeta_4}l_0(t,\rho_t,\tilde{\alpha}_t,\Xi_t^l)]\cdot\partial_{\bar{z}}\psi^l(\alpha_t)\Big]\mathrm{d}Y_t,\\
q_0 =-\partial_y\gamma(y_0).
\end{cases} \tag{5.3}
$$

According to Theorem 3.9, the necessary condition (3.14) for optimality will be

$$\begin{aligned}&\mathbb{E}\Big[\big\{p_{2,t}\partial_v f_0(x_t,u_t,\mathbb{E}[\varphi^f(x_t,u_t)]) - q_t\partial_v g_0(t,\alpha_t,\Xi^g_t) + k_{2,t}\partial_v\sigma_0(x_t,u_t,\mathbb{E}[\varphi^\sigma(x_t,u_t)])\\&\quad + \rho_t\partial_v l_0(t,\rho_t,\alpha_t,\Xi^l_t) + \tilde{\mathbb{E}}[\tilde p_{2,t}\partial_\zeta f_0(\tilde x_t,\tilde u_t,\mathbb{E}[\varphi^f(x_t,u_t)])]\cdot\partial_v\varphi^f(x_t,u_t)\\&\quad - \tilde{\mathbb{E}}[\tilde q_t\partial_{\zeta_5} g_0(t,\tilde\alpha_t,\Xi^g_t)]\cdot\partial_v\psi^g(\alpha_t) + \tilde{\mathbb{E}}[\tilde k_{2,t}\partial_\zeta\sigma_0(\tilde x_t,\tilde u_t,\mathbb{E}[\varphi^\sigma(x_t,u_t)])]\cdot\partial_v\varphi^\sigma(x_t,u_t)\\&\quad + \rho_t\tilde{\mathbb{E}}[\partial_{\zeta_5} l_0(t,\rho_t,\tilde\alpha_t,\Xi^l_t)]\cdot\partial_v\psi^l(\alpha_t)\big\}\cdot(v-u_t)\Big|\mathcal{F}^Y_t\Big]\geq 0,\quad \forall v\in U,\ \text{a.s. a.e.}\end{aligned}\tag{5.4}$$

Noticing that the equations (5.1) for $p_1(\cdot)$ and (5.2)) for $p_2(\cdot)$ are decoupled (due to the assumption $h(t,x,v,\xi)=h_t$). Moreover, one can find that $p_1(\cdot),k_1(\cdot)$ and $\bar k_1(\cdot)$ do not appear in (5.4).

5.2. Linear quadratic case

In this subsection, we will consider linear quadratic (LQ) partially observed optimal control problem. Let us consider the following linear forward-backward system with scalar interaction.

$$\begin{cases}\ \mathrm{d}x_t = (f_{1,t}x_t + f_{2,t}\mathbb{E}x_t + f_{3,t}v_t + f_{4,t}\mathbb{E}v_t)\mathrm{d}t + c_t\mathrm{d}W_t + \bar c_t\mathrm{d}\bar W_t,\\ -\mathrm{d}y_t = (g_{1,t}x_t + g_{2,t}\mathbb{E}x_t + g_{3,t}y_t + g_{4,t}\mathbb{E}y_t + g_{5,t}z_t + g_{6,t}\mathbb{E}z_t + g_{7,t}v_t + g_{8,t}\mathbb{E}v_t)\mathrm{d}t\\ \qquad\qquad - z_t\mathrm{d}W_t - \bar z_t\mathrm{d}Y_t,\\ \ x(0) = x_0,\quad y(T) = \phi_1 x_T + \phi_2\mathbb{E}x_T,\end{cases}\tag{5.5}$$

where the observation process Y is given by $\mathrm{d}Y_t = h_t\mathrm{d}t + \mathrm{d}\bar W_t$, with $Y_0 = 0$. We introduce $\rho_t = \exp\Big\{\int_0^t h_s\mathrm{d}Y_s - \frac12\int_0^t |h_s|^2\mathrm{d}s\Big\}$ which is the solution of SDE: $\mathrm{d}\rho_t = \rho_t h_t\mathrm{d}Y_t$, with $\rho_0 = 1$, and we define the probability measure $\mathbb{P}^v$ by $\mathrm{d}\mathbb{P}^v = \rho_T\mathrm{d}\mathbb{P}$.

Then we give the following cost functional

$$\begin{aligned}J(v(\cdot)) =& \mathbb{E}^v\Big[\int_0^T \frac{1}{\rho_t}\left[L_{1,t}x_t^2 + L_{2,t}(\mathbb{E}x_t)^2 + L_{3,t}y_t^2 + L_{4,t}(\mathbb{E}y_t)^2 + L_{5,t}v_t^2 + L_{6,t}(\mathbb{E}v_t)^2\right]\mathrm{d}t\\&+\frac{1}{\rho_T}(M_1x_T^2 + M_2(\mathbb{E}x_T)^2) + \gamma y_0^2\Big],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}J(v(\cdot)) =& \mathbb{E}\Big[\int_0^T \left[L_{1,t}x_t^2 + L_{2,t}(\mathbb{E}x_t)^2 + L_{3,t}y_t^2 + L_{4,t}(\mathbb{E}y_t)^2 + L_{5,t}v_t^2 + L_{6,t}(\mathbb{E}v_t)^2\right]\mathrm{d}t\\&+ M_1x_T + M_2(\mathbb{E}x_T)^2 + \gamma y_0\Big].\end{aligned}$$

Here, all the coefficients are uniformly bounded and deterministic, $L_{i,t}$ is positive function and $L_{i,t}$ is uniformly bounded, for $i = 1,2,3,4,5,6$, and M_1, M_2, γ are positive constants.

We notice that, in such case, we have

$$\begin{aligned}&l(\rho,x,y,z,\bar z,\eta) = \frac{1}{\rho}\big(L_1x^2 + L_2(\mathbb{E}x)^2 + L_3y^2 + L_4(\mathbb{E}y)^2 + L_5v^2 + L_6(\mathbb{E}v)^2\big),\\&L(X,y,z,\bar z,\eta) = L_1x^2 + L_2(\mathbb{E}x)^2 + L_3y^2 + L_4(\mathbb{E}y)^2 + L_5v^2 + L_6(\mathbb{E}v)^2 = l_1,\\&\chi(\rho,x,\mu_1) = \frac{1}{\rho}\big(M_1x^2 + M_2(\mathbb{E}x)^2\big),\qquad M(X,\mu_1) = M_1x^2 + M_2(\mathbb{E}x)^2 = \chi_1,\end{aligned}$$

where $X = \begin{pmatrix} \rho \\ x \end{pmatrix}$. It is easy to check that $L = l_1$ and $M = \chi_1$ satisfy assumption (H.2), but l and χ do not satisfy assumption (H.2). However, one can check that Lemma 3.6 still hold (the proof even become very simple since there is no term ρ in L and M), and then the maximum principle (see Thm. 3.9) works.

To apply Theorem 3.9, we rewrite the state as

$$\begin{cases} \mathrm{d}X_t = (F_{1,t}X_t + F_{2,t}\mathbb{E}X_t + F_{3,t}v_t + F_{4,t}\mathbb{E}v_t + F_t)\mathrm{d}t + C_t\mathrm{d}W_t + \bar{C}_t\mathrm{d}Y_t, \\ -\mathrm{d}y_t = (g_{1,t}x_t + g_{2,t}\mathbb{E}x_t + g_{3,t}y_t + g_{4,t}\mathbb{E}y_t + g_{5,t}z_t + g_{6,t}\mathbb{E}\bar{z}_t + g_{7,t}v_t + g_{8,t}\mathbb{E}v_t)\mathrm{d}t \\ \qquad\qquad - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\ X(0) = X_0, \quad y(T) = \phi_1 x_T + \phi_2\mathbb{E}x_T, \end{cases}$$

where $X_t := \begin{pmatrix} \rho_t \\ x_t \end{pmatrix}, F_{i,t} := \begin{pmatrix} 0 & 0 \\ 0 & f_{i,t} \end{pmatrix}, F_t := \begin{pmatrix} 0 \\ -\bar{c}_t h_t \end{pmatrix}, C_t := \begin{pmatrix} 0 \\ c_t \end{pmatrix}, \bar{C}_t := \begin{pmatrix} \rho_t h_t \\ \bar{c}_t \end{pmatrix}, X_0 := \begin{pmatrix} 1 \\ x_0 \end{pmatrix}$. In this setting, the Hamiltonian function is of the form

$$\begin{aligned} H(t, X, y, z, \bar{z}, v, p, q, k, \bar{k}) = & \langle F_{1,t}X + F_{2,t}\mathbb{E}X + F_{3,t}v + F_{4,t}\mathbb{E}v + F_t, p\rangle + \langle C_t, k\rangle + \langle \bar{C}_t, \bar{k}\rangle \\ & - q(g_{1,t}x + g_{2,t}\mathbb{E}x + g_{3,t}y + g_{4,t}\mathbb{E}y + g_{5,t}z + g_{6,t}\mathbb{E}z + g_{7,t}v + g_{8,t}\mathbb{E}v) \\ & + \rho\left[L_{1,t}x^2 + L_{2,t}(\mathbb{E}x)^2 + L_{3,t}y^2 + L_{4,t}(\mathbb{E}y)^2 + L_{5,t}v^2 + L_{6,t}(\mathbb{E}v)^2\right]. \end{aligned}$$

The adjoint equations will be given by

$$\begin{cases} -\mathrm{d}p_{1,t} = h_t\bar{k}_{1,t}\mathrm{d}t - k_{1,t}\mathrm{d}W_t - \bar{k}_{1,t}\mathrm{d}Y_t, \\ \quad p_{1,T} = 0, \end{cases}$$

$$\begin{cases} -\mathrm{d}p_{2,t} = \Big\{f_{1,t}p_{2,t} + f_{2,t}\mathbb{E}[p_{2,t}] - g_{1,t}q_t - g_{2,t}\mathbb{E}[q_t] + 2L_{1,t}x_t + 2L_{2,t}\mathbb{E}[x_t]\Big\}\mathrm{d}t \\ \qquad\qquad - k_{2,t}\mathrm{d}W_t - \bar{k}_{2,t}\mathrm{d}Y_t, \\ \quad p_{2,T} = 2M_1x_T + 2M_2\mathbb{E}[x_T] - \phi_1 q_T - \phi_2\mathbb{E}[q_T], \end{cases}$$

and

$$\begin{cases} \mathrm{d}q_t = \Big[g_{3,t}q_t + g_{4,t}\mathbb{E}[q_t] - 2L_{3,t}y_t - 2L_{4,t}\mathbb{E}[y_t]\Big]\mathrm{d}t + g_{5,t}q_t\mathrm{d}W_t + g_{6,t}\mathbb{E}[q_t]\mathrm{d}Y_t, \\ q_0 = -2\gamma y_0. \end{cases}$$

According to Theorem 3.9, if $U = \mathbb{R}$, the necessary condition for optimality (3.14) will be

$$\mathbb{E}\Big[f_{3,t}p_{2,t} + f_{4,t}\mathbb{E}[p_{2,t}] - g_{7,t}q_t - g_{8,t}\mathbb{E}[q_t] + 2L_{5,t}u_t + 2L_{6,t}\mathbb{E}[u_t]\Big|\mathcal{F}_t^Y\Big] = 0, \text{ a.s. a.e.}$$

Taking expectations, we obtain

$$\mathbb{E}[u_t] = -\frac{1}{2(L_{5,t} + L_{6,t})}\big\{(f_{3,t} + f_{4,t})\mathbb{E}[p_{2,t}] - (g_{7,t} + g_{8,t})\mathbb{E}[q_t]\big\},$$

and the optimal control should satisfy

$$
\begin{aligned}
u_t =& -\frac{1}{2L_{5,t}}\Big\{f_{3,t}\mathbb{E}\big[p_{2,t}\big|\mathcal{F}_t^Y\big] - g_{7,t}\mathbb{E}\big[q_t\big|\mathcal{F}_t^Y\big]\Big\} \\
&- \frac{1}{2L_{5,t}}\Big\{\frac{L_{5,t}f_{4,t}-L_{6,t}f_{3,t}}{L_{5,t}+L_{6,t}}\mathbb{E}[p_{2,t}] - \frac{L_{5,t}g_{8,t}-L_{6,t}g_{7,t}}{L_{5,t}+L_{6,t}}\mathbb{E}[q_t]\Big\}.
\end{aligned}
\tag{5.6}
$$

Finally, one can also check assumption (H.3) and (H.4) hold, then the verification theorem yields that the control $u(\cdot)$ given by (5.6) is indeed an optimal control.

Furthermore, by inserting (5.6) to the following Hamiltonian system

$$
\begin{cases}
\quad \mathrm{d}x_t =(f_{1,t}x_t + f_{2,t}\mathbb{E}x_t + f_{3,t}u_t + f_{4,t}\mathbb{E}u_t - \bar{c}_t h_t)\mathrm{d}t + c_t\mathrm{d}W_t + \bar{c}_t\mathrm{d}Y_t, \\
-\mathrm{d}y_t =(g_{1,t}x_t + g_{2,t}\mathbb{E}x_t + g_{3,t}y_t + g_{4,t}\mathbb{E}y_t + g_{5,t}z_t + g_{6,t}\mathbb{E}z_t + g_{7,t}u_t + g_{8,t}\mathbb{E}u_t)\mathrm{d}t \\
\qquad\qquad - z_t\mathrm{d}W_t - \bar{z}_t\mathrm{d}Y_t, \\
-\mathrm{d}p_{2,t} =\Big\{f_{1,t}p_{2,t} + f_{2,t}\mathbb{E}[p_{2,t}] - g_{1,t}q_t - g_{2,t}\mathbb{E}[q_t] + 2L_{1,t}x_t + 2L_{2,t}\mathbb{E}[x_t]\Big\}\mathrm{d}t \\
\qquad\qquad - k_{2,t}\mathrm{d}W_t - \bar{k}_{2,t}\mathrm{d}Y_t, \\
\quad \mathrm{d}q_t =\Big[g_{3,t}q_t + g_{4,t}\mathbb{E}[q_t] - 2L_{3,t}y_t - 2L_{4,t}\mathbb{E}[y_t]\Big]\mathrm{d}t + g_{5,t}q_t\mathrm{d}W_t + g_{6,t}\mathbb{E}[q_t]\mathrm{d}Y_t, \\
\quad x(0) =x_0, \quad y(T) = \phi_1 x_T + \phi_2\mathbb{E}x_T, \\
\quad q_0 =-2\gamma y_0, \quad p_{2,T} = 2M_1x_T + 2M_2\mathbb{E}[x_T] - \phi_1 q_T - \phi_2\mathbb{E}[q_T].
\end{cases}
\tag{5.7}
$$

one will obtain a fully coupled mean-field FBSDE, once we obtain $p_{2,t}$ and q_t, we can then get the optimal control through (5.6).

Moreover, one can also try to write the optimal control $u(\cdot)$ given in (5.6) as the feedback of the filtered state $\mathbb{E}[x_t|\mathcal{F}_t^Y], \mathbb{E}[y_t|\mathcal{F}_t^Y], \mathbb{E}[z_t|\mathcal{F}_t^Y], \mathbb{E}[\bar{z}_t|\mathcal{F}_t^Y]$ and the expectation of the state $\mathbb{E}[x_t], \mathbb{E}[y_t], \mathbb{E}[z_t], \mathbb{E}[\bar{z}_t]$ through Riccati equations. Due to the complicated coupled structure of (5.7), usually it is not easy to find such feedback optimal control, the interesting readers are referred to *e.g.* [28, 30, 49, 50] for some special cases.

Remark 5.1. If the cost functional is given by

$$
\begin{aligned}
J(v(\cdot)) =&\mathbb{E}^v\Big[\int_0^T \left[l_{1,t}x_t^2 + l_{2,t}(\mathbb{E}x_t)^2 + l_{3,t}y_t^2 + l_{4,t}(\mathbb{E}y_t)^2 + l_{5,t}v_t^2 + l_{6,t}(\mathbb{E}v_t)^2\right]\mathrm{d}t \\
&+ m_1x_T^2 + m_2(\mathbb{E}x_T)^2 + \gamma y_0\Big]
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
J(v(\cdot)) =&\mathbb{E}\Big[\int_0^T \rho_t\left[l_{1,t}x_t^2 + l_{2,t}(\mathbb{E}x_t)^2 + l_{3,t}y_t^2 + l_{4,t}(\mathbb{E}y_t)^2 + l_{5,t}v_t^2 + l_{6,t}(\mathbb{E}v_t)^2\right]\mathrm{d}t \\
&+ \rho_T(m_1x_T^2 + m_2(\mathbb{E}x_T)^2) + \gamma y_0^2\Big],
\end{aligned}
$$

one can also apply Theorem 3.9 to obtain that the optimal control should satisfy

$$
\begin{aligned}
u_t =& -\frac{1}{2\rho_t l_{5,t}}\Big\{f_{3,t}\mathbb{E}\big[p_{2,t}\big|\mathcal{F}_t^Y\big] - g_{7,t}\mathbb{E}\big[q_t\big|\mathcal{F}_t^Y\big]\Big\} \\
&- \frac{1}{2\rho_t l_{5,t}}\Big\{\frac{l_{5,t}f_{4,t}-l_{6,t}f_{3,t}}{l_{5,t}+l_{6,t}}\mathbb{E}[p_{2,t}] - \frac{l_{5,t}g_{8,t}-l_{6,t}g_{7,t}}{l_{5,t}+l_{6,t}}\mathbb{E}[q_t]\Big\},
\end{aligned}
\tag{5.8}
$$

where the adjoint equations will be

$$\begin{cases} -\mathrm{d}p_{2,t} = \Big\{ f_{1,t}p_{2,t} + f_{2,t}\mathbb{E}[p_{2,t}] - g_{1,t}q_t - g_{2,t}\mathbb{E}[q_t] + 2\rho_t l_{1,t}x_t + 2\rho_t l_{2,t}\mathbb{E}[x_t] \Big\}\mathrm{d}t \\ \qquad\qquad - k_{2,t}\mathrm{d}W_t - \bar{k}_{2,t}\mathrm{d}Y_t, \\ p_{2,T} = 2m_1\rho_T x_T + 2m_2\rho_T\mathbb{E}[x_T] - \phi_1 q_T - \phi_2\mathbb{E}[q_T]. \end{cases}$$

$$\begin{cases} \mathrm{d}q_t = \Big[g_{3,t}q_t + g_{4,t}\mathbb{E}[q_t] - 2\rho_t l_{3,t}y_t - 2\rho_t l_{4,t}\mathbb{E}[y_t] \Big]\mathrm{d}t + g_{5,t}q_t\mathrm{d}W_t + g_{6,t}\mathbb{E}[q_t]\mathrm{d}Y_t, \\ q_0 = -2\gamma y_0. \end{cases}$$

However, in this case, assumptions (H.3)-(H.4) fail, then we can not apply verification theorem to affirm that the control given by (5.8) is indeed an optimal control.

Appendix A.

A.1 Proof of Lemma 3.2

Step 1. Let us first show that $\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}|x_t^{\varepsilon,1}|^2 = 0$. We set

$$(\theta_t^{'})^{\varepsilon} = (x_t^{\varepsilon}, u_t^{\varepsilon}, \mathcal{L}(x_t^{\varepsilon}, u_t^{\varepsilon})), \quad (\theta_t^{'})^{\lambda,\varepsilon} = (x_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon}, \mathcal{L}(x_t^{\lambda,\varepsilon}, u_t^{\lambda,\varepsilon})),$$

where

$$x_t^{\lambda,\varepsilon} := x_t + \lambda(x_t^{\varepsilon} - x_t) = x_t + \lambda\varepsilon(x_t^1 + x_t^{\varepsilon,1}), \quad u_t^{\lambda,\varepsilon} := u_t + \lambda(u_t^{\varepsilon} - u_t) = u_t + \lambda\varepsilon v_t. \tag{A.1}$$

Then from (3.4) and the forward equation in (2.7), we have

$$\mathrm{d}x_t^{\varepsilon,1} = (v_t^f - v_t^{\bar{\sigma}}h(t,\theta_t^{'}) - v_t^h\bar{\sigma}(t,\theta_t^{'}) - v_t^{\bar{\sigma},h})\mathrm{d}t + v_t^{\sigma}\mathrm{d}W_t + v_t^{\bar{\sigma}}\mathrm{d}Y_t, \quad x_0^{\varepsilon,1} = 0, \tag{A.2}$$

where for $\varphi = f, h, \sigma, \bar{\sigma}$,

$$\begin{aligned} v_t^{\varphi} :=& \frac{\varphi(t,(\theta_t^{'})^{\varepsilon}) - \varphi(t,\theta_t^{'})}{\varepsilon} - \partial_x\varphi(t,\theta_t^{'})x_t^1 - \partial_v\varphi(t,\theta_t^{'})v_t \\ & \qquad - \tilde{\mathbb{E}}[\partial_{\mu_1}\varphi(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] - \tilde{\mathbb{E}}[\partial_{\mu_5}\varphi(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t], \\ v_t^{\bar{\sigma},h} :=& \frac{\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon}) - \bar{\sigma}(t,\theta_t^{'})}{\varepsilon}\Delta h^{\varepsilon}(t) := \frac{\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon}) - \bar{\sigma}(t,\theta_t^{'})}{\varepsilon}\left[h(t,(\theta_t^{'})^{\varepsilon}) - h(t,\theta_t^{'})\right]. \end{aligned} \tag{A.3}$$

Let us compute the term v_t^{φ}, for $\varphi = f, h, \sigma, \bar{\sigma}$. Recalling the notations in (3.5) and (A.1), and for $\varphi = f, h, \sigma, \bar{\sigma}$ and $\psi = v, x$, $i = 1, 5$, denoting

$$\begin{aligned} \Delta\varphi_{\psi}^{\lambda,\varepsilon}(t) :=& \partial_{\psi}\varphi(t,(\theta_t^{'})^{\lambda,\varepsilon}) - \partial_{\psi}\varphi(t,\theta_t^{'}) \\ \Delta\varphi_{\mu_i}^{\lambda,\varepsilon}(t) :=& \partial_{\mu_i}\varphi(t,(\theta_t^{'})^{\lambda,\varepsilon})(\widetilde{x_t^{\lambda,\varepsilon}}, \widetilde{u_t^{\lambda,\varepsilon}}) - \partial_{\mu_i}\varphi(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t). \end{aligned} \tag{A.4}$$

we obtain from the expansion (3.6) that

$$\begin{aligned} v_t^{\varphi} =& \frac{\varphi(t,(\theta_t^{'})^{\varepsilon})-\varphi(t,\theta_t^{'})}{\varepsilon} - \partial_x\varphi(t,\theta_t^{'})x_t^1 - \partial_v\varphi(t,\theta_t^{'})v_t \\ & \qquad - \tilde{\mathbb{E}}[\partial_{\mu_1}\varphi(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] - \tilde{\mathbb{E}}[\partial_{\mu_5}\varphi(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t] \\ =& \int_0^1 \partial_x\varphi(t,(\theta_t^{'})^{\lambda,\varepsilon})x_t^{\varepsilon,1}\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\partial_{\mu_1}\varphi(t,(\theta_t^{'})^{\lambda,\varepsilon})(\widetilde{x_t^{\lambda,\varepsilon}},\widetilde{u_t^{\lambda,\varepsilon}})\widetilde{x_t^{\varepsilon,1}}\right]\mathrm{d}\lambda \\ & + \int_0^1 \Delta\varphi_x^{\lambda,\varepsilon}(t)x_t^1\mathrm{d}\lambda + \int_0^1 \Delta\varphi_v^{\lambda,\varepsilon}(t)v_t\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\Delta\varphi_{\mu_1}^{\lambda,\varepsilon}(t)\tilde{x}_t^1\right]\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\Delta\varphi_{\mu_5}^{\lambda,\varepsilon}(t)\tilde{v}_t\right]\mathrm{d}\lambda. \end{aligned} \tag{A.5}$$

Thus, by using the inequality $\tilde{\mathbb{E}}[\tilde{X}\tilde{Y}] \le (\tilde{\mathbb{E}}|\tilde{X}|^2)^{\frac{1}{2}}(\tilde{\mathbb{E}}|\bar{Y}|^2)^{\frac{1}{2}}$ and uniformly boundedness of $\partial_x\varphi, \partial_v\varphi$ and $\int_{\mathbb{R}^{n+k}}|\partial_{\mu_i}\varphi(x,v,\xi)(x',v')|^2\mathrm{d}\xi(x',v')$, $i=1,5$, (see assumptions (H.1)), we get

$$\begin{aligned} |v_t^{\varphi}| \le& C|x_t^{\varepsilon,1}| + C(\mathbb{E}|x_t^{\varepsilon,1}|^2)^{\frac{1}{2}} + \left(\int_0^1 (|\Delta\varphi_x^{\lambda,\varepsilon}(t)| + |\Delta\varphi_v^{\lambda,\varepsilon}(t)|)\mathrm{d}\lambda\right)(|x_t^1|+|v_t|) \\ & + \left(\int_0^1 \left[\left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_1}^{\lambda,\varepsilon}(t)|^2\right)^{\frac{1}{2}} + \left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_5}^{\lambda,\varepsilon}(t)|^2\right)^{\frac{1}{2}}\right]\mathrm{d}\lambda\right)\left[(\mathbb{E}|x_t^1|^2)^{\frac{1}{2}} + (\mathbb{E}|v_t|^2)^{\frac{1}{2}}\right]. \end{aligned} \tag{A.6}$$

Next, let us analysis $v_t^{\bar{\sigma},h} := \frac{\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon})-\bar{\sigma}(t,\theta_t^{'})}{\varepsilon}\Delta h^{\varepsilon}(t)$. From the expansion (3.6), we have

$$\begin{aligned} v_t^{\bar{\sigma},h} =& \Delta h^{\varepsilon}(t)\left(\int_0^1 \partial_x\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon})\cdot(x_t^1+x_t^{\varepsilon,1})\mathrm{d}\lambda + \int_0^1 \partial_v\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon})\cdot v_t\mathrm{d}\lambda\right) \\ & + \Delta h^{\varepsilon}(t)\int_0^1 \tilde{\mathbb{E}}[\partial_{\mu_1}\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon})(\widetilde{x_t^{\lambda,\varepsilon}},\widetilde{u_t^{\lambda,\varepsilon}})\cdot(\tilde{x}_t^1+\widetilde{x_t^{\varepsilon,1}})]\mathrm{d}\lambda \\ & + \Delta h^{\varepsilon}(t)\int_0^1 \tilde{\mathbb{E}}[\partial_{\mu_5}\bar{\sigma}(t,(\theta_t^{'})^{\varepsilon})(\widetilde{x_t^{\lambda,\varepsilon}},\widetilde{u_t^{\lambda,\varepsilon}})\cdot\tilde{v}_t]\mathrm{d}\lambda, \end{aligned}$$

and similarly, we also have

$$|v_t^{\bar{\sigma},h}| \le C|\Delta h^{\varepsilon}(t)|(|v_t|+|x_t^1|+|x_t^{\varepsilon,1}| + ((\mathbb{E}|v_t|^2)^{\frac{1}{2}} + (\mathbb{E}|x_t^{\varepsilon,1}|^2)^{\frac{1}{2}} + (\mathbb{E}|x_t^1|^2)^{\frac{1}{2}}). \tag{A.7}$$

From (A.2), by using Burkholder-Davis-Gundy (BDG) inequality, we obtain for any $S\in[0,T]$,

$$\mathbb{E}\sup_{0\le t\le S}|x_t^{\varepsilon,1}|^2 \le C\mathbb{E}\int_0^S \left(|v_t^f|^2 + |v_t^{\bar{\sigma}}h(t,\theta_t^{'})|^2 + |v_t^h\bar{\sigma}(t,\theta_t^{'})|^2 + |v_t^{\bar{\sigma},h}|^2 + |v_t^{\sigma}|^2 + |v_t^{\bar{\sigma}}|^2\right)\mathrm{d}t,$$

and with the help of (A.6), (A.7) and the boundedness of $\bar{\sigma}$ and h, we have

$$\mathbb{E}\sup_{0\le t\le S}|x_t^{\varepsilon,1}|^2 \le C\mathbb{E}\int_0^T |\Delta^{\varepsilon}(t)|^2\mathrm{d}t + C\mathbb{E}\int_0^S |x_t^{\varepsilon,1}|^2\mathrm{d}t, \tag{A.8}$$

where

$$
\begin{aligned}
\Delta^\varepsilon(t) :=& \left(\int_0^1 \sum_{\varphi=f,\sigma,\bar\sigma,h} (|\Delta\varphi_x^{\lambda,\varepsilon}(t)| + |\Delta\varphi_v^{\lambda,\varepsilon}(t)|)\mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right)(|x_t^1| + |v_t|) \\
&+ \left(\int_0^1 \sum_{\varphi=f,\sigma,\bar\sigma,h} \left[\left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_1}^{\lambda,\varepsilon}(t)|^2\right)^{\frac12} + \left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_5}^{\lambda,\varepsilon}(t)|^2\right)^{\frac12}\right]\mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right) \\
&\qquad\qquad\qquad\qquad\cdot\left[(\mathbb{E}|x_t^1|^2)^{\frac12} + (\mathbb{E}|v_t|^2)^{\frac12}\right].
\end{aligned}
$$

Thus the Gronwall's inequality yields that

$$
\mathbb{E}\sup_{0\le t\le T}|x_t^{\varepsilon,1}|^2 \le C\mathbb{E}\int_0^T |\Delta^\varepsilon(t)|^2\mathrm{d}t. \tag{A.9}
$$

Now let us focus on Δ^ε. On the one hand, the uniformly bounded assumptions of $h, \partial_x\varphi, \partial_v\varphi$ and $\int_{\mathbb{R}^{n+k}} |\partial_{\mu_i}\varphi(x,v,\xi)(x',v')|^2\mathrm{d}\xi(x',v')$, $i=1,5$, yield that

$$
\begin{aligned}
&\left(\int_0^1 \sum_{\varphi=f,\sigma,\bar\sigma,h} (|\Delta\varphi_x^{\lambda,\varepsilon}(t)| + |\Delta\varphi_v^{\lambda,\varepsilon}(t)|)\mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right) \le C, \\
&\left(\int_0^1 \sum_{\varphi=f,\sigma,\bar\sigma,h} \left[\left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_1}^{\lambda,\varepsilon}(t)|^2\right)^{\frac12} + \left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_5}^{\lambda,\varepsilon}(t)|^2\right)^{\frac12}\right]\mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right) \le C.
\end{aligned} \tag{A.10}
$$

Thus from $\sup\limits_{0\le t\le T}\mathbb{E}(|x_t^1|^2 + |v_t|^2) < \infty$ (see Rem. 3.1), and (A.9) and (A.10), we obtain that, there exists a constant C independent of ε such that

$$
\mathbb{E}\sup_{0\le t\le T}|x_t^{\varepsilon,1}|^2 \le C. \tag{A.11}
$$

On the other hand, by using (A.11) and $\sup\limits_{0\le t\le T}\mathbb{E}|x_t^1|^2 < \infty$, we have

$$
\lim_{\varepsilon\to0}\mathbb{E}|x_t^{\lambda,\varepsilon} - x_t|^2 = \lambda^2\lim_{\varepsilon\to0}\mathbb{E}|x_t^\varepsilon - x_t|^2 \le 2\lim_{\varepsilon\to0}\lambda^2\varepsilon^2\mathbb{E}[|x_t^{\varepsilon,1}|^2 + |x_t^1|^2] = 0,
$$

By using that

$$
\lim_{\varepsilon\to0}\mathbb{E}|u_t^{\lambda,\varepsilon} - u_t|^2 = \lambda^2\lim_{\varepsilon\to0}\mathbb{E}|u_t^\varepsilon - u_t|^2 \le \lim_{\varepsilon\to0}\lambda^2\varepsilon^2\mathbb{E}|v_t|^2 = 0,
$$

and the continuity assumptions of $h, \partial_x\varphi, \partial_v\varphi$, as well as the continuity of the mappings, $i=1,5$,

$$
\begin{aligned}
\mathbb{R}^{n+k}\times L^2(\Omega;\mathbb{R}^{n+k}) \ni (x,v,(X,\beta)) \mapsto& \partial_{\mu_i}(f,\sigma,\bar\sigma,h)(t,x,v,\mathcal{L}(X,\beta))(X,\beta) \\
&\in L^2(\Omega;\mathbb{R}^{n\times n}\times\mathbb{R}^{l\times m\times n}\times\mathbb{R}^{l\times d\times n}\times\mathbb{R}^{d\times n}),
\end{aligned}
$$

we have that for each $t \in [0,T]$,

$$\begin{aligned}
&\left(\int_0^1 \sum_{\varphi=f,\sigma,\bar{\sigma},h} (|\Delta\varphi_x^{\lambda,\varepsilon}(t)| + |\Delta\varphi_v^{\lambda,\varepsilon}(t)|)\mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right) \xrightarrow{\mathbb{P}} 0, \text{ as } \varepsilon \to 0,\\
&\left(\int_0^1 \sum_{\varphi=f,\sigma,\bar{\sigma},h} \left[\left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_1}^{\lambda,\varepsilon}(t)|^2\right)^{\frac{1}{2}} + \left(\tilde{\mathbb{E}}|\Delta\varphi_{\mu_5}^{\lambda,\varepsilon}(t)|^2\right)^{\frac{1}{2}}\right] \mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right) \xrightarrow{\mathbb{P}} 0, \text{ as } \varepsilon \to 0.
\end{aligned} \tag{A.12}$$

Then by using (A.10), and (A.12) and $\sup\limits_{0\le t\le T} \mathbb{E}(|x_t^1|^2 + |v_t|^2) < \infty$, from dominated convergence theorem, we obtain $\lim\limits_{\varepsilon\to 0} \mathbb{E}\int_0^T |\Delta^\varepsilon(t)|^2 \mathrm{d}t = 0$. Consequently, noting (A.9), we get

$$\lim_{\varepsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |x_t^{\varepsilon,1}|^2 = 0. \tag{A.13}$$

Step 2. We can further show that for $2 \le p \le 4$,

$$\mathbb{E} \sup_{0\le t\le T} |x_t^{\varepsilon,1}|^p \le C, \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |x_t^{\varepsilon,1}|^p = 0. \tag{A.14}$$

Indeed, from (A.2), by using Burkholder-Davis-Gundy (BDG) inequality, we obtain for any $S \in [0,T]$,

$$\mathbb{E} \sup_{0\le t\le S} |x_t^{\varepsilon,1}|^p \le C\mathbb{E}\left[\int_0^S \left(|v_t^f|^2 + |v_t^{\bar{\sigma}} h(t,\theta_t')|^2 + |v_t^h \bar{\sigma}(t,\theta_t')|^2 + |v_t^{\bar{\sigma},h}|^2 + |v_t^\sigma|^2 + |v_t^{\bar{\sigma}}|^2\right) \mathrm{d}t\right]^{p/2},$$

and with the help of (A.6), (A.7) and the boundedness of $\bar{\sigma}$ and h, we have

$$\begin{aligned}
\mathbb{E} \sup_{0\le t\le S} |x_t^{\varepsilon,1}|^p \le& C\mathbb{E}\left(\int_0^T |\Delta^\varepsilon(t)|^2 \mathrm{d}t\right)^{p/2} + C\mathbb{E}\left(\int_0^S |x_t^{\varepsilon,1}|^2 \mathrm{d}t\right)^{p/2}\\
\le& C\mathbb{E}\left(\int_0^T |\Delta^\varepsilon(t)|^2 \mathrm{d}t\right)^{p/2} + C\mathbb{E}\int_0^S |x_t^{\varepsilon,1}|^p \mathrm{d}t,
\end{aligned}$$

and Gronwall's inequality yields that

$$\mathbb{E} \sup_{0\le t\le T} |x_t^{\varepsilon,1}|^p \le C\mathbb{E}\left(\int_0^T |\Delta^\varepsilon(t)|^2 \mathrm{d}t\right)^{p/2}. \tag{A.15}$$

Then on the one hand, from (A.10), (A.15) and $\sup\limits_{0\le t\le T} \mathbb{E}(|x_t^1|^p + |v_t|^p) < \infty$, for $2 \le p \le 4$, we have $\mathbb{E} \sup\limits_{0\le t\le T} |x_t^{\varepsilon,1}|^p \le C$. On the other hand, by using (A.10), (A.12) and $\sup\limits_{0\le t\le T} \mathbb{E}(|x_t^1|^p + |v_t|^p) < \infty$, for $2 \le p \le 4$, from dominated convergence theorem, we obtain $\lim\limits_{\varepsilon\to 0} \mathbb{E}\left(\int_0^T |\Delta^\varepsilon(t)|^2 \mathrm{d}t\right)^{p/2} = 0$. Consequently, noting (A.15), we get

$$\lim_{\varepsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |x_t^{\varepsilon,1}|^p = 0. \tag{A.16}$$

Step 3. Let us study $\rho_t^{\varepsilon,1}$. From (3.3) and the forward equation in (2.7) (or (2.3)), we have

$$\mathrm{d}\rho_t^{\varepsilon,1} = \alpha_t^{\rho,h}\mathrm{d}Y_t.$$

Here, we denote

$$\begin{aligned}\alpha_t^{\rho,h} :=& \frac{\rho_t^\varepsilon h(t,(\theta_t^{'})^\varepsilon) - \rho_t h(t,\theta_t^{'})}{\varepsilon} - \Big(\rho_t^1 h(t,\theta_t^{'}) + \rho_t \partial_x h(t,\theta_t^{'})x_t^1 + \rho_t \partial_v h(t,\theta_t^{'})v_t \\ &+ \rho_t \tilde{\mathbb{E}}[\partial_{\mu_1} h(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t)\tilde{x}_t^1] + \rho_t \tilde{\mathbb{E}}[\partial_{\mu_5} h(t,\theta_t^{'})(\tilde{x}_t,\tilde{u}_t)\tilde{v}_t]\Big) \\ =& \frac{(\rho_t^\varepsilon - \rho_t)(h(t,(\theta_t^{'})^\varepsilon) - h(t,\theta_t^{'}))}{\varepsilon} + \rho_t^{\varepsilon,1} h(t,\theta_t^{'}) + \rho_t v_t^h \\ =& (\rho_t^{\varepsilon,1} + \rho_t^1)\Delta h^\varepsilon(t) + \rho_t^{\varepsilon,1} h(t,\theta_t^{'}) + \rho_t v_t^h,\end{aligned}$$

where we recall $\Delta h^\varepsilon(t) := h(t,(\theta_t^{'})^\varepsilon) - h(t,\theta_t^{'})$ and v_t^h are given in (A.3) and (A.5). Then by using (A.6) and similar to the proof in Step 1, one can show that

$$\mathbb{E}\sup_{0\le t\le S}|\rho_t^{\varepsilon,1}|^2 \le C\mathbb{E}\int_0^S |\alpha_t^{\rho,h}|^2\mathrm{d}t \le C\mathbb{E}\int_0^T |\Delta_\rho^\varepsilon(t)|^2\mathrm{d}t + C\int_0^S \mathbb{E}|\rho_t^{\varepsilon,1}|^2\mathrm{d}t, \tag{A.17}$$

where (recalling (A.4))

$$\begin{aligned}\Delta_\rho^\varepsilon(t) :=& |\Delta h^\varepsilon(t)||\rho_t^1| + \rho_t\left(|x_t^{\varepsilon,1}| + (\mathbb{E}|x_t^{\varepsilon,1}|^2)^{\frac12}\right) \\ &+ \rho_t\left(\int_0^1 (|\Delta h_x^{\lambda,\varepsilon}(t)| + |\Delta h_v^{\lambda,\varepsilon}(t)|)\mathrm{d}\lambda + |\Delta h^\varepsilon(t)|\right)(|x_t^1| + |v_t|) \\ &+ \rho_t\left(\int_0^1\left[\left(\tilde{\mathbb{E}}|\Delta h_{\mu_1}^{\lambda,\varepsilon}(t)|^2\right)^{\frac12} + \left(\tilde{\mathbb{E}}|\Delta h_{\mu_5}^{\lambda,\varepsilon}(t)|^2\right)^{\frac12}\right]\mathrm{d}\lambda\right)\cdot\left[(\mathbb{E}|x_t^1|^2)^{\frac12} + (\mathbb{E}|v_t|^2)^{\frac12}\right].\end{aligned}$$

Thus from (A.17) and Gronwall's inequality, it follows that

$$\mathbb{E}\sup_{0\le t\le T}|\rho_t^{\varepsilon,1}|^2 \le C\mathbb{E}\int_0^T |\Delta_\rho^\varepsilon(t)|^2\mathrm{d}t.$$

On the one hand, from the boundedness assumption of $\partial_x h, \partial_v h, \partial_{\mu_1} h, \partial_{\mu_5} h$, and the following integrability condition (see Step 2, Rems. 2.2 and 3.1),

$$\begin{aligned}&\sup_{0\le t\le T}\mathbb{E}(|x_t^1|^p + |v_t|^p + |x_t^{\varepsilon,1}|^p) < \infty,\ 2\le p\le 4, \\ &\mathbb{E}[\sup_{t\in[0,T]}(|\rho_t|^p + |\rho_t|^{-p}] < +\infty,\ p\ge 1,\ \text{and}\ \mathbb{E}[\sup_{t\in[0,T]}|\rho_t^1|^p] < +\infty,\ 2\le p<4,\end{aligned} \tag{A.18}$$

we can show that $\mathbb{E}\int_0^T |\Delta_\rho^\varepsilon(t)|^2\mathrm{d}t \le C$ and thus

$$\mathbb{E}\sup_{0\le t\le T}|\rho_t^{\varepsilon,1}|^2 \le C. \tag{A.19}$$

On the other hand, similar to Step 1, by using (A.16) and dominated convergence theorem, we can prove that $\lim_{\varepsilon\to 0}\mathbb{E}\int_0^T|\Delta_\rho^\varepsilon(t)|^2\mathrm{d}t=0$ and thus $\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}|\rho_t^{\varepsilon,1}|^2=0$.

Finally, we also have

$$\mathbb{E}\sup_{0\le t\le S}|\rho_t^{\varepsilon,1}|^p\le C\mathbb{E}\left(\int_0^S|\alpha_t^{\rho,h}|^2\mathrm{d}t\right)^{p/2}\le C\mathbb{E}\left(\int_0^T|\Delta_\rho^\varepsilon(t)|^2\mathrm{d}t\right)^{p/2}+C\int_0^S\mathbb{E}|\rho_t^{\varepsilon,1}|^p\mathrm{d}t,$$

and Gronwall's inequality yields that

$$\mathbb{E}\sup_{0\le t\le T}|\rho_t^{\varepsilon,1}|^p\le C\mathbb{E}\left(\int_0^T|\Delta_\rho^\varepsilon(t)|^2\mathrm{d}t\right)^{p/2},$$

and from the integrability condition (A.18), similar to previous Steps, one can show that for any $2\le p<4$ (note that p is strictly less than 4), we have

$$\mathbb{E}\sup_{0\le t\le T}|\rho_t^{\varepsilon,1}|^p\le C, \tag{A.20}$$

and moreover $\lim_{\varepsilon\to 0}\mathbb{E}\sup_{0\le t\le T}|\rho_t^{\varepsilon,1}|^p=0$. We complete the proof.

A.2 Proof of Lemma 3.3

We use the same notations in the proof of Lemma 3.2. From the backward equations in (3.1) and (2.7), we have

$$-\mathrm{d}y_t^{\varepsilon,1}=v_t^G\mathrm{d}t-z_t^{\varepsilon,1}\mathrm{d}W_t-\bar{z}_t^{\varepsilon,1}\mathrm{d}Y_t,\qquad y_T^{\varepsilon,1}=v_T^\Phi, \tag{A.21}$$

where

$$\begin{aligned}v_t^G:=&\frac{G(t,(\Theta_t)^\varepsilon)-G(t,\Theta_t)}{\varepsilon}-\partial_xG(t,\Theta_t)x_t^1-\partial_yG(t,\Theta_t)y_t^1-\partial_zG(t,\Theta_t)z_t^1-\partial_{\bar z}G(t,\Theta_t)\bar{z}_t^1\\&-\partial_vG(t,\Theta_t)v_t-\tilde{\mathbb{E}}[\partial_{\mu_1}G(t,\Theta_t)(\tilde\alpha_t)\tilde{x}_t^1]-\tilde{\mathbb{E}}[\partial_{\mu_2}G(t,\Theta_t)(\tilde\alpha_t)\tilde{y}_t^1]\\&-\tilde{\mathbb{E}}[\partial_{\mu_3}G(t,\Theta_t)(\tilde\alpha_t)\tilde{z}_t^1]-\tilde{\mathbb{E}}[\partial_{\mu_4}G(t,\Theta_t)(\tilde\alpha_t)\tilde{\bar{z}}_t^1]-\tilde{\mathbb{E}}[\partial_{\mu_5}G(t,\Theta_t)(\tilde\alpha_t)\tilde{v}_t]\end{aligned}$$

and

$$v_T^\Phi:=\frac{\Phi(x_T^\varepsilon,\mathcal{L}(x_T^\varepsilon))-\Phi(x_T,\mathcal{L}(x_T))}{\varepsilon}-\partial_x\Phi(x_T,\mathcal{L}(x_T))x_T^1-\tilde{\mathbb{E}}[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\tilde{x}_T)\tilde{x}_T^1].$$

Let us first calculate v_t^G. Recall the notations at the beginning of subsection 3.1 and by using similar expansion as (3.6), we have

$$\begin{aligned}
v_t^G =& \int_0^1 \partial_x g(t,(\Theta_t^{'})^{\lambda,\varepsilon})x_t^{\varepsilon,1}\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\partial_{\mu_1} g(t,(\Theta_t^{'})^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})\widetilde{x_t^{\varepsilon,1}}\right]\mathrm{d}\lambda\\
&+ \int_0^1 \partial_y g(t,(\Theta_t^{'})^{\lambda,\varepsilon})y_t^{\varepsilon,1}\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\partial_{\mu_2} g(t,(\Theta_t^{'})^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})\widetilde{y_t^{\varepsilon,1}}\right]\mathrm{d}\lambda\\
&+ \int_0^1 \partial_z g(t,(\Theta_t^{'})^{\lambda,\varepsilon})z_t^{\varepsilon,1}\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\partial_{\mu_3} g(t,(\Theta_t^{'})^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})\widetilde{z_t^{\varepsilon,1}}\right]\mathrm{d}\lambda\\
&+ \int_0^1 \partial_{\bar z} g(t,(\Theta_t^{'})^{\lambda,\varepsilon})\bar z_t^{\varepsilon,1}\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\partial_{\mu_4} g(t,(\Theta_t^{'})^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}})\widetilde{\bar z_t^{\varepsilon,1}}\right]\mathrm{d}\lambda + G_t^{\varepsilon,1} + G_t^{\varepsilon,2},
\end{aligned}$$

where for $\psi = v, x, y, z, \bar z$, and $j = 1,2,3,4,5$, we denote

$$\begin{aligned}
\Delta g_\psi^{\lambda,\varepsilon}(t) :=& \partial_\psi g(t,(\Theta_t^{'})^{\lambda,\varepsilon}) - \partial_\psi g(t,\Theta_t^{'}),\\
\Delta g_{\mu_j}^{\lambda,\varepsilon}(t) :=& \partial_{\mu_j} g(t,(\Theta_t^{'})^{\lambda,\varepsilon})(\widetilde{\alpha_t^{\lambda,\varepsilon}}) - \partial_{\mu_j} g(t,\Theta_t^{'})(\tilde\alpha_t),\\
\Delta_1^\varepsilon(t) :=& \int_0^1 \left(|\Delta g_v^{\lambda,\varepsilon}(t)| + |\Delta g_x^{\lambda,\varepsilon}(t)| + |\Delta g_y^{\lambda,\varepsilon}(t)| + |\Delta g_z^{\lambda,\varepsilon}(t)| + |\Delta g_{\bar z}^{\lambda,\varepsilon}(t)|\right)\mathrm{d}\lambda,\\
\Delta_2^\varepsilon(t) :=& \int_0^1 \sum_{j=1}^5 \left(\tilde{\mathbb{E}}|\Delta g_{\mu_j}^{\lambda,\varepsilon}(t)|^2\right)^{1/2}\mathrm{d}\lambda,
\end{aligned}$$

and

$$\begin{aligned}
G_t^{\varepsilon,1} :=& \int_0^1 \left[\Delta g_v^{\lambda,\varepsilon}(t)v_t + \Delta g_x^{\lambda,\varepsilon}(t)x_t^1 + \Delta g_y^{\lambda,\varepsilon}(t)y_t^1 + \Delta g_z^{\lambda,\varepsilon}(t)z_t^1 + \Delta g_{\bar z}^{\lambda,\varepsilon}(t)\bar z_t^1\right]\mathrm{d}\lambda\\
\le& \Delta_1^\varepsilon(t)\left(|v_t| + |x_t^1| + |y_t^1| + |z_t^1| + |\bar z_t^1|\right),
\end{aligned}$$

and

$$\begin{aligned}
G_t^{\varepsilon,2} :=& \int_0^1 \tilde{\mathbb{E}}\left[\Delta g_{\mu_1}^{\lambda,\varepsilon}(t)\tilde x_t^1 + \Delta g_{\mu_2}^{\lambda,\varepsilon}(t)\tilde y_t^1 + \Delta g_{\mu_3}^{\lambda,\varepsilon}(t)\tilde z_t^1 + \Delta g_{\mu_4}^{\lambda,\varepsilon}(t)\tilde{\bar z}_t^1 + \Delta g_{\mu_5}^{\lambda,\varepsilon}(t)\tilde v_t^1\right]\mathrm{d}\lambda\\
\le& \Delta_2^\varepsilon(t)\left(\left(\tilde{\mathbb{E}}|\tilde x_t^1|^2\right)^{1/2} + \left(\tilde{\mathbb{E}}|\tilde y_t^1|^2\right)^{1/2} + \left(\tilde{\mathbb{E}}|\tilde z_t^1|^2\right)^{1/2} + \left(\tilde{\mathbb{E}}|\tilde{\bar z}_t^1|^2\right)^{1/2} + \left(\tilde{\mathbb{E}}|\tilde v_t|^2\right)^{1/2}\right).
\end{aligned}$$

Consequently, with the help of assumption (H.1), we have

$$\begin{aligned}
|v_t^G|^2 \le& C(|x_t^{\varepsilon,1}| + |y_t^{\varepsilon,1}| + |z_t^{\varepsilon,1}| + |\bar z_t^{\varepsilon,1}|)^2 + C\mathbb{E}\left[|x_t^{\varepsilon,1}|^2 + |y_t^{\varepsilon,1}|^2 + |z_t^{\varepsilon,1}|^2 + |\bar z_t^{\varepsilon,1}|^2\right]\\
&+ C|\Delta_1^\varepsilon(t)|^2(|x_t^1| + |y_t^1| + |z_t^1| + |\bar z_t^1| + |v_t|)^2\\
&+ C|\Delta_2^\varepsilon(t)|^2\mathbb{E}(|x_t^1|^2 + |y_t^1|^2 + |z_t^1|^2 + |\bar z_t^1|^2 + |v_t|^2).
\end{aligned} \tag{A.22}$$

Similarly, for v_T^Φ, we have

$$v_T^\Phi = \int_0^1 \partial_x\Phi(x_T,\mathcal{L}(x_T))x_T^{\varepsilon,1}\mathrm{d}\lambda + \int_0^1 \tilde{\mathbb{E}}\left[\partial_{\mu_1}\Phi(x_T,\mathcal{L}(x_T))(\widetilde{x_T^{\lambda,\varepsilon}})\widetilde{x_T^{\varepsilon,1}}\right]\mathrm{d}\lambda + \Phi_t^{\varepsilon,1} + \Phi_t^{\varepsilon,2},$$

where

$$\begin{aligned}
\Phi_t^{\varepsilon,1} &:= \int_0^1 \left[\partial_x \Phi(x_T^{\lambda,\varepsilon}, \mathcal{L}(x_T^{\lambda,\varepsilon})) - \partial_x \Phi(x_T, \mathcal{L}(x_T))\right] x_T^1 \mathrm{d}\lambda, \\
\Phi_t^{\varepsilon,2} &:= \int_0^1 \tilde{\mathbb{E}}\left[\left(\partial_{\mu_1} \Phi(x_T^{\lambda,\varepsilon}, \mathcal{L}(x_T^{\lambda,\varepsilon}))(\widetilde{x_T^{\lambda,\varepsilon}}) - \partial_{\mu_1} \Phi(x_T, \mathcal{L}(x_T))(\tilde{x}_T)\right) \tilde{x}_T^1\right] \mathrm{d}\lambda.
\end{aligned}$$

With the help of assumption (H.1), we can show that

$$|v_T^\Phi|^2 \le C|x_T^{\varepsilon,1}|^2 + C\mathbb{E}|x_T^{\varepsilon,1}|^2 + C|\Delta_3^\varepsilon|^2|x_T^1|^2 + C|\Delta_4^\varepsilon|^2\mathbb{E}|x_T^1|^2, \tag{A.23}$$

where

$$\begin{aligned}
\Delta_3^\varepsilon &:= \int_0^1 \left|\partial_x \Phi(x_T^{\lambda,\varepsilon}, \mathcal{L}(x_T^{\lambda,\varepsilon})) - \partial_x \Phi(x_T, \mathcal{L}(x_T))\right| \mathrm{d}\lambda, \\
\Delta_4^\varepsilon &:= \int_0^1 \left[\tilde{\mathbb{E}}\left|\partial_{\mu_1} \Phi(x_T^{\lambda,\varepsilon}, \mathcal{L}(x_T^{\lambda,\varepsilon}))(\widetilde{x_T^{\lambda,\varepsilon}}) - \partial_{\mu_1} \Phi(x_T, \mathcal{L}(x_T))(\tilde{x}_T)\right|^2\right]^{1/2} \mathrm{d}\lambda.
\end{aligned}$$

Step 1. We first show that there exists a constant C which does not depend on ε such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |y_t^{\varepsilon,1}|^2 + \int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right] \le C\mathbb{E}\left(|I_0|^2 + |I_1|^2 + |I_2|^2\right), \tag{A.24}$$

where

$$\begin{aligned}
I_0 &:= |x_T^{\varepsilon,1}| + \left(\mathbb{E}|x_T^{\varepsilon,1}|^2\right)^{1/2} + |\Delta_3^\varepsilon||x_T^1| + |\Delta_4^\varepsilon|\left(\mathbb{E}|x_T^1|^2\right)^{1/2}, \\
I_1 &:= \int_0^T |\Delta_1^\varepsilon(t)|(|x_t^1| + |y_t^1| + |z_t^1| + |\bar{z}_t^1| + |v_t|)\mathrm{d}t + \int_0^T |x_t^{\varepsilon,1}|\mathrm{d}t + \int_0^T (\mathbb{E}|x_t^{\varepsilon,1}|^2)^{\frac{1}{2}}\mathrm{d}t, \\
I_2 &:= \int_0^T |\Delta_2^\varepsilon(t)|\left[(\mathbb{E}|x_t^1|^2)^{\frac{1}{2}} + (\mathbb{E}|y_t^1|^2)^{\frac{1}{2}} + (\mathbb{E}|z_t^1|^2)^{\frac{1}{2}} + (\mathbb{E}|\bar{z}_t^1|^2)^{\frac{1}{2}} + (\mathbb{E}|v_t|^2)^{\frac{1}{2}}\right]\mathrm{d}t.
\end{aligned} \tag{A.25}$$

To prove this, we recall (A.21) and apply Itô's formula to $|y_t^{\varepsilon,1}|^2$, then we have for any $0 \le r \le T$,

$$|y_r^{\varepsilon,1}|^2 + \int_r^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t = |v_T^\Phi|^2 + 2\int_r^T y_t^{\varepsilon,1}v_t^G\mathrm{d}t - 2\int_r^T y_t^{\varepsilon,1}(z_t^{\varepsilon,1}\mathrm{d}W_t + \bar{z}_t^{\varepsilon,1}\mathrm{d}Y_t). \tag{A.26}$$

Thus

$$\mathbb{E}|y_r^{\varepsilon,1}|^2 + \mathbb{E}\int_r^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t = \mathbb{E}|v_T^\Phi|^2 + 2\mathbb{E}\int_r^T y_t^{\varepsilon,1}v_t^G\mathrm{d}t. \tag{A.27}$$

From (A.26) and BDG inequality, we have

$$\begin{aligned}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^2 \le{}& \mathbb{E}|v_T^{\Phi}|^2 + C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}||v_t^G|\mathrm{d}t \\ &+ C\mathbb{E}\left(\int_r^T |y_t^{\varepsilon,1}|^2|z_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{1/2} + C\mathbb{E}\left(\int_r^T |y_t^{\varepsilon,1}|^2|\bar{z}_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{1/2}.\end{aligned} \tag{A.28}$$

By noticing (A.22), and using

$$\mathbb{E}\int_r^T |y_t^{\varepsilon,1}||\psi_t|\mathrm{d}t \le \frac{\kappa}{2}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^2 + \frac{1}{2\kappa}\mathbb{E}\left(\int_r^T|\psi_t|\mathrm{d}t\right)^2,$$

we have

$$\begin{aligned}\mathbb{E}\int_r^T |y_t^{\varepsilon,1}||v_t^G|\mathrm{d}t \le{}& C\kappa\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^2 + \frac{C}{\kappa}\mathbb{E}\left(|I_1|^2+|I_2|^2\right) + \frac{C}{\kappa}\left(\int_0^T(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac12}\mathrm{d}t\right)^2 \\ &+ C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^2\mathrm{d}t + \frac{C}{\kappa}\mathbb{E}\left(\int_r^T(|z_t^{\varepsilon,1}|+|\bar{z}_t^{\varepsilon,1}|)\mathrm{d}t\right)^2 \\ &+ \frac{C}{\kappa}\left(\int_r^T(\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac12}\mathrm{d}t\right)^2 + \frac{C}{\kappa}\left(\int_r^T(\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac12}\mathrm{d}t\right)^2.\end{aligned} \tag{A.29}$$

Then by using

$$\mathbb{E}\left(\int_r^T|y_t^{\varepsilon,1}|^2|\phi_t|^2\mathrm{d}t\right)^{1/2} \le \frac{\kappa}{2}\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^2 + \frac{1}{2\kappa}\mathbb{E}\int_r^T|\phi_t|^2\mathrm{d}t,$$

and $\left(\int_r^T(\mathbb{E}|\varphi_t|^2)^{\frac12}\mathrm{d}t\right)^2 \le 2T\mathbb{E}\int_r^T|\varphi_t|^2\mathrm{d}t$, and then by taking κ small enough, we imply from (A.23), (A.28) and (A.29) that

$$\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^2 \le C\mathbb{E}\left(|I_0|^2+|I_1|^2+|I_2|^2+\int_0^T(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right) + C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^2\mathrm{d}t,$$

and then Gronwall's inequality yields

$$\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^2 \le C\mathbb{E}\left(|I_0|^2+|I_1|^2+|I_2|^2+\int_0^T(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right). \tag{A.30}$$

On the other hand, applying following inequality for $\psi_t = |\Delta_1^\varepsilon(t)|\sum_{a=x^1,y^1,z^1,\bar{z}^1,v}|a_t|$, $|\Delta_2^\varepsilon(t)|\sum_{a=x^1,y^1,z^1,\bar{z}^1,v}(\mathbb{E}|a_t|^2)^{\frac12}$, $|x_t^{\varepsilon,1}|$ and $(\mathbb{E}|x_t^{\varepsilon,1}|^2)^{\frac12}$,

$$\mathbb{E}\int_r^T|y_t^{\varepsilon,1}||\psi_t|\mathrm{d}t \le \mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|\int_r^T|\psi_t|\mathrm{d}t,$$

and for $\phi = z^{\varepsilon,1}, \bar{z}^{\varepsilon,1}, (\mathbb{E}|z^{\varepsilon,1}|^2)^{\frac{1}{2}}, (\mathbb{E}|\bar{z}^{\varepsilon,1}|^2)^{\frac{1}{2}}, (\mathbb{E}|y^{\varepsilon,1}|^2)^{\frac{1}{2}}$, (noting that $2ab \leq \kappa a^2 + \frac{1}{\kappa}b^2$)

$$\mathbb{E}\int_r^T |y_t^{\varepsilon,1}||\phi_t|\mathrm{d}t \leq \frac{1}{2\kappa}\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^2\mathrm{d}t + \frac{\kappa}{2}\mathbb{E}\int_r^T |\phi_t|^2\mathrm{d}t,$$

we have, by noticing (A.22),

$$\begin{aligned}\mathbb{E}\int_r^T |y_t^{\varepsilon,1}||v_t^G|\mathrm{d}t \leq & C\mathbb{E}\left[\sup_{r\leq t\leq T}|y_t^{\varepsilon,1}|\left(|I_1|+|I_2|\right)\right] + C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^2\mathrm{d}t \\ & + \kappa C\mathbb{E}\int_r^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t.\end{aligned} \tag{A.31}$$

From (A.23), (A.27) and (A.31) and by taking κ small enough, we have that

$$\begin{aligned}&\mathbb{E}|y_r^{\varepsilon,1}|^2 + \mathbb{E}\int_r^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t \\ \leq & C\mathbb{E}|I_0|^2 + C\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|\left(|I_1|+|I_2|\right)\right] + C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^2\mathrm{d}t\end{aligned}$$

and Gronwall's inequality yields that,

$$\sup_{0\leq t\leq T}\mathbb{E}|y_t^{\varepsilon,1}|^2 + \mathbb{E}\int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t \leq C\mathbb{E}|I_0|^2 + C\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|\left(|I_1|+|I_2|\right)\right].$$

By using $2ab \leq \kappa a^2 + \frac{1}{\kappa}b^2$ once again, we get for arbitrary $\kappa > 0$,

$$\sup_{0\leq t\leq T}\mathbb{E}|y_t^{\varepsilon,1}|^2 + \mathbb{E}\int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t \leq \kappa\mathbb{E}\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^2 + C\mathbb{E}(|I_0|^2 + |I_1|^2 + |I_2|^2). \tag{A.32}$$

By combining (A.30) and (A.32) and by choosing $\kappa > 0$ small enough we can obtain (A.24).
Step 2. Now, let us prove (3.7). Noticing (A.24), it only need to show that

$$\lim_{\varepsilon\to 0}\mathbb{E}(|I_0|^2 + |I_1|^2 + |I_2|^2) = 0.$$

On the one hand, from assumption (H.1), it follows that there exists a constant C independent of ε such that

$$\Delta_1^\varepsilon(t) + \Delta_2^\varepsilon(t) + \Delta_3^\varepsilon + \Delta_4^\varepsilon \leq C,$$

and by recalling (A.25) and the integrability of $x^1, y^1, z^1, \bar{z}^1, v, x^{\varepsilon,1}$ (see *e.g.* (A.18)), there exists a constant C independent of ε such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^2 + \int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right] \leq C. \tag{A.33}$$

Moreover, standard arguments for mean-field SDEs and BSDEs, (see *e.g.* Thm. 4.21 of [17] for SDEs and Lem. 3.1 of [13] for BSDEs), allow us to show the following continuous dependence on the paramaters/coefficients

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}(|X_t^\varepsilon - X_t|^2 + |y_t^\varepsilon - y_t|^2) + \int_0^T (|z_t^\varepsilon - z_t|^2 + |\bar{z}_t^\varepsilon - \bar{z}_t|^2)\mathrm{d}t\right] = 0.$$

Then, from the continuity assumption of $\partial_x g, \partial_v g, \partial_{\mu_j} g$, we have that (here $\mathbb{L}$ is the Lebsegue measure),

$$\begin{aligned}
&\Delta_1^\varepsilon(t) + \Delta_2^\varepsilon(t) \xrightarrow{\mathbb{P}\times\mathbb{L}} 0, \text{ as } \varepsilon \to 0,\\
&\Delta_3^\varepsilon + \Delta_4^\varepsilon \xrightarrow{\mathbb{P}} 0, \text{ as } \varepsilon \to 0.
\end{aligned}$$

By Lemma 3.2, the integrability of $x^1(\cdot), y^1(\cdot), z^1(\cdot), \bar{z}^1(\cdot), v(\cdot)$ and dominated convergence theorem, we can obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}(|I_0|^2 + |I_1|^2 + |I_2|^2) = 0.$$

A.3 Proof of Lemma 3.4

We mention that since g depends on the law of $z(\cdot), \bar{z}(\cdot)$, it will be a little complicated to obtain the related L^p estimate. This is mainly due to the fact that it will appear simultaneously the terms $\mathbb{E}\int_0^T |y_t|^{p-1}(\mathbb{E}|z_t|^2)^{\frac12}\mathrm{d}t$ and $\mathbb{E}\int_0^T |y_t|^{p-2}|z_t|^2\mathrm{d}t$. In fact, to give the L^p estimate of $y(\cdot), z(\cdot), \bar{z}(\cdot)$, usually we apply Itô's formula to $|y_t|^p$ (see *e.g.* Thm. 4.4.4 of Zhang [52] for the case without mean-field term), which will yield the terms $\mathbb{E}\int_0^T |y_t|^{p-2}|z_t|^2\mathrm{d}t$ and $\mathbb{E}\int_0^T |y_t|^{p-1}(\mathbb{E}|z_t|^2)^{\frac12}\mathrm{d}t$, see (A.36) and (A.38) below. With subtle analysis on such terms, we need first establish the L^2 estimate and then use it to obtain the desired L^p estimates.

We use the same notations in the proof of Lemma 3.2. Recalling (A.21) and applying Itô's formula to $|y_t^{\varepsilon,1}|^p$, where $2\le p\le 4$, (note that for simplicity, we assumed that y is one-dimensional, so the mapping $y\mapsto |y|^p$ belongs to $C^2(\mathbb{R};\mathbb{R})$. For multidimensional case, one should first consider $|\zeta + y_t^{\varepsilon,1}|^p$, and then let $\zeta\to 0$), we have for any $0\le r\le T$,

$$\begin{aligned}
&|y_r^{\varepsilon,1}|^p + \frac{p(p-1)}{2}\int_r^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\\
=&|v_T^\Phi|^p + p\int_r^T |y_t^{\varepsilon,1}|^{p-2}y_t^{\varepsilon,1}v_t^G\mathrm{d}t - p\int_r^T |y_t^{\varepsilon,1}|^{p-2}y_t^{\varepsilon,1}(z_t^{\varepsilon,1}\mathrm{d}W_t + \bar{z}_t^{\varepsilon,1}\mathrm{d}Y_t).
\end{aligned}\tag{A.34}$$

Then we have

$$\mathbb{E}|y_r^{\varepsilon,1}|^p + \frac{p(p-1)}{2}\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t = \mathbb{E}|v_T^\Phi|^p + p\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-2}y_t^{\varepsilon,1}v_t^G\mathrm{d}t. \tag{A.35}$$

From (A.34), and with the help of BDG inequality, we have

$$\begin{aligned}
\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^p \le{}& \mathbb{E}|v_T^\Phi|^p + C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-1}|v_t^G|\mathrm{d}t\\
&+ C\mathbb{E}\left(\int_r^T |y_t^{\varepsilon,1}|^{2p-2}|z_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{1/2} + C\mathbb{E}\left(\int_r^T |y_t^{\varepsilon,1}|^{2p-2}|\bar{z}_t^{\varepsilon,1}|^2\mathrm{d}t\right)^{1/2}.
\end{aligned}\tag{A.36}$$

By noticing (A.22), the term $\mathbb{E}\int_0^T |y_t|^{p-1}(\mathbb{E}|z_t|^2)^{\frac{1}{2}}\mathrm{d}t$ appear in the right hand side of (A.36).

We will first show that for any $2 \le p \le 4$,

$$\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + \left(\int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}\right] \le C\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right), \tag{A.37}$$

where I_0, I_1, I_2 defined by (A.25).

On the one hand, using similar arguments as in Step 1 of the proof of Lemma 3.3, we apply following inequality for $\psi_t = |\Delta_1^\varepsilon(t)|\sum_{a=x^1,y^1,z^1,\bar{z}^1,v}|a_t|,\ |\Delta_2^\varepsilon(t)|\sum_{a=x^1,y^1,z^1,\bar{z}^1,v}(\mathbb{E}|a_t|^2)^{\frac{1}{2}},\ |x_t^{\varepsilon,1}|,\ (\mathbb{E}|x_t^{\varepsilon,1}|^2)^{\frac{1}{2}}$ and $|\sum_{a=x^1,y^1,z^1,\bar{z}^1,v}(\mathbb{E}|a_t|^2)^{\frac{1}{2}}$, (noting the Young's inequality yields $ab = (a\kappa)(\frac{b}{\kappa}) \le \frac{p-1}{p}(a\kappa)^{\frac{p}{p-1}} + \frac{b^p}{p\kappa^p}$)

$$\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-1}|\psi_t|\mathrm{d}t \le \frac{p-1}{p}\kappa^{\frac{p}{p-1}}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^p + \frac{1}{p\kappa^p}\mathbb{E}\left(\int_r^T|\psi_t|\mathrm{d}t\right)^p$$

and for $\phi = z^{\varepsilon,1}, \bar{z}^{\varepsilon,1}$

$$\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-1}|\phi_t|\mathrm{d}t \le \frac{\kappa}{2}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^p + \frac{1}{2\kappa}\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-2}|\phi_t|^2\mathrm{d}t$$

and

$$\mathbb{E}\left(\int_0^T |y_t^{\varepsilon,1}|^{2p-2}|\phi_t|^2\mathrm{d}t\right)^{1/2} \le \frac{\kappa}{2}\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + \frac{1}{2\kappa}\mathbb{E}\left(\int_0^T |y_t^{\varepsilon,1}|^{p-2}|\phi_t|^2\mathrm{d}t\right),$$

choosing κ small enough, we obtain from (A.22), (A.23) and (A.36) that

$$\begin{aligned}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^p \le& C\mathbb{E}\left[|I_0|^p + |I_1|^p + |I_2|^p\right] + C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^p\mathrm{d}t\\ &+ C\left(\int_r^T \left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac{1}{2}} + (\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac{1}{2}} + (\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac{1}{2}}\right]\mathrm{d}t\right)^p\\ &+ C\mathbb{E}\int_r^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t.\end{aligned} \tag{A.38}$$

Noticing that from (A.24), we have

$$\begin{aligned}&\left(\int_r^T \left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac{1}{2}} + (\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac{1}{2}} + (\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac{1}{2}}\right]\mathrm{d}t\right)^p\\ \le& C\left(\mathbb{E}\int_r^T (|y_t^{\varepsilon,1}|^2 + |z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}\\ \le& C\left[\mathbb{E}\left(|I_0|^2 + |I_1|^2 + |I_2|^2\right)\right]^{p/2} \le C\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right),\end{aligned}$$

thus (A.38) yields that

$$\begin{aligned}\mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^p \le & C\mathbb{E}\left[|I_0|^p+|I_1|^p+|I_2|^p\right]+C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^p\mathrm{d}t\\ &+C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t.\end{aligned}$$

and using Gronwall's inequality, we obtain

$$\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p \le C\mathbb{E}\left[|I_0|^p+|I_1|^p+|I_2|^p\right]+C\mathbb{E}\int_0^T|y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t. \tag{A.39}$$

On the other hand, by using

$$\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-1}|\psi_t|\mathrm{d}t \le \mathbb{E}\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\int_r^T|\psi_t|\mathrm{d}t,$$

and for $\phi=z^{\varepsilon,1},\bar{z}^{\varepsilon,1}$, (noting that $2a^{p-1}b=2a^{\frac{p}{2}}(a^{\frac{p-2}{2}}b)\le\kappa a^p+\frac{1}{\kappa}a^{p-2}b^2$)

$$\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-1}|\phi_t|\mathrm{d}t \le \frac{1}{2\kappa}\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^p\mathrm{d}t+\frac{\kappa}{2}\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-2}|\phi_t|^2\mathrm{d}t,$$

from (A.35), we have

$$\begin{aligned}\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-1}|v_t^G|\mathrm{d}t \le\, & C\mathbb{E}\left[\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\left(|I_1|+|I_2|\right)\right]+C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^p\mathrm{d}t\\ &+C\mathbb{E}\left[\sup_{r\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\int_r^T\left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac12}+(\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac12}+(\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac12}\right]\mathrm{d}t\right]\\ &+\kappa C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t.\end{aligned} \tag{A.40}$$

From (A.23), (A.35) and (A.40) and by taking κ small enough, we have

$$\begin{aligned}&\mathbb{E}|y_r^{\varepsilon,1}|^p+\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2+|\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\\ \le & C\mathbb{E}\int_r^T|y_t^{\varepsilon,1}|^p\mathrm{d}t+C\mathbb{E}|I_0|^p+C\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\left(|I_1|+|I_2|\right)\right]\\ &+C\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\int_0^T\left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac12}+(\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac12}+(\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac12}\right]\mathrm{d}t\right],\end{aligned}$$

and Gronwall's inequality yields that, for any $0 \le r \le T$,

$$\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}|y_t^{\varepsilon,1}|^p + \mathbb{E}\int_0^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t \\
\le& C\mathbb{E}|I_0|^p + C\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\left(|I_1|+|I_2|\right)\right] \\
&+ C\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\int_0^T\left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac12} + (\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac12} + (\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac12}\right]\mathrm{d}t\right].
\end{aligned} \tag{A.41}$$

By using $ab = (\kappa a)(b/\kappa) \le \frac{(\kappa a)^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{(b/\kappa)^p}{p}$, we get for arbitrary $\kappa > 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\left(|I_1|+|I_2|\right)\right] \le \kappa\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + C_\kappa\mathbb{E}\left(|I_1|^p + |I_2|^p\right), \tag{A.42}$$

and by noticing (A.24), we also have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^{p-1}\int_0^T\left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac12} + (\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac12} + (\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac12}\right]\mathrm{d}t\right] \\
\le& \kappa\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + C_\kappa\mathbb{E}\left(\int_0^T\left[(\mathbb{E}|y_t^{\varepsilon,1}|^2)^{\frac12} + (\mathbb{E}|z_t^{\varepsilon,1}|^2)^{\frac12} + (\mathbb{E}|\bar{z}_t^{\varepsilon,1}|^2)^{\frac12}\right]\mathrm{d}t\right)^p \\
\le& \kappa\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + C_\kappa\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right).
\end{aligned} \tag{A.43}$$

Now inserting (A.42) and (A.43) into (A.41), we obtain

$$\sup_{0\le t\le T}\mathbb{E}|y_t^{\varepsilon,1}|^p + \mathbb{E}\int_0^T |y_t^{\varepsilon,1}|^{p-2}(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t \le \kappa\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p + C_\kappa\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right). \tag{A.44}$$

By combining (A.39), (A.44) and by choosing $\kappa > 0$ small enough, we can obtain

$$\mathbb{E}\sup_{0\le t\le T}|y_t^{\varepsilon,1}|^p \le C\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right). \tag{A.45}$$

Finally, setting $p = 2$ in (A.34), we have

$$\int_0^T(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t = |v_T^\Phi|^2 - |y_0^{\varepsilon,1}|^2 + 2\int_0^T y_t^{\varepsilon,1}v_t^G\mathrm{d}t - 2\int_0^T y_t^{\varepsilon,1}(z_t^{\varepsilon,1}\mathrm{d}W_t + \bar{z}_t^{\varepsilon,1}\mathrm{d}Y_t),$$

and with the help of BDG inequality and (A.45), we get

$$\begin{aligned}
\mathbb{E}\left(\int_0^T(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2} \le& C\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right) + C\mathbb{E}\left(\int_0^T|y_t^{\varepsilon,1}||v_t^G|\mathrm{d}t\right)^{p/2} \\
&+ C\mathbb{E}\left(\int_0^T|y_t^{\varepsilon,1}|^2(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/4}.
\end{aligned} \tag{A.46}$$

Let us first estimate $\mathbb{E}\left(\int_0^T |y_t^{\varepsilon,1}||v_t^G|\mathrm{d}t\right)^{p/2}$. By using

$$\int_0^T |y_t^{\varepsilon,1}||\psi_t|\mathrm{d}t \leq \frac{1}{2}\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^2 + \frac{1}{2}\left(\int_0^T |\psi_t|\mathrm{d}t\right)^2,$$

and for $\phi = z^{\varepsilon,1}, \bar{z}^{\varepsilon,1}$,

$$\int_0^T |y_t^{\varepsilon,1}||\phi_t|\mathrm{d}t \leq \frac{1}{2\kappa}\int_0^T |y_t^{\varepsilon,1}|^2\mathrm{d}t + \frac{\kappa}{2}\int_0^T |\phi_t|^2\mathrm{d}t \leq C\mathbb{E}\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^2 + \frac{\kappa}{2}\int_0^T |\phi_t|^2\mathrm{d}t,$$

we have, (by noticing (A.22) and (A.24))

$$\begin{aligned}\mathbb{E}\left(\int_0^T |y_t^{\varepsilon,1}||v_t^G|\mathrm{d}t\right)^{p/2} \leq & C\mathbb{E}\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^p + C\mathbb{E}(|I_0|^p + |I_1|^p + |I_2|^p) \\ & + \kappa C\mathbb{E}\left(\int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}.\end{aligned} \tag{A.47}$$

Now let us estimate $\mathbb{E}\left(\int_0^T |y_t^{\varepsilon,1}|^2(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/4}$. In fact, we have

$$\mathbb{E}\left(\int_0^T |y_t^{\varepsilon,1}|^2(|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/4} \leq \frac{1}{2\kappa}\mathbb{E}\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^p + \frac{\kappa}{2}\mathbb{E}\left(\int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}. \tag{A.48}$$

By inserting (A.47) and (A.48) into (A.46), and choosing κ small enough, we have

$$\mathbb{E}\left(\int_0^T (|z_t^{\varepsilon,1}|^2 + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2} \leq C\mathbb{E}\left(|I_0|^p + |I_1|^p + |I_2|^p\right) + C\mathbb{E}\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^p.$$

Then by noticing (A.45), we can obtain (A.37).

Finally, similar to Step 2 of the proof of Lemma 3.3, by using p-integrability of $x^1(\cdot), y^1(\cdot)$, $z^1(\cdot)$,$\bar{z}^1(\cdot)$, $v(\cdot)$, $x^{\varepsilon,1}(\cdot)$ (see (A.18)), we can prove that there exists a constant C independent of ε such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t^{\varepsilon,1}|^p + \left(\int_0^T (|z_t^{\varepsilon,1}|^2| + |\bar{z}_t^{\varepsilon,1}|^2)\mathrm{d}t\right)^{p/2}\right] \leq C. \tag{A.49}$$

Moreover, the continuity assumption of $\partial_x g, \partial_v g, \partial_{\mu_j} g$, and dominated convergence theorem as well Lemma 3.2 yield that

$$\lim_{\varepsilon\to 0}\mathbb{E}(|I_0|^p + |I_1|^p + |I_2|^p) = 0.$$

Finally, by noticing (A.37), the proof is complete.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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